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Article

Sumset Estimates in Convex Geometry

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Sumset estimates, which provide bounds on the cardinality of sumsets of finite sets in a group, form an essential part of the toolkit of additive combinatorics. In recent years, probabilistic or entropic forms of many of these inequalities were introduced. We study analogues of these sumset estimates in the context of convex geometry and the Lebesgue measure on $\mathbb{R}^n$. First, we observe that, with respect to Minkowski summation, volume is supermodular to arbitrary order on the space of convex bodies. Second, we explore sharp constants in the convex geometry analogues of variants of the Plünnecke-Ruzsa inequalities. In the last section of the paper, we provide connections of these inequalities to the classical Rogers-Shephard inequality.

1 Introduction

Minkowski summation is a basic and ubiquitous operation on sets. Indeed, the Minkowski sum $A+B = \{a+b : a \in A, b \in B\}$ of sets A and B makes sense as long as A and B are subsets of an ambient set in which a closed binary operation denoted by $+$ is defined. In particular, this notion makes sense in any group, and *additive combinatorics* (which arose out of exploring the additive structure of sets of integers, but then expanded to the consideration of additive structure in more general groups) is a field of mathematics that is preoccupied with studying what exactly this operation does in a quantitative way.

"Sumset estimates" are a collection of inequalities developed in additive combinatorics that provide bounds on the cardinality of sumsets of finite sets in a group. In this paper, we use $\#(A)$ to denote the cardinality of a countable set A, and $|A|$ to denote the volume (i.e., n-dimensional Lebesgue measure) of A when A is a measurable subset of $\mathbb{R}^n$. The simplest sumset estimate is the two-sided inequality $\#(A)\#(B) \geq \#(A+B) \geq \#(A) + \#(B) - 1$, which holds for finite subsets A, B of the integers; equality in the second inequality holds only for arithmetic progressions. A much more sophisticated sumset estimate is Kneser's theorem [32] (cf., [60, Theorem 5.5], [14]), which asserts that for finite, nonempty subsets A, B in any abelian group G, $\#(A+B) \geq \#(A+H) + \#(B+H) - \#(H)$, where H is the stabilizer of $A+B$, that is, $H = \{g \in G : A+B+g = A+B\}$. Kneser's theorem contains, for example, the Cauchy-Davenport inequality that provides a sharp lower bound on sumset cardinality in $\mathbb{Z}/p\mathbb{Z}$. In the reverse direction of finding upper bounds on cardinality of sumsets, there are the so-called Plünnecke-Ruzsa inequalities [49, 51]. One example of the latter states that if $\#(A+B) \leq \alpha\#A$, then $\#(A+k\cdot B) \leq \alpha^k\#A$, where $k \cdot B$ refers to the sum of k copies of B. Such sumset estimates form an essential part of the toolkit of additive combinatorics.

In the context of the Euclidean space $\mathbb{R}^n$, inequalities for the volume of Minkowski sums of convex sets, and more generally Borel sets, play a central role in geometry and functional analysis. For example, the well-known Brunn-Minkowski inequality can be used to deduce the Euclidean isoperimetric inequality, which identifies the Euclidean ball as the set of any given volume with minimal surface

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area. Therefore, it is somewhat surprising that in the literature, there has only been rather limited exploration of geometric analogues of sumset estimates. We work towards correcting that oversight in this contribution.

The goal of this paper is to explore a variety of new inequalities for volumes of Minkowski sums of convex sets, which have a combinatorial flavor and are inspired by known inequalities in the discrete setting of additive combinatorics. These inequalities are related to the notion of supermodularity: we say that a set function $F : 2^{[n]} \to \mathbb{R}$ is *supermodular* if $F(s \cup t) + F(s \cap t) \geq F(s) + F(t)$ for all subsets s, t of $[n]$, and that F is *submodular* if $-F$ is supermodular.

Our study is motivated by two relatively recent observations. The first observation motivating this paper, from [24, Theorem 4.5], states that given convex bodies A, B_1, B_2 in $\mathbb{R}^n$, $|A + B_1 + B_2| + |A| \geq |A + B_1| + |A + B_2|$. This inequality has a form similar to that of Kneser's theorem– indeed, observe that the latter can be written as $\#(A + B + H) + \#(H) \geq \#(A + H) + \#(B + H)$, since adding the stabilizer to $A + B$ does not change it. Furthermore, it implies that the function $v : 2^{[k]} \to \mathbb{R}$ defined, for given convex bodies $B_1, \ldots, B_k$ in $\mathbb{R}^n$, by $v(s) = \left|\sum_{i\in s} B_i\right|$ is supermodular. Foldes and Hammer [17] defined the notion of higher order supermodularity for set functions. In Section 3, we generalize their definition and main characterization theorem from [17] to functions defined on $\mathbb{R}^n_+$, and apply it to show that volumes and mixed volumes satisfy this higher order supermodularity.

The second observation motivating this paper, due to Bobkov and the second named author [9, Corollary 7.5], is that for any convex bodies A, B_1, B_2 in $\mathbb{R}^n$, one has

$$|A||A + B_1 + B_2| \leq 3^n |A + B_1||A + B_2|. \tag{1}$$

The above inequality is inspired by an inequality in information theory analogous to the Plünnecke-Ruzsa inequality (the most general version of which was proved by Ruzsa for compact sets in [53], and is discussed in Section 2.2 below). If not for the multiplicative factor of 3^n in (1), this inequality would imply that the *logarithm* of the volume of the Minkowski sum of convex sets is submodular. In this sense, it goes in the reverse direction to the supermodularity of volume and thus complements it. However, the constant 3^n obtained by [9] is rather loose. We take up the question of tightening this constant in Section 4.

Specifically, we obtain both upper and lower bounds for the optimal constant

$$c_n = \sup \frac{|A||A + B + C|}{|A + B||A + C|}, \tag{2}$$

where the supremum is taken over all convex bodies A, B, C in $\mathbb{R}^n$, in general dimension n. We get an upper bound of $c_n \leq \varphi^n$ in Section 4.2, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio, and an asymptotic lower bound of $c_n \geq (4/3 + o(1))^n$ in Section 4.4. In Section 4.3, we show that the optimal constant is 1 in dimension 2 and $\frac{4}{3}$ in dimension 3 (i.e., $c_2 = 1$ and $c_3 = 4/3$), and also that $c_4 \leq 2$. In Section 4.5, we improve inequality (1) in the special case where A is an ellipsoid, B_1 is a zonoid, and B_2 is any convex body: in this case, the optimal constant is 1. This result partially answers a question of Courtade, who asked (motivated by an analogous inequality in information theory) if $|A + B_1 + B_2|\,|A| \leq |A + B_1|\,|A + B_2|$ holds when A is the Euclidean ball and B_1, B_2 are arbitrary convex bodies. Finally, in Section 4.6, we prove that (1) cannot possibly hold in the more general setting of compact sets with any absolute constant, which signifies a sharp difference between the proof of this inequality compared with the tools used by Ruzsa in [53].

The last section of the paper is dedicated to questions surrounding Ruzsa's triangle inequality: if A, B, and C are finite subsets of an abelian group, then $\#(A)\#(B - C) \leq \#(A - B)\#(A - C)$. This inequality is also known to be true for volume of compact sets in $\mathbb{R}^n$: $|A|\,|B - C| \leq |A - B|\,|A - C|$. We investigate the best constant c such that the inequality

$$|A|\,|A + B + C| \leq c|A - B|\,|A - C|. \tag{3}$$

is true for all convex sets A, B and C in $\mathbb{R}^n$. For example, in the plane, we observe that it holds with the sharp constant $c = \frac{3}{2}$.

Again, it is interesting to note that (3) is different from Ruzsa's triangle inequality, and it is not true, with any absolute constant c, if one omits the assumption of convexity.

In a companion paper [25] written together with M. Meyer, we explore the question of reducing the constant in the Plünnecke-Ruzsa inequality for volumes from φ^n, when we restrict attention to the subclass of convex bodies known as zonoids. In another series of papers [19, 20] written together with D. Langharst, we explore "weighted" extensions of the preceding results for convex bodies, focusing on log-concave and in particular Gaussian measures.

We also mention that there are probabilistic or entropic analogues of many of the inequalities in this paper. For example, the afore-mentioned observation due to [9], that a Plünnecke-Ruzsa inequality for convex bodies holds with a constant 3^n, emerges as a consequence of Rényi entropy comparisons for convex measures on the one hand, and the submodularity of entropy of convolutions on the other. The submodularity of entropy of convolutions refers to the inequality $h(X)+h(X+Y+Z) \leq h(X+Y)+h(X+Z)$, where h denotes entropy, and X, Y, Z are independent $\mathbb{R}^n$-valued random variables, and may be thought of as an entropic analogue of the Plünnecke-Ruzsa inequality. This latter inequality was obtained in [35] as part of an attempt to develop an additive combinatorics theory for probability measures where cardinality or volume of a set is replaced by entropy of a random variable. A number of works have explored this avenue [30, 35, 37, 38] for probability measures on $\mathbb{R}^n$ and more general locally compact abelian groups; for discrete analogues (e.g., when the random variables take values in finite groups or the integers), see for example [1, 39, 40, 43, 54, 59].

2 Preliminaries

2.1 Mixed Volumes

In this section, we introduce basic notations and collect essential facts and definitions from convex geometry that are used in the paper. As a general reference on the theory we use [56].

We write $x \cdot y$ for the inner product of vectors x and y in $\mathbb{R}^n$ and by $|x|$ the length (Euclidean norm) of a vector $x \in \mathbb{R}^n$. The closed unit ball in $\mathbb{R}^n$ is denoted by B_2^n, and its boundary by S^{n-1}. We will also denote by $e_1, \dots, e_n$ the standard orthonormal basis in $\mathbb{R}^n$. Moreover, for any set in $A \subset \mathbb{R}^n$, we denote its boundary by ∂A. We write $|K|_m$ for the m-dimensional Lebesgue measure (volume) of a measurable set $K \subset \mathbb{R}^n$, where $m = 1, \dots, n$ is the dimension of the minimal affine space containing K, we will often use the shorten notation $|K|$ for the n-dimensional volume.

We write $\mathcal{K}_n$ for the collection of nonempty convex compact subsets of $\mathbb{R}^n$. A convex body in $\mathbb{R}^n$ is a convex, compact set with nonempty interior; we write $\mathcal{K}_n^{(o)}$ for the collection of convex bodies in $\mathbb{R}^n$. A polytope which is the Minkowski sum of finitely many line segments is called a zonotope. Limits of zonotopes in the Hausdorff metric are called zonoids; see [56], Section 3.2, for details. From [56, Theorem 5.1.6], for any compact convex sets $K_1, \dots K_r$ in $\mathbb{R}^n$ and any non-negative numbers $t_1, \dots, t_r$, one has

$$|t_1 K_1 + \cdots + t_r K_r| = \sum_{i_1,\dots,i_n=1}^{r} t_{i_1} \cdots t_{i_n} V(K_{i_1}, \dots, K_{i_n}), \tag{4}$$

for some non-negative numbers $V(K_{i_1}, \dots, K_{i_n})$, which are called the mixed volumes of $K_1, \dots, K_r$. One readily sees that the mixed volumes satisfy $V(K, \dots, K) = |K|$. Moreover, they satisfy a number of properties which are crucial for our study (see [56]) including the fact that a mixed volume is symmetric in its argument; it is multilinear, that is, for any $\lambda, \mu \geq 0$ we have $V(\lambda K+\mu L, K_2, \dots, K_n) = \lambda V(K, K_2, \dots, K_n)+ \mu V(L, K_2, \dots, K_n)$. Mixed volume is translation invariant, that is, $V(K + a, K_2, \dots K_n) = V(K, K_2, \dots, K_n)$, for $a \in \mathbb{R}^n$ and satisfy a monotonicity property, i.e $V(K, K_2, K_3, \dots, K_n) \leq V(L, K_2, K_3, \dots, K_n)$, for $K \subset L$. We will also often use a two body version of (4)—the Steiner formula:

$$|A + tB| = \sum_{k=0}^{n} \binom{n}{k} t^k V(A[n-k], B[k]), \tag{5}$$

for any $t \geq 0$ and any compact, convex sets A, B in $\mathbb{R}^n$, where for simplicity we use the notation $A[m]$ when a convex set A is repeated m times.

