AN ENRICHED COUNT OF NODAL ORBITS IN AN INVARIANT PENCIL OF CONICS

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ABSTRACT. This work gives an equivariantly enriched count of nodal orbits in a general pencil of plane conics that is invariant under a linear action of a finite group on $\mathbb{CP}^2$. This can be thought of as spearheading equivariant enumerative enrichments valued in the Burnside Ring, both inspired by and a departure from $R(G)$-valued enrichments such as Roberts' equivariant Milnor number and Damon's equivariant signature formula. Given a G-invariant general pencil of conics, the weighted sum of nodal orbits in the pencil is a formula in terms of the base locus considered as a G-set. We show this is true for all finite groups except $\mathbb{Z}/2 \times \mathbb{Z}/2$ and D_8 and give counterexamples for the two exceptional groups.

1. Introduction

Given a pair of conics in general position in $\mathbb{CP}^2$ defined by equations f and g, we can form a family of curves parameterized by $\mathbb{CP}^1$, $X = \{\mu f(x,y,z) + \lambda g(x,y,z) = 0 \colon [\mu, \lambda] \in \mathbb{CP}^1\}$. This is the the pencil of conics spanned by f and g, and choosing different $[\mu, \lambda]$ in $\mathbb{CP}^1$ specifies different conics in the pencil. The set $\Sigma := \{p \in \mathbb{CP}^2 \colon f(p) = g(p) = 0\}$, the base locus of X, contains 4 distinct points since f and g intersect generically. It is natural to ask how many conics in X are nodal. As long as f and g are in general position, there are always exactly $\#\Sigma - 1 = 3$ nodal conics in X. This is a case of Göttsche's conjecture [Got98], proved first by Y. Tzeng in 2010 in [Tze12], with another proof given by Kool, Schende, and Thomas [KST11].

Rather than ask for the number of nodal conics, one can ask if there is a description *orbits* of nodal conics under the presence of a finite group action on $\mathbb{CP}^2$ under which the pencil is invariant. This work gives a formula for the count of nodal orbits when the group G acting on $\mathbb{CP}^2$ is not isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8. We define the G-weight of a nodal orbit, a first case of a Burnside-valued Milnor number inspired by the $R(G)$-valued Milnor number of [Rob85] and signature formula of [Dam91]. The main result then shows that the sum of these G-weights of nodal orbits is a formula in Σ in the Burnside Ring, $A(G)$.

To be more precise, let G be a finite group that acts linearly on $\mathbb{CP}^2$, and let f and g be equations defining a general pair of conics in $\mathbb{CP}^2$ such that the corresponding pencil, X, is G-invariant. From this linear action on $\mathbb{CP}^2$, we obtain an action on $\mathrm{Sym}^2(\mathbb{C}^3)^\vee = \mathrm{Span}\{x^2, y^2, z^2, xy, xz, yz\}$. X being G-invariant means that $g \cdot C_t$ is another conic in X for all g in G and for all conics C_t in X, where we've written $t = [\mu : \lambda] \in \mathbb{CP}^1$ for simplicity so that C_t is the conic obtained by specializing at t. Equivalently, X is G-invariant if $\langle f, g \rangle$ is a G-invariant subspace of $\mathrm{Sym}^2(\mathbb{C}^3)^\vee$. With this setup, one can ask for a Burnside-valued formula of G-sets counting *orbits* of nodal conics in the G-space X. Given such a formula, we can take the cardinality of the H-fixed points of the formula for any subgroup H of G to obtain the integer count of nodal conics that are fixed by H.

The question of whether there is a Burnside-valued formula is answered affirmatively in Theorem 7. We state and the main theorem here, and prove it in Section 4:

Theorem 1. *Let G be a finite group not isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8, and assume G acts linearly on $\mathbb{CP}^2$. Let X be a G-invariant pencil spanned by a pair of conics in general position in $\mathbb{CP}^2$, and let $[\Sigma]$ in $A(G)$ represent the base locus of X. Then*

$$\sum_{\substack{G\cdot C_t, \\ C_t \in X \text{ is nodal}}} \mathrm{wt}^G(C_t) = [\Sigma] - \{*\} \tag{2}$$

in $A(G)$. That is, there is a weighted count of nodal orbits in X, valued in the Burnside ring of G.

For any subgroup H of G, the cardinality of

$$([\Sigma] - \{*\})^H$$

is equal to the number of nodal conics in X that are fixed by H. In particular, we recover the classical count of $\#\Sigma - 1 = 3$ nodal conics by taking H to be the trivial subgroup, and we recover the number of nodal conics that are fixed under the action on $\mathbb{CP}^2$ by taking G-fixed points. In this sense, the equivariant enrichment in Theorem 1 is a direct generalization of the classical result counting $\#\Sigma - 1$ nodal conics in a general pencil.

The Burnside Ring, $A(G)$, is the Grothendieck ring constructed from the monoid of G-isomorphism classes of finite G-sets, with addition given by disjoint union and ring structure given by Cartesian product. Equivariant formulas of G-sets should be valued in $A(G)$, as above, as $A(G)$ distinguishes equivariant homotopy classes of endomorphisms of G-representation spheres. Specifically,

$$\deg^G\colon\ \left[S^V, S^V\right]^G \xrightarrow{\sim} A(G)$$

is an isomorphism (see [Seg70]), analogous to $\deg\colon\ [S^n, S^n] \xrightarrow{\sim} \mathbb{Z}$ being an isomorphism, which motivates the replacement of $\mathbb{Z}$ by $A(G)$ as the ring of definition for equivariant enumerative results. Further description of the Burnside ring is given in Section 2.

The weighting convention for nodal orbits in X appearing in the left-hand side of equation (2) is defined in Section 4 before the main theorem is restated, and it generalizes the real sign of a node in the sense that the G-fixed point cardinality of the weight of a non-split node is $+1$, likewise -1 for a split node, when $G = \mathbb{Z}/2$ acts on $\mathbb{CP}^2$ by conjugation.

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2. Notation and definitions

This section will introduce all definitions related to the Burnside Ring so that this paper will be self-contained, following [tD79]. We will always assume G is a finite group, and all group actions are assumed to be left actions. Given G-sets S and T, a set map $f\colon S \to T$ is G-equivariant if $g \cdot f(s) = f(g \cdot s)$ for all g in G. Given a G-set S and a subset S' of S, we will say S' is G-invariant if $g \cdot s'$ is in S' for all s' in S' and g in G.

