Student Thinking with a Non-Traditional Linear Coordinate System

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This study explores different ways that linear algebra students reason with a non-traditional linear system, referred to as the Gulliver system, in a task-based clinical interview. Using the constructs of Naming and Locating developed in the conceptual framework, an a priori analysis outlines how students may engage in Locating and Naming tasks. The a priori analysis was used for data analysis as a basic framing. Students' engagement with the non-traditional linear system and the refined and extended a priori analysis will be presented. Students' adoption of their previous experience with the Cartesian coordinate system will be also discussed.

Keywords: Linear Algebra, Coordinate Systems, Conceptual Framework, RME, A priori analysis

Coordinate systems are widely used in secondary and collegiate mathematics. Students learn the Cartesian coordinate system in their early mathematics, where they associate an ordered pair (a, b) with a point and use it to graph equations in the plane. Later in Pre-Calculus and Calculus, many students progress to explore a new coordinate system, the Polar coordinate system. In Linear algebra, students encounter non-traditional linear coordinate systems that are similar to the Cartesian coordinate system but scaled and/or rotated. There are a decent number of studies which focus on student reasoning with the Polar coordinate system and how their understanding of the Cartesian coordinate system impacts their reasoning with the Polar system. (Montiel et al., 2008; Montiel et al., 2009; Montiel et al., 2012; Moore, Paoletti, & Musgrave, 2014; Sayre &Wittmann 2008) Despite the growing importance of linear algebra in STEM education (Tucker, 1993), there is a noticeable gap in study concerning student thinking of a non-traditional linear system and how students employ their understanding of the Cartesian coordinate system when engaging with a non-traditional linear system. This report foregrounds a non-traditional linear system that shares similarities with, yet is distinct from, the Cartesian coordinate system.

Literature

Some studies found that students' understanding of the Polar coordinate system is closely related to their understanding of the Cartesian coordinate system and sometimes students' familiarity with the Cartesian coordinate system delays the shift to other coordinate systems (Arcavi, 2003; Hillel & Sierpinska, 1993; Montiel et al., 2008; Montiel et al., 2009; Montiel et al., 2012; Sayre &Wittmann 2008). For example, Montiel and his colleagues found that students applied the *vertical line test* to a graph defined in the Polar coordinate system to check if it is a function over the Polar coordinate system even though the vertical line test is no longer useful. Similarly, Moore et al. (2014) found that the convention from the Cartesian coordinate system of using the ordered pair (input, output) may be problematic when constructing the Polar coordinate system which uses the reversed ordered pair (output, input).

In linear algebra, Wawro et al. (2013) created a lesson which includes a task that uses $y = x$ and $y = -3x$, as the new axes, to rename a location in a non-traditional linear system. Zandieh et al. (2017) found that students in the class using the task sequence symbolized locations in three different ways. (1) Some students renamed locations using geometry by identifying which new axis to treat as the x and y and the size and direction of a unit vector. (2) Other students used a matrix equation: setting up a matrix equation $A\boldsymbol{x}_{[a\ system]} = \boldsymbol{x}_{[another\ system]}$, solving for components in the two-by-two matrix A, and using A to convert names from a system to the other. (3) Another way that students renamed involves the idea of linear combination. Students found two vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\3\end{bmatrix}$ to match the two new axis directions of $y = x$ and $y = -3x$. Then, they determined c_1 and c_2, how much in each direction should travel along them in reaching a location in the plane. The sum of the scalar multiplications, $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\3\end{bmatrix}$, provided a coordinate pair with respect to the Cartesian system.

Other studies have discussed basis or other aspects of linear combinations but did not refer to these as coordinate systems. (Bernier & Zandieh, 2022; Bettersworth et al., 2022; Dogan, 2019; Dreyfus et al., 1999; Turgut et al., 2022; Wawro et al., 2012) Given that studies of non-standard linear coordinate systems are rare in the literature, this study intends to begin filling this absence.