Mixed volumes are also very useful for studying the volume of orthogonal projections of convex bodies. Let $P_H A$ be the orthogonal projection of a convex body A onto an m dimensional subspace H of

$\mathbb{R}^n$, then

$$|P_H A|_m |U|_{n-m} = \binom{n}{m} V(A[m], U[n-m]), \tag{6}$$

where U is any convex body in the subspace $H^\perp$ orthogonal to H. For example, if we denote by $\theta^\perp = \{x \in \mathbb{R}^n : x \cdot \theta = 0\}$ the hyperplane orthogonal to $\theta \in S^{n-1}$ and passing through the origin, we obtain

$$|P_{\theta^\perp} A|_{n-1} = nV(A[n-1], [0, \theta]). \tag{7}$$

Yet another useful formula is connected with the computation of the surface area and mixed volumes:

$$|\partial A| = nV(A[n-1], B_2^n), \tag{8}$$

where by $|\partial A|$ we denote the surface area of the compact set A in $\mathbb{R}^n$.

Mixed volumes satisfy a number of extremely useful inequalities. The first one is the Brunn-Minkowski inequality:

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{9}$$

whenever A, B and $A + B$ are measurable. The most powerful inequality for mixed volumes is the Alexandrov–Fenchel inequality:

$$V(K_1, K_2, K_3, \ldots, K_n) \geq \sqrt{V(K_1, K_1, K_3, \ldots, K_n) V(K_2, K_2, K_3, \ldots, K_n)}, \tag{10}$$

for any compact convex sets $K_1, \ldots K_r$ in $\mathbb{R}^n$. We will also use the following classical local version of Alexandrov-Fenchel's inequality that was proved by W. Fenchel (see [16] and also [56]) and further generalized in [4, 18, 57],

$$|A| V(A[n-2], B, C) \leq 2V(A[n-1], B) V(A[n-1], C), \tag{11}$$

for any convex compact sets A, B, C in $\mathbb{R}^n$. Moreover it was noticed in [57] that (11) is true with constant one instead of two in the case when A is a simplex. This inequality is one part of a rich class of Bézout-type inequalities proposed in [55, 57]. The core tool of our work is the following inequality of J. Xiao (Theorem 1.1 and Lemma 3.4 in [63])

$$\begin{aligned} |A| V(A[n-j-m], &B[j], C[m]) \\ &\leq \min\left(\binom{n}{j}, \binom{n}{m}\right) V(A[n-j], B[j]) V(A[n-m], C[m]), \end{aligned} \tag{12}$$

for any convex, compact sets $A, B, C \subset \mathbb{R}^n$.

2.2 Plünnecke-Ruzsa inequality

Plünnecke-Ruzsa inequalities (see, e.g., [60]) are an important class of inequalities in the field of additive combinatorics. These were introduced by Plünnecke [49] and generalized by Ruzsa [51], and a simpler proof was given by Petridis [47]; a more recent generalization is proved in [29], and entropic versions are developed in [40]. For illustration, the form of Plünnecke's inequality developed in [51] states that, if $A, B_1, \ldots, B_m$ are finite sets in a commutative group, then there exists an $X \subset A$, $X \neq \emptyset$, such that

$$\#(A)^m \#(X + B_1 + \ldots + B_m) \leq \#(X) \prod_{i=1}^{m} \#(A + B_i).$$

In [53], Ruzsa generalized the above inequality to the case of compact sets on a locally compact commutative group, written additively, with the Haar measure. The volume case of this deep theorem is one of our main inspirations: for any compact sets $A, B_1, \ldots, B_m$ in $\mathbb{R}^n$, with $|A| > 0$ and for every $\varepsilon > 0$, there exists a compact set $A' \subset A$ such that

$$|A|^m |A' + B_1 + \ldots + B_m| \leq (1+\varepsilon)|A'| \prod_{i=1}^{m} |A + B_i|. \tag{13}$$

It immediately follows that for any compact sets $A, B_1, \ldots, B_m$ in $\mathbb{R}^n$,

$$|A|^{m-1} |B_1 + \ldots + B_m| \leq \prod_{i=1}^{m} |A + B_i|. \tag{14}$$

2.3 Submodularity and supermodularity

Let us first recall the notion of a supermodular set function.

Definition 2.1. A set function $F : 2^{[n]} \to \mathbb{R}$ is *supermodular* if

$$F(s \cup t) + F(s \cap t) \geq F(s) + F(t), \tag{15}$$

for all subsets s, t of $[n]$.

One says that a set function F is submodular if $-F$ is supermodular. Submodularity is closely related to a partial ordering on hypergraphs as we will see below. This relationship is frequently attributed to Bollobas and Leader [11] (cf. [5]), where they introduced the related notion of "compressions". However, it appears to have origins that trace back much further—explicitly discussed by Emerson [15], where he says it is "well known".

To present this relationship, let us introduce some notation. Let $\mathcal{M}(n,m)$ be the following family of (multi)hypergraphs: each consists of non-empty (ordinary) subsets s_i of $[n]$, $s_i = s_j$ is allowed, and $\sum_i |s_i| = m$. Consider a given multiset $\mathcal{C} = \{s_1, \ldots, s_l\} \in \mathcal{M}(n,m)$. The idea is to consider an operation that takes two sets in $\mathcal{C}$ and replaces them by their union and intersection; however, note that

(i) if s_i and s_j are nested (i.e., either $s_i \subset s_j$ or $s_j \subset s_i$), then replacing (s_i, s_j) by $(s_i \cap s_j, s_i \cup s_j)$ does not change $\mathcal{C}$; and

(ii) if $s_i \cap s_j = \emptyset$, the empty set may enter the collection, which would be undesirable.

Thus, take any pair of non-nested sets $\{s_i, s_j\} \subset \mathcal{C}$ and let $\mathcal{C}' = \mathcal{C}(i,j)$ be obtained from $\mathcal{C}$ by replacing s_i and s_j by $s_i \cap s_j$ and $s_i \cup s_j$, keeping only $s_i \cup s_j$ if $s_i \cap s_j = \emptyset$. $\mathcal{C}'$ is called an *elementary compression* of $\mathcal{C}$. The result of a sequence of elementary compressions is called a *compression*.

Define a partial order on $\mathcal{M}(n,m)$ by setting $\mathcal{A} > \mathcal{A}'$ if $\mathcal{A}'$ is a compression of $\mathcal{A}$. To check that this is indeed a partial order, one needs to rule out the possibility of cycles, which can be done by noting that if $\mathcal{A}'$ is an elementary compression of $\mathcal{A}$ then

$$\sum_{s \in \mathcal{A}} |s|^2 < \sum_{s \in \mathcal{A}'} |s|^2.$$

Theorem 2.2. Suppose F is a supermodular function on the ground set $[n]$. Let $\mathcal{A}$ and $\mathcal{B}$ be finite multisets of subsets of $[n]$, with $\mathcal{A} > \mathcal{B}$. Then,

$$\sum_{s \in \mathcal{A}} F(s) \leq \sum_{t \in \mathcal{B}} F(t).$$

Proof. When $\mathcal{B}$ is an *elementary* compression of $\mathcal{A}$, the statement is immediate by definition, and transitivity of the partial order gives the full statement. ■

Note that for every multiset $\mathcal{A} \in \mathcal{M}(n,m)$ there is a unique minimal multiset $\mathcal{A}^{\#}$ dominated by $\mathcal{A}$, that is, $\mathcal{A}^{\#} < \mathcal{A}$, consisting of the sets $s_j^{\#} = \{i \in [n] : i \text{ lies in at least} j \text{ of the sets} s \in \mathcal{A}\}$. Thus, a particularly nice instance of Theorem 2.2 is for the special case of $\mathcal{B} = \mathcal{A}^{\#}$ (we refer to [5, page 132], for further discussion). We also have a notion of supermodularity on the positive orthant of the Euclidean space.

Definition 2.3. A function $f : \mathbb{R}_+^n \to \mathbb{R}$ is supermodular if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y),$$

for any $x, y \in \mathbb{R}_+^n$, where $x \vee y$ denotes the componentwise maximum of x and y and $x \wedge y$ denotes the componentwise minimum of x and y.

We note that Definition 2.3 can be viewed as an extension of Definition 2.1 if one consider set functions on $2^{[n]}$ as a function on $\{0, 1\}^n$. We record this connection in the next lemma, using $\mathbf{1}_s : [n] \to \{0, 1\}$ to denote the indicator function of s, that is, $\mathbf{1}_s(i) = 1$ if $i \in s$ and $\mathbf{1}_s(i) = 0$ if $i \notin s$.

Lemma 2.4. If $f : \mathbb{R}_+^n \to \mathbb{R}$ is supermodular, and we set $F(s) := f(\mathbf{1}_s)$ for each $s \subset [n]$, where we view $\mathbf{1}_s \in \{0, 1\}^{[n]}$ as a vector in $\mathbb{R}^n$, then F is a supermodular set function.

Proof. Observe that

$$\begin{aligned} F(s \cup t) + F(s \cap t) &= f(\mathbf{1}_{s\cup t}) + f(\mathbf{1}_{s\cap t}) \\ &= f(\mathbf{1}_s \vee \mathbf{1}_t) + f(\mathbf{1}_s \wedge \mathbf{1}_t) \\ &\geq f(\mathbf{1}_s) + f(\mathbf{1}_t) \\ &= F(s) + F(t). \end{aligned}$$

■

The fact that supermodular functions are closely related to functions with increasing differences is classical (see, e.g., [36] or [62], which describes more general results involving arbitrary lattices). We will denote by $\partial_i f$ the partial derivative of function f with the respect the i's coordinate and by $\partial^m_{i_1,\ldots,i_m}$ the mixed derivative with respect to coordinates $i_1, \ldots i_m$.

Proposition 2.5. Suppose a function $f : \mathbb{R}_+^n \to \mathbb{R}$ is in C^2, that is, it is twice-differentiable with a continuous Hessian matrix. Then, f is supermodular if and only if

$$\partial^2_{i,j} f(x) \geq 0,$$

for every distinct $i, j \in [n]$, and for any $x \in \mathbb{R}_+^n$.

We will prove Proposition 2.5 as a part of a more general statement on the mixed derivatives of the supermodular functions of higher order (Theorem 3.5 below).

Supermodular set functions also arise naturally in connection with convex functions. For instance, let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a convex function. Then for every $a_0, a_1, a_2 \in \mathbb{R}_+$, one has

$$\varphi(a_0 + a_1) + \varphi(a_0 + a_2) \leq \varphi(a_0 + a_1 + a_2) + \varphi(a_0).$$

This property can be seen as the supermodularity of the function $\Phi : 2^{[2]} \to \mathbb{R}$ defined by $\Phi(s) = \varphi(a_0 + \langle \mathbf{1}_s, a\rangle)$, where we set $a = (a_1, a_2)$.