Given any two G-sets S and T, there are natural set operations on S and T from which other G-sets can be obtained. We can take the disjoint union of S and T, $S \amalg T$, or the Cartesian product, $S \times T$, and obtain G-sets by letting G act diagonally in both cases. Let $A(G)^+$ denote the semi-ring of G-isomorphism classes of finite G-sets with addition given by disjoint union and multiplication by Cartesian product, where G-isomorphism means an isomorphism of sets that is G-equivariant.

Definition 3. *Given a group G, the Burnside ring of G is the Grothendieck ring associated to $A(G)^+$, denoted $A(G)$.*

Additively, $A(G)$ is the free abelian group on isomorphism classes of transitive G-sets of the form $[G/H]$ for subgroups H of G. Given any G-set S, we will denote its class in $A(G)$ by $[S]$. The set $\{*\}$ will always denote the one-point set, which can only be given the trivial action. Any G-set which comes from a genuine set with a group action will be called a genuine G-set. This is in contrast to a virtual G-set, for example, $-\{*\}$ denotes the virtual G-set that is the formal additive inverse of the G-set $\{*\}$. Further literature on the Burnside ring is rich, a standard reference being [tD79].

Any genuine finite G-set can be written as the disjoint union of its orbits $\amalg G/H_i$ where $\{H_i\}$ is some finite collection of subgroups of G. Furthermore, the isomorphism class of $[G/H]$ in $A(G)$ is determined by the conjugacy class of H in G. Thus every genuine G-set $[S]$ in $A(G)$ can be written as

$$[S] = \sum_{H_i \leq G} n_i [G/H_i]$$

for some positive integers n_i, uniquely up to (H_i) for each H_i. A well-known result in equivariant topology, see Proposition 1.2.2 in [tD79], says that two isomorphism classes of finite G-sets, $[S_1]$ and $[S_2]$, are equal in $A(G)$ if $|(S_1)^H| = |(S_2)^H|$ for all subgroups H of G. These facts are simple to state but useful in practice, and will be important in proving the main result and constructing counterexamples.

Given G-sets S and T, we have already described two ways of producing new G-sets with disjoint union and Cartesian product. Given a finite group G and a subgroup H of G, we use the inflation from H to G as a change of group method to obtain a G-set from an H-set.

Definition 4. *Given a subgroup H of G and an H-set X, we define a G-set with underlying set structure $(G \times X)/\sim$, where $(gh, x) \sim (g, hx)$ for all h in H, and x in X. The* inflation of X from H to G, *denoted* $\inf_H^G(X)$, *is the G-set $(G \times X)/\sim$ with G-action given by $g' \cdot (g, x) = (g'g, x)$ for all $g' \in G$ and $(g, x) \in \inf_H^G(X)$.*

Every genuine G-set we will encounter in this paper will already be represented as a formal sum of orbits, each equal to $[G/H]$ in $A(G)$ for some subgroup H of G. Thus it will be useful to have a description of the inflation of an H-set to G when represented by a sum of orbits of this form. The following lemma gives such a description.

Lemma 5. *Let H be a subgroup of a finite group G and let $[X]$ be a finite H-set the form*

$$[X] = \sum_{i=1}^{m} n_i [H/K_i]$$

in $A(H)$ for some $m \in \mathbb{N}$, some $n_i \in \mathbb{Z}$, and $K_i \leq H$ some finite collection of subgroups of H. Then $\inf_H^G(X) = \sum_{i=1}^m n_i [G/K_i]$ in $A(G)$.

Proof. First note

$$\inf_H^G\left(\sum_{i=1}^{m} n_i [H/K_i]\right) = \sum_{i=1}^{m} n_i \inf_H^G([H/K_i])$$

because Cartesian products commute with disjoint unions and the action on a disjoint union is assumed to be diagonal. Thus we only need to show that $\inf_H^G(H/K) = [G/K]$ in $A(G)$ for any $K \leq H$, i.e., by defining a set isomorphism $\inf_H^G(H/K) \to G/K$ and showing it is G-equivariant.

Define $f\colon \inf_H^G(H/K) \to G/K$ by $f((g, hK)) = ghK$. It is straightforward to check that f is well-defined, injective, and surjective as a set function. The last step to show $\inf_H^G(H/K) = [G/K]$ in $A(G)$ is to check f is G-equivariant. This is true by definition, as

$$g' \cdot f((g, hK)) = g'ghK = f((g'g, hK)) = f(g' \cdot (g, hK))$$

for all g' in G and (g, hK) in $\inf_H^G(H/K)$. □

3. Proof of the Classical Result

Before proving the main theorem giving an equivariant enrichment of the count of nodal orbits as equal to $\#\Sigma - 1 = 3$, we'll sketch a topological proof of the classical result. Let f and g be a general pair of conics in $\mathbb{CP}^2$, and let

$$X := \{\mu f + \lambda g = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\} \subseteq \mathbb{CP}^2$$

be the pencil of conics defined by f and g. Let

$$X_{tot} := \{(t, p)\colon \mu f(p) + \lambda g(p) = 0\} \subseteq \mathbb{CP}^1 \times \mathbb{CP}^2$$

be the total space of X. We have two projections from X_{tot}, $\pi_1\colon X_{tot} \to \mathbb{CP}^1$ by projecting onto the first coordinate and $\pi_2\colon X_{tot} \to \mathbb{CP}^2$ by projecting onto the second coordinate. We will compute $\chi(X_{tot})$ in two ways and set them equal to obtain the number of nodal conics.

First we will compute $\chi(X_{tot})$ using the projection $\pi_1 \colon X_{tot} \to \mathbb{CP}^1$. Let

$$D := \{[\mu, \lambda] \colon \mu f + \lambda g = 0\} \subseteq \mathbb{CP}^1$$

be the set of points in $\mathbb{CP}^1$ that specify a nodal conic in X. Note that $\#D$ is equal to the number of singular conics in X, which is what we want to find. The fibers of π_1 over D are singular conics, and the fibers over $\mathbb{CP}^1 - D$ are smooth conics. Using the fact that the compactly supported Euler characteristic is additive over X_{tot} as the disjoint union of fibers over D and fibers over $\mathbb{CP}^1 - D$, we have

$$\begin{aligned}\chi(X_{tot}) &= \chi(X_{tot}|D) + \chi(X_{tot}|\mathbb{CP}^1 - D)\\ &= \chi(C_{sing}) \cdot \chi(D) + \chi(C_{sm}) \cdot \chi(\mathbb{CP}^1 - D)\end{aligned}$$

where C_{sm} denotes any smooth conic in a fiber over $\mathbb{CP}^1 - D$ and C_{sing} denotes any singular conic in a fiber over D. This uses the topological Hurwitz formula: if $E \to B$ is a fiber bundle with fiber F and B is path connected, then $\chi(F) \cdot \chi(B) = \chi(E)$.

