Quantum geometric formulation of Brans-Dicke theory for Bianchi I spacetime

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We consider a formulation of the Brans-Dicke theory in Jordan's frame for Bianchi-I spacetime within the framework of loop quantum gravity. The robustness of singularity resolutions due to the quantum effects is explicitly verified in the context of two quantization schemes typically used in the literature. We present an exploration of the effects of quantum geometry on the background dynamics, which is also illustrated through some explicit numerical examples and showing the absence of the singularity.

I. INTRODUCTION

Loop quantum gravity (LQG) is a candidate of quantum gravity theory, which takes the premise of gravity as a manifestation of geometry of spacetime and systematically constructs a theory of quantum Riemanian geometry with rigor (see, e.g., Ref. [1] for a recent review). LQG stands out as a non-perturbative background independent approach to quantize gravity [2]. At its depth, this theory brings out a fundamental discreteness at Planck scale wherein the underlying geometric observables, such as areas of physical surfaces and volumes of physical regions, are discrete in nature [3–6]. At present, the studies in cosmology and black holes provide ones of the major avenues for applying and testing the ideas of the theory. The absence of the possibility to design a table top experiment, other than thought experiment, can be easily understood from the energy scale involved in LQG, which is beyond the reach of present day technology. However, it is feasible to apply LQG against observational physics in the context of cosmology, which serves a great purpose here. In fact, the early Universe is a great laboratory for this purpose, so it is relevant to investigate this epoch in this framework.

Loop quantum cosmology (LQC) is an application of LQG techniques to the symmetry reduced spacetime and for an homogeneous spacetime in particular [7]. In LQC the Big Bang singularity is resolved in the sense that it is replaced by a quantum bounce, across which physical macroscopic observables, such as energy density and curvature, which diverge at the Big Bang singularity in classical general relativity (GR), now all remain finite. This non-divergent behavior owes to the fact that the effective quantities in this scenario depends on the fundamental discrete parameters of the theory, in particular, on the fundamental area gap, whose smallest eigenvalue is nonzero [8–10]. LQC produces a contracting Friedmann-Lemaître-Robertson-Walker (FLRW) universe that bounces back to an expanding one, thus, avoiding the occurrence of a singularity. This is achieved without adding any nontrivial matter component, unlike in the case of classical or matter bounces (see, e.g., Refs. [11, 12]). This quantum bounce occurs purely due to quantum geometric effects, a novel repulsive effective force that is manifest in the quantum corrected Friedmann and Raychaudhuri equations. Also, it is to be noted that in all the different classes of spacetimes permitting different sets of symmetries [13], including the Bianchi and Gowdy models, the singularity is resolved in the framework of LQC [14–19] (for reviews, see also Refs. [20–24]).

An important issue that appears in the bouncing scenario is the possible instability in the growth of anisotropic density during the contracting phase. During contraction, the contribution from anisotropic stresses to the Friedmann equation grows much faster than the energy density of the usual fields, like radiation, baryonic matter and cold dark matter. Once the ratio of the anisotropic stresses to the total energy density becomes comparable to one, it is not possible any longer to assume that the background spacetime is approximately isotropic, and this ratio can even become larger than unity, eventually meeting an anisotropic collapse. When considering inhomogeneities, the situation is even more sensitive, since the ratio of the anisotropic stresses to the total energy density becoming larger than unity in this context can indicate the onset of the conjectured Belinski-Khalatnikov-Lifshitz (BKL) chaotic instability [25]. As it is well known, due to the high sensitivity of the dynamics to the initial conditions (neighboring points following very different dynamics), this scenario can lead to the loss of predictivity. In this case, the large anisotropies impact the density fluctuations in a highly inhomogeneous way, spoiling the prediction of a nearly isotropic cosmic microwave background (CMB) [26, 27].

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This can be avoided, for example, in some situations when the nearly scale-invariant perturbations are generated after the bounce [28]. Furthermore, in the special case of the Bianchi type I model [29], it was shown that isotropization is an attractor, which makes this scenario a natural choice to investigate viable candidates. This is especially promising in the case of LQC, where we can analyze the Bianchi I background in a bouncing scenario driven by modifications to the Einstein equations in the high energy regime. Great efforts have been devoted towards understanding the quantum corrected dynamics of anisotropic spacetime starting with the Einstein-Hilbert action (see, for example, the Refs. [30–36]).

A natural question to ask is what it will happen when one goes beyond the Einstein-Hilbert action. One of the salient features of this theory is all the geometrical quantities, such as distance, area and volume can be constructed from the knowledge of the metric. The metric and, hence, the geometry of the spacetime, being a dynamical quantity determined by the distribution of matter and energy. These fundamental properties of the theory remain intact beyond the minimally coupled Einstein-Hilbert action. This motivates us to consider an action with a non-minimal coupling term and to find the corresponding quantum corrected dynamics. As an example toward this goal, we consider the Brans-Dicke theory (BDT). In particular, the techniques of LQG are not limited to a particular action. Based on the loop quantization, some aspects of its implementation in the context of the BDT has been explored for the FLRW spacetime [37–39], and the presence of a quantum bounce is shown in both the Jordan and Einstein frames. Particularly, an anisotropic Bianchi-I spacetime was considered in LQC in Ref. [40], where a bounce dynamics was shown to emerge. One of the prime focus of the present paper is to formulate the quantum corrected effective dynamics of BDT in homogeneous and anisotropic Bianchi-I spacetime in the Jordan frame, whose study is still missing in the literature, as far as we are aware of. This is a crucial step towards establishing and checking the robustness of the major results of loop quantization in spacetimes with different symmetries and different theories of gravity. In particular, we aim to ultimately check the robustness of the singularity resolution and the consistency of the effective dynamics.

This paper is organized as follows. In Sec. II, we review the Hamiltonian formulation of BDT and present the complete constraint analysis and the associated Lie algebra. Following this, we cast the theory in connection formulation, the Asthekar variables, which is suitable for employing the loop quantization program. Our main results starts from Sec. III, where we present the symmetry reduction of the constraints. As one of our prime focus is to formulate the quantum corrected dynamics of BDT in Bianchi-I spacetime. In Sec. IV we project the full BDT dynamics in Bianchi-I spacetime at the classical level before quantization. The classical dynamics is found by taking the Poisson flow of the phase-space variables with the scalar Hamiltonian constraint. In Sec. V, we find the quantum corrected dynamics using two alternative and complementary $\bar{\mu}$ quantization schemes which have been considered in the literature in other contexts. In Sec. VI, we discuss the dynamics of the Brans-Dicke theory for Bianchi LQC models in the two quantization schemes considered in this paper. Our final considerations and conclusions are presented in Sec. VII. Throughout this work, we will adopt the convention $M_{Pl} = 1$, where $M_{Pl} = \frac{1}{8\pi G}$ is the reduced Planck mass.