3 Higher Order Supermodularity of Mixed Volumes

3.1 Local characterization of higher order supermodularity

We now present analogues of the above development for higher-order supermodularity. Let us notice that a set function $F : 2^{[n]} \to \mathbb{R}$ is supermodular if and only if for any $s_0, s_1, s_2 \in 2^{[n]}$ with $s_1 \cap s_2 = \emptyset$

one has

$$F(s_0 \cup s_1) + F(s_0 \cup s_2) \le F(s_0 \cup s_1 \cup s_2) + F(s_0).$$

Generalizing this property, Foldes and Hammer [17] defined the notion of higher order supermodularity. In this section, we will adapt their definition and study the following property:

Definition 3.1. Let $1 \le m \le n$. A function $F : 2^{[n]} \to \mathbb{R}$ is m-supermodular if for any $s_0 \in 2^{[n]}$ and for any mutually disjoint $s_1, \dots, s_m \in 2^{[n]}$ one has

$$\sum_{I \subset [m]} (-1)^{m-|I|} F\left(s_0 \cup \bigcup_{i \in I} s_i \right) \ge 0.$$

Note that for $m = 2$ in the above definition, we recover a supermodular set function. We also introduce the notion of higher-order supermodularity for functions defined on the positive orthant of a Euclidean space.

Definition 3.2. Let $1 \le m \le n$. A function $f : \mathbb{R}_+^n \to \mathbb{R}$ is m-supermodular if

$$\sum_{I \subset [m]} (-1)^{m-|I|} f\left(x_0 \vee \bigvee_{i \in I} x_i \right) \ge 0,$$

for any $x_0 \in \mathbb{R}_+^n$ and for any $x_1, \dots, x_m \in \mathbb{R}_+^n$, with mutually disjoint supports, that is such that $x_i \wedge x_j = 0$, for any $1 \le i < j \le m$.

Remark 3.3. Fix some $n \in \mathbb{N}$ and $m \in [n]$. Notice that, as in Lemma 2.4, if $f : \mathbb{R}_+^n \to \mathbb{R}$ is m-supermodular then $F : 2^{[n]} \to \mathbb{R}$ defined by $F(s) = f(\mathbf{1}_s)$ is m-supermodular.

For $m = 1$ in the above definition, we obtain that f is 1-supermodular if and only if it is non-decreasing in each coordinate. For $m = 2$, we recover a supermodular function on the orthant as we prove in the following lemma.

Lemma 3.4. Let $f : \mathbb{R}_+^n \to \mathbb{R}$. Then, f is supermodular if and only if for any $x, y, z \in \mathbb{R}_+^n$ such that $y \wedge z = 0$, one has

$$f(x \vee y \vee z) + f(x) \ge f(x \vee y) + f(x \vee z). \tag{16}$$

Proof. Suppose f is supermodular and one has $x, y, z \in \mathbb{R}_+^n$ such that $y \wedge z = 0$. Then, we set $a = x \vee y$ and $b = x \vee z$, then, $a \vee b = x \vee y \vee z$ and $a \wedge b = x$, since $y \wedge z = 0$. Thus,

$$f(x \vee y \vee z) + f(x) - f(x \vee y) - f(x \vee z) = f(a \wedge b) + f(a \vee b) - f(a) - f(b) \ge 0.$$

Now assume that f satisfies (16) and let $a, b \in \mathbb{R}_+^n$. We set $x = a \wedge b$ and we define y by putting $y_i = a_i$ for i such that $b_i < a_i$ and $y_i = 0$ otherwise. In the same way, we set $z_i = b_i$, for i such that $a_i < b_i$ and $z_i = 0$ otherwise. Then, $x \vee y = a$, $x \vee z = b$ and $x \vee y \vee z = a \vee b$, hence, we conclude similarly. ■

The next theorem generalizes Proposition 2.5 to higher order supermodularity.

Theorem 3.5. Let $n \in \mathbb{N}$ and $m \in [n]$. Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be a C^m function. Then, f is m-supermodular if and only if

$$\partial^m_{i_1,\dots,i_m} f(x) \ge 0$$

for every distinct $i_1, \dots, i_m \in [n]$, and for any $x \in \mathbb{R}_+^n$.

Proof. Let $x_0 \in \mathbb{R}^n_+$ and $x_1, \ldots, x_m \in \mathbb{R}^n_+$, with mutually disjoint supports.

$$\begin{aligned}
&\sum_{I\subset[m]} (-1)^{m-|I|} f\left(x_0 \vee \bigvee_{i\in I} x_i\right)\\
&\quad= \sum_{I\subset[m-1]} (-1)^{m-|I|-1} f\left(x_0 \vee \bigvee_{i\in I} x_i \vee x_m\right) + \sum_{I\subset[m-1]} (-1)^{m-|I|} f\left(x_0 \vee \bigvee_{i\in I} x_i\right)\\
&\quad= \sum_{I\subset[m-1]} (-1)^{m-1-|I|} \left[f\left(x_0 \vee \bigvee_{i\in I} x_i \vee x_m\right) - f\left(x_0 \vee \bigvee_{i\in I} x_i\right)\right]\\
&\quad= \sum_{I\subset[m-1]} (-1)^{m-1-|I|} g_{x_m}\left(x_0 \vee \bigvee_{i\in I} x_i\right),
\end{aligned} \tag{17}$$

where $g_z(x) = f(x \vee z) - f(x)$, for any $x, z \in \mathbb{R}^n_+$. Thus, f is m-supermodular if and only if $x \mapsto g_z(x)$ is $(m-1)$-supermodular for any $z \in \mathbb{R}^n_+$ as a function on the coordinate subspace $H_z = \{y \in \mathbb{R}^n_+ : y_i z_i = 0, \forall\, i = 1, \ldots, n\}$.

Now we are ready to prove the theorem for the case $m = 2$. In this case, the above equivalence (17) gives us that f is supermodular if and only if $x \mapsto g_z(x)$ is 1-supermodular for any $z \in \mathbb{R}^n_+$ as a function of $x \in H_z$ and thus g_z is non-decreasing in each coordinate direction of H_z, that is, for each coordinate index i such that $z_i = 0$. Thus, assuming differentiability, this is equivalent $\partial_i g_z \geq 0$ for all i which is a coordinate direction in H_z. Taking $z = z_j e_j, z_j > 0$, we get $\partial_i g_{z_j e_j} \geq 0$ for all $i \neq j$. Thus, $\partial_i f(x \vee z_j e_j) - \partial_i f(x) \geq 0$, and finally $\partial^2_{i,j} f(x) \geq 0$. Reciprocally, assuming that $\partial^2_{i,j} f(x) \geq 0$ for all $i \neq j$ and all x, we get $D_y \partial_i f(x) \geq 0$ for $y \in \mathbb{R}^n_+$ such that $y_i = 0$, where by $D_y f = y \cdot \nabla f$ we denote the directional derivative with respect to vector $y \in \mathbb{R}^n$. Thus $\partial_i f(x + y) - \partial_i f(x) \geq 0$ for all $y \in \mathbb{R}^n_+$ with $y_i = 0$. Thus, considering $y \in \mathbb{R}^n_+$, such that $y_i = 0$ and $y_j = x_j \vee z_j - x_j, j \neq i$, we get $\partial_i g(z) = \partial_i f(x \vee z) - \partial_i f(x) \geq 0$ for all i not in the support of z.

We will finish the proof applying an induction argument. Assume that the statement of the theorem is true for $m - 1$, for some $m \geq 3$. Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^m m-supermodular function. Then, applying (17) we get that $x \mapsto g_z(x)$ is $m - 1$-supermodular for any $z \in \mathbb{R}^n_+$ as a function of $x \in H_z$, which, applying inductive assumption, gives us

$$\partial_{i_1} \cdots \partial_{i_{m-1}} [f(x \vee z) - f(x)] \geq 0,$$

for every distinct $i_1, \ldots, i_{m-1}$, coordinates of H_z, applying it to $z = z_{i_m} e_{i_m}$, we get

$$\partial_{i_1} \cdots \partial_{i_m} f(x) \geq 0.$$

Now assume the partial derivative condition of the theorem. Then, for every $z \in \mathbb{R}^n_+$ and $i_1, \ldots, i_{m-1}$, coordinates of H_z, we have

$$\partial_{i_1} \cdots \partial_{i_{m-1}} g_z(x) = \partial_{i_1} \cdots \partial_{i_{m-1}} f(x \vee z) - \partial_{i_1} \cdots \partial_{i_{m-1}} f(x),$$

but $\partial_{i_1} \cdots \partial_{i_{m-1}} \partial_{i_m} f(x) \geq 0$ for every $i_m \neq i_k$, for $k = 1, \ldots, m-1$ and thus for every i_m for which $z_{i_m} \neq 0$. So $\partial_{i_1} \cdots \partial_{i_{m-1}} f(x)$ is an non-decreasing function in each coordinate i_m for which $z_{i_m} \neq 0$:

$$\partial_{i_1} \cdots \partial_{i_{m-1}} f(x \vee z) - \partial_{i_1} \cdots \partial_{i_{m-1}} f(x) \geq 0$$

and, applying the inductive assumption, we get that $x \mapsto g_z(x)$ is $(m-1)$-supermodular for any $z \in \mathbb{R}^n_+$ as a function of $x \in H_z$, which finishes the proof with the help of (17). ■

We remark in passing that the positivity of mixed partial derivatives and its global manifestation also arises in the theory of copulas in probability (see, e.g., [13]). In particular, it is well known there that for smooth functions $C : [0,1]^m \to [0,1]$, the condition $\partial^m_{1,2,\ldots,m} C \geq 0$ is equivalent to the condition that $\sum_{z\in\{x_i,y_i\}^m} (-1)^{N(z)} C(z) \geq 0$ for every box $\prod_{i=1}^m [x_i, y_i] \subset [0,1]^m$, where $N(z) = \#\{k : z_k = x_k\}$.

3.2 Higher order supermodularity of volume

Theorem 3.6. Let $n, k \in \mathbb{N}$. Let $B_1, \dots, B_k$ be convex compact sets of $\mathbb{R}^n$. Then the function $v : \mathbb{R}_+^k \to \mathbb{R}$ defined as

$$v(x) = \left|\sum_{i=1}^{k} x_i B_i\right| \tag{18}$$

is m-supermodular for any $m \in [k]$.

Proof. From the mixed volume formula (4), the function v is a polynomial with non-negative coefficients so its mixed derivatives of any order are non-negative on $\mathbb{R}_+^n$. By Theorem 3.5, we conclude that it is m-supermodular for any m. ■

Remark 3.7. Theorem 3.6 can be given in a more general form: for any natural number $l \le n$, any convex compact sets $C_1, \dots, C_l, B_1, \dots, B_k$ in $\mathbb{R}^n$, the function $v : \mathbb{R}_+^k \to \mathbb{R}$, defined as

$$v(x) = V\left(\left(\sum_{i=1}^{k} x_i B_i\right)[n-l], C_1, \dots, C_l\right)$$

is m-supermodular for any $m \in [k]$.

We notice that in Theorem 3.6 the convexity assumption is essential. Indeed, as was observed in [24], for $k = 3$, there exists non convex sets B_1, B_2, B_3 such that the function v defined above is not supermodular. We will discuss this issue in more details in Section 3.3 below. Using Theorem 3.6, Remark 3.7, Remark 3.3 and Theorem 2.2 we deduce the following corollary.

Corollary 3.8. Let $n, k \in \mathbb{N}$ and $B_1, \dots B_k$ be compact convex sets of $\mathbb{R}^n$. Let $0 \le l \le n$ and $C_1, \dots, C_l$ be convex bodies in $\mathbb{R}^n$. Then:

1) the function $\bar{v} : 2^{[k]} \to [0, \infty)$ defined by

$$\bar{v}(s) = V\left(\left(\sum_{i \in s} B_i\right)[n-l], C_1, \dots, C_l\right), \tag{19}$$

for each $s \subset [k]$, is a m-supermodular set function, for any $m \in [k]$.