The Euler characteristic of a smooth projective curve is $2 - 2g$ where g is the genus, and the Euler characteristic of a singular curve is $2 - 2g + \mu(C_{sing})$ where $\mu(C)$ is the Milnor number of a curve C. Since conics have genus 0 and the Milnor number of a nodal conic is 1, we have

$$\begin{aligned}\chi(X_{tot}) &= \chi(C_{sing}) \cdot \chi(D) + \chi(C_{sm}) \cdot \chi(\mathbb{CP}^1 - D)\\ &= \#D(2 + \mu(C_{sing})) + 2(2 - \#D)\\ &= \#D + 4.\end{aligned}$$

The second way to compute $\chi(X_{tot})$ is to use the fact that $\pi_2 \colon X_{tot} \to \mathbb{CP}^2$ is the blow-up of $\mathbb{CP}^2$ at the $d^2 = 4$ points of the base locus $\Sigma := \{p \in \mathbb{CP}^2 \colon f(p) = g(p) = 0\}$, where $d = 2$ is the degree of f and g as homogenous polynomials in three variables. Again using additivity for the compactly supported Euler characteristic, we have

$$\begin{aligned}\chi(X_{tot}) &= \chi(\mathbb{CP}^2) + \chi(\Sigma)(\chi(\mathbb{CP}^1) - \chi(pt))\\ &= 3 + 4(2 - 1)\\ &= 7.\end{aligned}$$

Combining the two calculations of $\chi(X_{tot})$ we get

$$\#D + 4 = 7,$$

and we conclude that the number of nodal conics in X is $\#D = 3$. This approach works equally well for generically intersecting curves in higher degree d. Another proof can obtained by taking the degree of the top chern class of the bundle of principle parts on $\mathcal{O}(d)$, both approaches can be found in detail in [EH16, Chapter 7]. As mentioned in the introduction, it is worth noting that another way to write the formula for $\#D$ is

$$\#D = \#\Sigma - 1,$$

which motivates the form of Equation (2) in Theorem 1.

Another proof of the same result can be described as follows. If f and g define a general pair of conics, then they intersect in exactly the four points of Σ. A nodal conic geometrically has irreducible components equal to a pair of lines, and the generic intersection assumption on f and g rules out the possibility that the two curves share a common line. Thus, asking how many conics in X are nodal is equivalent to asking how many ways there are to draw a pair of distinct lines through four points in $\mathbb{CP}^2$, which is three. Labeling the points of Σ as b_1, b_2, b_3, and b_4 and writing L_{ij} for the line through b_i and b_j, the three pairs of lines are $\{L_{12}, L_{34}\}$, $\{L_{13}, L_{24}\}$, and $\{L_{14}, L_{23}\}$:

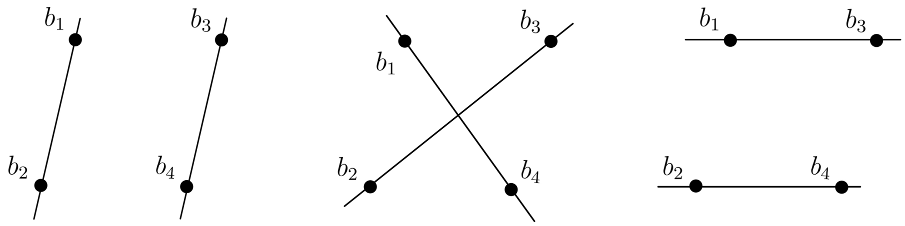

FIGURE 1. Disjoint lines through $\Sigma = \{b_1, b_2, b_3, b_4\}$

This way of thinking of the number of nodal conics in X will be useful to us going forward. When describing the G-orbits of nodal conics, we can instead look at orbits of configurations of disjoint lines through Σ.

It's worth noting that any set of four points in $\mathbb{P}^2$ with no three co-linear uniquely determines a pair of general conics. Indeed, the vector space of conics in $\mathbb{CP}^2$ is 5-dimensional, $\text{Span}_{\mathbb{C}}\{x^2, y^2, z^2, xy, xz, yz\}$. Requiring that a conic passes through a point imposes a 1-dimensional condition on the space of conics, so requiring that a conic passes through four points results in a 1-dimensional linear span of conics, or a 2-dimensional projective span of conics, i.e. a pencil of conics. Therefore any Σ which is a set of four points in $\mathbb{CP}^2$ with no three co-linear uniquely determines a pencil of conics.

4. An Equivariant Count of Orbits of Nodal Conics

Let f and g be a general pair of conics and let $X := \{\mu f + \lambda g = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\} \subseteq \mathbb{CP}^2$. Henceforth for simplicity of notation we will write $t = [\mu : \lambda]$ so that $C_t \in X$ denotes the element of X obtained by specifying $[\mu, \lambda]$ in $\mathbb{CP}^1$. Given a nodal conic C_t in X, we will write B_t to denote the irreducible components of C_t. Thus $B_t = \{L_1, L_2\}$ is the set of branches of C_t if C_t is a nodal conic that can be parameterized as the product of lines $L_1 \cdot L_2$ at the nodal point $p \in C_t$.

Definition 6. *Let C_t be a nodal conic in a G-invariant pencil of conics, and let $H \leq G$ be the stabilizer of C_t. Define the H-weight of C_t to be*

$$\text{wt}^H(C_t) := [B_t] - \{*\}$$

in $A(H)$ where $[B_t]$ denotes the branches of C_t as an H-set in $A(H)$. The G-weight of the orbit $G \cdot C_t$ is defined to be

$$\mathrm{wt}^G(C_t) := \inf_H^G(\mathrm{wt}^H(C_t))$$

in $A(G)$.

Given nodal conics C_s and C_t in the same orbit it is straightforward to check that $\mathrm{wt}^G(C_t) = \mathrm{wt}^G(C_s)$ in $A(G)$ using Lemma 5, so the weight of an orbit is well-defined. When the action of G on $\mathbb{CP}^2$ is trivial, $\mathrm{stab}(C_t) = G$ for all nodal C_t in X, and so $\mathrm{wt}^G(C_t) = [B_t] - \{*\}$. Since the action is trivial, the branches in $[B_t]$ are fixed and $[B_t] = \{*\}$ in $A(G)$. Thus $\mathrm{wt}^G(C_t) = 2\{*\} - \{*\}$ has cardinality $2 - 1 = 1$. Therefore the cardinality of $\sum \mathrm{wt}^G(C_t)$ is 3, recovering the classical result that there are 3 nodal conics in a pencil spanned by two conics in general position.

