Adomian decomposition method reformulated using dimensionless nonlinear perturbation theory

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Abstract

The Adomian decomposition method (ADM) is a universal approach to solving governing equations in various engineering and technological applications. The applicability of the ADM is almost limitless due to its universal applicability, but its convergence rate and numerical accuracy are sensitive to the number of truncated terms in series solutions. More importantly, Adomian formalism still holds unresolved issue regarding the mismatch of the order of the expansion parameter. The current work provides an in-depth analysis of Adomian's decomposition method, Lyapunov's stability theory, and the nonlinear perturbation theory to resolve the fundamental mismatch with physical interpretation.

Keywords: Adomian decomposition method; perturbation theory; normalization; Lyapunov's stability theory; commutative operation

1 Introduction

The Adomian Decomposition Method (ADM) is a mathematical technique that uses iterative operator theory to solve nonlinear differential equations, including stochastic differential equations [1–6]. One of the notable benefits of the ADM is its ability to be applied universally to a wide range of governing equations, such as ordinary and partial differential equations and multi-dimensional integrals in various physical and engineered systems [7–10]. Advanced ADM approaches were applied in various areas of applied mathematics and physics, as follows. The variational iteration method was developed to solve boundary value problems on finite and semi-infinite intervals, enabling the easy identification of the Lagrange multiplier and providing a computational advantage for converging approximate solutions on semi-infinite intervals [11]. In the dynamics of hydromagnetic ferro-nanofluid, the governing equations are transformed into ordinary differential equations (ODEs) and solved analytically using the ADM that was compared with numerical solutions obtained using the Runge-Kutta-Fehlberg method [12]. A time fractional partial integro-differential equation was solved using the ADM and compared with the homotopy perturbation method, providing an in-depth analysis of the convergence to the exact solution [13]. To overcome singularity at the origin, multi-point singular boundary value problems are converted into equivalent integral equations, and the ADM is applied with convergence analysis for approximate solutions [14]. A nonlinear partial differential equation in 4D Euclidean anti-de Sitter space was solved, providing perturbative series solutions for three specific modes of the equation, all valid in the probe approximation [15].

The ADM's application scopes and disciplinary areas can be further expanded by resolving its intrinsic limitations, as follows. First, the convergence rate of ADM's series solutions is impacted by the nonlinear nature of the governing equation. With increasing n, higher-order terms receive more low-order coefficients [16]. A new analytic method was proposed [17] which includes a convergence-control parameter to regulate the convergence region and rate; however, an additional numerical

algorithm is required to identify the optimal parameter value. Second, the ADM fails to provide reliable solutions for initial value problems at asymptotic limits [16], likely due to the limited availability of the maximum ADM polynomial ($n \leq 10$) [9, 10, 18]. The construction of extended tables becomes progressively more demanding in terms of effort and complexity as the order n is raised. Nevertheless, there have been additional methodologies that prioritize expediting evaluations of A_n (rather than incorporating higher order terms based on Adomian's original research), including modified [19–21], iterative [22], and multi-term [23, 24] ADMs. Third, the ADM is facing an unresolved issue concerning the inconsistency of the expansion parameter λ between the polynomial coefficient and the recurrence formula [25]. The main focus should be on meticulously resolving the Adomian's dilemma, the discrepancy issue addressed in this work.

The terms constituting A_n were obtained by Adomian and Rach [26–28] through the utilization of a series parameter, reminiscent of the method employed in conventional perturbation theory: nevertheless, they underscored the autonomy of the ADM from perturbation and analogous approaches. Nevertheless, the series expansions of the general nonlinear governing equation were not correctly segregated and synchronized with respect to the parameter order. The analysis of the available literature accentuates the divergence in how nonlinear perturbation theory and Lyapunov's stability theory might discern the mathematical formalism of ADM, potentially yielding physical interpretations when applied to physical and engineering systems. Dynamical systems in diverse fields are assessed for stability through the application of the LST. This analysis is crucial in determining the stability of a system's equilibrium point, categorizing it as stable, asymptotically stable, or unstable. Multiple mathematicians and physicists have developed perturbation theory over time [29], including Lagrange [30], Laplace [31], and Poincare [32] in celestial dynamics and Schrodinger [33, 34], Dirac [35], and Feynman [36] in quantum mechanics.

