



# Measure comparison problems for dilations of convex bodies

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*Abstract.* We study a version of the Busemann-Petty problem for log-concave measures with an additional assumption on the dilates of convex, symmetric bodies. One of our main tools is an analog of the classical large deviation principle applied to log-concave measures, depending on the norm of a convex body. We hope this will be of independent interest.

## 1 Introduction

We denote by  $\mathbb{R}^n$  the  $n$ -dimensional Euclidean space equipped with the inner product  $\langle \cdot, \cdot \rangle$  and the standard orthonormal basis  $\{e_1, \dots, e_n\}$ , and we denote by  $|\cdot|$  the standard Euclidean norm on  $\mathbb{R}^n$ . For a measurable set  $A \subset \mathbb{R}^n$ , we refer to its volume (the Lebesgue measure) by  $|A|$  and its boundary by  $\partial A$ . The notation  $B_2^n$  stands for the closed unit ball in  $\mathbb{R}^n$ , and  $\mathbb{S}^{n-1}$  for the unit sphere (i.e.,  $\mathbb{S}^{n-1} = \partial B_2^n$ ). A convex body is a convex, compact set with a nonempty interior. Furthermore, a convex body  $K$  is symmetric if  $K = -K$ . A measure  $\mu$  is log-concave on  $\mathbb{R}^n$  if for every pair of nonempty compact sets  $A$  and  $B$  in  $\mathbb{R}^n$  and  $0 < \lambda < 1$ , we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda},$$

where the addition is the Minkowski sum which is defined as the set  $A + B = \{a + b : a \in A, b \in B\}$ , and the constant multiple (dilation) of a set  $A \subset \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$  is defined as  $\alpha A = \{\alpha a : a \in A\}$ . It follows from the Prékopa-Leindler inequality [18, 19, 9] that if a measure  $\mu$  that is defined on the measurable subsets of  $\mathbb{R}^n$  is generated by a log-concave density, then  $\mu$  is also log-concave. Furthermore, Borell provides a characterization for log-concave measures [3]; precisely, a locally finite and regular Borel measure  $\mu$  is log-concave, if and only if its density (with respect to the Lebesgue on the appropriate subspace) is log-concave.

In 1956, Busemann and Petty [5] posed the following volume comparison problem: Let  $K$  and  $L$  be symmetric convex bodies in  $\mathbb{R}^n$  so that the  $(n - 1)$ -dimensional volume of every central hyperplane section of  $K$  is smaller than the same section for  $L$ . Does it follow that the  $n$ -dimensional volume of  $K$  is smaller than the  $n$ -dimensional volume of  $L$ ? In the late 1990s, the Busemann-Petty problem was solved as a result of many works including [8, 10, 13, 14, 21]. The answer is affirmative when  $n \leq 4$  and

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Received by the editors March 22, 2024; revised September 22, 2024; accepted October 3, 2024.

Both authors are supported in part by the U.S. National Science Foundation Grant DMS-1101636 and the United States - Israel Binational Science Foundation (BSF) Grant 2018115.

AMS Subject Classification: 52A20, 52A21, 46T12, 60F10.

Keywords: Busemann-Petty problem, log-concavity, large deviation.



negative whenever  $n \geq 5$ . It is natural to consider an analog of the Busemann-Petty problem for a more general class of measures. The first result in this direction was a solution of the Gaussian analog of the Busemann-Petty problem [22]. It turns out that the answer for the Busemann-Petty problem is the same if we replace the volume with the Gaussian measure. Moreover, it was proved in [23] that the answer is the same if we replace the volume with any measure with continuous, positive, and even density.

V. Milman [16] asked whether the answer to the Gaussian Busemann-Petty problem would change in a positive direction if we compared not only the Gaussian measure of sections of the bodies but also the Gaussian measure of sections of their dilates; that is, consider two convex symmetric bodies  $K, L \subset \mathbb{R}^n$ , such that

$$\gamma_{n-1}(rK \cap \xi^\perp) \leq \gamma_{n-1}(rL \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0,$$

where  $\xi^\perp$  denotes the central hyperplane perpendicular to  $\xi$ . Does it follow that

$$\gamma_n(K) \leq \gamma_n(L)?$$

Here,  $\gamma_n$  denotes the standard Gaussian measure on  $\mathbb{R}^n$ , and

$$\gamma_{n-1}(K \cap \xi^\perp) = \frac{1}{(\sqrt{2\pi})^{n-1}} \int_{K \cap \xi^\perp} e^{-\frac{|x|^2}{2}} dx.$$

The addition of dilation to the Busemann-Petty problem, clearly, would not change anything in the case of the volume measure. Still, in the case of more general log-concave measures, the behavior of the measure of a dilation of a convex body is very interesting; we refer to [1, 6, 12, 15] for just a few examples of such results.

Even though the dilation adds some strength to the condition of the bodies, the answer to the dilation problem for Gaussian measure is positive for  $n \leq 4$  and negative for  $n \geq 7$  (see [24]). That leaves the problem open for  $n = 5, 6$ .

To show the strength of the condition of the dilates, it was proved in [24] that the dilation problem has an affirmative answer when  $K$  is a dilate of a centered Euclidean ball: Consider a star body  $L \subset \mathbb{R}^n$  and assume there exists  $R > 0$ , such that

$$\gamma_{n-1}(rRB_2^n \cap \xi^\perp) \leq \gamma_{n-1}(rL \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0.$$

Then, it follows that  $RB_2^n \subseteq L$ .