Conceptual Framing

The conceptual framework was developed from the author's calculus and linear algebra textbook analysis (Author, year). It can serve as a useful framework when designing tasks that involve coordinate systems and analyzing students' mathematical activity. (Lee, year) The coordinate system framework includes two fundamental processes with representations: Naming and Locating. In Naming, a location in space is being measured following the convention imposed by a coordinate system and creates the measurement, a name. For example, a location in the 2D plane gets its name as (1,1) with the Cartesian coordinate system. On the other hand, in Locating, an existing name creates its location in space following the convention imposed by a coordinate system. An example of Locating is that the ordered pair (1,1) puts on a specific point in the Cartesian coordinate plane. Figure 1(left) illustrates that Naming and Locating are the reverse processes to each other. The two processes can be extended to represent an object with multiple coordinate systems: Re-Naming and Re-Locating. In Re-Naming, a location that has been measured by a coordinate system gets its new name measured in a new coordinate system that is laid atop the location. That is, the location previously paired with (1,1) is being renamed with $(\sqrt{2}, \frac{\pi}{4}) \approx (1.414, 0.785)$ in the same space using the Polar coordinate system (Figure 1, middle). On the other hand, in Re-Locating, an existing name creates two different locations in space depending on coordinate systems being used. The new location may appear different from the first, but they share the same name. For example, (1,1) corresponds to two locations: one defined by a horizontal and vertical distance of 1 each, and the other determined by a distance of 1 from the origin and an angle measure of 1 radian from the horizontal axis. (Figure1, right)

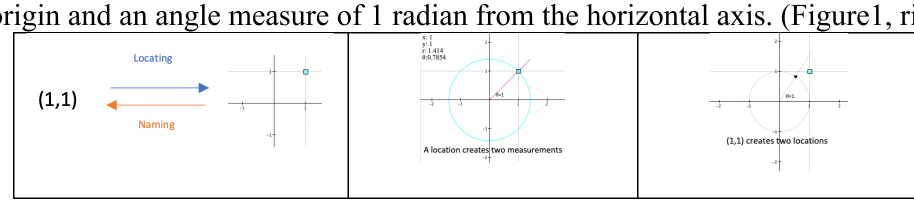


Figure 1. Naming and Locating (left), Re-Naming (middle), Re-Locating (right)

A priori analysis

Prior to conducting interviews with students, the author described an a priori analysis of how students might engage in Locating and Naming. Students' possible steps in Locating include (1) pairing known each component in an ordered pair with a proper axis, (2) identifying the location of each component by comparing it to the unit length imposed on each coordinate axis, (3) finding the intersection that comes from two locations on the axes. Reversely, in Naming, students would do (1) splitting the known location into two locations, one on the x-axis and one on the y-axis (if it is the Cartesian system), (2) measuring the length of each location by comparing it to the unit length imposed by the coordinate system, (3) expressing the measurements on the two axes symbolically. Figure 2 outlines the a priori analysis.

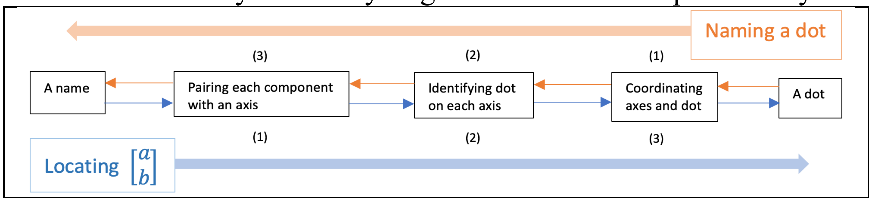


Figure 2. An a priori analysis for Locating and Naming

This study reports student reasoning with a non-traditional linear coordinate system in a task-based clinical interviews, designed to answer the research questions: (1) What are the different ways that linear algebra student reasons in a new linear coordinate system? (2) How do they employ their familiarity with the Cartesian coordinate system in working with the new system?

