The Hardy-Weyl algebra

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Abstract

We study the algebra A generated by the Hardy operator H and the operator M_x of multiplication by x on $L^2[0,1]$. We call A the Hardy-Weyl algebra. We show that its quotient by the compact operators is isomorphic to the algebra of functions that are continuous on Λ and analytic on the interior of Λ for a planar set $\Lambda = [-1,0] \cup \overline{\mathbb{D}(1,1)}$, which we call the lollipop. We find a Toeplitz-like short exact sequence for the C^*algebra generated by A.

We study the operator $Z = H - M_x$, show that its point spectrum is $(-1,0] \cup \mathbb{D}(1,1)$, and that the eigenvalues grow in multiplicity as the points move to 0 from the left.

1 Introduction

The classical Weyl algebra is generated by the operators of multiplication by x, denoted M_x, and differentiation, denoted by D. These operators satisfy the commutator relation

$$DM_x - M_xD = 1. \tag{1.1}$$

The algebra generated by these relations has been studied extensively in both algebra and operator theory—see e.g. the books [17, 15, 22, 10]. In operator theory, the study is complicated by the fact that no bounded operators satisfy (1.1). In this note, we shall study the associated algebras that arise when one replaces the differentiation operator D by a bounded integration operator.

The Hardy operator H is the bounded operator defined on $L^2[0,1]$ by

$$Hf(x) \ = \ \frac{1}{x}\int_0^x f(t)dt.$$

Let $V = M_xH$ denote the Volterra operator

$$V : f \mapsto \int_0^x f(t)dt,$$

which is a right inverse of D. These operators give rise to a new set of relations instead of (1.1), namely

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$$V M_x - M_x V = -V^2 \qquad (1.2)$$
$$HM_x - M_x H = -HM_x H = -HV. \qquad (1.3)$$

We shall use L^2 to mean $L^2[0,1]$ throughout.

Definition 1.4. Let $\mathcal{A}$ denote the closure of the unital algebra generated by H and M_x in the norm topology of $B(L^2)$. We call $\mathcal{A}$ the Hardy-Weyl algebra.

When is an operator in $\mathcal{A}$? Can all the elements be described? Since the commutator of H and M_x is compact, we first describe the quotient of $\mathcal{A}$ by the compact operators. Define Λ, a subset of the plane, by

$$\Lambda = [-1,0] \cup \overline{\mathbb{D}(1,1)}.$$

We call Λ the *lollipop*. The *lollipop algebra* $A(\Lambda)$ is the Banach algebra of functions that are continuous on Λ and analytic on $\mathrm{int}(\Lambda)$, equipped with the maximum modulus norm. Let $\mathcal{K}$ denote the compact operators on L^2, and let $\mathcal{K}_{\mathcal{A}} = \mathcal{K} \cap \mathcal{A}$.

Theorem 1.5. There is a Banach algebra isomorphism γ from $A(\Lambda)$ onto $\mathcal{A}/\mathcal{K}_{\mathcal{A}}$. It is given by $\gamma(f) = f|_{[-1,0]}(-M_x) + f|_{\mathbb{D}(1,1)}(H) - f(0)$.

Let $\theta : \mathcal{A} \to A(\Lambda)$ be defined by $\theta(T) = \gamma^{-1}([T])$, where $[T]$ is the projection of T onto $\mathcal{A}/\mathcal{K}_{\mathcal{A}}$. As a corollary to Theorem 1.5 we obtain that every element T of $\mathcal{A}$ can be written uniquely as

$$T = M_\varphi + g(H) + K, \qquad (1.6)$$

where $\varphi \in C([0,1])$, g is in $A(\overline{\mathbb{D}(1,1)})$, $\varphi(0) = g(0)$, and $K \in \mathcal{K}_{\mathcal{A}}$.

If we look at $C^*(\mathcal{A})$, the C^*-algebra generated by $\mathcal{A}$, we get something similar to the Toeplitz algebra short exact sequence. See [4, 9, 21] for some recent results on the Toeplitz algebra, and [3, 12, 18, 19] for some applications. The lollipop algebra is replaced by the functions that are continuous on the boundary.

Theorem 1.7. There is a short exact sequence of C^*-algebras

$$0 \to \mathcal{K} \to C^*(\mathcal{A}) \to C(\partial\Lambda) \to 0.$$

In addition to studying the algebra generated by H and M_x, one may also study the smaller algebra generated by V and M_x.

Definition 1.8. The algebra $\mathcal{A}_0$ is the norm-closed unital algebra generated by V and M_x.

One can see that $\mathcal{A}_0$ is a proper subalgebra of $\mathcal{A}$ by noting that its quotient by the compact operators is isomorphic to $C([0,1])$.

The lattice of closed invariant subspaces of the Volterra operator V was shown independently by Brodskii and Donoghue [5, 7] to be

$$\mathrm{Lat}(V) = \left\{ \{f \in L^2 : f = 0 \text{ on } [0,s]\} : s \in [0,1] \right\}.$$

We let AlgLat(V) denote the set of bounded operators on L^2 that leave invariant every element of LatV. This is a large algebra—it includes all right translation operators for example. It is described by the following theorem of Radjavi and Rosenthal [20, Example 9.26].

Theorem 1.9. The algebra AlgLat(V) is the weak operator topology closure of the algebra generated by V and M_x.

It follows from Theorem 1.9 that H is in the WOT closure of $\mathcal{A}_0$, and hence $\mathcal{A}$ and $\mathcal{A}_0$ have the same WOT closure. However, no extra compact operators are added in the WOT closure.

Theorem 1.10.

$$\mathrm{AlgLat}(V) \cap \mathcal{K} \subset \mathcal{A}_0.$$

In Section 6 we consider the operator $Z = H - M_x$. It follows from Theorem 1.5 that $[Z]$ generates $\mathcal{A}/\mathcal{K}_{\mathcal{A}}$. We show Z has a surprisingly rich collection of eigenvectors.

Theorem 1.11. Let $Z = H - M_x$. Then

$$\sigma_p(Z) = (-1,0] \cup \mathbb{D}(1,1).$$

The algebraic multiplicity of the eigenvalues of Z on the stick $(-1,0]$ increases as $\lambda \to 0^-$, and hence operators X in the closed algebra generated by Z have the property that $\theta(X)$ is not just in $A(\Lambda)$, but is smoother. We prove:

Theorem 1.12. Let X be in the norm-closed algebra generated by Z. Then $\theta(X)$ is C^m on $(-\frac{2}{2m+1}, 0)$.

2 Preliminaries

Definition 2.1. A *monomial operator* is a bounded linear operator $T : L^2[0,1] \to L^2[0,1]$ with the property that there exist constants c_n and p_n so that

$$T : x^n \mapsto c_n x^{p_n} \qquad \forall n \in \mathbb{N}. \tag{2.2}$$

We call it a *flat monomial operator* if there exists some τ so that $p_n = n + \tau$ for all n.

In [2] we showed that every flat monomial operator is in AlgLat(V), and hence by Theorem 1.9 in the weak closure of $\mathcal{A}_0$. The Volterra operator V is Hilbert-Schmidt, and hence compact. See e.g. [14] for a proof. Hardy proved that the Hardy operator is bounded [11].

Lemma 2.3. Equalities (1.2) and (1.3) hold.

Proof: As all the operators are bounded, it suffices to check on monomials. We get

$$(VM_x - M_xV)\ x^k = -V^2\ x^k = -\frac{1}{(k+1)(k+2)}x^{k+2}$$
$$(HM_x - M_xH)\ x^k = -HM_xH\ x^k = -\frac{1}{(k+1)(k+2)}x^{k+1}.$$

Let H^2 denote the Hardy space of the unit disk, and $k_w(z) = \frac{1}{1-\bar{w}z}$ the Szegő kernel. We shall let $S : f(z) \mapsto zf(z)$ denote the unilateral shift on H^2. Let

$$\beta(z) = \frac{1}{2-z},$$

and let $C_\beta : f \mapsto f \circ \beta$ denote the composition operator of composing with β.