2) Let $\mathcal{A}$ and $\mathcal{B}$ be finite multisets of subsets of $[k]$, with $\mathcal{A} > \mathcal{B}$. Then,

$$\sum_{s \in \mathcal{A}} \bar{v}(s) \le \sum_{t \in \mathcal{B}} \bar{v}(t). \tag{20}$$

Let us note that the above m-supermodularity of the function $\bar{v}$ is equivalent to the fact that for any convex bodies $B_0, B_1, \dots, B_k, C_1, \dots, C_l$ in $\mathbb{R}^n$,

$$\sum_{s \subset [k]} (-1)^{k-|s|} V\left(\left(B_0 + \sum_{i \in s} B_i\right)[n-l], C_1, \dots, C_l\right) \ge 0.$$

Applying the previous theorem to $l = 0$, we get

$$\sum_{s \subset [k]} (-1)^{k-|s|} \left|B_0 + \sum_{i \in s} B_i\right| \ge 0. \tag{21}$$

The above inequality for $k = n$ and $B_0 = \{0\}$ follows also directly from the following classical formula (see Lemma 5.1.4 in [56])

$$\sum_{s \subset [n]} (-1)^{n-|s|} \left| \sum_{i \in s} B_i \right| = n!\, V(B_1, \ldots, B_n).$$

In the same way, we can also give another proof of the general case of (21).

Theorem 3.9. Let $B_0, B_1, \ldots, B_m$ be convex bodies in $\mathbb{R}^n$.

$$\sum_{s \subset [m]} (-1)^{m-|s|} \left| B_0 + \sum_{i \in s} B_i \right| = \sum_{\substack{\sum_{i=0}^m k_i = n; \\ k_1, \ldots, k_m \geq 1}} \binom{n}{k_0, k_1, \ldots, k_m} V(B_0[k_0], B_1[k_1], \ldots, B_m[k_m])$$

for $m \leq n$ and zero otherwise.

Proof. Following the proof of Lemma 5.1.4 in [56]. Define

$$g(t_0, t_1, \ldots, t_m) = \sum_{s \subset [m]} (-1)^{m-|s|} \left| t_0 B_0 + \sum_{i \in s} t_i B_i \right|.$$

Observe that g is a homogeneous polynomial of degree n and note that $g(t_0, 0, t_2, \ldots, t_m) = 0$, which can be seen by noticing that, in this particular case, the sum is telescopic. This implies that, in the polynomial $g(t_0, \ldots, t_m)$, all monomials with non-zero coefficients must contain a non-zero power of t_1. The same being true for each t_i, $i \geq 1$, there is no non-zero monomials if $m > n$. If $m \leq n$ all non-zero monomials must come from the case $|s| = m$, that is, from

$$|t_0 B_0 + t_1 B_1 + \cdots + t_m B_m|,$$

which finishes the proof. ■

Thanks to the fact that supermodular set functions taking the value 0 at the empty set are fractionally superadditive (see, e.g., [42, 44]), we can immediately deduce the following inequality. Let $n \geq 1$, $k \geq 2$ be integers and let $A_1, \ldots, A_k$ be convex sets in $\mathbb{R}^n$. Then, for any fractional partition β using a hypergraph $\mathcal{C}$ on $[k]$,

$$\left| \sum_{i=1}^{k} A_i \right| \geq \sum_{s \in \mathcal{C}} \beta(s) \left| \sum_{j \in s} A_j \right|. \tag{22}$$

It was shown in [6] that (22) actually extends to all compact sets in $\mathbb{R}^n$, but supermodularity does not extend to compact sets as discussed in the next section. In fact, an even stronger inequality—fractional superadditivity of the functional $vr(A) = |A|^{1/n}$—had been conjectured for compact sets in [10] motivated by analogy to information theoretic inequalities [3, 36], but this conjecture was shown to be false in [23]. As observed in [41], the latter counterexample also shows that a natural conjecture about Schur-concavity of volume is false when applied to non-convex sets.

3.3 Going beyond convex bodies

Consider sets $A, B \subset \mathbb{R}^n$, such that $0 \in B$. Define $\Delta_B(A) = (A + B) \setminus A$, and note that $A + B$ is always a superset of A because of the assumption that $0 \in B$. The supermodularity of volume is also saying something about set increments. Indeed, for any sets A, B, C consider

$$\Delta_C \Delta_B(A) = \Delta_C\big((A + B) \setminus A\big) = \big(((A + B) \setminus A) + C\big) \setminus \big((A + B) \setminus A\big).$$

We have, if $0 \in B \cap C$:

$$\begin{aligned} |\Delta_C \Delta_B(A)| &= \left|\left((A+B) \setminus A\right) + C\right| - |(A+B) \setminus A| \\ &\geq |(A+B+C) \setminus (A+C)| - |A+B| + |A| \\ &= |A+B+C| - |A+C| - |A+B| + |A|, \end{aligned} \tag{23}$$

where the inequality follows from the general fact that $(K+C) \setminus (L+C) \subset (K \setminus L) + C$. Moreover, if A, B, C are convex, compact sets then the estimate is non-trivial, that is, using Theorem 3.9, we get that the right-hand side of the above quantity is non-negative. It is interesting to note that the Δ operation is not commutative, that is, $\Delta_C \Delta_B(A) \neq \Delta_B \Delta_C(A)$; this can be seen, for example, in $\mathbb{R}^2$ by taking A to be a square, B to be a segment, and C to be a Euclidean ball.

It is natural to ask if the higher-order analog of this observation remains true.

Question 3.10. Let $m \in \mathbb{N}$ and $B_1, \dots B_m \subset \mathbb{R}^n$ be compact sets containing the origin. Then, for any compact $B_0 \subset \mathbb{R}^n$, is it true that

$$\left|\Delta_{B_1} \dots \Delta_{B_m}(B_0)\right| \geq \sum_{s \subset [m]} (-1)^{m-|s|} \left|B_0 + \sum_{i \in s} B_i\right| ?$$

The inequality (23) gives a positive answer to the above question in the case $m = 2$. We also observe that if $B_0, B_1, \dots, B_m$ are convex, then the right-hand side is non-negative thanks to Theorem 3.6. We note that it was observed in [24], by considering $A = \{0, 1\}$ and $B = C = [0, 1]$ in $\mathbb{R}^1$, that the volume of Minkowski sums cannot be supermodular (even in dimension 1) if the convexity assumption on the set A is removed. Nonetheless, [24] observed that if $A, B, C \subset \mathbb{R}$ are compact, then

$$|A+B+C| + |\mathrm{conv}(A)| \geq |A+B| + |A+C|;$$

it is unknown if this extends to higher dimension. In particular, we do not know if the following conjecture is true for $n \geq 2$.

Conjecture 3.11. For any convex body A and any compact sets B and C in $\mathbb{R}^n$,

$$|A+B+C| + |A| \geq |A+B| + |A+C|.$$

We can confirm Conjecture 3.11 under the assumption that B is a zonoid.

Theorem 3.12. Assume A is a convex compact set, B is a zonoid and C is any compact set in $\mathbb{R}^n$. Then,

$$|A+B+C| + |A| \geq |A+B| + |A+C|.$$

Proof. By approximation, we may assume that B is a zonotope. Using the definition of mixed volumes (4) and (7) we get that for any convex compact set M in $\mathbb{R}^n$,

$$|M + [0, tu]| - |M| = t|P_{u^\perp} M|_{n-1}, \text{ for all} t > 0, u \in S^{n-1}.$$

The above formula can be also proved using a geometric approach and thus studied in the case of not necessarily convex M. Indeed, consider a compact set M in $\mathbb{R}^n$, $t > 0$ and $u \in S^{n-1}$, let $\partial_u M$ be the set of all $x \in \partial M$ such that $x \cdot u \geq y \cdot u$, for all $y \in M$ for which $P_{u^\perp} y = P_{u^\perp} x$. Note that

$$(\partial_u M + (0, tu]) \cap M = \emptyset, \text{ but} M \cup (\partial_u M + (0, tu]) \subseteq M + [0, tu].$$

Thus,

$$|M + [0, tu]| \geq |M| + t|P_{u^\perp} M|_{n-1}.$$

Now, we are ready to prove the theorem with $B = [0, tu]$,

$$|A + C + [0, tu]| - |A + C| \geq t|P_{u^\perp}(A + C)|_{n-1} \geq t|P_{u^\perp} A|_{n-1} = |A + [0, tu]| - |A|.$$

Thus, we proved that, for any $u \in \mathbb{R}^n$,

$$|A + C + [0, u]| - |A + [0, u]| \geq |A + C| - |A|. \tag{24}$$

Now, we can prove the theorem for the case of a zonotope. Indeed, let $Z_k = \sum_{i=1}^k [0, u_i]$ be a zonotope. Apply inequality (24) to the convex body $A + Z_{k-1}$ and the vector $u = u_k$ to get

$$|A + C + Z_k| - |A + Z_k| \geq |A + C + Z_{k-1}| - |A + Z_{k-1}|.$$

Iterate the above inequality to prove the theorem for the case of B being a zonotope. The theorem now follows from continuity of the volume and the fact that every zonoid is a limit of zonotopes. ■

4 Plünnecke-Ruzsa Inequalities for Convex Bodies

4.1 Existing Plünnecke-Ruzsa inequality for convex bodies

Bobkov and Madiman [9] developed a technique for going from entropy to volume estimates, by using certain reverse Hölder inequalities that hold for convex measures. Specifically, [9, Proposition 3.4] shows that if X_i are independent random variables with X_i uniformly distributed on a convex body $K_i \subset \mathbb{R}^n$ for each $i = 1, \ldots, m$, then $h(X_1 + \ldots + X_m) \geq \log|K_1 + \ldots + K_m| - n \log m$, where the entropy of a random variable X with density f on $\mathbb{R}^n$ is defined by

$$h(X) = -\int f(x) \log f(x) dx. \tag{25}$$

This is a reverse Hölder inequality in the sense that $h(X_1 + \ldots + X_m) \leq \log|K_1 + \ldots + K_m|$ may be seen by applying Hölder's inequality and then taking a limit. More general sharp inequalities relating Rényi entropies of various orders for measures having convexity properties are described in [22] (see also [7, 8, 26]). Applied to the submodularity of entropy of sums discovered in [35], the paper [9] uses this technique to demonstrate the following inequality.

Theorem 4.1. Let $\mathcal{C}_k$ denote the collection of all subsets of $[m] = \{1, \ldots, m\}$ that are of cardinality k. Let A and $B_1, \ldots, B_m$ be convex bodies in $\mathbb{R}^n$, and suppose

$$\left|A + \sum_{i \in s} B_i\right|^{\frac{1}{n}} \leq c_s |A|^{\frac{1}{n}}$$

for each $s \in \mathcal{C}_k$, with given numbers c_s. Then,

$$\left|A + \sum_{i=1}^m B_i\right|^{\frac{1}{n}} \leq (1 + m) \left[\prod_{s \in \mathcal{C}_k} c_s\right]^{\frac{1}{\binom{m-1}{k-1}}} |A|^{\frac{1}{n}}.$$

In particular, by choosing $k = 1$, one already obtains an interesting inequality for volumes of Minkowski sums.

Corollary 4.2. Let A and $B_1, \dots, B_m$ be convex bodies in $\mathbb{R}^n$. Then,

$$|A|^{m-1}\left|A + \sum_{i=1}^{m} B_i\right| \le (1+m)^n \prod_{i=1}^{m} |A + B_i|.$$

Thus, one may think of Corollary 4.2 as providing yet another continuous analogues of the Plünnecke-Ruzsa inequalities in the context of volumes of convex bodies in Euclidean spaces (compare with (13)), where going from the discrete to the continuous incurs the extra factor of $(1+m)$, but one does not need to bother with taking subsets of the set A. In particular, with $m = 2$, one gets "log-submodularity of volume up to an additive term" on convex bodies.