The resolution of the Adomian's dilemma is of paramount importance in rigorously improving the quality of the ADM solution in diverse areas. The present work provides a comprehensive summary of ADM formalism, followed by recent endeavors to tackle the parameter order mismatch issue. Subsequently, a theoretical resolution is presented to address the parameter order mismatch through the fundamental perturbation formalism.

2 Problem Statement

The operator equation in the Adomian literature can be written as

$$L[U] + R[U] + N[U] = g \tag{1}$$

where L, R, and N represent linear, remaining, and nonlinear operators, respectively, for a specific physical quantity U. In this work, time t is selected as an independent variable for an s-fold differential operator L, such as $L[\cdots] = \partial^s[\cdots]/\partial t^s$, which is assumed to be invertible, such as

$$L^{-1}[\cdots] = \int^{t} dt_1 \cdot \int^{t_1} dt_2 \cdots \int^{t_{s-1}} dt_s [\cdots] \tag{2}$$

The remaining linear operator R includes lower-order derivatives, such as $\partial^k[\cdots]/\partial t^k$ for k from 0 to $s-1$ and also r-th order spatial derivatives $\partial^r[\cdots]/\partial q_k^r$ for the k-th generalized coordinate q_k ($k = 1-d$) in a d-dimensions. For example, if a 3D thermal diffusion is involved, L contains a first-order time differential, and R includes a Laplacian operator (of $r = 2$) with generalized coordinates of $\boldsymbol{q} = (q_1, q_2, q_3) = (x, y, z)$, where U is temperature T [K]. The target variable U, especially for transport phenomena, can be also concentration C [mol/liter] in the molecular diffusion equations, fluid velocity vector $\boldsymbol{V}$ [m/s] in the Navier-Stokes equation, and probability distribution functions in the Fokker-Planck and Schrodinger equations. Throughout the present work, $O[U]$ means operator O operated to variable U, $g(t)$ specifies g as a function of t, and $f(\{x_n\})$ indicates $f(x_0, x_1, \cdots, x_n)$, i.e., a function of $n+1$ independent variables from x_0 to x_n.

Adomian suggested a decomposed series of U [9], such as

$$U = U_0 + U_1 + \cdots = \sum_{n=0}^{\infty} U_n \tag{3}$$

where U_n is denoted as the n-th component function of U. An identity operation was employed such as $L^{-1}[L[U]] = U - U(0)$, where $U(0)$ is a complementary solution of s-fold Eq. (1), such as

$$U(0) = \sum_{k=0}^{s-1} \frac{t^k}{k!} \left[\frac{\partial^k U}{\partial t^k}\right]_{t=0} \tag{4}$$

and the inverse operator L^{-1} was applied on the both sides of Eq. (1) from the left to result in

$$U - U_0 = -L^{-1}[R[U] + N[U]] \tag{5}$$

where $U_0 = U(0) + L^{-1}g$ is a trivial solution for $R = N = 0$. Eq. (3) was substituted into Eq. (5) to match the terms having the same appearing sequences on each side. Then, the recurrence formula was obtained, such as

$$U_{n+1} = -L^{-1}R[U_n] - L^{-1}A_n(\{U_n\}) \tag{6}$$

where $A_n(\{U_n\})$ is the Adomian polynomial, determined by letting $N[U]$ as a continuous function, expanded using Taylor's series, such as

$$N[U] = f(U) = \sum_{n=0} \frac{f_0^{(n)}}{n!}(U - U_0)^n = \sum_{n=0}^{\infty} A_n(\{U_n\}) \tag{7}$$

where $f_0^{(n)} = [d^n f(U)/dU^n]_{U=U_0}$ is the n^{th} derivative of $f(U)$ evaluated at U_0. To derive the series solution U, a generating function of A_n is crucial (instead of its manual calculation for specific n) derived in principle, such as

$$A_n = \frac{1}{n!}\left[\frac{\partial^n}{\partial \lambda^n} f(U(\lambda))\right]_{\lambda=0} = \sum_{k=0}^{n} f_0^{(k)} C_{k,n} \tag{8}$$

