Séminaire Lotharingien de Combinatoire, B91Bpref (2024), 1 p.

Marni Mishna, Brendon Rhoades and Raman Sanyal

Preface

FPSAC 2024 was held in Ruhr-Universität Bochum, Bochum (Germany) July 22-26, 2024.

It is our pleasure to have served as program committee co-chairs for FPSAC 2024. In this volume, you will find the extended abstracts for the accepted talks and posters, and in addition a tribute to the life and works of Ian G. Macdonald whose memory was honored by Arun Ram. Together these represent the rich program of the conference.

As co-chairs, we were deeply impressed by the exceptional quality and depth of the submissions, and the compelling talks and presentations by the participants. Such an event only comes together with the work of a dedicated community, and we wish to thank them now. We are grateful to the invited speakers and the participant contributors for bringing cutting edge research in combinatorics to the forefront. The members of the program committee and their secondary reviewers were an essential part of a careful review process, and considered a record number (225) of submissions. This work is difficult, and the high quality of the conference is reliant upon the generous work of these colleagues.

It is a significant logistical task to bring together such a diverse international community. Christian Stump and the organizing committee were meticulous in their organization, considering many small details. The quality of this volume would not have been possible without the work of the proceedings editor, Christian Gaetz.

Such an event relies on significant external funding to remain accessible. We also gratefully acknowledge our sponsors including:

- Ruhr-Universität Bochum
- Deutsche Forschungsgemeinschaft (DFG)
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Séminaire Lotharingien de Combinatoire **91B** (2024), Article 91B.0

Ian G. Macdonald: Works of Art

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Abstract

Ian Macdonald's works changed our perspective on so many parts of algebraic combinatorics and formal power series. This talk will display some selected works of the art of Ian Macdonald, representative of different periods of his œuvre, and analyze how they resonate, both for the past development of our subject and for its future.

Contents

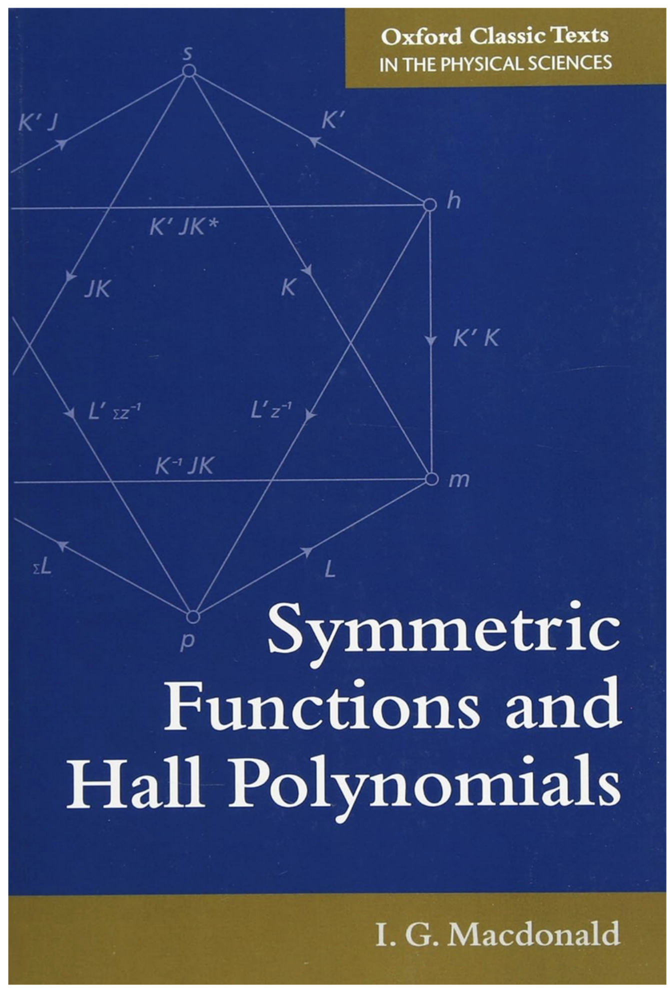
Acknowledgments. First and foremost my thanks go to Ian Macdonald for his teaching, companionship and for giving me bunches of handwritten notes and copies of his books over the years. I am so grateful that circumstances were such that I was able to convey these thanks directly to him in person in June 2023. I thank David Lumsden, Ziheng Zhou, Alex Shields and Dhruv Gupta for energy and insight as we worked through Ian Macdonald's unpublished manuscript on the n-line. I thank Chris Macdonald for reaching out to provide scans and files of the n-line manuscript. I am very grateful to Laura Colmenarejo, Persi Diaconis and Ole Warnaar for helpful suggestions and revisions on this tribute article.

AMS Subject Classifications: Primary 01A70; Secondary 05E05.

1 Preamble

This paper was prepared for the occasion of a lecture in tribute to Ian G. Macdonald, delivered at FPSAC 2024 in Bochum, Germany on 22 July 2024. I want to express thanks to the Executive Committee of FPSAC, the Organizing Committee of FPSAC 2024, and to the whole of our FPSAC 2024 community for making this lecture a possibility and for considering me for its delivery. Macdonald is my hero, and to be asked to play such a role in his legacy touches me deeply.

For this paper I have chosen a few selected topics from Macdonald's immense contributions to highlight (Macdonald polynomials, classification of affine root systems, cohomology and proof of the Weil conjectures for symmetric products of a curve, and the Clifford chain). I hope that you, as reader, will have your own favorites from Macdonald's œuvre and that your choices are equally stimulating, but different from mine. The final sections of this article highlight some of Macdonald's service contributions: as an influencer, as a translator, and as an expositor.



Ian G. Macdonald

The Symmetric Functions Bible

The image of Ian G. Macdonald is from https://sites.google.com/view/garsiafest/mementos

2 Tableaux and Macdonald polynomials

One of our favorite formulas is the formula for the Schur polynomial as a sum over semistandard Young tableaux (SSYTs),

$$s_\lambda = \sum_{T\in B(\lambda)} x^T, \qquad \text{where} \qquad \begin{array}{c} B(\lambda) = \{\text{SSYTs of shape } \lambda\} \\ \text{and} \\ x^T = x_1^{(\#1\text{s in } T)} \cdots x_n^{(\#n\text{s in } T)}. \end{array}$$

It is most amazing that if $\delta = (n-1, \ldots, 2, 1, 0)$ then

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}, \qquad \text{where} \qquad a_\mu = \sum_{w\in S_n} (-1)^{\ell(w)} w x^\mu$$

with $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ if $\mu = (\mu_1, \ldots, \mu_n)$. This second formula for the Schur polynomial is the "Weyl character formula", which (in this type A case) was one of the first definitions of the Schur function (Jacobi 1841, according to Macdonald).

Macdonald pointed out something spectacular. The first formula for the Schur polynomial is the special case $q = t$ of the formula

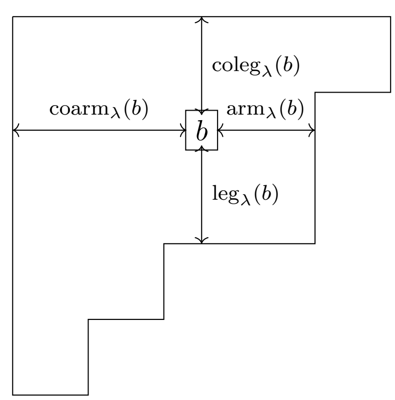
$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t), \qquad \text{where } \psi_T(q,t) \text{ is given by (2.2) below,}$$

and the second formula for the Schur polynomial is a special case of

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}, \qquad \text{where} \quad A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

with T_w and $E_\mu(q,t)$ as defined (2.3) and (2.4) below. Maybe we think Schur polynomials are cool, but the Macdonald polynomials $P_\lambda(q,t)$ are two parameters cooler.

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & 4 & \\ \hline 3 & 4 & & & \\ \hline \end{array}$$



$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t).$$

2.1 (q,t)-hooks and the bosonic Macdonald polynomials $P_\lambda(q,t)$

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$ so that λ is a partition of length at most n.

A *box in* λ is a pair $b = (r,c)$ with $r \in \{1,\ldots,n\}$ and $c \in \{1,\ldots,\lambda_r\}$.

Identify λ with its set of boxes so that

$$\lambda = \{(r,c) \mid (r,c) \text{ is a box in } \lambda\}.$$

For a box $b = (r,c)$ in λ define

$$\begin{aligned} \mathrm{arm}_\lambda(b) &= \mathrm{arm}_\lambda(r,c) = \{(r,c') \in \lambda \mid c' > c\} \qquad \text{and} \\ \mathrm{leg}_\lambda(b) &= \mathrm{leg}_\lambda(r,c) = \{(r',c) \in \lambda \mid r' > r\}. \end{aligned}$$

A *SSYT (semistandard Young tableau) of shape* λ *filled from* $\{1,\ldots,n\}$ is a function

$$T\colon \lambda \to \{1,\ldots,n\} \qquad \text{such that}$$

(a) If $(r,c),(r+1,c) \in \lambda$ then $T(r,c) < T(r+1,c)$, $\begin{array}{|c|}\hline \\ \wedge \\ \hline \\ \hline\end{array}$

(b) If $(r,c),(r,c+1) \in \lambda$ then $T(r,c) \leq T(r,c+1)$. $\begin{array}{|c|c|}\hline & \leq & \\ \hline\end{array}$

Let

$$B(\lambda) = \{\text{SSYTs of shape } \lambda \text{ filled from } \{1,\ldots,n\}\}.$$

Let $T \in B(\lambda)$ and let $b \in \lambda$. Let $T(b)$ denote the entry in box b of T. Let $i \in \{1,\ldots,n\}$ with $i > T(b)$. Define the i-*restricted arm length*, i-*restricted leg length*, and i-*restricted* (q,t)-*hook length* by

$$\begin{aligned} a(b,<i) &= \mathrm{Card}\{b' \in \mathrm{arm}_\lambda(b) \mid T(b') < i\}, \\ l(b,<i) &= \mathrm{Card}\{b' \in \mathrm{leg}_\lambda(b) \mid T(b') < i\}, \end{aligned} \qquad \text{and} \qquad h_T(b,<i) = \frac{1 - t\cdot q^{a(b,<i)}t^{l(b,<i)}}{1 - q\cdot q^{a(b,<i)}t^{l(b,<i)}}. \tag{2.1}$$

For a column strict tableau $T \in B(\lambda)$ define

$$\psi_T(q,t) = \prod_{b\in\lambda} \psi_T(b), \qquad \text{where} \qquad \psi_T(b) = \prod_{\substack{i > T(b), i \in T(\mathrm{arm}_\lambda(b)) \\ i \notin T(\mathrm{leg}\lambda(b))}} \frac{h_T(b,<i)}{h_T(b,<i+1)}. \tag{2.2}$$

The *bosonic Macdonald polynomial* is $P_\lambda(q,t) \in \mathbb{C}[x_1,\ldots,x_n]$ given by

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t), \qquad \text{where} \quad x^T = x_1^{(\#1\text{s in } T)} \cdots x_n^{(\#n\text{s in } T)}.$$

The *Schur polynomial* is

$$s_\lambda = P_\lambda(t,t) = P_\lambda(0,0) = \sum_{T\in B(\lambda)} x^T.$$

2.2 Electronic and fermionic Macdonald polynomials

For $i \in \{1, \ldots, n-1\}$ and $f \in \mathbb{C}[x_1, \ldots, x_n]$ define

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

and

$$T_i f = -t^{-\frac{1}{2}} f + (1+s_i) \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}} f.$$

If $w \in S_n$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w as a product of s_is then write

$$T_w = T_{i_1} \cdots T_{i_\ell} \qquad \text{and} \qquad w = s_{i_1} \cdots s_{i_\ell}. \tag{2.3}$$

For $i \in \{1 \ldots, n-1\}$ let

$$\partial_i = (1+s_i) \frac{1}{x_i - x_{i+1}}.$$

The *electronic Macdonald polynomial* $E_\mu = E_\mu(q,t)$ is recursively determined by

(E0) $E_{(0,\ldots,0)} = 1$,

(E1) $E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1}x_1)$,

(E2) If $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu} = \Big(\partial_i x_i - t x_i \partial_i + \frac{(1-t) q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \Big) E_\mu, \tag{2.4}$$

where $v_\mu \in S_n$ is the minimal length permutation such that $v_\mu \mu$ is weakly increasing.

The monomial x^μ is $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$. The world of Macdonald polynomials replaces the monomials x^μ with electronic Macdonald polynomials E_μ and replaces the action of permutations w by the operators T_w.

Let $\delta = (n-1, n-2, \ldots, 2, 1, 0)$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Let w_0 be the longest element of S_n so that $\ell(w_0) = \binom{n}{2}$. Then define

$$A_{\lambda+\delta}(q,t) = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w E_{\lambda+\delta}(q,t) \qquad \text{and} \qquad a_{\lambda+\delta} = \sum_{w \in S_n} (-1)^{\ell(w)-\ell(w_0)} w x^{\lambda+\delta}.$$

The $A_{\lambda+\delta}(q,t)$ are the *fermionic Macdonald polynomials* (see [CR22, Intro] for motivation for the 'electronic', 'bosonic', 'fermionic' terminology which is in parallel analogy with the isomorphism between Heisenberg algebra representations on fermionic Fock space (an exterior algebra) and bosonic Fock space (a symmetric algebra) which appears, for example, in [Kac, § 14.10]).

2.3 The Weyl character formula

The "Weyl character formula" in the next theorem gives a formula for the bosonic Macdonald polynomial as a quotient of two fermionic Macdonald polynomials. When $q = t = 0$ this formula becomes the formula for the Schur function as a quotient of two determinants.

Theorem 2.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$.*

(a)

$$P_\lambda(q, qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)} \qquad \textit{and} \qquad s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

(b)

$$A_\delta(q,t) = \prod_{i<j} (x_i - t x_j) \qquad \textit{and} \qquad a_\delta = \prod_{i<j} (x_i - x_j).$$

3 Can you do type B?

Having worked something out for type A, a natural next problem for our community is to work it out for type B. Here Macdonald has something interesting to say.

Which type B?

Because, as Macdonald worked out in his 1972 paper on affine root systems,

there are 9 different type Bs.

A diagram showing these is given in Section 3.1.

But, there is something wonderful here. The type $(C^\vee, C)$ root system is one of the type Bs and

all other type Bs are obtained by specializations from type $(C^\vee, C)$.

This means that, if one wants to compute Macdonald polynomials for any one of the 9 different type Bs, then all one has to do, is compute the Macdonald polynomials for type $(C^\vee, C)$ and then specialize parameters as appropriate.

Each of the affine root systems of classical type is a subset of the $\mathbb{Z}$-vector space spanned by symbols $\varepsilon_1, \ldots, \varepsilon_n$ and $\frac{1}{2}\delta$,

$$V_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n, \tfrac{1}{2}\delta\}.$$

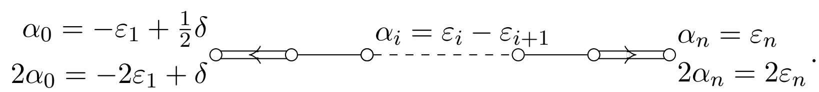
The affine Weyl group W is the group of $\mathbb{Z}$-linear transformations of $V_{\mathbb{Z}}$ generated by the transformations $s_0, s_1, \ldots, s_n$ given by: for $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n + \frac{k}{2}\delta$,

$$\begin{aligned}
s_0\lambda &= -\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n + \left(\tfrac{k}{2} + \lambda_1\right)\delta, \\
s_n\lambda &= \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} - \lambda_n\varepsilon_n + \tfrac{k}{2}\delta, \qquad \text{and} \\
s_i\lambda &= \lambda_1\varepsilon_1 + \cdots + \lambda_{i-1}\varepsilon_{i-1} + \lambda_{i+1}\varepsilon_i + \lambda_i\varepsilon_{i+1} + \lambda_{i+2}\varepsilon_{i+2} + \cdots + \lambda_n\varepsilon_n + \tfrac{k}{2}\delta,
\end{aligned}$$

for $i \in \{1, \ldots, n-1\}$. Each of the affine root systems of classical type is defined by which orbits of the affine Weyl group W that it contains. Let

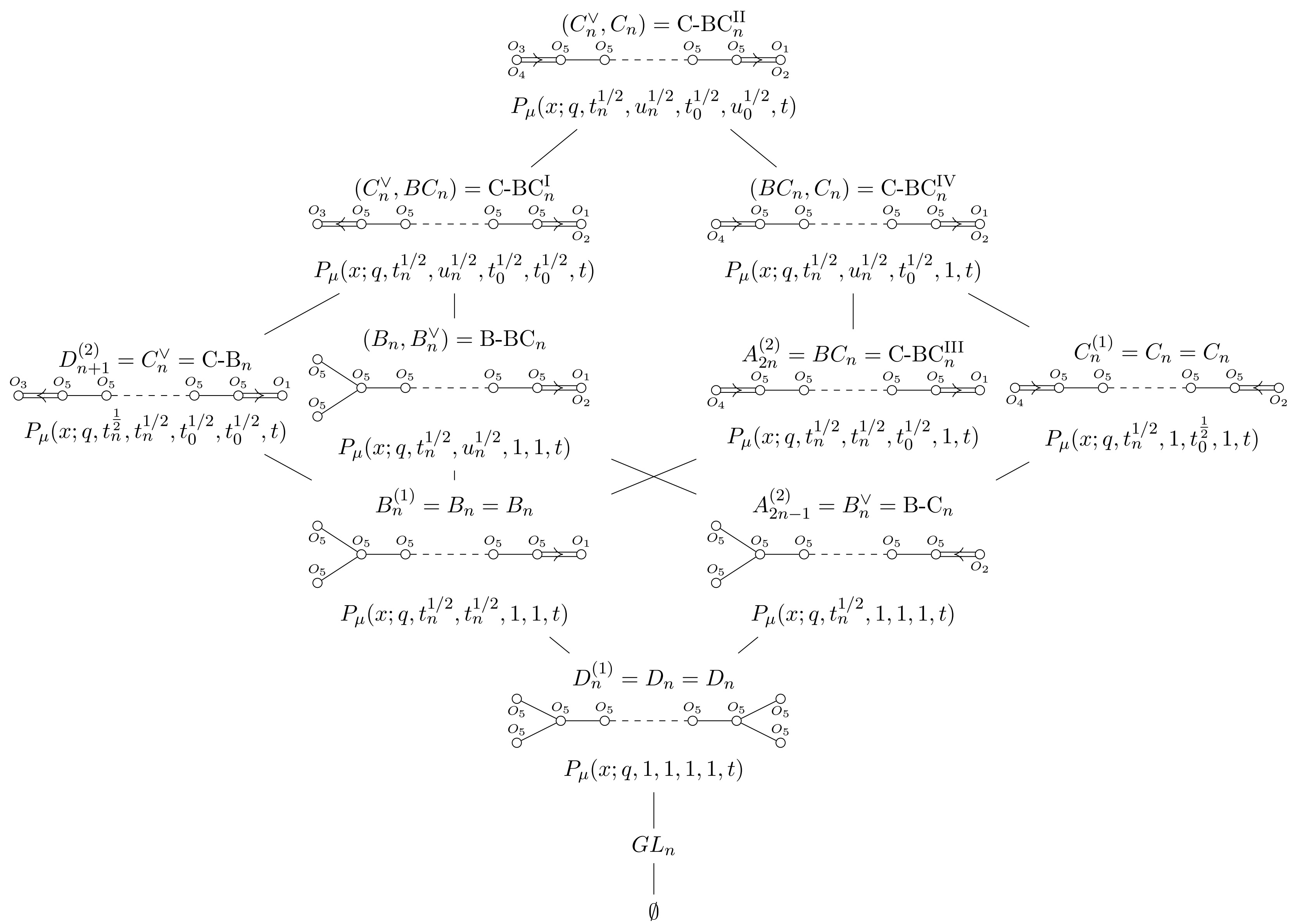
$$\begin{aligned}
O_1 &= W \cdot \alpha_n = W \cdot \varepsilon_n = \{\pm\varepsilon_i + r\delta \mid i \in \{1, \ldots, n\}, r \in \mathbb{Z}\}, \\
O_2 &= W \cdot 2\alpha_n = W \cdot 2\varepsilon_n = \{\pm 2\varepsilon_i + 2r\delta \mid i \in \{1, \ldots, n\}, r \in \mathbb{Z}\}, \\
O_3 &= W \cdot \alpha_0 = W \cdot (-\varepsilon_1 + \tfrac{1}{2}\delta) = \{\pm(\varepsilon_i + \tfrac{1}{2}(2r+1)\delta \mid i \in \{1, \ldots, n\}, r \in \mathbb{Z}\}, \\
O_4 &= W \cdot 2\alpha_0 = W \cdot (-2\varepsilon_1 + \delta) = \{\pm 2\varepsilon_i + (2r+1)\delta \mid i \in \{1, \ldots, n\}, r \in \mathbb{Z}\}, \\
O_5 &= W \cdot \alpha_1 = W \cdot (\varepsilon_1 - \varepsilon_2) = \left\{ \begin{array}{c} \pm(\varepsilon_i + \varepsilon_j) + r\delta \\ \pm(\varepsilon_i - \varepsilon_j) + r\delta \end{array} \;\middle|\; i, j \in \{1, \ldots, n\}, i < j, r \in \mathbb{Z} \right\},
\end{aligned}$$

where



With these notations the irreducible affine root systems of classical type (and the appropriate specializations for obtaining the Macdonald polynomials of each type from the Macdonald polynomials of type $(C^\vee, C)$) are given by the following diagram. The middle notation for each root system is the notation in Macdonald [Mac03, § 1.3], the right notation is that of Bruhat and Tits [BT72] and the left notation is that of Kac [Kac, Ch. 6].

3.1 The poset of affine root systems of classical type



4 Circles and Lines

Though I don't travel often to England, whenever a trip did bring me to England I liked to try to stop in and visit Ian and his wife Greta if I could manage it. Greta passed away in 2019, and I saw Ian at his place two times after that. The last time was in June of 2023. When I first arrived, Ian emphatically told me he hadn't thought about mathematics in 15 years. He pointed to the Sudoku puzzles and the newspapers on his table as evidence. We chatted about mutual friends in mathematics and other memories.