Corollary 4.3. Let A and B_1, B_2 be convex bodies in $\mathbb{R}^n$. Then,

$$|A|\,|A + B_1 + B_2| \le 3^n |A + B_1|\,|A + B_2|. \tag{26}$$

Unfortunately, the dimension-dependent additive term is a hindrance that one would like to remove or improve, which is the purpose of the next section.

Remark 4.4. We notice that in the case where $B_1 = B_2 = B$, the inequality holds with constant 1:

$$|A|\,|A + B + B| \le |A + B|^2.$$

by the Brunn-Minkowski inequality. In the next section, we shall see that it is no longer true for $B_1 \neq B_2$. Moreover, as we will see in Lemma 4.22, the above inequality is not true with any absolute constant if we only assume that the sets A and B_1 are compact, which exposes an essential difference of this inequality with (13).

4.2 Improved upper bounds in general dimension

In this section, we will present an improvements in the constant 3^n in the three body inequality from Corollary 4.3. We denote the best constant in such an inequality by c_n; this is defined by (2), or equivalently as the infimum of the constants $c > 0$ such that, for every convex compact sets A, B, C in $\mathbb{R}^n$,

$$|A|\,|A + B + C| \le c|A + B|\,|A + C|.$$

For convenience, let us define

$$c(A, B, C) = \frac{|A|\,|A + B + C|}{|A + B|\,|A + C|},$$

so that $c_n = \sup_{A,B,C \in \mathcal{K}_n^{(0)}} c(A, B, C)$. The following lemma finds repeated use.

Lemma 4.5. Let $n \ge 2$. For compact convex sets $A, B, C \subset \mathbb{R}^n$, let

$$v_{A,B,C}(j, m) = \frac{V(A[n-j-m], B[j], C[m])}{(n-j-m)!},$$

and

$$d(A, B, C) = \max_{j,m: 0 \le j+m \le n} \frac{v_{A,B,C}(0,0)\, v_{A,B,C}(j, m)}{v_{A,B,C}(j, 0)\, v_{A,B,C}(0, m)}.$$

Then $c(A, B, C) \le d(A, B, C)$. Consequently, $c_n \le d_n := \sup_{A,B,C \in \mathcal{K}_n^{(0)}} d(A, B, C)$.

Proof. We apply (4) to get

$$|A|\,|A+B+C| = \sum_{k+j+m=n} \binom{n}{k,j,m} |A| V(A[k], B[j], C[m])$$
$$= \sum_{0\le j+m\le n} \binom{n}{j,m,n-j-m} |A| V(A[n-j-m], B[j], C[m]),$$
$$|A+B|\,|A+C| = \sum_{j=0}^{n}\sum_{m=0}^{n} \binom{n}{j}\binom{n}{m} V(A[n-j], B[j]) V(A[n-m], C[m]).$$

The comparison of the above sums term by term shows that $c(A,B,C) \le d$ if d satisfies

$$|A| V(A[n-j-m], B[j], C[m]) \le d \frac{\binom{n}{j}\binom{n}{m}}{\binom{n}{j,m,n-j-m}} V(A[n-j], B[j]) V(A[n-m], C[m]),$$

for each $m, j \ge 0$ with $m + j \le n$. The above may be rewritten in a more symmetric way as

$$\frac{|A|}{n!} \frac{V(A[n-j-m], B[j], C[m])}{(n-j-m)!} \le d \frac{V(A[n-j], B[j])}{(n-j)!} \frac{V(A[n-m], C[m])}{(n-m)!}. \tag{27}$$

By definition, $d(A,B,C)$ is the best constant that satisfies all these conditions, and we deduce that $c(A,B,C) \le d(A,B,C)$. ■

We recall that $\varphi = (1+\sqrt{5})/2$ denotes the golden ratio.

Theorem 4.6. Let $n \ge 2$. Then, one has $1 = c_2 \le c_n \le \varphi^n$, that is, for every convex compact sets $A, B, C \subset \mathbb{R}^n$,

$$|A|\,|A+B+C| \le \varphi^n |A+B|\,|A+C|.$$

Proof. Observe that, taking $B = C = \{0\}$ we get $c_n \ge 1$. For the upper bound, we use Lemma 4.5.

Notice that for $m = 0$ or $j = 0$, (27) trivially holds for $d = 1$. Using inequality (12), we get that d will satisfy inequality (27) as long as

$$\min\left\{\binom{n-j}{m}, \binom{n-m}{j}\right\} \le d. \tag{28}$$

Note that the above is true with constant $d = 1$ if $m + j = n$.

We also note that, if $m = j = 1$, then the required inequality (27) becomes

$$|A| V(A[n-2], B, C) \le d \frac{n}{n-1} V(A[n-1], B) V(A[n-1], C). \tag{29}$$

Using (11), we see that in this case, it suffices to select $d = \frac{2(n-1)}{n}$. In particular, we get that $c_2 = d_2 = 1$.

For the more general case, we can provide a bound for $d_n^{1/n}$ using Stirling's approximation formula. Indeed,

$$\binom{p}{q} \le \frac{p^p}{(p-q)^{p-q} q^q} e^{\frac{1}{12p} - \frac{1}{12(p-q)+1} - \frac{1}{12q+1}} \sqrt{\frac{p}{2\pi(p-q)q}}$$

$$\binom{p}{q} \le \frac{p^p}{(p-q)^{p-q} q^q} e^{\frac{1}{12p} - \frac{12p+2}{(12(p-q)+1)(12q+1)}} \sqrt{\frac{p}{2\pi(p-1)}} \le \frac{p^p}{(p-q)^{p-q} q^q}.$$

Next, let $j = yn$ and $m = xn$, where $x, y \geq 0$, $x + y \leq 1$, then it is sufficient for d_n to satisfy

$$\max_{\substack{x,y\geq 0\\ x+y\leq 1}} \min\left\{\frac{(1-y)^{1-y}}{(1-x-y)^{1-x-y}x^x}, \frac{(1-x)^{1-x}}{(1-x-y)^{1-x-y}y^y}\right\} \leq d_n^{1/n}.$$

Without loss of generality we may assume that $|x - 1/2| \leq |y - 1/2|$ and thus $(1-x)^{1-x}x^x \leq (1-y)^{(1-y)}y^y$. Our next goal is to provide an upper estimate for

$$\max \frac{(1-x)^{1-x}}{(1-x-y)^{1-x-y}y^y},$$

where the maximum is taken over a set

$$\begin{aligned}\Omega &= \{(x,y) \in \mathbb{R}_+^2 : x + y \leq 1, |1/2 - x| \leq |1/2 - y|\}\\ &= \{(x, 1-x) : 0 \leq x \leq 1/2\} \cup \{(x,y) \in \mathbb{R}_+^2 : y \leq \min(x, 1-x)\}.\end{aligned}$$

We note that the function $y \mapsto (1-x-y)^{1-x-y}y^y$ is decreasing for $y \in [0, (1-x)/2]$ and increasing on $[(1-x)/2, (1-x)]$. So we may consider two cases, comparing x and $(1-x)/2$. Next,

$$\max_{\Omega\cap\{x\in[0,1/3]\}} \frac{(1-x)^{1-x}}{(1-x-y)^{1-x-y}y^y} = \max_{[0,1/3]} \frac{(1-x)^{1-x}}{(1-2x)^{1-2x}x^x} = \frac{1+\sqrt{5}}{2}.$$

The last equality follows from the fact that the maximum is achieved when $\frac{(1-2x)^2}{(1-x)x} = 1$, that is, at $x = (5-\sqrt{5})/10$. Finally,

$$\max_{\Omega\cap\{x\in[1/3,1]\}} \frac{(1-x)^{1-x}}{(1-x-y)^{1-x-y}y^y} \leq \max_{x\in[1/3,1]} \frac{(1-x)^{1-x}}{((1-x)/2)^{1-x}} = 2^{2/3} < \frac{1+\sqrt{5}}{2}.$$

■

Remark 4.7. Recently Cheikh Saliou Ndiaye [46] proved a very useful extension to the inequality of Xiao (12) and thus extended Theorem 4.6 by providing an estimate for the best constant $c_{n,m}$ such that, for any convex bodies $A, B_1, \ldots, B_m$ in $\mathbb{R}^n$, then

$$|A|\,|A + B_1 + \cdots + B_m| \leq c_{n,m} \prod_{i=1}^{m} |A + B_i|. \tag{30}$$

Ndiaye [46] established in particular that $c_{n,m} \leq 2^n$ and other sharper bounds depending on m.

The next proposition gives a different proof of (14) for a special case of three convex sets and, we hope, gives yet another example of how the methods of mixed volumes as well as the Bézout type inequality (12) can be applied in this context.

Proposition 4.8. Let A, B, C be convex bodies in $\mathbb{R}^n$, then

$$|A|\,|B + C| \leq |A + B||A + C|, \tag{31}$$

with equality if and only if $|A| = 0$.

Proof. The inequality follows from the proof of Theorem 4.6 and the observation that decomposing the left- and right-hand sides of the (31), we need to show that

$$\sum_{m=0}^{n}\binom{n}{m}|A|V(B[n-m],C[m]) \le \sum_{j=0}^{n}\sum_{m=0}^{n}\binom{n}{j}\binom{n}{m}V(A[n-j],B[j])V(A[n-m],C[m]).$$

It turns out that it is enough to consider only the terms with $m+j=n$ on the right-hand side, that is, to show that

$$\sum_{m=0}^{n}\binom{n}{m}|A|V(B[n-m],C[m]) \le \sum_{m=0}^{n}\binom{n}{n-m}\binom{n}{m}V(A[m],B[n-m])V(A[n-m],C[m]),$$

which is true term by term by using (12) with $m+j=n$. Now assume that there is equality. This implies that the term $j=m=0$ in the above double sum must vanish, that is, $|A|=0$. ■

Remark 4.9. Cheikh Saliou Ndiaye [46] used mixed volume techniques as well as determinant inequalities to give a proof of Ruzsa's inequality (14) in the case of convex sets $A, B_1, \ldots, B_m$, that is, the extension of Proposition 4.8 from $m=2$ to general m.

4.3 Improved constants in dimensions 3 and 4

Theorem 4.6 gives an optimal bound of 1 for the three body inequality in dimension 2. Next, we will show how we can get better bounds for c_n in dimension 3 and 4.

