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On the volume of the Minkowski sum of zonoids

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ABSTRACT

We explore some inequalities in convex geometry restricted to the class of zonoids. We show the equivalence, in the class of zonoids, between a local Alexandrov-Fenchel inequality, a local Loomis-Whitney inequality, the log-submodularity of volume, and the Dembo-Cover-Thomas conjecture on the monotonicity of the ratio of volume to the surface area. In addition to these equivalences, we confirm these conjectures in $\mathbb{R}^3$ and we establish an improved inequality in $\mathbb{R}^2$. Along the way, we give a negative answer to a question of Adam Marcus regarding the roots of the Steiner polynomial of zonoids. We also investigate analogous questions in the L_p-Brunn-Minkowski theory, and in particular, we confirm all of the above conjectures in the case $p = 2$, in any dimension.

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1. Introduction

The deep parallels between inequalities in Information Theory and Convex Geometry have been explored intensively. The recognition of these parallels goes back to at least 1984, when Costa and Cover [15] observed the formal resemblance between the Brunn-Minkowski inequality in Convex Geometry and the entropy power inequality in Information Theory. The Brunn-Minkowski inequality (see, e.g., [30,51]) states that for all compact sets A, B in $\mathbb{R}^n$, $|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$, where we write $|A|$ for the volume of A. The entropy power inequality (see, e.g., [17]) states that for any pair of independent random vectors X, Y in $\mathbb{R}^n$, $N(X+Y) \geq N(X) + N(Y)$, where $N(X) = e^{\frac{2h(X)}{n}}$ denotes the entropy power of X, and $h(X) = -\int f(x) \log f(x) dx$ is the entropy of X if X has density f (and $h(X) = -\infty$ if not). Despite the apparent distinctness of the settings, a compelling case can be made (see, e.g., [18,54,30,56,43]) that these inequalities are deeply connected – in particular, it is now well understood that the functional $A \mapsto |A|^{1/n}$ in the geometry of compact subsets of $\mathbb{R}^n$, and the functional $f_X \mapsto N(X)$ in probability are analogous in many (but not all) ways. In the last decade, several further developments have been made that link Information Theory to the Brunn-Minkowski theory, including entropy analogues of Blaschke-Santaló's inequality [38], reverse Brunn-Minkowski's inequality [7,8], Rogers-Shephard's inequality [10,41] and Busemann's inequality [5]. Indeed, volume inequalities and entropy inequalities (and also certain small ball inequalities [44]) can be unified using the framework of Rényi entropies; this framework and the relevant literature is surveyed in [43].

The analogies between Information Theory and Convex Geometry are not so direct when one moves to second-order functionals. In particular, convex geometry analogues of Fisher information inequalities hold sometimes but not always [24,2,29]. Motivated by

these analogies, Dembo, Cover and Thomas [18] proposed the inequality, called concavity of the ratio of volume to the surface area,

$$\frac{|A+B|}{|\partial(A+B)|_{n-1}} \geq \frac{|A|}{|\partial A|_{n-1}} + \frac{|B|}{|\partial B|_{n-1}} \tag{1}$$

as a natural analogue dual to the Fisher information inequality; here and in the sequel, $|A|_m$ will mean the m-dimensional Hausdorff measure of the set A. It was noticed already in [18] (see also [31]) that (1) is true when A (or B) is an Euclidean Ball, moreover, in [24], it was proved that this last inequality holds for any convex bodies A and B in $\mathbb{R}^2$. It is interesting to note that even a weaker conjecture, called monotonicity of the ratio of volume to the surface area,

$$\frac{|A+B|}{|\partial(A+B)|_{n-1}} \geq \frac{|A|}{|\partial A|_{n-1}} \tag{2}$$

is not trivial. The case of B being a segment already gives another natural conjecture

$$\frac{|\partial(P_{u^\perp}A)|_{n-2}}{|P_{u^\perp}A|_{n-1}} \leq \frac{|\partial A|_{n-1}}{|A|}, \tag{3}$$

where $P_{u^\perp}A$ denotes the orthogonal projection A onto the hyperplane with normal vector $u \in S^{n-1}$. We explain in details the relationships between the above conjectures as parts of Theorems 3.6, 3.9 and Remark 3.10 below. It was proved in [31] (see also Lemma 5.4 for a simple proof) that (3) holds with a multiplicative constant:

$$\frac{|\partial(P_{u^\perp}A)|_{n-2}}{|P_{u^\perp}A|_{n-1}} \leq \frac{2(n-1)}{n}\frac{|\partial A|_{n-1}}{|A|}. \tag{4}$$

Moreover, it was shown in [24] that the constant is sharp. Thus, there are counterexamples to (3) in $\mathbb{R}^n$ for any $n \geq 3$ and inequalities (1) and (2) are not true in general for the whole class of convex bodies in $\mathbb{R}^n$, $n \geq 3$.

Conjecture (1) is connected with the following conjecture for the volume of projections (see Remark 3.10 below):

$$\frac{|A+B|}{|P_{u^\perp}(A+B)|_{n-1}} \geq \frac{|A|}{|P_{u^\perp}A|_{n-1}} + \frac{|B|}{|P_{u^\perp}B|_{n-1}}. \tag{5}$$

Bonnesen proved in [12] (see [51, eq. (7.196)]) that, for any convex bodies A, B in $\mathbb{R}^n$,

$$|A+B| \geq \left(|P_{u^\perp}A|_{n-1}^{\frac{1}{n-1}} + |P_{u^\perp}B|_{n-1}^{\frac{1}{n-1}}\right)^{n-1}\left(\frac{|A|}{|P_{u^\perp}A|_{n-1}} + \frac{|B|}{|P_{u^\perp}B|_{n-1}}\right), \tag{6}$$

which is (5) for $n = 2$. The fact that (1) is true in dimension 2 and in the case when one of the bodies is an Euclidean ball inspires the natural conjecture that (1) holds for

zonoids. Recall that zonoids are Hausdorff limits of zonotopes and that zonotopes are Minkowski sums of segments. Our main goal, in this paper, is to study a weaker version of this conjecture: is it true that for two zonoids A, B in $\mathbb{R}^n$ we have

$$\frac{|A+B|}{|P_{u^\perp}(A+B)|_{n-1}} \geq \frac{|A|}{|P_{u^\perp}A|_{n-1}}? \tag{7}$$

Note that (7) is again not true for general convex bodies in dimension $n \geq 3$. We prove it for zonoids in $\mathbb{R}^3$ (Theorem 5.1). We also present a number of equivalent and useful restatements of (7) (Theorems 3.6, 3.9 and 5.3).

The second main contribution of this paper has to do with an analogue of the Plünnecke-Ruzsa [47,50] inequality for zonoids, or equivalently, the log-submodularity property of volume on the space of zonoids with respect to Minkowski summation. More precisely, we study the following conjecture from [28]: given zonoids A, B_1, B_2 in $\mathbb{R}^n$, one has

$$|A|\,|A+B_1+B_2| \leq |A+B_1|\,|A+B_2|. \tag{8}$$

This conjecture was inspired by Bobkov and the second-named author [8] who proved that for convex bodies A, B_1, B_2 in $\mathbb{R}^n$, one has

$$|A|\,|A+B_1+B_2| \leq 3^n |A+B_1|\,|A+B_2|.$$

Recently, it was proved in [28] that the constant 3^n in the preceding inequality may be replaced by φ^n, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio, that the best constant c_n is lower bounded by $(4/3+o(1))^n$ and that $c_2 = 1$ and $c_3 = 4/3$. These observations imply that, for general convex bodies in dimensions $n \geq 3$, it is impossible to have (8). In this paper, we prove that (8) holds in $\mathbb{R}^3$ for zonoids. We explore related questions for Gaussian and more general measures in [25,26].

The log-submodularity property of volume is tightly connected with the submodularity of entropy proved in [39], which asserts that

$$N(X)N(X+Y+Z) \leq N(X+Y)N(X+Z), \tag{9}$$

for independent $\mathbb{R}^n$-valued random variables X, Y, Z. This inequality may be thought of as an entropic analogue of the Plünnecke-Ruzsa inequality. It was observed in [41] that (9), applied to Gaussian random vectors, immediately implies log-submodularity of the determinant for positive-definite matrices T_1, T_2, T_3:

$$\det(T_1+T_2+T_3)\det(T_1) \leq \det(T_1+T_2)\det(T_1+T_3).$$

A much more general inequality, however, has a long history in functional analysis. The Grothendieck determinant inequality asserts that if T_1, T_2 are trace-class operators on a separable Hilbert space, then

$$\det(I + |T_1 + T_2|) \leq \det(I + |T_1|)\det(I + |T_2|),$$

where we use I for the identity operator on the Hilbert space, det for the Fredholm determinant, and $|T| = (T^*T)^{\frac{1}{2}}$. This inequality first appeared in unpublished notes of Grothendieck in 1955, and was rediscovered by Rotfel'd [49], and by Seiler and Simon [52] motivated by questions in quantum field theory (our approach to volume inequalities is akin to the proof of [52] in that they use the polynomial expansion of determinants). Another proof was given by Lieb [35] and a further generalization to an arbitrary von Neumann algebra equipped with a faithful semifinite trace (which had been conjectured by Grothendieck) was obtained by Fack [20].

Inequality (9) was later modified by Courtade in [16], who established that if X is a Gaussian random vector then one has the reinforced inequality

$$N(Y)N(Z) + N(X)N(X + Y + Z) \leq N(X + Y)N(X + Z).$$

Again, when applied for independent Gaussian random vectors, this immediately implies that for any symmetric positive semidefinite matrices T_1, T_2, T_3, one has

$$\det(T_2)^{\frac{1}{n}}\det(T_3)^{\frac{1}{n}} + \det(T_1)^{\frac{1}{n}}\det(T_1 + T_2 + T_3)^{\frac{1}{n}} \leq \det(T_1 + T_2)^{\frac{1}{n}}\det(T_1 + T_3)^{\frac{1}{n}}. \quad (10)$$

During the 2017 workshop on Information-Theoretic Inequalities at the University of Delaware, Courtade conjectured that the analogous inequality is true for any convex bodies B, C in $\mathbb{R}^n$

$$|B|^{1/n}|C|^{1/n} + |B_2^n|^{1/n}|B_2^n + B + C|^{1/n} \leq |B_2^n + B|^{1/n}|B_2^n + C|^{1/n},$$

where B_2^n is the Euclidean unit ball. In Theorem 4.1, we confirm this conjecture in the plane, moreover, we show that B_2^n in the above inequality may be replaced by a general planar convex body.

We would like to also illustrate an application of (10) to volume inequalities. For this, let us recall Firey's [22] one-parameter extension of the concept of Minkowski sum: for any real number $p \geq 1$, the so-called ℓ_p-sum $K \oplus_p L$ of convex bodies K and L containing the origin is defined by:

$$h_{K \oplus_p L}(x) = (h_K(x)^p + h_L(x)^p)^{\frac{1}{p}}, \quad \forall x \in \mathbb{R}^n,$$

where h_K is the support function of K. In particular, for the case $p = 2$, the class of ellipsoids is closed with respect of ℓ_2-sum and if $\mathcal{E}_1 = T_1 B_2^n$ and $\mathcal{E}_2 = T_2 B_2^n$, for some positive-definite symmetric matrices T_1, T_2, then $\mathcal{E}_1 \oplus_2 \mathcal{E}_2 = \sqrt{T_1^2 + T_2^2} B_2^n$. Thus (10), immediately gives

$$|\mathcal{E}_2|^{\frac{2}{n}}|\mathcal{E}_3|^{\frac{2}{n}} + |\mathcal{E}_1|^{\frac{2}{n}}|\mathcal{E}_1 \oplus_2 \mathcal{E}_2 \oplus_2 \mathcal{E}_3|^{\frac{2}{n}} \leq |\mathcal{E}_1 \oplus_2 \mathcal{E}_2|^{\frac{2}{n}}|\mathcal{E}_1 \oplus_2 \mathcal{E}_3|^{\frac{2}{n}}.$$

In addition to the above inequality, in this paper, we establish the ℓ_2-version of the Dembo-Cover-Thomas conjecture (1) for ellipsoids and discuss other possible ℓ_p- extensions of inequality (5) (see Theorem 6.2 and Corollary 6.7).

This paper is organized as follows. In Section 2, we collect background material on mixed volumes. Numerous equivalent descriptions of log-submodularity, on a given class of convex bodies, are explored in Section 3. In Section 4, we first prove the planar case of the conjecture of Courtade, and we then prove inequality (7) in the special case of paralleletopes. Section 5 explores log-submodularity of volume on the class of zonoids – we first prove that this holds in $\mathbb{R}^3$, and then discuss an array of inequalities that are equivalent to log-submodularity for zonoids in arbitrary dimension (all of which therefore now hold in $\mathbb{R}^3$). Along the way, we answer a question of Adam Marcus about Steiner polynomials of zonoids. Finally, in Section 6, inspired by a work of Brazitikos and McIntyre [13] on vector-valued Maclaurin inequalities and the inequality (10), we discuss possible extensions of our results to the more general class of ℓ_p-zonoids, which appear in the more general L_p-Brunn-Minkowski theory.

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2. Preliminaries on mixed volumes

In this section, we introduce basic notations and collect essential facts and definitions from convex geometry that are used in the paper. As a general reference on the theory, we use [51]. We write $\langle x, y\rangle$ for the inner product of vectors x and y in $\mathbb{R}^n$ and by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$. The closed Euclidean ball in $\mathbb{R}^n$ is denoted by B_2^n, and its boundary by S^{n-1}. We also denote by $e_1, \dots, e_n$ the standard orthonormal basis in $\mathbb{R}^n$. Moreover, for any set in $K \subset \mathbb{R}^n$, we denote its boundary by ∂K. A compact set K in $\mathbb{R}^n$ is called star-shaped if, for every $x \in K$, the segment $[0, x]$ is a subset of K; its radial function ρ_K is then defined by $\rho_K(x) = \sup\{a; ax \in K\}$. When 0 belongs to the interior of K then $\|x\|_K = \rho_K^{-1}(x)$ is the Minkowski functional of K. A convex body is a convex, compact set with non-empty interior. For a convex body K, we define its support function by $h_K(x) = \max_{y\in K}\langle x, y\rangle$.