There is a unitary $U : L^2 \to H^2$ that is defined on monomials by

$$U : x^s \mapsto \frac{1}{s+1}k_{\frac{\bar{s}}{\bar{s}+1}}, \tag{2.4}$$

and extended by linearity and continuity to the whole space. If $T \in B(L^2)$, we shall let $\widehat{T}$ denote UTU^*. It is easy to see that U is unitary, as it preserves inner products. In [1] we prove that U is given by the formula

$$Uf(z) = \frac{1}{1-z}\int_0^1 f(x)x^{\frac{z}{1-z}}dx, \tag{2.5}$$

and show that

$$\widehat{M_x} = S^*C_\beta^*$$
$$\widehat{V} = (1-S^*)C_\beta^*$$
$$\widehat{H} = 1 - S^*.$$

The fact that $1 - H$ is unitarily equivalent to the backward shift was proved in [6]; see also [13].

If X is a compact subset of $\mathbb{C}$, we shall let $C(X)$ denote the Banach algebra of functions that are continuous on X, with the maximum modulus norm. We shall let $A(X)$ denote the subalgebra of functions that are continuous on X and analytic on the interior of X, and $P(X)$ denote the closure of the polynomials in $C(X)$. A theorem of Mergelyan [16] says that if the complement of X is connected, then $A(X) = P(X)$.

3 The Calkin Hardy-Weyl Algebra

We let $\mathcal{K}$ denote the ideal of compact operators acting on L^2 and set

$$\mathcal{K}_0 = \mathcal{A} \cap \mathcal{K}.$$

Evidently, K_0 is a 2-sided ideal in A. Consequently, we may define an algebra C, the *Calkin Hardy-Weyl algebra*, by

$$C = A/K_0$$

If $T \in A$ we let $[T]$ denote the coset of T in C, i.e.,

$$[T] = \{T + K \mid K \in K_0\}.$$

Proposition 3.1. C is an abelian Banach algebra.

Proof. That C is a Banach algebra follows from the fact that K_0 is closed in A. To see that C is abelian, observe that as M_x and H generate A, $[M_x]$ and $[H]$ generate C. Furthermore, as $M_x H = V \in K_0$,

$$[M_x][H] = 0. \tag{3.2}$$

Likewise, as $HM_x = (1 - H)V \in K_0$,

$$[H][M_x] = 0, \tag{3.3}$$

so that in particular we have that

$$[M_x][H] = [H][M_x].$$

As $[M_x]$ and $[H]$ commute and generate C, C is abelian. □

3.1 A Uniform Algebra Homeomorphically Isomorphic to C

We begin by defining an algebra by gluing together two simpler algebras whose maximal ideal spaces overlap at a single point. Let

$$P = \{ f = (f_-, f_+) : f_- \in C([-1,0]), f_+ \in A(\overline{D(1,1)}) \text{ and } f_-(0) = f_+(0) \}$$ where we view P

as an algebra with the operations $cf = (cf_-, cf_+)$, $f + g = (f_- + g_-, f_+ + g_+)$, and $fg = (f_-g_-, f_+g_+)$,

and the norm

$$\|f\| = \max\left\{ \max_{t\in[-1,0]} |f_-(t)|, \max_{z\in\overline{\mathbb{D}(1,1)}} |f_+(z)| \right\}.$$

We abuse notation by letting

$$f(0) = f_-(0)$$

when $f \in P$.

We note that if $f \in C([-1,0])$, then as $-M_x$ is self-adjoint and has spectrum equal to $[-1,0]$, we may form the operator $f(-M_x)$. Likewise, as H is cosubnormal and has spectrum equal to $\overline{D(1,1)}$, if $g \in A(\overline{D(1,1)})$, then we may form the operator $g(H)$. Concretely,

$$f(-M_x) = M_{f(-x)} \text{ and } \widehat{g(H)} = M_h^*,$$

where $h(z) = \overline{g(1-\bar{z})}$, and M_h denotes multiplication by h.

Lemma 3.4. If $f \in C([-1,0])$ and $g \in A(\overline{\mathbb{D}(1,1)})$, then

$$[f(-M_x)][g(H)] = g(0)[f(-M_x)] + f(0)[g(H)] - f(0)g(0).$$

Proof. Since $[-1,0]$ is a spectral set for $-M_x$ and $\mathbb{D}(1,1)$ is a spectral set for H it suffices to prove the lemma in the special case when f and g are polynomials. Let $f(x) = f(0)+xf_1(x)$ and $g(x) = g(0) + xg_1(x)$. Using (3.2) and (3.3) we see that

$$[f(-M_x)][g(H)] = \big([f(0)] + [-M_x][f_1(-M_x)]\big)\big([g(0)] + [H][g_1(H)]\big)$$

$$= f(0)g(0) + g(0)[-M_x][f_1(-M_x)] + f(0)[H][g_1(H)] =$$

$$f(0)g(0) + g(0)[f(-M_x) - f(0)] + f(0)[g(H) - g(0)] =$$

$$g(0)[f(-M_x)] + f(0)[g(H)] - f(0)g(0).$$

□

If $f \in \mathcal{P}$ we define $\gamma(f) \in \mathcal{A}$ by the formula

$$\gamma(f) = f_-(-M_x) + f_+(H) - f(0).$$ We also

define $\Gamma : \mathcal{P} \to \mathcal{C}$ by the formula

$$\Gamma(f) = [\gamma(f)]$$

Proposition 3.5. Γ is a continuous unital homomorphism.

Proof. γ is linear and $\gamma(1) = 1$. Therefore, Γ is linear and $\Gamma(1) = 1$. Also,

$$\begin{aligned}
\|\Gamma(f)\| &= \|[\gamma(f)]\| \\
&\leq \|\gamma(f)\| \\
&= \|f_-(-M_x) + f_+(H) - f(0)\| \\
&\leq \|f_-(-M_x)\| + \|f_+(H)\| + |f(0)| \\
&= \max_{t\in[-1,0]} |f_-(t)| + \max_{z\in\overline{\mathbb{D}(1,1)}} |f_+(z)| + |f(0)| \\
&\leq 3\|f\|,
\end{aligned}$$

so Γ is continuous.

Finally, to see that Γ preserves products, fix $f,g \in \mathcal{P}$.

$$\Gamma(f)\Gamma(g) = [\gamma(f)]\,[\gamma(g)]$$

$$= [f_-(-M_x) + f_+(H) - f(0)]\,[g_-(-M_x) + g_+(H) - g(0)]$$

$$= \Big([f_-(-M_x)][g_-(-M_x)] + [f_+(H)][g_+(H)]\Big)$$

$$+ \Big([f_-(-M_x)][g_+(H)] + [g_-(-M_x)][f_+(H)]\Big)$$

$$- \Big(f(0)[g_-(-M_x) + g_+(H)] + g(0)[f_-(-M_x) + f_+(H)]\Big)$$

$$+ f(0)g(0)$$

$$= \quad A + B - C + f(0)g(0).$$

But

$$\begin{aligned} A &= \Big([f_-(-M_x)][g_-(-M_x)] + [f_+(H)][g_+(H)]\Big) \\ &= \Big([f_-g_-(-M_x)] + [f_+g_+(H)] - f(0)g(0)\Big) + f(0)g(0) \\ &= [\gamma(fg)] + f(0)g(0) \\ &= \Gamma(fg) + f(0)g(0), \end{aligned}$$

and using Lemma 3.4, we see that

$$\begin{aligned} B &= \Big([f_-(-M_x)][g_+(H)]\Big) + \Big([g_-(-M_x)][f_+(H)]\Big) \\ &= \Big(g(0)[f_-(-M_x)] + [f(0)g_+(H)] - f(0)g(0)\Big) + \Big(f(0)[g_-(-M_x)] + [g(0)f_+(H)] - f(0)g(0)\Big) \\ &= C - 2f(0)g(0). \end{aligned}$$

Therefore,

$$\begin{aligned} \Gamma(f)\Gamma(g) &= A + B - C + f(0)g(0) \\ &= (\Gamma(fg) + f(0)g(0)) + (C - 2f(0)g(0)) - C + f(0)g(0) = \Gamma(fg). \end{aligned}$$

□

Lemma 3.6. If p is a polynomial in two variables and we define $f \in$ P by letting

$$f_-(t) = p(t,0),\ t \in [-1,0] \qquad \text{and} \qquad f_+(z) = p(0,z),\ z \in \overline{\mathrm{D}(1,1)},$$

then $p([-M_x],[H]) = \Gamma(f)$.