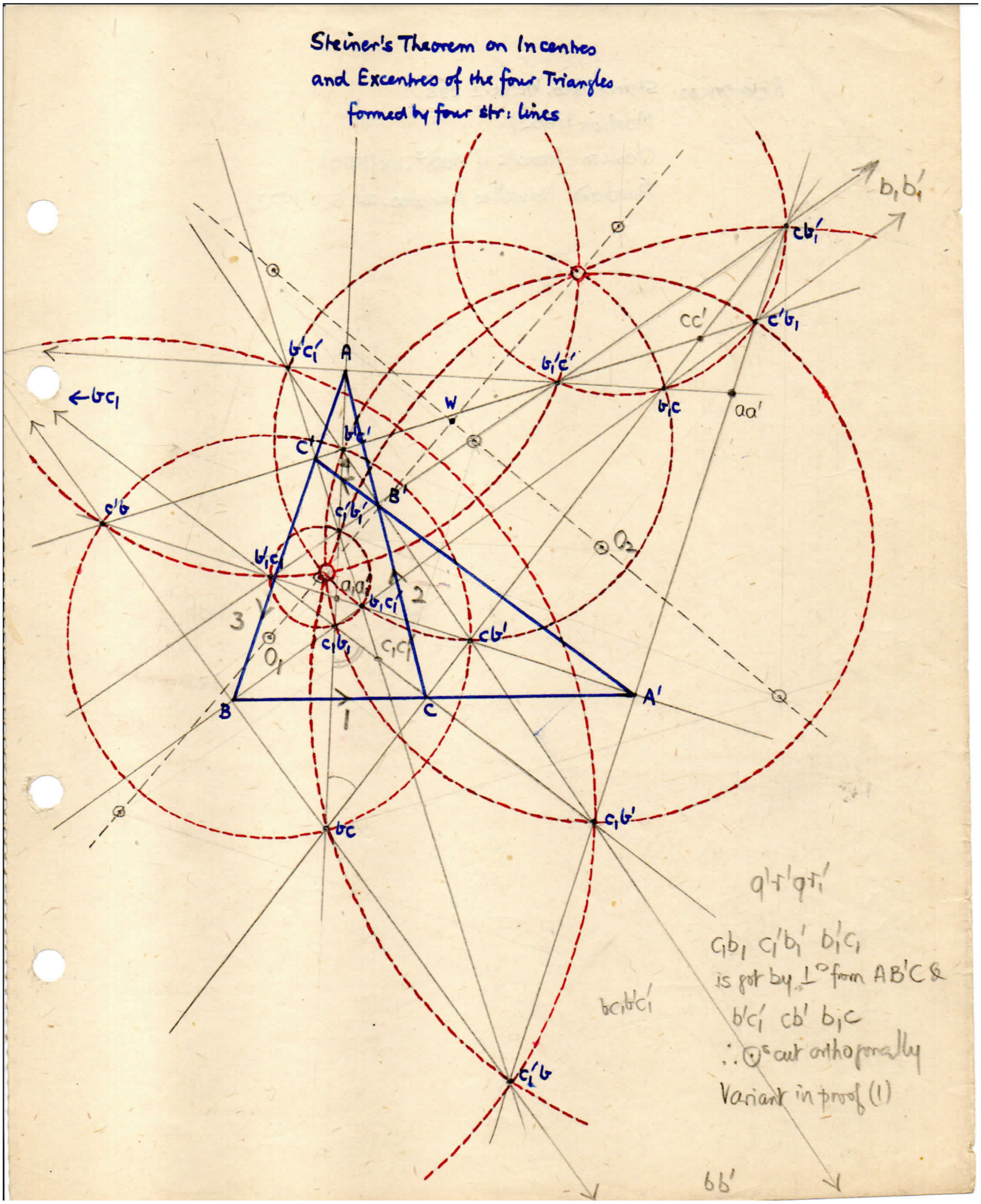
One evening during my visit, Ian and his sister and I went across the road for dinner – fish and chips and beer. During that dinner it came out in conversation with Ian's sister – Ian had indeed recently been fiddling with some mathematics, and Ian told me about the Clifford circle for the n-line. After dinner, when I was back at his place chatting; at some point, Ian lifted himself out of his chair, walked over to the other side of the room, picked up a manuscript, and gifted it to me. He explained that that was what he had been fiddling with and that it was a supplementary chapter to a book he had written just after high school on circles and triangles. It seems that the manuscript to the book was lost, but I was being given the supplementary chapter. I didn't quite know what to make of that, but I carefully packed it in my suitcase for my trip home.

After Ian passed away, his son kindly sent me scans of the original handwritten manuscript of the supplementary chapter and the tex source of the printed copy that Ian gave to me. Since the topic was lines and circles in the plane, I got a few undergraduates together to work through the manuscript.

The author of this manuscript was a talented math student right out of high school. He clearly had not read our key reference for symmetric functions – his notations for symmetric functions are certainly nonstandard for anyone that has read the Symmetric Functions Bible. This student shows a penchant for thorough work and thinking. For the first main theorem appearing in the manuscript he gives 6 or 7 different proofs, all from different points of view, before moving on to generalizations. This high school student is incredibly deft with classical and projective geometry and complex numbers (linear equations, determinants, lemniscates, cardiods, deltoids, Euler lines, coaxal systems, Newton identities for symmetric functions, etc.). Some of the induction proofs are a little bit clumsy – it seems that this student has not been formally taught 'proof by induction' like we might do in a first proof course for undergraduates. The command and thoroughness that this student exhibits extends to his referencing of the literature – in our modern times most of our community has no idea of the main players of classical intersection geometry any more. But this high school student was on top of this literature. If there were one piece of advice that I'd give to this student, it would be to read the books of Ian Macdonald and improve his writing style by emulating the master (admittedly, these books were not yet available).

After getting a feel for the contents of this high school student's manuscript I began to understand Macdonald's early trajectory in mathematics. He did Tripos at Cambridge and had some exposure to the professors there. Particularly from the vantage of Hodge, Pedoe and Todd, intersection theory and its connection to cohomology was "in the air" but not fully developed. Indeed, in his first published paper [Mac58], Macdonald thanks "Dr. J.A. Todd for his interest and helpful advice". By the time of his 1962 papers, Macdonald was clearly following the work of Grothendieck, and had understood that cohomology was an efficient way to compute intersections of the type that he had been computing in high school. In his paper [Mac62b] he already wields the tools of sheaves and cohomology like a master. It is truly amazing to see how this high school student's interest in intersections in classical geometry led him to the very forefront of the technology of cohomology and algebraic geometry that was being vigorously developed at the time.

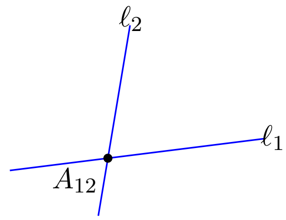
By 1962, without a Ph.D., Ian Macdonald was no longer a high school student, but had followed his nose to already become a mature mathematician of the highest caliber and a great expositor.



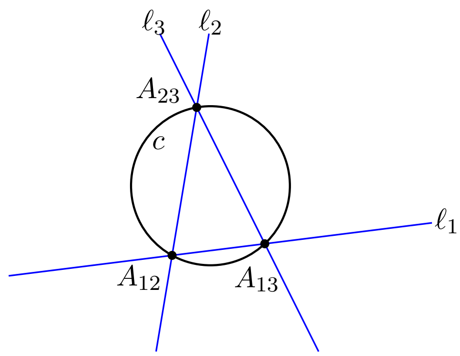
A diagram from Ian Macdonald's 1947 manuscript on the n-line

4.1 Clifford's n-line chain

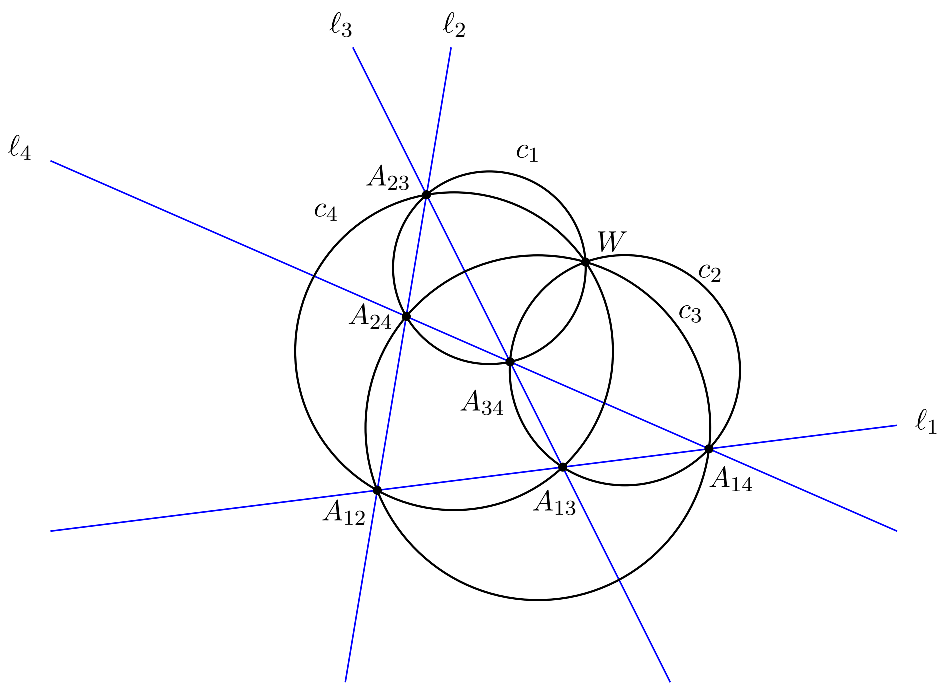
Two generic lines ℓ_1 and ℓ_2 intersect in a point A_{12}. The point A_{12} is the *Clifford point* of the 2-line.



Each pair of lines in a generic 3-line $\{\ell_1, \ell_2, \ell_3\}$ intersect in a point, and these three points determine a circle c. The circle c is the *Clifford circle of the* 3*-line*.



Each triple of lines in a generic 4-line $\{\ell_1, \ell_2, \ell_3, , \ell_4\}$ determines a Clifford circle, giving the circles c_1, c_2, c_3, c_4. These four circles intersect in a point W. The point W is the *Clifford point of the* 4*-line*.

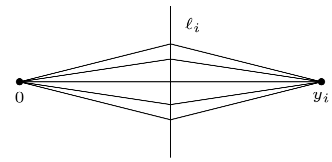


Each 4-tuple of lines in a generic 5-line $\{\ell_1, \ell_2, \ell_3, , \ell_4, \ell_5\}$ determines a Clifford point, giving the points p_1, p_2, p_3, p_4, p_5. These five points lie on a circle C. The circle C is the *Clifford circle of the* 5*-line*.
... and so on ...

4.2 Ian Macdonald's general formulation

Let $y_1, \ldots, y_n \in \mathbb{C}^\times$. For $i \in \{1, \ldots, n\}$ let ℓ_i be the line consisting of the points in $\mathbb{C}$ that are equidistant from 0 and y_i. The set of n lines is *the n-line* $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$, where

$$\ell_i = \{z \in \mathbb{C} \mid \bar{z} = t_i(z - y_i)\}, \qquad \text{where} \quad t_i = \frac{-\overline{y_i}}{y_i}.$$



For $k \in \{0, 1, \ldots, n-1\}$, define

$$c_k(\mathcal{L}) = \frac{y_1 t_1^{n-1-k}}{g_1(\mathcal{L})} + \frac{y_2 t_2^{n-1-k}}{g_2(\mathcal{L})} + \cdots + \frac{y_n t_n^{n-1-k}}{g_n(\mathcal{L})},$$

where

$$g_j(\mathcal{L}) = (t_j - t_1)(t_j - t_2) \cdots (t_j - t_{j-1})(t_j - t_{j+1})(t_j - t_{j+2}) \cdots (t_j - t_n),$$

for $j \in \{1, \ldots, n\}$.

Theorem 4.1 (Clifford's chain). *Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ be an n-line (satisfying an appropriate genericity condition).*

Case n even: Each $(n-1)$-subset of the n-line determines a Clifford circle, and these n Clifford circles intersect in a unique point $p(\mathcal{L})$. Let $k \in \mathbb{Z}_{>0}$ such that $n = 2k$ and let $a_1, \ldots, a_{k-1} \in \mathbb{C}$ be given by

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} c_2(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}^{-1} \begin{pmatrix} -c_1(\mathcal{L}) \\ -c_2(\mathcal{L}) \\ \vdots \\ -c_{k-1}(\mathcal{L}) \end{pmatrix}.$$

Then

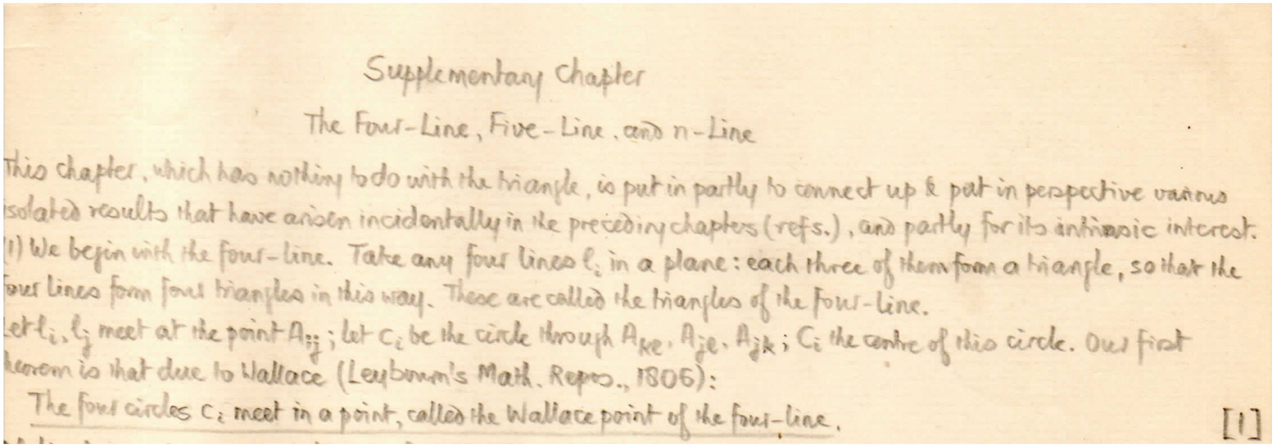
$$p(\mathcal{L}) = c_0(\mathcal{L}) + a_1 c_1(\mathcal{L}) + a_2 c_2(\mathcal{L}) + \cdots + a_{k-1} c_{k-1}(\mathcal{L})$$

is the Clifford point of the n-line $\mathcal{L} = \{\ell_1, \ldots, \ell_{2k}\}$.

Case n odd: Each $(n-1)$-subset of the n-line determines a Clifford point, and these n Clifford points lie on a unique circle $C(\mathcal{L})$. Let $k \in \mathbb{Z}_{>0}$ such that $n = 2k+1$. The Clifford circle $C(\mathcal{L})$ is given by

$$C(\mathcal{L}) = \{A(\mathcal{L}) - \theta B(\mathcal{L}) \mid \theta \in U_1(\mathbb{C})\}, \qquad \textit{where} \qquad U_1(\mathbb{C}) = \{\theta \in \mathbb{C} \mid \theta\bar{\theta} = 1\},$$

$$A(\mathcal{L}) = \frac{\det \begin{pmatrix} c_0(\mathcal{L}) & \cdots & c_{k-1}(\mathcal{L}) \\ c_1(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_{k-1}(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}}{\det \begin{pmatrix} c_2(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}} \qquad \textit{and} \qquad B(\mathcal{L}) = \frac{\det \begin{pmatrix} c_1(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-1}(\mathcal{L}) \end{pmatrix}}{\det \begin{pmatrix} c_2(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}}.$$



Supplementary Chapter

The Four-Line, Five-Line, and n-Line

This chapter, which has nothing to do with the triangle, is put in partly to connect up & put in perspective various solated results that have arisen incidentally in the preceding chapters (refs.), and partly for its intrinsic interest.
(1) We begin with the four-line. Take any four lines l_i in a plane: each three of them form a triangle, so that the our lines form four triangles in this way. These are called the triangles of the four-line.
et l_i, l_j meet at the point A_{ij}; let c_i be the circle through A_{kl}, A_{jl}, A_{jk}; C_i the centre of this circle. Our first heorem is that due to Wallace (Leybourn's Math. Repos., 1806):
The four circles c_i meet in a point, called the Wallace point of the four-line. [1]

The first page of the high school student's manuscript

5 The symmetric product of a curve Σ

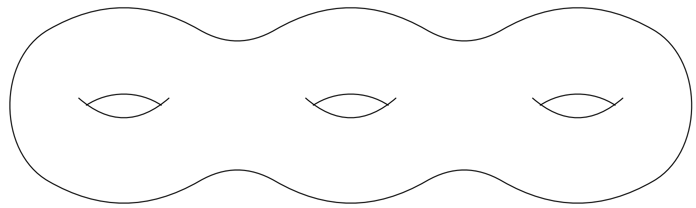
After high school Macdonald served in the military and then did the Mathematical Tripos at Trinity. After finishing at Cambridge in 1952, at the insistence of his father, Macdonald took competitive exams for a civil service job (i.e., a government job). He "stuck it out for five years" in a civil service job before leaving his good secure job for a temporary (1957–1960) position at Manchester and then another temporary position (1960–1963) at Exeter University. Then he became "Fellow and Tutor in Mathematics" at Magdalen College at Oxford until 1972.

In 1958, Macdonald's first paper appeared in Proceedings of the Cambridge Philosophical Society. Very likely this study arose as a continuation of his study of intersections of lines and circles. The paper is entitled *"Some enumerative formulae for algebraic curves"*. In Part I, Macdonald gives a generalization of de Jonquières formula and Part II makes contact with Schur functions and Schubert conditions in intersection theory. It shows a mastery of the methods of the classical Italian algebraic geometry school. This paper is a significant development of his high school knowledge of intersection theory. Even so, it hardly gives any hint of the amazing achievement that was to come next.

By 1962, Macdonald had understood that intersection numbers of families of curves could be computed by using cohomology as a tool. In his paper on the cohomology of symmetric products of an algebraic curve [Mac62b] he states "in particular we obtain natural proofs of the results of an earlier paper [Mac58] which were there obtained laboriously by classical methods."

5.1 The cohomology of a symmetric product of a curve

Let Σ be a curve.

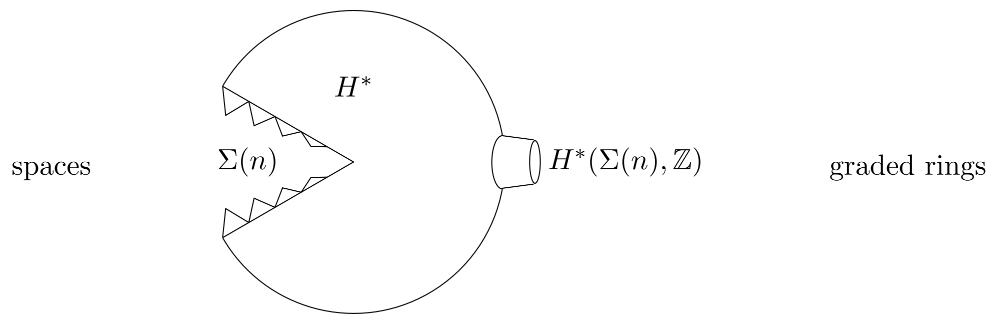


The nth symmetric power of Σ is

$$\Sigma(n) = \Sigma^n / S_n, \qquad \text{where} \quad w \cdot (p_1, \ldots, p_n) = (p_{w^{-1}(1)}, \ldots, p_{w^{-1}(n)}),$$

for $w \in S_n$ and $(p_1, \ldots, p_n) \in \Sigma^n$.

Cohomology is a creature (more precisely, a functor) that eats spaces and outputs graded rings.



In spite of the frightening teeth, cohomologies are really quite friendly (it is the spaces that are dangerous). How does one write down the cohomology $H^*(X;\mathbb{Z})$ of a space X? Well, $H^*(X,\mathbb{Z})$ is a graded ring and a graded ring is written down in a presentation by generators and relations. Macdonald's 1962 paper [Mac62b] gives an elegant presentation of the graded ring $H^*(\Sigma(n),\mathbb{Z})$, the cohomology of the nth symmetric product of a curve Σ.

Theorem 5.1. *Let Σ be a curve of genus g. The cohomology ring $H^*(\Sigma(n),\mathbb{Z})$ is the $\mathbb{Z}$-algebra presented by generators*

$$\xi_1, \ldots, \xi_g, \quad \xi'_1, \ldots, \xi'_g, \quad \eta$$

and relations

(a) If $i, j \in \{1, \ldots, g\}$ then

$$\xi_i \xi_j = -\xi_j \xi_i, \qquad \xi'_i \xi'_j = -\xi'_j \xi'_i, \qquad \xi_i \xi'_j = -\xi'_j \xi_i,$$

$$\xi_i \eta = \eta \xi_i, \qquad\qquad \xi'_i \eta = \eta \xi'_i,$$

(b) If $a, b, c, q \in \mathbb{Z}_{\geq 0}$ and $a + b + 2c + q = n + 1$ and $i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c$ are distinct elements of $\{1, \ldots, g\}$ then

$$\xi_{i_1} \cdots \xi_{i_a} \xi'_{j_1} \cdots \xi'_{j_b} (\xi_{k_1} \xi'_{k_1} - \eta) \cdots (\xi_{k_c} \xi'_{k_c} - \eta) \eta^q = 0.$$

5.2 The Weil conjectures for the symmetric product $\Sigma(n)$ of a curve

Weil's famous conjectures about zeta functions of algebraic varieties are from his paper of 1949 [We49]. These conjectures stimulated a huge effort which included the development of étale cohomology and ℓ-adic cohomology. The Weil conjectures were proved in the 1960s and 70s: the proof of the rationality conjecture came in 1960 (Dwork), the proof of the functional equation and Betti numbers connection in 1965 (Grothendieck school) and the analogue of the Riemann hypothesis in 1974 (Deligne). In 1962 [Mac62b], as an application of his description of the cohomology of $\Sigma(n)$, Macdonald proved Weil's conjectures in an important special case: "... we calculate the zeta function of $\Sigma(n)$ and verify Weil's conjectures in this case."

The zeta function $Z(t)$ of an algebraic variety X is an exponential generating function for the number of points of X over the finite fields $\mathbb{F}_{q^n}$,

$$\frac{d}{dt}\log Z(t) = \sum_{n\in\mathbb{Z}_{>0}} \mathrm{Card}(X(\mathbb{F}_{q^n}))t^{n-1}.$$

Let Σ be a curve of genus g and assume that $\rho_1,\ldots,\rho_{2g}\in\mathbb{C}$ are such that

$$Z_1(t) = \frac{(1-\rho_1 t)\cdots(1-\rho_{2g}t)}{(1-t)(1-qt)} \qquad \text{is the zeta function of } \Sigma.$$

Let $\phi_0(t) = 1-t$ and, for $k\in\{1,\ldots,2g\}$, let

$$\phi_k(t) = \prod_{1\le i_1<\cdots<i_k\le 2g} (1-\rho_{i_1}\cdots\rho_{i_k}t).$$

Then define

$$F_k(t) = \begin{cases} \phi_k(t)\phi_{k-2}(t)\phi_{k-4}(t)\cdots, & \text{if } k\in\{0,1,\ldots,n\},\\ F_{2n-k}(q^{k-n}t), & \text{if } k\in\{n+1,\ldots,2n\}.\end{cases}$$

Corollary 5.2. *The Weil conjectures hold for $\Sigma(n)$. More specifically,*

(a) The zeta function of $\Sigma(n)$ is

$$Z_n(t) = \frac{F_1(t)F_3(t)\cdots F_{2n-1}(t)}{F_0(t)F_2(t)\cdots F_{2n}(t)}.$$

(b) The Riemann hypothesis for $\Sigma(n)$ holds:

$$\textit{All roots of } Z_n(t) \textit{ have absolute value in } \quad \{q^{-\frac12\cdot 0}, q^{-\frac12\cdot 1}, q^{-\frac12\cdot 2},\ldots,q^{-\frac12\cdot 2n}\}.$$

(c) The functional equation for $\Sigma(n)$ is

$$Z_n\big(\tfrac{1}{q^n t}\big) = (-q^{-\frac12 n}t)^{(-1)^n\binom{2g-2}{n}} Z_n(t).$$

6 I.G. Macdonald as influencer

6.1 Deligne–Lusztig 1976

In Lecture Notes in Math. **131**, T. Springer precisely states conjectures of Macdonald about complex representations of finite groups of Lie type. Looking back at these references, one gathers that the notes of Macdonald on Hall polynomials that were circulating in the late 1960's eventuallly became Chapter IV of his book on Symmetric functions and Hall polynomials. T. Springer's expositions appearing in [Spr70] make it clear that, by 1968, Ian Macdonald had understood how the type GL_n story from J.A. Green's 1955 paper could be reshaped for a statement for general Lie types. Macdonald's conjectures were proved by Deligne and Lusztig in 1976.

Annals of Mathematics, **103** (1976), 103-161

Representations of reductive groups over finite fields

By P. Deligne and G. Lusztig

Introduction

Let us consider a connected, reductive algebraic group G, defined over a finite field $\mathbf{F}_q$, with Frobenius map F. We shall be concerned with the representation theory of the finite group G^F, over fields of characteristic 0.

