Linear Algebra Students' Reasoning with Compositions of Linear Transformations

Lorna Headrick
Arizona State University

Michelle Zandieh
Arizona State University

We report on preliminary findings from a study of the ways in which linear algebra students used function composition to describe the result of applying two linear transformations, defined using symbolic vector notation, to a graphical region. Our current results from even a small set of data suggest that the students engage in function composition in a variety of ways. These results suggest areas for future exploration regarding how students engage in these forms of composition with linear transformations, and how their conceptualizations of individual transformations inform their understandings of the composition.

Linear transformation is an important idea within linear algebra. Furthermore, several areas of linear algebra rely on the idea of composing transformations, particularly in relation to multiplying matrices. These include the invertible matrix theorem (e.g., Wawro, 2014), and the diagonalization equation (e.g., Zandieh, Wawro, et al., 2017). Currently a few studies have investigated students' understandings of linear transformations, a small subset of which have included composition as one of the focal ideas (e.g., Bagley et al., 2015). Thus, a study of how linear algebra students conceptualize compositions of linear transformations as a form of function composition could have useful implications for the teaching and learning of linear algebra. This paper reports on our initial findings from pursuing this line of research.

Background and Literature Review

A large body of research has investigated students' understandings of function (early examples include Breidenbach et al., 1992; Carlson, 1998; Sfard, 1992). In contrast, few studies have had an explicit focus on function composition (e.g., Bagley et al., 2015; Bowling, 2014; Chen et al., 2023; Engelke et al., 2005; Headrick, 2023a; Kimani, 2008; Modabbernia et al., 2023). Most of these studies have focused on high-school or early college students (Bagley et al., 2015 being an exception). Furthermore, these studies have mainly used students' reasoning in the context of function composition to draw conclusions about their understandings of function. Some research has revealed evidence that students' reasoning with function composition extends beyond their reasoning with individual functions, suggesting a need for research that focuses on what is unique and important about students' reasoning with function composition specifically (e.g., Headrick, 2023a; Bowling, 2014).

Function composition has hardly been studied in linear algebra students' applications of linear transformations. A few studies have investigated students' understandings of linear transformations as functions (e.g., Andrews-Larson et al., 2017; Bagley et al., 2015; Oktaç, 2019; Turgut, 2019; Zandieh, Ellis, et al., 2017). One such study with a focus on composition of transformations (Bagley et al., 2015) found that students constructed the idea of an identity transformation as a "do-nothing function" and composing a transformation with its inverse as "doing" and "undoing." Our study will add to this research by investigating the reasoning linear algebra students use to describe the result of composing two distinct linear transformations.

Theoretical Framework

The theoretical framework currently guiding our data analysis in this study has two components. The first and primary component, which we refer to as *function composition* *reasoning*, is intended to explain students' reasoning unique to function composition specifically, accounting for ways students conceptualize multiple linear transformations and consider the result of applying them in sequence or in some other combination (Headrick, 2023b). The second component, called *clusters of metaphorical expressions*, focuses on the language students use to describe how they imagine individual transformations occurring (Zandieh, Ellis, et al., 2017).

Function Composition Reasoning

By function composition reasoning, we refer not only to previously described conceptualizations of function composition in research (e.g., Ayers et al., 1988; Breidenbach et al., 1992), but also other forms of reasoning students use to combine or apply multiple functions in response to a situation. Reviewing prior research led to identifying four distinct types of function composition reasoning high school and early college students engage in (Headrick, 2023b). Our current data suggests that these four types could also explain linear algebra students' reasoning with compositions of transformations. An example of each type of reasoning from our data is presented in the preliminary results.

The first type, *modifying a function with another function* (in short, *modifying*), involves conceptualizing a specific transformation and subsequently applying small tweaks or changes to this transformation via another, modifying transformation. The second type, *applying an operation on two functions* (in short, *operation)*, involves conceptualizing multiple transformations individually, and subsequently imagining the result of applying all of them together. The third type, *chaining input/output relationships*, involves taking some starting point (or input), applying one transformation to it, producing a particular result (or output), applying another transformation to the output of the first transformation, producing another output, and so forth. The fourth type, *chaining relationships between variables*, is an application of the third type in which the 'input' and 'output' of each transformation being composed is a variable.