II. HAMILTONIAN FORMULATION OF BRANS-DICKE THEORY AND CONSTRAINT ANALYSIS

The action for the Brans-Dicke theory is given by

$$S_{\rm BDT} = \frac{1}{2}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \quad (2.1)$$

where R is the Ricci scalar, $g_{\mu\nu}$ is the metric and g is its determinant, the scalar field ϕ is the Brans-Dicke (BD) scalar and ω is a coupling constant. The equations of motion are obtained by varying the action, Eq. (2.1), independently, with respect to the metric $g_{\mu\nu}$ and the scalar field ϕ, which, in natural units, give, respectively,

$$\begin{aligned} \phi G_{\mu\nu} &= \boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu\phi - g_{\mu\nu}\Box\phi + \\ &+ \frac{\omega}{\phi}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\boldsymbol{\nabla}\phi)^2\right], \quad (2.2)\\ R &+ \frac{2\omega}{\phi}\Box\phi - \frac{w}{\phi^2}(\partial_\mu\phi)\partial^\mu\phi = 0, \quad (2.3)\end{aligned}$$

where $R_{\mu\nu}$ is the Ricci tensor and $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. However, the present format of covariant formulations of the theory is not suitable for loop quantization. In the Lagrangian formulation, space and time are treated on equal footing, but the construction of the canonical formula demands splitting of spacetime into space and time. It is the Arnowitt-Deser-Misner (ADM) formalism [41–43] with the $3+1$ decomposition that comes to rescue the description in terms of the first-order Hamilton's equation of the phase space variables. In brief, this can be achieved by decomposing the spacetime manifold endowed with a metric, designated by the pair $(\mathcal{M}, g_{\mu\nu})$, into $(\Sigma, \mathbb{R})$, where Σ represents the spatial three-dimensional hypersurface and $\mathbb{R}$ stands for the real valued time. Algebraically, it is represented by splitting the metric as

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \quad (2.4)$$

where $q_{\mu\nu}$ is the intrinsic spatial metric that defines the hypersurface Σ and n_μ is the normal vector to the hypersurface, with the property $n_\mu n^\nu = -1$.

A careful observation of the splitting of the metric into its intrinsic spatial metric $q_{\mu\nu}$ and the normal vector n_μ

to the hypersurface naturally leads to two notions of curvature [44]: the intrinsic curvature, $\mathcal{R}$, purely defined in terms of commutator of covariant derivatives strictly sitting on the three-dimensional hypersurface, and the extrinsic curvature, $\mathbf{K}$, defined in terms of the covariant derivative of the normal vector to the hypersurface. Due to the Gauss-Codazzi equation [44], a relation between the two is given in an elegant way,

$$\begin{aligned} R &= K_{ab}K^{ab} - K^2 + \mathcal{R} + \frac{2}{\sqrt{-g}}\partial_\mu(\sqrt{-g}n^\mu K) \\ &- \frac{2}{N\sqrt{q}}\partial_a(\sqrt{q}q^{ab}\partial_b N). \end{aligned} \tag{2.5}$$

The set of corresponding conjugate momenta, in terms of the newly defined geometrical objects, for the BDT action given by Eq. (2.1), are

$$p^{ab} \equiv \frac{\delta\mathcal{L}}{\delta\dot{q}_{ab}} = \frac{\sqrt{q}}{2}\left[\phi(K^{ab} - Kq^{ab}) - \frac{q^{ab}}{N}(\dot{\phi} - N^c\partial_c\phi)\right], \tag{2.6}$$

$$\pi \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = -\sqrt{q}\left[K - \frac{w}{N\phi}(\dot{\phi} - N^c\partial_c\phi)\right], \tag{2.7}$$

$$p_N(x) \equiv \frac{\delta\mathcal{L}}{\delta\dot{N}(x)} = 0, \tag{2.8}$$

$$p_a(x) \equiv \frac{\delta\mathcal{L}}{\delta\dot{N}^a(x)} = 0. \tag{2.9}$$

Thus, given the Lagrangian, the Hamiltonian can be obtained by the Legendre transformation,

$$H_{\text{total}} = \int_\Sigma d^3x\sqrt{q}(\dot{q}_{ab}p^{ab} + \dot{\phi}\pi - \mathcal{L}(q_{ab},\dot{q}_{ab})), \tag{2.10}$$

which can be cast in the form

$$\begin{aligned} H_{\text{total}} &= \int_\Sigma d^3x\, N^a\left[-2D^b p_{ab} + \pi\partial_a\phi\right] \\ &+ \int_\Sigma d^3x\, N\left\{\frac{2}{\sqrt{q}\phi}\left[p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{(p-\pi\phi)^2}{2(3+2\omega)}\right]\right. \\ &+ \left.\frac{\sqrt{q}}{2}\left[-\phi R + \frac{w}{\phi}(D_a\phi)D^a\phi + 2D_aD^a\phi\right]\right\}. \end{aligned} \tag{2.11}$$

In terms of the Asthekar variables, with the densetized triads $E^a_i \equiv \sqrt{e}e^a_i$, curvature tensor $F^i_{ab} \equiv \partial_{[a}A^i_{b]} + \epsilon^i_{kl}A^k_l A^k_b$, with the Yang-Mills connection $A^i_a \equiv \Gamma^i_a + \gamma K^i_a$, the constraints of the BDT in the Yang-Mills phase space read

$$\mathcal{G}_i = D_a E^a_i \approx 0, \tag{2.12}$$

$$\mathcal{V}_a = \frac{1}{\gamma}F^i_{ab}E^b_i \approx 0, \tag{2.13}$$

$$\begin{aligned} \mathcal{H} &= \frac{\phi}{2}\left[F^j_{ab} - \left(\gamma^2 + \frac{1}{\phi^2}\right)\epsilon_{jmn}\tilde{K}^m_a\tilde{K}^n_b\right]\frac{\epsilon_{jkl}E^a_kE^b_l}{\sqrt{q}} \\ &+ \frac{1}{3+2\omega}\left[\frac{(\tilde{K}^i_aE^a_i)^2}{\phi\sqrt{q}} + \frac{\tilde{K}^i_aE^a_i}{\sqrt{q}}\Pi + \frac{\Pi^2\phi}{\sqrt{q}}\right] \approx 0, \end{aligned} \tag{2.14}$$

which are the Gauss, diffeomorphism and scalar constraints, respectively. In the above expressions, γ is the Barbero-Immirzi parameter and $\tilde{K}^i_a = e^b_i\tilde{K}K_{ab}$ with

$$\tilde{K}^{ab} = \phi K^{ab} + \frac{q^{ab}}{2N}(\dot{\phi} - N^c\partial_c\phi). \tag{2.15}$$

In Eqs. (2.12)-(2.14), the symbol "≈ 0" indicates that the constraints hold only on the shell (hypersurface). Thus, the total Hamiltonian of BDT can be expressed as

$$C_{\text{total}} = \int d^3x(N^a\mathcal{V}_a + N\mathcal{H} + \mathcal{G}_iN^i). \tag{2.16}$$

III. SYMMETRY REDUCTION OF BDT FOR THE BIANCHI I MODEL

The Bianchi Class A model, in general, represents all possible homogeneous spacetimes with different degrees of symmetry. The four-dimensional Bianchi spacetime has a topology of the $\Sigma \times \mathbb{R}$, where the intrinsic metric q_{ab} defined on the spatial manifold Σ is homogeneous with three-dimensional symmetry group that is simply transitive. The symmetry of the spatial hypersurface is determined by the Killing vectors. Essentially, the killing vector fields give the direction along which Lie dragging the metric does not change the metric [45]. The fiducial triads and co-triads, denoted, respectively, by $\mathring{e}^a_i$ and $\mathring{e}^i_a$, satisfy the relations $\mathring{e}^a_i\mathring{e}^i_b = \delta^a_b$ and $\mathring{e}^a_i\mathring{e}^j_a = \delta^j_i$. The geometry of Bianchi I, which is one of the simplest types of Bianchi Class A model, is such that it admits the universe to have different scale factors in each principal direction, but with zero spatial curvature.