Methods

This proposal includes the first two tasks of a clinical interview, part of a longer dissertation study that consists of clinical interviews and teaching experiments. The clinical interview tasks were designed based on the central idea of Realistic Mathematics Education (RME), which emphasizes tasks to be experientially real starting point to students informed by Freudenthal (1991). A motive from the famous book "Gulliver's Travels" (Jonathan Swift, 1726) is combined with a treasure hunt. (Figure 3) Task 1 is a *Locating* task to place a dot with the number pair provided. Task 2 is a *Naming* task to name the treasure location. Both the tasks are built in the Gulliver system that is a new linear coordinate system different from the Cartesian system.

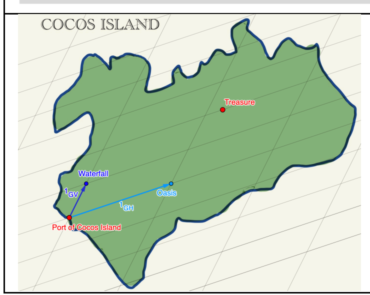
<table>
<tr><td colspan="2">Gulliver discovered an ancient treasure map of Cocos Island in his hometown.
A grid and two arrows pointing to the location of an Oasis and a Waterfall were drawn on the map. Gulliver named the arrows $1_{GH} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $1_{GV} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.</td></tr>
<tr><td rowspan="2">
</td><td>Task 1:
Place a dot on the map indicating where $A=\begin{bmatrix} 1.3 \\ 0.5 \end{bmatrix}$ is located.</td></tr>
<tr><td>Task 2:
Describe the location of the Treasure.</td></tr>
</table>

Figure 3. Problem setting with Task 1(Locating) & Task 2(Naming)

Data Source and Analytic Method

The author conducted face-to-face clinical interviews with five students who have taken linear algebra at a large public university of the Southwestern United States. The students were STEM majors who had taken Calculus 1 or 2 as a prerequisite. Both their written work and interview conversations were recorded. The interview data were transcribed into spreadsheet and coded line by line, based on the author's a priori analysis. Whenever students engaged with the steps outlined in the a priori analysis, their quotes were noted and examined to characterize their reasoning. Additionally, the steps were refined and extended, resulting in the separation of one step into two distinct steps. All the names used in Results are pseudonyms reflecting their ethnicity.

Result

As shown in Figure 3, each task has its own goal aimed at examining student reasoning that corresponds to Locating and Naming. In this section, the different ways that linear algebra student reasons with the Gulliver coordinate system within steps outlined in the a priori analysis will be explored.

Locating: Determining a location in space

The students have demonstrated their ways of determining location in space. They coordinated the components in $\begin{bmatrix}1.3\\0.5\end{bmatrix}$ with the appropriate axes and identified the intersecting point corresponding to 1.3 and 0.5 on the axes.

Pairing each component with an axis

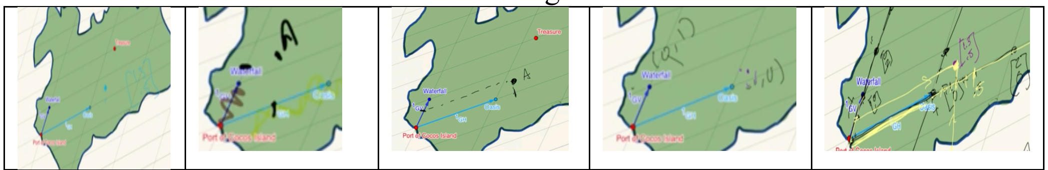
Given the components of A as 1.3 and 0.5, the participants positioned them on either the axes of the Cocos Island map or the axes separately created in a blank space. Positioning the number values requires the students' two-step commitments. First, they figure out which component of the first (1.3) or second (0.5) is paired with each direction of the horizontal or vertical axis. Second, they compare each number value to the unit length of 1. The following excerpts illustrate how they engaged with coordinating and locating on the axes.