This is also true even when the action isn't trivial, though a nodal orbit $[C_t]$ might contain multiple conics with branches that are not fixed. Rather than taking the cardinality of $\sum \mathrm{wt}^G(C_t)$, we could also take the cardinality of the H-fixed points of $\sum \mathrm{wt}^G(C_t)$ for any subgroup of H of G, and this is not guaranteed to be 3. If we let $\mathbb{Z}/2$ act on $\mathbb{CP}^2$ by pointwise conjugation and take $\mathbb{Z}/2$ fixed points, we recover the weighted count of nodal conics in X defined over $\mathbb{R}$ up to a sign, weighting a split node by -1 and a non-split node as $+1$. Non-equivariantly, the number of real conics in a pencil is $-(\#\Sigma(\mathbb{R}) - 1)$ rather than $\#\Sigma - 1$.

The formula relating the base locus with the weighted sum of nodal orbits is stated in the main theorem:

Theorem 7. *Let G be a finite group not isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8, and assume G acts linearly on $\mathbb{CP}^2$. Let X be a G-invariant pencil spanned by a pair of conics in general position in $\mathbb{CP}^2$, and let $[\Sigma]$ in $A(G)$ represent the base locus of X. Then*

$$\sum_{\{G \cdot C_t \,:\, C_t \in X \text{ is nodal}\}} \mathrm{wt}^G(C_t) = [\Sigma] - \{*\} \tag{8}$$

in $A(G)$. That is, there is a weighted count of orbits of nodal conics in X, valued in the Burnside ring of G.

This can be proved directly by explicitly checking that the formula holds for all possible invariant pencils of conics.

Proof. We will prove the theorem is true for each finite group G that can act linearly on $\mathbb{CP}^2$ and invariantly on a pencil of conics. Any such group must be a finite subgroup of $PGL(3, \mathbb{C})$, a reference for which can be found in [HL88]. If G is a finite group that acts linearly on $\mathbb{CP}^2$ and invariantly on a pencil of conics, then G must fix the base locus of the pencil, i.e., G must act bijectively on a set of four distinct points. Thus we only need to consider linear group actions of subgroups of S_4, which is indeed a subgroup of $PGL(3, \mathbb{C})$.

It is well known that if $H_1, H_2 \leq G$ are conjugate subgroups of a finite group G, then $A(H_1) \cong A(H_2)$, a proof can be found in [Bou00]. Thus we will only check that the theorem is true for each conjugacy class of subgroups of S_4. These are:

$$\langle()\rangle \qquad \mathbb{Z}/2 \cong \langle(12)\rangle \qquad S_3 \cong \langle(123),(12)\rangle$$
$$A_3 = \{(),(123),(132)\} \qquad \mathbb{Z}/4 \cong \langle(1234)\rangle \qquad A_4 = \langle(123),(12)(34)\rangle$$
$$\mathbb{Z}/2\times\mathbb{Z}/2 \cong \langle(12)(34),(13)(24)\rangle \qquad D_8 \cong \langle(1234),(13)\rangle \qquad S_4.$$

For each of these groups except $\mathbb{Z}/2\times\mathbb{Z}/2$ and D_8, one can directly show that the theorem is true by computing weights of orbits of lines through $[\Sigma]$. In the next section, we will provide counterexamples for $\mathbb{Z}/2\times\mathbb{Z}/2$ and D_8 and an explanation for why these cases fail.

We will write $[\Sigma] = \{b_1, b_2, b_3, b_4\} \in A(G)$ for the base locus of a pencil, and the line through any b_i and b_j will be denoted by L_{ij}. Any nodal conic through $[\Sigma]$ has irreducible components given by the union of a pair of lines $\{L_{ij}, L_{kl}\}$, which will be denoted $[L_{ij}, L_{kl}]$ in $A(G)$.

The set of G-invariant general pencils of conics in $\mathbb{CP}^2$ is in bijection with the set of G-invariant collections of four points in $\mathbb{CP}^2$ with no three co-linear by a vector space argument: Every G-invariant pencil of general conics in $\mathbb{CP}^2$ uniquely determines a G-set of four points satisfying the linearity condition. Separately, every G-set $[\Sigma]$ of four points in $\mathbb{CP}^2$ satisfying the linearity condition uniquely determines at most one pencil of general conics, which is G-invariant as the unique 1-dimensional subspace of $\mathbb{P}\mathrm{Sym}^2((\mathbb{C}^3)^\vee)$ corresponding to Σ is G-invariant. Showing for each subgroup G of S_4 that equation (8) holds for any possible configuration of $[\Sigma] \in A(G)$ will prove the theorem. We will show all of the details for $\mathbb{Z}/2$, S_3, and an interesting case for A_4. The same methods can be used verbatim for A_3, $Z/4$, and S_4.

If $G = \langle()\rangle$ is the trivial group, then any group action on $\mathbb{CP}^2$ is trivial. Thus this is simply the classical result over $\mathbb{C}$.

If $G = \mathbb{Z}/2 \cong \langle(12)\rangle$, the only genuine size four G-sets, and therefore the only possible choices for $[\Sigma]$ in $A(G)$, are the following:

(1) $[\Sigma] = 4\{*\}$
(2) $[\Sigma] = 2[G]$
(3) $[\Sigma] = 2\{*\} + [G]$

The fact that $[\Sigma]$ must be one of these cases relies on the fact that any genuine G-set $[S] \in A(G)$ has the form

$$[S] = \sum_{(H_i):\ H_i \leq G} n_i[G/H_i] = n_0[G/G] + n_1[G/\langle()\rangle]$$

with $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ being the number of orbits with stabilizer equal to G or $()$ respectively. Since $[\Sigma]$ is a genuine G-set, it must have one of the three configurations listed above.

Given a configuration of $[\Sigma]$, if there is a G-invariant pencil of conics X determined by $[\Sigma]$, then the set of irreducible components of any nodal conic in X is determined by one of the three configurations of a pair of distinct lines through $[\Sigma]$. Thus to see that the theorem is true for every configuration of $[\Sigma]$, and therefore true for $G = \mathbb{Z}/2$, we will compute the weight of each orbit of lines through any configuration of $[\Sigma]$ and show that the sum of the weights is equal to $[\Sigma] - \{*\}$ in $A(G)$.

First consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12}, L_{34}]) = \mathrm{stab}([L_{13}, L_{24}]) = \mathrm{stab}([L_{14}, L_{23}]) = G$$

and each branch is fixed. Hence $\mathrm{wt}^G([L_{ij}, L_{kl}]) = \{[L_{ij}, L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any all $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (8) is $\sum \mathrm{wt}^G(B_t) = 3\{*\}$, and the right-hand side of equation (8) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

Consider the second case where $[\Sigma] = 2[G]$, and say that $\{b_1, b_2\}$ and $\{b_3, b_4\}$ are the orbits of $[\Sigma]$. In this case, $()$ is the element that acts trivially and (12) is the element that acts nontrivially on each orbit, i.e., swaps b_1 and b_2 and swaps b_3 and b_4. Then for $g \in G$,

$$g \cdot \{L_{12}, L_{34}\} = \begin{cases} L_{12}, L_{34}, & \text{for } g = () \\ L_{21}, L_{43}, & \text{for } g = (12) \end{cases} \qquad g \cdot \{L_{13}, L_{24}\} = \begin{cases} L_{13}, L_{24}, & \text{for } g = () \\ L_{24}, L_{13}, & \text{for } g = (12) \end{cases}$$

$$g \cdot \{L_{14}, L_{23}\} = \begin{cases} L_{14}, L_{23}, & \text{for } g = () \\ L_{23}, L_{14}, & \text{for } g = (12) \end{cases}.$$

The stabilizer of each nodal orbit is G, and so $\mathrm{wt}^G([L_{ij}, L_{kl}]) = [L_{ij}, L_{kl}] - \{*\}$. Note that $[L_{12}, L_{34}] = 2\{*\}$ because the branches are fixed by G, but $[L_{13}, L_{24}]$ and $[L_{14}, L_{23}]$ are both equal to $[G]$ in $A(G)$ because the branches are swapped by G. Hence

$$\mathrm{wt}^G([L_{12}, L_{34}]) = [L_{12}, L_{34}] - \{*\} = 2\{*\} - \{*\} = \{*\},$$

$$\mathrm{wt}^G([L_{13}, L_{24}]) = [L_{13}, L_{24}] - \{*\} = [G] - \{*\}, \text{ and}$$

$$\mathrm{wt}^G([L_{14}, L_{23}]) = [L_{14}, L_{13}] - \{*\} = [G] - \{*\}.$$

Thus the left-hand side of equation (8) is $\{*\} + 2[G] - 2\{*\} = 2[G] - \{*\}$, and the right-hand side of equation (8) is $[\Sigma] - \{*\} = 2[G] - \{*\}$, as desired.

The last configuration of $[\Sigma]$ is $2\{*\} + [G]$. Say that b_1 and b_2 are the fixed points and $\{b_3, b_4\}$ are an orbit with (12) swapping b_3 and b_4. Thus

$$g \cdot \{L_{12}, L_{34}\} = \begin{cases} L_{12}, L_{34}, & \text{for } g = () \\ L_{12}, L_{43}, & \text{for } g = (12) \end{cases} \qquad g \cdot \{L_{13}, L_{24}\} = \begin{cases} L_{13}, L_{24}, & \text{for } g = () \\ L_{14}, L_{23}, & \text{for } g = (12) \end{cases}$$

$$g \cdot \{L_{14}, L_{23}\} = \begin{cases} L_{14}, L_{23}, & \text{for } g = () \\ L_{13}, L_{24}, & \text{for } g = (12) \end{cases}.$$

Here, $\mathrm{stab}([L_{12}, L_{34}]) = G$ and both lines are fixed, so

$$\mathrm{wt}^G([L_{12}, L_{34}]) = [L_{12}, L_{34}] - \{*\} = 2\{*\} - \{*\} = \{*\}.$$

Note that $\text{stab}([L_{13}, L_{24}]) = \text{stab}([L_{14}, L_{23}]) = \langle () \rangle$. Furthermore, $(12) \cdot \{L_{13}, L_{24}\} = \{L_{14}, L_{23}\}$ and $(12) \cdot \{L_{14}, L_{23}\} = \{L_{13}, L_{24}\}$, so they are both in the same orbit. Therefore we only need to count one of $G \cdot \{L_{13}, L_{24}\}$ or $G \cdot \{L_{14}, L_{23}\}$ in the weighted sum of nodal curves in the pencil determined by $[\Sigma]$. Making an arbitrary choice and using Lemma 5,

$$\begin{aligned} \text{wt}^G([L_{13}, L_{24}]) = \inf^G_{\langle () \rangle}(\text{wt}^{\langle () \rangle}([L_{13}, L_{24}])) &= \inf^G_{\langle () \rangle}(2\{*\} - \{*\}) \\ &= \inf^G_{\langle () \rangle}(\{*\}) \\ &= [G/\langle () \rangle] = [G]. \end{aligned}$$

Finally, the left-hand side of equation (8) is $\text{wt}^G([L_{12}, L_{34}]) + \text{wt}^G([L_{13}, L_{24}]) = \{*\} + [G]$ and the right-hand side of equation (8) is $[\Sigma] - \{*\} = [G] + 2\{*\} - \{*\} = [G] + \{*\}$, as desired. Therefore the theorem is true for $\mathbb{Z}/2$.

If $G = S_3 \cong \langle (123), (12) \rangle$, the only possibilities for $[\Sigma]$ in $A(G)$ are:

(1) $[\Sigma] = 4\{*\}$
(2) $[\Sigma] = \{*\} + [G/\langle (12) \rangle]$
(3) $[\Sigma] = 2\{*\} + [G/\langle (123) \rangle]$

The first case has been covered before, and is the same as the $\mathbb{Z}/2$ case when $[\Sigma] = 4\{*\}$.

Consider the second case where $[\Sigma] = \{*\} + [G/\langle (12) \rangle]$. Say b_4 is fixed and $\{b_1, b_2, b_3\}$ are an orbit so that $\{b_1, b_2, b_3\} = [G/\langle (12) \rangle] = \{[()], [(123)], [(132)]\}$ in $A(G)$. Using the same method as for $\mathbb{Z}/2$ to find the stabilizer and orbit of each node, we observe that $\langle (12) \rangle$ is the stabilizer of all three sets of branches through $[\Sigma]$. Furthermore, all nodes are in the same orbit because $(123) \cdot \{L_{12}, L_{34}\} = \{L_{14}, L_{23}\}$, $(123) \cdot \{L_{14}, L_{23}\} = \{L_{13}, L_{24}\}$, and $(123) \cdot \{L_{13}, L_{24}\} = \{L_{12}, L_{34}\}$.