In 1968, Macdonald conjectured, on the basis of the character tables known at the time (GL_n, Sp_4), that there should be a well defined correspondence which, to any F-stable maximal torus T of G and a character θ of T^F in general position, associates an irreducible representation of G^F; moreover, if T modulo the centre of G is anisotropic over $\mathbf{F}_q$, the corresponding representation of G^F should be cuspidal (see Seminar on algebraic groups and related finite groups, by A. Borel et al., Lecture Notes in Mathematics, 131, pp. 117 and 101). In this paper we prove Macdonald's conjecture. More precisely, for T as above and θ an arbitrary character of T^F we construct virtual representations R_T^θ which have all the required properties.

6.2 Maulik–Yun 2013

It is still the case that most topologists and geometers view Macdonald's computation of the cohomology of the symmetric product of a curve [Mac62b] as his most well known achievement. In recent years the study of moduli spaces of curves and related cohomological Hall algebras has become an important part of geometry and mathematical physics, and Macdonald's study of symmetric products of curves continues to be an important stimulus for research in this direction today.

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Journal für die reine und angewandte Mathematik

Macdonald formula for curves with planar singularities

By *Davesh Maulik* at Columbia and *Zhiwei Yun* at Stanford

Abstract. We generalize Macdonald's formula for the cohomology of Hilbert schemes of points on a curve from smooth curves to curves with planar singularities: we relate the cohomology of the Hilbert schemes to the cohomology of the compactified Jacobian of the curve. The new formula is a consequence of a stronger identity between certain perverse sheaves defined by a family of curves satisfying mild conditions. The proof makes essential use of Ngô's support theorem for compactified Jacobians and generalizes this theorem to the relative Hilbert scheme of such families. As a consequence, we give a cohomological interpretation of the numerator of the Hilbert-zeta function of curves with planar singularities.

1. Introduction

Let C be a smooth projective connected curve over an algebraically closed field k. Let $\mathrm{Sym}^n(C)$ be the n-th symmetric product of C. Macdonald's formula [21] says there is a canonical isomorphism between graded vector spaces,

$$\mathrm{H}^*(\mathrm{Sym}^n(C)) \cong \mathrm{Sym}^n(\mathrm{H}^*(C)) = \bigoplus_{i+j\le n, i,j\ge 0} \bigwedge^i(\mathrm{H}^1(C))[-i-2j](-j). \tag{1.1}$$

Here [?] denotes the cohomological shift and (?) denotes the Tate twist. This formula respects

6.3 Casselman 2012

An announcement of Macdonald's computation of the spherical function for p-adic groups appeared in 1968, and the full details appeared in his book published by the University of Madras in 1971. From the point of view of symmetric function theory, Macdonald proved that the favorite formula [Mac, Ch. III, (2.1)] for the Hall-Littlewood polynomial

$$P_\lambda(x;t) = \frac{1}{v_\lambda(t)} \sum_{w\in S_n} w\Big(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big)$$

generalizes to all Lie types and is a formula for the spherical function for G/K where G is the corresponding p-adic group $G = G(\mathbb{Q}_p)$ and $K = G(\mathbb{Z}_p)$ is a maximal compact subgroup of G.

1:17 p.m. August 25, 2012 [Macdonald]

Remarks on Macdonald's book on p-adic spherical functions

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When Ian Macdonald's book **Spherical functions on a group of *p*-adic type** first appeared, it was one of a very small number of publications concerned with representations of $\mathfrak{p}$-adic groups. At just about that time, however, the subject began to be widely recognized as indispensable in understanding automorphic forms, and the literature on the subject started to grow rapidly. Since it has by now grown so huge, in discussing here the subsequent history of some of Macdonald's themes I shall necessarily restrict myself only to things closely related to them. This will be no serious restriction since some of the most interesting problems in all of representation theory—among others, those connected with Langlands' 'fundamental lemma'—are concerned with $\mathfrak{p}$-adic spherical functions. Along the way I'll reformulate from a few different perspectives what his book contains. I'll begin, in the next section, with a brief sketch of the main points, postponing most technical details until later.

Throughout, suppose k to be what I call a $\mathfrak{p}$-adic field, which is to say that it is either a finite extension of some $\mathbb{Q}_p$ or the field of Laurent polynomials in a single variable with coefficients in a finite field. Further let

- $\mathfrak{o}$ = the ring of integers of k;
- $\mathfrak{p}$ = the maximal ideal of $\mathfrak{o}$;
- ϖ = a generator of $\mathfrak{p}$;
- $q = |\mathfrak{o}/\mathfrak{p}|$, so that $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$.

Let $\mathbb{D}$ be a field of characteristic 0, which will play the role of coefficient field in representations. The minimal requirement on $\mathbb{D}$ is that it contain $\sqrt{q}$, but it will in the long run be convenient to assume that it is algebraically closed. It may usually be taken to be $\mathbb{C}$, but I want to emphasize that special properties of $\mathbb{C}$ are rarely required.

In writing this note I had one major decision to make about what class of groups I would work with. What made it difficult was that there were conflicting goals to take into account. On the one hand, I wanted to be able to explain a few basic ideas without technical complications. For this reason, I did not want to deal with arbitrary reductive groups, because even to state results precisely in this case would have required much distracting effort—effort, moreover, that would have just duplicated things explained very well in Macdonald's book. On the other, I wanted to illustrate some of the complexities that Macdonald's book confronts. In the end, I chose to restrict myself to **unramified** groups. *I will suppose throughout this account that G is a reductive group defined over k arising by base extension from a smooth reductive scheme over $\mathfrak{o}$.* I hope that the arguments I present here are clear enough that generalization to arbitrary reductive groups will be straightforward once one understands their fine structure. I also hope that the way things go with this relatively simple class of groups will motivate the geometric treatment in Macdonald's book, which although extremely elegant is somewhat terse and short of examples. I'll say something later on in the section on root data about their structure.

Upon learning that I was going to be writing this essay, Ian Macdonald asked me to mention that Axiom V in Chapter 2 of his book is somewhat stronger than the corresponding axiom of Bruhat-Tits, and not valid for the type C–B_2 in their classification. Deligne pointed this out to him, and made the correction:

> Axiom V. *The commutator group $[U_\alpha, U_\beta]$ for $\alpha, \beta > 0$ is contained in the group generated by the U_γ with $\gamma > 0$ and not parallel to α or β.*

6.4 V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1982

Macdonald's work on p-adic groups drew him into the combinatorics of affine root systems and he made a thorough classification and study of affine root systems and affine Weyl groups, resulting in his 1972 paper entitled *"Affine root systems and Dedekind's η-function"*. This study brought him into contact with affine Kac–Moody Lie algebras and formulas for characters of their representations.

At about the same time Macdonald [1972] obtained his remarkable identities. In this work he undertook to generalize the Weyl denominator

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identity to the case of affine root systems. He remarked that a straightforward generalization is actually false. To salvage the situation he had to add some "mysterious" factors, which he was able to determine as a result of lengthy calculations. The simplest example of Macdonald's identities is the famous Jacobi triple product identity:

$$\prod_{n\geq 1}(1-u^nv^n)(1-u^{n-1}v^n)(1-u^nv^{n-1}) = \sum_{m\in\mathbb{Z}}(-1)^m u^{\frac{1}{2}m(m+1)}v^{\frac{1}{2}m(m-1)}.$$

The "mysterious" factors which do not correspond to affine roots are the factors $(1-u^nv^n)$.

After the appearance of the two works mentioned above very little remained to be done: one had to place them on the desk next to one another to understand that Macdonald's result is only the tip of the iceberg—the representation theory of Kac–Moody algebras. Namely, it turned out that a simplified version of Bernstein–Gelfand–Gelfand's proof may be applied to the proof of a formula generalizing Weyl's formula, for the formal character of the representation π_Λ of an arbitrary Kac–Moody algebra $\mathfrak{g}'(A)$ corresponding to a symmetrizable matrix A. In the case of the simplest 1-dimensional representation π_0, this formula becomes the generalization of Weyl's denominator identity. In the case of an affine Lie algebra, the generalized Weyl denominator identity turns out to be equivalent to the Macdonald identities. In the process, the "mysterious" factors receive a

7 I.G. Macdonald as translator

One of Ian Macdonald's great silent contributions to the mathematical community was his work as a translator.

7.1 I.G. Macdonald as translator: Bourbaki

I.G. Macdonald was the first translator of Bourbaki into English. It is not clear how much of Bourbaki Macdonald translated as the publisher did not list the translator in the published English versions. A best guess is that the volumes which appeared in English between 1966 and 1974 were translated by Macdonald. These volumes comprise more than 2500 pages.

Bourbaki, General Topology Parts I and II 1966, vii+437 pp. and iv+363 pp.

Bourbaki, Theory of Sets 1968, viii+414 pp.

Bourbaki, Commutative Algebra 1972, xxiv+625 pp.

Bourbaki, Algebra 1974, xxiii+709 pp.

7.2 I.G. Macdonald as translator: Dieudonné

I.G. Macdonald's work as a translator of Dieudonné's Treatise on Analysis is documented in [Mar] and he is explicitly listed as translator in the English version of Dieudonné's Panorama of Pure Mathematics, which appeared in 1982. Together, these volumes amount to more than 2300 pages.

Dieudonné, Foundations of Modern Analysis 1960 and 1969, xiv+361 pp.

Dieudonné, Treatise on Analysis Vol. II 1970 and 1976, xviii+387 pp.

Dieudonné, Treatise on Analysis Vol. III 1972, xvii+388 pp.

Dieudonné, Treatise on Analysis Vol. IV 1974, xiv+444 pp.

Dieudonné, Treatise on Analysis Vol. V 1977, xiv+243 pp.

Dieudonné, Treatise on Analysis Vol. VI 1978, xi+239 pp

Dieudonné, A panorama of pure mathematics 1982, x+289 pp.

8 I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Lie groups**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

Algebraic structure of Lie groups, Cambridge University Press, 1980.
https://doi.org/10.1017/CBO9780511662683.005

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **algebraic groups**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

Linear algebraic groups,
in Lectures on Lie Groups and Lie Algebras, Cambridge University Press 1995
https://doi.org/10.1017/CBO9781139172882

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **reflection groups**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

Reflection groups, unpublished notes 1991.
Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **algebraic geometry**, do you have a reference that you can recommend?" I usually find myself saying, "How about the book of Macdonald?"

Algebraic Geometry - Introduction to schemes, published by W.A. Benjamin 1968.
Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Haar measure, spherical functions and harmonic analysis**, do you have a reference that you can recommend?" I usually find myself saying, "How about the book of Macdonald?"

Spherical functions on a group of p-adic type, University of Madras 1971.
Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Kac–Moody Lie algebras**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

Kac–Moody Lie algebras, unpublished notes 1983.
Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **flag varieties and Schubert varieties**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

Notes on Schubert polynomials: Appendix: Schubert varieties.
Published by LACIM 1991.
Available at http://math.soimeme.org/~arunram/resources.html

9 I.G. Macdonald as an author of books

"If you see a gap in the literature, write a book to fill it." – I.G. Macdonald

Atiyah-Macdonald, Introduction to commutative algebra 1969

Spherical functions on a group of p-adic type 1971

Symmetric functions and Hall polynomials First Edition 1979

Kac–Moody Lie algebras: unpublished notes 1983

Hypergeometric functions: unpublished notes 1987

Reflection groups: unpublished notes 1991

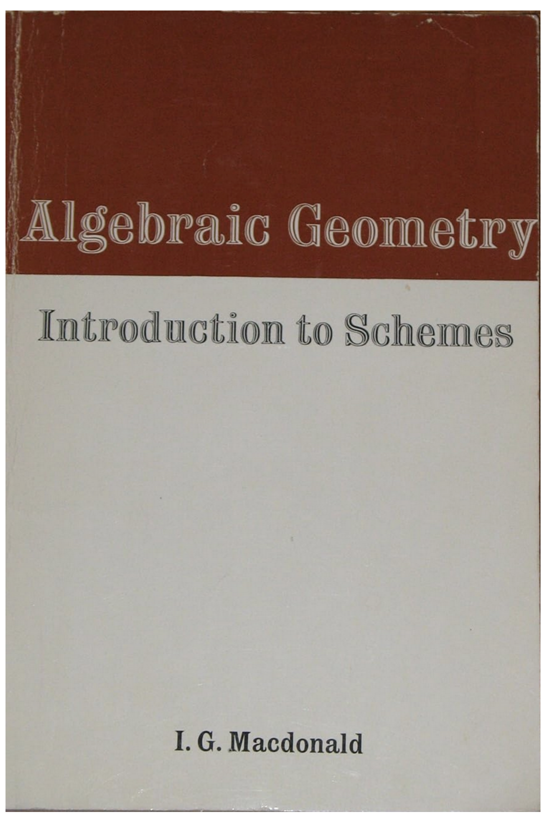
Schubert polynomials 1991

Symmetric functions and Hall polynomials Second Edition 1995

Linear algebraic groups: in Lectures on Lie groups and Lie algebras 1995

Affine Hecke algebras and orthogonal polynomials 2003

9.1 The first book: Algebraic Geometry - Introduction to Schemes 1968



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Séminaire Lotharingien de Combinatoire **91B** (2024)
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Hochschild polytopes

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Abstract. The (m, n)-multiplihedron is a polytope whose faces correspond to m-painted n-trees. Deleting certain inequalities from its facet description, we obtain the (m, n)-Hochschild polytope whose faces correspond to m-lighted n-shades. Moreover, there is a natural shadow map from m-painted n-trees to m-lighted n-shades, which defines a meet semilattice morphism of rotation lattices. In particular, when $m = 1$, our Hochschild polytope is a deformed permutahedron realizing the Hochschild lattice.

Résumé. Le (m, n)-multiplièdre est un polytope dont les faces correspondent aux n-arbres m-peints. En retirant certaines inégalités de sa description par facettes, nous obtenons le (m, n)-polytope de Hochschild dont les faces correspondent aux n-ombres m-illuminées. De plus, il existe une fonction d'ombre naturelle des n-arbres m-peints vers les n-ombres m-illuminées, qui définit un morphisme de semi-treillis supérieur entre les treillis de rotations correspondants. En particulier, quand $m = 1$, notre polytope de Hochschild est un permutaèdre déformé qui réalise le treillis de Hochschild.

Keywords: Multiplihedron, Freehedron, Hochschild lattice, Quotient

Introduction

We present a remake of the famous combinatorial, geometric, and algebraic interplay between permutations and binary trees. In the original story, the central character is the surjective map from permutations to binary trees (given by successive binary search tree insertions [19, 9]). This map enables us to construct the Tamari lattice [18] as a lattice quotient of the weak order, the sylvester fan as a quotient fan of the braid fan, Loday's associahedron [10] as a removahedron of the permutahedron, and the Loday–Ronco Hopf algebra as a Hopf subalgebra of the Malvenuto–Reutenauer Hopf algebra. Many variations of this saga have been further investigated, notably for other lattice quotients of the weak order and for generalized associahedra arizing from finite type cluster algebras. See [13] for a recent survey on this topic, in particular for a bibliography.

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In the present remake, permutations are replaced by binary m-painted n-trees (binary trees on n nodes with m horizontal labeled edge cuts), while binary trees are replaced by unary m-lighted n-shades (compositions of n with m labels inside their gaps). The precise definitions are delayed to Section 1, but the reader can already glance at Figure 8 for $m = 1$ and $n = 3$. The m-painted n-trees already appeared in [5, Sect. 3.1], inspired from the case $m = 1$ studied in [17, 8, 2]. They are mixtures (in the sense of [5]) between the permutations of $[m]$ and the binary trees with n nodes. The m-lighted n-shades are introduced in this paper, inspired from the case $m = 1$ studied in [1, 14, 4, 6, 11]. Here again, the central character is a natural surjective map from the former to the latter. Namely, the shadow map sends an m-painted n-tree to the m-lighted n-shade obtained by collecting the arity sequence along the right branch. In other words, this map records the shadow projected on the right of the tree when the sun sets on the left of the tree.

We first use this map for lattice purposes. It was proved in [5] that the right rotation digraph on binary m-painted n-trees (a mixture of the simple transposition digraph on permutations and the right rotation digraph on binary trees) defines a lattice. We consider here the right rotation digraph on unary m-lighted n-shades. We prove that it defines as well a lattice by showing that the shadow map is a meet semilattice morphism (but not a lattice morphism). When $m = 0$, this gives an unusual meet semilattice morphism from the Tamari lattice to the Boolean lattice (distinct from the usual lattice morphism given by the canopy map). When $m = 1$, this gives a connection, reminiscent of [14], between the painted tree rotation lattice and the Hochschild lattice [4, 6, 11].

We then use the shadow map for polytopal purposes. The refinement poset on all m-painted n-trees is isomorphic to the face lattice of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$. This polytope is a deformed permutahedron (*a.k.a.* polymatroid [7], or generalized permutahedron [15]) obtained as the shuffle product [5] of an m-permutahedron with an n-associahedron of [10]. Oriented in a suitable direction, the skeleton of the (m, n)-multiplihedron is isomorphic to the right rotation digraph on binary m-painted n-trees [5]. Similarly, we show that the refinement poset on all m-lighted n-shades is isomorphic to the face lattice of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$. We obtain this polytope by deleting some inequalities in the facet description of the (m, n)-multiplihedron. We also work out the vertex description of the (m, n)-Hochschild polytope. We obtain a deformed permutahedron whose oriented skeleton is isomorphic to the right rotation digraph on unary m-lighted n-shades. When $m = 0$, the $(0, n)$-multiplihedron is the n-associahedron and the $(0, n)$-Hochschild polytope is a skew cube (which is not a parallelotope). When $m = 1$, the $(1, n)$-multiplihedron is the classical multiplihedron [17, 8, 2], and the $(1, n)$-Hochschild polytope is a deformed permutahedron realizing the Hochschild lattice [4, 6, 11], answering an open question of F. Chapoton.

We refer to [12] for many details and all proofs omitted in this extended abstract due to space limitations. The interested reader will in particular find enumerative formulas and cubic coordinates for multiplihedra and Hochschild polytopes.

1 Painted trees, lighted shades, and the shadow map

1.1 m-painted n-trees

We start with the combinatorics of m-painted n-trees already studied in detail in [5, Sect. 3.1]. It was inspired from the case $m = 1$ studied in [17, 8, 2].

An *n-tree* is a rooted plane tree with $n+1$ leaves. As usual, we orient such a tree towards its root and label its vertices in inorder. Namely, each node with ℓ children is labeled by an $(\ell-1)$-subset $\{x_1, \dots, x_{\ell-1}\}$ of $[n]$ such that all labels in its ith subtree are larger than x_{i-1} and smaller than x_i (where by convention $x_0 = 0$ and $x_\ell = n+1$). Note in particular that unary nodes receive an empty label. A *cut* of an n-tree T is a subset c of nodes of T containing precisely one node along the path from the root to any leaf of T. A cut c is *below* a cut c' if the unique node of c is after the unique node of c' along any path from the root to a leaf of T (note that we draw trees growing downward).

Definition 1 ([5, Def. 105]). *An m-painted n-tree $\mathbb{T} := (T, C, \mu)$ is an n-tree T together with a sequence $C := (c_1, \dots, c_k)$ of k cuts of T and an ordered partition μ of $[m]$ into k parts for some $k \in [m]$, such that*

- *c_i is below c_{i+1} for all $i \in [k-1]$,*
- *$\bigcup C := c_1 \cup \dots \cup c_k$ contains all unary nodes of T.*

We represent an m-painted n-tree $\mathbb{T} := (T, C, \mu)$ as a downward growing tree T, where the cuts of C are red horizontal lines, labeled by the corresponding parts of μ. As there is no ambiguity, we write 12 for the set $\{1, 2\}$. See Figures 1 to 3 for illustrations.

We now associate to each m-painted n-tree a preposet (*i.e.* a reflexive and transitive binary relation) on $[m+n]$. See Figure 1.

Definition 2. *Consider an m-painted n-tree $\mathbb{T} := (T, C, \mu)$. Orient T towards its root, label each node x of T by the union of the part in μ corresponding to the cut of C passing through x (empty set if x is in no cut of C) and the inorder label of x in T shifted by m, and merge all nodes contained in each cut. Define $\preccurlyeq_{\mathbb{T}}$ as the preposet on $[m+n]$ where $i \preccurlyeq_{\mathbb{T}} j$ if there is a (possibly empty) oriented path from the node containing i to the node containing j in the resulting oriented graph.*

We now use these preposets to define the refinement poset on m-painted n-trees.

Definition 3 ([5, Def. 108]). *The m-painted n-tree refinement poset is the poset on m-painted n-trees ordered by refinement of their corresponding preposets, that is, $\mathbb{T} \le \mathbb{T}'$ if $\preccurlyeq_{\mathbb{T}} \supseteq \preccurlyeq_{\mathbb{T}'}$.*

In the following statement, we denote by $|T|$ the number of nodes of a tree T (including unary nodes), and define $|C| := k$ and $|\bigcup C| := |c_1 \cup \dots \cup c_k|$ for $C = (c_1, \dots, c_k)$.

Proposition 4 ([5, Props. 107 & 116]). *The m-painted n-tree refinement poset is a meet semi-lattice ranked by $m + n - |T| - |C| + |\bigcup C|$.*

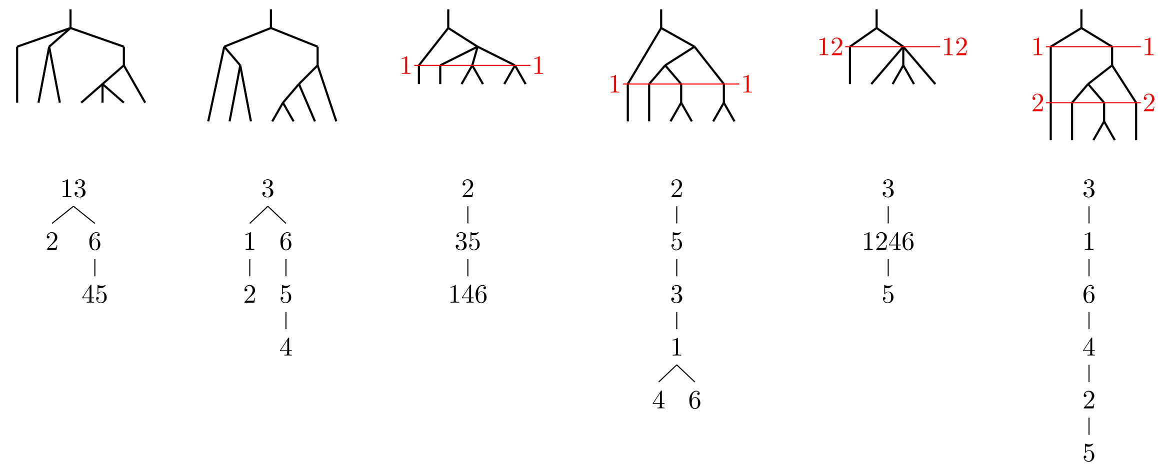

Figure 1: Some m-painted n-trees (top) and their preposets (bottom). Here $m + n = 6$.

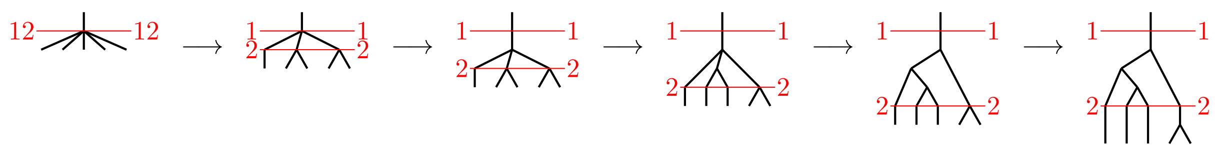

Figure 2: Refinements of some 2-painted 4-trees.