Hann determined that the first coordinate should be in the direction of the Oasis and then it is positioned to the right of it. "*1.3, that's going to be in the Oasis direction, so that would be somewhere here* [points to the right of the 1]*"*. Wilson also attempted to mark the number values, 1.3 and 0.5, on each direction of axis, however his pairing of the first and second component with the horizontal and vertical axis went opposite. *"to go and establish where that 1.3 , 0.5 is, 1.3 roughly there* [marks on the vertical axis, above the 1] *and 0.5 there* [marks on the horizontal axis, to the left of 1]".

Jeraldo, Wanita, and Neeman marked on axes indicating where the size of the number values is located by comparing them to the size of units. Wanita said, "*I'm in going a little bit further* [than 1 for 1.3], *and then 0.5 is a little bit less from the 1*". Neeman also mentioned "*the y* [0.5] *is the midpoint here* [points on the vertical axis]…*and then so this is 1*[points to the Oasis location, the unit], *then 1.3*[marks on the horizontal axis the right of 1]."

Coordinating axes and dot; Intersecting two locations from axes. Once the students determined the location of each coordinate on the respective axis, some of them drew auxiliary lines. For instance, Jeraldo indicated a dotted line starting at 0.5 on the vertical axis and is parallel to the horizontal axis, then made some extended portion at 1.3 vertically. He finally placed a dot A by intersecting the auxiliary segments (Figure 4. Jeraldo). Neeman also indicated some auxiliary lines in yellow that pass the two locations representing 1.3 and 0.5 on the axes he marked earlier (Figure 4. Neeman). He constructed a dot for A where the two lines intersect. The written work of Jeraldo and Neeman shows that they think of the location of [1.3 0.5] as the intersection of the extended two locations from axes. Even though the other students did not draw auxiliary lines, they seemed to engage with the intersecting process in that the final dot was marked once they determine each coordinate on its corresponding axis.

I note that the auxiliary lines are not always parallel to the Gulliver coordinate system's grids; rather the vertical part of the auxiliary lines appears somewhat perpendicular to the horizontal axis. This demonstrates that the students tend to think of perpendicular grids even when it does not precisely match the appropriate projections of measurements from the axes. The students' final answers to this task are shown in Figure 4.



Hann	Wilson	Jeraldo	Wanita	Neeman

Figure 4. Location of $\begin{bmatrix}1.3\\0.5\end{bmatrix}$

Naming: Obtaining coordinates-pair

Similar to Task 1, the students utilized their understanding of the rectangular coordinate system to determine the coordinates of the Treasure location in Task 2 (Figure 3). One of the participants, Hann, seemed to first look for measurement on each axis and then coordinate them with their corresponding axis. "*The location of the treasure looks to be probably about 1.3 in that* [Oasis] *direction and then exactly 2 in the waterfall or y direction*".

Some of the participants described how to obtain the coordinates-pair from the Treasure location given on the map. I present two distinct ways that students have engaged in obtaining the coordinates: Projecting the location onto axes and Over-and-up reaching the location.

Coordinating axes and dot; Projecting a given location onto axes. Jeraldo and Neeman indicated that they need some kind of projection to obtain measurements of the Treasure location. Jeraldo noted, *"If we follow, like, the parallel line, it's almost at the same exact place* [refers to the Task 1's horizontal component, 1.3]. *So, I'd probably say it's around the same as the previous at 1.3".* From his description, he seemed to be looking at the slanted projection from the Treasure location onto the Gulliver horizontal axis, recognizing that the slanty line passes through location A that he had placed earlier with 1.3 in Task 1. Similarly, Neeman's written work depicted reasoning with a slanted projection indicated from the auxiliary lines in black (Figure 5. left). He drew the lines to pass through the Treasure location and to be parallel to the Gulliver coordinate grids. The measurements that he obtained result from the slanted projection of the Treasure location onto the two axes. "*This* [where the slanted projection meets on the horizontal axis] *is a little bit before the 1.3, that is like 1.2. And it's* [where the slanted projection meets on the vertical axis] *on this one* [points the Waterfall]*, which is 2. So, let's say it's a 1.2 over 2".*