It is to be noted that a fiducial triad is introduced here at each point of the spatial manifold to carry out the integration as Σ is non-compact and they commute with the killing vector fields denoted by $\mathring{\xi}^a_i$, $[\mathring{e}_i, \mathring{\xi}_j]^a = 0$. The underlying structure of the spatial hypersurface is dictated by the commutator relations, given by $[\mathring{e}_i, \mathring{e}_j]^a = C^k_{ij}\mathring{e}^a_k$. The values of the structure constants C^k_{ij} completely define all the symmetry of the spatial sector. The Bianchi I model considered in this work is specified by $C^i_{jk} = 0$. To be specific, the Bianchi I hypersurface Σ is flat and, therefore, $\Gamma^i_a = 0$. Also, the homogeneity implies the contribution to the Hamiltonian from the spatial derivative is identically zero. This, in turn, it greatly simplifies the expression of the curvature tensor. The diagonal Bianchi I universe in terms of the Asthekar variables and densetized triads are parameterized by

$$A_a^i \;=\; \tilde{c}_i V_o^{-1/3} \mathring{e}_i^a, \tag{3.1}$$

$$E_j^b \;=\; p_j V_o^{-2/3} \sqrt{q} \mathring{e}_j^b, \tag{3.2}$$

where there is no summation over indexes i or j. Let us note that Bianchi I is a homogeneous but anisotropic spacetime. Therefore, $\tilde{c}_i$ and p_j only depend on time. Also, with the information that there is no spatial curvature for Bianchi I, $\Gamma_a^i = 0$, we shall show that the only constraint that survives off-shell is the Hamiltonian constraint. Below, we investigate the Gauss, diffeomorphism and Hamiltonian constraints in the given context.

The Gauss constraint is

$$\mathcal{G}_i = D_a E_i^a, \tag{3.3}$$

which is identically zero because Bianchi-I is a homogeneous and, therefore, the derivative with respect to spatial coordinate vanishes identically. The diffeomorphism constraint is

$$\mathcal{V}_a \;=\; \frac{1}{\gamma} F_{ab}^i E_i^b = \frac{1}{\gamma} \epsilon_{kl}^i A_a^k A_b^l E_i^b \propto \epsilon_{kl}^i \delta_i^l. \tag{3.4}$$

Therefore, the diffeomorphism constraints vanish identically for the Bianchi I model. Thus, we are only left with the scalar (or Hamiltonian) constraint, which is given by

$$\begin{aligned}\mathcal{H} \;&=\; \frac{\phi}{2}\left[F_{ab}^j - \left(\gamma^2 + \frac{1}{\phi^2}\right)\epsilon_{jmn}\tilde{K}_a^m \tilde{K}_b^n\right]\frac{\epsilon_{jkl}E_k^a E_l^b}{\sqrt{q}} \\ &+\; \frac{1}{3+2\omega}\left[\frac{(\tilde{K}_a^i E_i^a)^2}{\phi\sqrt{q}} + 2\frac{(\tilde{K}_a^i E_i^a)\pi_\phi}{\sqrt{q}} + \frac{\pi_\phi^2}{\sqrt{q}}\right] \\ &+\; \frac{\omega}{2\phi}\sqrt{q}(D_a\phi)D^a\phi + \sqrt{q}D_a D^a \phi.\end{aligned} \tag{3.5}$$

Now, substituting the curvature tensor, $F_{ab}^j = \epsilon_{kl}^i(\Gamma_a^k + \gamma\tilde{K}_a^k)(\Gamma_b^l + \gamma\tilde{K}_b^l)$, due to the fact that Bianchi I is spatially flat, we find that $\Gamma_a^k = 0$ and, hence, we are left with

$$F_{ab}^j = \gamma^2 \epsilon_{kl}^j \tilde{K}_a^k \tilde{K}_b^l. \tag{3.6}$$

We also notice that

$$\epsilon_{jkl}\epsilon_{jmn}E_k^a E_l^b \tilde{K}_a^m \tilde{K}_b^n \;=\; 2E_m^a E_n^b \tilde{K}_{[a}^m \tilde{K}_{b]}^n, \tag{3.7}$$

where $[a,b] \equiv (ab - ba)/2$ denotes the antisymmetrization, with which, the expression for $\mathcal{H}$ can be further reduced to

$$\begin{aligned}\mathcal{H} \;&=\; -\frac{1}{\phi\sqrt{q}}\left[E_m^a E_n^b \tilde{K}_{[a}^m \tilde{K}_{b]}^n + \right. \\ &-\; \left.\frac{1}{(3+2\omega)}\left(\tilde{K}_a^i E_i^a + \pi_\phi \phi\right)^2\right].\end{aligned} \tag{3.8}$$

In the limiting case of $\omega \to \infty$ and ϕ being a constant, the scalar constraint of the BDT gets reduced to the scalar constraint of Einstein's GR in the vacuum.

The next important step is to cast the theory in terms of the Asthekar's variables. The canonical structure is given by

$$\{A_a^i(x), E_j^b(y)\} = \gamma\delta_j^i \delta_a^b \delta(x-y). \tag{3.9}$$

It is to be noted that the Bianchi-I spacetime is noncompact and in order to carry out the integrations, we consider a fiducial volume V_0. Evaluating each term of the Hamiltonian individually by using the parameterization of Eq. (3.1), we have

$$\begin{aligned}E_m^a E_n^b \tilde{K}_{[a}^m \tilde{K}_{b]}^n \;&=\; \frac{\mathring{q}}{\gamma^2 l_0^6}\left(\tilde{c}_1\tilde{c}_2 p_1 p_2 + \tilde{c}_2 p_2 \tilde{c}_3 p_3\right. \\ &+\; \left.\tilde{c}_1 p_1 \tilde{c}_3 p_3\right),\end{aligned} \tag{3.10}$$

$$\tilde{K}_a^i E_i^a \;=\; \frac{\sqrt{\mathring{q}}}{\gamma l_0^3}(\tilde{c}_1 p_1 + \tilde{c}_2 p_2 + \tilde{c}_3 p_3). \tag{3.11}$$

Now, combining all the terms and using that $\sqrt{q} = \sqrt{\mathring{q}}\sqrt{p_1 p_2 p_3}/l_0^3$ in Eq. (3.8), we find

$$\begin{aligned}\mathcal{H} \;&=\; -\frac{\sqrt{\mathring{q}}}{l_0^3}\frac{1}{\sqrt{p_1 p_2 p_3}}\frac{1}{\gamma^2\phi}\Big[\tilde{c}_1\tilde{c}_2 p_1 p_2 + \tilde{c}_2 p_2 \tilde{c}_3 p_3 + \tilde{c}_1 p_1 \tilde{c}_3 p_3 \\ &-\; \frac{1}{\beta}\left(\tilde{c}_1 p_1 + \tilde{c}_2 p_2 + \tilde{c}_3 p_3 + \gamma\pi_\phi\phi\right)^2\Big],\end{aligned} \tag{3.12}$$

where we have redefined the conjugate momentum corresponding to the scalar field as $\pi_\phi \equiv p_\phi\sqrt{\mathring{q}}/l_0^3$ and $\beta = 3 + 2\omega$. The smeared Hamiltonian is obtained by integrating the symmetry reduced Hamiltonian, Eq. (3.12), over the fiducial volume V_0, leading to

$$\begin{aligned}C_{\mathcal{H}} \;&=\; \int d^3x N\mathcal{H} \\ &=\; -\frac{N}{\sqrt{p_1 p_2 p_3}}\frac{1}{\gamma^2\phi}\Big[\tilde{c}_1\tilde{c}_2 p_1 p_2 + \tilde{c}_2 p_2 \tilde{c}_3 p_3 + \tilde{c}_1 p_1 \tilde{c}_3 p_3 \\ &-\; \frac{1}{\beta}\left(\tilde{c}_1 p_1 + \tilde{c}_2 p_2 + \tilde{c}_3 p_3 + \gamma\pi_\phi\phi\right)^2\Big],\end{aligned} \tag{3.13}$$

where we have used that $\int d^3x\sqrt{\mathring{q}} = l_0^3$. The Poisson flow of a Dirac observable $\dot{\mathcal{O}} = \{\mathcal{O}, C_H\}$ with the Hamiltonian gives the equation of motion.