Theorem 4.10. Let A, B, C be convex compact sets in $\mathbb{R}^3$. Then,

$$|A|\,|A+B+C| \le \frac{4}{3}|A+B|\,|A+C|$$

and the constant is best possible: $c_3 = \frac{4}{3}$. Moreover, if A is a simplex, then

$$|A|\,|A+B+C| \le |A+B|\,|A+C|.$$

Proof. We follow the same strategy as in the proof of Theorem 4.6 and arrive to the inequality (27) with $m, j \ge 0$ and $m+j \le 3$:

$$\frac{|A|}{3!}\frac{V(A[3-j-m],B[j],C[m])}{(3-j-m)!} \le d\frac{V(A[3-j],B[j])}{(3-j)!}\frac{V(A[3-m],C[m])}{(3-m)!}.$$

Again, the inequality is trivially true for $m=0$ or $j=0$ with constant $d=1$. Thus, we are left with the two following inequalities

$$|A|V(C,B[2]) \le 3d\cdot V(A,B[2])V(A[2],C), \tag{32}$$

$$|A|V(A,B,C) \le \frac{3}{2}d\cdot V(B,A[2])V(C,A[2]). \tag{33}$$

We note that the inequality (32) with $d=1$ follows from (12). Next we note that (33) is true with $d=1$ when A is a simplex (see [57]). The general case of (33) follows from (11) with $d=4/3$. Thus, the maximal constant d_3 is 4/3, and it follows from Lemma 4.5 that $c_3 \le 4/3$. The optimality of this bound follows from Section 4.4, where we establish more generally that $c_n \ge 2-\frac{2}{n}$. ■

Theorem 4.11. Let A, B, C be convex compact sets in $\mathbb{R}^4$, then

$$|A|\,|A+B+C| \le 2|A+B|\,|A+C|.$$

Thus, $c_4 \le 2$. Moreover, if A is a simplex, then

$$|A|\,|A+B+C| \le |A+B|\,|A+C|.$$

Proof. We will check inequality (27) for $n = 4$, $0 \le m \le j \le 4$ and $m + j \le 4$:

$$\frac{|A|}{4!}\frac{V(A[4-j-m], B[j], C[j])}{(4-j-m)!} \le d\frac{V(A[4-j], B[j])}{(4-j)!}\frac{V(A[4-m], C[m])}{(4-m)!}. \tag{34}$$

The inequality is trivially true for $m = 0$ or $j = 0$ with a constant $d = 1$. Taking into account that the inequality is symmetric with respect to m and j and to B and C we get that it is enough to obtain cases $(j, m) = \{(1, 1); (1, 2); (1, 3); (2, 2)\}$. For $(j, m) = (1, 1)$, we require

$$|A|V(A[2], B, C) \le \frac{4}{3}d \cdot V(A[3], B)V(A[3], C).$$

If A is a simplex, then the above is true with $\frac{4}{3}d = 1$ (see [57]), and the general case follows from (11) with $\frac{4}{3}d = 2$ (or $d = \frac{3}{2}$). For $(j, m) = (1, 2)$, we require

$$|A|V(A, B, C[2]) \le 2d \cdot V(A[3], B)V(A[2], C[2]).$$

We again observe that if A is a simplex then the above is true with $2d = 1$. To resolve the general case, we apply (12) with $n = 4$ and $(j, m) = (1, 2)$ to get $2d = 4$, which will satisfy the requirement. When $(j, m) = (1, 3)$ and we may use the remark after (28) to claim that $d = 1$ will satisfy the condition (34). Thus, the maximal constant $d_4 = 2$, and Lemma 4.5 yields that $c_4 \le 2$. ■

Remark 4.12. We conjecture that actually $c_4 = 3/2$.

Remark 4.13. We conjecture that, for $1 \le j \le n$,

$$|A|V(L_1, \ldots, L_j, A[n-j]) \le jV(L_j, A[n-1])V(L_1, \ldots, L_{j-1}, A[n-j+1]).$$

This conjectured inequality would improve a special case of (12). It would also provide an improved constant for the $(j, m) = (1, 2)$ case in the proof of Theorem 4.11, and thereby show that $d_4 = 3/2$, which together with the lower bound from the next section, would yield the conjectured best constants ($c_4 = d_4 = 3/2$) in $\mathbb{R}^4$.

4.4 Lower bounds in general dimension

In this section, we provide a lower bound for the Plünnecke-Ruzsa inequality for convex bodies. A weaker lower bound was also independently obtained by Nayar and Tkocz [45]. We first observe that the best constant c_n in the Plünnecke-Ruzsa inequality:

$$|A|\,|A+B+C| \le c_n|A+B|\,|A+C|, \tag{35}$$

satisfies $c_{n+m} \ge c_n c_m$. Indeed, this follows immediately by considering critical examples of A_1, B_1, C_1 in $\mathbb{R}^n$ and A_2, B_2, C_2 in $\mathbb{R}^m$ together with their direct products $A_1 \times A_2, B_1 \times B_2, C_1 \times C_2$ in $\mathbb{R}^{n+m}$. Next, we notice that if (35) is true in a class of convex bodies closed by linear transformations, then

$$|P_{E\cap H}K||K| \le c_n|P_E K|\,|P_H K|, \tag{36}$$

for any K in this class and any subspaces E, H of $\mathbb{R}^n$, such that $\dim E = i$, $\dim H = j$, $i + j \geq n + 1$ and $E^\perp \subset H$. To see this consider $B = U$, with $\dim U = n - i$, $|U| = 1$ and $C = V$, with $\dim V = n - j$, $|V| = 1$ and U, V belong to orthogonal subspaces of $\mathbb{R}^n$. Let $A = tK$, where $t > 0$ and set $k = n - (n - i) - (n - j) = i + j - n$. Then, (35) yields together with (4) and (6),

$$t^n|K|\Big(\sum_{m=k}^{n}\binom{n}{m}V(K[m],(U+V)[n-m])t^m\Big)$$
$$\leq c_n\Big(\sum_{m=i}^{n}\binom{n}{m}V(K[m],U[n-m])t^m\Big)\Big(\sum_{m=j}^{n}\binom{n}{m}V(K[m],V[n-m])t^m\Big).$$

Dividing the above inequality by t^{n+k} and taking $t = 0$, we get

$$|K|\binom{n}{k}V(K[k],(U+V)[n-k]) \leq c_n\binom{n}{i}V(K[i],U[n-i])\binom{n}{j}V(K[j],V[n-j]).$$

Finally, using (6), we get (36). It was proved in [28] that

$$|P_{\{u,v\}^\perp}K||K| \leq \frac{2(n-1)}{n}|P_{u^\perp}K|\,|P_{v^\perp}K|, \tag{37}$$

for any convex body $K \subset \mathbb{R}^n$ and a pair of orthogonal vectors $u, v \in S^{n-1}$. It was also shown in [28] that the constant $2(n-1)/n$ is optimal. Thus, $c_n \geq 2 - \frac{2}{n}$ and this estimate gives a sharp constant in $\mathbb{R}^3$: $c_3 = 4/3$. In the case when $n = 4$, we get $c_4 \geq 3/2$.

The inequalities analogous to (37) and (36) were studied in many other works, including [2, 4, 18, 57]. In particular, it was proved in [2] that (36) is sharp with

$$c_n \geq c_n(i,j,k) = \frac{\binom{i}{k}\binom{j}{k}}{\binom{n}{k}}.$$

Thus, to find a lower bound on c_n, one may maximize over $c_n(i,j,k)$ with restriction that $i + j \geq n + 1$ and $k = i + j - n$. One may use Stirling's approximation, with $i = j = 2n/3$ and $k = n/3$ (when n is a multiple of 3, with minor modifications if not) to obtain the following theorem.

Theorem 4.14. For sufficiently large n, we have that $c_n \geq \left(\frac{4}{3} + o(1)\right)^n$.

4.5 Improved upper bound for subclasses of convex bodies

The goal of this section is to prove the following theorem:

Theorem 4.15. Let $n \geq 1$ and K be a convex body in $\mathbb{R}^n$. Let B be an ellipsoid and Z be a zonoid in $\mathbb{R}^n$. Then,

$$|B|\,|B + K + Z| \leq |B + K|\,|B + Z|.$$

Theorem 4.15 motivates us to pose the following conjecture.

Conjecture 4.16. Let $n \geq 1$ and A, B, C be zonoids in $\mathbb{R}^n$. Then,

$$|A|\,|A + B + C| \leq |A + B|\,|A + C|.$$

A detailed study of this conjecture is undertaken in the paper [25].

Before proving Theorem 4.15, we will prove a theorem that would help us to verify Plünnecke-Ruzsa inequality for convex bodies when the body A is fixed.

Theorem 4.17. Let $n \geq 1$ and A, B be a convex bodies in $\mathbb{R}^n$ such that for every $1 \leq k \leq n$ and any subspace E of $\mathbb{R}^n$ of dimension k one has

$$\frac{|P_E(A+B)|_k}{|P_{E\cap u^\perp}(A+B)|_{k-1}} \geq \frac{|P_E A|_k}{|P_{E\cap u^\perp} A|_{k-1}}.$$

Then, for any zonoid Z in $\mathbb{R}^n$, one has

$$|A|\,|A+B+Z| \leq |A+B|\,|A+Z|.$$

Proof. Notice that it is enough to prove the inequality for Z being a zonotope and use an approximation argument. In fact, we prove by induction on k that for any $1 \leq k \leq n$ and any subspace E of $\mathbb{R}^n$ of dimension k and zonoid Z in E one has

$$|P_E A|_k |P_E(A+B)+Z|_k \leq |P_E(A+B)|_k |P_E A + Z|_k. \tag{38}$$

This statement is true for $k = 1$ so let us assume that it's true for $k-1$, for some $1 \leq k \leq n$ and let's prove it for k. Let E be a subspace of dimension k. To prove that inequality (38) holds for any zonotope in E, we proceed by induction on the number of segments in Z. Notice that the inequality holds as an equality for $Z = 0$. Let us assume that inequality (38) holds for some fixed zonotope Z in E and prove it for $Z + t[0,u]$ where $t > 0$ and $u \in S^{n-1} \cap E$. Using (5), we get

$$|P_E(A+B)+Z+t[0,u]|_k = |P_E(A+B)+Z|_k + t|P_{E\cap u^\perp}(A+B+Z)|_{k-1}$$

and

$$|P_E A + Z + [0,tu]|_k = |P_E A + Z|_k + t|P_{E\cap u^\perp}(A+Z)|_{k-1}.$$

Applying the induction hypothesis, it is enough to prove that

$$|P_E A|_k |P_{E\cap u^\perp}(A+B+Z)|_{k-1} \leq |P_{E\cap u^\perp}(A+Z)|_{k-1} |P_E(A+B)|_k. \tag{39}$$

But the inequality in the $k-1$-dimensional subspace $E \cap u^\perp$ for the zonotope $P_{u^\perp} Z$ gives

$$|P_{E\cap u^\perp}(A)|_{k-1} |P_{E\cap u^\perp}(A+B+Z)|_{k-1} \leq |P_{E\cap u^\perp}(A+B)|_{k-1} |P_{E\cap u^\perp}(A+Z)|_{k-1}.$$

Multiplying this inequality by the assumption of the theorem:

$$\frac{|P_E A|_k}{|P_{E\cap u^\perp} A|_{k-1}} \leq \frac{|P_E(A+B)|_k}{|P_{E\cap u^\perp}(A+B)|_{k-1}},$$

we get (39). ■

Next, we will prove that B_2^n satisfies the conditions of Theorem 4.17.