Given that all nodes are in the same orbit, as in the third case for $\mathbb{Z}/2$ we only need to count one weighted node in the orbit to obtain the left-hand side of equation (8). Arbitrarily choosing $[L_{12}, L_{34}]$, the branches of $[L_{12}, L_{34}]$ are equal to $2\{*\}$ in $A(\langle (12) \rangle)$. Thus $\text{wt}^G([L_{12}, L_{34}]) = \inf^G_{\langle (12) \rangle}(2\{*\} - \{*\}) = [G/\langle (12) \rangle]$ Therefore, the left-hand side of equation (8) is $[G/\langle (12) \rangle]$ and the right-hand side of equation (8) is $[\Sigma] - \{*\} = [G/\langle (12) \rangle]$, as desired.

The last case to consider for S_3 is when $[\Sigma] = 2\{*\} + [G/\langle (123) \rangle]$. Say that b_3 and b_4 are fixed and $\{b_1, b_2\}$ is an orbit with $b_1 = [()]$ and $b_2 = [(12)]$. Then $\text{stab}([L_{12}, L_{34}]) = G$ and $\text{wt}^G([L_{12}, L_{34}]) = 2\{*\} - \{*\} = \{*\}$.

We can use the same method used for $\mathbb{Z}/2$ to find the stabilizer and orbit of each remaining node. In this case only one of $[L_{13}, L_{24}]$ or $[L_{14}, L_{23}]$ needs to be counted in the left-hand side of equation (8) because they are in the same orbit with stabilizer $\langle (123) \rangle$. Arbitrarily choosing $[L_{13}, L_{24}]$, we see that $\text{wt}^G([L_{13}, L_{24}]) = \inf^G_{\langle (123) \rangle}(\{*\}) = [G/\langle (123) \rangle]$. Therefore, the left-hand side of equation (8) is $[G/\langle (123) \rangle] + \{*\}$ and the right-hand side of equation (8) is $[\Sigma] - \{*\} = 2\{*\} + [G/\langle (123) \rangle] - \{*\} = [G/\langle (123) \rangle] + \{*\}$, as desired. Therefore the theorem is true for S_3.

If $G = A_3 = \{(), (123), (132)\}$, the only possibilities for $[\Sigma]$ in $A(G)$ are:

(1) $[\Sigma] = 4\{*\}$
(2) $[\Sigma] = \{*\} + [G]$

Both cases can be checked using similar methods as $G = S_3$, no new ideas appear for A_3. The same is true for $G = \mathbb{Z}/4$ and $G = S_4$.

We will show one case for $G = A_4$ to illustrate how to use Proposition 1.2.2 of [tD79] to show two G-sets are equal by showing they have the same number of H-fixed points for all subgroups H of G.

When $G = A_4$, the possible options for $[\Sigma]$ are:

(1) $[\Sigma] = 4\{*\}$
(2) $[\Sigma] = \{*\} + [G/(\mathbb{Z}/2)^2]$, with $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{(), (12)(34), (13)(24), (14)(23)\}$
(3) $[\Sigma] = [G/A_3]$

Consider the last case, $[\Sigma] = [G/A_3]$, and write $[\Sigma] = \{b_1, b_2, b_3, b_4\}$ where $b_1 = [()], b_2 = [(124)], b_3 = [(142)]$, and $b_4 = [(243)]$. Using a similar method as for $\mathbb{Z}/2$ to find the stabilizer and orbit of each node, observe all of L_{12}, L_{34}, L_{13}, L_{24}, and L_{14}, L_{23} are all in the same orbit. Therefore, we only need to count one of $[L_{12}, L_{34}]$, $[L_{13}, L_{24}]$, or $[L_{14}, L_{23}]$ in the left-hand side of equation (8). Making an arbitrary choice, we will count $[L_{12}, L_{34}]$.

The stabilizer of $[L_{12}, L_{34}]$ is $H := \{(), (12)(34), (13)(24), (14)(23)\}$ and is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. One can check directly that the branches of $[L_{12}, L_{34}]$ as an H-set are $[H/\langle(14)(23)\rangle]$. Hence the weight of $[L_{12}, L_{34}]$, and therefore the left-hand side of equation (8), is

$$\begin{aligned} \mathrm{wt}^G([L_{12}, L_{34}]) &= \mathrm{inf}_H^G([\frac{H}{\langle(14)(23)}\rangle] - \{*\}) \\ &= [G/\langle(14)(23)\rangle] - [G/H]. \end{aligned}$$

Since the right-hand side of equation (8) is $[\Sigma] - \{*\} = [G/A_3] - \{*\}$, we need to show that $[G/\langle(14)(23)\rangle] - [G/H]$ and $[G/A_3] - \{*\}$ are equal in $A(G)$. In order to show both sides are equal, we will use Proposition 1.2.2 from [tD79] by showing that for each $K \leq G$, the number of K-fixed points of each side of equation 8 are equal. We will only need to check this for each conjugacy class of subgroups of A_4 since conjugate subgroups have isomorphic Burnside Rings.

Writing S_1 for $[G/\langle(14)(23)\rangle] - [G/H]$ and S_2 for $[G/A_3] - \{*\}$, we record cardinalities of fixed points in the table below:

conjugacy class representative of $K \leq G$	$\lvert (S_1)^K \rvert$	$\lvert (S_2)^K \rvert$
$\langle () \rangle$	3	3
$\mathbb{Z}/2 = \{(), (12)(34)\}$	-1	-1
$H = \{(), (12)(34), (13)(23), (14)(23)\}$	-1	-1
$A_3 = \{(), (123), (132)\}$	0	0
$G = A_4$	-1	-1

Since for each $K \leq G$, the number of K-fixed points of $[G/\langle (14)(23) \rangle] - [G/H]$ and $[G/A_3] - \{*\}$ are equal, the two G-sets are equal in $A(G)$ by [tD79] Proposition 1.2.2. Therefore, equation (8) is true for $G = A_3$ and $[\Sigma] = [G/A_3]$. □

5. Counterexamples

This section will give counterexamples where equation (8) does not hold, which is for groups isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8.

First consider the case when

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \{(), (12)(34), (13)(24), (14)(23)\}.$$

We have an action of S_4, and therefore of G, on $\mathbb{CP}^2$ using the standard $PGL(3, \mathbb{C})$-representation of S_4 given by

$$g_1 := () \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad g_2 := (12)(34) \mapsto \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$g_3 := (13)(24) \mapsto \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad , \text{ and } \quad g_4 := (14)(23) \mapsto \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Consider the point $p = [1, 2, 3] \in \mathbb{CP}^2$. Using the g_i above to also denote the action on $\mathbb{CP}^2$, define the G-set

$$\begin{aligned} [\Sigma] &:= \{b_1 := g_1 \cdot p, b_2 := g_2 \cdot p, b_3 := g_3 \cdot p, b_4 := b_4 \cdot p\} \\ &= \{[1:2:3], [1:2:-1], [1:-2:-1], [-3:-2:-1]\}. \end{aligned}$$

We will show that $[\Sigma]$ is the G-invariant base locus of a general pencil, i.e., that no three points in $[\Sigma]$ are colinear, but that (8) does not hold for the pencil of conics associated to $[\Sigma]$.