In this paper, we review some generalizations of the above fact. In particular, we study measures  $\mu$  for which we have an affirmative answer for the following problem:

**Question 1** Consider a convex, symmetric body  $K \subset \mathbb{R}^n$  such that for every  $t$  large enough and for some  $R > 0$ ,

$$\mu(tRB_2^n) \leq \mu(tK).$$

Does it follow that  $RB_2^n \subseteq K$ ?

In Section 2, we will present a solution for Question 1 for the case of a log-concave rotation invariant probability measure  $\mu$ .

In Section 3, we consider a more general case. Instead of comparing  $K$  with  $B_2^n$ , we will compare  $K$  with another convex, symmetric body  $L$ . Let us denote by  $\|x\|_L$  the Minkowski functional of  $L$  which is defined to be  $\|x\|_L = \min\{\lambda > 0 : x \in \lambda L\}$ .

**Question 2** Let  $K, L \subset \mathbb{R}^n$  be convex, symmetric bodies, and let  $\mu$  be a log-concave probability measure with density  $e^{-\phi(\|x\|_L)}$ , where  $\phi: [0, \infty) \rightarrow [0, \infty)$  is a nonconstant, convex function. If for every  $t$  large enough and some  $R > 0$ ,

$$\mu(tRL) \leq \mu(tK),$$

does it follow that  $RL \subseteq K$ ?

One of the core steps in answering Questions 1 and 2 is a generalization of the classical large deviation principle, which is provided in Lemma 2.6 and Equation (3.2) below: Consider two symmetric, convex bodies  $K, L \subset \mathbb{R}^n$ , and let  $r(K, L) = \max\{R > 0 : RL \subset K\}$ . Then,

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(r(K, L)t)} = -1,$$

where  $\mu$  is a log-concave probability measure with density  $e^{-\phi(\|x\|_L)}$ , and by  $A^c$  we denote a complement of a set  $A \subset \mathbb{R}^n$  (i.e.,  $A^c = \mathbb{R}^n \setminus A$ ).

Finally, in Section 4, we will discuss the generalization of the dilation problem for Gaussian measures:

**Question 3** Consider a measure  $\mu$  with continuous positive density  $f$ . Let  $\mu_{n-1}(K \cap \xi^\perp) = \int_{K \cap \xi^\perp} f(x) dx$ . Consider two convex symmetric bodies  $K, L \subset \mathbb{R}^n$ , such that

$$\mu_{n-1}(rK \cap \xi^\perp) \leq \mu_{n-1}(rL \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0.$$

Does it follow that

$$\mu(K) \leq \mu(L)?$$

We show that, in general, the answer is still negative in dimension  $n \geq 5$ , even under the assumption that  $f$  is a nonconstant log-concave function. We also prove that if we add the requirement for the measure to be rotation invariant, the answer will be negative in dimension  $n \geq 7$ , which leaves the case of rotation invariant log-concave measures open in dimension  $n = 5, 6$ .

## 2 The case of rotation invariant measures

In this section, we consider a rotation invariant probability log-concave measure  $\mu$  with nonconstant density – that is,

$$\mu(A) = \int_A e^{-\phi(\|x\|)} dx,$$

where  $\phi: [0, \infty) \rightarrow [0, \infty)$  is a nonconstant, convex function. We will denote by  $\phi'(t)$  the left derivative in the case when the convex function  $\phi(t)$  is not differentiable at  $t$ .

**Theorem 2.1** Consider a convex, symmetric body  $K \subset \mathbb{R}^n$  such that for every  $t$  large enough and some  $R > 0$ ,

$$\mu(tRB_2^n) \leq \mu(tK).$$

Then,  $RB_2^n \subseteq K$ .

In order to prove the above theorem, we will need two lemmas.

**Lemma 2.2** Consider  $R > 0$ . Then,

$$(2.1) \quad \limsup_{t \rightarrow \infty} \frac{\ln \mu((tRB_2^n)^c)}{\phi(tR)} = -1.$$

**Proof** Without loss of generality, we may assume that  $R = 1$ . Let us first show that the left-hand side of equality (2.1) is less or equal to  $-1$ . Writing the integral in polar coordinates, we get

$$\limsup_{t \rightarrow \infty} \frac{\ln \int_{S^{n-1}} \int_t^\infty e^{-\phi(r)} r^{n-1} dr d\theta}{\phi(t)} = \limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} r^{n-1} dr}{\phi(t)}.$$

We remind that  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ . Let  $\eta(t) = -(n-1) \ln t + \phi(t)$ . Using that  $\phi$  is a convex and nonconstant function, we get that there exists  $t_0 \geq 0$  such that  $\phi'(t_0) > 0$ . Thus,  $\phi'(t) > 0$  for all  $t > t_0$ , and there exists a constant  $a > 0$  such that  $\eta'(t) > a$  for all  $t > t_0$ . Thus,  $\eta(r) \geq \eta(t) + a(r-t)$  for  $r > t > t_0$ , and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} r^{n-1} dr}{\phi(t)} &\leq \limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\eta(t)-a(r-t)} dr}{\phi(t)} \\ &= \limsup_{t \rightarrow \infty} \frac{\ln e^{-\eta(t)} + \ln \int_t^\infty e^{-a(r-t)} dr}{\phi(t)} \\ &= \limsup_{t \rightarrow \infty} \frac{\ln(t^{n-1} e^{-\phi(t)}) + \ln \frac{1}{a}}{\phi(t)} \\ &= -1 + \limsup_{t \rightarrow \infty} \frac{\ln \frac{1}{a}}{\phi(t)} \\ &= -1. \end{aligned}$$

Next, we will show that the right-hand side of equality (2.1) is greater or equal to  $-1$ . Since  $r > t$ , we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} r^{n-1} dr}{\phi(t)} &\geq \limsup_{t \rightarrow \infty} \frac{\ln(t^{n-1} \int_t^\infty e^{-\phi(r)} dr)}{\phi(t)} \\ &= \limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} dr}{\phi(t)}. \end{aligned}$$

To finish proving the lemma, we prove the following claim.