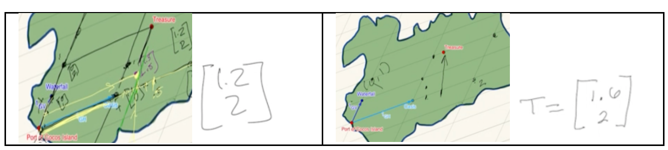


Figure 5. Written work of Neeman (left) and Wanita (right) in Naming

Coordinating axes and dot; Over-and-up reaching a location. Another student, Wanita, also demonstrated her reasoning with the rectangular coordinate system to obtain the coordinates of the Treasure. She illustrated the process of moving along the Gulliver horizontal axis and then turning up towards the Treasure location (Figure 5. right).

Wanita: It would be like 1.5 and then 2. I'm supposing the port is at the zero. It would be like 1.5, 1.6 or so. It's going up [indicates the horizontally positive direction] and then 2.

Interviewer: Why do you say 1.5 or 1.6?

Wanita: …2 is over here [points to (2, 0) location] <u>1.5 should be like, right in the middle, and then you just go up [draws an arrow from the middle to the Treasure] and that's where the treasure would be…</u>T is equal to the 1.6 and then 2.

I note that the two distinct ways of finding the coordinates-pair, slanted projection onto axes and over-and-up process, are not exclusive to each other. For example, Jeraldo, one of the students who engaged in the slanted projection, initially answered that the horizontal coordinate has to be 1.5 by over-and-up process, similar to Wanita. He was looking at the locations of one unit, two units, and midpoint on the Gulliver horizontal axis, estimating the right location to be 1.5 in order to reach the Treasure. Soon after, he realized that the slanted projection is not precisely matching the upward movement from 1.5 and switched over to the projection onto axes way from the over-and-up process.

Types of names in Naming

Once the participants determine the measurements for the Treasure location, some students attempted to represent the coordinates-pair in different ways, whether in response to my request or without any prompting. There were three different ways of representing the same coordinate found in students written expression: Vector form $\begin{bmatrix} a \\ b \end{bmatrix}$, Linear combination with opaque symbolics of 1_{GH} and 1_{GV} provided in the problem, and Linear combination with actual vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hann's written answer includes all three representations. He first wrote $\begin{bmatrix} 1.3 \\ 2 \end{bmatrix}$ in a vector form and represented it using the linear combination format employing the provided opaque symbolics, 1_{GH} and 1_{GV}. (Figure 6) He commented that the symbolics could be substituted with the actual vectors consisting of 1's and 0's.

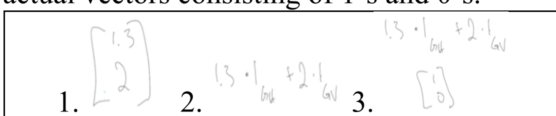
	1. Vector 2. Linear combination with opaques 3. Linear combination with actual vectors

Figure 6. Hann's symbolic representations for the same coordinates-pair

Wilson chose to represent the determined coordinates using the linear combination format as well in addition to the vector form. (Figure 7) The distinction between Wilson and Hann's representation is that Wilson made a slightly different modification to the opaque symbolics. That is, GH and GV have been used instead of 1GH and 1GV. This is an indication that Wilson conceives of the number '1' in the opaque symbolics of 1GH and 1GV as an actual measurement rather than as a symbol emphasizing a unit.