Proof. If $p = p(x,y)$ is a polynomial in two variables and we let

$$q(x,y) = p(x,y) - p(x,0) - p(0,y) + p(0,0),$$

then $p(x,y) = p(x,0) + p(0,y) - p(0,0) + q(x,y)$

and

$$q([-M_x],[H]) = 0.$$

Therefore,

$$\begin{aligned} p([-M_x],[H]) &= p([-M_x],0) + p(0,[H]) - p(0,0) \\ &= f_-([M_x]) + f_+([h]) - f(0) \\ &= [\gamma(f)] = \Gamma(f). \end{aligned}$$

□

Corollary 3.7. The range of Γ is dense in C.

Proof. This follows immediately from Lemma 3.6 by recalling that $[-M_x]$ and $[H]$ generate C (cf. proof of Proposition 3.1). □

Lemma 3.6 suggests that we consider the subset P_0 of P defined by

$$P_0 = \{f \in P \mid f_- \text{ and } f_+ \text{ are polynomials}\}.$$

We note that it follows from the facts that the polynomials are dense in both $C([-1,0])$ and $A(D(1,1))$ that P_0 is dense in P.

Lemma 3.8. If $s \in [-1,0]$, then

$$|f_-(s)| \le \|\Gamma(f)\| \tag{3.9}$$

for all $f \in P$.

Proof. As f is continuous, it suffices to prove the lemma under the assumption that $s \in (-1,0)$. For n satisfying $1/n < \min\{s,1-s\}$ we define a unit vector $\chi_n \in L^2$ by the formula

$$\chi_n(t) = \begin{cases} \sqrt{\frac{n}{2}} & \text{if } |t-s| \le 1/n \\ 0 & \text{if } |t-s| > 1/n \end{cases}$$

We observe that the mean value theorem for integrals implies that

$$\lim_{n\to\infty} \langle g\chi_n,\chi_n\rangle = g(s)$$

whenever $g \in C([0,1])$. Also, as $\chi_n \to 0$ weakly,

$$\lim_{n\to\infty} \|K\chi_n\| = 0$$

whenever K is a compact operator acting on L^2. In particular, as V is compact and $V\chi_n(t) = 0$ when $t \in [0,s-1/n)$,

$$\lim_{n\to\infty} \|H\chi_n\| = \lim_{n\to\infty} \|M_{1/x}V\chi_n\| = 0.$$

More generally, if q is a polynomial and $q(0) = 0$, write $q(z) = zr(z)$, and we get

$$\lim_{n\to\infty} \|q(H)\chi_n\| = \lim_{n\to\infty} \|r(H)\ H\chi_n\| = 0.$$

Now fix $f \in \mathcal{P}_0$ and a compact operator K acting on L^2. Using the observations in the previous paragraph we have that

$$\begin{aligned} \langle(\gamma(f) + K)\,\chi_n,\chi_n\rangle &= \langle(f_-(-M_x) + f_+(H) - f(0) + K)\,\chi_n,\chi_n\rangle \\ &= \langle f_-(-x)\chi_n,\chi_n\rangle + \langle(f_+ - f_+(0))(H)\,\chi_n,\chi_n\rangle + \langle K\,\chi_n,\chi_n\rangle \\ &\to \quad f_-(s) \quad + \quad 0 \quad + \quad 0 \\ &= f_-(s). \end{aligned}$$

Therefore, as $\|\chi_n\| = 1$, $|f_-(s)| \le \|\gamma(f) + K\|$
for all $f \in \mathcal{P}_0$ and K any compact operator acting on L^2. Hence,

$$|f_-(s)| \le \inf_{K\in\mathcal{K}_0} \|\gamma(f) + K\| = \|\Gamma(f)\|$$

for all $f \in \mathcal{P}_0$. As Γ is continuous and $\mathcal{P}_0$ is dense in $\mathcal{P}$, it follows that (3.9) holds for all $f \in \mathcal{P}$.
□

Lemma 3.10. If $z \in \overline{\mathsf{D}(1,1)}$, then

$$|f_+(z)| \le \|\Gamma(f)\| \tag{3.11}$$

for all $f \in \mathcal{P}$.

Proof. We first observe that as $f_+ \in A(\overline{\mathsf{D}(1,1)})$, by the Maximum Modulus Theorem it suffices to prove the lemma under the assumption that $z = 1 + \tau$ where $\tau \in \mathsf{T} \setminus \{-1\}$. For $\alpha \in \mathsf{D}$, let

$$\Upsilon_\alpha = U^* \frac{k_{-\bar\alpha}}{\|k_{-\bar\alpha}\|},$$

where U is as in (2.4). Clearly, as $k_{-\bar\alpha}/\|k_{-\bar\alpha}\|$ is a unit vector and U^* is unitary, Υ_α is a unit vector. Also, as

$$(1 - S^*)\frac{k_{-\bar\alpha}}{\|k_{-\bar\alpha}\|} = (1 + \alpha)\frac{k_{-\bar\alpha}}{\|k_{-\bar\alpha}\|},$$

it follows that $H\Upsilon_\alpha = (1 + \alpha)\Upsilon_\alpha$, and more generally,

$$f_+(H)\Upsilon_\alpha = f_+(1 + \alpha)\Upsilon_\alpha \tag{3.12}$$

for all $f \in \mathcal{P}$.

Now notice that (2.4) implies that

$$\Upsilon_\alpha = \frac{\sqrt{1 - |\alpha|^2}}{1 + \alpha} x^{-\frac{\alpha}{1+\alpha}}.$$

Claim 3.13. If $\rho > 0$ and $\tau \in \mathbb{T} \setminus \{-1\}$, then

$$\lim_{\alpha\to\tau}\langle x^{\rho}\Upsilon_\alpha,\Upsilon_\alpha\rangle = 0. \qquad (3.14)$$

Proof. First note that

$$\rho - \left(\frac{\alpha}{1+\alpha} + \frac{\bar\alpha}{1+\bar\alpha}\right) + 1 = \rho + \frac{1-|\alpha|^2}{|1+\alpha|^2},$$

so that

$$\int_0^1 x^{\rho-(\frac{\alpha}{1+\alpha}+\frac{\bar\alpha}{1+\bar\alpha})}dx = \left(\rho + \frac{1-|\alpha|^2}{|1+\alpha|^2}\right)^{-1}.$$

Hence,

$$\begin{aligned}\langle x^{\rho}\Upsilon_\alpha,\Upsilon_\alpha\rangle &= \frac{1-|\alpha|^2}{|1+\alpha|^2}\int_0^1 x^{\rho-(\frac{\alpha}{1+\alpha}+\frac{\bar\alpha}{1+\bar\alpha})}dx\\ &= \frac{1-|\alpha|^2}{|1+\alpha|^2}\left(\rho + \frac{1-|\alpha|^2}{|1+\alpha|^2}\right)^{-1}\\ &= \frac{1-|\alpha|^2}{|1+\alpha|^2\rho\ +1-|\alpha|^2}.\end{aligned}$$

Therefore, if $\rho > 0$ and $\tau \in \mathbb{T} \setminus \{-1\}$, (3.14) holds. □

Observe that if q is a polynomial and $\tau \in \mathbb{T} \setminus \{-1\}$, then Claim 3.13 implies that $\langle q\Upsilon_\alpha,\Upsilon_\alpha\rangle \to q(0)$ as $\alpha \to \tau$. In particular,

$$\lim_{\alpha\to\tau}\langle q(-M_x)\Upsilon_\alpha,\Upsilon_\alpha\rangle = 0 \qquad (3.15)$$

whenever $\tau \in \mathbb{T} \setminus \{-1\}$ and q is a polynomial satisfying $q(0) = 0$.

