On the existence of EFX under picky or non-differentiative agents

Extended Abstract

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ABSTRACT

In this paper, we consider the fair division of indivisible goods under arguably the strongest envy-based fairness notion of *envy-free up to any item (EFX).* Extending the long line of work on special cases of additive valuations, we show existence of EFX for the following two cases: (i) instances where agents are very picky, *i.e.,* each agent likes at most four items positively. (ii) ternary instances where the value of an agent for an item is 0, a, or b for $0 < a < b \leq 2a$. In both cases, the existence is shown by designing an efficient algorithm to find an EFX allocation.

KEYWORDS

Fair division; envy-free; envy-free up to any item (EFX)

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1 INTRODUCTION

Fair division of scarce resources [19] is a fundamental problem in many disciplines, including computer science, economics, operations research, and social choice theory, with numerous contemporary applications [3, 5, 10, 17, 20].[1] This paper considers the discrete fair division problem, where a set M of indivisible goods needs to be allocated to a set N of n agents. Each agent i's preference over bundles of goods is represented by a valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$. One of the most well-studied classes of valuation functions are *additive*, *i.e.,* for any subset $S \subseteq M$ of goods, $v_i(S) = \sum_{j \in S} v_i(j)$. The goal is to find a *fair* allocation/partition $X = (X_1, \ldots, X_n)$ of the goods set M where agent i receives bundle X_i.

Envy-freeness (EF), is arguably one of the most sought-after fairness notion which dictates that no agent should envy another agent's allocation over their own, *i.e.,* for each agent i, $v_i(X_i) \geq v_i(X_{i'}), \forall i' \in N$. It has been extensively studied when the items are divisible like cake, land, and milk [2, 9]. However, it ceases to exist when items are indivisible – if we want to allocate an iPhone among two agents who both value it, then no allocation is EF. Therefore, the focus shifted to the relaxations of EF. Arguably, the strongest among these is *Envy-free up to any item (EFX)* [6], where no agent envies another agent's bundle after removal of any (non-trivial) good from it, *i.e.,* for any agent i, $v_i(X_i) \geq v_i(X_{i'} \setminus \{j\})$, $\forall i' \in N, \forall j \in X_{i'}$ such that $v_i(j) > 0$.[2] Despite extensive work in recent years, the existence of EFX remains unknown and is considered one of the most important questions of fair-division [18]. Towards this question, we study the existence of EFX allocations when agents are either picky, i.e., value fewer items, or non-differentiative, i.e., have only a few different values for the items.

[6] introduced EFX as the closest analog of EF for handling indivisible goods. [16] showed the existence of EFX for the cases of (i) two agents, and (ii) for n agents with identical valuations. In a breakthrough, [7] showed the existence of EFX with three agents under additive valuations. However, for four or more agents, the problem still remains open.[3] [4] showed existence of EFX for binary valuations where every $v_i(j) \in \{0, 1\}$. This was extended to restricted-additive valuations by [13] where every agent values good j at 0 or $v_j > 0$, *i.e.,* $v_i(j) \in \{0, v_j\}$, $\forall i \in N$. Extending the case of the identical valuations, [15] showed that EFX exists if all the agents have one of two given valuation functions. [11] considered the case where all but two agents have identical valuation functions. Additionally, [14] showed that EFX exists for n agents when there are at most $n + 3$ items. Recently, [8] considered instances called graphical instances where every item is valued by at most two agents. [12] showed that EFX exists when there are 2 types of objects and all agents have the same value for objects of the same type. This is incomparable to ternary instances because when there are two types of objects, $v_i(j) = v_i(j') \implies v_{i'}(j) = v_{i'}(j')$ which need not be the case for ternary instances.

Our Results. Continuing this line of work, we ask the following:

- Q1. What if every agent likes only a few goods? For example, while comparing courses, students may have positive value for only a few that align with their interests.
- Q2. What if every agent has only a few different values for the goods? For example, every agent gives items one of three different ratings.

Towards these questions, we show the existence of EFX under the following two cases:

1. Every agent likes at most 4 goods with strictly positive value.
2. Every agent has a *ternary valuation* function with values from $\{0, a, b\}$ for $0 < a < b \leq 2a$. This generalizes the binary case and is incomparable to the restricted additive case.

[1]Such problems find very early historical mentions, for instance, in ancient Greek mythology and the Bible.

[2]Many works consider a stronger version: for any agent i, $v_i(X_i) \geq v_i(X_{i'} \setminus \{j\})$, $\forall i' \in N, \forall j \in X_{i'}$. That is, the removed good may be trivial for agent i in the sense that her value for it may be zero.

[3]Under general monotone valuations, the problem remains open for three or more agents.

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Figure 1: (Left) Envy graph. (Right) EFX graph when object d is being assigned in Example 2.1.

For both cases, existence is derived by designing a polynomial-time algorithm to find an EFX allocation. In terms of techniques, we introduce the notion of the EFX graph with respect to a particular good and use it in combination with repeated matching.

Finally, we note that [1] showed two results that relate to our ternary result. Their first result, that EFX exists if all agents value each object as either a or b for some $a, b \in \mathbb{R}$ seems to cover our ternary result. However, we provide a different algorithm. Their algorithm relies on freezing agents for a number of rounds dependent on the common a and b, while ours could be extended to the case where every agent i values each object as $0, a_i$ or b_i for $0 \leq a_i \leq b_i \leq 2a_i$. Our algorithm is more similar to their second result that EFX exists for instances where the values of an agent i are in the interval $[x_i, 2x_i]$ for $x_i \in \mathbb{R}_{>0}$ as we also use a variation on a round robin, but account for the fact that the inclusion of 0 as a value means that an agent could run out of valued objects in any round, not just the last round.