where $C_{k,n}$ are a product (or a sum of products) of k component functions, whose subscripts sum to n, divided by the factorial of the repeated number of subscripts [26–28] (See Appendix for an exemplary description), and $U(\lambda)$ is a perturbation-style series as an alternative representation of Eq. (3), such as

$$U(\lambda) = U_0 + \lambda U_1 + \lambda^2 U_2 + \cdots = \sum_{n=0}^{\infty} \lambda^n U_n \tag{9}$$

where λ is often defined as an expansion parameter. Adomian's original work independently use Eq. (3) and (9) for separate purposes, as follows. The decomposed series U of Eq. (3) was directly substituted into Eq. (1) (without using λ) to create the recurrence formula of Eq. (6); but, the perturbation-style series U of Eq. (9) was employed only to collect terms for A_n comprising U_0, U_1, ..., and U_n using λ. However, direct substituting Eq. (9) into (1) does not reproduce the same recurrence formula of Eq. (6), which is denoted as Adomian's dilemma in this work.

Related topics to the efficient expansion were discussed to provide more robust representations and efficient computing algorithms for Adomian polynomials [37–39]. In a new algorithm [37], U was used as a sum of its components, U_n for $n \geq 1$ (instead of those for $n \geq 0$), which can be elegantly used without any extra formulas other than elementary operations. The orthogonality of complex variable e^{inx} for integer n was introduced to compute Adomian polynomials for various nonlinear operators [39] without continuously storing the index sum of U_n [37]. Alternatively the final solution U (to be sought) is pre-expanded as a Taylor series of the independent variable (such as t, used in this study) and substituted into the nonlinear term of $N[U]$, which reduces computational costs for mere integration and simple arithmetic operations [38, 40]. The above-mentioned approaches [37–40] provided new insight and faster algorithms to solve nonlinear differential equations. However, the order of expansion parameter λ remained mismatched, providing series solutions at the same level of accuracy and convergence as Adomian's original work.

The parameter order mismatch has not been theoretically discussed until Zhang and Liang [25] initiated the fundamental issue. Rach [27] emphasized that "λ is not in any sense a perturbation parameter but a convenient device for collecting terms and is dropped at the end of the calculation."

In fact, Adomian and Rach [26] introduced λ as an intermediate parameter to collect terms satisfying conditions in Eq. (8). At the end, λ was replaced by 0 to nullify terms having higher orders than n because terms of lower orders than n were already eliminated by n-time differentiation. Adomian's explicit expression A_n of Eq. (8) originates from the perturbation-style series of Eq. (9). In this case, the nonlinear term can be rigorously rewritten, such as

$$N[U] = \lim_{\lambda \to 1} \sum_{n=0}^{\infty} \lambda^n A_n (\{U_n\}) = \lim_{\lambda \to 1} \sum_{n=0}^{\infty} \lambda^n \frac{1}{n!} \left[\frac{\partial^n}{\partial \lambda^n} f(U(\lambda)) \right]_{\lambda=0} \tag{10}$$

where λ^n inside the summation will be replaced by 1 after A_n is obtained. Zhang and Liang [25] multiplied the small parameter λ to the remainder and nonlinear operator terms, such as

$$L[U] + \lambda (R[U] + N[U]) = g \tag{11}$$

and indicated that the ADM must be a special case of LASPM, which seemed successfully resolving the parameter order mismatch by providing some clues to the "mysterious relationship." The adjusted Eq. (11) differs from physical governing equations that do not explicitly contain λ. Introducing the additional λ in Eq. (11) is at the same level of Adomian's intuitive ordering used in Eq. (5), and the source-term g remained unmodified.

Lyapunov's theory, primarily assessing system stability [41], uses an artificially small parameter, known as ε, having a specific dimension that can be determined empirically [42, 43]. Adomian's expansion parameter, λ, was used as a temporary derivation tool to derive the analytical expression of A_n and the recurrence Eq. (6) without explicitly including λ at the end. Therefore, it is challenging to fit one theory into the framework of another, and a detailed physical interpretation is important for achieving more rigorous resolution. We used the standard perturbation theory to provide a physics-based resolution of Adomian's dilemma.

3 Perturbation-based analysis of Adomian polynomial

We reformulate Adomian's governing equation in the perturbation-style formalism and provide a physics-based resolution of the Adomian's dilemma.