We now conclude the proof of the lemma. We need to show that if $f \in \mathcal{P}$ and $\tau \in \mathbb{T}\setminus\{-1\}$ then (3.11) holds with $z = 1 + \tau$. First assume that $f \in \mathcal{P}_0$ and fix $K \in \mathcal{K}_0$. Since $\Upsilon_\alpha \to 0$ weakly as $\alpha \to \tau$, using (3.12) and (3.14) we have

$$\begin{aligned}\langle(\gamma(f) + K)\Upsilon_\alpha,\Upsilon_\alpha\rangle &= \langle(f_- - f_-(0))(-M_x)\Upsilon_\alpha,\Upsilon_\alpha\rangle + \langle f_+(H)\Upsilon_\alpha,\Upsilon_\alpha\rangle + \langle K\Upsilon_\alpha,\Upsilon_\alpha\rangle\\ &\to \quad 0 \quad + \quad f_+(1+\tau) \quad + \quad 0\\ &= f_+(1+\tau).\end{aligned}$$

as $\alpha \to \tau$. Therefore, if $f \in \mathcal{P}_0$ and $\tau \in \mathbb{T} \setminus \{-1\}$,

$$|f_+(1+\tau)| \le \|\gamma(f) + K\|.$$

Hence, if $f \in \mathcal{P}_0$,

$$|f_+(1+\tau)| \le \inf \|\gamma(f) + K\| = \|\Gamma(f)\|.$$

$K \in K_0$

As Γ is continuous and P_0 is dense in P, it follows that (3.11) holds with $z = 1 + \tau$ for all $f \in P$. □

Lemma 3.16. Γ is a homeomorphism.

Proof. In the proof of Proposition 3.5 we showed that

$$\|\Gamma(f)\| \leq 3\|f\|$$

for all $f \in P$. On the other hand, Lemma 3.8 implies that

$$\max_{t\in[-1,0]} |f_-(t)| \leq \|\Gamma(f)\|$$

for all $f \in P$ and Lemma 3.10 implies that

$$\max_{z\in\overline{\mathbb{D}(1,1)}} |f_+(z)| \leq \|\Gamma(f)\|$$

for all $f \in P$. Therefore,

$$\|f\| = \max\left\{ \max_{t\in[-1,0]} |f_-(t)|, \max_{z\in\overline{\mathbb{D}(1,1)}} |f_+(z)| \right\} \leq \|\Gamma(f)\|$$

for all $f \in P$. □

Putting together the results of Subsection 3.1 we get the following theorem.

Theorem 3.17. The map Γ is a homeomorphic unital isomorphism from P onto C.

3.2 Some Observations on the Gelfand Theory of C

If $\Lambda = [-1,0] \cup \overline{\mathbb{D}(1,1)}$, then there is an isometric isomorphism from P onto the lollipop algebra $A(\Lambda)$ given by

$$A(\Lambda) \ni f \mapsto \left(f|_{[-1,0]}\ , f|_{(\overline{\mathbb{D}(1,1)})} \right).$$

So one could just as well state Theorem 3.17 with P replaced by $A(\Lambda)$ and Γ replaced with the map $\Gamma^{\sim} : A(\Lambda) \to C$ defined by

$$\Gamma^{\sim}(f) = \left[f\big|_{[-1,0]}(-M_x) + f\big|_{\overline{\mathbb{D}(1,1)}}(H) - f(0) \right].$$

Definition 3.18. Define $\theta : A \to A(\Lambda)$ by

$$\theta(X) = (\Gamma^{\sim})^{-1}([X]).$$

Then Theorem 3.17 says that there is a short exact sequence

$$0 \to K_A \to A \to^{\theta} A(\Lambda) \to 0.$$

Remark 3.19. By Mergelyan's theorem, $A(\Lambda) = P(\Lambda)$, and since z generates $P(\Lambda)$, it follows that

$$\Gamma^{\sim}(z) = [H - M_x]$$

generates C. We shall examine $H - M_x$ in Section 6.

4 C^*(A)

We shall let B = C^*(A) denote the C^*-algebra generated by A. Since it is irreducible, B contains all the compact operators.

The Toeplitz C^*-algebra T is the C^* algebra generated by the shift S. There is a short exact sequence

$$0 \to \mathrm{K} \to \mathrm{T} \to^{\alpha} A(\overline{\mathrm{D}}) \to 0.$$

(See e.g. [8, 7.23]). A cross-section of α is the map that sends a function m to the Toeplitz operator T_m on H^2 with symbol m.

Since $H = S^* + 1$, the C^*-algebra generated by H is unitarily equivalent to T . We wish to think of it as living on D(1,1), so we must shift things over. Let $\tau(z) = z + 1$. For any function f defined on some domain in C, let $f^{\cup}(z) = \overline{f(\bar{z})}$ be its reflection in the real axis.

Definition 4.1. Let $\psi \in C(\partial\mathrm{D}(1,1))$. Let $H_\psi \in B(L^2)$ be defined by

$$H_\psi = U^* T^*_{(\psi\circ\tau)^{\cup}} U.$$

The map $\psi \mapsto H_\psi$ is unital and linear, but not multiplicative. One checks that if $\psi(z) = z^n$, then $H_{z^n} = H^n$, and if $\psi(z) = \bar{z}^n$, then $H_{\bar{z}^n} = (H^*)^n$.

Theorem 4.2. There is a short exact sequence

$$0 \to \mathrm{K} \to \mathrm{B} \to^{\pi} C(\partial\Lambda) \to 0. \tag{4.3}$$

For every X in B, its coset in B/K can be written uniquely as

$$[X] = [g(-M_x) + H_\psi - g(0)] \tag{4.4}$$

where $g \in C[-1,0]$, $\psi \in C(\partial\mathrm{D}(1,1))$, and $g(0) = \psi(0)$. The essential spectrum of X as in (4.4) is $g([0,1]) \cup \psi(\partial\mathrm{D}(1,1))$. If $\lambda \notin \sigma_e(X)$, then the Fredhom index is given by the winding number of ψ about λ: $\mathrm{ind}(X - \lambda) = \mathrm{ind}_\psi(\lambda)$.

Proof: For $X \in$ B, we shall let $[X]$ denote its equivalence class in B/K. We have

$$[HM_x] = [M_xH] = 0, \tag{4.5}$$

since $M_xH = V$ which is compact, and (1.3) shows HM_x is also compact. Moreover

$$[HH^*] = [H^*H] = [H + H^*], \tag{4.6}$$

which can be seen by noting that all three of HH^*,H^*H and $H + H^*$ take x^k to a constant minus $\frac{1}{k(k+1)}x^k$, for $k \geq 1$. So B/K is abelian. Moreover, for any polynomial q in 3 variables, there are polynomials p_1,p_2,p_3 in one variable so that

$$[q(M_x,H,H^*)] = [p_1(M_x) + p_2(H) + p_3(H^*)]. \tag{4.7}$$

Indeed, by (4.5), any term that has both M_x and either H or H^* in it can be removed. An induction argument on the total degree using (4.6) shows that any term that has factors of both H and H^* can be reduced to a linear combination of terms in just powers of H and powers of H^*. Therefore operators of the form (4.7) are dense in B/K.