2 TECHNICAL OVERVIEW

2.1 Picky Agents: Agents with limited liking

We discuss our algorithm to handle the case where every agent likes a limited number of goods. Let us define an l-limited instance.

[l-**limited**] We say that a valuation function v_i is l-limited if $|\{j \in M \mid v_i(j) > 0\}| \leq l$. We say that instance (N, M, V) is l-limited if for every agent $i \in N$, v_i is l-limited.

We next define the notion of *EFX* with respect to partial allocations. In a partial allocation $X = (X_1, \dots X_n)$, an agent *i envies* an agent i' if $v_i(X_i) < v_i(X_{i'})$. In a partial allocation $X = (X_1, \dots X_n)$, an agent *i EFX-envies* an agent i' if $v_i(X_i) < v_i(X_{i'} \setminus \{j\})$ for some $j \in X_{i'}$ with $v_i(j) > 0$. A partial allocation $X = (X_1, \dots X_n)$ is *EFX* if no agent i EFX-envies another agent i'.

We then define two graph notions crucial for our algorithm: kth value graph and the EFX graph. Fix a 4-limited instance (N, M, V).
kth Value Graph. Given a subset $S \subseteq N$ of agents and $T \subseteq M$ of objects, the kth value graph denoted by $G_k(S, T)$ is defined as the bipartite graph with vertices $S \cup T$ and an edge between $i \in S$ and $j \in T$ if and only if j is the kth highest ranked object for agent i.
EFX Graph. For good $j \in M$, set $S \subseteq N$ of agents, and partial allocation X, define the EFX graph $G_{\text{efx}}(j, S, X)$ as follows: it is a directed graph with nodes for each $i \in S$ with $v_i(j) > 0$, and an edge from $i \to i' \iff v_i(X_i) < v_i((X_{i'} \cup \{j\}) \setminus \{j'\})$ for some $j' \in (X_{i'} \cup \{j\})$ s.t. $v_i(j') > 0$.

Example 2.1. Consider an instance with agents A, B, C and objects a, b, c, d with valuation functions $v_A(a) = 2, v_A(b) = 1, v_A(c) = 3$, and $v_A(d) = 1$; $v_B(a) = 1$, $v_B(b) = 3$, $v_B(c) = 2$, and $v_B(d) = 4$; $v_C(a) = 1$, $v_C(b) = 2$, $v_C(c) = 3$, and $v_C(d) = 4$. Let the partial allocation be $A \mapsto a$, $B \mapsto b$, and $C \mapsto c$. Figure 1 shows the envy graph and the EFX Graph for assigning object d. □

Algorithm for 4-limited instances.

We initialize remaining agents $N^r = N$, remaining objects $M^r = M$, and the current partial allocation X to be such that no object is assigned yet. The algorithm runs in phases. In *phase I*, we define a subset of agents, A, that strongly prefer their highest ranked object, i.e. value their highest ranked object more than their second and third highest ranked objects combined. We assign a 1st choice object to as many agents in A as possible using a maximum matching in the 1*st value graph* on A and all objects, namely $G_1(A, M^r)$. Remove A from N^r and the assigned objects from M^r.

In *phase II*, we assign at most one object to every agent that doesn't have an object, in the best way possible. We run the maximum matching algorithm 4 times in sequence. For $k = 1, 2, 3, 4$, find a maximum matching M_k in graph $G_k(N^r, M^r)$. For $(i, j) \in M_k$, assign object j to agent i, and remove the agents and objects in the matching from N^r and M^r. Let N_k denote agents with their kth ranked object. Agents in N^r received nothing, while the others received exactly one item. Thus this partial allocation is EFX.

In *phase III*, we look at agents, starting with those that have low-ranked objects. Note that all the objects valued by an $i \in N_4 \cup N^r$ have been assigned, and therefore i is *taken care of* in the sense that i will not EFX-envy any agent no matter how the allocation is extended. We then assign a 4th choice object to as many of the agents in N_3 as possible. To assign a second object to as many of the agents in N_2 as possible, we first assign a 3rd choice object to as many agents in N_2 as possible and then assign a 4th choice object to as many of the agents in N_2 who have only one object, as possible. We call the set of agents in N_2 with their 3rd choice, N_2^r. Now, the only agents that may have valued objects not yet assigned are in N_1 and N_2^r. At this stage, we are able to prove that for every unassigned object j valued by someone, there will be a source in the EFX graph $G_{\text{efx}}(j, N_1 \cup N_2^r, X)$ where X is the current partial allocation. Assigning j to that source ensures that the partial allocation stays EFX. Finally, we assign remaining objects, which must be valued zero by all agents, to an arbitrary agent.

2.2 Agents with ternary valuations

We discuss our algorithm to handle the case where every agent likes a limited number of goods. Let us define a ternary instance.

We say that instance (N, M, V) is ternary for $\{a, b\}$, $a, b \in \mathbb{R}$ such that $0 < a < b$, if $\forall v_i \in V, \forall j \in M, v_i(j) \in \{0, a, b\}$.
Algorithm for ternary instances for $\{a, b\}$ where $0 \leq a \leq b \leq 2a$. Our algorithm consists of 3 steps repeated until all goods have been assigned. In the first step, it modifies the current allocation to remove all cycles from the envy graph. The resulting envy graph (called G in the algorithm) is acyclic, and therefore its vertices can be ordered topologically. In the second step, it finds a topological ordering, L, of the vertices of envy graph G. This ensures that in each iteration, an agent i is always processed before all agents it envies. Then, in the third step, it goes through L and assigns the best possible object to each agent in L.

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