Theorem 4.18. Let $n \geq 1$ and K be a compact set in $\mathbb{R}^n$. Let $u \in S^{n-1}$. Then,

$$\frac{|K + B_2^n|}{|P_{u^\perp}(K+B_2^n)|_{n-1}} \geq \frac{|B_2^n|}{|B_2^{n-1}|_{n-1}}.$$

Proof. We will use a trick from [4] to reduce the proof to the case of $K \subset u^{\perp}$. Indeed, let S_u be the Steiner symmetrization with respect to $u \in S^{n-1}$ (see [56] and [12] remark 9.3.2). Then one has

$$S_u(K + B_2^n) \supset S_u(K) + S_u(B_2^n) = S_u(K) + B_2^n.$$

Hence,

$$|K + B_2^n| = |S_u(K + B_2^n)| \geq |S_u(K) + B_2^n|.$$

Moreover, $P_{u^{\perp}}(S_u(K)) = P_{u^{\perp}}K$ and $P_{u^{\perp}}K \subset K$, hence,

$$|K + B_2^n| \geq |P_{u^{\perp}}K + B_2^n|.$$

It follows that we are reduced to the case when $K \subset u^{\perp}$, that is, $K = P_{u^{\perp}}K$. Without loss of generality, we may assume that $u = e_n$ and we write $B_2^n \cap u^{\perp} = B_2^{n-1}$. In this case, we can describe the set $K + B_2^n$ by its slices by the hyperplanes orthogonal to e_n denoted $H_t = \{x \in \mathbb{R}^n : x_n = t\}$, $t \in \mathbb{R}$. We have

$$(K + B_2^n) \cap H_t = K + B_2^n \cap H_t = K + \sqrt{1 - t^2} B_2^{n-1} + te_n.$$

Using (5), we get that for all $t \in [-1, 1]$,

$$|(K + B_2^n) \cap H_t|_{n-1} = \left| K + \sqrt{1 - t^2} B_2^{n-1} \right|_{n-1}.$$

It follows from Proposition 2.1 [27] (see also [58]) that

$$\left| K + \sqrt{1 - t^2} B_2^{n-1} \right|_{n-1} \geq \left| \sqrt{1 - t^2} K + \sqrt{1 - t^2} B_2^{n-1} \right|_{n-1}.$$

Using Fubini's theorem, we get

$$|K + B_2^n| = \int_{-1}^{1} |(K + B_2^n) \cap H_t|_{n-1} dt \geq \left| K + B_2^{n-1} \right|_{n-1} \int_{-1}^{1} (1 - t^2)^{\frac{n-1}{2}} dt.$$

We finish the proof by noticing that

$$\int_{-1}^{1} (1 - t^2)^{\frac{n-1}{2}} dt = \frac{|B_2^n|}{|B_2^{n-1}|}.$$

■

Proof of Theorem 4.15. Let T be the affine transform such that $B = T(B_2^n)$. If B lives in an hyperplane then $|B| = 0$ and the inequality holds. If not, then T is invertible and since the affine image of a zonoid is a zonoid by applying T^{-1}, we may assume that $B = B_2^n$. Now the theorem follows immediately from Theorems 4.18 and 4.17. ■

Remark 4.19. By applying a linear transform, it is possible to show that, more generally, for any compact set K and for any ellipsoid B,

$$\frac{|K + B|}{|P_{u^{\perp}}(K + B)|_{n-1}} \geq \frac{|B|}{|P_{u^{\perp}}B|_{n-1}}.$$

Indeed, let $T \in GL_n$, such that $B = TB_2^n$. Denoting $H = (T^*u)^{\perp}$, one has, that for any compact K, $P_{u^{\perp}}(TK) = TP_{H,T^{-1}u}K$, where $P_{H,T^{-1}u}K$ is the linear projection of K onto H along $T^{-1}u$. Then, the remark follows by applying Theorem 4.18 to TK.

Corollary 4.20. Let $n \geq 1$ and K be a convex body in $\mathbb{R}^n$. Let B be an ellipsoid and $Z_1, \ldots, Z_m$, $m \geq 1$ be zonoids in $\mathbb{R}^n$. Then,

$$|B|^m |B + K + Z_1 + \ldots Z_m| \leq |B + K| \prod_{i=1}^{m} |B + Z_i|.$$

Proof. The corollary follows immediately applying induction on $m \geq 1$ together with Theorem 4.15:

$$|B|^m |B + K + Z_1 + \ldots Z_m| \leq |B|^{m-1} |B + K + Z_1 + \ldots Z_{m-1}|\, |B + Z_m|.$$

■

Remark 4.21. Theorem 4.6 was inspired by the following inequality

$$V(B_2^n[n-1], Z) V(B_2^n[n-1], K) \geq \frac{n-1}{n} c_n |B_2^n| V(B_2^n[n-2], Z, K), \tag{40}$$

where $c_n = |B_2^{n-1}|^2 / (|B_2^{n-2}|\, |B_2^n|) > 1$. This inequality was proved by Artstein-Avidan, Florentin, and Ostrover [4] when Z is a zonoid and K is an arbitrary convex body, as a generalisation of a result of Hug and Schneider [31] who proved it for K and Z zonoids. It is an interesting question if one can prove directly Theorem 4.15 by using the decomposition into mixed volumes as it was done in the proof of Theorem 4.6 and applying inequality (40). Inequality (40) is a sharp improvement of (12) in the case when $A = B_2^n$ and $m = j = 1$, and one of the bodies is a zonoid. Unfortunately, there seems not to be a direct way to apply (40) to prove Theorem 4.15 due to the lack of a sharp analog of this inequality for $V(B_2^n[n-m-j], Z[m], K[j])$ when $m, j > 1$.

4.6 The case of compact sets

Let us note that the inequality

$$|A|\, |A + B + C| \leq |A + B|\, |A + C| \tag{41}$$

is valid when A, B are intervals and C is any compact set in $\mathbb{R}$. Indeed, by approximation we may assume that C is a finite union of closed intervals, $A = [0, a]$ and $B = [0, b]$, for some $a, b \geq 0$. Then, we may assume that $A + C = \cup_{i=1}^m [\alpha_i, \beta_i + a]$ where intervals $[\alpha_i, \beta_i + a]$ are mutually disjoint. Then,

$$|A + C| = ma + \sum_{i=1}^{m} (\beta_i - \alpha_i),$$

$$|A + B + C| \leq \sum_{i=1}^{m} (\beta_i - \alpha_i + a + b) = m(a + b) + \sum_{i=1}^{m} (\beta_i - \alpha_i),$$

and (41) follows from

$$a\Big(m(a + b) + \sum_{i=1}^{m} (\beta_i - \alpha_i)\Big) \leq \Big(ma + \sum_{i=1}^{m} (\beta_i - \alpha_i)\Big)(a + b).$$

We also note that, as discussed before, inequality (13) as well as inequality (31) and Theorem 4.18 are valid without additional convexity assumptions.

Still, we now show that there is a sharp difference to those inequalities– the convexity assumption in Theorem 4.6 can not be removed. The construction is inspired by the proof of [52, Theorem 7.1]:

Lemma 4.22. Fix $n \geq 1$, then for any $\beta > 0$ there exist two compact sets $A, B \subset \mathbb{R}^n$ such that

$$|A|\, |A + B + B| > \beta |A + B|^2.$$

Proof. It is enough to prove the theorem for the case $n = 1$, indeed, for any other dimension $n > 1$, one can consider $A \times [-1, 1]^{n-1}$, $B \times [-1, 1]^{n-1}$ where A, B are the example constructed in $\mathbb{R}$.

To construct the sets $A, B \subset \mathbb{R}$, we fix $m, l \in \mathbb{N}$ large enough and define first two discrete sets A' and B' in $\mathbb{R}$ to establish the analogue result for cardinality instead of volume. Let

$$A' = \{x + y\sqrt{2} : x, y \in \{0, 1, \ldots, m-1\}\} \cup \{z\sqrt{3} : z \in \{1, \ldots, m\}\},$$

thus $\#A' = m(m+1)$. We also define

$$B' = \{x : x \in \{0, 1, \ldots, l\}\} \cup \{y\sqrt{2} : y \in \{1, \ldots, l\}).$$

Thus, one has

$$B' + B' = \{x + y\sqrt{2} : x, y \in \{0, 1, \ldots, l\}\} \cup \{x : x \in \{l+1, \ldots, 2l\}\} \cup \{y\sqrt{2} : y \in \{l+1, \ldots, 2l\}\}.$$

It follows that $\#(B' + B') = l^2 + 4l + 1$. It may help to imagine A' and B' as 3-dimensional subsets which are linear combinations of vectors $e_1, \sqrt{2}e_2$ and $\sqrt{3}e_3$. Then, it is easy to see that $A' + B'$ consists of an $m \times m$ square of integer points united with two $m \times l$ rectangles of integer points in the $\{e_1, e_2\}$ plane and two additional rectangles: one of size $m \times (l+1)$ in the $\{e_1, e_3\}$ plane and another of size $m \times (l+1)$ in the $\{e_2, e_3\}$ plane, where the last two rectangles overlap by m points. Thus,

$$\#(A' + B') = m^2 + 2ml + 2(m(l+1) - m) = m(m + 4l + 1).$$

Finally, we note that $A' + B' + B'$ contains the set

$$\{z\sqrt{3} : z \in \{1, \ldots, m\}\} + B' + B',$$

thus $\#(A' + B' + B') \geq l^2 m$. Now consider any $\beta' > 0$. Our goal is to select $m, l \in \mathbb{N}$ such that

$$\#(A')\#(A' + B' + B') > \beta'(\#(A' + B'))^2. \tag{42}$$

For this, it is enough to pick $m, l \in \mathbb{N}$ such that

$$m(m+1)l^2 m \geq \beta' m^2 (m + 4l + 1)^2$$

or

$$\sqrt{m+1}\, l \geq \sqrt{\beta'}(m + 4l + 1),$$

which is true as long as $m + 1 > 16\beta'$ and l is large enough.

Now we are ready to construct our continuous example in $\mathbb{R}$, for volume. For the fixed $\beta > 0$ consider $\beta' = \frac{4}{3}\beta$ and sets A' and B' satisfying (42). Define $A = A' + [-\varepsilon, \varepsilon]$ and $B = B' + [-\varepsilon, \varepsilon]$ where $\epsilon > 0$ small enough such that

$$|A| = 2\varepsilon\#(A'); |A + B| = 4\varepsilon\#(A' + B') \text{ and } |A + B + B| = 6\varepsilon\#(A' + B' + B'),$$

which, together with (42) gives the required inequality. ∎

It turns out (see, for example, [21]) that some sumset estimates can still be proved if the convexity assumption is relaxed by an assumption that the body is star-shaped. The next lemma shows that this is still not the case for Theorem 4.6.

Lemma 4.23. Fix $n \geq 3$, then for any $\beta > 0$, there exists a compact star-shaped symmetric body $A \subset \mathbb{R}^n$ such that

$$|A|\,|A+A+A| \geq \beta |A+A|^2.$$

Proof. Let $n = 3$, consider a cube $Q = [-1,1]^3$ and a set $X = m([-e_1,e_1] \cup [-e_2,e_2] \cup [-e_3,e_3])$, that is, the union of 3 orthogonal segments of length $2m$. Let $A = Q \cup X$. Then,

$$|A+A| \leq |X+X| + |Q+Q| + |X+Q| \leq 0 + 4^3 + 3 * (2m+2) * 2$$

but $|A+A+A| \geq |X+X+X| = 8m^3$. We note that in dimension $n > 3$ one can consider the direct sum of the above three dimensional example with $[-1,1]^{n-3}$. ■

5 Playing With Signs

5.1 Ruzsa's triangle inequality

In additive combinatorics, the Ruzsa distance is defined by $d(A,B) = \log \frac{\#(A-B)}{\sqrt{\#(A)\#(B)}}$, where A and B are subsets of an abelian group. We refer to [60] for more information and properties of this object, which is useful even though it is *not* a metric (since typically $d(A,A) > 0$). The Ruzsa distance satisfies the triangle inequality, which is equivalent to $\#(C-B) \cdot \#A \leq \#(A-C)\#(B-A)$.

An analogue of Ruzsa's triangle inequality holds for compact sets in $\mathbb{R}^n$.