We write $|K|_m$ for the m-dimensional Hausdorff measure of a measurable set $K \subset \mathbb{R}^n$. In the case where K is a convex set, the typical use of this notation will involve the dimension $m \in \{1, \dots, n\}$ of the minimal affine space containing K, so that $|K|_m$ is just the Lebesgue measure (volume) of the "correct" dimension. We often use the shorter notation $|K|$ for n-dimensional volume; also, for the sake of convenience, we define $|K|_0 = 1$. From [51, Theorem 5.1.6], for any compact convex sets $K_1, \dots, K_r$ in $\mathbb{R}^n$ and any non-negative numbers $t_1, \dots, t_r$, one has

$$|t_1K_1 + \cdots + t_rK_r| = \sum_{i_1,\ldots,i_n=1}^{r} t_{i_1}\cdots t_{i_n} V(K_{i_1},\ldots,K_{i_n}), \tag{11}$$

for some non-negative numbers $V(K_{i_1},\ldots,K_{i_n})$, which are the mixed volumes of $K_{i_1},\ldots,K_{i_n}$.

We also often use the version of (11) for two bodies:

$$|A + tB| = \sum_{k=0}^{n} \binom{n}{k} t^k V(A[n-k], B[k]), \tag{12}$$

for any $t > 0$ and compact, convex sets A, B in $\mathbb{R}^n$, where for simplicity we use notation $A[m]$ for a convex set A repeated m times. Mixed volumes satisfy a number of extremely useful inequalities. The first one is the Brunn-Minkowski inequality $|K+L|^{1/n} \geq |K|^{1/n} + |L|^{1/n}$, whenever K, L and $K + L$ are measurable. A direct consequence is Minkowski's first inequality

$$V(L, K[n-1]) \geq |L|^{1/n}|K|^{(n-1)/n}, \tag{13}$$

and Minkowski's second inequality

$$V(L, K[n-1])^2 \geq |K|V(L[2], K[n-2]), \tag{14}$$

for two convex, compact subsets K and L in $\mathbb{R}^n$. We will use the classical integral representation for the mixed volume:

$$V(L, K[n-1]) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS_K(u), \tag{15}$$

where S_K is the surface area measure of K [51]. Mixed volumes are also useful to study the volume of the orthogonal projections of convex bodies. Let E be an m-dimensional subspace of $\mathbb{R}^n$, for $1 \leq m \leq n$ and let $P_E : \mathbb{R}^n \to E$ be the orthogonal projection onto E. Then for any convex body K we have (see [51] Theorem 5.3.1)

$$|U|_{n-m}|P_E K|_m = \binom{n}{m} V(K[m], U[n-m]), \tag{16}$$

where U is any convex body in the subspace $E^\perp$ orthogonal to E. It follows from (16) that for any orthonormal system $u_1,\ldots,u_r$, $1 \leq r \leq n$ we get

$$\left|P_{[u_1,\ldots,u_r]^\perp} K\right|_{n-r} = \frac{n!}{(n-r)!} V(K[n-r], [0,u_1],\ldots,[0,u_r]). \tag{17}$$

For example, denote by $u^\perp = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$ the hyperplane orthogonal to a vector $u \in S^{n-1}$, we obtain

$$|P_{u^\perp}K|_{n-1} = nV(K[n-1],[0,u]). \tag{18}$$

Another useful formula is connected with the computation of surface area and mixed volumes:

$$|\partial K| = nV(K[n-1],B_2^n), \tag{19}$$

where by $|\partial K|$ we denote the surface area of the convex body K in $\mathbb{R}^n$.

A polytope which is the Minkowski sum of finitely many line segments is called a zonotope. Limits of zonotopes in the Hausdorff metric are called zonoids, see [51, Section 3.2] for details. Consider zonotopes $Z_j = \sum_{i_j=1}^{N_j}[0,w_{i_jj}]$, where, $j=1,\dots,n$ and $w_{i_jj} \in \mathbb{R}^n$. Using the linearity of mixed volumes, we get that

$$V(Z_1,\dots,Z_n) = \sum V([0,w_{i_11}],\dots[0,w_{i_nn}]) = \frac{1}{n!}\sum |\det(\{w_{i_jj}\}_{j=1}^n)|, \tag{20}$$

where the sums run over the integers $i_j \in \{1,\dots,N_j\}$, for $j \in \{1,\dots,n\}$. Notice that, if $Z_1 = [0,u]$, with $u \in S^{n-1}$, we may use the basic properties of determinants to show that

$$V([0,u],Z_2,\dots,Z_n) = \frac{1}{n!}\sum |\det(\{P_{u^\perp}w_{i_jj}\}_{j=2}^n)|,$$

where the sum is taken over all integer $i_j \in \{1,\dots,N_j\}$, for $j \in \{2,\dots,n\}$. We mainly use the following particular case: for $Z = [0,u_1]+\cdots+[0,u_m]$, then

$$\begin{aligned} |Z| &= \sum_{1\le i_1<\cdots<i_n\le m} |\det(u_{i_1},\dots,u_{i_n})| \\ |P_{e^\perp}Z| &= \sum_{1\le i_2<\cdots<i_n\le m} |\det(P_{e^\perp}u_{i_2},\dots,P_{e^\perp}u_{i_n})|. \end{aligned} \tag{21}$$

3. Equivalent forms of the log-submodularity of volume

3.1. Submodularity

Let us recall the notion of submodular set functions and some of their basic properties; see [40,55,23,28] for more information on this subject and the proofs of the theorems. We denote $[n] = \{1,\cdots,n\}$ and let $2^{[n]}$ be the family of subsets of $[n]$.

Definition 3.1. A set function $F : 2^{[n]} \to \mathbb{R}$ is *submodular* if

$$F(S\cup T) + F(S\cap T) \le F(S) + F(T) \quad \text{for all subsets } S,T \text{ of } [n]. \tag{22}$$

Submodularity is closely related to a partial ordering on hypergraphs [19,11,4]. Let $\mathcal{M}(n,m)$ be the following family of multi-hypergraphs: each consists of non-empty subsets S_i of $[n]$, $\sum_i |S_i| = m$, with $S_i = S_j$ allowed. Consider a given multi-hypergraph $\mathcal{C} = \{S_1, \ldots, S_l\} \in \mathcal{M}(n,m)$. Take any pair of non-nested sets $\{S_i, S_j\} \subset \mathcal{C}$ and let $\mathcal{C}' = \mathcal{C}(i,j)$ be obtained from $\mathcal{C}$ by replacing S_i and S_j by $S_i \cap S_j$ and $S_i \cup S_j$, keeping only $S_i \cup S_j$ if $S_i \cap S_j = \emptyset$. $\mathcal{C}'$ is called an elementary compression of $\mathcal{C}$. The result of a sequence of elementary compressions is called a compression. Define a partial order on $\mathcal{M}(n,m)$ by setting $\mathcal{A} > \mathcal{A}'$ if $\mathcal{A}'$ is a compression of $\mathcal{A}$. Using transitivity of the partial order and reducing to elementary compressions, we get the following theorem (see [4]).

Theorem 3.2. *Suppose F is a submodular function on $[n]$. Let $\mathcal{A}$ and $\mathcal{B}$ be finite multi-hypergraph of subsets of $[n]$, with $\mathcal{A} > \mathcal{B}$. Then*

$$\sum_{S \in \mathcal{A}} F(S) \geq \sum_{T \in \mathcal{B}} F(T).$$

For every multi-hypergraph $\mathcal{A} \in \mathcal{M}(n,m)$ there is a unique minimal multi-hypergraph $\mathcal{A}^{\#}$ dominated by $\mathcal{A}$ consisting of the sets $S_j^{\#} = \{i \in [n] : i \text{ lies in at least } j \text{ of the sets } S \in \mathcal{A}\}$. One implication of Theorem 3.2 is that submodular functions F with $F(\varphi) = 0$ are "fractionally subadditive" (see, e.g., [45])– this property has also been investigated in connection with volumes of Minkowski sums [9,27,6].

We also have a notion of submodularity on the positive octant of the Euclidean space; such functions also arise naturally in probability, see, e.g., [42].

Definition 3.3. A function $f : \mathbb{R}_+^n \to \mathbb{R}$ is submodular if, for any $x, y \in \mathbb{R}_+^n$,

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y),$$

where $x \vee y$ (resp. $x \wedge y$) denotes the componentwise maximum (resp. minimum) of x and y.

The next very simple lemma connects submodularity for functions defined on $\mathbb{R}_+^n$ to submodularity for set functions. For a set $S \subset \{1, \ldots, n\}$, let $\mathbf{1}_S$ be the vector in $\mathbb{R}^n$ such that for each $i \in \{1, \ldots, n\}$, the i-th coordinate of $\mathbf{1}_S$ is 1, for $i \in S$ and 0, for $i \notin S$.

Lemma 3.4. *If $f : \mathbb{R}_+^n \to \mathbb{R}$ is submodular, and we set $F(S) := f(\mathbf{1}_S)$ for each $S \subset \{1, \ldots, n\}$, then F is a submodular set function.*

The fact that submodular functions are closely related to functions with decreasing differences is classical (see, e.g., [40] or [55], which describes more general results involving arbitrary lattices). We denote by $\partial^2_{i,j}$ the second derivative with respect to coordinates i, j.

Proposition 3.5. *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^2 function. Then f is submodular if and only if $\partial^2_{i,j} f \leq 0$ on $\mathbb{R}^n_+$, for every $i \neq j$.*

3.2. Classes closed under sums and dilation

Let us now show the various forms that the notions of submodularity and log-submodularity can have applied to volume of convex compact sets.

Theorem 3.6. *Consider a collection $\mathcal{K}$ of compact convex sets in $\mathbb{R}^n$ stable by sums and dilation. Then the following statements are equivalent:*

1. *For every $m \geq 1$ and every $A, B_1, \dots, B_m$ in $\mathcal{K}$*

$$|A|^{m-1} \left| A + \sum_{i=1}^m B_i \right| \leq \prod_{i=1}^m |A + B_i|. \tag{23}$$

2. *For every $m \geq 1$ and $A, B_1, \dots, B_m \in \mathcal{K}$, the function $\bar{w} : 2^{[m]} \to \mathbb{R}$ defined by*

$$\bar{w}(S) = \log \left| A + \sum_{i \in S} B_i \right| \tag{24}$$

   for each $S \in 2^{[m]}$, is a submodular.
3. *For every $m \geq 1$ and any multi-hypergraphs $\mathcal{A}, \mathcal{B}$ on $[m]$ with $\mathcal{A} > \mathcal{B}$,*

$$\sum_{S \in \mathcal{A}} \bar{w}(S) \geq \sum_{T \in \mathcal{B}} \bar{w}(T),$$

   where $\bar{w}$ is defined by (24).
4. *For every $m \geq 1$ and every $A, B_1, \dots, B_m \in \mathcal{K}$, the function $w : \mathbb{R}^m_+ \to \mathbb{R}$ defined, for $x = (x_1, \dots, x_m) \in \mathbb{R}^m_+$, by*

$$w(x) = \log \left| A + \sum_{i=1}^m x_i B_i \right|, \tag{25}$$

   is submodular.
5. *For every $A, B_1, B_2 \in \mathcal{K}$*

$$|A| V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1) V(A[n-1], B_2).$$

Proof. 2. $\implies$ 1. This follows directly from the definition.
1. $\implies$ 2.: We use (23) with $A + \sum_{i \in S \cap T} B_i$, $\sum_{i \in S \setminus T} B_i$ and $\sum_{i \in T \setminus S} B_i$.
2. $\iff$ 3.: This follows from Theorem 3.2 applied to $\bar{w}$.
2. $\iff$ 4.: This comes from Lemma 3.4 and the fact that $\mathcal{K}$ is dilation invariant.

4. $\iff$ 5.: We apply Proposition 3.5 to $f = w$. Let $K_x = A + \sum_i x_i B_i$ and $v(x) = |K_x|$. Then,

$$\partial_j v(x) = \lim_{\varepsilon \to 0} \frac{|K_x + \varepsilon B_j| - |K_x|}{\varepsilon} = nV(K_x[n-1], B_j),$$

and, for $k \neq j$,

$$\begin{aligned}\partial^2_{j,k} v(x) &= n\partial_k V(K_x[n-1], B_j) \\ &= n \lim_{\varepsilon \to 0} \frac{V((K_x + \varepsilon B_k)[n-1], B_j) - V(K_x[n-1], B_j)}{\varepsilon} \\ &= n(n-1)V(K_x[n-2], B_k, B_j).\end{aligned}$$

By Proposition 3.5, the submodularity of w is equivalent to $\partial^2_{j,k} w(x) \leq 0$. But $w = \log(v)$ so

$$\partial^2_{j,k} w(x) = \frac{v\partial_{jk} v - \partial_j v \partial_k v}{v^2},$$

which is non-positive if and only if

$$|K_x| V(K_x[n-2], B_j, B_k) \leq \frac{n}{n-1} V(K_x[n-1], B_j) V(K_x[n-1], B_k).$$

Thus plugging $x = 0$ we get 4. $\implies$ 5. and using 4. with K_x instead of A gives 5. $\implies$ 4. $\square$

We note that it is enough to check property 1. in Theorem 3.6, just in the case $m = 2$. Indeed the case of general $m > 2$ follows by an iteration argument.

We also note that, in dimension 2, property 5. of Theorem 3.6 holds for any convex bodies by the classical local version of Alexandrov's inequality that was proved by W. Fenchel (see [21], also [51] and discussion in Section 4.1 below) and further generalized in [24,2,53]: for any convex compact sets A, B_1, B_2 in $\mathbb{R}^n$ we have

$$|A| V(A[n-2], B_1, B_2) \leq 2V(A[n-1], B_1) V(A[n-1], B_2). \tag{26}$$

The constant 2 is sharp in any dimension (see [31] and [28]). This shows also that log-submodularity doesn't hold in the set of compact convex sets in $\mathbb{R}^n$, for $n \geq 3$. From (26) and Theorem 3.6, the following theorem holds.