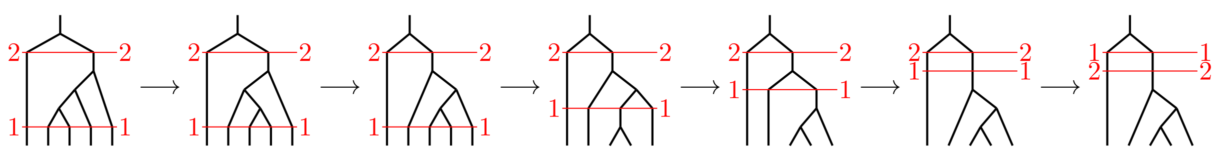

Figure 3: Rotations of some binary 2-painted 4-trees.

We now define another lattice, but on rank 0 m-painted n-trees. See Figures 3 and 7.

Definition 5 ([5, Def. 112])**.** *An m-painted n-tree $\mathbb{T} := (T, C, \mu)$ is binary if it has rank 0, meaning that all nodes in $\bigcup C$ are unary, while all nodes not in $\bigcup C$ are binary. The binary m-painted n-tree right rotation digraph is the directed graph on binary m-painted n-tree with an edge $(\mathbb{T}, \mathbb{T}')$ if and only if there exists $1 \le i < j \le m + n$ such that $\preccurlyeq_{\mathbb{T}} \smallsetminus \{(i, j)\} = \preccurlyeq_{\mathbb{T}'} \smallsetminus \{(j, i)\}$.*

Proposition 6 ([5, Def. 119])**.** *The binary m-painted n-tree right rotation digraph is the Hasse diagram of a lattice.*

Example 7. *When $m = 0$, the 0-painted n-tree rotation lattice is the Tamari lattice [18]. When $m = 1$, the 1-painted n-tree rotation lattice is the multiplihedron lattice introduced in [5].*

Remark 8. *Note that the m-painted n-tree rotation lattice is meet semidistributive, but not join semidistributive when $m \geq 1$.*

Let us finally mention that m-painted n-trees have interesting enumerative properties. See [12, Sect. 1.1] for formulas for some m-painted n-trees generating functions.

1.2 m-lighted n-shades

We now introduce the main new characters of this paper, which will later appear as certain shadows of m-painted n-trees.

Definition 9. *An n-shade is a sequence of (possibly empty) tuples of integers, whose total sum is n. An m-lighted n-shade* $\mathbb{S} := (S, C, \mu)$ *is an n-shade S together with a set C of k distinguished positions in S, containing all positions of empty tuples of S, and an ordered partition* μ *of* $[m]$ *into k parts for some* $k \in [m]$.

We represent an m-lighted n-shade $\mathbb{S} := (S, C, \mu)$ as a vertical line, with the tuples of the sequence S in black on the left, and the cuts of C in red on the right, all from top to bottom. As there is no ambiguity, we write 12 for the tuple $(1, 2)$ or the set $\{1, 2\}$. See Figures 4 to 6 for illustrations.

We now associate to each m-lighted n-shade a preposet on $[m + n]$. See Figure 4

Definition 10. *Consider an m-lighted n-shade* $\mathbb{S} := (S, C, \mu)$*. The preceeding sum* $ps(x)$ *of an entry x in a tuple of S is m plus the sum of all entries that appear weakly before x in S (meaning either the entries in a strictly earlier tuple of S, or the weakly earlier entries in the same tuple as x). Define* $\preccurlyeq_{\mathbb{S}}$ *as the preposet on* $[m + n]$ *given by the relations*

- $i \preccurlyeq_{\mathbb{S}} j$ *if* $i, j \in [m]$ *and i appears weakly after j in* μ,
- $k \preccurlyeq_{\mathbb{S}} ps(y)$ *if x and y are elements of tuples of S such that the tuple of x appears weakly after the tuple of y, and* $ps(x) - x < k \le ps(x)$,
- $i \preccurlyeq_{\mathbb{S}} ps(x)$ *if* $i \in [m]$ *and x is an element of a tuple of S which appears weakly before the cut containing i,*
- $k \preccurlyeq_{\mathbb{S}} i$ *if* $i \in [m]$ *and* $ps(x) - x < k \le ps(x)$ *for some element x of a tuple of S which appears weakly after the cut containing i.*

We now use these preposets to define the refinement poset on m-lighted n-shades.

Definition 11. *The m-lighted n-shade refinement poset is the poset on m-lighted n-shades defined by refinement of their corresponding preposets, that is,* $\mathbb{S} \le \mathbb{S}'$ *if* $\preccurlyeq_{\mathbb{S}} \supseteq \preccurlyeq_{\mathbb{S}'}$.

For a sequence $S := (s_1, \dots, s_\ell)$ of tuples, we define $|S| := \ell$ and $\|S\| := \sum_{i \in [\ell]} |s_i|$, where $|s_i|$ is the length of the tuple s_i.

Proposition 12. *The m-lighted n-shade refinement poset is a meet semilattice ranked by* $m - |S| + \|S\|$.

We now define another lattice, but on rank 0 m-lighted n-shades. See Figures 6 and 7.

Definition 13. *An m-lighted n-shade* $\mathbb{S} := (S, C, \mu)$ *is unary if it has rank 0, meaning that all tuples in* $\bigcup C$ *are empty tuples, while all tuples not in* $\bigcup C$ *are singletons. The unary m-lighted n-shade right rotation digraph is the directed graph on unary m-lighted n-shades with an edge* $(\mathbb{S}, \mathbb{S}')$ *if and only if there exists* $1 \le i < j \le m + n$ *such that* $\preccurlyeq_{\mathbb{S}} \smallsetminus \{(i, j)\} = \preccurlyeq_{\mathbb{S}'} \smallsetminus \{(j, i)\}$.

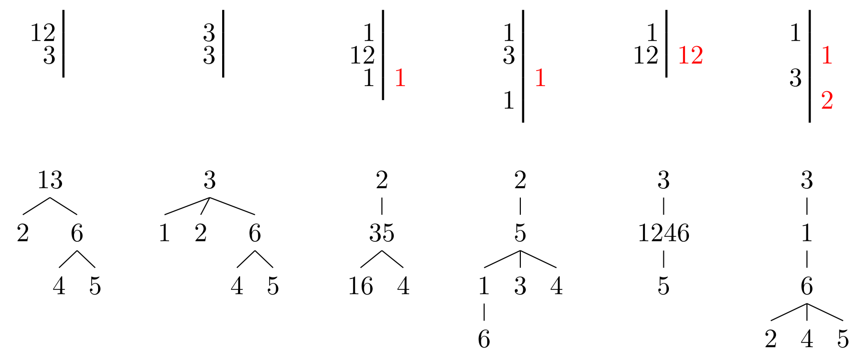

Figure 4: Some m-lighted n-shades (top) and their preposets (bottom). Here $m + n = 6$.

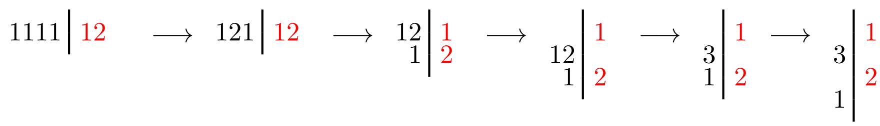

Figure 5: Refinements of some 2-lighted 4-shades.

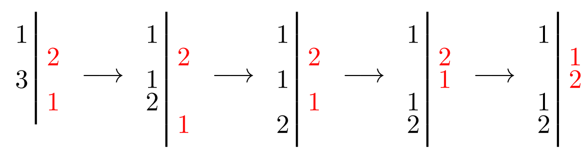

Figure 6: Rotations of some unary 2-lighted 4-shades.

Remark 14. *We observe that any unary m-lighted n-shade $\mathbb{S}$ with singleton tuples $s_1, \dots, s_k$ admits $m + k - 1 + \sum_{i \in [k]} (s_i - 1) = m + n - 1$ (left or right) rotations. In other words, the (undirected) rotation graph is regular of degree $m + n - 1$.*

Proposition 15. *The unary m-lighted n-shade right rotation digraph is the Hasse diagram of a lattice.*

Example 16. *When $m = 0$, the 0-lighted n-shade rotation lattice is boolean. When $m = 1$, the 1-lighted n-shade rotation lattice is the Hochschild lattice studied in [4, 6, 11].*

Remark 17. *Computational experiments indicate that the m-lighted n-shade rotation lattice is constructible by interval doubling (hence semidistributive and congruence uniform). However, in contrast to the case when $m \leq 1$, it is not extremal (see [11] for context), and its Coxeter polynomial is not a product of cyclotomic polynomials (see [3] and [6, Appendix] for context). Nevertheless, its subposet induced by unary m-lighted n-shades where the labels of the lights are ordered seems to enjoy all these nice properties.*

Let us finally mention that m-lighted n-shades have interesting enumerative properties. See [12, Sect. 1.1] for formulas for some m-lighted n-shades generating functions.

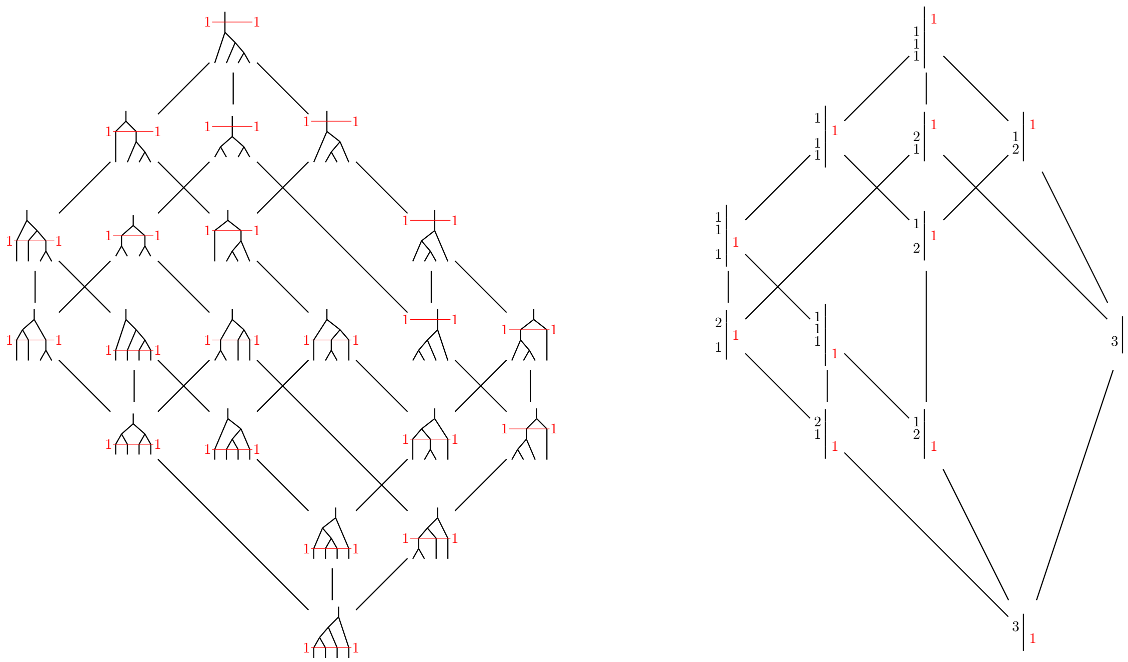

Figure 7: The 1-painted 3-tree (left) and 1-lighted 3-shade (right) rotation lattices.

1.3 Shadow map

We now describe the shadow map sending an m-painted n-tree to an m-lighted n-shade. Intuitively, the shadow is what you see on the right of the tree when the sun sets on its left. For instance, the m-painted n-trees of Figure 1 are sent to the m-lighted n-shade of Figure 4. We call *right branch* of a tree T the path from the root to the rightmost leaf of T.

Definition 18. *The shadow of an n-tree T is the n-shade $\mathrm{Sh}(T)$ obtained by*

- *contracting all edges joining a child to a parent which does not lie on the right branch of T,*
- *replacing each node on the right branch of T by the tuple of the arities of its children except its rightmost.*

The shadow of a cut c in T is the position $\mathrm{Sh}(c)$ in $\mathrm{Sh}(T)$ of the unique node of the right branch of T contained in c. For a sequence $C = (c_1, \ldots, c_k)$, define $\mathrm{Sh}(C) := (\mathrm{Sh}(c_1), \ldots, \mathrm{Sh}(c_k))$. The shadow of an m-painted n-tree $\mathbb{T} := (T, C, \mu)$ is the m-lighted n-shade $\mathrm{Sh}(\mathbb{T}) := (\mathrm{Sh}(S), \mathrm{Sh}(C), \mu)$.

Given two meet semilattices $(M, \wedge)$ and $(M', \wedge')$, a map $f : M \to M'$ is a *meet semilattice morphism* if $f(x \wedge y) = f(x) \wedge' f(y)$ for all $x, y \in M$.

Theorem 19. *The shadow map is a surjective meet semilattice morphism from the binary m-painted n-tree rotation lattice to the unary m-lighted n-shade rotation lattice. See Figure 7.*

Remark 20. *Note that the shadow map is not a join semilattice morphism. For instance,*

2 Multiplihedra and Hochschild polytopes

2.1 Multiplihedra

We now consider the (m,n)-multiplihedron which realizes the m-painted n-tree refinement lattice. It is illustrated for $m = 1$ and $n = 3$ in Figure 8. Although they were previously constructed when $m = 1$ in [17, 8, 2], we use here the construction of [5, Sect. 3]. This construction is just an example of the shuffle product on deformed permutahedra, introduced in [5, Sect. 2]. However, we do not need the generality of this operation and define the (m,n)-multiplihedron using its vertex and facet descriptions.

Definition 21. *Consider a binary m-painted n-tree $\mathbb{T} := (T, C, \mu)$. We associate to $\mathbb{T}$ a point $\boldsymbol{a}(\mathbb{T})$ whose pth coordinate is*

- *if $p \le m$, the number of binary nodes and cuts weakly below the cut labeled by p,*
- *if $p \ge m+1$, the number of cuts below plus the product of the numbers of leaves in the left and right subtrees of the node of T labeled by $p - m$ in inorder.*

See Figure 9 for some examples.

Definition 22. *Consider the hyperplane $\mathbb{H}_{m+n}$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{[m+n]} \,\rangle = \binom{m+n+1}{2}$. Moreover, for each rank $m+n-2$ m-painted n-tree $\mathbb{T} := (T, C, \mu)$, consider the halfspace $\boldsymbol{H}(\mathbb{T})$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{A \cup B} \,\rangle \ge \binom{|A|+1}{2} + \binom{|B_1|+1}{2} + \dots + \binom{|B_k|+1}{2} + |A| \cdot |B|$, where*

- *A denotes the set of elements of $[m]$ which label the cut of C not containing the root of T (hence, $A = \varnothing$ if C has only one cut, which contains the root of T),*
- *$B := B_1 \cup \dots \cup B_k$ where $B_1, \dots, B_k$ are the inorder labels shifted by m of the non-unary nodes of T distinct from the root of T.*

See Figure 9 for some examples.

Theorem 23 ([5, Props. 116, 122, 123])**.** *The m-painted n-tree refinement lattice is anti-isomorphic to the face lattice of the (m,n)-multiplihedron $\mathbb{M}\mathrm{ul}(m,n)$, defined equivalently as*

(i) the convex hull of the vertices $\boldsymbol{a}(\mathbb{T})$ for all binary m-painted n-trees $\mathbb{T}$,

(ii) the intersection of the hyperplane $\mathbb{H}_{m+n}$ with the halfspaces $\boldsymbol{H}(\mathbb{T})$ for all rank $m+n-2$ m-painted n-trees $\mathbb{T}$.

Proposition 24 ([5, Prop. 118])**.** *The normal fan of the (m,n)-multiplihedron $\mathbb{M}\mathrm{ul}(m,n)$ is the fan whose cones are the preposet cones of the preposets $\preccurlyeq_{\mathbb{T}}$ of all m-painted n-trees $\mathbb{T}$.*

Proposition 25 ([5, Prop. 119])**.** *The skeleton of the (m,n)-multiplihedron $\mathbb{M}\mathrm{ul}(m,n)$ oriented in the direction $\boldsymbol{\omega}_{m+n} := (m+n, \dots, 1) - (1, \dots, m+n)$ is isomorphic to the right rotation digraph on binary m-painted n-trees.*

Example 26. *When $m = 0$, the $(0,n)$-multiplihedron is Loday's associahedron [10]. When $m = 1$, the $(1,n)$-multiplihedron is the classical multiplihedron alternatively constructed in [8, 2].*

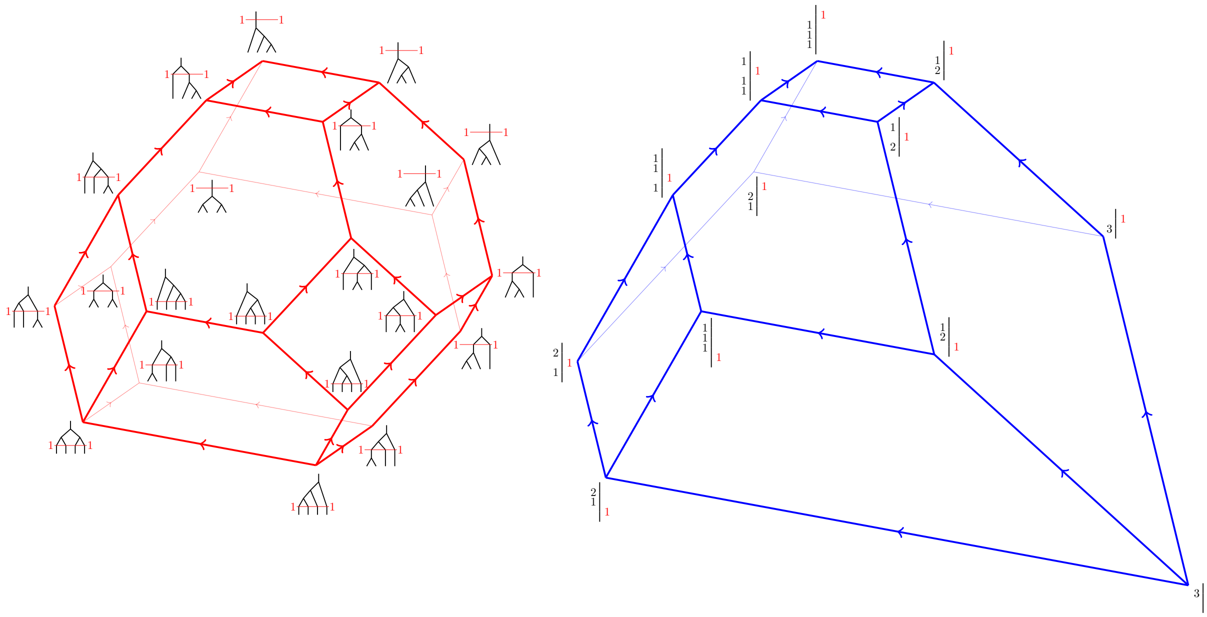

Figure 8: Multiplihedron $\mathbb{M}\mathrm{ul}(1,3)$ (left) and Hochschild polytope $\mathbb{H}\mathrm{och}(1,3)$ (right).

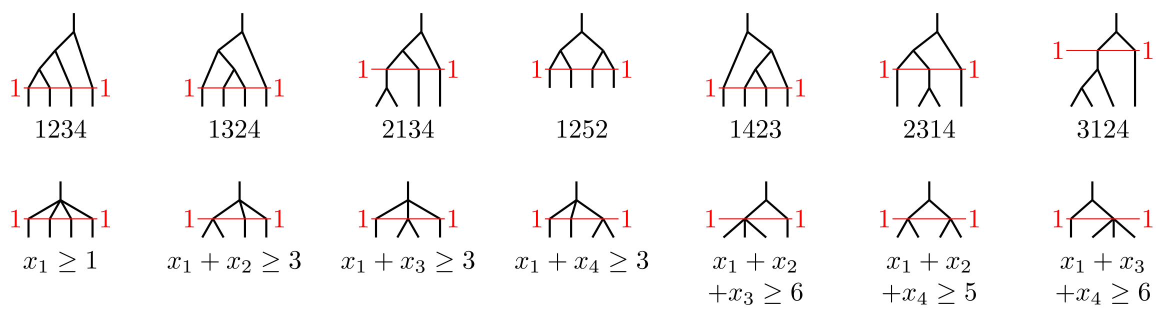

Figure 9: Some vertices (top) and facet defining inequalities (bottom) of $\mathbb{M}\mathrm{ul}(1,3)$.

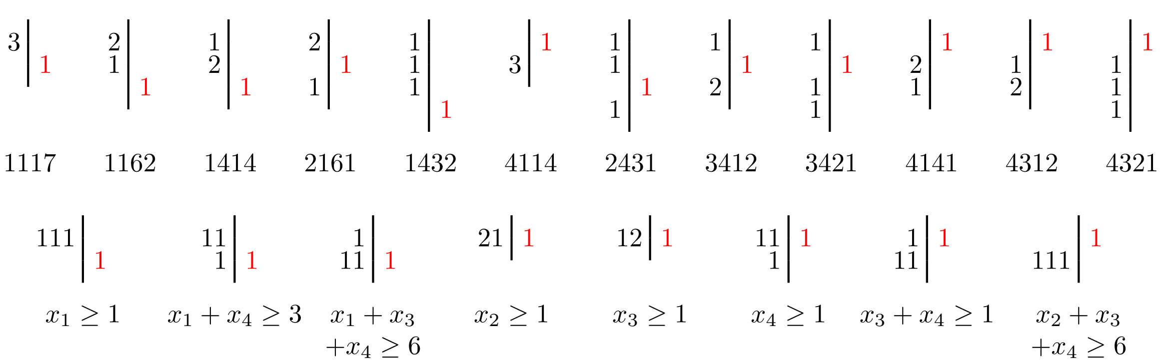

Figure 10: Vertices (top) and facet defining inequalities (bottom) of $\mathbb{H}\mathrm{och}(1,3)$.

2.2 Hochschild polytopes

We now construct the (m, n)-Hochschild polytope which realizes the m-lighted n-shade refinement lattice. It is illustrated for $m = 1$ and $n = 3$ in Figure 8. Recall that we denote by $ps(x)$ the preceeding sum of an entry x in an m-lighted n-shade (see Definition 10).

Definition 27. *Consider a unary m-lighted n-shade $\mathbb{S} := (S, C, \mu)$ and denote by $s_1, s_2, \ldots, s_k$ the values of the singleton tuples of S. We associate to $\mathbb{S}$ a point $\boldsymbol{a}(\mathbb{S})$ whose pth coordinate is*

- *if $p \le m$, then the number of cuts plus the sum of the entries s_i which are weakly below the cut labeled p,*
- *if there is $j \in [k]$ such that $p = ps(s_j)$, then $1 + s_j\big(m + n - p + c_p\big) + \binom{s_j}{2}$ where c_p is the number of cuts below s_j,*
- *1 otherwise.*

See Figure 10 for some examples.

Definition 28. *Consider the hyperplane $\mathbb{H}_{m+n}$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{[m+n]} \,\rangle = \binom{m+n+1}{2}$. Moreover, for each rank $m + n - 2$ m-lighted n-shade $\mathbb{S} := (S, C, \mu)$, consider the halfspace $\boldsymbol{H}(\mathbb{S})$ of $\mathbb{R}^{m+n}$ defined by $\langle\, \boldsymbol{x} \mid \mathbf{1}_{A \cup B} \,\rangle \ge \binom{|A|+|B|+1}{2}$, where*

- *A denotes the set of elements of $[m]$ which label the cut of C not containing the first tuple of S (hence, $A = \varnothing$ if C has only one cut, which contains the first tuple of S),*
- *$B = \{m + q\}$ if S is a single tuple with 2 in position q, and $B = \{m + q + 1, \ldots, m + n\}$ if $S = (s_1, s_2)$ is a pair of tuples with $|s_1| = q$.*

See Figure 10 for some examples.