Clusters of Metaphorical Expressions

Zandieh, Ellis, and Rasmussen (2017) presented five types of metaphorical expressions they found students to use when reasoning with linear transformations. Our current data suggests that students used these metaphors when composing transformations. The first, *input/output*, involves a transformation taking in or accepting some initial input, and giving some output in return. The second, *morphing*, involves an entity changing from one form to another through the transformation. The third, *machine*, involves the transformation doing something or acting on an initial entity to produce a result. The fourth, *traveling*, involves an entity moving from one location to another through the transformation. The fifth, *mapping*, involves an assignment of a one entity or value to another through some rule of correspondence.

Research Question

In light of existing research, we consider the question: In what ways do linear algebra students use function composition reasoning when composing two vector-defined linear transformations in a graphical context?

Method

Data Collection

The first author conducted one-on-one task-based clinical interviews with six students. The students had recently finished a Linear Algebra course taught by the second author and in which the first author served as a teaching assistant, at a large, public university in the United States. The Linear Algebra course was designed from a research-based curriculum (Wawro et al., 2013). The interviews were audio and video-recorded, and all written work was retained for additional evidence of the students' thinking.

Each task involved describing the result of applying a transformation to a 1-unit by 1-unit square region in the standard Cartesian plane, with its bottom-left vertex at the origin. Each task was structured identically with a three-fold structure: (i) predicting the result of applying a particular transformation, (ii) explaining the role of the notation used in the task to define the transformation in making the prediction, and (iii) sketching the transformed region, which for some (but not all) students, involved more precise calculations for determining points in the transformed region. Students applied these prompts to a single transformation T, another single transformation U, and the results of composing T and U: T(U(original square)), and U(T(original square)). Transformations in the tasks were defined using two different notations: vector notation, and matrix notation (different transformations were defined for each notation). Our preliminary results are from compositions of transformations T and U from the vector notation tasks.

Data Analysis

Since there are few prior studies on how students make sense of compositions of linear transformations, we began with an open-ended analysis in which we examined the video data to determine what themes might emerge from the students' reasoning. Upon further examination, we determined that existing frameworks for function composition reasoning (Headrick, 2023b) and for students' reasoning with transformations as clusters of metaphorical expressions (Zandieh, Ellis, et al., 2017) could enable us to develop a useful coding scheme.

Preliminary Results

In this section we present three distinct examples from our data to illustrate how the four types of function composition reasoning (underlined and italicized) appear to have emerged so far. Within each of these examples, the students appeared to use at least one cluster of metaphorical expressions (underlined). These examples suggest that linear algebra students could imagine composing linear transformations from a single problem context in a variety of ways. The data presented in this section are from students' predictions of how a composition of two transformations, defined as $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \end{bmatrix}$ and $U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ y \end{bmatrix}$, will transform a 1-unit by 1-unit square region drawn in the standard Cartesian plane with its bottom-left vertex at (0, 0).

Luna: Chaining Input/Output Relationships with Input/Output Transformation Metaphor

To consider the potential result of T(U(original square)), Luna evaluated the vector-components definition of U, then T at the original square's top-right vertex: 1, 1.

Luna: Yeah, the T, U. Then, yeah, it would be--you would use the U which would turn it--the 1, 1 into 3, 1 and then you plug 3, 1 into T, which would give you 6, 1…

Luna described first applying U to the point, getting particular coordinates as a result, and then applying T to the resulting coordinates, suggesting she was *chaining input/output relationships*. She appeared to use the input/output metaphor when describing how she would apply individual transformations ("then you plug 3, 1 into T, which would give you…"), and perhaps a morphing metaphor ("turn it—the 1, 1 into 3, 1").

James: Modifying and Operation Function Composition Reasoning with Morphing

James appeared to focus on describing the entire square as a completed shape. When initially describing his prediction for the result of T(U(original square)), James said,

James: So if I do U on the original square that's going to turn it into the--that parallelogram that I--I drew. But if I do the T after that, that's just gonna stretch that parallelogram twice its x-value. So if I would just like to take that side and just pull it 'til it's twice it's x-value, I think that's what would--oh, wait, is that what would happen? 'Cause that 2 (pauses) Oo, it's also italicizing it further.