Theorem 3.7. *For every convex compact A, B_1, B_2 in $\mathbb{R}^2$ it holds*

$$|A|\,|A + B_1 + B_2| \leq |A + B_1|\,|A + B_2|.$$

Note that the inequality from Theorem 3.7 holds also in $\mathbb{R}$. Moreover, the inequality in $\mathbb{R}$ and $\mathbb{R}^2$ can be improved as shown in Theorem 4.1, in particular in the class of convex bodies the inequality is strict.

It is easy to see that 1. from Theorem 3.6 works well with direct sums, more precisely if $A^1, B_1^1, \dots, B_m^1 \subset \mathbb{R}^{n_1}$ and $A^2, B_1^2, \dots, B_m^2 \subset \mathbb{R}^{n_2}$ satisfy (23), then $A = A^1 \times A^2 \subset \mathbb{R}^{n_1+n_2}$ and $B_i = B_i^1 \times B_i^2$ in $\mathbb{R}^{n_1+n_2}$, $i = 1, \dots, m$ satisfy (23). This fact can be used to create different classes of compact convex sets stable by sum and dilations which satisfy the properties of Theorem 3.6.

Remark 3.8. Minkowski's second inequality gives that, for any compact convex sets A and B, one has

$$|A|V(A[n-2], B[2]) \leq V(A[n-1], B)^2. \tag{27}$$

Theorem 3.6 implies that if $\mathcal{K}$ is a class of convex bodies on which log-submodularity holds (property 1. of Theorem 3.6), then for bodies in $\mathcal{K}$, a Fenchel type inequality similar to (26) holds with a dimensional factor $\frac{n}{n-1}$ instead of 2.

Let us, also, note that the same proof shows that for a fixed compact convex set A not necessarily belonging to $\mathcal{K}$, if, for every B_1, B_2 in $\mathcal{K}$, one has

$$|A|\,|A + B_1 + B_2| \leq |A + B_1|\,|A + B_2|, \tag{28}$$

then, for every $B_1, B_2 \in \mathcal{K}$, one has

$$|A|V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1)V(A[n-1], B_2).$$

3.3. Classes closed under linear transformations

Theorem 3.9. *Let $\mathcal{L}$ be a class of a compact convex sets in $\mathbb{R}^n$ stable under any linear transformations. The following are equivalent:*

1. $|A|\,|\partial(A + [0,u])| \leq |\partial A|\,|A + [0,u]|$, *for any $A \in \mathcal{L}$ and any $u \in \mathbb{R}^n$.*
2. $|A|\,|\partial(P_{u^\perp} A)| \leq |\partial A|\,|P_{u^\perp} A|_{n-1}$, *for any $A \in \mathcal{L}$ and any $u \in S^{n-1}$.*
3. $|A|\,|P_{[u,v]^\perp} A|_{n-2}\sqrt{1 - \langle u, v\rangle^2} \leq |P_{u^\perp} A|_{n-1}|P_{v^\perp} A|_{n-1}$, *for any $A \in \mathcal{L}$ and any $u, v \in S^{n-1}$.*
4. $|A + [0,u] + [0,v]|\,|A| \leq |A + [0,u]|\,|A + [0,v]|$, *for any $A \in \mathcal{L}$ and any $u, v \in \mathbb{R}^n$.*
5. $|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1} V(A[n-1], Z_1)V(A[n-1], Z_2)$, *for any $A \in \mathcal{L}$ and any zonoids Z_1, Z_2.*
6. *For any $A \in \mathcal{L}$ and any $u, v \in \mathbb{R}^n$, let us define $P(t) = |A + t([0,u] + [0,v])|$, for $t > 0$. Then $P(t)$ is the restriction to $\mathbb{R}_+$ of a polynomial on $\mathbb{R}$, which has only real roots.*

Proof. 1. $\iff$ 2.: This was observed in [2]. It is true even for fixed A and u. For $u \neq 0$, it follows from the identities $|A + [0,u]| = |A| + |u||P_{u^\perp} A|$ and

$$|\partial(A+[0,u])| = nV((A+[0,u])[n-1], B_2^n) = |\partial A| + |u||\partial(P_{u^\perp}A)|.$$

2. $\Longrightarrow$ 3.: Define $T_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by $T_\varepsilon x = \varepsilon x + \langle x, v\rangle v$. Notice that $T_0(B_2^n) = [-v, v]$. From (2) applied to $T_\varepsilon^{-1}A$ and $T_\varepsilon^{-1}u$ we have

$$\begin{aligned}&|T_\varepsilon^{-1}A|V(T_\varepsilon^{-1}A[n-2], [0, T_\varepsilon^{-1}u], B_2^n)\\ \leq &\frac{n}{n-1}V(T_\varepsilon^{-1}A[n-1], B_2^n)V(T_\varepsilon^{-1}A[n-1], [0, T_\varepsilon^{-1}u]).\end{aligned}$$

Thus

$$|A|V(A[n-2], [0,u], T_\varepsilon B_2^n) \leq \frac{n}{n-1}V(A[n-1], T_\varepsilon B_2^n)V(A[n-1], [0,u]).$$

When $\varepsilon \to 0$, we get

$$|A|V(A[n-2], [0,u], [0,v]) \leq \frac{n}{n-1}V(A[n-1], [0,v])V(A[n-1], [0,u]). \tag{29}$$

From (16) and (17), we get $|P_{u^\perp}A|_{n-1} = nV(A[n-1], [0,u])$ and

$$|P_{[u,v]^\perp}A|_{n-2}\sqrt{1-\langle u,v\rangle^2} = n(n-1)V(A[n-2], [0,u], [0,v]).$$

3. $\Longleftrightarrow$ 4.: We may assume that u is not colinear with v. Applying a linear transformation to A, u and v, we may assume that u, v are orthonormal. Expanding both sides of the inequality in 4. and using (11), we get 3.

3. $\Longrightarrow$ 5.: As noticed above, 3. is equivalent to (29). From the linearity of mixed volumes, we deduce that for every zonotopes Z_1 and Z_2, one has

$$|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1}V(A[n-1], Z_1)V(A[n-1], Z_2). \tag{30}$$

Taking limits, we conclude that (30) is valid for every zonoids Z_1, Z_2.

5. $\Longrightarrow$ 2.: Applying (30) to $Z_1 = [0,u]$ and $Z_2 = B_2^n$ and using that $V(A[n-2], [0,u], B_2^n) = \frac{1}{n(n-1)}|\partial(P_{u^\perp}A)|$, $V(A[n-1], [0,u]) = \frac{1}{n}|P_{u^\perp}A|_{n-1}$ and (19), we conclude.

3. $\Longleftrightarrow$ 6.: We may assume that u is not colinear with v. Applying a linear transformation, to (3) and (6), it is enough to assume that u, v are orthonormal. Then

$$P(t) = |A| + t(|P_{u^\perp}A|_{n-1} + |P_{v^\perp}A|_{n-1}) + |P_{[u,v]^\perp}A|_{n-2}t^2.$$

The equation

$$|A| + t(|P_{u^\perp}A|_{n-1} + |P_{v^\perp}A|_{n-1}) + |P_{[u,v]^\perp}A|_{n-2}t^2 = 0$$

has real roots is equivalent to

$$\left(\frac{|P_{u^\perp}A|_{n-1}+|P_{v^\perp}A|_{n-1}}{2}\right)^2 \geq |A||P_{[u,v]^\perp}A|_{n-2}, \tag{31}$$

which follows immediately from 3. To show that 6. $\implies$ 3., assume (31) is true for all A in $\mathcal{L}$ and $u,v\in S^{n-1}$. Consider the linear operator T such that $Tu=tu$ and $Tv=t^{-1}v$ for $t>0$ and $Tx=x$ for $x\in[u,v]^\perp$. Taking $t=(|P_{u^\perp}A|_{n-1}/|P_{v^\perp}A|_{n-1}|)^{1/2}$, we get

$$|P_{v^\perp}TA|_{n-1}=|P_{u^\perp}TA|_{n-1}.$$

Applying (31) to TA, we get

$$|P_{u^\perp}TA|_{n-1}|P_{v^\perp}TA|_{n-1}\geq |TA||P_{[u,v]^\perp}TA|_{n-2}.$$

Since $|TA|=|A|$, $|P_{[u,v]^\perp}TA|_{n-2}=|P_{[u,v]^\perp}A|_{n-2}$ and

$$|P_{u^\perp}TA|_{n-1}|P_{v^\perp}TA|_{n-1}=|P_{u^\perp}A|_{n-1}|P_{v^\perp}A|_{n-1},$$

we get 3. □

Remark 3.10. Notice that, if $\mathcal{L}$ is the class of zonoids, then 1. from Theorem 3.9

$$\frac{|A+[0,u]|}{|\partial(A+[0,u])|_{n-1}}\geq\frac{|A|}{|\partial A|_{n-1}}$$

for every zonoid A and every $u\in\mathbb{R}^n$ is equivalent to

$$\frac{|A+B|}{|\partial(A+B)|_{n-1}}\geq\frac{|A|}{|\partial A|_{n-1}}$$

for all zonoids A and B. Similarly, from 4., we deduce that these inequalities are equivalent to

$$\frac{|A+B|}{|P_{u^\perp}(A+B)|_{n-1}}\geq\frac{|A|}{|P_{u^\perp}A|_{n-1}}$$

for all zonoids A and B, as stated in the introduction. Similarly, using also the methods of the proof of Corollary 6.7, we deduce that the stronger conjectures

$$\frac{|A+B|}{|\partial(A+B)|_{n-1}}\geq\frac{|A|}{|\partial A|_{n-1}}+\frac{|B|}{|\partial B|_{n-1}}$$

for all zonoids A and B, and

$$\frac{|A+B|}{|P_{u^\perp}(A+B)|_{n-1}}\geq\frac{|A|}{|P_{u^\perp}A|_{n-1}}+\frac{|B|}{|P_{u^\perp}B|_{n-1}}$$

for all zonoids A and B, are equivalent.

Remark 3.11. It was proved in [53] that 5. in Theorem 3.9 is satisfied when A is a simplex (actually, even without constant $n/(n-1)$), thus all of the properties in Theorem 3.9 are true for simplices. Actually, the inequalities of statements 2. and 3. hold with an extra factor $\frac{n-1}{n}$ on the right hand side.

Remark 3.12. Notice that the inequality (26) shows that the property 5. of Theorem 3.9 holds true for the class $\mathcal{L}$ of compact convex sets in $\mathbb{R}^2$ and doesn't hold in the class of compact convex sets in $\mathbb{R}^n$, for $n \geq 3$.

4. Some special cases

4.1. An improved inequality in $\mathbb{R}^2$

Inspired by an analogous result [16, Theorem 3] in Information Theory, T. Courtade asked if

$$|B|^{1/n}|C|^{1/n} + |A|^{1/n}|A+B+C|^{1/n} \leq |A+B|^{1/n}|A+C|^{1/n} \tag{32}$$

for $A = B_2^n$ being the Euclidean ball, and any convex bodies B, C in $\mathbb{R}^n$. Here we confirm Courtade's conjecture in $\mathbb{R}^2$ in a more general setting.

Theorem 4.1. *Consider convex bodies* $A, B, C \subset \mathbb{R}^2$*, then*

$$|A|^{1/2}|A+B+C|^{1/2} + |B|^{1/2}|C|^{1/2} \leq |A+B|^{1/2}|A+C|^{1/2}. \tag{33}$$

Proof. The main tool to prove the above inequality is the following classical inequality of Fenchel that we have already used. We will need now to use the most general form of this inequality (see [51, (7.69) pp. 401]):

$$(|A|V(B,C) - V(A,B)V(A,C))^2 \leq (V(A,B)^2 - |A||B|)(V(A,C)^2 - |A||C|).$$

Note that the above can be rewritten as

$$|C|V(A,B)^2 + |B|V(A,C)^2 + |A|V^2(B,C) - |A||B||C| - 2V(A,B)V(A,C)V(B,C) \leq 0. \tag{34}$$

Squaring both sides of (33), we get

$$2\big(|A||A+B+C||B||C|\big)^{1/2} + |A||A+B+C| + |B||C| \leq |A+B||A+C|.$$

We use that $|A+B| = |A| + 2V(A,B) + |B|$ to rewrite the above as

$$(|A||B||C||A+B+C|)^{1/2} + |A|V(B,C) \leq 2V(A,B)V(A,C) + V(A,B)|C| + |B|V(A,C).$$

Using (26), in order to prove the above inequality, it is enough to establish

$$|A||B||C||A+B+C| \le \left(2V(A,B)V(A,C)+V(A,B)|C|+|B|V(A,C)-|A|V(B,C)\right)^2.$$

Consider rB, $r \ge 0$, instead of B:

$$|A||B||C||A+rB+C| \le \left(2V(A,B)V(A,C)+V(A,B)|C|+r|B|V(A,C)-|A|V(B,C)\right)^2.$$

The above represents a quadratic inequality $\alpha r^2+\beta r+\gamma \ge 0$, with

$$\alpha = |B|^2V^2(A,C)-|A||B|^2|C|,$$
$$\beta = 2\big(2V(A,B)V(A,C)+V(A,B)|C|-|A|V(B,C)\big)|B|V(A,C)-2|A||B||C|V(B,A+C),$$
$$\gamma = \big(2V(A,B)V(A,C)+V(A,B)|C|-|A|V(B,C)\big)^2-|A||B||C||A+C|.$$

It follows from (13) that $\alpha \ge 0$. It turns out β may be negative and thus we need to show that $D=\beta^2-4\alpha\gamma \le 0$, which after division by $|B|^2$ becomes

$$\Big[\Big(2V(A,B)V(A,C)+V(A,B)|C|-|A|V(B,C)\Big)V(A,C)-|A||C|V(B,A+C)\Big]^2$$
$$-\big(V^2(A,C)-|A||C|\big)\Big[\Big(2V(A,B)V(A,C)+V(A,B)|C|-|A|V(B,C)\Big)^2-|A||B||C||A+C|\Big] \le 0.$$

Simplifying the above inequality and dividing it by $|A||C|$ we may rearrange the terms to get that our goal is to show that

$$\Big(|C|V(A,B)^2-2V(A,B)V(A,C)V(B,C)+|A|V^2(B,C)\Big)\Big(|A|+2V(A,C)+|C|\Big)$$
$$+\big(V^2(A,C)-|A||C|\big)|B|(|A+C|) \le 0.$$

Factoring out $|A+C|$ we get that our goal is to show that

$$\Big(|C|V(A,B)^2-2V(A,B)V(A,C)V(B,C)+|A|V^2(B,C)+|B|V^2(A,C)-|A||C||B|\Big)|A+C| \le 0.$$

Finally, the above inequality follows from (34). □

From the failure of log-submodularity on the space of convex bodies for $n \ge 3$ (observed independently by Nayar and Tkocz [46] and a subset of the authors [28]), we know that inequality (32) cannot possibly hold for $n \ge 3$ if A is an arbitrary convex body. Of course, Courtade's conjecture could still be true since it only considers the case $A = B_2^n$. We note that a weaker version of the conjecture, namely,

$$|B_2^n||B_2^n + B + C| \le |B_2^n + B||B_2^n + C|$$

was proved in [28, Theorem 4.12] in the special case when B is a zonoid and C is an arbitrary convex body.