Theorem 29. *The m-lighted n-shade refinement lattice is anti-isomorphic to the face lattice of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$, defined equivalently as*

(i) the convex hull of the vertices $\boldsymbol{a}(\mathbb{S})$ for all unary m-lighted n-shades $\mathbb{S}$,

(ii) the intersection of the hyperplane $\mathbb{H}_{m+n}$ with the halfspaces $\boldsymbol{H}(\mathbb{S})$ for all rank $m + n - 2$ m-lighted n-shades $\mathbb{S}$.

Proposition 30. *The normal fan of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$ is the fan whose cones are the preposet cones of the preposets $\preccurlyeq_{\mathbb{S}}$ of all m-lighted n-shades $\mathbb{S}$.*

Proposition 31. *The skeleton of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$ oriented in the direction $\boldsymbol{\omega}_{m+n} := (m + n, \ldots, 1) - (1, \ldots, m + n)$ is isomorphic to the right rotation digraph on unary m-lighted n-shades.*

Remark 32. *It follows from Remark 14 that the (m, n)-Hochschild polytope is simple and the m-lighted n-shade fan is simplicial.*

Remark 33. *As mentioned in the introduction, there are deep similarities between the behaviors of*

- *the permutahedron $\mathbb{P}\mathrm{erm}(d)$ and the associahedron $\mathbb{A}\mathrm{sso}(d)$,*
- *the multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$ and the Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$.*

Some comments on the behavior of the latter for the reader familiar with the behavior of the former:

- *The (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$ can be obtained by deleting inequalities in the facet description of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$.*
- *The common facet defining inequalities of $\mathbb{M}\mathrm{ul}(m, n)$ and $\mathbb{H}\mathrm{och}(m, n)$ are precisely those that contain a common vertex of $\mathbb{M}\mathrm{ul}(m, n)$ and $\mathbb{H}\mathrm{och}(m, n)$.*
- *In contrast, the vertex barycenters of the (m, n)-multiplihedron $\mathbb{M}\mathrm{ul}(m, n)$ and of the (m, n)-Hochschild polytope $\mathbb{H}\mathrm{och}(m, n)$ do not coincide.*
- *When $m = 0$, the $(0, n)$-Hochschild polytope $\mathbb{H}\mathrm{och}(0, n)$ is a skew cube distinct from the parallelepiped obtained by considering the canopy congruence on binary trees.*

Example 34. *When $m = 0$, the $(0, n)$-Hochschild polytope is a skew cube, distinct from the parallelotope $\sum_{i \in [n-1]} [\boldsymbol{e}_i, \boldsymbol{e}_{i+1}]$. When $m = 1$, the $(1, n)$-Hochschild polytope gives a realization of the Hochschild lattice [4, 6, 11]. Note that the unoriented rotation graph on 1-lighted n-shades was already known to be isomorphic to the unoriented skeleton of a deformed permutahedron called freehedron and obtained as a truncation of the standard simplex [16], or more precisely as the Minkowski sum $\sum_{i \in [n]} \triangle_{\{1,\dots,i\}} + \sum_{i \in [n]} \triangle_{\{i,\dots,n\}}$ of the faces of the standard simplex corresponding to initial and final intervals, see Figure 11. However, orienting the skeleton of the freehedron in direction $\boldsymbol{\omega}_{m+n}$, we obtain a poset different from the Hochschild lattice, and which is not even a lattice. Indeed, in Figure 11 (left) the two blue vertices have no join while the two red vertices have no meet. In fact, the Hasse diagram of the Hochschild lattice cannot be obtained as a Morse orientation given by a linear functional on the freehedron. Finally, observe that the freehedron cannot be obtained by removing inequalities in the facet description of the permutahedron or of the multiplihedron. See Figure 11 where the removahedra have the wrong combinatorics.*

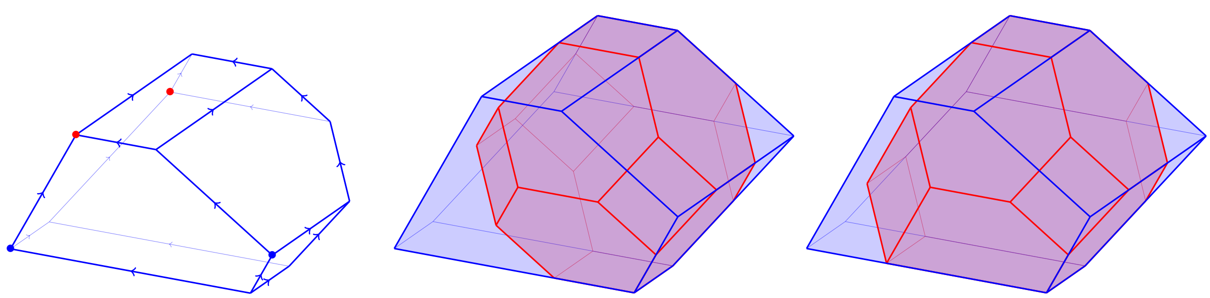

Figure 11: The freehedron obtained as Minkowski sum of the faces of the standard simplex corresponding to initial or final intervals (left), and failed attempts to obtain it as a removahedron of the permutahedron (middle) or of the multiplihedron (right).

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Toric and Permutoric Promotion

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Abstract. We introduce *toric promotion* as a cyclic analogue of Schützenberger's promotion operator. Toric promotion acts on the set of labelings of a graph G; it is defined as the composition of certain toggle operators, listed in a natural cyclic order. We provide a surprisingly simple description of the orbit structure of toric promotion when G is a forest. We then consider more general permutoric promotion operators, which are defined as compositions of the same toggle operators, but in permuted orders. When G is a path graph, we provide a complete description of the orbit structures of all permutoric promotion operators, showing that they satisfy the cyclic sieving phenomenon.

Keywords: promotion, cyclic analogue, cyclic sieving, toggle operator

This is an extended abstract for the articles [2] and [4]. The first of these articles—written by the first author—focuses on toric promotion, while the second article—written by all three authors—concerns the more general permutoric promotion operators.

1 Introduction

In his study of the Robinson–Schensted–Knuth correspondence, Schützenberger [9, 10] introduced a beautiful bijective operator called *promotion*, which acts on the set of linear extensions of a finite poset. Haiman [6] and Malvenuto–Reutenauer [7] found that promotion could be defined as a composition of local *toggle operators* (also called *Bender–Knuth involutions*). Promotion is now one of the most extensively studied operators in the field of dynamical algebraic combinatorics.

Following the approach first considered by Malvenuto and Reutenauer [7], we define promotion on labelings of graphs instead of linear extensions of posets. Let $G = (V, E)$ be a graph with n vertices. A *labeling* of G is a bijection $V \to \mathbb{Z}/n\mathbb{Z}$. We denote the set

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of labelings of G by Λ_G. Given distinct $a, b \in \mathbb{Z}/n\mathbb{Z}$, let $(a\ b)$ be the transposition that swaps a and b. For $i \in \mathbb{Z}/n\mathbb{Z}$, the *toggle* operator $\tau_i \colon \Lambda_G \to \Lambda_G$ is defined by

$$\tau_i(\sigma) = \begin{cases} (i\ i+1) \circ \sigma & \text{if } \{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \notin E; \\ \sigma & \text{if } \{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \in E. \end{cases}$$

In other words, τ_i swaps the labels i and $i+1$ if those labels are assigned to nonadjacent vertices of G, and it does nothing otherwise. Define *promotion* to be the operator $\mathrm{Pro} \colon \Lambda_G \to \Lambda_G$ given by

$$\mathrm{Pro} = \tau_{n-1} \cdots \tau_2 \tau_1.$$

Here and in the sequel, concatenation of operators represents composition.

A recent trend in algebraic combinatorics aims to find cyclic analogues of more traditional "linear" objects (see [1, 5] and the references therein). In the same vein, we introduce a cyclic analogue of promotion called *toric promotion*; this is the operator $\mathrm{TPro} \colon \Lambda_G \to \Lambda_G$ given by

$$\mathrm{TPro} = \tau_n \tau_{n-1} \cdots \tau_2 \tau_1 = \tau_n \,\mathrm{Pro}\,.$$

Our first main result reveals that toric promotion has remarkably nice dynamical properties when G is a forest.

Theorem 1 ([2])**.** *Let G be a forest with $n \geq 2$ vertices, and let $\sigma \in \Lambda_G$ be a labeling. The orbit of toric promotion containing σ has size $(n-1)t/\gcd(t,n)$, where t is the number of vertices in the connected component of G containing $\sigma^{-1}(1)$. In particular, if G is a tree, then every orbit of* $\mathrm{TPro} \colon \Lambda_G \to \Lambda_G$ *has size $n-1$.*

Theorem 1 stands in stark contrast to the wild dynamics of promotion on most forests. For example, even when G is a path graph with 7 vertices, the order of $\mathrm{Pro} \colon \Lambda_G \to \Lambda_G$ is 3224590642072800, whereas all orbits of $\mathrm{TPro} \colon \Lambda_G \to \Lambda_G$ have size 6.

We now consider a generalization of toric promotion in which the toggle operators $\tau_1, \ldots, \tau_n$ can be composed in any order. Let $[n] = \{1, \ldots, n\}$, and let $\pi \colon [n] \to \mathbb{Z}/n\mathbb{Z}$ be a bijection. The *permutoric promotion* operator $\mathrm{TPro}_\pi \colon \Lambda_G \to \Lambda_G$ is defined by

$$\mathrm{TPro}_\pi = \tau_{\pi(n)} \cdots \tau_{\pi(2)} \tau_{\pi(1)}.$$

One would ideally hope to have an extension of Theorem 1 to arbitrary permutoric promotion operators. Unfortunately, trying to completely describe the orbit structure of $\mathrm{TPro}_\pi \colon \Lambda_G \to \Lambda_G$ for arbitrary forests G and arbitrary permutations π seems to be very difficult. However, it turns out that we *can* do this when G is a path. To state our main result, we need a bit more terminology.

Let $[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1}$ and $[k]_q! = [k]_q [k-1]_q \cdots [1]_q$. The *q-binomial coefficient* $\left[{k \atop r}\right]_q$ is the polynomial $\dfrac{[k]_q!}{[r]_q![k-r]_q!} \in \mathbb{C}[q]$.

Let X be a finite set, and let $f\colon X \to X$ be an invertible map of order ω (i.e., ω is the smallest positive integer such that $f^{\omega}(x) = x$ for all $x \in X$). Let $F(q) \in \mathbb{C}[q]$ be a polynomial in the variable q. Following [8], we say the triple $(X, f, F(q))$ *exhibits the cyclic sieving phenomenon* if for every integer k, the number of elements of X fixed by f^k is $F(e^{2\pi ik/\omega})$.

Although we view the set $\mathbb{Z}/n\mathbb{Z}$ as a "cyclic" object, it will often be convenient to identify $\mathbb{Z}/n\mathbb{Z}$ with the "linear" set $[n]$ and consider the total ordering of its elements given by $1 < 2 < \cdots < n$. If $\pi\colon [n] \to \mathbb{Z}/n\mathbb{Z}$ is a bijection, then a *cyclic descent* of π^{-1} is an element $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\pi^{-1}(i) > \pi^{-1}(i+1)$ (note that n is permitted to be a cyclic descent).

Let Path_n and Cycle_n be the n-vertex path graph and cycle graph, respectively. In [2], the first author conjectured that for every bijection $\pi\colon [n] \to \mathbb{Z}/n\mathbb{Z}$, the order of $\mathrm{TPro}_\pi\colon \Lambda_{\mathsf{Path}_n} \to \Lambda_{\mathsf{Path}_n}$ is $d(n-d)$, where d is the number of cyclic descents of π^{-1}. Our next main theorem not only proves this conjecture, but also determines the entire orbit structure of TPro_π in this case.

Theorem 2 ([4])**.** *Let $\pi\colon [n] \to \mathbb{Z}/n\mathbb{Z}$ be a bijection, and let d be the number of cyclic descents of π^{-1}. The order of the permutoric promotion operator $\mathrm{TPro}_\pi\colon \Lambda_{\mathsf{Path}_n} \to \Lambda_{\mathsf{Path}_n}$ is $d(n-d)$. Moreover, the following triple exhibits the cyclic sieving phenomenon:*

$$\left(\Lambda_{\mathsf{Path}_n}, \mathrm{TPro}_\pi, n(d-1)!(n-d-1)![n-d]_{q^d}\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right).$$

Note that when $d = 1$, the sieving polynomial in Theorem 2 is $n(n-2)![n-1]_q$, which agrees with Theorem 1.

Remark 1. Theorem 1 determines the orbit structure of toric promotion when G is a forest. It is still open to understand the dynamics of toric promotion for other graphs, including cycles. Theorem 2 determines the orbit structure of any permutoric promotion operator when G is a path. It would be interesting to gain a better understanding of TPro_π when G is another type of tree, even when π^{-1} has just 2 cyclic descents.

In Section 2, we summarize some of the main ideas that go into the proof of Theorem 2, referring the reader to our full article [4] for the (quite involved) details that we have omitted. We also briefly summarize a proof of Theorem 1 in Section 3, though we refer the reader to [2] for a full proof.

2 Dynamics of Permutoric Promotion

As before, fix a bijection $\pi\colon [n] \to \mathbb{Z}/n\mathbb{Z}$, and let d be the number of cyclic descents of π^{-1}. We assume from now on that G is the path graph Path_n so that TPro_π is an operator on $\Lambda_{\mathsf{Path}_n}$. Given a finite set X and an invertible map $f\colon X \to X$, we write Orb_f for the set of orbits of f.

2.1 A Reduction

Let $\mathsf{Comp}_d(n)$ denote the set of compositions of n with d parts (i.e., d-tuples of positive integers that sum to n). There is a natural *rotation* operator $\mathrm{Rot}_{n,d}\colon \mathsf{Comp}_d(n) \to \mathsf{Comp}_d(n)$ defined by $\mathrm{Rot}_{n,d}(a_1, a_2, \ldots, a_d) = (a_2, \ldots, a_d, a_1)$. Reiner, Stanton, and White [8] proved that the triple $\left(\mathsf{Comp}_d(n), \mathrm{Rot}_{n,d}, \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)$ exhibits the cyclic sieving phenomenon. As it turns out, this result is responsible for the factor of $\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q$ in the sieving polynomial in Theorem 2.

Let $\mathrm{cyc}\colon \Lambda_{\mathsf{Path}_n} \to \Lambda_{\mathsf{Path}_n}$ be the *cyclic shift* operator given by $(\mathrm{cyc}(\sigma))(v) = \sigma(v) + 1$. Let $\Phi_{n,d}\colon \Lambda_{\mathsf{Path}_n} \to \Lambda_{\mathsf{Path}_n}$ be the operator

$$\mathrm{cyc}^d \prod_{i=n-d}^{1} (\tau_i \tau_{i+1} \cdots \tau_{i+d-1}) = \mathrm{cyc}^d (\tau_{n-d}\tau_{n-d+1} \cdots \tau_{n-1}) \cdots (\tau_2\tau_3 \cdots \tau_{d+1})(\tau_1\tau_2 \cdots \tau_d).$$

Using the identity $\mathrm{cyc}\,\tau_i = \tau_{i+1}\,\mathrm{cyc}$ together with the fact that τ_i and τ_j commute whenever $j \notin \{i-1, i+1\}$, one can show (see [4] for details) that

$$\Phi_{n,d}^{n/\gcd(n,d)} = \mathrm{TPro}_\pi^{\mathrm{lcm}(d,n-d)}. \tag{2.1}$$

Using a result about *friends-and-strangers graphs* from [3], one can prove that every orbit of $\Phi_{n,d}$ has size divisible by $n/\gcd(n,d)$ (see [4, Lemma 6.3]). A substantial portion of our full article is devoted to proving that every orbit of TPro_π has size divisible by $\mathrm{lcm}(d, n-d)$ (see [4, Proposition 5.1]). Together with (2.1), these divisibility results allow us to transfer the problem of describing the orbit structure of TPro_π to that of describing the orbit structure of $\Phi_{n,d}$. Thus, we deduce Theorem 2 from the following proposition and the fact that $\left(\mathsf{Comp}_d(n), \mathrm{Rot}_{n,d}, \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)$ exhibits the cyclic sieving phenomenon.

Proposition 1. *There is a map $\Omega\colon \mathrm{Orb}_{\Phi_{n,d}} \to \mathrm{Orb}_{\mathrm{Rot}_{n,d}}$ such that $|\Omega(\mathcal{O})| = \frac{d}{n}|\mathcal{O}|$ for every $\mathcal{O} \in \mathrm{Orb}_{\Phi_{n,d}}$ and $|\Omega^{-1}(\widehat{\mathcal{O}})| = d!(n-d)!$ for every $\widehat{\mathcal{O}} \in \mathrm{Orb}_{\mathrm{Rot}_{n,d}}$.*

2.2 Sliding Stones and Colliding Coins

We now discuss how to construct the map Ω from Proposition 1. Code implementing several of the combinatorial constructions described in this section can be found at `https://cocalc.com/hrthomas/permutoric-promotion/implementation`.

For each integer k, let $\theta_k = \tau_{q+d+1-r}$, where q and r are the unique integers satisfying $k = qd + r$ and $1 \le r \le d$. Let

$$\nu_\ell = \theta_{d\ell}\theta_{d\ell-1} \cdots \theta_{d(\ell-1)+2}\theta_{d(\ell-1)+1}.$$

Observe that $\theta_{k+dn} = \theta_k$ for all integers k. We have

$$\Phi_{n,d} = \mathrm{cyc}^d\, \theta_{d(n-d)} \cdots \theta_2\theta_1 = \mathrm{cyc}^d\, \nu_{n-d} \cdots \nu_2\nu_1.$$

By combining the identity $\operatorname{cyc}\tau_i = \tau_{i+1}\operatorname{cyc}$ with the fact that cyc^n is the identity map, one can easily verify that $\Phi_{n,d}^m = \theta_{md(n-d)}\cdots\theta_2\theta_1 = \nu_{m(n-d)}\cdots\nu_2\nu_1$ whenever m is a positive multiple of $n/\gcd(n,d)$.

Define a *state* to be a pair $(\sigma, t) \in \Lambda_{\mathsf{Path}_n} \times \mathbb{Z}$; we call σ the *labeling* of the state and say that the state is at *time* t. A *timeline* is a bi-infinite sequence $\mathcal{T} = (\sigma_t, t)_{t\in\mathbb{Z}}$ of states such that $\sigma_t = \nu_t(\sigma_{t-1})$ for all $t \in \mathbb{Z}$. Note that every state belongs to a unique timeline. For $\sigma \in \Lambda_{\mathsf{Path}_n}$, let $\mathcal{T}_\sigma$ be the unique timeline containing the state $(\sigma, 0)$.

Let $v_1, \ldots, v_n$ be the vertices of Path_n, listed from left to right. For each $\ell \in [n]$, let $\mathbf{v}_\ell$ be a formal symbol associated to v_ℓ; we will call $\mathbf{v}_\ell$ a *replica*. Let $\mathsf{s}_1, \ldots, \mathsf{s}_d$ be stones of different colors. We define the *stones diagram* of a state (σ, t) as follows. Start with a copy of Cycle_n, whose vertices we identify with $\mathbb{Z}/n\mathbb{Z}$. Place $\mathsf{s}_1, \ldots, \mathsf{s}_d$ on the vertices $t+d, \ldots, t+1$, respectively. Then place each replica $\mathbf{v}_\ell$ on the vertex $\sigma(v_\ell)$ of Cycle_n; if this vertex already has a stone sitting on it, then we place the replica on top of the stone.

Suppose we have a timeline $\mathcal{T} = (\sigma_t, t)_{t\in\mathbb{Z}}$. We want to describe how the stones diagrams of the states evolve as we move through the timeline. We will imagine transforming the stones diagram of $(\sigma_{t-1}, t-1)$ into that of (σ_t, t) via a sequence of d *small steps*. The i-th small step moves s_i one space clockwise. Now, $(\theta_{d(t-1)+i}\cdots\theta_{d(t-1)+1})(\sigma_{t-1})$ is obtained from $(\theta_{d(t-1)+i-1}\cdots\theta_{d(t-1)+1})(\sigma_{t-1})$ by applying the operator $\theta_{d(t-1)+i} = \tau_{t+d-i}$. If this operator has no effect, then we do not move any of the replicas $\mathbf{v}_1, \ldots, \mathbf{v}_n$ during the i-th small step (in this case, the stone s_i slides from underneath one replica to underneath a different replica). Otherwise, $\theta_{d(t-1)+i}$ has the effect of swapping the labels $t+d-i$ and $t+d-i+1$, so we swap the replicas that were sitting on the vertices $t+d-i$ and $t+d-i+1$ (in this case, the stone s_i carries the replica sitting on it along with it as it slides). Figure 1 illustrates these small steps for a particular example with $n = 8$, $d = 3$, and $t = 1$.

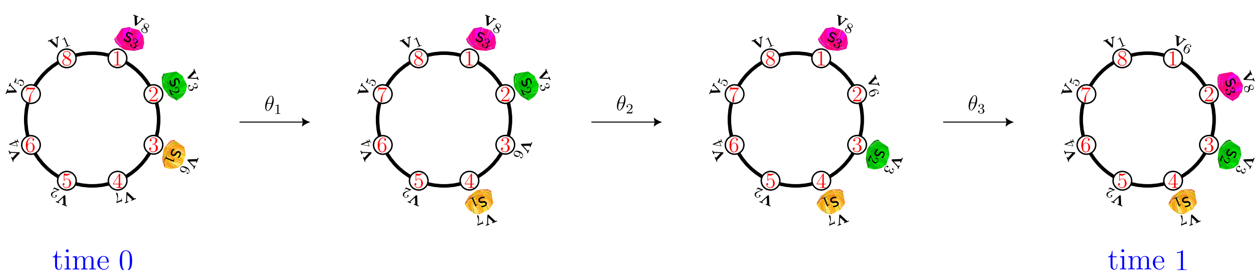

Figure 1: The $d = 3$ small steps transforming the stones diagram of a state at time 0 into the stones diagram of the next state at time 1.

Now consider d coins of different colors such that the set of colors of the coins is the same as the set of colors of the stones. We define the *coins diagram* of a state (σ, t) as follows. Start with a copy of Path_n. For each $i \in [d]$, there is a replica $\mathbf{v}_\ell$ sitting on the stone s_i in the stones diagram of (σ, t); place the coin with the same color as the stone s_i on the vertex v_ℓ (see Figures 2 and 3). Note that the set of vertices of Path_n occupied by coins is $\{\sigma^{-1}(t+1), \ldots, \sigma^{-1}(t+d)\}$.

Consider how the coins diagrams evolve as we move through a timeline $\mathcal{T} = (\sigma_t, t)_{t \in \mathbb{Z}}$. Let us transform the stones diagram of $(\sigma_{t-1}, t-1)$ into that of (σ_t, t) via the d small steps described above. Let $\mathbf{v}_\ell$ be the replica sitting on s_i right before the i-th small step, and let $\mathbf{v}_{\ell'}$ be the replica sitting on the vertex one step clockwise from s_i right before the i-th small step. When s_i moves in the i-th small step, it will either carry its replica $\mathbf{v}_\ell$ along with it or slide from underneath $\mathbf{v}_\ell$ to underneath $\mathbf{v}_{\ell'}$; the latter occurs if and only if $\ell' = \ell \pm 1$. In the former case, no coins move during the i-th small step; in the latter case, a coin moves from v_ℓ to the adjacent vertex $v_{\ell'}$ (which did not have a coin on it right before this small step).