**Claim 2.3**  $\limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} dr}{\phi(t)} \geq -1.$  ■

**Proof of Claim 2.3.** Assume the result is not true. Then, there exists  $\alpha > 1$  such that

$$\limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} dr}{\phi(t)} < -\alpha.$$

Thus, there exists  $t_0 > 0$  such that for all  $t > t_0$ , we have

$$(2.2) \quad \int_t^\infty e^{-\phi(r)} dr \leq e^{-\alpha\phi(t)}.$$

Let  $F(t) = \int_t^\infty e^{-\phi(r)} dr$ , and note that  $F'(t) = -e^{-\phi(t)}$ , and thus, (2.2) is equivalent to  $F(t)^{\frac{1}{\alpha}} \leq -F'(t)$ . Therefore, for  $t > t_0$ , we have

$$1 \leq -\frac{F'(t)}{F(t)^{\frac{1}{\alpha}}}.$$

Integrating both sides of the above inequality over  $t \in [t_0, \infty)$ , we get that  $\frac{1}{1-\frac{1}{\alpha}} F(t)^{1-\frac{1}{\alpha}}$  is unbounded, which gives a contradiction, and the claim is proved. This finishes the proof of Lemma 2.2. ■

**Remark 2.4** We note that in Claim 2.3, we have proved a stronger statement. Indeed, fix  $\alpha > 1$  and let

$$E = \left\{ t : \ln \int_t^\infty e^{-\phi(r)} dr < -\alpha\phi(t) \right\}.$$

Then,  $|E| < \infty$ . This follows from the fact that for all  $t \in E$ , we have that

$$1 < -\frac{F'(t)}{F(t)^{\frac{1}{\alpha}}}.$$

Thus,

$$(2.3) \quad |E| \leq \int_E -\frac{F'(t)}{F(t)^{\frac{1}{\alpha}}} dt \leq \int_{t_0}^\infty -\frac{F'(t)}{F(t)^{\frac{1}{\alpha}}} dt < \infty.$$

**Remark 2.5** It is tempting to replace limit superior by the actual limit in the statement of Lemma 2.2. This may be done in many particular cases of measure  $\mu$ , but it is not true in general. Indeed, if we assume that

$$\lim_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(r)} dr}{\phi(t)} = -1,$$

then there exists  $T > 0$  such that for all  $t > T$ , we have

$$\left| \frac{\ln \int_t^\infty e^{-\phi(r)} dr}{\phi(t)} + 1 \right| \leq 1.$$

In particular,

$$(2.4) \quad \int_t^\infty e^{-(\phi(r)-\phi(t))} dr \geq e^{-\phi(t)}.$$

Using convexity of  $\phi$ , we get that  $\phi(r) - \phi(t) \geq \phi'(t)(r - t)$ , and thus, combining this with (2.4), we get that

$$(2.5) \quad \phi'(t) \leq e^{\phi(t)}, \text{ for all } t > T.$$

Let us show that there is an increasing, positive, convex, piecewise quadratic function  $\phi$  which has sufficiently large derivative at a sequence of points  $t_k \rightarrow \infty$ ; such  $\phi$  would contradict (2.5).

We define function  $\phi$  to be quadratic on each interval  $[k, k+1]$  and show that there exist  $t_k \in (k, k+1)$ , for all  $k \in \{0, 1, \dots\}$ , which would contradict (2.5). Let  $\phi(0) = \phi'(0) = 1$ . Assume we have constructed desired function  $\phi$  on interval  $[0, k]$  with  $\phi(k) = a_k$ ,  $\phi'(k) = b_k$ . Consider an auxiliary quadratic function  $\phi_k : [k, \infty) \rightarrow [a_k, \infty)$ , such that  $\phi'_k(t) = \alpha_k(t-k) + b_k$ , where  $\alpha_k > 0$  to be selected later. Thus,  $\phi_k(t) = \alpha_k(t-k)^2/2 + b_k(t-k) + a_k$ . Our goal is to find  $t_k \in (k, k+1)$  and  $\alpha_k$  such that  $\alpha_k(t-k) + b_k > e^{\alpha_k(t-k)^2/2 + b_k(t-k) + a_k}$ . Let  $t_k = k + 1/\sqrt{\alpha_k}$ . Then, the previous inequality becomes  $\sqrt{\alpha_k} + b_k > e^{1/2 + b_k/\sqrt{\alpha_k} + a_k}$ , which is true for all  $\alpha_k$  large enough (and in particular allows us to guarantee that  $t_k \in (k, k+1)$ ). We now set  $\phi(t) = \phi_k(t)$  for  $t \in [k, k+1]$  and repeat the process for the interval  $[k+1, k+2]$ .

We remind that for two convex, symmetric bodies  $K, L \subset \mathbb{R}^n$ , we define  $r(K, L) = \max\{R > 0 : RL \subset K\}$ . The next lemma may be seen as a generalization of the classical large deviation principle (see, for example, Corollary 4.9.3 in [2]).