Theorem 5.1. (see, e.g., [61, Lemma 3.12]) For any compact sets $A, B, C \subset \mathbb{R}^n$,

$$|A|\,|B-C| \leq |A-C|\,|A-B|. \tag{43}$$

This inequality has a short proof that we provide here for the sake of completeness. Indeed

$$|B-A||A-C| = \int_{\mathbb{R}^n} 1_{B-A} * 1_{A-C}(x)dx \geq \int_{B-C} 1_{B-A} * 1_{A-C}(x)dx,$$

where 1_M is a characteristic function of a set $M \subset \mathbb{R}^n$ and $f*g$ is the convolution of functions $f, g : \mathbb{R}^n \to \mathbb{R}$. Now let $x \in B-C$, there are $b \in B$ and $c \in C$ such that $x = b-c$. Thus, changing variable, one has

$$\begin{aligned} 1_{B-A} * 1_{A-C}(b-c) &= \int_{\mathbb{R}^n} 1_{B-A}(z)1_{A-C}(b-c-z)dz = \int_{\mathbb{R}^n} 1_{B-A}(b-y)1_{A-C}(y-c)dy \\ &\geq \int_A 1_{B-A}(b-y)1_{A-C}(y-c)dy = |A|. \end{aligned}$$

5.2 An inequality with signed sums

In view of Ruzsa's triangle inequality, it is natural to try to generalize Theorem 4.6 to the case of the difference of convex bodies. We recall that $\varphi = (1+\sqrt{5})/2$ denotes the golden ratio and for $n \geq 2$ the constant c_n was defined in Theorem 4.6 satisfying $1 = c_2 \leq c_n \leq \varphi^n$.

Theorem 5.2. Let A, B, C be convex bodies in $\mathbb{R}^n$. Then,

$$|A|\,|A+B+C| \leq \frac{1}{2^n}\binom{2n}{n} c_n \min\{|A-B|\,|A+C|, |A-B|\,|A-C|, |A+B|\,|A-C|\}. \tag{44}$$

Proof. We first recall Litvak's observation (see [56, pp. 534]) that

$$|A+B| \leq \frac{1}{2^n}\binom{2n}{n}|A-B|, \tag{45}$$

for any convex bodies A, B in $\mathbb{R}^n$, with equality for $A = -B$ being simplices. Litvak obtained this by simply combining the Rogers-Shephard inequality ([56], Theorem 10.1.4) and the Brunn-Minkowski inequality. Indeed, applying the Rogers-Shephard inequality we get

$$|(A+B)-(A+B)| = |(A-B)-(A-B)| \leq \binom{2n}{n}|A-B|,$$

and the Brunn-Minkowski inequality gives us

$$|(A+B)-(A+B)| \geq 2^n|A+B|.$$

Now, we can write

$$|A|\,|A+B+C| \leq c_n|A+B|\,|A+C| \leq \frac{1}{2^n}\binom{2n}{n}c_n|A-B|\,|A+C|,$$

where the first inequality follows from Theorem 4.6, and the second from Litvak's observation. This gives the first half of the inequality (44) (i.e., the inequality with the first term inside the minimum). The third inequality is proved in a similar way.

Next, we notice that changing B to $-B$ and C to $-C$, Theorem 4.6 becomes equivalent to

$$|A|\,|A-(B+C)| \leq c_n|A-B|\,|A-C|,$$

and we finish the proof of the inequality (44) (i.e., we get the inequality with the second term inside the minimum) by applying Litvak's observation on the left-hand side. ■

Remark 5.3. We note that the constant in (44) is sharp in the case $n = 2$, no matter which term inside the minimum we consider. Indeed, if $C = \{0\}$, then (44) becomes $|A+B| \leq \frac{3}{2}|A-B|$, which is sharp for triangles A, B such that $A = -B$.

We can actually improve Theorem 5.2 in $\mathbb{R}^2$ via the following inequality, which the next section is dedicated to proving: if A, C are convex sets in $\mathbb{R}^2$, then $|A-C| \leq |A+C| + 2\sqrt{|A|\,|C|}$. This inequality is an intriguing improvement (in dimension 2) of Litvak's observation (45). To see this, observe that by the Brunn-Minkowski inequality,

$$|A-B| \geq (\sqrt{|A|}+\sqrt{|-B|})^2 = |A|+|B|+2\sqrt{|A|\,|B|} = (\sqrt{|A|}-\sqrt{|B|})^2 + 4\sqrt{|A|\,|B|},$$

and hence

$$4\sqrt{|A|\,|B|} \leq |A-B| - (\sqrt{|A|}-\sqrt{|B|})^2.$$

Thus, in dimension 2, since $c_2 = 1$, we obtain

$$\begin{aligned}|A|\,|A+B+C| &\leq |A+B|\,|A+C| \\ &\leq (|A-B| + 2\sqrt{|A|\,|B|})\,|A+C| \\ &\leq \left[\frac{3}{2}|A-B| - \frac{1}{2}(\sqrt{|A|}-\sqrt{|B|})^2\right]|A+C|,\end{aligned}$$

which is an improvement of Theorem 5.2 in dimension 2.

5.3 A sum-difference inequality in the plane

For subsets A and C of an abelian group, comparison of cardinality of $A+C$ and $A-C$ has been of interest in additive combinatorics (see, e.g., [48, 60]). Analogously, in the probabilistic setting, various works have explored the comparison of the entropies of sums and differences of independent random variables in different abelian groups (see, e.g., [1, 33, 34, 37]). Inspired by this background, we develop a sharp sum-difference inequality when the abelian group in question is the Euclidean plane.

Theorem 5.4. Let A, C be two convex sets in $\mathbb{R}^2$, then

$$|A-C| \le |A+C| + 2\sqrt{|A|\,|C|}. \tag{46}$$

Moreover, the equality in the above inequality is only possible in the following cases:

- One of the sets A or C is a singleton or a segment and the other one is any convex body.
- A is a triangle and $C = tA + b$, for some $t > 0$ and $b \in \mathbb{R}^2$.

Proof. Let us first prove the inequality. We note that (46) is equivalent to

$$V(A, -C) \le V(A, C) + \sqrt{|A|\,|C|}. \tag{47}$$

Assume $A = A_1 + A_2$ and

$$V(A_1, -C) \le V(A_1, C) + \sqrt{|A_1|\,|C|}\ \text{and} V(A_2, -C) \le V(A_2, C) + \sqrt{|A_2|\,|C|}.$$

Then,

$$V(A, -C) = V(A_1, -C) + V(A_2, -C) \le V(A, C) + \sqrt{|A_1|\,|C|} + \sqrt{|A_1|\,|C|}.$$

Using the Brunn-Minkowski inequality in the plane: $\sqrt{|A_1|} + \sqrt{|A_1|} \le \sqrt{|A|}$, we get

$$V(A, -C) \le V(A, C) + \sqrt{|A|\,|C|}.$$

Thus, to prove (47) we may assume that both A and C are triangles. Indeed, any planar convex body can be approximated by a polygon and any planar polygon can be written as a Minkowski sum of triangles [56] (here we will treat a segment as a degenerate triangle). If A or C is a segment, then the inequality becomes an equality. Thus, we may assume that A and C are not degenerate triangles. We notice that (47) is invariant under dilation and shift of the convex body C, thus we may assume that $C \subset A$ and has common points with all three edges of A. Thus, $V(A, C) = |A|$ (see, for example, [57]). Finally, our goal is to show that for any triangles A, C, such that $C \subset A$ and C touches all edges of A, we have

$$V(A, -C) \le |A| + \sqrt{|A|\,|C|}. \tag{48}$$

To prove the above inequality, one may use the technique of shadow systems ([50] or [56], Section 10.4). In this particular case, the method can be applied directly. Indeed, if $C = A$, then (48) becomes an equality. Otherwise, let $C = \operatorname{conv}\{c_1, c_2, c_3\}$ and one of the c_i's is not a vertex of A. Assume, without loss of generality, that c_1 is not a vertex of A. Then, there exists a vertex a_1 of A such that the segment (a_1, c_1) does not intersect C. Let $c_t = tc_1 + (1-t)a_1$ for $t \in [t_0, t_1]$, where $[t_0, t_1]$ is the largest interval such that $c_t \in A$ and c_t is not aligned with c_2 and c_3, for all $t \in (t_0, t_1)$. Notice that c_{t_0} and c_{t_1} are either vertices of A or belong to the line containing $[c_2, c_3]$. Let $C_t = \operatorname{conv}\{c_t, c_2, c_3\}$. Then,

$$V(A, -C_t) = V(-A, C_t) = \frac{1}{2}\sum_{i=1}^{3} h_{C_t}(-u_i)|F_i|,$$

where u_i, $i = 1, 2, 3$ is a normal vector to the edge F_i of A. The function $h_{C_t}(-u_i)$ is convex in t and thus the same is true for $V(A, -C_t)$. We also notice that $|C_t|$ is an affine function of t on $[t_0, t_1]$ and thus

$$g(t) = V(A, -C_t) - \sqrt{|A|\,|C_t|}$$

is a convex function of $t \in [t_0, t_1]$ and $\max_{t\in[t_0,t_1]} = \max\{g(t_0), g(t_1)\}$. Thus, the maximum of $g(t)$ achieved when either when C_t becomes a segment (and the proof is complete in this case) or c_t reaches a vertex of A. Then, either $C = A$ or there is a vertex of C that is not a vertex of A and we repeat the procedure.

Now let us consider the equality case. Assume

$$V(A, -C) = V(A, C) + \sqrt{|A|\,|C|}. \tag{49}$$

First, let us assume that A is not a triangle. Then, A is decomposable: it can be written as $A = A_1 + A_2$ where A_1 is not homothetic to A_2. Then, applying (47) we get

$$V(A_1, -C) \le V(A_1, C) + \sqrt{|A_1|\,|C|} \text{ and } V(A_2, -C) \le V(A_2, C) + \sqrt{|A_2|\,|C|}.$$

From (49), we have equality in the above inequalities and

$$\sqrt{|A_1|\,|C|} + \sqrt{|A_2|\,|C|} = \sqrt{|A_1 + A_2|\,|C|}.$$

The above equality is only possible in two cases: first, when there is an equality in Brunn-Minkowski inequality, which would result A_2 to be homothetic to A_1, and we assumed that this is not the case, and the second case, when $|C| = 0$, that is, C is the singleton or a segment. If C is not a triangle, then, the above discussion shows that $|A| = 0$.

Now, we assume that A and C are non degenerate triangles. Using homogeneity of equality (49), we may assume that the triangle C touches all edges of A and equality (49) becomes

$$V(A, -C) - \sqrt{|A|\,|C|} = |A|. \tag{50}$$

Assume towards the contradiction that $C \neq A$. Then, there is a vertex c_1 of the triangle $C = \mathrm{conv}(c_1, c_2, c_3)$ that is not a vertex of A. We reproduce the same shadow system $(C_t)_{t\in[t_0,t_1]}$ as in the proof of the inequality. We only need to prove that the function $g(t) = V(A, -C_t) - \sqrt{|A|\,|C_t|}$ is not constant on $[t_0, t_1]$. To do this, we prove that, among the two functions, $V(A, -C_t)$ and $\sqrt{|C_t|}$ at least one is not affine. There are two cases. If $|C_t|$ is not constant then $\sqrt{|C_t|}$ is strictly concave, thus g is not a constant. If $|C_t|$ is constant then all vertices of C are different from the vertices of A. Recall that $V(A, -C_t) = \frac{1}{2}\sum_{i=1}^{3} h_{C_t}(-u_i)|F_i|$. Let u_2, u_3 be the normal of the edges of A that do not contain c_1. Then it is easy to see that $h_{C_t}(-u_2) + h_{C_t}(-u_3)$ is not affine. Thus, $V(A, -C_t)$ is not constant. This is a contradiction. ■

Let us define the *additive asymmetry* of the pair (A, C) by

$$\mathrm{asym}(A, C) = \left|\frac{|A + C|}{|A - C|} - 1\right|,$$

and note that $\mathrm{asym}(A, C)$ is trivially 0 if either A or C is symmetric. Observe that inequality (46) may be rewritten as

$$\mathrm{asym}(A, C) \le 2e^{-d(A,C)}, \tag{51}$$

where $d(A, C) = \log \frac{|A-C|}{\sqrt{|A|\cdot|C|}}$ is the Euclidean analogue of the Ruzsa distance defined at the beginning of this section. One wonders if this inequality extends to dimension higher than 2.

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