If we watch the coins diagrams evolve as we move through the timeline, then by the previous paragraph, the coins will move around on Path_n, but they will never move through each other. Therefore, it makes sense to name the coins $\mathsf{c}_1, \ldots, \mathsf{c}_d$ in the order they appear along the path from left to right, and this naming only depends on the timeline (not the specific state in the timeline). Define a *traffic jam* to be a maximal nonempty collection of coins that occupy a contiguous block of vertices (so the vertices occupied by the coins in a particular traffic jam induce a connected subgraph of Path_n). Note that a traffic jam could have just a single coin. We say a traffic jam *touches a wall* if it contains a coin that occupies v_1 or v_n.

At any time, a coin has an idea of the direction in which it expects to move next (our coins are conscious now). Note that this is not necessarily the direction in which it will move next because it may change its mind before it moves. The way that a coin c decides which direction it expects to move is as follows. Suppose c currently occupies vertex v_j, and suppose the coins in the traffic jam containing c occupy the vertices $v_r, v_{r+1}, \ldots, v_s$. The coin c looks at the stones diagram and reads ahead in the clockwise direction, starting from the stone of its color, and it determines whether it first sees $\mathbf{v}_{r-1}$ or $\mathbf{v}_{s+1}$. If it first sees $\mathbf{v}_{r-1}$, it expects to move left; if it first sees $\mathbf{v}_{s+1}$, it expects to move right. If $r-1$ is not the index of a replica (because $r = 1$), the first replica that c sees will be $\mathbf{v}_{s+1}$; similarly, if $s+1$ is not the index of a replica (because $s = n$), the first replica c sees will be $\mathbf{v}_{r-1}$.

Figure 2 shows several stones diagrams and coins diagrams. In each coins diagram, an arrow has been placed over each coin to indicate which direction it expects to move.

Lemma 1 ([4])**.** *When a coin moves, it moves in the direction that it expects to move.*

The importance of understanding the direction in which a coin expects to move is that it will enable us to understand *collisions*. There are *two-coins collisions*, which involve two coins that occupy adjacent vertices of Path_n; there are *left-wall collisions*, which can occur when c_1 occupies v_1; and there are *right-wall collisions*, which can occur when c_d

occupies v_n. The prototypical examples of collisions are when two non-adjacent coins move to become adjacent or when a coin moves to become adjacent to a wall, but other examples are possible when traffic jams of size greater than 1 are involved.

The precise definition of a two-coins collision that occurs in a traffic jam that does not touch a wall is as follows. We say coins c_i and c_{i+1} are *butting heads* if they occupy adjacent vertices and c_i expects to move right while c_{i+1} expects to move left. We say c_i and c_{i+1} are involved in a two-coins collision at a small step if they are not butting heads immediately before the small step and they are butting heads immediately after the small step. This can happen either because the two coins were not adjacent prior to the small step, but it can also happen because the two coins were adjacent but one of them changed its mind about the direction it expected to move.

The definition of a collision has to be slightly modified when considering a traffic jam that touches a wall; the reader may refer to [4] for details.

We say a collision occurs at time t if it occurs during a small step between times $t-1$ and t.

Example 1. Suppose $n = 6$ and $d = 3$. Figure 2 shows some stones diagrams and coins diagrams evolving over time. At each stage, the arrow over a coin points in the direction that the coin expects to move. Collisions are indicated in the coins diagrams by stars, and each star is colored to indicate which stone moves in the small step during which the collision occurs. $\diamondsuit$

Example 2. Suppose $n = 6$ and $d = 3$. Figure 3 shows the stones diagrams and coins diagrams of a particular timeline at times $0, 1, \ldots, 17$. For brevity, we have not shown the individual small steps. All of the collisions the occur at time t (i.e., during the small steps between time $t-1$ and time t) are indicated in the coins diagram at time t. The color of the star can be used to determine the small step during which the collision occurs. One can check that the states in this timeline are periodic with period 18. $\diamondsuit$

Let $\mathrm{Coll}_{\mathcal{T}}$ be the set of all collisions that take place in the coins diagrams of the states of the timeline $\mathcal{T}$. We define a directed graph with vertex set $\mathrm{Coll}_{\mathcal{T}}$ by drawing an arrow from a collision κ to a collision κ' whenever there is a coin involved in both κ and κ' and the collision κ occurs before κ'. Let $(\mathrm{Coll}_{\mathcal{T}}, \leq_{\mathcal{T}})$ be the transitive closure of this directed graph. Let $\mathbf{H}_{\mathcal{T}}$ be the Hasse diagram of $(\mathrm{Coll}_{\mathcal{T}}, \leq_{\mathcal{T}})$. This Hasse diagram, which is one of our primary tools, has the shape of a bi-infinite chain link fence (see Figure 4). Suppose $\kappa_1 \lessdot_{\mathcal{T}} \kappa_2$ is an edge in $\mathbf{H}_{\mathcal{T}}$. Then κ_1 and κ_2 are collisions that both use some coin c; we define the *energy* of this edge, denoted $\mathcal{E}(\kappa_1 \lessdot_{\mathcal{T}} \kappa_2)$, to be the number of different vertices that c occupies between these two collisions, including the vertices occupied by c when the collisions occur. More generally, if $\kappa_1 \lessdot_{\mathcal{T}} \kappa_2 \lessdot_{\mathcal{T}} \cdots \lessdot_{\mathcal{T}} \kappa_r$ is a saturated chain in $\mathbf{H}_{\mathcal{T}}$, then we write $\mathcal{E}(\kappa_1 \lessdot_{\mathcal{T}} \kappa_2 \lessdot_{\mathcal{T}} \cdots \lessdot_{\mathcal{T}} \kappa_r)$ for the tuple $(\mathcal{E}(\kappa_1 \lessdot_{\mathcal{T}} \kappa_2), \ldots, \mathcal{E}(\kappa_{r-1} \lessdot_{\mathcal{T}} \kappa_r))$ of energies of the edges in the chain.

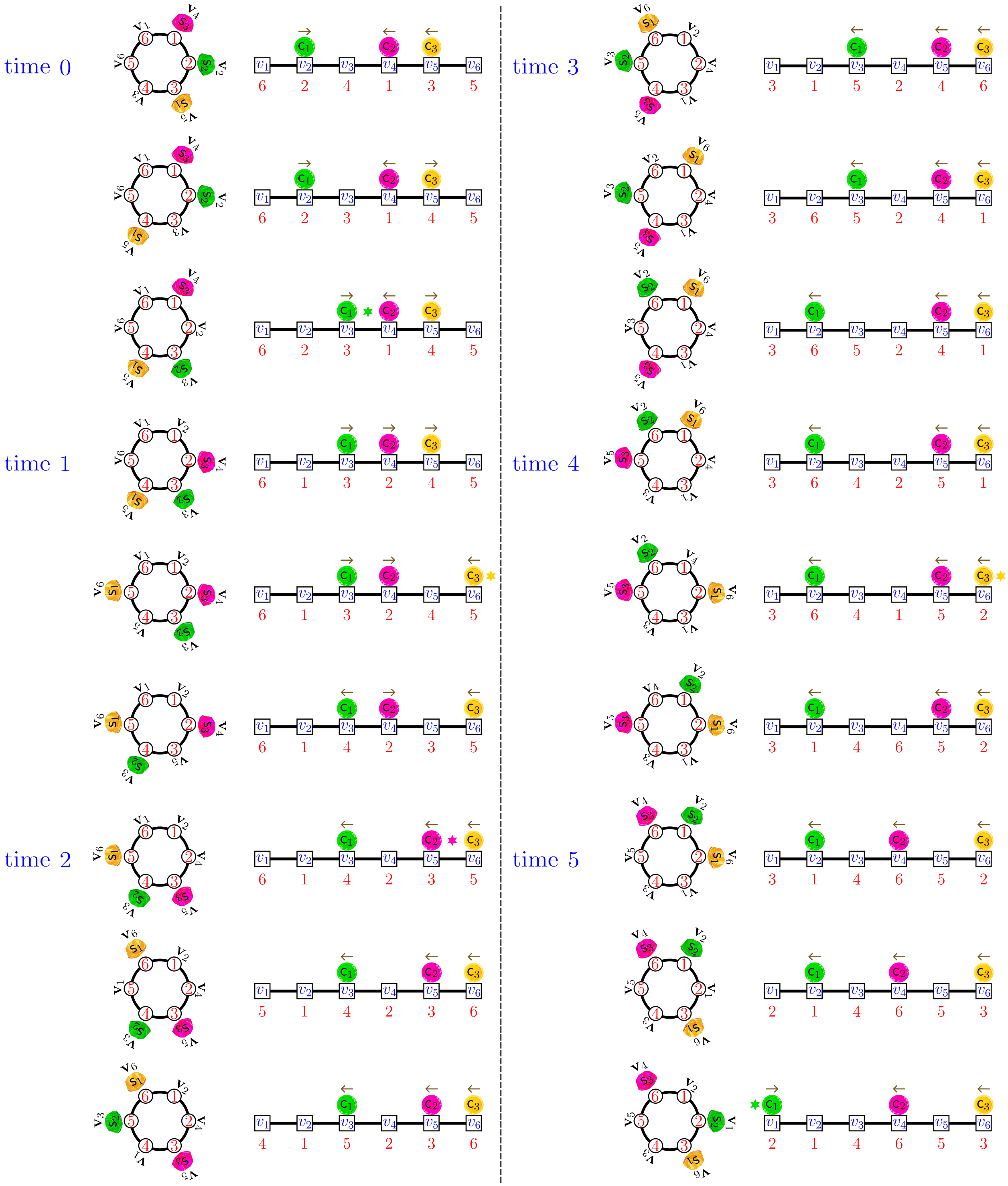

Figure 2: The evolution of stones diagrams and coins diagrams over time, with each individual small step illustrated. At each moment, we have drawn an arrow over each coin to indicate which direction it expects to move. Each collision is indicated by a star whose color is the same as that of the stone that moved to cause the collision. Each labeling is depicted in red numbers below the path.

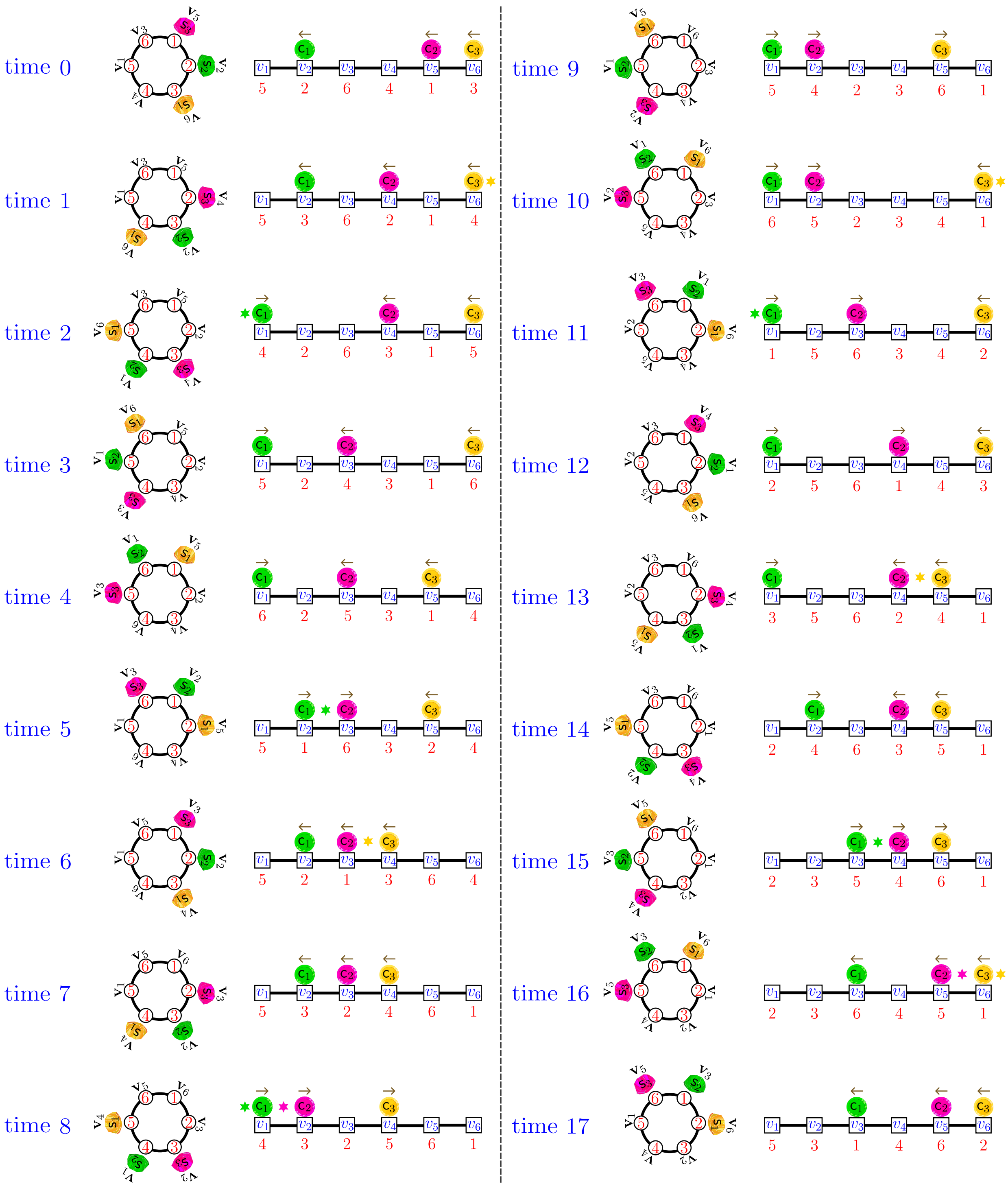

Figure 3: The stones diagrams and coins diagrams of the states in a timeline at times $0, 1, \ldots, 17$. Here, $n = 6$ and $d = 3$. The collisions that occur during the small steps between times $t-1$ and t are represented by color-coded stars in the coins diagram at time t. Each labeling is depicted by the red numbers below the path.

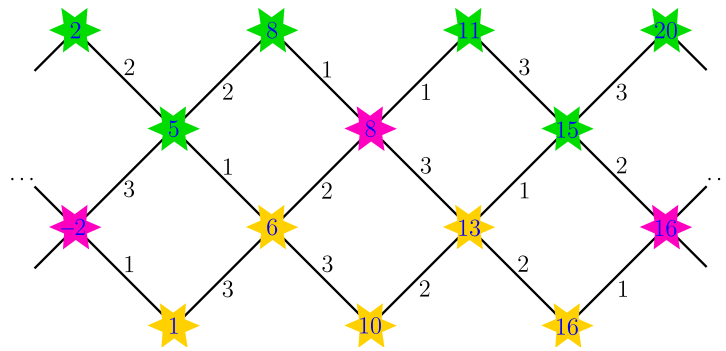

Figure 4: The Hasse diagram $\mathbf{H}_{\mathcal{T}}$, where $\mathcal{T}$ is the timeline containing the states whose stones diagrams and coins diagrams are shown in Figure 3. We have drawn the Hasse diagram sideways (to save vertical space), so each cover relation $\kappa \lessdot_{\mathcal{T}} \kappa'$ is drawn with κ to the left of κ'. Each collision is represented by a star whose color is the same as that of the stone that moved to cause the collision. Blue numbers indicate the times when the collisions occur. Edges are labeled by their energies.

A *diamond* in $\mathbf{H}_{\mathcal{T}}$ consists of collisions $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ together with four edges given by cover relations $\kappa_1 \lessdot_{\mathcal{T}} \kappa_2$, $\kappa_1 \lessdot_{\mathcal{T}} \kappa_3$, $\kappa_2 \lessdot_{\mathcal{T}} \kappa_4$, $\kappa_3 \lessdot_{\mathcal{T}} \kappa_4$. A *half-diamond* in $\mathbf{H}_{\mathcal{T}}$ consists of collisions $\kappa_1', \kappa_2', \kappa_3'$, where κ_1' and κ_3' are either both left-wall collisions or both right-wall collisions, together with two edges given by cover relations $\kappa_1' \lessdot_{\mathcal{T}} \kappa_2'$ and $\kappa_2' \lessdot_{\mathcal{T}} \kappa_3'$. Our arguments rely crucially on the following lemma.

Lemma 2 ([4]). *In any half-diamond in the Hasse diagram $\mathbf{H}_{\mathcal{T}}$, the two edges have the same energy. In any diamond in the Hasse diagram $\mathbf{H}_{\mathcal{T}}$, opposite edges have the same energy.*

For each collision $\kappa \in \mathrm{Coll}_{\mathcal{T}}$, let $\varphi(\kappa)$ be the collision involving the same set of coins as κ that occurs next after κ. In other words, if κ is the bottom element of a diamond (respectively, half-diamond), then $\varphi(\kappa)$ is the top element of that same diamond (respectively, half-diamond). We extend this notation to saturated chains in $\mathbf{H}_{\mathcal{T}}$ (including edges) by letting $\varphi(\kappa_1 \lessdot_{\mathcal{T}} \kappa_2 \lessdot_{\mathcal{T}} \cdots \lessdot_{\mathcal{T}} \kappa_m) = \varphi(\kappa_1) \lessdot_{\mathcal{T}} \varphi(\kappa_2) \lessdot_{\mathcal{T}} \cdots \lessdot_{\mathcal{T}} \varphi(\kappa_m)$. We define the *period* of $\mathbf{H}_{\mathcal{T}}$ to be the smallest positive integer p such that e and $\varphi^p(e)$ have the same energy for every edge e of $\mathbf{H}_{\mathcal{T}}$. A *transversal* of $\mathbf{H}_{\mathcal{T}}$ is a saturated chain $\mathscr{T} = (\kappa_0 \lessdot_{\mathcal{T}} \kappa_1 \lessdot_{\mathcal{T}} \cdots \lessdot_{\mathcal{T}} \kappa_d)$ such that κ_0 is a left-wall collision, κ_d is a right-wall collision, and κ_i involves the stones c_i and c_{i+1} for every $i \in [d-1]$. In other words, a transversal is a saturated chain that moves from left to right across $\mathbf{H}_{\mathcal{T}}$. We define the *energy composition* of $\mathscr{T}$ to be the tuple $\mathcal{E}(\mathscr{T}) = (\varepsilon_1, \ldots, \varepsilon_d)$, where ε_i is the energy of the edge $\kappa_{i-1} \lessdot_{\mathcal{T}} \kappa_i$; note that $\mathcal{E}(\mathscr{T}) \in \mathsf{Comp}_d(n)$.

Lemma 3 ([4]). *Let $\mathcal{T}$ be a timeline, and let $\mathscr{T}$ be a transversal of $\mathbf{H}_{\mathcal{T}}$. Then $\mathcal{E}(\varphi(\mathscr{T})) = \mathrm{Rot}_{n,d}(\mathcal{E}(\mathscr{T}))$. The period of $\mathbf{H}_{\mathcal{T}}$ is equal to the size of the orbit of $\mathrm{Rot}_{n,d}$ containing $\mathcal{E}(\mathscr{T})$.*

Proof. The second statement follows from the first because, by Lemma 2, the energies of all edges in $\mathbf{H}_{\mathcal{T}}$ are determined by the energy composition of a single transversal of $\mathbf{H}_{\mathcal{T}}$. The first statement is also immediate from Lemma 2. $\square$

Example 3. Suppose $n = 6$ and $d = 3$. Let $\mathbf{H}_{\mathcal{T}}$ be the Hasse diagram from Figure 4, and let $\mathscr{T} = (\kappa_0 \lessdot_{\mathcal{T}} \kappa_1 \lessdot_{\mathcal{T}} \kappa_2 \lessdot_{\mathcal{T}} \kappa_3)$ be the transversal consisting of the collisions that occur at times $2, 5, 6, 10$. Then $\mathcal{E}(\mathscr{T}) = (2, 1, 3) \in \mathsf{Comp}_3(6)$. The period of $\mathbf{H}_{\mathcal{T}}$ is 3, which is the size of the $\mathrm{Rot}_{6,3}$-orbit containing $(2, 1, 3)$. The transversal $\varphi(\mathscr{T})$ consists of both the collisions that occur at time 8 along with the collisions at times 13 and 16. We have $\mathcal{E}(\varphi(\mathscr{T})) = (1, 3, 2) = \mathrm{Rot}_{6,3}(\mathcal{E}(\mathscr{T}))$. Similarly, $\mathcal{E}(\varphi^2(\mathscr{T})) = (3, 2, 1) = \mathrm{Rot}_{6,3}^2(\mathcal{E}(\mathscr{T}))$. $\Diamond$

For $k, t \in \mathbb{Z}$, let $\sigma_t^{(k)} = \mathrm{cyc}^{-k}(\sigma_{t+k})$. It follows immediately from the definition of a timeline that the sequence $\mathcal{T}^{(k)} = (\sigma_t^{(k)}, t)_{t \in \mathbb{Z}}$ is also a timeline; that is, $\nu_t(\sigma_{t-1}^{(k)}) = \sigma_t^{(k)}$ for all $t \in \mathbb{Z}$. Furthermore, the stones diagram of $(\sigma_t^{(k)}, t)$ is obtained from that of $(\sigma_{t+k}, t + k)$ by moving all stones and replicas k positions counterclockwise. It follows that the coins diagrams of $(\sigma_t^{(k)}, t)$ and $(\sigma_{t+k}, t + k)$ are identical. Therefore, if κ is a collision in $\mathrm{Coll}_{\mathcal{T}^{(k)}}$ that occurs at time t, then there is a collision $\psi_k(\kappa) \in \mathrm{Coll}_{\mathcal{T}}$ that occurs at time $t + k$. The resulting map $\psi_k \colon \mathrm{Coll}_{\mathcal{T}^{(k)}} \to \mathrm{Coll}_{\mathcal{T}}$ is an isomorphism from $(\mathrm{Coll}_{\mathcal{T}^{(k)}}, \leq_{\mathcal{T}^{(k)}})$ to $(\mathrm{Coll}_{\mathcal{T}}, \leq_{\mathcal{T}})$; furthermore, under this isomorphism, corresponding edges of the Hasse diagrams $\mathbf{H}_{\mathcal{T}^{(k)}}$ and $\mathbf{H}_{\mathcal{T}}$ have the same energy.

Recall that we write $\mathcal{T}_\sigma$ for the unique timeline containing the state $(\sigma, 0)$. It follows from Lemma 3 that the energy compositions of the transversals of $\mathbf{H}_{\mathcal{T}_\sigma}$ form a single orbit $\widetilde{\Omega}(\sigma)$ of $\mathrm{Rot}_{n,d}$. If $\mathcal{T}_\sigma = (\sigma_t, t)_{t \in \mathbb{Z}}$ (so $\sigma_0 = \sigma$), then $\Phi_{n,d}(\sigma_0) = \sigma_0^{(n-d)}$, so $\mathcal{T}_{\Phi_{n,d}(\sigma_0)} = \mathcal{T}_\sigma^{(n-d)}$. Using the isomorphism ψ_{n-d}, we find that $\widetilde{\Omega}(\sigma_0) = \widetilde{\Omega}(\Phi_{n,d}(\sigma_0))$. Thus, we obtain a map

$$\Omega \colon \mathrm{Orb}_{\Phi_{n,d}} \to \mathrm{Orb}_{\mathrm{Rot}_{n,d}}$$

that sends the $\Phi_{n,d}$-orbit containing a labeling μ to $\widetilde{\Omega}(\mu)$. In [4], we prove that this map satisfies the conditions in Proposition 1; we omit the proof here.

3 Toric Promotion on a Forest

Let us briefly mention how the perspective of stones and coins diagrams can be used to prove Theorem 1. Let $G = (V, E)$ be a forest. Let $v_1, \ldots, v_n$ be the vertices of G, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be their replicas. We can represent a labeling $\sigma \in \Lambda_G$ by placing each replica $\mathbf{v}_k$ on the vertex $\sigma(v_k)$ of Cycle_n. We again place a stone on a vertex of Cycle_n to indicate which toggle we are about to apply, and we put a coin on the vertex of G whose replica sits on the stone.