Note first that $[\Sigma]$ is G-invariant by construction, with $g \cdot b_i = b_{i+1}$ for $1 \leq i \leq 3$ and $g \cdot b_4 = b_1$ for all $g \neq ()$ in G. Furthermore, $[\Sigma]$ was defined to be isomorphic to G as a G-set, with the isomorphism being given by $b_i \mapsto g_i$ for $1 \leq i \leq 4$. Therefore $[\Sigma] = [G]$ in $A(G)$. It is straightforward to check that no three points in $[\Sigma]$ lie on a line.

Now we will show that equation (8) does not hold for $[\Sigma]$. By observing where each element of G maps each line, we can see that each pair of lines through $[\Sigma]$ has stabilizer equal to G. Each node has branches equal to $[G/H]$ for $H \le G$ the subgroup of G that fixes both branches in addition to the union. Thus one can check that

$$\mathrm{wt}^G([L_{12}, L_{34}]) = [G/\langle(12)(34)\rangle] - \{*\},$$
$$\mathrm{wt}^G([L_{13}, L_{24}]) = [G/\langle(13)(24)\rangle] - \{*\}, \text{ and}$$
$$\mathrm{wt}^G([L_{14}, L_{23}]) = [G/\langle(14)(23)\rangle] - \{*\}.$$

Therefore the left-hand side of (8) is $[G/\langle(12)(34)\rangle] + [G/\langle(13)(24)\rangle] + [G/\langle(14)(23)] - 3\{*\}$. The right-hand side of (8) is $[\Sigma] - \{*\} = [G] - \{*\}$.

We will use Proposition 1.2.2 in [tD79] to determine whether the left-hand and right-hand sides of equation (8) are equal in $A(G)$ as we did to prove Theorem 7 for $G = A_4$. In particular, we need to compute for each $K \le G$ the number of K-fixed points of the left-hand side and the right-hand side of (8). Writing

$$S_1 = [G/\langle(12)(34)\rangle] + [G/\langle(13)(24)\rangle] + [G/\langle(14)(23)] - 3\{*\}$$

and

$$S_2 = [\Sigma] - \{*\} = [G] - \{*\}$$

for the left and right-hand sides of (8) cardinalities of fixed points of subgroups of G are :

conjugacy class of $K \le G$	$\lvert(S_1)^K\rvert$	$\lvert(S_2)^K\rvert$
$\langle()\rangle$	3	3
$\langle(12)(34)\rangle$	-2	-1
$\langle(13)(24)\rangle$	-2	-1
$\langle(14)(23)\rangle$	-2	-1
G	-3	-1

The fact that there are subgroups of G for which the number of fixed points of the LHS and RHS are not equal implies that the two sets are not equal in $A(G)$. Therefore equation (8) fails for $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $[\Sigma] = [G]$.

Finally, we'll construct a counterexample for $G = D_8$ using a different approach. We will start with a 3-dimensional representation of D_8 on $(\mathbb{C}^3)^\vee$ to obtain a 6-dimensional representation of D_8 on $V := \mathrm{Sym}^2((\mathbb{C}^3)^\vee)$. The G-invariant vector space V has a decomposition into irreducible sub-representations using the common eigenspaces of the generators of D_8, and from these irreducible sub-representations the pencils of conics correspond to the spans of irreducible 1-dimensional sub-representations.

Write $r := (13)$ and $s := (1234)$ so that $D_8 = \langle r, s \colon r^2 = s^4 = 1, rxr^{-1} = s^{-1}\rangle$. For reference, the character table of D_8 is given below, where χ_1, χ_2, χ_3, and χ_4 are the four 1-dimensional representations of D_8 and σ is the unique 2-dimensional representation of D_8. The character of any 3-dimensional representation of D_8 is given by $\chi = \sigma + \chi_i$ or $\chi = \chi_i + \chi_j + \chi_k$, $i, j, k \in \{1, 2, 3, 4\}$. We will produce two counterexamples to Theorem 7 using a 3-dimensional representation of W with character $\chi = \sigma + \chi_i$.

Character table of D_8

	e	r^2	r	s	sr
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
σ	2	-2	0	0	0

The unique 2-dimensional representation of D_8 is given by

$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore a 3-dimensional D_8 representation of $(\mathbb{C}^3)^\vee$ with basis $\{x, y, z\}$ and with character $\sigma + \chi_i$ is given by

$$r \mapsto \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} =: M_r, \qquad s \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & b \end{bmatrix} =: M_s$$

where $a, b \in \{\pm 1\}$ are equal to the values of $\operatorname{tr} \chi_i(r)$ and $\operatorname{tr} \chi_i(s)$ respectively. Using the basis $\{x^2, y^2, z^2, yz, xz, xy\}$ for V, observe that the 6-dimensional representation of V obtained from the symmetric power of W is given by

$$r \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \mathrm{Sym}^2(M_r)\ , \quad s \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \mathrm{Sym}^2(M_s).$$

The common 1-dimensional G-invariant eigenspaces of $\mathrm{Sym}^2(M_r)$ and $\mathrm{Sym}^2(M_s)$ are

$$z^2,\ xyx^2 - y^2, \text{ and } x^2 + y^2.$$

There is also a 2-dimensional common G-eigenspace with basis $\{yz, xz\}$. Therefore, the possible G-invariant pencils of conics in $\mathbb{P}^2$ with action coming from the representation of D_8 on V with character $\mathrm{Sym}^2(\sigma + \chi_i)$ are:

(1) $\{\mu YZ + \lambda XZ = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$
(2) $\{\mu Z^2 + \lambda(X^2 - Y^2) = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$
(3) $\{\mu Z^2 + \lambda(X^2 + Y^2) = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$
(4) $\{\mu Z^2 + \lambda XY = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$
(5) $\{\mu(X^2 - Y^2) + \lambda(X^2 + Y^2) = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$
(6) $\{\mu(X^2 - Y^2) + \lambda XY = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$
(7) $\{\mu(X^2 + Y^2) + \lambda XY = 0\colon [\mu, \lambda] \in \mathbb{CP}^1\}$

(8) $\{\mu(X^2 - Y^2) + \lambda(a(X^2+Y^2)+bZ^2) = 0\colon [\mu,\lambda] \in \mathbb{CP}^1\}$
(9) $\{\mu XY + \lambda(a(X^2+Y^2)+bZ^2) = 0\colon [\mu,\lambda] \in \mathbb{CP}^1\}$

In the first 7 cases, one can check that the conics defining the pencil are not in general position. We will show that Theorem 7 doesn't hold for (8) above, and case (9) is similar.