3.1 Nondimensionalization of Adomian Polynomial

Adomian considered the nonlinear term $N[U]$ as a Taylor series of U evaluated at U_0, where U_0 is a trivial solution in the absence of $R[U]$ and $N[U]$. In transport phenomena, nonlinear terms are often found as products of U and its derivatives, such as, $U\frac{\partial U}{\partial t}$, $U\frac{\partial U}{\partial x}$, and $U\frac{\partial^2 U}{\partial x^2}$, which require to extend $N[U]$ to $N\left[U, \dot{U}, U'\right] = f\left(U, \dot{U}, U'\right)$, where $\dot{U} = \partial U/\partial t$ and $U' = \partial U/\partial x$. The coupling between U, $\dot{U}$, and U' can be represented by including three parts separately, such as

$$f\left(U(\lambda), \dot{U}(\lambda), U'(\lambda)\right) = \sum_{n=0}^{\infty} \lambda^n A_n \left(U_0, \cdots, U_p, \dot{U}_0, \cdots, \dot{U}_q, U'_0, \cdots, U'_r\right) \tag{12}$$

where $p + q + r = n$. For dimensionless analysis, one can select representative parameters of U, $\dot{U}$, and U', denoted as $\overline{U}$, $\overline{V}$, and $\overline{U'}$, respectively, to propose dimensionless variables, such as $u = U/\overline{U}$, $\dot{u} = \dot{U}/\overline{V}$, and $u' = U'/\overline{U'}$. Because A_n is a sum of component products, the scalability of A_n is straightforward, such as

$$A_n \left(U_0, \cdots, U_p, \dot{U}_0, \cdots, \dot{U}_q, U'_0, \cdots, U'_r\right) = \alpha_n^* A_n (u_0, \cdots, u_p, \dot{u}_0, \cdots, \dot{u}_q, u'_0, \cdots, u'_r) \tag{13}$$

where $\alpha_n^* = \left(\overline{U}\right)^p \left(\overline{V}\right)^q \left(\overline{U'}\right)^r$ is a scaling parameter of A_n being a function of dimensionless variables. Including higher orders of time and spatial derivatives in the nonlinear term does not alter the scaling property of A_n.

3.2 Ordering mismatch resolved

Partial differential equations often used in the transport and wave phenomena can be generally written as

$$\sum_{k=1}^{s} a_k D_t^k U = \sum_{m=1}^{r} \sum_{i=1}^{d} b_m \mathscr{D}_{q_i}^m U + N\left(U, \left\{\mathscr{D}_{q_i}^m U\right\}, \left\{D_t^{s-1} U\right\}\right) + g\left(t, \{q_i\}\right) \tag{14}$$

where $D_t^n U$ is the partial or ordinary time differentials, such as $\partial^n U/\partial t^n$ or $d^n U/dt^n$, respectively, $\mathscr{D}_{q_i}^m U$ is the m^{th} order spatial gradient, q_i is the i^{th} generalized coordinate in d-dimensions, and a_k and b_m are coefficients for the time and spatial derivatives, respectively. For example, one-dimensional unsteady convection-diffusion-reaction equation with a source $g\left(x,t\right)$ [44] is written as

$$\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} - kC + g\left(x,t\right) \tag{15}$$

where C is the concentration of solutes of diffusion coefficient D_0, convective velocity v, and the reaction coefficient k. Mathematical parameters are determined, such as $s=1$, $a_1=1$, and $a_{k\neq 1}=0$ for time derivatives; $r=2$, and $d=1$ giving $q_1=x$ for the spatial derivative; and $D_0=b_2$, $v=-b_1$, and $k=-b_0$ for the physical parameters. The remainder operator is determined, such as

$$R\left[U\right] = \left[\sum_{j=0}^{s-1} a_j D_t^j - \sum_{k=0}^{r} b_k \mathscr{D}_q^k\right] U \tag{16}$$

and substituted into the governing equation (14) to create a perturbation series, such as

$$\sum_{k=0}^{\infty} \lambda^k \frac{\partial^s U_k}{\partial t^s} + \sum_{k=0}^{\infty} \lambda^k R\left[U_k\right] + \sum_{k=0}^{\infty} \lambda^k A_k = g \tag{17}$$