**Lemma 2.6** Consider a symmetric body  $K \subset \mathbb{R}^n$ . Then,

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(r(K, B_2^n)t)} = -1.$$

**Proof** Let  $R = r(K, B_2^n)$ . Then,  $(tK)^c \subset (tRB_2^n)^c$ . Using Lemma 2.2, we get

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(tR)} \leq \limsup_{t \rightarrow \infty} \frac{\ln \mu((tRB_2^n)^c)}{\phi(tR)} = -1.$$

To obtain the reverse inequality, we denote by  $P$  a plank of width  $2R$  which contains  $K$ . More precisely, using the maximality of  $R$ , there exist at least two tangent points  $y, -y \in R\mathbb{S}^{n-1} \cap \partial K$ . Thus, we may consider  $P = \{x \in \mathbb{R}^n : |\langle x, y \rangle| \leq R\}$ . Next,

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(tR)} \geq \limsup_{t \rightarrow \infty} \frac{\ln \mu((tP)^c)}{\phi(tR)}.$$

By the rotation invariant of  $\mu$ , we may assume that  $y = Re_n$ , and so

$$\mu((tP)^c) = 2 \int_{tR}^{\infty} \int_{\mathbb{R}^{n-1}} e^{-\phi(|ze_n + x|)} dx dz.$$

Using the triangle inequality and the polar coordinates, we get

$$\begin{aligned} \mu((tP)^c) &\geq 2 \int_{tR}^{\infty} \int_{\mathbb{R}^{n-1}} e^{-\phi(z+|x|)} dx dz \\ &= 2 \int_{tR}^{\infty} \int_{\mathbb{S}^{n-2}} \int_0^{\infty} e^{-\phi(z+r)} r^{n-2} dr d\theta dz \\ &= 2|\mathbb{S}^{n-2}| \int_0^{\infty} r^{n-2} \int_{tR}^{\infty} e^{-\phi(z+r)} dz dr \\ &= 2|\mathbb{S}^{n-2}| \int_0^{\infty} r^{n-2} \int_{tR+r}^{\infty} e^{-\phi(z)} dz dr \end{aligned}$$

$$\begin{aligned}
&= 2|\mathbb{S}^{n-2}| \int_{tR}^{\infty} e^{-\phi(z)} \int_0^{z-tR} r^{n-2} dr dz \\
&= 2 \frac{|\mathbb{S}^{n-2}|}{n-1} \int_{tR}^{\infty} (z-tR)^{n-1} e^{-\phi(z)} dz.
\end{aligned}$$

Now to finish the proof of Lemma 2, we need to prove the following claim.

**Claim 2.7**

$$\limsup_{t \rightarrow \infty} \frac{\ln \int_{tR}^{\infty} (z-tR)^m e^{-\phi(z)} dz}{\phi(Rt)} \geq -1,$$

for any nonnegative integer  $m$ .

**Proof of Claim 2.7.** Making the change of variables, we get

$$\limsup_{t \rightarrow \infty} \frac{\ln \int_{tR}^{\infty} (z-tR)^m e^{-\phi(z)} dz}{\phi(Rt)} = \limsup_{t \rightarrow \infty} \frac{\ln \int_t^{\infty} (r-t)^m e^{-\phi(r)} dr}{\phi(t)}.$$

We will first prove the following inductive step: fix a nonnegative integer  $m$ , and let

$$F_m(t) = \int_t^{\infty} (r-t)^m e^{-\phi(r)} dr.$$

Then,

$$(2.6) \quad \liminf_{t \rightarrow \infty} \frac{\ln F_m(t)}{\ln F_{m-1}(t)} = 1, \text{ for all } m \in \mathbb{N}.$$

We note that  $F_m(t) \leq 1$ , for  $t$  large enough, and thus, the denominator and numerator are negative. It is a bit easier to work with a fraction when both the denominator and numerator are nonnegative. So we will prove that  $\liminf_{t \rightarrow \infty} \frac{-\ln F_m(t)}{-\ln F_{m-1}(t)} = 1$ . Using integration by parts, we get

$$F_{m-1}(t) = \frac{1}{m} \int_t^{\infty} (r-t)^m \phi'(r) e^{-\phi(r)} dr \geq \frac{1}{m} \phi'(t) \int_t^{\infty} (r-t)^m e^{-\phi(r)} dr,$$

where, again, we denote by  $\phi'(t)$  the left derivative of  $\phi$ . Thus,

$$-\ln F_{m-1}(t) \leq -\ln(\phi'(t)/m) - \ln F_m(t),$$

and

$$\liminf_{t \rightarrow \infty} \frac{-\ln F_m(t)}{-\ln F_{m-1}(t)} \geq \liminf_{t \rightarrow \infty} \frac{\ln(\phi'(t)/m) - \ln F_{m-1}(t)}{-\ln F_{m-1}(t)}.$$

Now we may use that  $\phi'(t) > a > 0$  for  $t$  large enough and  $\lim_{t \rightarrow \infty} \ln F_{m-1}(t) = -\infty$  to claim that

$$(2.7) \quad \liminf_{t \rightarrow \infty} \frac{-\ln F_m(t)}{-\ln F_{m-1}(t)} \geq \liminf_{t \rightarrow \infty} \frac{\ln(a/m) - \ln F_{m-1}(t)}{-\ln F_{m-1}(t)} \geq 1.$$

To prove the reverse inequality, we note that

$$F'_m(t) = -m \int_t^{\infty} (r-t)^{m-1} e^{-\phi(r)} dr.$$

Assume that

$$\liminf_{t \rightarrow \infty} \frac{-\ln F_m(t)}{-\ln(-\frac{1}{m}F'_m(t))} > \alpha > 1,$$

but then, again there exists  $t_0 > 0$  such that for all  $t > t_0$ , we have

$$-\ln F_m(t) > -\alpha \ln(-\frac{1}{m}F'_m(t)),$$

and thus,

$$m < \frac{-F'_m(t)}{F_m(t)^{\frac{1}{\alpha}}}.$$

Take an integral over  $t \in [x, \infty)$  from both sides to get  $F_m^{1-\frac{1}{\alpha}}(x) = \infty$ , which is a contradiction. This finishes the proof of the inductive step, but we actually need a bit stronger statement, which is similar to Remark 2.4. Indeed, consider any  $m \in \mathbb{N}$  and  $\alpha > 1$ . Let

$$E_{m,\alpha} = \{t : -\ln F_m(t) > -\alpha \ln F_{m-1}(t)\}.$$

Then, using the same ideas as in (2.3), we get  $|E_{m,\alpha}| < \infty$ .