In the 8th case, $[\Sigma] = \{b_1, b_2, b_3, b_4\}$ where

$$b_1 = \left[1 : 1 : i\sqrt{\frac{2a}{b}}\right],\ b_2 = \left[1 : -1 : i\sqrt{\frac{2a}{b}}\right],$$

$$b_3 = \left[1 : 1 : -i\sqrt{\frac{2a}{b}}\right],\ \text{and } b_4 = \left[1 : -1 : -i\sqrt{\frac{2a}{b}}\right].$$

Since the representation on V is the symmetric power of the representation on W given by $r \mapsto M_r$ and $s \mapsto M_s$, with M_r and M_s depending on the values of $a = \operatorname{tr}\chi_i(r)$ and $b = \operatorname{tr}\chi_i(s)$ respectively, the four cases we need to consider are $a = b = 1$, $a = 1$ and $b = -1$, $a = -1$ and $b = 1$, and $a = b = -1$.

We will look at the case when $a = b = 1$, as the others are similar. In this case, using the matrices M_r and M_s one can check that:

$$\begin{array}{cc} () \cdot b_1 = b_1 & (14)(23)\cdot b_1 = b_1 \\ (13)\cdot b_1 = b_2 & (1432)\cdot b_1 = b_2 \\ (13)(24)\cdot b_1 = b_3 & (12)(34)\cdot b_1 = b_3 \\ (1234)\cdot b_1 = b_4 & (24)\cdot b_1 = b_4 \end{array}$$

so that $[\Sigma] = [G/\langle(14)(23)\rangle] = \{b_1 = [()], b_2 = [(13)], b_3 = [(13)(24)], b_4 = [(24)]\}$.

Using the method in the proof of Theorem 7 for finding stabilizers and orbits of each node,

$$\operatorname{stab}([L_{12}, L_{34}]) = \operatorname{stab}([L_{14}, L_{23}]) = \{(), (13)(24), (13), (24)\} := H_1,$$

and $H_1 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Furthermore, L_{12}, L_{34} and L_{14}, L_{23} are in the same orbit in $A(G)$. Therefore we only need to count one of $[L_{12}, L_{34}]$ or $[L_{14}, L_{23}]$ in the left-hand side of equation (8). Arbitrarily choosing $[L_{12}, L_{34}]$, observe that $[L_{12}, L_{34}] = [H_1/\langle(13)\rangle]$ in $A(H_1)$. Therefore,

$$\begin{aligned} \operatorname{wt}^G([L_{12}, L_{34}]) &= \inf{}^G_{H_1}([H_1/\langle(13)\rangle] - \{*\}) \\ &= \left[\frac{H_1}{\langle(13)\rangle}\right] \cdot \left[\frac{G}{H_1}\right] - [G/H_1] \\ &= [G/\langle(13)\rangle] - [G/H_1] \end{aligned}$$

in $A(G)$. We can also observe that $\operatorname{stab}([L_{13}, L_{24}]) = G$, and $[L_{13}, L_{24}] = [G/H_2]$ in $A(G)$ where $H_2 := \{(), (12)(34), (13)(24), (14)(23)\}$. Thus $\operatorname{wt}^G([L_{13}, L_{24}]) = [G/H_2] - \{*\}$.

It is worth noting that $H_1 \cong H_2$ in S_4, but H_1 and H_2 are *not* conjugate in D_8. Therefore the two G-sets $[G/H_1]$ and $[G/H_2]$ are not equal in $A(G)$. The left-hand side of equation (8) is

$$\mathrm{wt}^G([L_{12}, L_{34}]) + \mathrm{wt}^G([L_{13}, L_{24}]) = [G/\langle(13)\rangle] - [G/H_1] + [G/H_2] - \{*\}.$$

Given that $[\Sigma] = [G/\langle(14)(23)\rangle]$, the right-hand side of equation (8) is $[G/\langle(14)(23)\rangle] - \{*\}$.

As with the the counter example for $\mathbb{Z}/2 \times \mathbb{Z}/2$, we will use [tD79] Proposition 1.2.2 to show that Theorem 7 is not true for this case. In particular, we can show that for some $K \leq G$, the number of K-fixed points of the left and right-hand sides of are not equal. Writing $S_1 = [G/\langle(13)\rangle] - [G/H_1] + [G/H_2] - \{*\}$ and $S_2 = [G/\langle(14)(23)\rangle] - \{*\}$ for the right and left-hand sides of (8), fixed point cardinalities are:

conjugacy class of $K \leq G$	$\lvert(S_1)^K\rvert$	$\lvert(S_2)^K\rvert$
$\langle()\rangle$	3	3
G	-1	-1
H_1	-2	-1
H_2	-2	-1
$\langle(1234)\rangle$	-1	-1
$\langle(13)\rangle$	1	-1
$\langle(24)\rangle$	-1	-1
$\langle(13)(24)\rangle$	-1	-1
$\langle(12)(34)\rangle$	-1	-1
$\langle(14)(23)\rangle$	-1	3

The fact that the number of K-fixed points of the left-hand and right-hand sides of equation (8) are not equal for $H_1, H_2, \langle(13)\rangle$, and $\langle(14)(23)\rangle$ implies that the left-hand side and right-hand side are not equal in $A(G)$. Therefore Theorem 7 is not true for D_8.

It is worth noting that even if $[G/H_1] = [G/H_2]$ in $A(G)$, the left-hand side and right-hand side would still not be equal. In that case, the left-hand side of equation (8) would be $[G/\langle(13)\rangle] - \{*\}$ and the right-hand side would be $[\Sigma] - \{*\} = [G/\langle(14)(23)\rangle] - \{*\}$. The same issue arises, $[D_8/\langle(13)\rangle] = [D_8/\langle(14)(23)\rangle]$ in $A(S_4)$ because $\langle(13)\rangle$ and $\langle(14)(23)\rangle$ are conjugate in S_4, but not in D_8. The fact that D_8 has subgroups which are conjugate in S_4 but not in D_8 is the crux of why Theorem 7 fails in this case.

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