We wish to prove that B/K is isomorphic to the abelian C^*-algebra $C(\partial\Lambda)$. We will use a similar strategy to the proof of Theorem 3.17. Let

$Q = \{f = (f_-,f_+) : f_- \in C([-1,0]), f_+ \in C(\partial D(1,1)), f_-(0) = f_+(0)\}$. The algebra Q is just $C(\partial\Lambda)$, but it is easier to define the functional calculus on it. Define

$$\delta : Q \to B \; f \mapsto f_-(-M_x) + H_{f_+} - f(0).$$

Let $\Delta(f) = [\delta(f)]$. The following lemma is straightforward to prove.

Lemma 4.8. (i) Let $\psi,\varphi \in C(\partial D(1,1))$. Then $[H_\psi H_\varphi] = [H_{\psi\varphi}]$.

(ii) Let $g \in C([-1,0])$ and $\psi \in C(\partial D(1,1))$. Then

$$[g(-M_x)][H_\psi] = [H_\psi][g(-M_x)] = [g(0)H_\psi + \psi(0)g(-M_x) - g(0)\psi(0)].$$

Using Lemma 4.8, one can check that Δ is a unital *-homomorphism from Q into B/K. Its range is dense, so if we can show it has no kernel, then it is a C*-isomorphism.

Lemma 4.9. If $s \in [-1,0]$, then

$$|f_-(s)| \leq \|\Delta(f)\| \tag{4.10}$$

for all $f \in Q$.

Proof. As f is continuous, it suffices to prove the lemma under the assumption that $s \in (-1,0)$, and as Δ is continuous, we can assume that f_- is a polynomial, and that $f_+(z) = f(0) + zp_2(z) + \overline{zp_3(z)}$ where p_2 and p_3 are polynomials.

As in Lemma 3.8, for n satisfying $1/n < \min\{s, 1-s\}$ we define a unit vector $\chi_n \in L^2$ by the formula

$$\chi_n(t) = \begin{cases} \sqrt{\frac{n}{2}} & \text{if } |t-s| \leq 1/n \\ 0 & \text{if } |t-s| > 1/n \end{cases}$$

Let K be compact.

$$\begin{aligned} \mathrm{h}(\delta(f) + K)\, \chi_n,\chi_n\mathrm{i} &= \mathrm{h}(f_-(-M_x) + H_{zp_2} + H_{\bar{z}\bar{p}_3} + K)\, \chi_n,\chi_n\mathrm{i} \\ &= \mathrm{h}f_-(-x)\chi_n,\chi_n\mathrm{i} + \mathrm{h}p_2(H)H\, \chi_n,\chi_n\mathrm{i} + \mathrm{h}\chi_n,p_3(H)H\, \chi_n\mathrm{i} + \mathrm{h}K\, \chi_n,\chi_n\mathrm{i} \\ &\to \quad f_-(s) \quad + \quad 0 \quad + \quad 0 \quad + \quad 0 \\ &= \quad f_-(s). \end{aligned}$$

Therefore,

$$|f_-(s)| \leq \inf_{K \in \mathcal{K}} \mathrm{k}\delta(f) + K\mathrm{k} = \mathrm{k}\Delta(f)\mathrm{k}.$$

□

If $\Delta(f) = 0$, by Lemma 4.9 we must have $f_- = 0$. So $\delta(f) = H_{f_+}$ must be compact. But H_{f_+} is unitarily equivalent to a Toeplitz operator, and there are no non-zero compact Toeplitz operators. Therefore Δ has a trivial kernel, and hence is a $*$-isomorphism.

The claim about the spectrum of $[x]$ now follows from the fact that the spectrum of a function in $C(\partial\Lambda)$ equals its range. Finally, the claim about the Fredholm index follows from the fact that the Fredholm index at λ will be unchanged under any homotopy of f that keeps λ outside its range. Then f can be homotoped to (f_-,f_+) where $f_+(z) = (z-1-\lambda)^n$ and $f_-(x) = (-1-\lambda)^n$ for some integer n, and the Fredholm index of $\delta(f)$ is n.

5 Compact operators in the little algebra $\mathcal{A}_0$

Recall from Definition 1.8 that $\mathcal{A}_0$ is the norm-closed algebra generated by M_x and V. We shall prove that every compact operator in AlgLat(V) lies not just in $\mathcal{A}$ but in $\mathcal{A}_0$.

For I an interval in [0,1], let us write $L^2(I)$ for the subspace of L^2 that vanishes a.e. off I, and let P_I denote projection onto $L^2(I)$. For $\varphi,\psi \in L^2$ we write $\varphi\otimes\psi$ to denote the rank one operator

$$\varphi \otimes \psi : f \mapsto \mathrm{h}f,\psi\mathrm{i}\varphi.$$

The key observation is the following:

Lemma 5.1. Suppose $\varphi \in L^2[t,1]$ and $\psi \in L^2[0,t]$. Then $\varphi \otimes \psi = M_\varphi V M_\psi^*$.

Proof: We have

$$M_\phi V M_\psi^* f(x) \;=\; \phi(x) \int_0^x f(s)\overline{\psi(s)}ds.$$

The right-hand side is 0 if $x < t$, and $\varphi(x)\mathrm{h}f,\psi\mathrm{i}$ if $x > t$.

Lemma 5.2. Every finite rank operator on $L^2[0,1]$ can be written as an integral operator whose kernel is in $L^2([0,1] \times [0,1])$.

Proof: Let $K = \sum_{j=1}^n \phi_j \otimes \psi_j$. Define

$$k(x,s) \ = \ \sum_{j=1}^{n} \phi_j(x)\overline{\psi_j(s)} .$$

Then $Kf(x) = \int_0^1 k(x,s)f(s)ds$, and k is in $L^2([0,1] \times [0,1])$.

Lemma 5.3. Let k be in $L^2([0,1] \times [0,1])$, and let $Tf(x) = \int_0^1 k(x,s)f(s)ds$. Then T is in AlgLat(V) if and only if $k(s,x) = 0$ for $s > x$.

Proof: Sufficiency is clear. To prove necessity, assume that for some $0 < t < 1$, the kernel

$$k(s,x)\chi_{[0,t](x)}\chi_{[t,1]}(s)$$

is not 0 a.e. As an integral operator is zero if and only if the kernel is 0 a.e., this means that the corresponding integral operator is non-zero, and hence T maps a function in $L^2(t,1)$ to a function that is not 0 a.e. on $[0,t]$. **Lemma 5.4.** Let $T = \varphi \otimes \psi$ be a rank-one operator. Then T is in AlgLat(V) if and only if for some $0 < t < 1$, the support of φ is in $[t,1]$ (i.e. $\varphi = 0$ a.e. on $[0,t]$) and the support of ψ is in $[0,t]$. In this case, $T \in A_0$.

Proof: The first part follows from Lemma 5.3. For the second part, observe that if the supports of φ,ψ are in $[t,1]$ and $[0,t]$ respectively, then $\varphi \otimes \psi = M_\varphi V M_\psi^*$. If φ and ψ are both in $C([0,1])$, this proves that $\varphi \otimes \psi \in A_0$.

For the general case, choose continuous functions f_n and g_n that converge to φ and ψ respectively in L^2. It follows from Lemma 5.1 that $M_{f_n} V M_{g_m}^*$ converges to $M_\phi V M_{g_m}^*$ in norm as $n \to \infty$, and that $M_\phi V M_{g_m}^*$ converges to $M_\varphi V M_\psi^*$. Therefore $\phi \otimes \psi \in \mathcal{A}_0$. □

Theorem 5.5. Let K be a compact operator in AlgLat(V). Then $K \in A_0$, and can be approximated in norm by finite rank operators in A_0.