To complete our proof, let

$$X_i(t) = \frac{\ln F_i(t)}{\ln F_{i-1}(t)} \text{ and } Y(t) = \frac{\ln F_0(t)}{\phi(t)}.$$

Using (2.6), we get  $\liminf_{t \rightarrow \infty} X_i(t) = 1$ , and using (2.1), we get  $\limsup_{t \rightarrow \infty} Y(t) = -1$ . Now

let  $X(t) = \prod_{i=1}^m X_i(t)$ . Our goal is to prove that

$$\limsup_{t \rightarrow \infty} X(t)Y(t) \geq -1.$$

Assume that this is not true. Then, there exists  $\alpha > 1$  such that

$$\limsup_{t \rightarrow \infty} X(t)Y(t) < -\alpha < -1.$$

Therefore, there exists  $t_0$  such that for all  $t > t_0$ ,

$$(2.8) \quad X(t)Y(t) \leq -\alpha.$$

Using (2.7), we may also assume that  $X_i(t) > 0$  for all  $t > t_0$ . Next, consider the set

$$A := \left\{t > t_0 : X(t) > \frac{\alpha+1}{2}\right\}.$$

We claim that  $|A| < \infty$ . Note that

$$\begin{aligned} \left|\left\{t : X(t) > \frac{\alpha+1}{2}\right\}\right| &\leq \left|\left\{t : X_i(t) > \left(\frac{\alpha+1}{2}\right)^{\frac{1}{m}}, \text{ for some } i \in \{1, \dots, m\}\right\}\right| \\ &< \sum_{i=1}^m \left|\left\{t : X_i(t) > \left(\frac{\alpha+1}{2}\right)^{\frac{1}{m}}\right\}\right| < \infty. \end{aligned}$$



We also note that  $\frac{2\alpha}{\alpha+1} > 1$ , and thus,

$$\left| \left\{ t : Y(t) < -\frac{2\alpha}{\alpha+1} \right\} \right| < \infty.$$

Finally,

$$\begin{aligned} |\{t > t_0 : X(t)Y(t) < -\alpha\}| &= \left| \left\{ t > t_0 : Y(t) < -\frac{\alpha}{X(t)} \right\} \right| \\ &\leq |A| + \left| \left\{ t > t_0 : Y(t) < -\frac{\alpha}{X(t)} \text{ and } X(t) < \frac{\alpha+1}{2} \right\} \right| \\ &\leq |A| + \left| \left\{ t : Y(t) < -\frac{2\alpha}{\alpha+1} \right\} \right| < \infty, \end{aligned}$$

which contradicts with (2.8). The claim is proved, and this finishes the proof of Lemma 2.6. ■

We are now ready to prove Theorem 2.1.

**Proof** Let  $K \subset \mathbb{R}^n$  be a convex, symmetric body such that  $\mu(tRB_2^n) \leq \mu(tK)$  holds for some fixed  $R > 0$  and every  $t$  large enough, but  $RB_2^n \notin K$ . Thus, the maximal Euclidean ball in  $K$  has radius  $rR$ , with  $r \in (0, 1)$ . From the assumption, it follows that

$$\mu((tRB_2^n)^c) \geq \mu((tK)^c),$$

which implies that

$$\frac{\ln \mu((tRB_2^n)^c)}{\phi(tR)} \geq \frac{\ln \mu((tK)^c)}{\phi(tR)} \frac{\phi(trR)}{\phi(tR)}.$$

From the convexity of  $\phi$  and  $r \in (0, 1)$ , we get that

$$\phi(trR) = \phi(trR + (1-r)0) \leq r\phi(tR) + (1-r)\phi(0).$$

Using that  $\phi(tR) \rightarrow \infty$ , we get that there exists  $r' \in (0, 1)$  and  $t_0 > 0$  such that  $\frac{\phi(trR)}{\phi(tR)} \leq r'$  for all  $t > t_0$ . Thus,

$$(2.9) \quad \frac{\ln \mu((tRB_2^n)^c)}{\phi(tR)} \geq r' \frac{\ln \mu((tK)^c)}{\phi(trR)}$$

for all  $t > t_0$ . Taking the limit superior, as  $t \rightarrow \infty$ , from both sides of the inequality (2.9), we obtain  $-1 \geq -r'$ . But this contradicts the fact that  $r'$  is less than 1. Therefore, our assumption that  $RB_2^n \notin K$  must be false. ■

**Remark 2.8** The rotation invariant assumption on  $\mu$  in Theorem 2.1 is necessary. Indeed, one can construct an example of a log-concave probability measure that is not rotation invariant in  $\mathbb{R}^2$  which does not satisfy the statement of Theorem 2.1. Consider the rectangle  $\Omega = \{(x, y) : |x| \leq \frac{\pi}{2}, |y| \leq \frac{1}{2}\}$ , and define the measure  $\mu$  as

$\mu(K) = \frac{|K \cap \Omega|}{|\Omega|}$ . Taking  $K = \Omega$ , we have

$$B_2^2 \not\subset \Omega, \text{ but } |B_2^2| = |\Omega| = \pi \text{ and } |tB_2^2| = |t\Omega|, \forall t > 0.$$

Note that  $\mu(tB_2^2) \leq \mu(t\Omega)$ ;  $\forall t > 0$ ; indeed, this is equivalent to  $|tB_2^2 \cap \Omega| \leq |t\Omega \cap \Omega|$ . If  $t \leq 1$ , then we have  $|tB_2^2 \cap \Omega| \leq |tB_2^2| = |t\Omega|$ , and if  $t \geq 1$ , we get  $|tB_2^2 \cap \Omega| \leq |\Omega|$ . So, we provided an example where

$$\mu(tB_2^2) \leq \mu(tK); \quad \forall t > 0,$$

but  $B_2^2 \not\subset K$ .