IV. CLASSICAL DYNAMICS OF BDT IN BIANCHI-I MODEL

The symplectic structure of the complete phase space in terms of the new variables reads

$$\{\tilde{c}_I, p_J\} \;=\; \gamma\delta_{IJ}, \tag{4.1}$$

$$\{\phi, \pi_\phi\} \;=\; 1. \tag{4.2}$$

The dynamics is given by the Poisson flow of the variable with the smeared Hamiltonian. Then, the equations of motion for c_I and its conjugate variables p_I are

$$\dot{\tilde{c}}_I \;=\; \gamma\,\delta_{IJ}\frac{\delta C_{\mathcal{H}}}{\delta p_J}, \tag{4.3}$$

$$\dot{p}_I \;=\; -\gamma\,\delta_{IJ}\frac{\delta C_{\mathcal{H}}}{\delta \tilde{c}_J}, \tag{4.4}$$

whereas the equations of motion for ϕ and its conjugate variable π_ϕ are

$$\dot{\phi} = \frac{\delta C_{\mathcal{H}}}{\delta \pi_\phi}, \tag{4.5}$$

$$\dot{\pi}_\phi = -\frac{\delta C_{\mathcal{H}}}{\delta \phi}. \tag{4.6}$$

The cosmic time interval can be written as $dt = N dt'$, whereas the time variable t' will be determined by the choice of the lapse function N. In this work, we choose $N = \sqrt{p_1 p_2 p_3}$, then we have

$$dt = \sqrt{p_1 p_2 p_3} dt'. \tag{4.7}$$

Using the Hamiltonian constraint, Eq. (3.13), we can now derive the equations of motion. These will be obtained with respect to the time t' using Eq. (4.7). From Eq. (4.3), we obtain the following equations of motion for the c_I variables,

$$\begin{aligned} \tilde{c}_I' &= -\frac{\tilde{c}_I}{\gamma\phi}\Big[\tilde{c}_J p_J + \tilde{c}_K p_K \\ &- \frac{2}{\beta}\left(\tilde{c}_I p_I + \tilde{c}_J p_J + \tilde{c}_K p_K + \gamma \pi_\phi \phi\right)\Big], \end{aligned} \tag{4.8}$$

where $'$ refers to the derivative with respect to t'. Similarly, from Eq. (4.4), we obtain the equations of motion for the conjugate variables p_I,

$$\begin{aligned} p_I' &= \frac{p_I}{\gamma\phi}\Big[\tilde{c}_J p_J + \tilde{c}_K p_K \\ &- \frac{2}{\beta}\left(\tilde{c}_I p_I + \tilde{c}_J p_J + \tilde{c}_K p_K + \gamma \pi_\phi \phi\right)\Big]. \end{aligned} \tag{4.9}$$

Finally, from Eqs. (4.5) and (4.6), respectively, the equations of motion for the BD scalar field ϕ and its conjugate momentum π_ϕ are given by

$$\phi' = \frac{2}{\beta\gamma}\left(\tilde{c}_1 p_1 + \tilde{c}_2 p_2 + \tilde{c}_3 p_3 + \gamma \pi_\phi \phi\right), \tag{4.10}$$

$$\begin{aligned} \pi_\phi' &= -\frac{1}{\gamma^2\phi^2}\Big\{\left(\tilde{c}_1 p_1 \tilde{c}_2 p_2 + \tilde{c}_2 p_2 \tilde{c}_3 p_3 + \tilde{c}_1 p_1 \tilde{c}_3 p_3\right) \\ &- \frac{1}{\beta}\left[\left(\tilde{c}_1 p_1 + \tilde{c}_2 p_2 + \tilde{c}_3 p_3\right)^2 - \left(\phi\gamma\pi_\phi\right)^2\right]\Big\}. \end{aligned} \tag{4.11}$$

The set of Eqs. (4.8)-(4.11) completely specifies the dynamics of BDT in the Bianchi-I spacetime at the classical level. Also, even at the level of equations of motion, it is straightforward to see that in the limit $\omega \to \infty$ and constant ϕ, the equations of motion for the Einstein-Hilbert action are recovered [45].

V. QUANTUM CORRECTED DYNAMICS OF BDT IN BIANCHI-I SPACETIME

The quantum corrected effective dynamics of LQC have been studied in detail in [46], which incorporate the leading-order quantum geometric effects [47]. These effective models work very well in comparison with the full quantum dynamics of LQC even in the deep quantum regime [20], especially for the states that are sharply peaked on a classical trajectory at late times [48]. In the homogeneous and isotropic FLRW universe, it is obtained by the replacement $c \to \sin(\mu c)/\mu$, where c denotes the conjugate momentum of the area operator p ($\propto a^2$, where a is the expansion factor of the universe), and the lattice spacing, μ, is called the polymerization parameter. Different choices of the parameter μ give rise to different schemes of quantization with distinct effective dynamics. The expectation is that this would act as an ultraviolet regulator and mitigate the initial singularity [49, 50].

In the literature, various choices of μ have been considered [20, 51]. In the original scheme, the area gap was implemented as a kinematic feature on the comoving frame, which is equivalent to introducing a fixed lattice of constant spacing. On the other hand, the improved scheme is equivalent to introducing the minimum area eigenvalue on the physical frame, in contrast to the original. In this paper we shall adopt two different $\bar{\mu}$ schemes in order to explore different aspects of the model. We denominate them as $\bar{\mu}$ type A and B, to be denoted as $\bar{\mu}_A$ and $\bar{\mu}_B$, respectively, for our purpose. Next, we describe these two quantization schemes in more details.

A. The $\bar{\mu}_A$ quantization scheme

The $\bar{\mu}_A$ scheme, proposed in Ref. [52], for the homogeneous and anisotropic Bianchi I model is specified by

$$\bar{\mu}_I = \sqrt{\frac{\Delta}{p_I}}, \tag{5.1}$$

where $\Delta = 4\pi\sqrt{3}\gamma l_{Pl}^2$ is the area gap in the full theory of LQC.

Clearly, when $p_I \gg \Delta$, we have $\bar{\mu}_I \to 0$, and expect that the quantum effects are very small, and the classical limit is obtained. However, near the singular point, $p_I \simeq 0$, we have $\bar{\mu}_I \gg 1$, such that the quantum effects are expected to be very large. In the following, we shall consider whether such effects are larger enough such that the spacetime singularity that used to appear at $p_I = 0$ will now be regulated in the current setup.