Dimensionless time and concentration are defined, such as $\tau = t/t_0$ and $u = U/\overline{U}$, respectively, where t_0 is a representative time scale. Eq. (17) is rewritten as

$$\sum_{k=0}^{\infty} \lambda^k \frac{\partial^s u_k}{\partial \tau^s} + \sum_{k=0}^{\infty} \lambda^k t_0^s R\left[u_k\right] + \sum_{k=0}^{\infty} \lambda^k t_0^s \overline{U}^{k-1} A_k\left(\{u_k\}\right) = g t_0^s \tag{18}$$

where $t_0^s R$ and $t_0^s \overline{U}^{k-1} A_k$ are unconditionally dimensionless. Eq. (18) is equivalent to employing perturbation-style u, such as

$$u\left(\lambda\right) = u_0 + \lambda u_1 + \lambda^2 u_2 + \cdots = \sum_{n=0}^{\infty} \lambda^n u_n \tag{19}$$

Without loosing generality, we let

$$\overline{U} = 1 \tag{20}$$

$$t_0^s = \lambda \tag{21}$$

to rewrite Eq. (18), such as

$$\sum_{k=0}^{\infty} \lambda^k \left[\frac{\partial^s u_k}{\partial \tau^s} + \lambda R\left[u_k\right] + \lambda A_k\left(\{u_k\}\right)\right] = g\lambda \tag{22}$$

which is similar to a standard perturbation series having λ as an expansion parameter. Eq. (22) specifically gives

$$\frac{\partial^s u_0}{\partial \tau^s} - g\lambda = 0 \tag{23}$$

for $k = 0$ and

$$\frac{\partial^s u_k}{\partial \tau^s} = -\lambda R\left[u_k\right] - \lambda A_k\left(\{u_{j\leq k}\}\right) \tag{24}$$

for $k > 0$. By applying the inverse operator L^{-1} from the left on the both side of Eq. (24), we obtain a new recurrence formula of Eq. (6), such as

$$u_{k+1} = -\lambda R\left[L^{-1}\left(u_k\right)\right] - \lambda A_k\left(L^{-1}\left\{u_{j\leq k}\right\}\right) \tag{25}$$

using the commutative relationships of

$$L^{-1} R\left[u_k\right] = R\left[L^{-1} u_k\right] \tag{26}$$

$$L^{-1} A_k\left[\{u_{j\leq k}\}\right] = A_k\left[L^{-1}\left\{u_{j\leq k}\right\}\right] \tag{27}$$

where $u_{j\leq k}$ is obtained when u_{k+1} is sought. In our view, Eq. (25) resolves the Adomian's dilemma of the order mismatch of the expansion parameter by employing nonlinear perturbation theory with a physically dimensioned expansion parameter. Once sufficient component functions with a large n are obtained, the truncated form of u is finally obtained by letting $\lambda = 1$ in Eq. (19) as an analytical series solution.

4 Concluding Remarks

Applying Adomian's decomposition led to a recurrence formula that efficiently provides series solutions for nonlinear governing equations, regardless of the physical dimensions. Lyapunov's technique utilizes a small artificial parameter, presenting a physical quantity of a finite dimension. While intuitively proven, the Adomian decomposition method was perceived intuitively as a special case of Lyapunov's model instead of the perturbation theory having a closer similarity. The present work employed the nonlinear perturbation theory with a physically dimensioned expansion parameter and resolved the order mismatch dilemma of the expansion parameter. To have rigorous recursion processes for a large n, a new formalism is of great necessity for advanced ADM applications, ensuring faster convergence and higher accuracy of truncated solutions.