### 3 The cases where density depends on the norm

In this section, we would like to give a proof Theorem 2.1 in a more general case, which would answer Question 2. The main idea and computation are in the same spirit as in the proof of Theorem 2.1.

**Theorem 3.1** *Let  $K, L \subset \mathbb{R}^n$  be convex, symmetric bodies, and let  $\mu$  be a log-concave probability measure, with density  $e^{-\phi(\|x\|_L)}$ , where  $\phi: [0, \infty) \rightarrow [0, \infty)$  is an increasing, convex function. If for every  $t$  large enough and some  $R > 0$ ,*

$$\mu(tRL) \leq \mu(tK),$$

*then  $RL \subseteq K$ .*

**Proof** We have to check Lemma 2.2 and Lemma 2.6 (i.e., to prove yet another generalization of the classical large deviation principle (see (3.2) below)).

We claim that for any  $R > 0$ ,

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tRL)^c)}{\phi(tR)} = -1.$$

We can assume  $R = 1$ . Moreover, as before, using convexity of  $\phi$ , we may assume that  $\phi(t)$  is a strictly increasing function for large enough  $t$ . Thus,

$$\begin{aligned} \mu((tL)^c) &= \int_{(tL)^c} e^{-\phi(\|x\|_L)} dx = \int_{(tL)^c} \int_{\phi(\|x\|_L)}^{\infty} e^{-u} du dx \\ &= \int_{\mathbb{R}^n} \int_{\phi(\|x\|_L)}^{\infty} \chi_{(tL)^c}(x) e^{-u} du dx = \int_0^{\infty} \int_{\{x: \phi(\|x\|_L) < u\}} \chi_{(tL)^c}(x) e^{-u} dx du \\ &= \int_0^{\infty} e^{-u} |\{x: \|x\|_L \in [t, \phi^{-1}(u)]\}| du = |L| \int_{\phi(t)}^{\infty} ((\phi^{-1}(u))^n - t^n) e^{-u} du \\ &= |L| \int_t^{\infty} (v^n - t^n) \phi'(v) e^{-\phi(v)} dv = -|L| \int_t^{\infty} (v^n - t^n) de^{-\phi(v)} \\ (3.1) \quad &= n|L| \int_t^{\infty} v^{n-1} e^{-\phi(v)} dv. \end{aligned}$$

Note that  $\phi(t)$  may be a constant function on some interval  $[0, t_0]$  and strictly increasing on  $[t_0, \infty)$ . In such a case, we define  $\phi^{-1}(\phi(0)) = t_0$ . So, we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln \mu((tL)^c)}{\phi(t)} &= \limsup_{t \rightarrow \infty} \frac{\ln(n|L| \int_t^\infty e^{-\phi(v)} v^{n-1} dv)}{\phi(t)} \\ &= \limsup_{t \rightarrow \infty} \frac{\ln \int_t^\infty e^{-\phi(v)} v^{n-1} dv}{\phi(t)}, \\ &= -1, \end{aligned}$$

where the last equality follows from the proof of Lemma 2.2.

To finish the proof, we must check Lemma 2.6. In particular, we want to show that

$$(3.2) \quad \limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(r(K, L)t)} = -1$$

for symmetric, convex bodies  $K, L \subset \mathbb{R}^n$ , convex, increasing function  $\phi: [0, \infty) \rightarrow [0, \infty)$  and measure  $\mu$  with density  $e^{-\phi(\|x\|_L)}$ .

Let  $R = r(K, L)$ . Then, we have  $(tK)^c \subset (tRL)^c$  from the assumption. Thus, using Lemma 2.2, we get

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(tR)} \leq \limsup_{t \rightarrow \infty} \frac{\ln \mu((tRL)^c)}{\phi(tR)} = -1.$$

Using that  $RL$  is the maximal dilate of  $L$  inside  $K$ , we get that there is a pair of points  $v, -v \in \partial RL \cap \partial K$ . Let  $P$  be a plank created by tangent planes to  $RL$  and  $K$  at  $v$  and  $-v$ . Let  $n_v$  be a normal vector to  $\partial RL$  at  $v$ . Then, the width of the plank  $P$  is  $2Rh_L(n_v) = 2h_K(n_v)$ :  $P = \{x \in \mathbb{R}^n : |\langle x, n_v \rangle| \leq Rh_L(n_v)\}$ , where  $h_L(x) = \sup\{\langle x, y \rangle : y \in L\}$  is the support function of  $L$  (see [20] for basic definitions and properties). Next,

$$\limsup_{t \rightarrow \infty} \frac{\ln \mu((tK)^c)}{\phi(tR)} \geq \limsup_{t \rightarrow \infty} \frac{\ln \mu((tP)^c)}{\phi(tR)}.$$