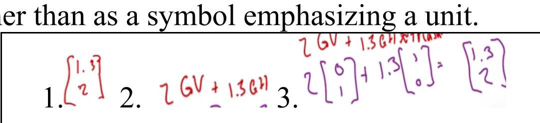
	1. Vector 2. Linear combination with opaques 3. Linear combination with actual vectors

Figure 7. Wilson's symbolic representations for the same coordinates

The other three participants, Jeraldo, Wanita, and Neeman represented the Treasure location as a vector form only using the measurements obtained from either slanted projection or over-and-up. I note that Neeman read the vector form of $\begin{bmatrix} 1.1 \\ 2 \end{bmatrix}$ "*1.1 over 2*". His treating the vector like a fraction sometimes comes a long later in the interview when he writes $\begin{bmatrix} \frac{14}{6} \end{bmatrix}$ to mean $\begin{bmatrix} 14 \\ 6 \end{bmatrix}$.

Discussion

The students have adopted their previous experience with the rectangular coordinate system to answer the first task, even though the Gulliver coordinate system is a non-Cartesian rectangular system. This aspect is well represented by one of the interview participants, Hann's comment "*Even though it's not rectangular, there's no reason not to act like it is*". The following lists different pieces of the rectangular coordinate system understanding that the participants brought to bear: identifying units as two directional line segments, labeling with 'x' and 'y', employing perpendicular axes, and pivoting a reference point. While some of them assisted the students in addressing the tasks, others were applied even though they were no longer useful.

Jeraldo called the 1_{GH} and 1_{GV} as matrices and noted that they represent one unit in each direction of x and y. The letters x and y, widely used in the mathematical community for coordinates in the rectangular coordinate system, have been used to denote the first coordinate as x and the second coordinate as y. *"I remember in back in the class that the matrices they obviously defined <u>that the first one represents the x coordinates, the second one equals the y.</u> So, x y x y* [writes x's and y's next to 1 0 and 0 1]". (Figure 8. Left) Wanita drew perpendicular axes on a blank space, and then placed two dots one on the horizontal axis and one on the vertical axis to coordinate them with $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. She indicated that these dots are pointing to the Oasis and Waterfall locations. (Figure 8. Middle) Neeman indicated the Port of Cocos Island on the map where the two axes intersect as the coordinate pair [0, 0]. That is, the reference point was identified as two components of null. He brought the letters x and y to indicate the first coordinate and the second coordinate, respectively and noted x to be corresponding to the horizontal axis and y to be the vertical axis. Additionally, he drew the perpendicular axes labeled with x and y. (Figure 8, right)

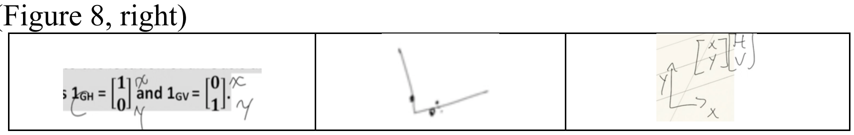


Figure 8. Employing rectangular coordinate system: Jeraldo(left), Wanita(middle), Neeman(right)

The Locating and Naming activities are reverse processes to each other. The processes were outlined in an a priori analysis in Figure 2, and it has been further refined in two ways: variations within a step and separation of one step into two distinct steps. Students have been engaged differently with the remaining steps when they progress to the subsequent set of tasks following Task 1 and 2.

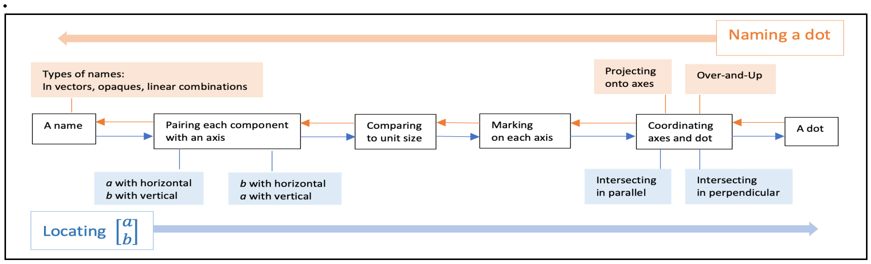


Figure 9. Locating and Naming in a linear coordinate system

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