After further consideration, James concluded that:

James: …It basically doubles the size, but it also becomes twice as italicized.

James' overarching reasoning, including a description of applying U "to the original square", producing a result ("that parallelogram"), and applying T to the result ("if I do the T after that") suggest he imagined the composition as a *chain of input/output relationships*. He appeared to engage in *modifying* reasoning when he initially predicted the result of T(U(original square)). He began by focusing on the parallelogram he described as resulting from applying U to the original square ("that parallelogram I drew"). Then, he described how applying T would change the shape of the parallelogram ("stretch…twice its x-value"; "italicizing it further"). Thus, when forming his initial prediction, James appeared to think of applying U as the emphasis and applying T as tweaking the results of applying U. However, when drawing a final conclusion about the transformed region, James also appeared to put greater emphasis on how T as an individual transformation would transform the parallelogram resulting from applying U to the square ("doubles the size"; "twice as italicized"), suggesting potential *operation* reasoning.

Throughout his description, James' language suggests he imagined a region of interest changing from one shape to another through the transformations ("turn...into"; "stretch"; "italicizing"; etc.). Thus, James appeared to use predominantly morphing metaphors.

Olivia: All Four Types of Function Composition Reasoning with Machine and Morphing

Olivia seemed to imagine applying transformations to a collection of points that she said comprised the square. She illustrated this idea by using vector notation to denote the points being transformed (later she said, "x, y is really just the same as the whole bunch of points that make up our square"). Olivia's orientation to the task and written work (Figure 1) are given below.

Olivia: So T of U of--we're gonna call it x y, 'cause the original square freaked me out. So that means U is gonna transform x y first, and then T will do it.

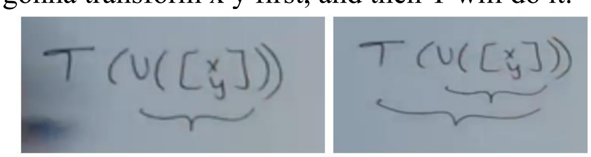

Figure 1. Olivia's Illustration of Applying U, then T to a Set of Points in the Cartesian Plane

Olivia's initial description of applying U, then T to the set of points suggests she imagined a *chain of input/output relationships*. When predicting the result of T(U(original square)) in more detail, she drew diagrams for the individual result of applying each transformation to the square and the combined result of applying both (see Figure 2). Meanwhile, she said,

Olivia: So this is our whatever x y is we call the original shape. This is my square. U took the square and made it like a parallelogram. That's what U did. And if we remember what T did, T took it and made it like, boop, it stretched it. So then, T would take this U thing, and stretch that to be like twice as long…

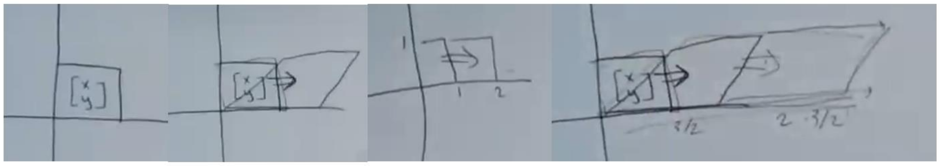

Figure 2. Olivia's Diagrams to Illustrate (left to right) the Original Square as a Set of Points, Applying U to the Square, Applying T to the Square, and Applying U, then T to the Square

Olivia appeared to imagine both U and T as having some individual effect on the square and T(U(original square)) as applying a combination of their individual effects (suggesting *operation* reasoning). There is also subtle evidence of her engaging in *modifying* when she described how T would change the result of U ("take this U thing, and stretch that…"). When asked what would happen to the corner points of the square "when you do T of U", Olivia described the transformation affecting the x-coordinates of multiple points. This reasoning, along with her earlier references to the square as a set of points suggests Olivia conceptualized T(U(original square)) as a *chain of relationships between variables*.

Olivia: ...you stretch everything, all the x-coordinates by 2. So I guess that would include these, like, these points [points to the edges of parallelogram for T(U(original square))]. Their x-coordinates have to get shifted by 2 too, so then the diagonals would have to.