Selecting a proper system of coordinates, we may assume that  $n_v = e_n$ . Let  $a = tRh_L(e_n)$ . Then,

$$\begin{aligned} \mu((tP)^c) &= 2 \int_a^\infty \int_{e_n^\perp} e^{-\phi(\|ze_n + x\|_L)} dx dz \\ &= 2 \int_a^\infty \int_{e_n^\perp} \int_{\phi(\|ze_n + x\|_L)}^\infty e^{-u} du dx dz \\ &= 2 \int_a^\infty \int_0^\infty \int_{\{x \in e_n^\perp : \phi(\|ze_n + x\|_L) < u\}} e^{-u} dx du dz \\ &= 2 \int_a^\infty \int_0^\infty e^{-u} |\{x \in e_n^\perp : \|ze_n + x\|_L \leq \phi^{-1}(u)\}| du dz. \end{aligned}$$

Now note that

$$|\{x \in e_n^\perp : \|ze_n + x\|_L \leq \phi^{-1}(u)\}| = |\{x \in e_n^\perp : ze_n + x \in \phi^{-1}(u)L\}|.$$

The above volume is zero if  $z > \phi^{-1}(u)h_L(e_n)$  (or  $\phi(z/h_L(e_n)) > u$ ). For  $z \in [0, \phi^{-1}(u)h_L(e_n)]$ , we note that  $\phi^{-1}(u)L$  is a convex body and thus contains

inside a pyramid  $\Delta$  with base  $L \cap e_n^\perp$  and the height  $\phi^{-1}(u)h_L(e_n)$  (with apex  $\phi^{-1}(u)h_L(e_n)v/R$ ). Then,

$$\begin{aligned} |\{x \in e_n^\perp : ze_n + x \in \phi^{-1}(u)L\}| &\geq |\Delta \cap (e_n^\perp + ze_n)| \\ &= (\phi^{-1}(u)h_L(e_n) - z)^{n-1} |L \cap e_n^\perp|. \end{aligned}$$

Thus,

$$\begin{aligned} \mu((tP)^c) &\geq 2|L \cap e_n^\perp| \int_a^\infty \int_{\phi(z/h_L(e_n))}^\infty e^{-u} (\phi^{-1}(u)h_L(e_n) - z)^{n-1} du dz \\ &= 2|L \cap e_n^\perp| \int_a^\infty \int_{z/h_L(e_n)}^\infty e^{-\phi(u)} \phi'(u) (uh_L(e_n) - z)^{n-1} du dz \\ &= -2|L \cap e_n^\perp| \int_a^\infty \int_{z/h_L(e_n)}^\infty (uh_L(e_n) - z)^{n-1} d e^{-\phi(u)} dz \\ &= 2(n-1)|L \cap e_n^\perp| \int_a^\infty \int_{z/h_L(e_n)}^\infty e^{-\phi(u)} (uh_L(e_n) - z)^{n-2} du dz \\ &= 2(n-1)|L \cap e_n^\perp| \int_{a/h_L(e_n)}^\infty \int_a^{h_L(e_n)u} e^{-\phi(u)} (uh_L(e_n) - z)^{n-2} dz du \\ &= 2|L \cap e_n^\perp| \int_{a/h_L(e_n)}^\infty e^{-\phi(u)} (uh_L(e_n) - a)^{n-1} du \\ &= 2h_L^{n-1}(e_n)|L \cap e_n^\perp| \int_{tR}^\infty e^{-\phi(u)} (u - tR)^{n-1} du. \end{aligned}$$

So, we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln \mu((tP)^c)}{\phi(tR)} &\geq \limsup_{t \rightarrow \infty} \frac{\ln (2h_L^{n-1}(e_n)|L \cap e_n^\perp| \int_{tR}^\infty e^{-\phi(u)} (u - tR)^{n-1} du)}{\phi(tR)} \\ &= \limsup_{t \rightarrow \infty} \frac{\ln \int_{tR}^\infty e^{-\phi(u)} (u - tR)^{n-1} du}{\phi(tR)}. \end{aligned}$$

By Claim 2, the above quantity is greater than or equal to  $-1$ ; thus, Lemma 2.6 is applied here, which finishes the proof for our main result. ■

**Remark 3.2** The proofs for Theorem 2.1 and Theorem 3.1 apply similarly to an asymmetric convex body  $K$  with the origin as an interior point of it. The only difference is that instead of dealing with a plank  $P$  in Lemma 2.6, we need to work with a half-space. Specifically, for Theorem 2.1, one would use the half-space  $H = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq R\}$ , where  $y \in \mathbb{R}S^{n-1} \cap \partial K$ . For Theorem 3.1, one may use the half-space defined by  $H = \{x \in \mathbb{R}^n : \langle x, n_v \rangle \leq Rh_L(n_v)\}$ , where  $n_v$  is the normal vector to  $\partial RL$  at a tangent point  $v$ .

## 4 The Busemann - Petty type problems

In this section, we will discuss Question 3. We first note that one must make some additional assumptions on the measure  $\mu$  to avoid a trivial answer. Indeed, if a measure

$\mu$  has a homogeneous density (i.e.,  $f(rx) = r^p f(x)$ , for  $r > 0$  and  $p > 1 - n$ ), then the answer is identical to the one given in [23].