Proof: Note that $K \in$ AlgLat(V) means that for all $0 < s < 1$, we have $P_{[0,s]}KP_{[s,1]} = 0$. Let $\varepsilon > 0$. First, consider $P_{[1/2,1]}KP_{[0,1/2]}$. This can be approximated within $\varepsilon/2$ by a finite rank operator that is a sum of rank one operators that map $L^2(0,1/2)$ to $L^2(1/2,1)$. By Lemma 5.4, this means that this finite rank operator is in A_0.

A similar argument shows that $P_{[1/4,1/2]}KP_{[0,1/4]}$ and $P_{[3/4,1]}KP_{[1/2,3/4]}$ can both be approximated by finite rank operators in A_0 within $\varepsilon/8$. Iterating, we get that if n is a power of 2, we can approximate

$$K - \sum_{j=1}^{n} P_{[(j-1)/n,j/n]}KP_{[(j-1)/n,j/n]}$$

within ε by a finite rank operator in A_0.

Finally we observe that

$$\| \sum_{j=1}^{n} P_{[(j-1)/n,j/n]}KP_{[(j-1)/n,j/n]}\| \ = \ \max_{1\le j\le n} \|P_{[(j-1)/n,j/n]}KP_{[(j-1)/n,j/n]}\| . \qquad (5.6)$$

Since K is compact,

$$\lim_{n\to\infty} \sup_{|I|=1/n} \|KP_I\| = 0 ,$$

so (5.6) tends to 0.

6 The operator $H - M_x$

Let us write Z for the operator $H - M_x$. We know that $[Z]$ generates the Calkin Hardy-Weyl algebra C. By Theorem 3.17 we know that the spectrum of $[Z]$ in A/K_0 is Λ. It is not surprising that $D(1,1)$ are eigenvalues of Z, since they are eigenvalues of H. It is perhaps surprising that every point in the stick, except −1, is also an eigenvalue. Moreover as we move up the stick to the bulb of the lollipop, the eigenvalues increase in multiplicity.

Theorem 6.1. (i) $\sigma_p(Z) = D(1,1) \cup (-1,0]$.

(ii) The point spectrum of Z^* is empty.

(iii) The spectrum of Z is Λ.

Proof: (i) Suppose $(Z-\lambda)f = 0$. Let $F(x) = Vf(x) = \int_0^x f(t)dt$. Then we have

$$\frac{1}{x}F(x) \;=\; (x+\lambda)f(x).$$

As $F'(x) = f(x)$, we get the equation

$$\frac{1}{x}F(x) = (x+\lambda)F'(x), \tag{6.2}$$

with the boundary condition

$$F(0) = 0. \tag{6.3}$$

The function F is continuous. Let Ω denote the relatively open subset of $[0,1]$ on which it is non-zero.

We get that the solution of (6.2), with $\lambda \neq 0$, is

$$F(x) \;=\; c\left(\frac{x}{x+\lambda}\right)^{1/\lambda}\chi_\Omega(x), \tag{6.4}$$

where the constant c can a priori be different on different components of Ω (though we show below that Ω is actually connected). Hence

$$f(x) = cx^{\frac{1}{\lambda}-1}(x+\lambda)^{-1-\frac{1}{\lambda}}\chi_\Omega(x). \tag{6.5}$$

Case: $\Omega = [0,1]$.

For f to be in L^2 with $c \neq 0$, we need

$$\Re\left(\frac{1}{\lambda}-1\right) \;>\; -\frac{1}{2},$$

which is the same as $\lambda \in \mathbb{D}(1,1)$. For $\lambda \in \mathbb{D}(1,1)$, we get the eigenfunctions

$$f(x) = x^{\frac{1}{\lambda}-1}(x+\lambda)^{-1-\frac{1}{\lambda}}. \tag{6.6}$$

When $\lambda = 0$, we get

$$F(x) = ce^{-1/x},$$

and

$$f(x) \;=\; c\frac{1}{x^2}e^{-\frac{1}{x}}. \tag{6.7}$$

Now suppose that Ω is not all of $[0,1]$. Decompose Ω as a union of disjoint non-empty intervals. On each interval, we have that f is given by (6.5), with some constant c that can depend on the interval. Choose such an interval, I. Then 0 cannot be an end-point of I, or we would have that F is given by (6.4), and this is discontinuous at the right-hand end-point of I.

So assume that the left-hand end-point of I is $t > 0$. Then the boundary condition (6.3) is replaced by $F(t) = 0$. On I, we have

$$F(x) = c\left(\frac{x}{x+\lambda}\right)^{1/\lambda}$$

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so to have $F(t) = 0$ we need $\lambda = -t$. By continuity, F cannot vanish again, so we conclude that $I = (t,1]$ and that for $\lambda \in (-1,0)$ the function

$$f(x) = x^{\frac{1}{\lambda}-1}(x+\lambda)^{-1-\frac{1}{\lambda}}\chi_{[-\lambda,1]}(x) \tag{6.8}$$

is an eigenvector of Z with eigenvalue λ.

(ii) As $H^*f(x) = \int_x^1 \frac{1}{t}f(t)dt$, the eigenvalue equation becomes

$$G(x) = (x+\lambda)f(x),$$

where

$$G(x) = \int_x^1 \frac{1}{t}f(t)dt$$

As $G'(x) = -\frac{1}{x}f(x)$, we get the differential equation

$$G(x) = -x(x+\lambda)G'(x), \qquad G(1) = 0. \tag{6.9}$$

Solving for G, we get

$$G(x) = cx^{-\frac{1}{\lambda}}(x+\lambda)^{\frac{1}{\lambda}}\chi_{\{G\neq 0\}}.$$

The only solution on an interval that vanishes on the right end-point is the zero solution. (iii) We know

$$\sigma_e(Z) = \partial\Lambda \subseteq \Lambda = \overline{\sigma_p(Z)} \subseteq \sigma(Z).$$

If λ is a point in $\mathbb{C} \setminus \Lambda$, it must be a Fredholm point. By (i), it is not an eigenvalue of Z, and by (ii) it is not an eigenvalue of Z^*. Therefore $Z - \lambda$ has trivial kernel and cokernel, and closed range. Therefore it is invertible, and λ is in the resolvent of Z.

Not only are points on the stick of the lollipop eigenvalues, there is some additional smoothness. By a generalized eigenvector of order n we mean a vector f that satisfies $(Z-\lambda)^{n+1}f = 0$ but $(Z-\lambda)^n f \neq 0$.

We shall prove the case $\lambda = 0$ first.

Lemma 6.10. At 0, the operator Z has generalized eigenvectors of all orders.

Proof: We want to show that if we let $f_0(z) = \frac{1}{x^2}e^{-\frac{1}{x}}$ from (6.7), then for every $n \in \mathbb{N}$ there exists $f \in L^2$ so that

$$Zf_{n+1} = f_n. \tag{6.11}$$

Claim 6.12. For every $n \in \mathbb{N}$ there exists a polynomial p_n of degree $2n+2$, with lowest order term of degree $n+2$, so that the functions

$$f_n(x) = p_n(\frac{1}{x})e^{-\frac{1}{x}}$$

satisfy (6.11).