Let T be the connected component of G containing $\sigma^{-1}(1)$; this is the connected component on which the coin always sits. Let t be the number of vertices of T. As we apply the sequence of toggles $\tau_1, \tau_2, \ldots$ (repeating cyclically), the coin will move around to all of the vertices in T. One can show that for all vertices $v_j, v_{j'} \in V$ such that v_j is in T and $j \neq j'$, there is a unique time in the interval $[1, t(n-1)]$ during which $\mathbf{v}_j$ sits on the stone and $\mathbf{v}_{j'}$ sits one space clockwise of the stone. This implies that if v_k is a vertex of degree δ in T, then there are $n - \delta - 1$ times in the interval $[1, t(n-1)]$ when $\mathbf{v}_k$ rides clockwise one space on the stone, and there are $t - \delta - 1$ times in the interval $[1, t(n-1)]$ when $\mathbf{v}_k$ moves counterclockwise one space because the stone slides through it. On the other hand, if v_k is a vertex that is not in T, then $\mathbf{v}_k$ never rides on the stone, and there are t times in the interval $[1, t(n-1)]$ when $\mathbf{v}_k$ moves counterclockwise one space because the stone slides through it. It follows that applying $t(n-1)$ toggles has the effect of rotating the stone and all of the replicas counterclockwise by t. This implies the desired result.

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Smirnov words and the Delta Conjectures

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Abstract. We provide a combinatorial interpretation for the symmetric function $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$ in terms of Smirnov words, which are words where adjacent letters are distinct. The motivation for this work is the study of a diagonal coinvariant ring with one set of commuting and two sets of anti-commuting variables, whose Frobenius characteristic is conjectured to be the symmetric function in question. It is intimately related to the two Delta conjectures, as our work is a step towards a unified formulation of these.

Résumé. Nous donnons une interprétation combinatoire à la fonction symétrique $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$ en termes de mots de Smirnov, qui sont les mots dont les lettres adjacentes sont distinctes. La motivation de ce travail est l'étude de l'anneau des coinvariants diagonaux avec un jeu de variables commutatives et deux jeux de variables anticommutatives, dont la caractéristique de Frobenius est, conjecturalement, la fonction symétrique en question. Elles est intimement liée aux conjectures Delta, ce travail constituant un pas vers une formulation unifiée de ces dernières.

Keywords: Delta conjecture, coinvariant ring, Smirnov words

1 Introduction

This work is mainly concerned with a combinatorial expansion and its consequences. It is motivated by a circle of problems in representation theory, which we briefly survey in this introduction.

In the 1990s, Garsia and Haiman introduced the ring of *diagonal coinvariants* DR_n. The study of the structure of this $\mathfrak{S}_n$-module and its generalizations has been an important research topic in algebra and combinatorics ever since. The ring is defined as follows: consider the space $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] \coloneqq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ and define an $\mathfrak{S}_n$-action as

$$\sigma \cdot f(x_1, \dots, x_n, y_1, \dots, y_n) \coloneqq f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)})$$

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for all $f \in \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$ and $\sigma \in \mathfrak{S}_n$. Let $I(\mathbf{x}_n, \mathbf{y}_n)$ be the ideal generated by the $\mathfrak{S}_n$-invariants with vanishing constant term. Then the ring of diagonal coinvariants is defined as

$$DR_n := \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/I(\mathbf{x}_n, \mathbf{y}_n).$$

The space has a natural bi-grading: let $DR_n^{(i,j)}$ be the component of DR_n with homogeneous $\mathbf{x}$-degree i and homogeneous $\mathbf{y}$-degree j. This grading is preserved by the $\mathfrak{S}_n$-action. Garsia and Haiman conjectured, and Haiman later proved [9], a formula for the graded Frobenius characteristic of the diagonal harmonics:

$$\operatorname{grFrob}(DR_n; q, t) := \sum_{i,j \in \mathbb{N}} q^i t^j \operatorname{Frob}(DR_n^{(i,j)}) = \nabla e_n, \tag{1.1}$$

where e_n is the n-th elementary symmetric function and ∇ is the operator introduced in [1]. In [7], the authors gave a combinatorial formula for this graded Frobenius character ∇e_n, in terms of *labelled Dyck paths*, called the *shuffle conjecture*. It is now a theorem by Carlsson and Mellit [2].

The *Delta conjecture* is a pair of combinatorial formulas for the symmetric function $\Delta'_{e_{n-k-1}} e_n$ in terms of *decorated* labelled Dyck paths, stated in [8] –we detail the combinatorics in Section 5. Here $\Delta'_{e_{n-k-1}}$ is a certain symmetric function operator (depending on q, t). These conjectures reduce to the shuffle theorem when $k = 0$.

This extension of the combinatorial setting led Zabrocki, D'Adderio, Iraci and Vanden Wyngaerd to introduce extensions of DR_n [15, 3]. Consider the ring $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]$ where the $\mathbf{x}_n, \mathbf{y}_n$ are the usual commuting (or *bosonic*) variables, while the $\boldsymbol{\theta}_n$, $\boldsymbol{\xi}_n$ are anti-commuting (or *fermionic*): $\theta_i\theta_j = -\theta_j\theta_i$ and $\xi_i\xi_j = -\xi_j\xi_i$ for all $1 \leq i, j \leq n$.

Again, consider the $\mathfrak{S}_n$-action that permutes all the variables simultaneously. If $I(\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n)$ now denotes the ideal generated by the $\mathfrak{S}_n$-invariants without constant term, define $TDR_n := \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]/I(\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n)$. This ring is naturally quadruply graded: let $TDR_n^{(i,j,k,l)}$ denote the component of TDR_n of homogeneous (i, j, k, l)-degrees.

In [15] Zabrocki conjectured

$$\sum_{i,j \in \mathbb{N}} q^i t^j \operatorname{Frob}(TDR_n^{(i,j,k,0)}) \stackrel{?}{=} \Delta'_{e_{n-k-1}} e_n. \tag{1.2}$$

Note that the symmetric function of the Delta conjectures occurs on the right-hand side. In [3], D'Adderio with the first and third named authors introduced operators Θ_f (depending on q, t), for any symmetric function f, and showed that $\Delta'_{n-k-1} e_n = \Theta_{e_k} \nabla e_{n-k}$. This permitted them to extend Zabrocki's conjecture as follows:

$$\sum_{i,j \in \mathbb{N}} q^i t^j \operatorname{Frob}(TDR_n^{(i,j,k,l)}) \stackrel{?}{=} \Theta_{e_l} \Theta_{e_k} \nabla e_{n-k-l}. \tag{1.3}$$

Special cases of the conjecture have been studied over the years. Let us call the "(a, b)-case" the structures linked to the diagonal coinvariant ring with a sets of bosonic variables and b sets of fermionic variables. The $(2, 1)$- and the $(2, 2)$-cases thus occur in (1.2) and (1.3) respectively, and the $(2, 0)$-case is the known case (1.1). The $(1, 0)$ and $(0, 1)$ cases are classical rings and the conjecture is known to hold in this case. The $(1, 1)$-case, or the *superspace coinvariant ring*, is still open, but Rhoades and Wilson in [12] showed that its Hilbert series agrees with the expected formula. The $(0, 2)$-case, or *fermionic Theta* case, was proved by Iraci, Rhoades, and Romero in [10].

In this abstract, we will turn our interest to the combinatorics that (conjecturally) occur in the $(1, 2)$-case. Following Conjecture (1.3), we thus are led to study the symmetric function $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$.

Our combinatorial model is that of *segmented Smirnov words*. A Smirnov word is a word in the alphabet of positive integers such that adjacent letters are distinct. A segmented Smirnov word is the concatenation of Smirnov words with prescribed lengths (see Definition 2.1). The main result of this paper (Theorem 2.5) is an expansion in terms of segmented Smirnov words.

Theorem. *For any n, k, l, we have the identity between symmetric functions in $(x_i)_{i\geq 1}$*

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0} = \sum_{w\in \mathsf{SW}(n,k,l)} q^{\mathsf{sminv}(w)} x_{w_1}x_{w_2}\cdots x_{w_n}.$$

Here $\mathsf{SW}(n, k, l)$ is the set of segmented Smirnov words with k descents and l ascents, while the power of q is given by a new *sminversion* statistic on these words (see Definition 2.3). This expansion can be expressed more compactly in terms of fundamental quasisymmetric functions (Proposition 2.7).

The proof of the main theorem relies on an algebraic recursion (Proposition 2.4) for the symmetric function under study. We show in Section 3 that the combinatorial expansion satisfies indeed the same recursion.

In Section 4, we focus on the special case $k + l = n - 1$ which turns out to be linked to various topics in the literature. In Section 5, we describe an explicit bijection between segmented Smirnov words and "doubly decorated labelled Dyck paths" (Theorem 5.1), motivated by a potential unified Delta conjecture.

2 Preliminaries and main result

Combinatorics. In this work $\mathbb{Z}_+$ is the set of positive integers, and we will fix $n \in \mathbb{Z}_+$. We write $\mu \vDash_0 n$ if μ is a *weak composition* of n, that is $\mu = (\mu_1, \mu_2, \ldots)$ where the μ_i are nonnegative integers that sum to n. A composition $\alpha \vDash n$ is a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_t)$ of positive integers that sums to n.

Definition 2.1. A *Smirnov word* of length n is an element $w \in \mathbb{Z}_+^n$ such that $w_i \neq w_{i+1}$ for all $1 \leq i < n$. A *segmented Smirnov word* is a word $w \in \mathbb{Z}_+^n$ together with a composition $\alpha = (\alpha_1, \dots, \alpha_t) \vDash n$ such that if w is written as the concatenation $w^1 \cdots w^t$ where each w^i has length α_i, then each w^i is a Smirnov word.

Let $\mathsf{SW}(n)$ be the set of segmented Smirnov words of length n. We say that α is the *shape* of w. We call $w^1, \dots, w^t$ *segments* of w. We usually simply denote a segmented Smirnov word by w, and omit the shape α. In examples, we separate segments by vertical bars. Segmented Smirnov words of shape (n) are naturally identified with Smirnow words of length n.

Given $\mu \vDash_0 n$, we denote by $\mathsf{SW}(\mu)$ the set of segmented Smirnov words with content μ, that is they contain μ_1 occurrences of 1, μ_2 occurrences of 2, and so on. We clearly have $\mathsf{SW}(n) = \bigcup_{\mu \vDash_0 n} \mathsf{SW}(\mu)$. We call *segmented permutation* a segmented Smirnov word in $\mathsf{SW}(1^n)$. Note that these can be identified with pairs (σ, α) with $\sigma \in S_n$ and $\alpha \vDash n$.

Example 2.2. If $\mu = (2,1)$, then $\mathsf{SW}(\mu)$ has 8 elements: $1|1|2, 1|2|1, 2|1|1$ with shape $(1,1,1)$; $1|12, 1|21$ with shape $(1,2)$; $21|1, 12|1$ with shape $(2,1)$ and 121 with shape (3).

Given a Smirnov word w, we say that i is an *ascent* of w if $w_{i+1} > w_i$, and a *descent* otherwise. If $w \in \mathsf{SW}(n)$, we say that i is an ascent (resp. descent) of w if it is an ascent (resp. descent) of one of its segments. Let us denote by $\mathsf{SW}(n,k,l)$ the set of segmented Smirnov words with k descents and l ascents; note that these words have $n-k-l$ segments. For $\mu \vDash_0 n$, we also define $\mathsf{SW}(\mu,k,l)$ as the intersection $\mathsf{SW}(\mu) \cap \mathsf{SW}(n,k,l)$.

We can now define the main new statistic of this work. An index $i \in \{1, \dots, n\}$ is called *initial* (resp. *final*) if it corresponds to the first (resp. last) position of a segment, *i.e.* if it has the form $i = \alpha_1 + \cdots + \alpha_{m-1} + 1$ (resp. $i = \alpha_1 + \cdots + \alpha_m$) for some $t \in \{1, \dots, t\}$.

Definition 2.3 (The sminv statistic)**.** For a segmented Smirnov word w of shape $\alpha \vDash n$, we say that (i,j) with $1 \leq i < j \leq n$ is a *sminversion* if $w_i > w_j$ and one of the following holds:

1. j is initial in w;
2. $w_{j-1} > w_i$;
3. $i \neq j-1$, $w_{j-1} = w_i$, and $j-1$ is initial in w;
4. $i \neq j-1$ and $w_{j-2} > w_{j-1} = w_i$.

We let $\mathsf{sminv}(w)$ be the number of sminversions of w. The segmented Smirnov word $w = 321|2131$, has sminv equal to 4, since $(1,4)$, $(2,5)$, $(2,7)$ and $(4,7)$ are its sminversions. Finally, define

$$\mathsf{SW}_q(\mu,k,l) = \sum_{w \in \mathsf{SW}(\mu,k,l)} q^{\mathsf{sminv}(w)}.$$

In view of Example 2.2, we can compute that $\mathsf{SW}_q((2,1),0,0) = 1 + q + q^2$; $\mathsf{SW}_q((2,1),1,0) = 1 + q$; $\mathsf{SW}_q((2,1),1,1) = 1$; and $\mathsf{SW}_q((2,1),0,1) = 1 + q$.

Let us note two important cases where the statistic sminv simplifies:

- When w is a segmented permutation σ, only cases (1) and (2) occur.
- When w has shape (n), *i.e.* w is a Smirnov word, only cases (2) and (4).

Symmetric functions. We refer to [14, Ch. 7] for undefined terminology. Consider the ring Λ of symmetric functions in $(x_i)_{i\in\mathbb{Z}_+}$ with coefficients in $\mathbb{Q}(q)$. Let us define

$$\mathsf{SF}(n,k,l) := \Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0} \in \Lambda \tag{2.1}$$

to simplify notations. Here $h_j^\perp$ is the operator dual to multiplication by h_j, with respect to the standard duality on Λ given by $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$.

The following proposition is the key to the combinatorial interpretation:

Proposition 2.4. *For any n,k,l with $k+l<n$, $\mathsf{SF}(n,k,l)$ satisfies*

$$\begin{aligned} h_j^\perp \mathsf{SF}(n,k,l) = \sum_{r=0}^{j}\sum_{a=0}^{j}\sum_{i=0}^{j} q^{\binom{r-i}{2}} q^{\binom{a-i}{2}} \begin{bmatrix} n-k-l \\ j-r-a+i \end{bmatrix}_q \begin{bmatrix} n-k-l-(j-r-a)-1 \\ i \end{bmatrix}_q \\ \times \begin{bmatrix} n-k-l-(j-r-a+i) \\ r-i \end{bmatrix}_q \begin{bmatrix} n-k-l-(j-r-a+i) \\ a-i \end{bmatrix}_q \mathsf{SF}(n-j,k-r,l-a) \end{aligned}$$

for any $j\geq 1$. Moreover $\mathsf{SF}(0,k,l)=\delta_{k,0}\delta_{l,0}$ and $\mathsf{SF}(n,k,l)=0$ if $n<0$.

We omit the proof of this proposition in this abstract: it comes from the specialization $t=0$ of [6, Theorem 8.2], with some extra elementary computations.

Main result. Define

$$\mathsf{SW}_{x;q}(n,k,l) = \sum_{\mu\vDash_0 n} \mathsf{SW}_q(\mu,k,l)x^\mu = \sum_{w\in\mathsf{SW}(n,k,l)} q^{\mathsf{sminv}(w)}x_w, \tag{2.2}$$

where $x_w = \prod_{i=1}^n x_{w_i}$.

Theorem 2.5. *For any n,k,l with $k+l<n$, we have the identity*

$$\mathsf{SF}(n,k,l) = \mathsf{SW}_{x;q}(n,k,l). \tag{2.3}$$

Expansion into fundamental quasisymmetric functions. Let w be a segmented Smirnov word. For $1\leq i\leq n$, we say that i is *thick* if i is initial or $w_{i-1}>w_i$, and *thin* otherwise.

Definition 2.6. Let σ be a segmented permutation of size n, and $i\in\{1,\dots,n\}$. Let j be such that $\sigma_j=\sigma_{i+1}$. We say that i is *splitting* for σ if either of the following holds:

- i and j are in the same segment of σ, and $|i-j|=1$;
- i is thick and j is thin;
- i and j are both thin and $i<j$;
- i and j are both thick and $j<i$.

Let $\mathsf{Split}(\sigma) = \{1 \leq i \leq n-1 \mid i \text{ is splitting for } \sigma\}$. For any subset $S \subseteq [n-1]$, let $Q_{S,n}$ be the fundamental quasisymmetric function associated to S (see [14, Sec. 7.19]).

Proposition 2.7.

$$\mathsf{SW}_{x;q}(n,k,l) = \sum_{\sigma \in \mathsf{SW}(1^n,k,l)} q^{\mathsf{sminv}(\sigma)} Q_{\mathsf{Split}(\sigma),n}.$$

The proof relies on grouping terms in the right-hand side of (2.2) using a certain "reading order". We omit it in this abstract.

3 Proof of Theorem 2.5

The proof consists in showing that the series $\mathsf{SW}_{x;q}(n,k,l)$ satisfies the relations encoded in Proposition 2.4.

In detail, fix $\mu \vDash_0 n$ nonzero, let F_μ be the coefficient of x^μ in the power series $\mathsf{SF}(n,k,l)$, and let the last nonzero part of μ be $\mu_m = j$. Then by taking the inner product of $\mathsf{SF}(n,k,l)$ with h_μ in Proposition 2.4, we obtain a recurrence for F_μ. Theorem 2.5 then claims that $\mathsf{SW}_q(\mu,k,l)$ obeys the same recurrence. Explicitly, let μ^- be equal to μ except that $\mu^-_m = 0$, and let $s := n-k-l$ be the number of segments, then one has to show:

$$\begin{aligned}\mathsf{SW}_q(\mu,k,l) = \sum_{i=0}^{j}\sum_{r=i}^{j}\sum_{a=i}^{j} q^{\binom{r-i}{2}} \begin{bmatrix} s-(j-r-a+i) \\ r-i \end{bmatrix}_q q^{\binom{a-i}{2}} \begin{bmatrix} s-(j-r-a+i) \\ a-i \end{bmatrix}_q \\ \times \begin{bmatrix} s \\ j-r-a+i \end{bmatrix}_q \begin{bmatrix} s-j+r+a-1 \\ i \end{bmatrix}_q \mathsf{SW}_q(\mu^-,k-r,l-a). \qquad (3.1)\end{aligned}$$

We will sketch a bijective proof below. Since it is quite technical, let us first give the simpler proof in the case $\mu = 1^n$, which boils down to the following proposition:

Proposition 3.1. *For any n,k,l with $k+l<n$, the polynomials $\mathsf{SW}_q(1^n,k,l)$ satisfy*

$$\begin{aligned}\mathsf{SW}_q(1^n,k,l) = [n-k-l]_q \Big(\mathsf{SW}_q(1^{n-1},k,l) + \mathsf{SW}_q(1^{n-1},k-1,l) \\ + \mathsf{SW}_q(1^{n-1},k,l-1) + \mathsf{SW}_q(1^{n-1},k-1,l-1)\Big).\end{aligned}$$

Proof. Given a segmented permutation on $n-1$ elements, we want to insert n in all possible ways. It can be done in four different manners:

1. as a new singleton segment. This keeps the number of ascents and descents the same, and increases the number of segments by one;
2. at the beginning of a segment. This creates no ascent and one descent, and keeps the number of segments the same;
3. at the end of a segment. This creates one ascent and no descents, and keeps the number of segments the same;

4. as an element merging two adjacent segments $\cdots w_1|w_2\cdots \to \cdots w_1 n w_2 \cdots$. This creates an ascent and a descent, and decreases the number of segments by one.

Each of these insertions can be done in s different ways, if s is the number of segments in the final segmented permutation. Moreover, the construction is injective: if i is such that $\sigma_i = n$ for some $\sigma \in \mathsf{SW}(1^n)$, then by looking whether i is initial and/or final, one knows which of the four types of insertion was performed.

From this one sees that the proposition holds at $q = 1$. As for sminversions, one checks that inserting n does not modify the number of those involving letters in $\{1, \dots, n-1\}$. Moreover, the value n is part of a sminversion with all initial letters to its right. In each case, this increases sminv by all possible amounts between 0 and $s-1 = n-k-l-1$. The recursion of Proposition 3.1 follows.

□

Sketch of the proof of (3.1). The idea is the same as in the standard case above. Starting with a word in $w \in \mathsf{SW}(\mu^-)$, we want to insert j occurrences of the letter m (larger than all letters of w) to create a word w' in $\mathsf{SW}(\mu, k, l)$. As in the standard case, we distinguish if the occurrences of m are initial and/or final. The complication comes from inserting several occurrences of m.

Pick $i, a, r \geq 0$ such that $i \leq a \leq j$ and $i \leq r \leq j$. Then we insert *successively*:

- i is occurrences of m that are neither initial nor final (this is done by merging adjacent segments as in the standard case);
- $r - i$ occurrences of m that initial but not final;
- $a - i$ occurrences of m that are final but not initial;
- and finally $j - r - a + i$ singletons equal to m.

Note that the total number of occurrences of m is indeed j. Since we want $s = n - k - l$ segments in the end, we must have $s + i - (j - r - a + i) = s - j + r + a$ segments in w. Also, w must have $k - r$ descents and $l - a$ ascents so that the final word has k descents and l ascents.

The claim is that the number of ways to insert m is given by the coefficient of $\mathsf{SW}_q(\mu^-, k-r, l-a)$ in (3.1) at $q = 1$: each of the four binomial coefficients can be whown to correspond naturally to one of the cases above. To complete the proof, one needs to check that then number of sminversions behaves as wanted. We omit the details in this abstract. □

4 The maximal case $k + l = n - 1$

We focus in this section on various aspects of the case $k + l = n - 1$ of Theorem 2.5. The combinatorial side now involves only Smirnov words. It is also conjecturally giving the graded Frobenius characteristic of the subspace of the $(1, 2)$-coinvariant space of maximum total degree in the fermionic variables $\boldsymbol{\zeta}_n, \boldsymbol{\xi}_n$ (cf. (1.3)).

Chromatic symmetric function interpretation. Given a graph $G = (V, E)$, a *proper coloring* is a function $c\colon V \to \mathbb{Z}_+$ such that $\{i, j\} \in E \implies c(i) \neq c(j)$. If $V = [n]$, a *descent* of a coloring is an edge $\{i, j\} \in E$ such that $i < j$ and $c(i) > c(j)$. The chromatic quasisymmetric function of G is defined as

$$X_G(\mathbf{x}; q) = \sum_{\substack{c\colon V \to \mathbb{Z}_+ \\ c \text{ proper}}} q^{\mathsf{des}(c)} \prod_{v \in V} x_{c(v)},$$

where $\mathsf{des}(c)$ is the number of descents of c.

For the path graph $G_n = 1 - 2 - \cdots - n$, if c is a proper coloring then $c(1)c(2)\ldots c(n)$ is a Smirnov word of length n, and vice versa, if w is a Smirnov word of length n, then $c(i) = w_i$ is a proper coloring of G_n. It follows from Theorem 2.5

$$X_{G_n}(\mathbf{x}; u) = \sum_{k=0}^{n-1} u^k \Theta_{e_k} \Theta_{e_{n-k-1}} e_1 \Big|_{q=1, t=0}.$$

This suggests also the existence of an extra q-grading on the cohomology of the permutahedral toric variety $\mathcal{V}_n$: indeed the graded Frobenius characteristic of that cohomology is known to be given by $\omega X_{G_n}(\mathbf{x}; u)$, see [13].