4.2. Parallelotopes in general dimension

It is clear that 3. from Theorem 3.9 holds for ellipsoids. Indeed, after applying a linear transformation, it reduces to the following inequality: $|B_2^n||B_2^{n-2}| \le |B_2^{n-1}|^2$, which follows from the log-convexity of the Gamma function. In the next theorem, we prove that 3. from Theorem 3.9 holds for the class of parallelotopes.

Theorem 4.2. *Let A be parallelotope in $\mathbb{R}^n$ and $u, v \in S^{n-1}$, then*

$$|A|\,|P_{[u,v]^\perp} A|_{n-2}\sqrt{1-\langle u,v\rangle^2} \le |P_{u^\perp} A|_{n-1}\,|P_{v^\perp} A|_{n-1}. \tag{35}$$

Moreover, suppose A is full dimensional and $A = a + \sum_{i=1}^n [0, w_i]$, for some $a, w_1, \dots, w_n$ in $\mathbb{R}^n$. Then there is equality in (35) if and only if $u \in \operatorname{span}\{w_i, i \in I\}$ and $v \in \operatorname{span}\{w_i, i \notin I^c\}$ for some $I \subset \{1, \dots, n\}$.

Proof. We use the representation of the volume of projections using mixed volumes (16) to restate the above statement as

$$|A|V(A[n-2],[0,u],[0,v]) \le \frac{n}{n-1}V(A[n-1],[0,u])V(A[n-1],[0,v]).$$

Applying an affine transformation, we may assume $A = [0,1]^n = \sum_{i=1}^n [0, e_i]$. Thus,

$$V\left(A[n-1],[0,u]\right) = \frac{1}{n}\sum_{|I|=n-1} |\det(u,(e_i)_{i\in I})| = \frac{1}{n}\sum_{i=1}^n |u_i|,$$

$$V\left(A[n-2],[0,u],[0,v]\right) = \frac{1}{n(n-1)}\sum_{|I|=n-2} |\det(u,v,(e_i)_{i\in I})| = \frac{1}{n(n-1)}\sum_{i<j} |u_i v_j - u_j v_i|.$$

Finally we need to show

$$\sum_{i<j} |u_i v_j - u_j v_i| \le \sum |u_i| \sum |v_j|, \tag{36}$$

which follows from the triangle inequality. The equality in (36) is only possible if and only if $u_i v_i = 0$ for every $i \in \{1, \dots, n\}$, which implies the desired equality case. □

We note that every zonotope A can be seen as an orthogonal projection of a high dimensional cube. Unfortunately, Theorem 4.2 can not be generalized directly to the

case of projection of higher co-dimensions (as it is done in 3. from Theorem 5.3 below), indeed such direct generalization requires Theorem 4.2 for all zonotopes in place of the cube. Thus we prove this property directly for the case of parallelotopes in the next theorem, which extends the previous one.

Theorem 4.3. *Let A be a full dimensional parallelotope in $\mathbb{R}^n$ and let E, F be two subspaces of $\mathbb{R}^n$ such that $E^\perp \subset F$, then*

$$|A||P_{E\cap F}A| \leq |P_E A||P_F A|, \tag{37}$$

where we understand $|\cdot|$ to mean Lebesgue measure on the subspace of appropriate dimension in each instance. Moreover, there is equality in (37) if and only if there exists two complementary subspaces G_1, G_2 of $\mathbb{R}^n$ such that $A_1 := A \cap G_1$ and $A_2 := A \cap G_2$ are two lower dimensional parallelotopes, $A = A_1 + A_2$, $E^\perp \subset G_1$ and $F^\perp \subset G_2$.

Proof. Notice that (37) is invariant under application of rotation $S \in O(n)$ to the parallelotope A and subspaces E and F. Thus, without loss of generality, we may assume $A = T(\sum_{i=1}^n [0, e_i])$, for some $T \in GL(n)$ and $E = \{e_1, \dots, e_m\}^\perp$ and $F = \{e_{m+1}, \dots e_{m+j}\}^\perp$, with $m + j \leq m$. Then

$$\left|P_E\left(T\sum_{i=1}^n [0,e_i]\right)\right|_{n-m} = \left|\sum_{i=1}^n [0, P_E(Te_i)]\right|_{n-m} = \left|\sum_{i=1}^n [0, P_E Te_i] + \sum_{k=1}^m [0,e_k]\right|_n,$$

where the last equality follows from $\sum_{i=1}^n [0, P_E(Te_i)]|_{n-m} \subset E = \{e_1, \dots, e_m\}^\perp$. Thus, using (21),

$$\begin{aligned}
\left|P_E\left(T\sum_{i=1}^n [0,e_i]\right)\right|_{n-m} &= \sum_{|I|=n-m} |\det(e_1, \dots, e_m, \{P_E Te_i\}_{i\in I}| \\
&= \sum_{|I|=n-m} |\det(e_1, \dots, e_m, \{Te_i\}_{i\in I}| \\
&= |\det(T)| \sum_{|I|=n-m} |\det(T^{-1}e_1, \dots, T^{-1}e_m, \{e_i\}_{i\in I}| \\
&= |\det(T)| \sum_{|J|=m} |\det(\{w_i^J\}_{i=1}^m)|,
\end{aligned} \tag{38}$$

where we denote by $w_i = T^{-1}e_i$, $i = 1, \dots, n$ and by u^J we denote the orthogonal projection of vector u onto $\operatorname{span}\{e_i\}_{i\in J}$. We apply (38) to get that

$$\left|P_{E\cap F}\sum_{i=1}^n [0, Te_i]\right|_{n-j-m}$$

$$= |\det(T)| \sum_{|N|=m+j} |\det(\{w_i^N\}_{i=1}^{m+j})|$$

$$= |\det(T)| \sum_{|N|=m+j} \left| \sum_{|I|=m, I\subset N} \varepsilon(N,I) \det(\{w_i^I\}_{i\le m}) \det(\{w_i^{N\cap I^c}\}_{i>m}) \right|,$$

where in the last step we have used the Laplace formula. Finally,

$$\begin{aligned}
\left|P_{E\cap F} \sum_{i=1}^n [0,Te_i]\right|_{n-j-m} &\le |\det(T)| \sum_{N=m+j} \sum_{|I|=m, I\subset N} |\det(\{w_i^I\}_{i\le m})||\det(\{w_i^{N\cap I^c}\}_{i>m})| \\
&= |\det(T)| \sum_{|I|=m} \sum_{N=m+j, I\subset N} |\det(\{w_i^I\}_{i\le m})||\det(\{w_i^{N\cap I^c}\}_{i>m})| \\
&= |\det(T)| \sum_{|I|=m} \sum_{I\cup J, |J|=j, I\cap J=\emptyset} |\det(\{w_i^I\}_{i\le m})||\det(\{w_i^J\}_{i>m})| \\
&\le |\det(T)| \sum_{|I|=m} |\det(\{w_i^I\}_{i\le m})| \sum_{|J|=j} |\det(\{w_i^J\}_{i>m})| \\
&= |\det(T)|^{-1} \left| P_E\left(T\sum_{i=1}^n [0,e_i]\right)\right| \left| P_F\left(T\sum_{i=1}^n [0,e_i]\right)\right|,
\end{aligned}$$

where the last equality, again, follows from (38). If there is equality in (37), then there is equality in the last inequality above, thus for every $I, J \subset \{1,\dots,n\}$ such that $I\cap J \neq \emptyset$ one has

$$\det(\{w_i^I\}_{i\le m}) \det(\{w_i^J\}_{i>m}) = 0 \tag{39}$$

Since $w_i = T^{-1}e_i$, $i=1,\dots,n$ and $T\in GL(n)$, the vectors $w_1,\dots,w_{m+j}$ are linearly independent. Hence, there exists $I_1, I_2 \subset \{1,\dots,n\}$, with $|I_1| = m$, $|I_2| = j$ such that $\det(\{w_i^{I_1}\}_{i\le m}) \neq 0$ and $\det(\{w_i^{I_2}\}_{i>m}) \neq 0$. Hence, $\det(\{w_i^{I_1}\}_{i\le m})\det(\{w_i^{I_2}\}_{i>m}) \neq 0$, which implies, from (39), that $I_1 \cap I_2 = \emptyset$. Let us prove that, for any $k\in I_1$, one has $(w_i^{\{k\}})_{i>m} = 0$. By contradiction if, for some $k\in I_1$, one has $(w_i^{\{k\}})_{i>m} \neq 0$ then, by the Steinitz exchange lemma, there exists $I_2' \subset I_2$, with $|I_2'| = j-1$ such that $\det(\{w_i^{I_2'\cup\{k\}}\}_{i>m}) \neq 0$. Since $(I_2'\cup\{k\})\cap I_1 \neq \emptyset$, this contradicts (39). Hence, for any $k\in I_1$, one has $(w_i^{\{k\}})_{i>m} = 0$. In the same way, for any $k\in I_2$, one has $(w_i^{\{k\}})_{i\le m} = 0$. Now, let $k\in\{1,\dots,n\}\setminus(I_1\cup I_2)$. If $(w_i^{\{k\}})_{i>m} \neq 0$ then, by the Steinitz exchange lemma, there exists $I_2'\subset I_2$, with $|I_2'| = j-1$ such that $\det(\{w_i^{I_2'\cup\{k\}}\}_{i>m}) \neq 0$. Applying the preceding result to $I_2'\cup\{k\}$ instead of I_2, we deduce that $(w_i^{\{k\}})_{i\le m} = 0$. We conclude that, for any $k\in\{1,\dots,n\}$, one has either $(w_i^{\{k\}})_{i\le m} = 0$ or $(w_i^{\{k\}})_{i>m} = 0$. Denoting $I = \{k\in\{1,\dots,n\}; (w_i^{\{k\}})_{i\le m} \neq 0\}$, we get that $\mathrm{span}(w_1,\dots,w_m) \subset \mathrm{span}(e_i)_{i\in I}$ and $\mathrm{span}(w_{m+1},\dots,w_{m+j}) \subset \mathrm{span}(e_i)_{i\notin I}$. Thus denoting $G_1 = \mathrm{span}(T(e_i))_{i\in I}$, $A_1 = \sum_{i\in I}[0,T(e_i)]$, $G_2 = \mathrm{span}(T(e_i))_{i\notin I}$ and $A_1 = \sum_{i\notin I}[0,T(e_i)]$, we conclude. □

5. Inequalities for zonoids

5.1. Zonoids in $\mathbb{R}^3$

Zonoids form a natural class of bodies which is stable under addition and linear transformations. In this section, we confirm property (4) from Theorem 3.9 in the class of 3-dimensional zonoids. Thus, using Theorem 5.1, we get that all properties described in Theorems 3.6 and 3.9 are true for this class.

Theorem 5.1. *Let A be a zonoid in $\mathbb{R}^3$ and $u, v \in S^2$. Then*

$$|A|_3 |P_{[u,v]^\perp} A|_1 \sqrt{1 - \langle u, v\rangle^2} \leq |P_{u^\perp} A|_2 |P_{v^\perp} A|_2. \tag{40}$$

Proof. Using (17), inequality (40) is equivalent to

$$|A|_3 V(A, [0,u], [0,v]) \leq \frac{3}{2} V(A[2], [0,u]) V(A[2], [0,v]). \tag{41}$$

We assume that u, v are linearly independent (otherwise the inequality is trivial) and note that it is enough to prove (41) in the case of $u = e_1$, $v = e_2$ and any zonoid A. Indeed, the more general case then follows by applying the inequality to $T^{-1}A$, where $T \in GL(3)$ is such that $Te_1 = u$ and $Te_2 = v$. Thus our goal is to prove that, for any zonoid $A \subset \mathbb{R}^3$,

$$|A|_3 |P_{[e_1,e_2]^\perp} A|_1 \leq |P_{e_1^\perp} A|_2 |P_{e_2^\perp} A|_2. \tag{42}$$

By approximation, it is enough to prove (42) when A is a zonotope. Suppose that $A = \sum_{i=1}^M [0, u_i]$, where $u_i = (x_i, y_i, z_i) \in \mathbb{R}^3$. Using (21), we get that (42) is equivalent to

$$\sum_{1 \leq i < j < k \leq M} \left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right| \sum_{i=1}^M |z_i| \leq \sum_{1 \leq i < j \leq M} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \sum_{1 \leq i < j \leq M} \left| \det \begin{pmatrix} x_i & x_j \\ z_i & z_j \end{pmatrix} \right| \tag{43}$$

We consider $y_1, \ldots, y_M$ and $z_1, \ldots, z_M$ as fixed, we write $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$ and we define $f, g : \mathbb{R}^M \to \mathbb{R}$ by

$$f(x) = \sum_{1 \leq i < j \leq M} \left| \det \begin{pmatrix} x_i & x_j \\ z_i & z_j \end{pmatrix} \right| \quad \text{and} \quad g(x) = \sum_{1 \leq i < j < k \leq M} \left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right|.$$

We note that f and g are piecewise affine and convex with respect to x_i, for any $1 \leq i \leq M$. We use the following elementary lemma.

Lemma 5.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a convex function. Fix some positive numbers $\{a_j\}_{j=1}^K$, $d \in \mathbb{R}$ and real numbers $\{c_j\}_{j=1}^K$. To prove that, for all $t \in \mathbb{R}$,*

$$\psi(t) \le \varphi(t) := \sum_{j=1}^{K} a_i |t + c_i| + d,$$

it is enough to prove the inequality at all critical points $t = -c_i$ of φ and at the limit $t \to \pm\infty$.