We prove this by induction on n. It is true when $n = 0$. Assume we have proved it up to level n, and we want to prove it for $n + 1$. So we wish to solve the equation

$$Zf_{n+1} = f_n = p_n(\frac{1}{x})e^{-\frac{1}{x}} \tag{6.13}$$

and show that the solution is of the form

$$f_{n+1}(x) = p_{n+1}(\frac{1}{x})e^{-\frac{1}{x}}. \tag{6.14}$$

Writing F_{n+1} for Vf_{n+1}, equation (6.13) is

$$\begin{aligned} (H - M_x)f_{n+1}(x) &= \frac{1}{x}\int_0^x f_{n+1}(t)dt - xf_{n+1}(x) \\ &= \frac{1}{x}F_{n+1}(x) - xF'_{n+1}(x) \\ &= f_n(x). \end{aligned}$$

This gives us the linear differential equation

$$F'_{n+1}(x) - \frac{1}{x^2}F_{n+1}(x) = -\frac{1}{x}f_n(x).$$

Multiply by the integrating factor $e^{\frac{1}{x}}$ to get

$$\begin{aligned} \frac{d}{dx}\left[e^{\frac{1}{x}}F_{n+1}(x)\right] &= -\frac{1}{x}e^{\frac{1}{x}}f_n(x) \\ &= -\frac{1}{x}p_n(\frac{1}{x}). \end{aligned}$$

Therefore

$$e^{\frac{1}{x}}F_{n+1}(x) = q_n(\frac{1}{x}),$$

where q_n is a polynomial of degree $2n + 2$ that may have a constant term, and whose next lowest order term is of degree $n + 2$. This gives

$$\begin{aligned} f_{n+1}(x) &= \frac{d}{dx}\left[e^{-\frac{1}{x}}q_n(\frac{1}{x})\right] \\ &= e^{-\frac{1}{x}}\left[\frac{1}{x^2}q_n(\frac{1}{x}) - \frac{1}{x^2}q'_n(\frac{1}{x})\right] \end{aligned}$$

Let

$$p_{n+1}(x) = x^2 q_n(x) - x^2 q'_n(x).$$

The degree of p_{n+1} is two higher than q_n, so it is $2(n+1)+2$. There may be a term of order 2; the next lowest order term is $n + 3 = (n + 1) + 2$. But as $Zf_0 = 0$, one can subtract a multiple of f_0 from f_{n+1} without changing (6.13), so we can assume that p_{n+1} has no term of order 2. So we have proved Claim 6.12. As any function of the form (6.14) is in L^2, we are done.

For points in the stick, a similar method works, but there are restrictions when requiring the generalized eigenvectors to be in L^2. Here is one result.

Lemma 6.15. Let $\lambda \in (-1,0)$. Then Z has a generalized eigenvalue of order 1 at λ if and only if $-\frac{2}{3} < \lambda < 0$.

Proof: Let $s = -\lambda$.

$$f_0(x) = \left(\frac{x-s}{x}\right)^{\frac{1}{s}} \frac{1}{x(x-s)} \chi_{[s,1]}. \tag{6.16}$$

All the functions below are supported on $[s,1]$. We wish to find a function f_1 that satisfies

$$(Z+s)f_1 = f_0. \tag{6.17}$$

Writing F_1 for $Vf_1(x) = \int_s^x f_1(t)dt$, this becomes

$$\frac{1}{x}F_1 - (x-s)F_1' = f_0.$$

or

$$F_1' - \frac{1}{x(x-s)}F_1 = -\frac{1}{x-s}f_0. \tag{6.18}$$

An integrating factor for (6.18) is $\left(\frac{x}{x-s}\right)^{\frac{1}{s}}$. This yields

$$\begin{aligned}\frac{d}{dx}\left[\left(\frac{x}{x-s}\right)^{\frac{1}{s}} F_1\right] &= -\left(\frac{x}{x-s}\right)^{\frac{1}{s}} \frac{1}{x-s} f_0 \\ &= -\frac{1}{x(x-s)^2}.\end{aligned}$$

Integrating, we get

$$\left(\frac{x}{x-s}\right)^{\frac{1}{s}} F_1 = \frac{1}{s^2}\log\frac{x-s}{x} + \frac{1}{s}\frac{1}{x-s} + c.$$

Dividing through by the integrating factor and differentiating, we get

$$\begin{aligned}f_1(x) = \;& \frac{1}{s^2}\left[\left(\frac{x-s}{x}\right)^{\frac{1}{s}-1} \frac{1}{x^2}\log\frac{x-s}{x} + \left(\frac{x-s}{x}\right)^{\frac{1}{s}} \frac{s}{x(x-s)}\right] \\ & + \frac{1}{s}\left[-\frac{1}{(x-s)^2}\left(\frac{x-s}{x}\right)^{\frac{1}{s}} + \frac{1}{x-s}\left(\frac{x-s}{x}\right)^{\frac{1}{s}-1}\frac{1}{x^2}\right] \\ & + c\left[\left(\frac{x-s}{x}\right)^{\frac{1}{s}-1}\frac{1}{x^2}\right].\end{aligned}$$

We can choose $c = 0$, since it is the coefficient of f_0. This gives

$$f_1(x) = \left(\frac{x-s}{x}\right)^{\frac{1}{s}} \left[\frac{1}{s^2}\frac{1}{x(x-s)}\log\frac{x-s}{x} + \frac{1-s}{s}\frac{1}{x(x-s)^2}\right]. \quad (6.19)$$

Examining this expression, we see that f_1 is smooth on $(s,1]$, and the first term

$$\frac{1}{s^2}\frac{(x-s)^{\frac{1}{s}-1}}{x^{\frac{1}{s}+1}}\log\frac{x-s}{x}$$

vanishes at s for every $s < 1$. However the second term

$$\frac{1-s}{s}\ \frac{(x-s)^{\frac{1}{s}-2}}{x^{\frac{1}{s}+1}}$$

has a singularity that grows like $(x-s)^{\frac{1}{s}-2}$. This is integrable for every $s < 1$, but it is only in $L^2[s,1]$ for $s < \frac{2}{3}$. So we have shown that (6.17) has a solution f_1 in L^2 if and only if $\lambda > -\frac{2}{3}$.

One can repeat the argument of Lemma 6.15 to get higher order generalized eigenvectors, as λ gets closer to 0.

Lemma 6.20. Let $m \geq 1$. Let λ lie in the interval $(-\frac{2}{2m+1}, 0)$. Then Z has generalized eigenvectors up to order m at λ.

Proof: We shall inductively find functions f_n satisfying $(Z - \lambda)f_{n+1} = f_n$, with f_0 as in (6.16). Let $s = -\lambda$, and write

$$\Phi(x) = \left(\frac{x-s}{x}\right)^{\frac{1}{s}} \chi_{[s,1]}(x).$$

Then we have

$$\begin{aligned} f_0(x) &= \Phi(x)\frac{1}{x(x-s)} \\ f_1(x) &= \Phi(x)\left[\frac{1}{s^2}\frac{1}{x(x-s)}\log\frac{x-s}{x} + \frac{1-s}{s}\frac{1}{x(x-s)^2}\right]. \end{aligned}$$

Writing F_{n+1} for $V f_{n+1} = \int_s^x f_{n+1}(t)dt$, we want to solve

$$\frac{1}{x}F_{n+1}(x) - (x-s)F'_{n+1}(x) = f_n(x). \quad (6.21)$$

After multiplying by the integrating factor $1/\Phi$, we have

$$\frac{d}{dx}\left[\frac{1}{\Phi}F_{n+1}\right] = -\frac{1}{\Phi}\frac{1}{x-s}f_n(x). \quad (6.22)$$

Claim 6.23. There are constants M_n such that the functions f_n satisfy

$$|f_n(x)| \leq M_n(x-s)^{\frac{1}{s}-n-1} \qquad \forall x \in (s,1]. \quad (6.24)$$

Proof of Claim 6.23: By induction on n. It is true when $n = 0$. Assume it is true up to n. From (6.22) we get for $x \in (s,1]$:

$$\frac{1}{\Phi(x)} F_{n+1}(x) \;=\; \int_x^1 \frac{1}{\Phi(t)} \frac{1}{t-s} f_n(t)dt + c_n . \tag{6.25}$$

By the inductive hypothesis, the integrand in (6.25) is $O(t-s)^{-n-2}$, so the integral is $O(x-s)^{-n-1}$. So F_{n+1} satisfies

$$F_{n+1}(x) = c_n \Phi(x) + O(x-s)^{\frac{1}{s}-n-1}. \tag{6.26}$$

From (6.21) we have

$$f_{n+1}(x) \;=\; \frac{1}{x(x-s)} F_{n+1}(x) - \frac{1}{x-s} f_n(x) . \tag{6.27}$$

When we use (6.26) for F_{n+1}, we get

$$f_{n+1}(x) \;=\; c_n f_0(x) - \frac{1}{x-s} f_n(x) + O(x-s)^{\frac{1}{s}-n-2} .$$

Now the claim follows from the inductive hypothesis on f_n.