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Appendix A Understanding A_n coefficient

In the ADM, the true solution u is defined as

$$u = \sum_{n=0}^{\infty} u_n \tag{A1}$$

and the nonlinear term is assumed to be an analytic function of u, i.e., $N[u] = f(u)$, such as

$$f(u) = \sum_{n=0}^{\infty} A_n (u_0, \cdots, u_n) \tag{A2}$$

where u_0 is the trivial solution in the absence of R and N, and A_n is known as the n^{th}-order Adomian polynomial. The A_n is given as

$$A_n = \sum_{\nu=1}^{n} f_0^{(\nu)} C(\nu, n) \tag{A3}$$

where

$$f_0^{(\nu)} = \left[\frac{\mathrm{d}^{\nu} f(u)}{\mathrm{d}u^{\nu}}\right]_{u=u_0} \tag{A4}$$

is the ν^{th}-derivative of $f(u)$ with respect to u at $u = u_0$, and

$$C(\nu, n) = \sum_{\nu_i} \prod_{k_i}^{\nu} \frac{1}{k_i!} u_{\nu_i}^{k_i} \tag{A5}$$

include product sums of ν components of u whose subscripts sum to n, divided by the factorial of the number of repeated subscripts. For n^{th} order, the following conditions should be satisfied, such as

$$\sum_{i=1} k_i = \nu \tag{A6}$$

and

$$\sum_{i=1}^{\nu_i} k_i \nu_i = n \tag{A7}$$

where k_i is the number of repetitions of u_{ν_i} with $0 \leq i \leq n$. For example, A_4 consists of products of four u's, such as

$$\frac{u_i^a}{a!}\frac{u_j^b}{b!}\frac{u_k^c}{c!}\frac{u_l^d}{d!} \tag{A8}$$

where a is the number of repetition of u_i, and similarly for b, c, and d, satisfying conditions of

$$a+b+c+d=\nu \tag{A9}$$

$$a\cdot i+b\cdot j+c\cdot k+d\cdot l=n \tag{A10}$$

and

$$1\leq i<j<k<l\leq n \tag{A11}$$

For $n=10$, $C\left(4,10\right)$ can be written explicitly, such as

$$\begin{aligned}
C\left(4,10\right)=&\left(\frac{1}{2!}u_2^2\right)\left(\frac{1}{2!}u_3^2\right)+\left(\frac{1}{1!}u_1^1\right)\left(\frac{1}{3!}u_3^3\right)\\
&+\left(\frac{1}{1!}u_1^1\right)\left(\frac{1}{1!}u_2^1\right)\left(\frac{1}{1!}u_3^1\right)\left(\frac{1}{1!}u_4^1\right)+\left(\frac{1}{3!}u_2^3\right)\left(\frac{1}{1!}u_4^1\right)\\
&+\left(\frac{1}{2!}u_1^2\right)\left(\frac{1}{2!}u_4^2\right)+\left(\frac{1}{1!}u_1^1\right)\left(\frac{1}{2!}u_2^2\right)\left(\frac{1}{1!}u_5^1\right)\\
&+\left(\frac{1}{2!}u_1^2\right)\left(\frac{1}{1!}u_3^1\right)\left(\frac{1}{1!}u_5^1\right)+\left(\frac{1}{2!}u_1^2\right)\left(\frac{1}{1!}u_2^1\right)\left(\frac{1}{1!}u_6^1\right)\\
&+\left(\frac{1}{3!}u_1^3\right)\left(\frac{1}{1!}u_7^1\right)
\end{aligned} \tag{A12}$$

having each term as a product of four components, satisfying

$$\sum_{i=1}k_i=a+b+c+d=4 \tag{A13}$$

$$\sum_{i=1}^{\nu_i}k_i\nu_i=a\cdot i+b\cdot j+c\cdot k+d\cdot l=10 \tag{A14}$$

For example, the first term of $C\left(4,10\right)$ can be written as

$$\left(\frac{1}{2!}u_2^2\right)\left(\frac{1}{2!}u_3^2\right)=\frac{1}{2!}\frac{1}{2!}u_2^1u_2^1u_3^1u_3^1 \tag{A15}$$

where the sum of superscripts is $1+1+1+1=4$, the product sum of subscripts and superscripts is $1\cdot 2+1\cdot 2+1\cdot 3+1\cdot 3=10$, and each of u_2 and u_3 are repeated twice providing $1/\left(2!2!\right)$. All other terms follow the same rules for the subscript and superscript sums. When n is large even if ν is small (e.g., $\nu=4$), finding all the possible combinations satisfying Eqs. (A6) and (A7) is a challenging task. The highest order of A_n reported in the literature is 10 from one of Adomian's initial work [10].