Returning to our model under investigation, from Eq. (3.13), the quantum deformed Hamiltonian constraint becomes

$$
\begin{aligned}
C_{\rm eff} &= -\frac{1}{\gamma^2\phi}\left[\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1\frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2+\frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3+\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1\right]\\
&+ \frac{1}{\beta\gamma^2\phi}\left[\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1+\frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2+\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3+\gamma\pi_\phi\phi\right]^2+\frac{P_\Phi^2}{2}. \qquad (5.2)
\end{aligned}
$$

In order to study the effective theory of loop quantum Brans-Dicke cosmology, we also want to know the effect of matter fields on the dynamical evolution. Hence, we have included in Eq. (5.2) an extra massless scalar matter field Φ, with constant conjugate momenta P_Φ, which acts as a clock and has energy density $\rho = P_\Phi^2/(2p_1p_2p_3)$. Quantum corrected equations of motion for the symmetry reduced connection parameterized by $\tilde c$ for the $\bar\mu_A$ scheme given by Eq. (5.1) are given by

$$
\begin{aligned}
\tilde c_I' &= -\frac{1}{\gamma\phi}\left[\frac{3\sin(\bar\mu_I\tilde c_I)}{2\bar\mu_I}-\frac{c_I\cos(\bar\mu_I\tilde c_I)}{2}\right]\left\{\frac{\sin(\bar\mu_J\tilde c_J)}{\bar\mu_J}p_J+\frac{\sin(\bar\mu_K\tilde c_K)}{\bar\mu_K}p_K\right.\\
&- \left.\frac{2}{\beta}\left[\frac{\sin(\bar\mu_I\tilde c_I)}{\bar\mu_I}p_I+\frac{\sin(\bar\mu_J\tilde c_J)}{\bar\mu_J}p_J+\frac{\sin(\bar\mu_K\tilde c_K)}{\bar\mu_K}p_K+\gamma\pi_\phi\phi\right]\right\}\\
&+ \gamma p_Jp_K\left(\rho+p_I\frac{\partial\rho}{\partial p_I}\right). \qquad (5.3)
\end{aligned}
$$

For other choices of schemes for $\bar\mu_I$, the change will be in the terms between the first square brackets. Similarly, quantum corrected equations of motion for the symmetry reduced densetized triads parameterized by p_I are given by

$$
\begin{aligned}
p_I' &= \frac{p_I}{\phi\gamma}\cos(\bar\mu_I\tilde c_I)\left\{\frac{\sin(\bar\mu_J\tilde c_J)}{\bar\mu_J}p_J+\frac{\sin(\bar\mu_K\tilde c_K)}{\bar\mu_K}p_K\right.\\
&- \frac{2}{\beta}\left[\frac{\sin(\bar\mu_I\tilde c_I)}{\bar\mu_I}p_I+\frac{\sin(\bar\mu_J\tilde c_J)}{\bar\mu_J}p_J\right.\\
&+ \left.\left.\frac{\sin(\bar\mu_K\tilde c_K)}{\bar\mu_K}p_K+\gamma\pi_\phi\phi\right]\right\}. \qquad (5.4)
\end{aligned}
$$

Finally, the equations of motion for ϕ and its conjugate momentum π_ϕ read

$$
\begin{aligned}
\phi' &= \frac{2}{\gamma\beta}\Big[\frac{\sin(\tilde c_1\bar\mu_1)}{\bar\mu_1}p_1+\frac{\sin(\tilde c_2\bar\mu_2)}{\bar\mu_2}p_2+\frac{\sin(\tilde c_3\bar\mu_3)}{\bar\mu_3}p_3\\
&+ \gamma\pi_\phi\phi\Big], \qquad (5.5)\\
\pi_\phi' &= -\frac{1}{\gamma^2\phi^2}\left\{\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1\frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2\right.\\
&+ \frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3+\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3\\
&+ \frac{(\gamma\pi_\phi\phi)^2}{\beta}\\
&- \left.\frac{1}{\beta}\left(\frac{\sin(\tilde c_1\bar\mu_1)}{\bar\mu_1}p_1+\frac{\sin(\tilde c_2\bar\mu_2)}{\bar\mu_2}p_2+\frac{\sin(\tilde c_3\bar\mu_3)}{\bar\mu_3}p_3\right)^2\right\}. \qquad (5.6)
\end{aligned}
$$

The set of Eqs. (5.3)-(5.6) completely define the quantum corrected dynamics of the gravitational sector of the Bianchi-I spacetime for BDT in the Jordan's frame in $\bar\mu_A$ scheme. At this point, we can compute the average Hubble parameter H for the quantum Bianchi-I spacetime,

$$
H \equiv \frac{H_1+H_2+H_3}{3}, \qquad (5.7)
$$

where we have defined the average scale factor as $a = \sqrt{a_1a_2a_3}$, $p_I = a_Ja_Kl_0^2$ for non repeated indexes I, J, K and $H_I = \dot a_I/a_I$. The average Hubble parameter reads

$$
\begin{aligned}
H &= \frac{1}{6\gamma\phi\sqrt{p_1p_2p_3}}\left\{\left[\frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2+\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3\right]\cos(\bar\mu_1\tilde c_1)\right.\\
&+ \left[\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1+\frac{\sin(\bar\mu_3\tilde c_3)}{\bar\mu_3}p_3\right]\cos(\bar\mu_2\tilde c_2)\\
&+ \left.\left[\frac{\sin(\bar\mu_2\tilde c_2)}{\bar\mu_2}p_2+\frac{\sin(\bar\mu_1\tilde c_1)}{\bar\mu_1}p_1\right]\cos(\bar\mu_3\tilde c_3)\right\}\\
&- \frac{1}{6\sqrt{p_1p_2p_3}}\frac{\dot\phi}{\phi}\left[\cos(\bar\mu_1\tilde c_1)+\cos(\bar\mu_2\tilde c_2)+\cos(\bar\mu_3\tilde c_3)\right], \qquad (5.8)
\end{aligned}
$$

where $\sqrt{p_1p_2p_3}$ comes from Eq. (4.7). It should be noted that at each point of the evolution of the universe dictated by Eq. (5.8), the condition for non-singular quantum bounce, $H = 0$, holds when we impose that $p_1, p_2, p_3 \neq 0$ simultaneously. One can notice from Eq. (5.8) that this condition will be satisfied only for $\cos(\bar\mu_I\tilde c_I) = 0$ for $I = 1, 2, 3$. However, this is a stringent condition as $\tilde c_I$ are all independent and one can not guarantee all of them to be zero simultaneously. In order to guarantee that a non-singular behavior happens indeed, we now follow the perspective of Ref. [36].

Defining

$$\mathcal{G}_I(t') = p_I \frac{\sin(\bar{\mu}_I \tilde{c}_I)}{\bar{\mu}_I}, \quad (5.9)$$

and using Eqs. (5.3)-(5.4), one can show that

$$\mathcal{G}'_I(t') = \gamma \cos(\bar{\mu}_I \tilde{c}_I) p_1 p_2 p_3 \left(\rho + p_I \frac{\partial \rho}{\partial p_I}\right). \quad (5.10)$$

Now setting the quantum effective constraint, Eq. (5.2), to zero, one obtains

$$\gamma^2 p_1 p_2 p_3 \rho_{\rm eff} = \mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_3, \quad (5.11)$$

where we have used Eq. (5.5) and defined the effective energy density as

$$\rho_{\rm eff} = \frac{\beta\dot{\phi}^2}{4} + \rho\phi. \quad (5.12)$$

Now, one remembers that $\rho = P_\Phi^2/(2p_1p_2p_3)$ is the energy density for a massless scalar field, then P_Φ is a constant of motion. Therefore, the term $\left(\rho + p_I \frac{\partial\rho}{\partial p_I}\right)$ in Eq. (5.10) vanishes, which means that $\mathcal{G}_I$ are constants. For constant $\mathcal{G}_I$, from Eq. (5.9), given that the sines have a maximum value of unit, it sets a lower bound on p_I,

$$p_I \geq (|\mathcal{G}|\sqrt{\Delta})^{2/3}. \quad (5.13)$$

This is to say that the results of Ref. [36] also hold in the present case of BDT, provided that $\rho \to \rho_{\rm eff}$. Then Eq. (5.13) shows that all scale factors are bounded from below, which proves that a non singular behavior is present.