We shall apply the above argument inductively to f and g as functions of x_i, for $i \in \{1, \dots, M\}$, successively, with x_j fixed for $j \neq i$.

We start with x_1 and first check the limiting behavior at infinity (we also prove the limit at infinity argument as a part of a more general statement below). We note that, as $x_1 \to \infty$, the left hand side of (43) behaves like

$$|x_1| \left(\sum_{1<j<k\le M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right) \left(\sum_{j=1}^{M} |z_j| \right)$$

and the right hand side of (43) behaves like

$$|x_1| \left(\sum_{1\le j<k\le M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right) \left(\sum_{j=2}^{M} |z_j| \right).$$

Thus, (43) becomes

$$\left(\sum_{1<j<k\le M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right) \left(\sum_{j=1}^{M} |z_j| \right) \le \left(\sum_{1\le j<k\le M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right) \left(\sum_{j=2}^{M} |z_j| \right)$$

or

$$\left(\sum_{1<j<k\le M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right) |z_1| \le \left(\sum_{1<k\le M} \left| \det \begin{pmatrix} y_1 & y_k \\ z_1 & z_k \end{pmatrix} \right| \right) \left(\sum_{j=2}^{M} |z_j| \right).$$

The above equation is exactly the $\mathbb{R}^2$ analog of (40), with $A = \sum_{i=2}^{M} [0, (y_i, z_i)]$; $u = [1, 0]$ and $v = (y_1, z_1)/|(y_1, z_1)|$, thus it holds.

Our next goal is to study the critical points of f with respect to x_1, which satisfy

$$\det \begin{pmatrix} x_1 & x_j \\ z_1 & z_j \end{pmatrix} = 0, \text{ for } z_j \neq 0, \ \ 2 \le j \le M.$$

When x_1 is a solution of the above equation, then (x_j, z_j) is parallel to (x_1, z_1). Assume (without loss of generality) that $j = 2$ and $(x_1, z_1) = \lambda(x_2, z_2)$. We study (43), with

respect to x_2, under the assumption that $(x_1, z_1) = \lambda(x_2, z_2)$ and continue this algorithm inductively (we will show the general step below).

We must also consider the case when $z_j = 0$, for all $j \geq 2$. In this case (43) becomes

$$\left(|z_1| \sum_{1<j<k\leq M} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \right) |z_1| \leq \left(|z_1| \sum_{j=2}^{M} |y_j| \right) \left(|z_1| \sum_{j=2}^{M} |x_j| \right)$$

or

$$\left(\sum_{1<j<k\leq M} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \right) \leq \left(\sum_{j=2}^{M} |y_j| \right) \left(\sum_{j=2}^{M} |x_j| \right),$$

which follows immediately from

$$\left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| \leq |x_j y_k| + |x_k y_j|. \tag{44}$$

Continuing this process, we arrive to the case

$$(x_1, z_1) = \lambda_1(x_m, z_m), \ldots, (x_{m-1}, z_{m-1}) = \lambda_{m-1}(x_m, z_m), \tag{45}$$

for some $2 \leq m \leq M$. We also denote $\lambda_m = 1$ and we study f and g as functions of x_m.

Again, our first step is to confirm (43), when $x_m \to \pm\infty$. To do so, let us see how the functions f and g changed under (45). Let us first consider the terms appearing in g:

$$\left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right|,$$

when $i < j < k$. For $m < i$, the above determinant doesn't depend on x_m, so we only consider the case when $m \geq i$.

- If $m \geq k$, then the determinant is zero.
- if $i \leq m < j$ then, when $|x_m| \to \infty$

$$\left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right| \sim |x_i| \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| = |\lambda_i||x_m| \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right|$$

- $j \leq m < k$ then, when $|x_m| \to \infty$, observing that $\lambda_i = z_i/z_m$ and $\lambda_j = z_j/z_m$ we get

$$\left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right| \sim \left| \lambda_i x_m \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} - \lambda_j x_m \det \begin{pmatrix} y_i & y_k \\ z_i & z_k \end{pmatrix} \right|$$

$$= |x_m| \frac{|z_k|}{|z_m|} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right|.$$

We also need to compute the behavior of the terms appearing in f:

$$\left| \det \begin{pmatrix} x_i & x_j \\ z_i & z_j \end{pmatrix} \right|,$$

for $i < j$. When $m \geq j$, it is zero and when $m < i$, it does not dependent on x_m. So we assume that $i \leq m < j$. We get

$$\left| \det \begin{pmatrix} x_i & x_j \\ z_i & z_j \end{pmatrix} \right| \sim |\lambda_i| \, |x_m| \, |z_j|.$$

Thus to show that the (43) is true, as $|x_m| \to \infty$, we need to prove that

$$\begin{aligned} &\left[(\sum_{i \leq m} |\lambda_i|) \sum_{m<j<k\leq M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right. \\ &\left. + \frac{1}{|z_m|} \left(\sum_{m<k} |z_k| \right) \left(\sum_{1\leq i<j\leq m} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \right) \right] \left(\sum_{i=1}^{M} |z_i| \right) \\ \leq &\left(\sum_{1\leq i<j\leq M} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \right) \left(\sum_{i\leq m} |\lambda_i| \right) \sum_{j>m} |z_j|. \end{aligned}$$

Multiplying both sides by $|z_m|$, we are reduced to

$$\begin{aligned} &\left[(\sum_{i \leq m} |z_i|) \sum_{m<j<k<M} \left| \det \begin{pmatrix} y_j & y_k \\ z_j & z_k \end{pmatrix} \right| \right. \\ &\left. + \left(\sum_{m<k} |z_k| \right) \left(\sum_{1\leq i<j\leq m} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \right) \right] \left(\sum_{i=1}^{M} |z_i| \right) \\ \leq &\left(\sum_{1\leq i<j\leq M} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \right) \left(\sum_{i\leq m} |z_i| \right) \sum_{j>m} |z_j| \end{aligned}$$

Let $A' = \sum_{i=m+1}^{M} [0, (y_i, z_i)]$ and $B' = \sum_{i=1}^{m} [0, (y_i, z_i)]$, then the above becomes:

$$\left[|P_{e_1^\perp} B'| |A'| + |P_{e_1^\perp} A'| |B'| \right] |P_{e_1^\perp} (A' + B')| \leq |A' + B'| |P_{e_1^\perp} B'| |P_{e_1^\perp} A'|.$$

This is Bonnesen's inequality (6) in $\mathbb{R}^2$

$$\frac{|A'|_2}{|P_{e_1^\perp} A|_1} + \frac{|B'|_2}{|P_{e_1^\perp} B'|_1} \leq \frac{|A'+B'|_2}{|P_{e_1^\perp}(A'+B')|_1}.$$

Our next step (if $m < M$) is to consider the critical points of f, as a function of x_m; they are given by the equations

$$\det \begin{pmatrix} x_m & x_j \\ z_m & z_j \end{pmatrix} = 0,$$

for $z_j \neq 0$, $j \geq m+1$, if such z_j exists and repeat the process for all $m < M$. If $z_j = 0$, for all $j \geq m+1$, let us confirm the inequality directly (as for the case $m = 1$). To calculate

$$\left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right|$$

we may consider cases: if $i, j, k \leq m$, then the rank of the above matrix is at most to 2 and the determinant is zero; if $i, j, k \leq m$, the matrix has a row of zeros and the determinant is again zero. In the case when $i \leq m < j < k$, we get

$$\left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right| = |z_i| \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right|.$$

When $i < j \leq m < k$, we use that (x_i, x_j) is parallel to (z_i, z_j) to get that

$$\left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right| = |x_k| \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right|.$$

We make a similar analysis on the right hand side of (43) which becomes

$$\left(\sum_{i=1}^m |z_i| \right) \left[\left(\sum_{i=1}^m |z_i| \right) \sum_{m<j<k} \left| \det \begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} \right| + \sum_{k>m} |x_k| \sum_{i<j\leq m} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \right]$$
$$\leq \left[\left(\sum_{i=1}^m |z_i| \right) \left(\sum_{j>m} |y_j| \right) + \sum_{i<j\leq m} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right| \right] \left[\left(\sum_{i=1}^m |z_i| \right) \left(\sum_{j>m} |x_j| \right) \right]$$

The above inequality follows directly by simplification and application of (44).

We repeat the above process until we have $m = M$, thus $(x_i, z_i) = \lambda_i (x_M, z_M)$ for each $i = 1, \ldots, M-1$. Thus, the left hand side of the inequality (43) is equal to zero. Indeed, each of the following matrices

$$\begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix}$$

has rank less or equal then 2. □

5.2. More equivalent formulations for zonoids

Let us repeat again that the class of zonoids is closed under the linear transformations and addition. Thus all equivalences proved in Theorems 3.6 and 3.9 hold for the class of zonoids. In this section, we show some additional equivalences (which thus hold in dimension 3).

It should be noted that when we write $|P_E A|$ for a subspace E below, this refers to the use of Lebesgue measure on the subspace E (typically of lower dimension than n though we sometimes skip the subscript).

Theorem 5.3. *Let $n \in \mathbb{N}$, then the following are equivalent.*

1. *For every $k \geq n$ and for every family of vectors $u_1, \dots, u_k$ in $\mathbb{R}^n$ the function $f : 2^{[k]} \to \mathbb{R} \cup \{-\infty\}$ defined, for $S \subset [k]$, by*

$$f(S) = \log \left| \sum_{i \in S} [0, u_i] \right| = \log \sum_{I \subset S, |I|=n} |\det(\{u_i\}_{i \in I})|$$

is submodular: for all zonoids A, B, C one has

$$|A||A + B + C| \leq |A + B||A + C|.$$

2. *For every $k \geq n$ and for every family of vectors $u_1 \dots, u_k$ in $\mathbb{R}^n$, for every $u, v \in \mathbb{R}^n$*

$$\sum_{|I|=n} |\det(\{u_i\}_{i \in I})| \sum_{|I|=n-2} |\det(u, v, (u_i)_{i \in I})| \leq \sum_{|I|=n-1} |\det(u, (u_i)_{i \in I})| \sum_{|I|=n-1} |\det(v, (u_i)_{i \in I})|.$$

3. *For every zonoid A in $\mathbb{R}^n$ and all subspaces E, F of $\mathbb{R}^n$ such that $E^\perp \subseteq F$ we have*

$$|A||P_{E \cap F} A| \leq |P_E A||P_F A|.$$

4. *For every $m = 1, \dots, n$, for every zonoid A and every orthonormal sequence $u_1, \dots, u_m$, one has*

$$|A|^{m-1} \left| P_{[u_1, \dots, u_m]^\perp} A \right|_{n-m} \leq \prod_{i=1}^{m} \left| P_{u_i^\perp} A \right|_{n-1}.$$

5. For every $m = 1, \dots, n$ and all zonoids $A, B_1, \dots, B_m$ in $\mathbb{R}^n$, one has

$$|A|^{m-1} V(A[n-m], B_1, \dots, B_m) \leq \frac{n^m (n-m)!}{n!} \prod_{i=1}^{m} V(A[n-1], B_i).$$

Proof. The proof is based on translations of the properties described in Theorems 3.6 and 3.9 to the properties of zonoids.

We first show that 1. in Theorem 5.3 is equivalent to 1. in Theorem 3.6 in the class of zonotopes. Indeed, f is submodular if and only if for every $S, T \subset \{1, \dots, k\}$

$$\left| \sum_{i \in S \cup T} [0, u_i] \right| \left| \sum_{i \in S \cap T} [0, u_i] \right| \leq \left| \sum_{i \in S} [0, u_i] \right| \left| \sum_{i \in T} [0, u_i] \right|,$$

which is 1. in Theorem 5.3 for zonotopes $A = \sum_{i \in S \cap T} [0, u_i]$, $B_1 = \sum_{i \in S \setminus T} [0, u_i]$ and $B_2 = \sum_{i \in T \setminus S} [0, u_i]$. We use (21) to finish the proof.

We next note that 2. in Theorem 5.3 is equivalent to 3. of Theorem 3.9 in the case of zonotopes. Assume by homogeneity that $u, v \in S^{n-1}$. Then we apply (21) to get:

$$\left| \sum_{i=1}^{m} [0, u_i] \right| = \sum_{I \subset [m], |I| = n} |\det(u_i)_{i \in I}| \text{ and}$$

$$\left| P_{v^\perp} \left(\sum_{i=1}^{m} [0, u_i] \right) \right| = \sum_{I \subset [m], |I| = n-1} |\det(v, (u_i)_{i \in I})|$$

and the similar formula for the volume of $P_{[u,v]^\perp} (\sum_{i=1}^{m} [0, u_i])$.

Next, we show that 3. is equivalent to 3. from Theorem 3.9, which can be restated as

$$|A| |P_{[e_i, e_j]^\perp} A|_{n-2} \leq |P_{e_i^\perp} A|_{n-1} |P_{e_j^\perp} A|_{n-1}, \tag{46}$$

for any zonoid A and $i \neq j$, where $(e_1, \dots, e_n)$ is any orthonormal basis. Moreover 3. from Theorem 5.3 is equivalent to

$$|A| |P_{[e_1, \dots e_k]^\perp} A|_{n-k} \leq |P_{[e_1, \dots, e_i]^\perp} A|_{n-i} |P_{[e_{i+1}, \dots, e_k]^\perp} A|_{n-(k-i)}. \tag{47}$$

for any $i < k \leq n$ and any zonoid A in $\mathbb{R}^n$. Thus (46) is a particular case of (47). To prove the reverse, we first notice that if (46) holds for any zonoid A in $\mathbb{R}^n$, then it also must hold for zonoids in any dimension $m \leq n$. Indeed, for any zonoid A in $\mathbb{R}^m$, the cylinder $A \times [0,1]^{n-m}$ is a zonoid. Next, we may prove property (47) by induction. Indeed, using (46), it is true for $k = 2$ and $i = 1$, any $n \in \mathbb{N}$ and any zonoid A in $\mathbb{R}^n$. Assume the statement is true for some $k \in \mathbb{N}$ any $i < k \leq n$. Let us apply the statement to the zonoid $P_{e_{k+1}^\perp} A$ to get

$$|P_{e_{k+1}^{\perp}}A|_{n-1}|P_{[e_1,\dots,e_k,e_{k+1}]^{\perp}}A|_{n-(k+1)} \tag{48}$$
$$\leq |P_{[e_1,\dots,e_i,e_{k+1}]^{\perp}}A|_{n-(i+1)}|P_{[e_{i+1},\dots,e_k,e_{k+1}]^{\perp}}A|_{n-(k+1-i)}.$$

In addition, we apply the inductive hypothesis to A and the subspace spanned by $\{e_1,\dots,e_i,e_{k+1}\}$ and the subspace spanned by e_{k+1} to get

$$|A||P_{[e_1,\dots,e_i,e_{k+1}]^{\perp}}A|_{n-(i+1)} \leq |P_{[e_1,\dots,e_i]^{\perp}}A|_{n-i}|P_{e_{k+1}^{\perp}}A|_{n-1}. \tag{49}$$

Finally, we multiply (48) and (49) to finish the proof.