It follows from (6.27) that that f_m is continuous on $(s,1]$, and Claim 6.23 shows that its singularity at s is of order $(x-s)^{\frac{1}{s}-m-1}$. This means f_m is in L^2 provided $\frac{1}{s} - m - 1 \;>\; -\frac{1}{2}$, which is the same as $s < \frac{2}{2m+1}$.

For later use, let us note that if you track the constants M_n in Claim 6.23, you can show:

Lemma 6.28. In Claim 6.23 one can take $M_0 = \frac{1}{(s^{1+1/s})}$ and the consants M_n satisfy

$$M_{n+1} \leq M_n \left(1 + \frac{M_0}{n+1}\right) .$$

Lemma 6.29. Let $m \geq 1$. On the interval $(-\frac{2}{2m+1}, 0)$ one can choose the generalized eigenvectors of order n of $Z - \lambda$ continuously in λ, for every $n \leq m$, and satisfying $(Z-\lambda)f_{\lambda,n} = f_{\lambda,n-1}$, for every $1 \leq n \leq m$.

Proof: Let us write $f_{\lambda,n}$ for the choice of generalized eigenvector of order n at λ. Write

$$\Phi(\lambda, x) \;=\; \left(\frac{x+\lambda}{x}\right)^{-\frac{1}{\lambda}} \chi_{[-\lambda,1]}(x) .$$

We have

$$f_{\lambda,0}(x) \;=\; \frac{1}{x(x+\lambda)} \Phi(\lambda, x) .$$

On every compact subset K, of $(-1,0)$ the functions $\{f_{\lambda,0} : \lambda \in K\}$ are uniformly bounded, and $\lim_{\lambda_0 \to \lambda} f_{\lambda_0,0}(x) = f_{\lambda,0}(x)$ a.e., so the map $\lambda \mapsto f_{\lambda,0}$ is continuous as a map from $(-\frac{2}{2m+1}, 0)$ into L^2.

For higher n, we find f_{n+1} as in Lemma 6.20. At each stage, we take the constant c_n in (6.26) to be 0. (We can do this because $c_n f_{\lambda,0}$ will be in the kernel of $Z - \lambda$.) This gives us

$$f_{\lambda,n+1}(x) \;=\; \frac{1}{x(x-s)}\Phi(\lambda,x)\int_x^1 \frac{1}{\Phi(\lambda,t)}\frac{1}{t-s}f_{\lambda,n}(t)dt - \frac{1}{x-s}f_{\lambda,n}(x).$$

Moreover we have $f_{\lambda,n}(x) = 0$ if $x < -\lambda$ and

$$|f_{\lambda,n}(x)| \leq M_{n,\lambda}(x+\lambda)^{-\frac{1}{\lambda}-n-1}, \qquad x > -\lambda.$$

By Lemma 6.28, we have that each $M_{n,\lambda}$ can be chosen uniformly in λ for λ in $(-\frac{2}{2m+1},0)$. For any interval I of length δ, we get

$$\begin{aligned}\int_I |f_{\lambda,n}(x)|^2 dx &\leq M_n \int_0^\delta |x+\lambda|^{-\frac{2}{\lambda}-2n-2} \\ &= M_n \frac{1}{-\frac{2}{\lambda}-2n-1}\delta^{-\frac{2}{\lambda}-2n-1}.\end{aligned}$$

Therefore, as λ ranges over any compact subset of $(-\frac{2}{2m+1},0)$, the functions $|f_{\lambda,n}(x)|^2$ are uniformly integrable in x. So, by the Vitali convergence theorem, the map

$$\begin{aligned}(-\frac{2}{2m+1},0) &\rightarrow L^2 \\ \lambda &\mapsto f_{\lambda,n}\end{aligned}$$

is continuous.

These lemmas say that operators in the closed algebra generated by Z have certain smoothness properties when mapped by σ into $A(\Lambda)$. The functions get smoother as we get closer to 0.

Theorem 6.30. Let X be in the norm-closed algebra generated by Z. Then $\theta(X)$ is C^m on $(-\frac{2}{2m+1},0)$.

Proof: Let $\theta(X) = \varphi \in A(\Lambda)$. Let p_j be a sequence of polynomials so that $\|p_j(Z) - X\| \to 0$. It follows from Theorem 3.17 that p_j converges to φ uniformly on Λ.

Case: $m = 1$. Let $f_{\lambda,n}$ be as in Lemma 6.29. For any polynomial p, we have

$$\langle p(Z)f_{\lambda,0}, f_{\lambda,0}\rangle = \|f_{\lambda,0}\|^2 p(\lambda).$$

Moreover, $p(Z)f_{\lambda,1} = p(\lambda)f_{\lambda,1} + p'(\lambda)f_{\lambda,0}$.

Let g_λ^1 be the linear combination of $f_{\lambda,0}$ and $f_{\lambda,1}$ that satisfies $\langle f_{\lambda,0}, g_\lambda^1\rangle = 1$ and $\langle f_{\lambda,1}, g_\lambda^1\rangle = 0$. Then

$$\langle p(Z)f_{\lambda,1}, g_\lambda^1\rangle = p'(\lambda).$$

So as functions on $(-\frac{2}{3}, 0)$, we get that $p_j'(\lambda)$ converges to some function $\psi(\lambda) = \langle Xf_{\lambda,1}, g_\lambda^1\rangle$.

Claim 6.31. For all x in $(-\frac{2}{3}, 0)$, we have $\varphi^0(x) = \psi(x)$, and ψ is continuous.

We have $g_\lambda^1 = a_{\lambda,0}f_{\lambda,0}+a_{\lambda,1}f_{\lambda,1}$, where the coefficients $a_{\lambda,0}$ and $a_{\lambda,1}$ solve the linear system

$$\begin{pmatrix} \langle f_{\lambda,0}, f_{\lambda,0}\rangle & \langle f_{\lambda,0}, f_{\lambda,1}\rangle \\ \langle f_{\lambda,1}, f_{\lambda,0}\rangle & \langle f_{\lambda,1}, f_{\lambda,1}\rangle \end{pmatrix} \begin{pmatrix} a_{\lambda,0} \\ a_{\lambda,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

By Lemma 6.29, since $f_{\lambda,0}$ and $f_{\lambda,1}$ are continuous in λ, so as they are linearly independent, we have that g_λ^1 is also continuous in λ. Therefore ψ is continuous, and $p^0{}_j$ converges to ψ locally uniformly on $(-\frac{2}{3}, 0)$. As $\int_{-\frac{1}{3}}^{x} p_j'(t)dt = \phi(x) - \phi(-\frac{1}{3})$ converges to $\int_{-\frac{1}{3}}^{x} \psi(t)dt$, we get that $\varphi^0(x) = \psi(x)$.

We have shown that ϕ is in $C^1(-\frac{2}{3}, 0)$. A similar argument with higher derivatives proves that ϕ is in $C^m(-\frac{2}{2m+1}, 0)$.

Of course one can also find generalized eigenvectors for Z of all orders at points in D(1,1), but we already know that $\theta(X)$ is analytic on D(1,1) for every $X \in$ A.

7 Open Questions

Question 7.1. Is there a good description of K_A, the compact operators in A?

Question 7.2. Is $C^*(Z) = C^*(A)$? To prove this, it is sufficient to show that $C^*(Z)$ is irreducible, since then it would contain all the compacts, and its quotient by the compacts would be all of $C(\partial\Lambda)$.

Question 7.3. Do the eigenvectors of Z span L^2?

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