B. The $\bar{\mu}_B$ quantization scheme

In order to express the average Hubble parameter in a closed form and to see the singularity resolution more directly, the use of the alternative $\bar{\mu}_B$ quantization scheme becomes more appropriate. In the $\bar{\mu}_B$ quantization scheme, we have that [40]

$$\bar{\mu}_I = \sqrt{\frac{\Delta\, p_I}{p_J p_K}}, \quad (5.14)$$

where I, J and K are non repeated indexes.

More explicilty, we begin by rewriting Eq. (5.5) as

$$\begin{aligned}\dot{\phi}^2 &= \frac{4}{\gamma^2\beta^2(p_1p_2p_3)}\left[\frac{p_1\sin(\tilde{c}_1\bar{\mu}_1)}{\bar{\mu}_1} + \frac{p_2\sin(\tilde{c}_2\bar{\mu}_2)}{\bar{\mu}_2}\right.\\ &+ \left.\frac{p_3\sin(\tilde{c}_3\bar{\mu}_3)}{\bar{\mu}_3} + \gamma\pi_\phi\phi\right]^2, \quad (5.15)\end{aligned}$$

which is now written with respect to the cosmic time t. Now setting once again the quantum effective Hamiltonian constraint, Eq. (5.2), to zero, one obtains

$$\begin{aligned}&- \frac{1}{\gamma^2\phi}\left[\frac{\sin(\bar{\mu}_1\tilde{c}_1)}{\bar{\mu}_1}p_1\frac{\sin(\bar{\mu}_2\tilde{c}_2)}{\bar{\mu}_2}p_2 + \frac{\sin(\bar{\mu}_2\tilde{c}_2)}{\bar{\mu}_2}p_2\frac{\sin(\bar{\mu}_3\tilde{c}_3)}{\bar{\mu}_3}p_3 + \frac{\sin(\bar{\mu}_3\tilde{c}_3)}{\bar{\mu}_3}p_3\frac{\sin(\bar{\mu}_1\tilde{c}_1)}{\bar{\mu}_1}p_1\right]\\ &+ \frac{1}{\beta\gamma^2\phi}\left[\frac{\sin(\bar{\mu}_1\tilde{c}_1)}{\bar{\mu}_1}p_1 + \frac{\sin(\bar{\mu}_2\tilde{c}_2)}{\bar{\mu}_2}p_2 + \frac{\sin(\bar{\mu}_3\tilde{c}_3)}{\bar{\mu}_3}p_3 + \gamma\pi_\phi\phi\right]^2 + p_1p_2p_3\rho = 0. \quad (5.16)\end{aligned}$$

Note that in the second line of the above equation the term in the square brackets can be written in terms of Eq. (5.15), then one obtains

$$\begin{aligned}\frac{\beta\dot{\phi}^2}{4\phi} + \rho &= \frac{1}{\gamma^2\phi}\left[\frac{\sin(\bar{\mu}_1\tilde{c}_1)\sin(\bar{\mu}_2\tilde{c}_2)}{\bar{\mu}_1\bar{\mu}_2 p_3}\right.\\ &+ \left.\frac{\sin(\bar{\mu}_2\tilde{c}_2)\sin(\bar{\mu}_3\tilde{c}_3)}{\bar{\mu}_2\bar{\mu}_3 p_1} + \frac{\sin(\bar{\mu}_3\tilde{c}_3)\sin(\bar{\mu}_1\tilde{c}_1)}{\bar{\mu}_1\bar{\mu}_3 p_2}\right]. \quad (5.17)\end{aligned}$$

At this point, we apply the aforementioned alternative scheme, given by Eq. (5.14), in Eq. (5.17), which can now be suitably written such as to define an effective energy density $\rho_{\rm eff}$ as

$$\begin{aligned}\frac{\rho_{\rm eff}}{\rho_c} &= \frac{\gamma^2\Delta}{3}\left(\frac{\beta\dot{\phi}^2}{4} + \rho\phi\right)\\ &= \frac{1}{3}\Big[\sin(\bar{\mu}_1\tilde{c}_1)\sin(\bar{\mu}_2\tilde{c}_2) + \sin(\bar{\mu}_2\tilde{c}_2)\sin(\bar{\mu}_3\tilde{c}_3)\\ &+ \sin(\bar{\mu}_1\tilde{c}_1)\sin(\bar{\mu}_3\tilde{c}_3)\Big]. \quad (5.18)\end{aligned}$$

Therefore, the effective energy density $\rho_{\rm eff}$, already defined by Eq. (5.12), is bounded from above by its maximal value ρ_c, given by $\rho_c = 3/(\gamma^2\Delta)$. These results are in agreement with Ref. [53] in the isotropic limit.

Now, a maximal value of energy density implies a minimum value of spatial volume. Therefore, initializing the universe with a contracting phase, a big bounce is predicted in this scenario. This confirms the removal of an initial singularity in the framework of LQG owing to the fact that geometry is quantized and there exists a non-zero minimal area. Starting with an initially contracting phase, in a classical setup, the collapse is inevitable. Thus, once again, through this analysis we conclude that quantum geometric effects at the Planck scale provide repulsive forces and replace the big bang singularity by a non-singular quantum bounce even beyond GR, with the non-trivial spacetime, the Bianchi I universe.

For completeness, let us now write a closed form of the Friedmann equation and that can be derived using the current scheme. From Eqs. (5.8) and (5.15), we obtain that

$$\left(H+\frac{\dot{\phi}}{2\phi}\right)^2 = \left\{\frac{1}{6\gamma\sqrt{\Delta}\phi}\left[\sin(\tilde{c}_1\bar{\mu}_1)\cos(\tilde{c}_2\bar{\mu}_2)+\sin(\tilde{c}_2\bar{\mu}_2)\cos(\tilde{c}_1\bar{\mu}_1)+\sin(\tilde{c}_1\bar{\mu}_1)\cos(\tilde{c}_3\bar{\mu}_3)+\sin(\tilde{c}_3\bar{\mu}_3)\cos(\tilde{c}_1\bar{\mu}_1)+\right.\right.$$
$$\left.\left.+\ \sin(\tilde{c}_2\bar{\mu}_2)\cos(\tilde{c}_3\bar{\mu}_3)+\sin(\tilde{c}_3\bar{\mu}_3)\cos(\tilde{c}_2\bar{\mu}_2)\right]+\frac{\dot{\phi}}{2\phi}\left[1-\frac{\cos(\tilde{c}_1\bar{\mu}_1)+\cos(\tilde{c}_2\bar{\mu}_2)+\cos(\tilde{c}_3\bar{\mu}_3)}{3}\right]\right\}^2 . \quad (5.19)$$

Inspired by Eq. (5.18), we define the directional effective energy density in the I-direction as

$$\rho_{\rm eff}^{(I)}=\frac{3\sin^2(\tilde{c}_I\bar{\mu}_I)}{\gamma^2\Delta}. \quad (5.20)$$