To prove that 3. implies 4., we apply (47) to $k=m$ and $i=1$, we get that

$$\frac{|P_{[u_1,\dots,u_m]^{\perp}}A|}{|P_{[u_2,\dots,u_m]^{\perp}}A|} \leq \frac{|P_{u_1^{\perp}}A|}{|A|}.$$

In the same way, for every $1\leq i\leq m$, one has

$$\frac{|P_{[u_i,\dots,u_m]^{\perp}}A|}{|P_{[u_{i+1},\dots,u_m]^{\perp}}A|} \leq \frac{|P_{u_i^{\perp}}A|}{|A|}.$$

Taking the product of these inequalities, we get the result.

To prove that 4. implies 5., we use (17). Thus (4) gives that for every orthonormal family of vectors $u_1,\dots,u_m$, one has

$$|A|^{m-1}V(A[n-m],[0,u_1],\dots,[0,u_m]) \leq \frac{n^m(n-m)!}{n!}\prod_{i=1}^{m}V(A[n-1],[0,u_i]).$$

Since this inequality is invariant with respect to any linear image of A by an invertible map, it holds also for any independent $u_1,\dots,u_m$. Then, we deduce that 5. holds for any sums of segments. The inequality for zonoids follows by taking limits.

Finally, 5. with $m=2$ is equivalent to 5. in Theorem 3.9. □

Notice that 3. of Theorem 5.3 can be rephrased by saying that, for any zonoid A in $\mathbb{R}^n$, the function $f:2^{[n]}\to\mathbb{R}$ defined by $f(S)=\log(|P_{[e_i;i\in S]}A|)$ is submodular.

The inequalities analogous to 3. in Theorems 3.9 and 5.3 belong to the class of local Loomis-Whitney type inequalities and were studied in many works, including [31,24,53, 2,1], for general classes of convex bodies.

In the next lemma, we present a new proof of a result from [31], which uses the same approach as our proof of Theorem 3.9.

Lemma 5.4. *Consider a convex body K in $\mathbb{R}^n$ and a pair of orthogonal vectors $u,v\in S^{n-1}$, then*

$$|K||P_{[u,v]^{\perp}}K| \leq \frac{2(n-1)}{n}|P_{u^{\perp}}K||P_{v^{\perp}}K|.$$

Proof. Let $L = [0,u] + \alpha[0,v]$, where $\alpha = |P_{u^\perp}K|/|P_{v^\perp}K|$, noticing that the case $|P_{v^\perp}K| = 0$ is trivial. Then

$$nV(K[n-1],L) = |P_{u^\perp}K|_{n-1} + \alpha|P_{v^\perp}K|_{n-1} \text{ and}$$
$$\frac{n(n-1)}{2}V(K[n-2],L[2]) = \alpha|P_{[u,v]^\perp}K|.$$

Using Minkowski's second inequality (14), we get

$$\left(\frac{|P_{u^\perp}K| + \alpha|P_{v^\perp}K|}{n}\right)^2 \geq \frac{2\alpha}{n(n-1)}|K||P_{[u,v]^\perp}K|$$

we substitute $\alpha = |P_{u^\perp}K|/|P_{v^\perp}K|$ to finish the proof. □

The main tool in the proof of Lemma 5.4 is Minkowski's second inequality, which relies on the fact that the polynomial $Q(t) = |K + tL|$ raised to the power $1/n$ is a concave function for $t \geq 0$. We conjecture that the concavity properties of this polynomial can be improved for Z being a zonoid and L being a finite sum of $m \leq n$ segments:

Conjecture 5.5. *Let $1 \leq m \leq n$, let Z be a zonoid in $\mathbb{R}^n$ and let $u_1, \dots, u_m \in \mathbb{R}^n$. Set $P(t) = |Z + t\sum_{i=1}^m [0,u_i]|$, then $t \mapsto P^{1/m}(t)$ is concave on $\mathbb{R}_+$.*

Conjecture 5.5 clearly holds for $m = 1$ and $m = n$. In particular, the conjecture holds in $\mathbb{R}^2$. Conjecture 5.5 would follow if the statements of Theorem 5.3 or Theorem 3.9 were true. Moreover, the following proposition shows that the conjecture also holds in $\mathbb{R}^3$.

Proposition 5.6. *Let Z be a zonoid in $\mathbb{R}^3$ and u, v be two vectors from $\mathbb{R}^3$. Let $P(t) = |Z + t([0,u] + [0,v])|$, then $t \mapsto P^{1/2}(t)$ is concave on $\mathbb{R}_+$.*

Proof. We may assume that the vectors u, v are linearly independent. Then, applying a linear transformation T to Z and u, v we may assume that vectors u, v are orthogonal to each other and belong to S^2. Using that $Z + t([0,u] + [0,v])$ is again a zonoid, it is enough for us to show that $(P^{1/2}(0))'' \leq 0$, or $2P(0)P''(0) \leq P'(0)^2$. Using that

$$P(0) = |Z|, \quad P'(0) = |P_{u^\perp}Z| + |P_{v^\perp}Z| \quad \text{and} \quad P''(0) = 2|P_{[u,v]^\perp}Z|$$

and Theorem 5.1, we get

$$2P(0)P''(0) = 4|Z||P_{[u,v]^\perp}Z| \leq 4|P_{u^\perp}Z||P_{v^\perp}Z| \leq (|P_{u^\perp}K| + |P_{v^\perp}K|)^2 = P'(0)^2. \quad \square$$

It follows from part 6. of Theorem 3.9 and Theorem 5.1 that if $P(t) = |Z + t([0,u] + [0,v])|$, then $P(t)$, as a polynomial on $\mathbb{R}$, has only real roots, for any zonoid Z and any $u, v \in \mathbb{R}^3$. Thus another way to prove Proposition 5.6 is to notice the following simple

property. Consider a quadratic polynomial P with positive coefficients, then $\sqrt{P(t)}$ is concave for $t \geq 0$ if and only if P has only real roots. Thus, to study concavity of $\sqrt{P_C(t)} = |Z + tC|^{1/2}$, when C is two dimensional, one may study the roots of P. The subject of roots of Steiner-type polynomials has attracted a fair bit of attention in the literature, see, e.g., [33] and references therein. Since zonoids in dimension 2 are precisely the symmetric convex bodies, Proposition 5.6 makes the following conjecture plausible.

Question 5.7. *Let Z be a zonoid in $\mathbb{R}^n$. Then is it true that the polynomial $P_C(t) = |Z + tC|$ has only real roots for every symmetric convex body C of dimension 2?*

This question is directly connected to a question of Adam Marcus that we learned from Guillaume Aubrun [3] about the roots of Steiner polynomials of zonoids, and, in fact, it was one of the starting points of this investigation.

Question 5.8. *Let Z be a zonoid in $\mathbb{R}^n$. Then is it true that the Steiner polynomial $P_Z(t) = |Z + tB_2^n|$ has only real roots?*

Observe that, in the plane, even more is true: for any convex bodies K, L, the polynomial $P_{K,L}(t) = |K + tL|$ has only real roots. Indeed, $\sqrt{P_{K,L}(t)}$ is concave for $t \geq 0$ by Brunn-Minkowski inequality. This can be also seen from the computing the discriminant and noticing that $V(K, L)^2 - |K||L| \geq 0$.

Analogously, Question 5.7 is equivalent to the following question for mixed volumes: fix $n \geq 3$, and let K a zonoid in $\mathbb{R}^n$ and L be a two dimensional zonoid, is it true that

$$|K|V(K[n-2], L[2]) \leq \frac{n}{2(n-1)} V(K[n-1], L)^2?$$

The above inequality is true for L being a parallelogram, as follows from Theorem 3.6. However, it is not true for general zonoids, as we show in the following proposition inspired by work of V. Katsnelson [34].

Proposition 5.9. *Let $n \geq 3$. Then there exists a zonoid Z in $\mathbb{R}^n$ such that the Steiner polynomial $P_Z(t) = |Z + tB_2^n|$ and the polynomial $Q_Z(t) = |tZ + B_2^n|$ has roots which are not real.*

Proof. Noticing that $Q_Z(t) = t^n P_Z(1/t)$, it is enough to show that $P_Z(t)$ has a non-real root. Consider $Z = B_2^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^{n-2}$. Integrating on sections parallel to $\mathbb{R}^2 \times \{0\}$ and changing variable, we get that, for $t \geq 0$,

$$\begin{aligned} P_Z(t) = |Z + tB_2^n| = |B_2^2| \int_{tB_2^{n-2}} \left(1 + \sqrt{t^2 - |x|^2}\right)^2 dx \\ = t^{n-2}|B_2^2| \int_{B_2^{n-2}} \left(1 + t\sqrt{1 - |x|^2}\right)^2 dx. \end{aligned}$$

Since this last expression is the restriction of a polynomial to $t \geq 0$, the equality between $P_Z(t)$ and the last term is valid on whole $\mathbb{R}$. Let $t \in \mathbb{R}$ be fixed. For almost all $x \in B_2^{n-2}$, $(1 + t\sqrt{1-|x|^2})^2 > 0$, thus $P_Z(t) > 0$, for $t \neq 0$, so the only real root of P_Z is 0 and it is of order $n-2$. Hence P_Z has exactly 2 non-real roots. □

Proposition 5.9 shows that the answers to both Question 5.7 and Question 5.8 are negative.

Note that the example of the zonoid Z created in Proposition 5.9 is flat (i.e., has an affine hull of lower dimension than the ambient space), but one can replace it by a non-flat zonoid by using a small perturbation and the continuity of the roots of polynomials.

6. Inequalities for L_p-zonoids

Firey [22] extended the concept of Minkowski sum and introduced, for each real $p \geq 1$, a p-linear combination of convex bodies, the so-called ℓ_p-sum $K \oplus_p L$ of the convex bodies K and L containing the origin by:

$$h_{K \oplus_p L}(x) = (h_K(x)^p + h_L(x)^p)^{\frac{1}{p}}, \quad \forall x \in \mathbb{R}^n.$$

For any linear transform T on $\mathbb{R}^n$, one has $T(K \oplus_p L) = (TK) \oplus_p (TL)$. In a series of papers, Lutwak [36,37] showed that the Firey sums lead to a Brunn-Minkowski theory for each $p \geq 1$, including L_p-Brunn-Minkowski inequality, definition and inequalities for L_p-mixed volumes, L_p-Minkowski problem, as well as many other applications. In this section, we show the connection of the discussion from previous sections to this theory.

An L_p-zonotope is the ℓ_p-sum of centered segments and an L_p-zonoid is the Hausdorff limit of L_p-zonotopes. For $p = 2$, an L_2-zonoid is always an ellipsoid, possibly living in a lower-dimensional subspace, thus it can be written as the sum of $m \leq n$ orthogonal segments and is therefore an L_2-zonotope. The following extension of Conjecture 5 is thus natural.

Question 6.1. *Let $p \geq 1$ and consider L_p-zonoids A, B in $\mathbb{R}^n$ is it true that*

$$\left(\frac{|A \oplus_p B|}{|P_{u^\perp}(A \oplus_p B)|_{n-1}}\right)^p \geq \left(\frac{|A|}{|P_{u^\perp} A|_{n-1}}\right)^p + \left(\frac{|B|}{|P_{u^\perp} B|_{n-1}}\right)^p ? \tag{50}$$

6.1. The case $p = 2$

The next theorem gives an affirmative answer to this question in the case $p = 2$.

Theorem 6.2. *Let A, B be a pair of full dimensional L_2-zonoids in $\mathbb{R}^n$ and let u in S^{n-1}. Then*

$$\left(\frac{|A \oplus_2 B|}{|P_{u^\perp}(A \oplus_2 B)|_{n-1}}\right)^2 \geq \left(\frac{|A|}{|P_{u^\perp} A|_{n-1}}\right)^2 + \left(\frac{|B|}{|P_{u^\perp} B|_{n-1}}\right)^2,$$

with equality if and only if A and B have parallel tangent hyperplanes at $\rho_A(u)u$ and $\rho_B(u)u$.

We first give two proofs of the inequality and then prove the equality case.