Let us first show that in dimension  $n \geq 5$ , one can always find a pair of convex, symmetric bodies  $K$  and  $L$  and measure  $\mu$ , such that the answer to Question 3 is negative. The main idea follows from the construction in [24]. We begin with the following fact:

**Fact** If  $d\mu = e^{-\phi(\|x\|_L)} dx$  is a log-concave measure and  $K, L \subset \mathbb{R}^n$  are convex, symmetric bodies such that  $|K| \leq |RL|$  for some  $R > 0$ , then

$$\mu(K) \leq \mu(RL).$$

**Proof** Using calculations similar to (3.1), we get

$$\mu(K) = \int_0^\infty e^{-u} |K \cap \phi^{-1}(u)L| du$$

and

$$\mu(RL) = \int_0^\infty e^{-u} |RL \cap \phi^{-1}(u)L| du.$$

To get  $\mu(K) \leq \mu(RL)$ , we only need to check that  $|K \cap \phi^{-1}(u)L| \leq |RL \cap \phi^{-1}(u)L|$ . Indeed, if  $R \leq \phi^{-1}(u)$ , then we have

$$|K \cap \phi^{-1}(u)L| \leq |K| \leq |RL| = |RL \cap \phi^{-1}(u)L|,$$

and if  $R \geq \phi^{-1}(u)$ , then we get

$$|K \cap \phi^{-1}(u)L| \leq |\phi^{-1}(u)L| = |RL \cap \phi^{-1}(u)L|.$$

Hence,  $\mu(K) \leq \mu(RL)$  for any  $R > 0$ . ■

Next, we show that Question 3 has a negative answer for  $n \geq 5$ .

**Theorem 4.1** For  $n \geq 5$ , there are convex symmetric bodies  $K, L \subset \mathbb{R}^n$  and log-concave measure  $\mu$  with density  $e^{-\phi(\|x\|_L)}$ , such that

$$(4.1) \quad \mu(rK \cap \xi^\perp) \leq \mu(rL \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0,$$

but  $\mu(K) > \mu(L)$ .

**Proof** Let us assume, toward the contradiction, that Question 3 has an affirmative answer in  $\mathbb{R}^n$  for some fixed  $n \geq 5$ . So, for any pair of convex symmetric bodies  $K, L$  that satisfy (4.1), we would get  $\mu(K) \leq \mu(L)$ . The condition on sections (4.1) will be also satisfied for the dilated bodies  $tK$  and  $tL$ , for all  $t > 0$ . Therefore, we have

$$(4.2) \quad \mu(tK) \leq \mu(tL), \quad \forall t > 0,$$

which, by definition of  $\mu$ , means

$$\int_{tK} e^{-\phi(\|x\|_L)} dx \leq \int_{tL} e^{-\phi(\|x\|_L)} dx,$$

or equivalently, applying the change of variables  $x = tx$ , we have

$$\int_K e^{-\phi(t\|x\|_L)} dx \leq \int_L e^{-\phi(t\|x\|_L)} dx.$$

Using the continuity of  $\phi$  and compactness of  $K$  and  $L$ , we can take the limit for the above inequality as  $t \rightarrow 0^+$  to obtain

$$|K| \leq |L|.$$

Therefore, we have a relation between the dilation problem for a log-concave probability measure, with the Busemann-Petty problem for volume measure, which is if

$$\mu(rK \cap \xi^\perp) \leq \mu(rL \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0,$$

then  $|K| \leq |L|$ .

A number of very interesting counterexamples to the Busemann-Petty problem were shown by Papadimitrakakis [17]; Gardner [7]; Gardner, Koldobsky, and Schlumprecht [10]: there are convex symmetric bodies  $K, L$  in  $\mathbb{R}^n$  for  $n \geq 5$  such that

$$(4.3) \quad |K \cap \xi^\perp| \leq |L \cap \xi^\perp|, \quad \forall \xi \in \mathbb{S}^{n-1},$$

but

$$(4.4) \quad |K| > |L|.$$

Note that because the volume measure is homogeneous, the condition on sections (4.3) is also true for dilates of  $K$  and  $L$ , so we have

$$(4.5) \quad |rK \cap \xi^\perp| \leq |rL \cap \xi^\perp|, \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0.$$

Now, applying the fact to (4.5), we get that

$$\mu(rK \cap \xi^\perp) \leq \mu(rL \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0.$$

Thus, using (4.2), we have

$$\mu(tK) \leq \mu(tL), \quad \forall t > 0.$$

Dividing by  $t^n$  and taking the limit of the above inequality as  $t \rightarrow 0^+$ , we get

$$|K| \leq |L|,$$

and this contradicts (4.4). ■

It is interesting to note that the measure  $\mu$  constructed above is very specific. For example, we cannot use this construction directly with the assumption that  $\mu$  is rotation invariant.

Still, we can show that the answer to Question 3 is negative in  $\mathbb{R}^n$  for  $n \geq 7$  even when  $\mu$  is a log-concave measure with rotation invariant density.

**Theorem 4.2** *For dimension  $n \geq 7$ , and  $d\mu = e^{-\phi(|x|)} dx$ , there is a convex symmetric body  $K \subset \mathbb{R}^n$  such that*

$$\mu(rK \cap \xi^\perp) \leq \mu(rB_2^n \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}, \quad \forall r > 0,$$

but  $\mu(K) > \mu(B_2^n)$ .

**Proof** Giannopoulos [11] and Bourgain [4] constructed an example in  $\mathbb{R}^n$  for  $n \geq 7$  of convex body  $K \subset \mathbb{R}^n$  that satisfies

$$|K \cap \xi^\perp| \leq |B_2^n \cap \xi^\perp|, \quad \forall \xi \in \mathbb{S}^{n-1},$$

but  $|K| > |B_2^n|$ . To prove Theorem 4.2, one may take the same convex body  $K$  and  $B_2^n$  as provided in [11, 4] and repeat the proof of Theorem 4.1. ■

**Acknowledgements** We are grateful to Matthieu Fradelizi, Dylan Langharst, Fedor Nazarov, and Mokshay Madiman for a number of valuable discussions and suggestions. Finally, we thank the two anonymous referees, whose remarks and corrections were an enormous help!

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