Using this definition, Eq. (5.19) can now be expressed as

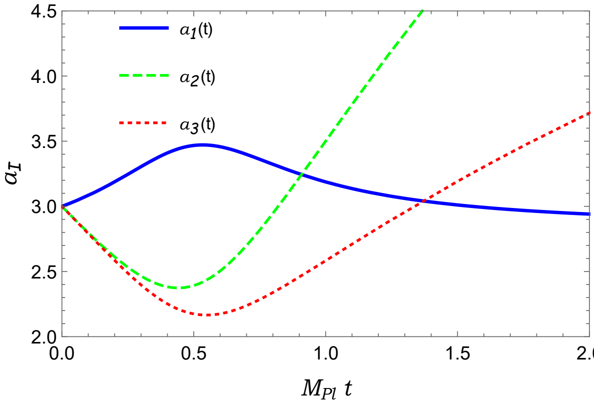
$$\left(H+\frac{\dot{\phi}}{2\phi}\right)^2 = \left\{\frac{1}{6\phi}\left[\sqrt{\frac{1}{3}\rho_{\rm eff}^{(1)}\left(1-\frac{\rho_{\rm eff}^{(2)}}{\rho_c}\right)}+\sqrt{\frac{1}{3}\rho_{\rm eff}^{(2)}\left(1-\frac{\rho_{\rm eff}^{(1)}}{\rho_c}\right)}+\sqrt{\frac{1}{3}\rho_{\rm eff}^{(1)}\left(1-\frac{\rho_{\rm eff}^{(3)}}{\rho_c}\right)}+\right.\right.$$
$$+\ \sqrt{\frac{1}{3}\rho_{\rm eff}^{(3)}\left(1-\frac{\rho_{\rm eff}^{(1)}}{\rho_c}\right)}+\sqrt{\frac{1}{3}\rho_{\rm eff}^{(2)}\left(1-\frac{\rho_{\rm eff}^{(3)}}{\rho_c}\right)}+\sqrt{\frac{1}{3}\rho_{\rm eff}^{(3)}\left(1-\frac{\rho_{\rm eff}^{(2)}}{\rho_c}\right)}\,\right]$$
$$\left.+\ \frac{\dot{\phi}}{2\phi}\left[1-\frac{1}{3}\left(\sqrt{1-\frac{\rho_{\rm eff}^{(1)}}{\rho_c}}+\sqrt{1-\frac{\rho_{\rm eff}^{(2)}}{\rho_c}}+\sqrt{1-\frac{\rho_{\rm eff}^{(3)}}{\rho_c}}\right)\right]\right\}^2 . \quad (5.21)$$

The above equation reduces to the isotropic BD theory [53] in the limit where the directional effective energy densities are equal. Also, the effective Friedmann equation of LQC is recovered for $\phi=1$, when $\rho_{\rm eff}\to\rho$.

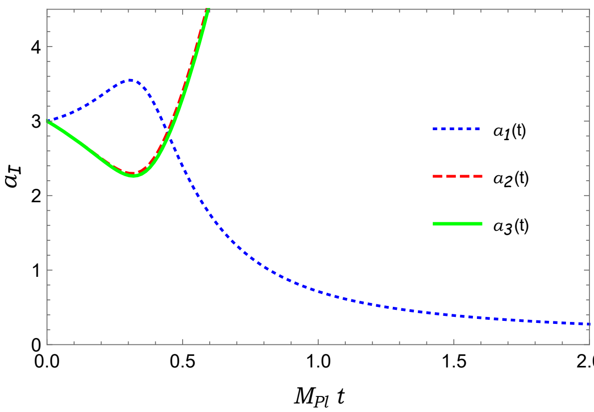
VI. DYNAMICS OF THE MODELS

In this section we present the results for the effective dynamics of the models considered in this work. We show the evolution of the scale factor and Hubble parameter for Bianchi LQC and BD Bianchi LQC models in each quantization scheme that has been discussed in the previous section.

In Figs. 1 and 2 we show the results for the directional scale factor a_I, $I=1,2,3$, in a Bianchi-I LQC model for the $\bar{\mu}_A$ and $\bar{\mu}_B$ schemes, given by Eqs. (5.1) and (5.14), respectively, with the parameters described in the figure captions. In Figs. 1(a) and 2(a), we show the results for the pure Bianchi LQC, while in Figs. 1(b) and 2(b) we have the corresponding results in the context of BDT. The initial conditions considered, which are described in the figure caption, are set at the contracting pre-bounce phase and at an instant we denote by $t=0$ (as will be the case for all the results presented in this Section). We can see from the figure that the behavior of the scale factor differs in each direction, as expected for Bianchi spacetimes. Especially, the behavior of a_1 is very different from a_2 and a_3 in both models considered. This type of dynamics is consistent with a Kasner-like phase, i.e., expansion followed by contraction in one direction and contraction followed by expansion in the others. It is important to observe that even though there appears contraction in one of the directions, we have checked explicitly that it never reaches zero (e.g., reaches a singularity) in that direction, but it reaches a nonnull value at very large times and then increases. The results shown in Figs. 1 and 2 are in agreement with the overall behavior obtained in Refs. [36] and [54] for the standard (non-BD, i.e. Einstein-Hilbert) Bianchi LQC case. From the numerical studies that we have performed, we have also verified that the inclusion of a massless scalar field does not change the results of the vacuum case significantly, although the exact behavior can depend on the initial conditions and on the choice of matter.



(a)

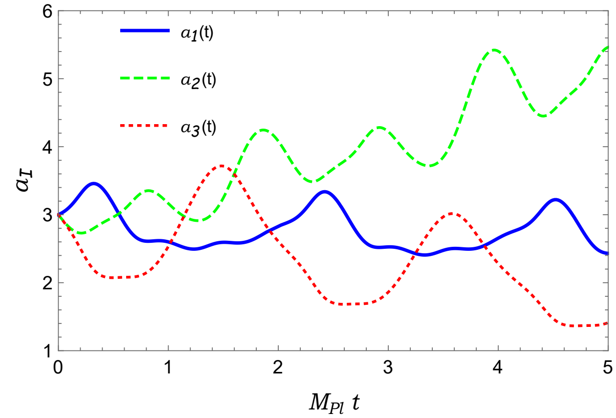


(b)

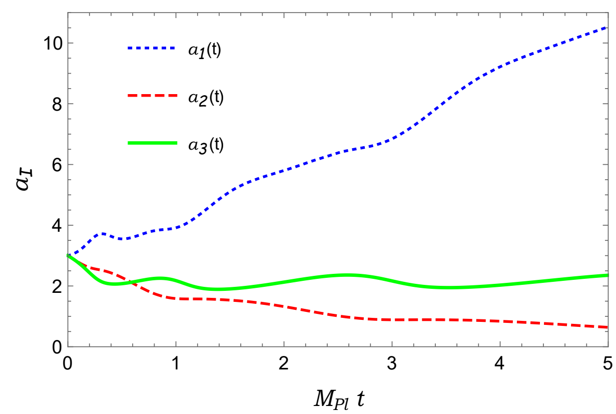
FIG. 1. The directional scale factors a_I, $I = 1, 2, 3$, in a Bianchi-I LQC model (panel a) and for the BD Bianchi-I LQC model (panel b), for the initial conditions $c_1 = 0.6$, $c_2 = -3.0$, $c_3 = 5.5$, $p_1 = 9$, $p_2 = 9.0001$, $p_3 = 8.9999$, and for the BD Bianchi-I LQC model, with the BD initial conditions $\phi = 1$, $p_\phi = 0$ and for $\omega = 0.1$ for the BD parameter. Both plots correspond to the $\bar{\mu}_A$ scheme.