Throughout her description, Olivia used machine metaphors, describing transformations as doing particular actions to points or figures ("U took…and made it"; "U did"; ect.), and morphing metaphors ("made it…a parallelogram"; stretched it; etc.).

Discussion

All three students in the results presented above appeared to use chaining of input/output relationships in some form. They all described some starting shape or value, applying the inner transformation to that starting shape or value to produce a result, and then applying the outer transformation to the result of applying the inner transformation. The pervasiveness of this reasoning in the data thus far could be partly due to the nature of the tasks posed; the two transformations were pre-defined and the task prompts involved the notation T(U(original square)) and U(T(original square)). This is an area for further exploration.

On the other hand, each student described the chain of input/output relationships they appeared to conceptualize quite differently. Chaining input/output relationships as a form of function composition reasoning was most prominent for Luna, and her reasoning with each individual transformations appeared to be largely connected to input/output metaphors. James' focus on transforming entire shapes and morphing seemed to naturally give rise to his modification reasoning. Olivia's focus on transformations as machines performing actions on a set of points seemed to inform her operation reasoning (i.e., first describing how each transformation would act on points individually and then constructing the composition), and chaining relationships between variables (i.e., describing how multiple points were transformed). Our results thus far lead us to consider the following questions: (a) What contexts for linear transformation problems might lead to specific types of function composition reasoning; and (b) How might different types of function composition reasoning with linear transformations support students in studying related ideas in linear algebra, such as inverses or diagonalization?

References

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology*, *48*(6), 809–829.

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education*, *19*(3), 246–259.

Bagley, S., Rasmussen, C., & Zandieh, M. (2015). Inverse, composition, and identity: The case of function and linear transformation. *The Journal of Mathematical Behavior*, *37*, 36–47.

Bowling, S. (2014). *Conceptions of Function Composition in College Precalculus Students* (Publication No. 3619278) [Doctoral Dissertation, Arizona State University]. ProQuest Dissertations Publishing.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Carlson, M. P. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education. III. CBMS issues in mathematics education* (Vol. 7, pp. 114–162).

Chen, Y., Wasserman, N. H., & Paoletti, T. (2023). Exploring Geometric Reasoning with Function Composition. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. SIGMAA on RUME, Omaha, NE.

Engelke, N., Oehrtman, M., Carlson, M., & Carlson, M. (2005). Composition of functions: Precalculus students' understandings. *Proceedings of the 27th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 20–23.

Headrick, L. (2023a). A Student's Application of Function Composition when Solving Unfamiliar Problems: The Case of Zander. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*, 744–751.

Headrick, L. (2023b). *High School Students' Early Function Composition Reasoning in the Context of Transformations of Polynomial Functions* [Unpublished manuscript]. School of Mathematical and Statistical Sciences, Arizona State University.

Kimani, P. M. (2008). *Calculus students' understandings of the concepts of function transformation, function composition, function inverse and the relationships among the three concepts* (Publication No. 3333570) [Doctoral Dissertation, Syracuse University]. ProQuest Dissertations Publishing.

Modabbernia, N., Yan, X., & Zazkis, R. (2023). When algebra is not enough: A dialogue on the composition of even and odd functions. *Educational Studies in Mathematics*, *112*(3), 397–414.

Oktaç, A. (2019). Mental constructions in linear algebra. *ZDM*, *51*(7), 1043–1054.

Sfard, A. (1992). Operational origins of mathematical notions and the quandary of reification—The case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. Mathematical Association of America.

Turgut, M. (2019). Sense-making regarding matrix representation of geometric transformations in $\mathbb{R}^2$: A semiotic mediation perspective in a dynamic geometry environment. *ZDM*, *51*(7), 1199–1214.

Wawro, M. (2014). Student reasoning about the invertible matrix theorem in linear algebra. *ZDM*, *46*(3), 389–406.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A Characterization of a Unified Notion of Mathematical Function: The Case of High School Function and Linear Transformation. *Educational Studies in Mathematics*, *95*(1), 21–38.
Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An Example of Inquiry in Linear Algebra: The Roles of Symbolizing and Brokering. *PRIMUS*, *27*(1), 96–124.