Proof 1. For $m \geq n$ and $u_1, \dots, u_m \in \mathbb{R}^n$, let U be the $n \times m$ matrix whose columns are $u_1, \dots, u_m$. One has $[-u_1, u_1] \oplus_2 \cdots \oplus_2 [-u_m, u_m] = UB_2^m = \sqrt{UU^*}B_2^n$. Indeed,

$$\begin{aligned} h^2_{[-u_1,u_1]\oplus_2\cdots\oplus_2[-u_m,u_m]}(x) &= \sum_{i=1}^m \langle u_i, x\rangle^2 = \sum_{i=1}^m \langle Ue_i, x\rangle^2 = \sum_{i=1}^m \langle e_i, U^*x\rangle^2 = |U^*x|^2 \\ &= h^2_{UB_2^m}(x) = \langle U^*x, U^*x\rangle = \langle \sqrt{UU^*}x, \sqrt{UU^*}x\rangle \\ &= |\sqrt{UU^*}x|^2 = h^2_{\sqrt{UU^*}B_2^n}(x). \end{aligned}$$

Using this and the Cauchy-Binet formula, one has

$$|[-u_1, u_1] \oplus_2 \cdots \oplus_2 [-u_m, u_m]|^2 = |UB_2^m|_n^2 = \det(UU^*)|B_2^n|^2 = |B_2^n|^2 \sum_{|I|=n} (\det(u_i)_{i\in I})^2. \tag{51}$$

Next, using

$$P_{u^\perp}([-u_1, u_1] \oplus_2 \cdots \oplus_2 [-u_m, u_m]) = [-P_{u^\perp}u_1, P_{u^\perp}u_1] \oplus_2 \cdots \oplus_2 [-P_{u^\perp}u_m, P_{u^\perp}u_m],$$

we get that

$$|P_{u^\perp}([-u_1, u_1] \oplus_2 \cdots \oplus_2 [-u_m, u_m])|^2 = |B_2^{n-1}|^2 \sum_{|I|=n-1} (\det(P_{u^\perp}u_i)_{i\in I})^2.$$

Thus our goal is to prove that, for $m \geq k$, with $m - k \geq n$, $k \geq n$ and $A = [-u_1, u_1] \oplus_2 \cdots \oplus_2 [-u_k, u_k]$ and $B = [-u_{k+1}, u_{k+1}] \oplus_2 \cdots \oplus_2 [-u_m, u_m]$ we have

$$\begin{aligned} &\frac{\sum\limits_{M\subset\{1,\dots,k\},|M|=n} |\det(\{u_m\}_{m\in M}|^2}{\sum\limits_{M\subset\{1,\dots,k\},|M|=n-1} |\det(\{P_{e_n^\perp}u_i\}_{i\in M})|^2} + \frac{\sum\limits_{L\subset\{k+1,\dots,m\},|L|=n} |\det(\{u_l\}_{l\in L}|^2}{\sum\limits_{L\subset\{k+1,\dots,m\},|L|=n-1} |\det(\{P_{e_n^\perp}u_l\}_{l\in L}|^2} \\ &\qquad \leq \frac{\sum\limits_{M\subset\{1,\dots,m\},|M|=n} |\det(\{u_i\}_{i\in M}|^2}{\sum\limits_{M\subset\{1,\dots,m\},|M|=n-1} |\det(\{P_{e_n^\perp}(u_i)\}_{i\in M}|^2}. \end{aligned} \tag{52}$$

For $I \subset \{1, \dots, m\}$ let U_I be the $n \times n$ submatrix built from U by taking the columns with indices in I. Denote by $z_1, \dots, z_n \in \mathbb{R}^m$ the rows of the matrix U. Since $n \leq m$, the set $\sum_{i=1}^n [0, z_i]$ is a parallelotope leaving in a n-dimensional subspace of $\mathbb{R}^m$. Thus,

$$\left|\sum_{i=1}^{n}[0,z_i]\right|_n^2 = \sum_{|I|=n}(\det U_I)^2.$$

Therefore, we get that, for $m \geq n$, an $n \times m$ matrix U whose rows are $z_1, \dots, z_n \in \mathbb{R}^m$ and columns $u_1, \dots, u_m \in \mathbb{R}^n$:

$$|[-u_1,u_1] \oplus_2 \cdots \oplus_2 [-u_m,u_m]|_n = |B_2^n| \left|\sum_{i=1}^{n}[0,z_i]\right|_n.$$

Let us write U as a block matrix with two blocks $U = (U_1|0) + (0|U_2)$, with U_1 being an $n \times k$ matrix and U_2 an $n \times (m-k)$ matrix and we denote $V = (U_1|0)$ and $W = (0|U_2)$. Moreover, denote by U', V', W' the matrices obtained from U, V, W by erasing the n^{th} row. Then we only need to prove that

$$\frac{\sum_{|I|=n}(\det U_I)^2}{\sum_{|I|=n-1}(\det U'_I)^2} \geq \frac{\sum_{|I|=n}(\det V_I)^2}{\sum_{|I|=n-1}(\det V'_I)^2} + \frac{\sum_{|I|=n}(\det W_I)^2}{\sum_{|I|=n-1}(\det W'_I)^2}.$$

Recall that $z_1, \dots, z_n \in \mathbb{R}^m$ are the rows of U. Thus the rows of V are $P_E z_1, \dots, P_E z_n$, where P_E denotes the projection on the first k coordinates and the rows of W are $P_{E^\perp} z_1, \dots, P_{E^\perp} z_n$, where $P_{E^\perp}$ denotes the projection on the $n-k$ last coordinates. Thus we only need to show the following relationship for low-dimensional parallelotopes

$$\left(\frac{|\sum_{i=1}^{n}[0,z_i]|}{\left|\sum_{i=1}^{n-1}[0,z_i]\right|}\right)^2 \geq \left(\frac{|\sum_{i=1}^{n}[0,P_E z_i]|}{\left|\sum_{i=1}^{n-1}[0,P_E z_i]\right|}\right)^2 + \left(\frac{|\sum_{i=1}^{n}[0,P_{E^\perp} z_i]|}{\left|\sum_{i=1}^{n-1}[0,P_{E^\perp} z_i]\right|}\right)^2.$$

Using that the volume of a parallelotope is the product of the volume of one of its face and its height, we get that, if $H_n = \mathrm{span}(z_1, \dots, z_{n-1})$, then

$$\frac{|\sum_{i=1}^{n}[0,z_i]|}{\left|\sum_{i=1}^{n-1}[0,z_i]\right|} = d(z_n, H_n).$$

So we are reduced to prove that

$$d(z_n,H_n)^2 \geq d(P_E z_n, P_E H_n)^2 + d(P_{E^\perp} z_n, P_{E^\perp} H_n)^2.$$

Let $h_n \in H_n$ such that $d(z_n, H_n) = |z_n - h_n|$. By Pythagoras' theorem,

$$d(z_n,H_n)^2 = |z_n - h_n|^2 = \|P_E z_n - P_E h_n\|^2 + \|P_{E^\perp} z_n - P_{E^\perp} h_n\|^2.$$

Since $P_E h_n \in P_E H_n$ and $P_{E^\perp} z_n \in P_{E^\perp} H_n$, we conclude. □

Second proof of the inequality. We give another proof of Theorem 6.2, using the comparison of the ℓ_2 sum and the radial 2-sum. For any symmetric convex bodies K and L in $\mathbb{R}^n$, one has

$$K \oplus_2 L \supset K \widetilde{+}_2 L := \cup_{u\in S^{n-1}} \left[0, \sqrt{\rho_K^2(u) + \rho_L^2(u)}u\right]. \tag{53}$$

Indeed, using support functions, the fact that $K \supset \rho_K(u)[-u,u]$ implies that $h_K(x) \geq \rho_K(u)|\langle x,u\rangle|$, for any $x \in \mathbb{R}^n$ and thus

$$\begin{aligned} h_{K\oplus_2 L}(x) = \sqrt{h_K(x)^2 + h_L(x)^2} &\geq \sqrt{\rho_K(u)^2 + \rho_L(u)^2}|\langle x,u\rangle| \\ &= \sqrt{\rho_K(u)^2 + \rho_L(u)^2} h_{[-u,u]}(x). \end{aligned}$$

The formula (53) can be also restated in the language of radial functions:

$$\rho_{K\oplus_2 L}(u) \geq \sqrt{\rho_K^2(u) + \rho_L^2(u)}, \text{ for all } u \in S^{n-1}. \tag{54}$$

Next, we notice a formula for the volume of the orthogonal hyperplane projection of an ellipsoid (see, for example, [14,48]), to which we give a very simple proof. Let $\mathcal{E} = TB_2^n$, for some positive definite T then

$$\frac{|P_{u^\perp}\mathcal{E}|}{|\mathcal{E}|\,\|u\|_{\mathcal{E}}} = \frac{nV(TB_2^n[n-1],[0,u])}{|TB_2^n|\,|T^{-1}u|} = \frac{nV(B_2^n[n-1],[0,T^{-1}u])}{|B_2^n|\,|T^{-1}u|} = \frac{|B_2^{n-1}|}{|B_2^n|}. \tag{55}$$

Using the above we get

$$\frac{|\mathcal{E}|}{|P_u\mathcal{E}|} = \frac{|B_2^n|}{|B_2^{n-1}|}\rho_{\mathcal{E}}(u). \tag{56}$$

Thus, using this formula and (53), we deduce that for any ellipsoids A, B

$$\begin{aligned} \frac{|A\oplus_2 B|^2}{|P_u(A\oplus_2 B)|^2} &= \frac{|B_2^n|^2}{|B_2^{n-1}|^2}\rho_{A\oplus_2 B}(u)^2 \\ &\geq \frac{|B_2^n|^2}{|B_2^{n-1}|^2}(\rho_A(u)^2 + \rho_B(u)^2) = \frac{|A|^2}{|P_uA|^2} + \frac{|B|^2}{|P_uB|^2}. \quad \square \end{aligned} \tag{57}$$

Proof of the equality case. This second proof also helps us to treat the equality case. From (56) there is equality if and only if

$$\rho_{A\oplus_2 B}(u)^2 = \rho_A(u)^2 + \rho_B(u)^2.$$

The above is equivalent to the $(\rho_A(u)^2 + \rho_B(u)^2)^{1/2}u \in \partial(A \oplus_2 B)$. From here we get that, if n is a normal vector to $\partial(A \oplus_2 B)$ at $(\rho_A(u)^2 + \rho_B(u)^2)^{1/2}u$ then

$$h_{A\oplus_2 B}(n) = (\rho_A(u)^2 + \rho_B(u)^2)^{1/2}\langle u, n\rangle.$$

The above is equivalent to $h_A(n) = \rho_A(u)\langle u, n\rangle$ and $h_B(n) = \rho_B(u)\langle u, n\rangle$, or simply to say that the normal vector to ∂A at $\rho_A(u)u$ is the normal vector to ∂B at $\rho_B(u)u$. □

Remark 6.3. In the proof of the equality case in the Theorem 6.2, suppose we represent $A = T_1 B_2^n$ and $B = T_2 B_2^n$, where T_1, T_2 are symmetric positive-definite matrices. Then $\|x\|_A^2 = \langle T_1^{-2}x, x\rangle$, and the normal vector to ∂A at $\rho_A(u)u$ is parallel to $T_1^{-2}u$; a similar statement is true for B. Thus our condition on parallel tangent hyperplanes is equivalent to the fact that there is a $\lambda > 0$ such that $(T_1^{-2} - \lambda T_2^{-2})u = 0$ or simply that u is an eigenvector for matrix $T_1^2 T_2^{-2}$.

Let us now present some consequences or Theorem 6.2.

Corollary 6.4. *Let n be a positive integer. Let A and B be full dimensional L_2-zonoids in $\mathbb{R}^n$ and u in S^{n-1}. Then, the function h defined, for $t \geq 0$, by*

$$h_2(t) = \frac{|A \oplus_2 (\sqrt{t}B)|^2}{|P_{u^\perp}(A \oplus_2 (\sqrt{t}B))|_{n-1}^2}$$

is concave on $\mathbb{R}_+$.

Proof. For any $\lambda \in [0, 1]$, one has $A = (\sqrt{1-\lambda}A) \oplus_2 (\sqrt{\lambda}A)$. Thus one deduces that

$$A \oplus_2 \left(\sqrt{(1-\lambda)s + \lambda t}B\right) = \left(\sqrt{1-\lambda}(A \oplus_2 \sqrt{s}B)\right) \oplus_2 \left(\sqrt{\lambda}(A \oplus_2 \sqrt{t}B)\right).$$

Using Theorem 6.2 and the homogeneity of volume, we deduce that h_2 is concave. □

Next we show that Theorem 6.2 has the following additional applications:

Theorem 6.5. *Let n be an integer, then for any $1 \leq k \leq n$ and for every pair of full dimensional L_2-zonoids A and B in $\mathbb{R}^n$ and any $(n-k)$-dimensional subspace E of $\mathbb{R}^n$ one has*

$$\left(\frac{|A \oplus_2 B|}{|P_E(A \oplus_2 B)|_{n-k}}\right)^{\frac{2}{k}} \geq \left(\frac{|A|}{|P_E A|_{n-k}}\right)^{\frac{2}{k}} + \left(\frac{|B|}{|P_E B|_{n-k}}\right)^{\frac{2}{k}}. \tag{58}$$

Proof. The proof goes by induction on k. Theorem 6.2 establishes the case $k = 1$ and any $n \geq 1$. We assume that the inequality holds for some $1 \leq k \leq n-1$ for all L_2-zonoids A, B in $\mathbb{R}^n$ and all $n-k$ dimensional subspace of $\mathbb{R}^n$. Let E be a $n-k-1$ dimensional subspace of $\mathbb{R}^n$. Then one may write $E = F \cap u^\perp$, for some $n-k$ dimensional subspace F and $u \in F^\perp$. Then, applying Theorem 6.2 to $P_F A$ and $P_F B$ and using that $P_{u^\perp} \circ P_F = P_E$ we get

$$\left(\frac{|P_F(A\oplus_2 B)|_{n-k}}{|P_E(A\oplus_2 B)|_{n-k-1}}\right)^2 \geq \left(\frac{|P_FA|_{n-k}}{|P_EA|_{n-k-1}}\right)^2 + \left(\frac{|P_FB|_{n-k}}{|P_EB|_{n-k-1}}\right)^2. \tag{59}$$

Applying (58) to $E=F$, raising the equation to power $k/2$ and taking the product with (59), we get

$$\frac{|A\oplus_2 B|}{|P_E(A\oplus_2 B)|_{n-k-1}} \geq \tag{60}$$
$$\left(\left(\frac{|P_FA|_{n-k}}{|P_EA|_{n-k-1}}\right)^2 + \left(\frac{|P_FB|_{n-k}}{|P_EB|_{n-k-1}}\right)^2\right)^{\frac{1}{2}} \left(\left(\frac{|A|}{|P_FA|_{n-k}}\right)^{\frac{2}{k}} + \left(\frac{|B|}{|P_FB|_{n-k}}\right)^{\frac{2}{k}}\right)^{\frac{k}{2}}.$$