Besides of checking the behavior of the scale factor in each direction, the important quantity to investigate, in order to analyze the effective cosmological dynamics in these models, is the average scale factor and the average Hubble parameter (where the average is taken with respect to the three directions).

We show in Fig. 3 the behavior of the average scale factor for the Bianchi LQC (+BD) in the two quantization schemes. We can see that the evolution is different in each scheme, as previously predicted in the work of Ref. [54]. By comparing the results within the same quantization scheme, we can see that the slopes of the curves are different depending on whether the model is



(a)



(b)

FIG. 2. The directional scale factors a_I, $I = 1, 2, 3$, in a Bianchi-I LQC model (panel a) and for the BD Bianchi-I LQC model (panel b), for the initial conditions $c_1 = -10$, $c_2 = -15$, $c_3 = 20$, $p_1 = 9$, $p_2 = 9.00001$, $p_3 = 8.99999$, and for the BD Bianchi-I LQC model, with the BD initial conditions $\phi = 1$, $p_\phi = 0$ and for $\omega = 0.1$ for the BD parameter. Both plots correspond to the $\bar{\mu}_B$ scheme.

in the context of the Brans-Dicke theory or not. In order to further investigate the dynamics in each case, it is worth to investigate the evolution of the Hubble parameter.

In Fig. 4, we present the evolution of the Hubble parameter in each model considered. For the models in the context of the $\bar{\mu}_A$-scheme, we can observe the presence of a single bounce. On the other hand, the curves representing the models in the $\bar{\mu}_B$-scheme crosses the value $H = 0$ more than once. By comparing each curve within the same quantization scheme, it becomes clear how the BD parameters affect the effective dynamics of the models. It is also possible to verify that the green (dashed) curve in Fig. 4, representing the dynamics of BD Bianchi LQC in the $\bar{\mu}_A$-scheme, shows an overall behavior consistent with the results obtained in Ref. [39].

We conclude that, for the representative parameters and initial conditions considered, all the results obtained consistently show that a bounce occurs around

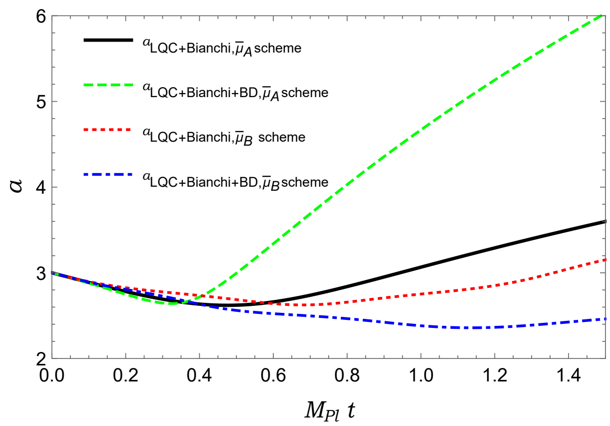


FIG. 3. The mean scale factor a. The initial conditions for Bianchi-I LQC model are $c_1 = 0.6$, $c_2 = -3.0$, $c_3 = 5.5$, $p_1 = 9$, $p_2 = 9.0001$, $p_3 = 8.9999$, whereas for the BD Bianchi-I LQC model we used the same ones with the addition of $\phi = 1$ and $p_\phi = 0$. We chose the value $\omega = 0.1$ for the BD parameter.

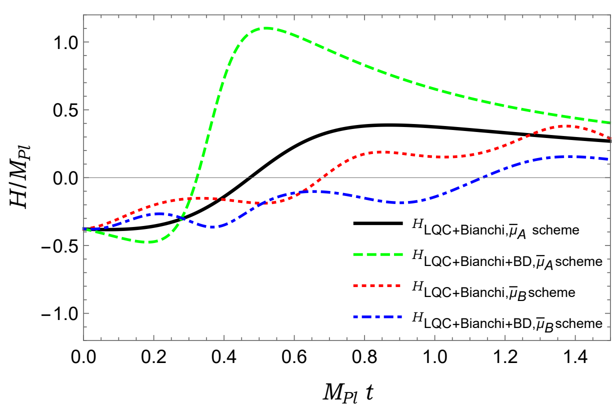


FIG. 4. The mean Hubble parameters H. The initial conditions for Bianchi-I LQC model are $c_1 = 0.6$, $c_2 = -3.0$, $c_3 = 5.5$, $p_1 = 9$, $p_2 = 9.0001$, $p_3 = 8.9999$, whereas for the BD Bianchi-I LQC model we used the same ones with the addition of $\phi = 1$ and $p_\phi = 0$. We chose the value $\omega = 0.1$ for the BD parameter.

$t \sim 0.1/M_{\rm pl}$, although the precise instant varies little within each model. The results present a behavior consistent with the ones previously obtained in the literature (for Bianchi LQC or BD LQC) in the respective limits. Although the detailed dynamics varies with the quantization scheme and the choice of initial conditions, the avoidance of the singularity is verified for all the cases considered.

In this section we did not include the presence of stiff matter since we verified that it does not change the general results presented here, as the qualitative behavior shown in the figures remains the same. We have also explicitly verified that the values of the parameters and initial conditions chosen are quite generic in the sense that, by changing them to other reasonable values, we obtain the same generic behavior, such that main conclusions and results discussed above are maintained.

VII. CONCLUSIONS

In this paper we have derived the quantum-corrected effective background dynamics for BDT in the Bianchi I model in the Jordan's frame in the framework of LQG. We have cast the theory in terms of the Yang-Mills phase space variables suitable for loop quantization and we have presented the quantum-corrected effective dynamics that resolve the spacetime singularity. The quantization is made by considering two different schemes, as given by Eqs. (5.1) and (5.14), respectively. The former set provides lower bounds on the scale factor, Eq. (5.13), proving the existence of a non-singular quantum behavior of spacetime. On the other hand, the latter scheme gives us an upper bound to the effective energy density, i.e., the critical density ρ_c, implying a minimal value of spatial volume. Additionally, we have also derived the quantum corrected Friedmann equation for this model inspired by the idea of a directional energy density. We have explicitly shown that the effective dynamics of the Bianchi I universe must undergo a smooth transition from a contracting phase to a expanding phase through the quantum bounce. Our analytical results, therefore, show robustness of the singularity resolution and the consistency of the effective dynamics beyond general relativity.

As a perspective of future study in the context of the model here studied, it would be of interest to explore any possible observational features it might lead and that can be detected in the forthcoming CMB experiments. An extension of the analysis like the one performed in earlier studies in the context of LQC for homogeneous and isotropic models [55–59] would be of interest. Likewise, the analysis of the perturbations in the model presented here and the possibility of an inflationary phase following the quantum bounce could be eventually done. Concerning any possible observational signatures in LQC scenarios with an anisotropic background, one can mention, for instance, the work of Ref. [28], where an analysis of the possible observational features in such models was performed. Hence, an extension of such analysis of perturbations in the context of the BDT anisotropic LQC will be necessary in order to elaborate the observable predictions of the model. Issues related to particle production due to the bounce can also be important [60–68] and their analysis and consequences in the context of anisotropic loop quantum BDT would be worthwhile to be pursued in the future as well.

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