From Hölder's inequality, we conclude that

$$\frac{|A\oplus_2 B|}{|P_E(A\oplus_2 B)|_{n-k-1}} \geq \left(\left(\frac{|A|}{|P_EA|_{n-k-1}}\right)^{\frac{2}{k+1}} + \left(\frac{|B|}{|P_EB|_{n-k-1}}\right)^{\frac{2}{k+1}}\right)^{\frac{k+1}{2}},$$

which is (58) for $k+1$. □

Remark 6.6. Note that the projection of a L_p zonoid is again a L_p zonoid. Thus, the proof of Theorem 6.5 can be used directly to prove the following claim: If the answer to Question 6.1 is positive for some $p\geq 1$, then for any $1\leq k\leq n$ and for every pair of L_p-zonoids A and B in $\mathbb{R}^n$ and any $(n-k)$-dimensional subspace E of $\mathbb{R}^n$ one has

$$\left(\frac{|A\oplus_p B|}{|P_E(A\oplus_p B)|_{n-k}}\right)^{\frac{p}{k}} \geq \left(\frac{|A|}{|P_EA|_{n-k}}\right)^{\frac{p}{k}} + \left(\frac{|B|}{|P_EB|_{n-k}}\right)^{\frac{p}{k}}.$$

Corollary 6.7. *Let k,n be a integer with $1\leq k\leq n$. Then, for all L_2-zonoids A and B in $\mathbb{R}^n$, and all zonoids $Z_1,\dots,Z_k$ in $\mathbb{R}^n$,*

$$\left(\frac{|A\oplus_2 B|}{V((A\oplus_2 B)[n-k],Z_1,\dots,Z_k)}\right)^{\frac{2}{k}} \geq \tag{61}$$
$$\left(\frac{|A|}{V(A[n-k],Z_1,\dots,Z_k)}\right)^{\frac{2}{k}} + \left(\frac{|B|}{V(B[n-1],Z_1,\dots,Z_k)}\right)^{\frac{2}{k}},$$

and thus the function f defined, for $t\geq 0$, by

$$f(t) = \frac{|A\oplus_2 \sqrt{t}B|^{\frac{2}{k}}}{|V((A\oplus_2\sqrt{t}B)[n-k],Z_1,\dots,Z_k)|_{n-k}^{\frac{2}{k}}}$$

is concave on $\mathbb{R}_+$. Moreover

$$\frac{|A\oplus_2 B|^2}{|\partial(A\oplus_2 B)|^2} \geq \frac{|A|^2}{|\partial A|^2} + \frac{|B|^2}{|\partial B|^2}, \tag{62}$$

and thus the function g defined, for $t \geq 0$, by

$$g(t) = \frac{|A \oplus_2 \sqrt{t}B|^2}{|\partial(A \oplus_2 \sqrt{t}B)|_{n-1}^2}$$

is concave on $\mathbb{R}_+$.

Proof. First notice that Theorem 6.5 may be reformulated in the following way. Let $u_1, \dots, u_k$ be an orthonormal system in $\mathbb{R}^n$. Then using (17) we have

$$\left(\frac{|A \oplus_2 B|}{V((A \oplus_2 B)[n-k], [0,u_1], \dots, [0,u_k])}\right)^{\frac{2}{k}} \geq \left(\frac{|A|}{V(A[n-k], [0,u_1], \dots, [0,u_k])}\right)^{\frac{2}{k}} + \left(\frac{|B|}{V(B[n-k], [0,u_1], \dots, [0,u_k])}\right)^{\frac{2}{k}}. \tag{63}$$

Applying a linear transform, (63) holds for any linearly independent system $u_1, \dots, u_k$. Then, for $x, y \geq 0$, define

$$\varphi(x,y) = (x^{-\frac{2}{k}} + y^{-\frac{2}{k}})^{-\frac{k}{2}} = \|(x,y)\|_{-\frac{2}{k}}. \tag{64}$$

For a compact convex set A and $u_1, \dots, u_k \in \mathbb{R}^n$, let

$$\psi_A(u_1, \dots, u_k) = \frac{V(A[n-k], [0,u_1], \dots, [0,u_k])}{|A|}. \tag{65}$$

From (63) we know that, if $u_1, \dots, u_k$ are linearly independent, then

$$\psi_{A \oplus_2 B}(u_1, \dots, u_k) \leq \varphi(\psi_A(u_1, \dots, u_k), \psi_B(u_1, \dots, u_k)).$$

For $i = 1, \dots, k$, let $Z_i = [0, u_{i,1}] \oplus_2 \cdots \oplus_2 [0, u_{i,m_i}]$ be a 2-zonotope. Assume that for any set of distinct k vectors from the set $\{u_{i,j}\}$ is an independent sequence (this can be achieved by a small perturbation of vectors $u_{i,j}$). Using that φ, being a $-\frac{2}{k}$-norm, satisfies the reverse Minkowski inequality, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{k} \sum_{j_i=1}^{m_i} \psi_{A \oplus_2 B}(u_{1,j_1}, \dots, u_{k,j_k}) \\
&\qquad \leq \sum_{i=1}^{k} \sum_{j_i=1}^{m_i} \varphi(\psi_A(u_{1,j_1}, \dots, u_{k,j_k}), \psi_B(u_{1,j_1}, \dots, u_{k,j_k})) \\
&\qquad \leq \varphi\left(\sum_{i=1}^{k} \sum_{j_i=1}^{m_i} \psi_A(u_{1,j_1}, \dots, u_{k,j_k}), \sum_{i=1}^{k} \sum_{j_i=1}^{m_i} \psi_B(u_{1,j_1}, \dots, u_{k,j_k})\right).
\end{aligned}$$

Thus we get

$$\frac{V((A \oplus_2 B)[n-k], Z_1, \dots, Z_k)}{|A \oplus_2 B|} \leq \varphi\left(\frac{V(A[n-k], Z_1, \dots, Z_k)}{|A|}, \frac{V(B[n-k], Z_1, \dots, Z_k)}{|B|}\right),$$

which is (61), for $Z_1, \dots, Z_k$ being a zonotopes. The result for zonoids follows by approximation. We apply (61) with $k = 1$ and $Z_1 = B_2^n$ to prove (62). The concavity of f and g is proved with the method used for Corollary 6.4. □

6.2. The case $p \neq 2$

Proposition 6.8. *The answer to Question 6.1 is negative when $p > 2$ and $n \geq 2$: a counterexample is given by $A = B_2^n$ and $B = \varepsilon^{\frac{1}{p}}[-v, v]$, for some $\varepsilon > 0$.*

Proof. Let $p > 2$. We disprove the weaker statement

$$\frac{|A \oplus_p B|}{|P_{u^\perp}(A \oplus_p B)|_{n-1}} \geq \frac{|A|}{|P_{u^\perp} A|_{n-1}}. \tag{66}$$

We restate (66) with $B = \varepsilon^{\frac{1}{p}}[-v, v]$, $v \in S^{n-1}$ and $\varepsilon \to 0$. For this we use the extension of the classical notion of the mixed volumes introduced by Lutwak [36,37], who proved that

$$\lim_{\varepsilon \to 0} \frac{|A \oplus_p (\varepsilon^{\frac{1}{p}} B)| - |A|}{\varepsilon} = \frac{1}{p} \int_{S^{n-1}} h_B^p(u) h_A^{1-p} dS_A(u),$$

for $p > 1$ and all convex compact sets A, B, containing the origin and defined

$$V_p(A[n-1], B) = \frac{1}{n} \int_{S^{n-1}} h_B^p(u) h_A^{1-p} dS_A(u).$$

Taking $B = \varepsilon^{\frac{1}{p}}[-v, v]$ for some $v \in S^{n-1}$, we get

$$\left|A \oplus_p B\right| = |A| + \frac{\varepsilon}{p} \int_{S^{n-1}} |\langle v, x\rangle|^p h_A^{1-p} dS_A(x) + o(\varepsilon)$$

Assuming v in $u^\perp$, we get

$$\left|P_{u^\perp} A \oplus_p P_{u^\perp} B\right| = |P_{u^\perp} A| + \frac{\varepsilon}{p} \int_{S^{n-1} \cap u^\perp} |\langle v, x\rangle|^p h_A^{1-p}(x) dS_{P_{u^\perp} A}(x) + o(\varepsilon).$$

Assume, by way of contradiction, that (66) is true for all L_p-zonoids $A \subset \mathbb{R}^n$. Then,

$$|A| \int\limits_{S^{n-1}\cap u^\perp} |\langle v,x\rangle|^p h_A^{1-p}(x) dS_{P_{u^\perp}A}(x) \le |P_{u^\perp}A| \int\limits_{S^{n-1}} |\langle v,x\rangle|^p h_A^{1-p}(x) dS_A(x). \tag{67}$$

Now we take $A = B_2^n$, $u = e_2$ and $v = e_1$. Notice that B_2^n is an L_p zonoid for all $p \ge 1$. Next (67) becomes

$$|B_2^n| \int\limits_{S^{n-2}} |x_1|^p dS_{B_2^{n-1}}(x) \le |B_2^{n-1}| \int\limits_{S^{n-1}} |x_1|^p dS_{B_2^n}(x). \tag{68}$$

Using polar coordinates and Fubini's theorem, we get

$$\int\limits_{S^{n-1}} |x_1|^p dS_{B_2^n}(x) = (n+p) \int\limits_{B_2^{n-1}} |z_1|^p dz = \frac{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)}.$$

Thus (68) becomes

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{p+n}{2}\right) \le \Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{p+n-1}{2}\right).$$

Using the strict log-convexity of Γ function the above is only true if and only if $p \le 2$. □

Proposition 6.9. *Fix $p > 1$. Then (50) does not hold in the class of all convex symmetric bodies in $\mathbb{R}^n$.*

Proof. Let us first construct an example in $\mathbb{R}^2$. We note that

$$\left| P_{u^\perp}A \oplus_p P_{u^\perp}\varepsilon^{\frac{1}{p}}[-v,v] \right| = 2h_A(v) + \frac{2\varepsilon}{p} h_A(v)^{1-p} + o(\varepsilon).$$

Thus (66), for $n = 2$, would imply

$$|A| \le h_A(v)^p \int\limits_{S^1} |\langle v,x\rangle|^p h_A^{1-p}(x) dS_A(x). \tag{69}$$

Consider $a \in (0,1)$, let $A = \{(x_1,x_2) \in \mathbb{R}^2, |x_i| \le 1, |x_1 \pm x_2| \le 2-a\}$. Then $|A| = 4-2a^2$. We check (69) with $v = e_1$. We first note that $h_A(e_1) = 1$. Next we compute

$$f(x) = |\langle e_1,x\rangle|^p h_A^{1-p}(x) S_A(x)$$

for different normal vectors x of A. We first note that $f(\pm e_2) = 0$, $f(\pm e_1) = 2(1-a)$ and $f((\pm 1/\sqrt{2}, \pm 1/\sqrt{2})) = a(2-a)^{1-p}$. Thus to contradict (69) we must select a such that

$$4 - 2a^2 > 4(1-a) + 4a(2-a)^{1-p},$$

or $a < 2 - 2^{1/p}$, which is possible for every $p > 1$.

To build a counterexample in $\mathbb{R}^n$ for $n \geq 3$, we use the fact that if K is a convex body in $\operatorname{span}\{e_1, \dots, e_{n_1}\}$ and L is a convex body in $\operatorname{span}\{e_{n_1+1}, \dots, e_{n_1+n_2}\}$, then

$$|K \oplus_p L| = \frac{\Gamma(\frac{n_1}{q}+1)\Gamma(\frac{n_2}{q}+1)}{\Gamma(\frac{n_1+n_1}{q}+1)}|K||L| = c_{n_1,n_2,q}|K||L|, \tag{70}$$

where $1/p + 1/q = 1$. Let $A_n = B_q^{n-2} \oplus_p A_2$, where $A_2 \subset \operatorname{span}\{e_1, e_2\}$ is the counterexample created above, and $B_q^{n-2} \subset \operatorname{span}\{e_3, \dots, e_n\}$. Then (70) gives

$$\frac{|A_n \oplus_p [-v,v]|}{|P_{e_1^\perp}(A_n \oplus_p [-v,v])|_{n-1}} = \frac{|B_q^{n-2} \oplus_p (A_2 \oplus_p [-v,v])|}{|B_q^{n-2} \oplus_p P_{e_1^\perp}(A_2 \oplus_p [-v,v])|_{n-1}} < \frac{|A_n|}{|P_{e_1^\perp} A_n|_{n-1}}. \quad \square$$

Let us note that the direct interpretation of the volume of L_p-zonotopes in terms of determinants is only possible in the case when $p = 1$ or $p = 2$ [32]. Thus it is natural to ask if the determinant inequality that we proved in the case $p = 2$ is still true in the case $p \neq 2$.

Question 6.10. *Let $p \in [1,2)$. Consider $N \geq N' \geq n$ and a sequence of vectors $\{u_i\}_{i=1}^N$ in $\mathbb{R}^n$. Is it true that*

$$\frac{\sum\limits_{M\subset\{1,\dots,N'\},|M|=n} |\det(\{u_m\}_{m\in M}|^p}{\sum\limits_{M\subset\{1,\dots,N'\},|M|=n-1} |\det(\{P_{e_1^\perp} u_m\}_{m\in M})|^p} + \frac{\sum\limits_{L\subset\{N'+1,\dots,N\},|L|=n} |\det(\{u_l\}_{l\in L}|^p}{\sum\limits_{L\subset\{N'+1,\dots,N\},|L|=n-1} |\det(\{P_{e_1^\perp} u_l\}_{l\in L}|^p}$$
$$\leq \frac{\sum\limits_{M\subset\{1,\dots,N\},|M|=n} |\det(\{u_m\}_{m\in M}|^p}{\sum\limits_{M\subset\{1,\dots,N\},|M|=n-1} |\det(\{P_{e_1^\perp}(u_m)\}_{m\in M}|^p}? \tag{71}$$

The reason that we only ask Question 6.10 for $p \in [1,2)$ is because we have already proved it for $p = 2$, and the question has a negative answer when $p > 2$. To see why, let $N = n = 2$ and $N' = 1$, and consider the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then (71) becomes

$$\frac{2^p}{1+1} \leq \frac{2^p+1+1}{1+1+1},$$

which is false for $p > 2$.

Data availability

No data was used for the research described in the article.

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