Parallelogram polyominoes. A *parallelogram polyomino* of size $m \times n$ is a pair of north-east lattice paths on a $m \times n$ grid, such that first is always strictly above the second, except on the endpoints $(0, 0)$ and (m, n). A labelling of a parallelogram polyomino is an assignment of positive integer labels to each cell that has a north step of the first path as its left border, or an east step of the second path as its bottom border, such that columns are strictly increasing bottom to top and rows are strictly decreasing left to right. In [5] it is conjectured that $\Theta_{e_{m-1}} \Theta_{e_{n-1}} e_1$ enumerates labelled parallelogram polyominoes of size $m \times n$ with respect to two statistics, one of which is (a labelled version of) the area, and the other is unknown.

It is immediate to see that parallelogram polyominoes of size $(n - k) \times (k + 1)$ and area 0 are again in bijection with Smirnov words of length n with k descents, and proper colorings of G_n with k descents. Indeed, reading the labels of such a polyomino bottom to top, left to right, yields a Smirnov word of size n with k descents, and the correspondence is bijective. In particular, sminversions on Smirnov words define a statistic on this subfamily of parallelogram polyominoes, proving the conjectural identity and partially answering Problem 7.13 from [5] in the case when the area is 0.

The case $q = 0$. Note that in this case, it is known [10] that the symmetric function in Theorem 2.5 is the Frobenius characteristic of the $(0, 2)$-case. It was also shown that the high-degree part of this module has a basis indexed by noncrossing partitions. In particular, this means that there is a bijection between segmented permutations with one segment (that is, permutations) with zero sminv, and noncrossing partitions.

Lemma 4.1. *Permutations with zero* sminv *are exactly* 231*-avoiding permutations, that is permutations* σ *with no* $i < j < k$ *such that* $\sigma_k < \sigma_i < \sigma_j$.

Proof. Let σ be a permutation, and suppose that it has a 231 pattern, that is, that there exist indices $i < j < k$ such that $\sigma_k < \sigma_i < \sigma_j$. Let $m = \min j < a \leq k \mid \sigma_m < \sigma_i$; by definition, $i < j \leq m-1$, and $\sigma_{m-1} > \sigma_i$, so (i,m) is a sminversion of σ. It follows that permutations with zero sminv are 231-avoiding permutations. Since a sminversion in a permutation corresponds to a 231 pattern, this concludes the proof. $\square$

Let π be a noncrossing partition, and let $\phi(\pi)$ be the permutation that, in one line notation, is written by listing the blocks of π sorted by their smallest element, with the elements of each segment sorted in decreasing order. Let us call *decreasing run* of a permutation σ a maximal subsequence of consecutive decreasing entries of σ (in one line notation): then the blocks of π correspond to the decreasing runs of $\phi(\pi)$. For instance, if $\pi = \{\{1,2,5\},\{3,4\},\{6,8,9\},\{7\}\}$, then $\phi(\pi) = 521439867$.

The map ϕ defines a classical bijection between noncrossing partitions of size n with $k+1$ blocks and 231-avoiding permutations with k descents. This recovers known numerology about the $(0,2)$-case.

Remark 4.2. More generally, standard segmented permutations with zero sminv can be characterized as 231-avoiding permutations where letters of a segment are smaller all than letters of the segments to its right. These can be easily counted, and we recover the total dimension of the $(0,2)$-coinvariant ring given by $\binom{2n+1}{n}$.

5 Connection with the Delta conjectures

Let us first note that we recover known combinatorial interpretations when setting $k=0$ (resp. $l=0$) in Theorem 2.5. Indeed this gives an expansion over segmented Smirnov words with no descents (resp. ascents), and these are easily identified with ordered multiset partitions [11]. In each case, the sminv statistic can moreover be seen to be distributed as the inv statistic on ordered set partitions.

The two different Delta conjectures are as follows:

$$\Delta'_{e_{n-k-1}} e_n = \Theta_{e_k} \nabla e_{n-k} = \sum_{D \in \mathsf{LD}(n)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D \tag{5.1}$$

$$\stackrel{?}{=} \sum_{D \in \mathsf{LD}(n)^{\bullet k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D. \tag{5.2}$$

The sets $\mathsf{LD}(n)^{*k}$ and $\mathsf{LD}(n)^{\bullet k}$ denote labelled Dyck paths of size n with k decorations on *rises* or *valleys*, respectively; and the statistics dinv and area depend on the decorations. So

(5.1) is referred to as the *rise version* and (5.2) as the *valley version* of the Delta conjecture. The rise version was recently proved in [4].

Let us make some of the combinatorics explicit. A *Dyck path* of size n is a lattice path starting at $(0,0)$, ending at (n,n), using only unit North (N) and East (E) steps, and staying weakly above the line $x=y$. A *labelled Dyck path* is a Dyck path together with a positive integer label on each of its vertical steps such that labels on consecutive vertical steps must be strictly increasing (from bottom to top).

A *rise* of a labelled Dyck path is a North step that is preceded by another North step. A *(contractible) valley* of a labelled Dyck path is a vertical step v that is preceded either by two horizontal steps, or by a horizontal step that is preceded by a vertical step whose label is strictly smaller than v's label.

A *decorated labelled Dyck path* D is a labelled Dyck path, together with a choice of rises and (contractible) valleys, which are *decorated*. Let $\mathsf{DRise}(D)$, resp. $\mathsf{DValley}(D)$, be the set of $i \in [n]$ such that the i-th vertical step of D is a decorated rise, resp. a decorated valley. We decorate rises with a $*$ and valleys with a $\bullet$. The set of decorated labelled Dyck paths with k decorated rises and l decorated valleys, is denoted by $\mathsf{LD}(n)^{*k,\bullet l}$. The sets $\mathsf{LD}(n)^{*k}$, resp. and $\mathsf{LD}(n)^{\bullet l}$, above correspond to setting l, resp. k, to 0.

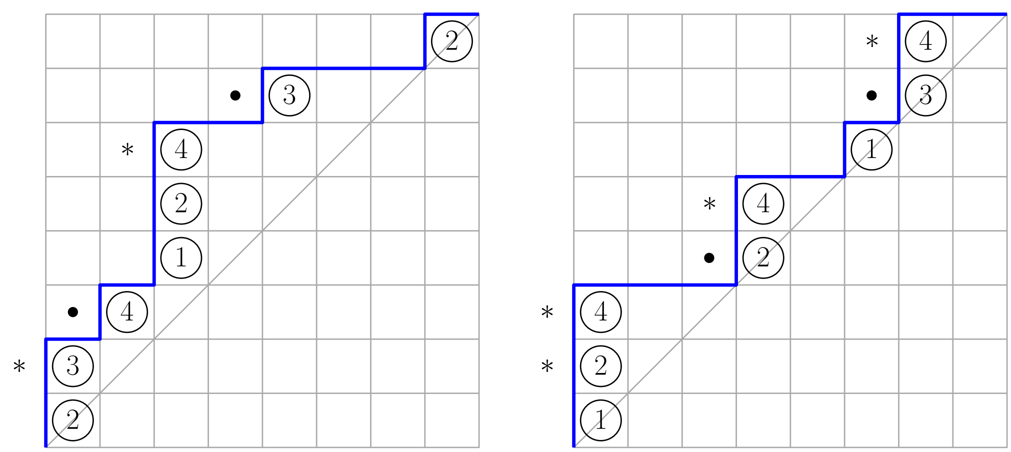

Figure 1: Elements of $\mathsf{LD}(8)^{*2,\bullet 2}$ (left) and $\mathsf{LD}_0(8)^{*4,\bullet 2}$ (right).

Given a decorated labelled Dyck path D of size n, its *area word* is the word of non-negative integers whose i-th letter equals the number of whole squares between the i-th vertical step of the path and the line $x=y$. If a is the area word of D, the *area* of D is

$$\mathsf{area}(D) := \sum_{i \in [n] \setminus \mathsf{DRise}(D)} a_i. \tag{5.3}$$

Take D to be the left path in Figure 1. We have $\mathsf{DRise}(D) = \{2,6\}$, $\mathsf{DValley} = \{3,7\}$. Its area word of D is 01112320, and so its area equals 6.

The statistic $\mathsf{dinv}(D)$ counts “diagonal inversions” minus the number of decorated valleys; we omit its precise definition in this abstract.

In [3], the authors conjectured a partial formula for a possible *unified Delta conjecture*, for which they have significant computational evidence:

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}\big|_{q=1} \stackrel{?}{=} \sum_{D\in\mathsf{LD}(n)^{*k,\bullet l}} t^{\mathsf{area}(D)}x^D, \tag{5.4}$$

The goal would thus be to find a statistic $\mathsf{qstat}\colon \mathsf{LD}(n)^{*k,\bullet l}\to\mathbb{N}$ so that

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l} \stackrel{?}{=} \sum_{D\in\mathsf{LD}(n)^{*k,\bullet l}} q^{\mathsf{qstat}(D)}t^{\mathsf{area}(D)}x^D; \tag{5.5}$$

and such that when $k=0$ or $l=0$, the formula reduces to (5.1) or (5.2), respectively.

Let us now come back to our setting. Comparing our main Theorem 2.5 at $q=1$ with (5.4) at $t=0$, we get the conjectural existence of a bijection between labelled Dyck paths of area zero and segmented Smirnov words. This bijection exists indeed: Let $\mathsf{LD}_0(n)^{*k,\bullet l}$ be the subset of area zero Dyck paths in $\mathsf{LD}(n)^{*k,\bullet l}$.

Theorem 5.1. *For any n,k,l, there is a bijection ϕ between $\mathsf{SW}(n,k,l)$ and $\mathsf{LD}_0(n)^{*k,\bullet l}$ such that $x_w = x^{\phi(w)}$.*

Sketch of the proof. Paths in $\mathsf{LD}_0(n)^{*k,\bullet l}$ have a very specific shape: they are the concatenation of paths of the form N^iE^i, where all rises are decorated; see Figure 1, right. This precisely ensures that the area is zero, cf. Formula (5.3).

For $\mu\vDash_0 n$, and let $\mathsf{LD}_0(\mu)^{*k,\bullet l}$ be the subset of $\mathsf{LD}_0(n)^{*k,\bullet l}$ such that $x^D=x^\mu$. Using the special structure detailed above, one can then show bijectively that the cardinalities of the sets of $\mathsf{LD}_0(\mu)^{*k,\bullet l}$ decompose as $\mathsf{SW}_{q=1}(\mu,k,l)$: namely, they satisfy (3.1) at $q=1$. By matching with the bijective decomposition of $\mathsf{SW}(\mu,k,l)$ in Section 3, we can obtain a recursively defined bijection ϕ between the two sets. We omit the details in this abstract. □

What about q ? By transporting the sminv statistic through the bijection ϕ, we get a q-statistic on $\mathsf{LD}_0(n)^{*k,\bullet l}$. Now this statistic will *not* satisfy the unified Delta conjecture (5.5) at $t=0$, because it does not match the dinv-statistic coming from the rise Delta conjecture.

It is however possible to fix this –thus we do have a unified Delta conjecture at $t=0$– by recursively defining a different q-statistic on $\mathsf{SW}(n)$. Roughly put, this is done by ordering segments in *ad hoc* ways when proving the recursion for $\mathsf{SW}(\mu,k,l)$ (for sminv we simply order segments right to left).

Added in revision: this is done explicitly in the long version of this work.

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The Poincaré-extended **ab**-index

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Abstract. Motivated by a conjecture concerning Igusa local zeta functions for intersection posets of hyperplane arrangements, we introduce and study the *Poincaré-extended* **ab**-*index*, which generalizes both the **ab**-index and the Poincaré polynomial. For posets admitting R-labelings, we give a combinatorial description of the coefficients of the extended **ab**-index, proving their nonnegativity. In the case of intersection posets of hyperplane arrangements, we prove the above conjecture of the second author and Voll.

Keywords: poset, matroid, oriented matroid, ab-index, hyperplane arrangement, R-labeling, quasisymmetric function

Grunewald, Segal, and Smith introduced the subgroup zeta function of finitely-generated groups [14], and Du Sautoy and Grunewald gave a general method to compute such zeta functions using p-adic integration and resolution of singularities [25]. This motivated Voll and the second author to examine the setting where the multivariate polynomials factor linearly. They found that the p-adic integrals are specializations of multivariate rational functions depending only on the combinatorics of the corresponding hyperplane arrangement [19]. After a natural specialization, its denominator greatly simplifies, and they conjecture that the numerator polynomial has nonnegative coefficients.

In this work, we prove their conjecture, which is related to the poles of these zeta functions; see Remark 1.19. Specifically, we reinterpret these numerator polynomials by introducing and studying the *(Poincaré-)extended* **ab**-*index*, a polynomial generalizing both the *Poincaré polynomial* and **ab**-*index* of the *intersection poset* of the arrangement. These polynomials have been studied extensively in combinatorics, although from different perspectives. The coefficients of the Poincaré polynomial have interpretations in terms of the combinatorics and the topology of the arrangement [8, Section 2.5]. The **ab**-index, on the other hand, carries information about the order complex of the poset and is particularly well-understood in the case of face posets of oriented matroids—or, more generally, Eulerian posets. In those settings, the **ab**-index encodes topological data via the *flag f-vector* [2].

We study the extended **ab**-index in the generality of graded posets admitting R-labelings. This class of posets includes intersection posets of hyperplane arrangements

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and, more generally, geometric lattices and geometric semilattices. We show that the extended **ab**-index has nonnegative coefficients by interpreting them in terms of a combinatorial statistic. This generalizes statistics given for the **ab**-index by Billera, Ehrenborg, and Readdy [6] and for the pullback **ab**-index (defined below) by Bergeron, Mykytiuk, Sottile and van Willigenburg [5]. This interpretation proves the aforementioned conjecture [19], as well as a related conjecture from Kühne and the second author [18].

Motivated by the proofs of these conjectures, we describe a close relationship between the Poincaré polynomial and the **ab**-index by showing that the extended **ab**-index can be obtained from the **ab**-index by a suitable substitution. This recovers, generalizes and unifies several results in the literature. Concretely, special cases of this substitution were observed by Billera, Ehrenborg and Readdy for lattices of flats of *oriented matroids* [6], by Saliola and Thomas for lattices of flats of *oriented interval greedoids* [24], and by Ehrenborg for *distributive lattices* [11].

1 The Poincaré-Extended ab-index

1.1 Main definitions

Unless otherwise specified, P is a finite ***graded poset*** of rank n, that is, P is a finite poset with unique minimum element $\hat{0}$ of rank 0 and unique maximum element $\hat{1}$ of rank n such that $\mathsf{rank}(X)$ is equal to the length of any maximal chain from $\hat{0}$ to X. The ***Möbius function*** μ of P is given by $\mu(X,X) = 1$ for all $X \in P$ and $\mu(X,Y) = -\sum_{X \leq Z < Y} \mu(X,Z)$ for all $X < Y$ in P. The ***Poincaré polynomial*** of P is

$$\mathsf{Poin}(P;y) = \sum_{X \in P} |\mu(\hat{0},X)| \cdot y^{\mathsf{rank}(X)} \ \in \mathbb{Z}[y].$$

The ***chain Poincaré polynomial*** of a chain $\mathcal{C} = \{\mathcal{C}_1 < \cdots < \mathcal{C}_k\}$ in $P \setminus \{\hat{1}\}$ is

$$\mathsf{Poin}_{\mathcal{C}}(P;y) = \prod_{i=1}^{k} \mathsf{Poin}([\mathcal{C}_i, \mathcal{C}_{i+1}];y) \ \in \mathbb{Z}[y],$$

where we set $\mathcal{C}_{k+1} = \hat{1}$. By taking the singleton chain $\{\hat{0}\}$, we recover the usual Poincaré polynomial, $\mathsf{Poin}(P;y) = \mathsf{Poin}_{\{\hat{0}\}}(P;y)$. The ranks of a given chain $\mathcal{C}$ is given by

$$\mathsf{Rank}(\mathcal{C}) = \{\mathsf{rank}(\mathcal{C}_i) \mid 1 \leq i \leq k\}.$$

We often consider polynomials in noncommuting variables $\mathbf{a}$ and $\mathbf{b}$ with coefficients being polynomials in $\mathbb{Z}[y]$. For a subset $S \subseteq \{i, i+1, \dots, j\}$, we write $\mathsf{m}_S = m_i \dots m_j$ for the monomial with $m_k = \mathbf{b}$ if $k \in S$ and $m_k = \mathbf{a}$ if $k \notin S$ and we similarly write $\mathsf{wt}_S = w_i \dots w_j$ for the polynomial with

$$w_k = \begin{cases} \mathbf{b} & \text{if } k \in S, \\ \mathbf{a} - \mathbf{b} & \text{if } k \notin S. \end{cases} \tag{1.1}$$

The supersets $\{i, i+1, \ldots, j\}$ are understood from the context as the set of all indices that can possibly be contained in the set S. In case of ambiguity, we in addition identify the considered superset. For a chain $\mathcal{C}$ in P, we also set $\mathsf{m}_{\mathcal{C}} = \mathsf{m}_{\mathsf{Rank}(\mathcal{C})}$ and $\mathsf{wt}_{\mathcal{C}} = \mathsf{wt}_{\mathsf{Rank}(\mathcal{C})}$. The following is the main object of study of this paper.

Definition 1.1. The (***Poincaré-)extended*** **ab-*index*** of P is

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C} \text{ chain in } P\setminus\{\hat{1}\}} \mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}_{\mathcal{C}} \ \in \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle\,,$$

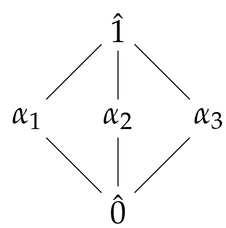
where $\mathsf{wt}_{\mathcal{C}} = w_0 \cdots w_{n-1}$ is given in Equation (1.1).

Since P has a unique minimum, we always have $\mathsf{Poin}(P; 0) = 1$, implying

$$_{\mathsf{ex}}\Psi(P; 0, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C} \text{ chain in } P\setminus\{\hat{1}\}} \mathsf{wt}_{\mathcal{C}}\,.$$

This recovers the **ab-*index*** $\Psi(P; \mathbf{a}, \mathbf{b}) = {}_{\mathsf{ex}}\Psi(P; 0, \mathbf{a}, \mathbf{b})$.

Example 1.2. We compute the extended **ab**-index of the poset $\mathcal{L}$ drawn below on the left.



$\mathcal{C}$	$\mathsf{Poin}_{\mathcal{C}}(\mathcal{L}; y)$	$\mathsf{Rank}(\mathcal{C})$	$\mathsf{wt}_{\mathcal{C}}$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0} < \alpha_i\}$	$(1+y)^2$	$\{0,1\}$	$\mathbf{b}^2$

The extended **ab**-index and its specialization to the **ab**-index are thus

$$\begin{aligned}
_{\mathsf{ex}}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\
&= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2, \\
\Psi(\mathcal{L}; \mathbf{a}, \mathbf{b}) &= \mathbf{a}^2 + 2\mathbf{ab}\,.
\end{aligned}$$

Remark 1.3. Taking chains $\mathcal{C}$ in $P \setminus \{\hat{1}\}$, rather than in P, is a harmless reduction in the definition of the extended **ab**-index since $\mathsf{Poin}_{\mathcal{C}}(P; y) = \mathsf{Poin}_{\mathcal{C}\cup\{\hat{1}\}}(P; y)$. If we consider both $\mathcal{C}$ and $\mathcal{C} \cup \{\hat{1}\}$ separately as summands of $_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b})$, we would need to consider weights $\mathsf{wt}^+_{\mathcal{C}} = w_0 \cdots w_n$ taking also the n-th position into account. We would have the two terms $\mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}^+_{\mathcal{C}}$ and $\mathsf{Poin}_{\mathcal{C}\cup\{\hat{1}\}}(P; y) \cdot \mathsf{wt}^+_{\mathcal{C}\cup\{\hat{1}\}}$, differing only in the last entry of the weight, so their sum is $\mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}_{\mathcal{C}} \cdot \mathbf{a}$. This holds for all chains, proving

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) \cdot \mathbf{a} = \sum_{\mathcal{C} \text{ chain in } P} \mathsf{Poin}_{\mathcal{C}}(P; y) \cdot \mathsf{wt}^+_{\mathcal{C}}. \tag{1.2}$$

The fact that $\hat{1}$ is included in every chain in the computation of the chain Poincaré polynomial is inspired by the setting of hyperplane arrangements; see [1, 22] for more details. A (central, real) ***hyperplane arrangement*** $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{R}^d$, all of which have a common intersection. The ***lattice of flats*** $\mathcal{L}$ of $\mathcal{A}$ is the poset of subspaces of $\mathbb{R}^d$ obtained from intersections of subsets of the hyperplanes, ordered by reverse inclusion. The open, connected components of the complement $\mathbb{R}^d \setminus \mathcal{A}$ are called (open) ***chambers***. The set of (closed) ***faces*** Σ is the set of *closures* of chambers of $\mathcal{A}$, together with all possible intersections of closures of chambers (ignoring intersections which are empty). This set comes equipped with a natural partial order by reverse inclusion, and for this reason we refer to Σ as the ***face poset*** of $\mathcal{A}$. There is an order-preserving, rank-preserving surjection $\mathsf{supp} : \Sigma \twoheadrightarrow \mathcal{L}$ sending a face to its affine span [8, Proposition 4.1.13]. This map extends to chains, and the fiber sizes are given, for $\mathcal{C} = \{\mathcal{C}_1 < \cdots < \mathcal{C}_k\} \subseteq \mathcal{L}$, by

$$\#\mathsf{supp}^{-1}(\mathcal{C}) = \prod_{i=1}^{k} \mathsf{Poin}([\mathcal{C}_i, \mathcal{C}_{i+1}]; 1) = \mathsf{Poin}_{\mathcal{C}}(P; 1), \tag{1.3}$$

with $\mathcal{C}_{k+1} = \hat{1}$; see [8, Proposition 4.6.2]. This is the key motivation for the next definition.

Definition 1.4. The ***pullback*** $\mathbf{ab}$***-index*** of P is

$$\Psi_{\mathsf{pull}}(P; \mathbf{a}, \mathbf{b}) = {}_{\mathsf{ex}}\Psi(P; 1, \mathbf{a}, \mathbf{b}).$$

Let Σ be the face poset and $\mathcal{L}$ the lattice of flats of a real central hyperplane arrangement. Since Σ may not have a unique minimum element, we formally add a minimum element $\hat{0}$ and let $\Sigma \cup \{\hat{0}\}$ be the resulting poset. Now, Equation (1.3) relates the $\mathbf{ab}$-index of the face poset and the pullback $\mathbf{ab}$-index of the lattice of flats by

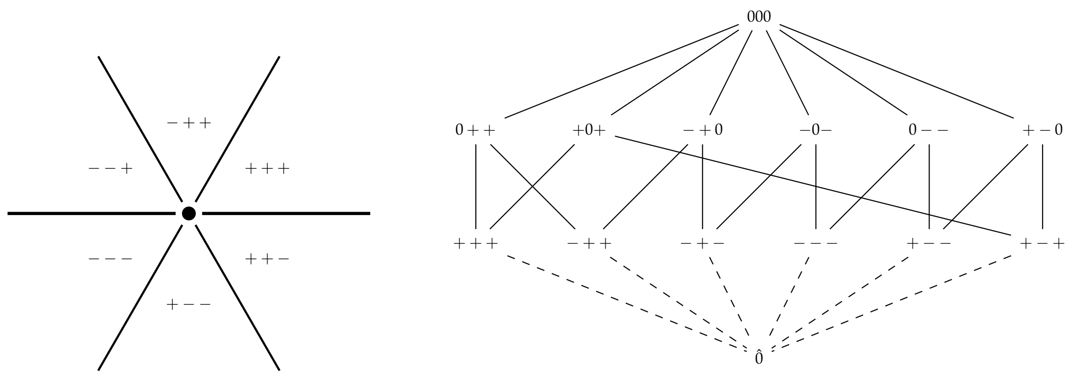
$$\Psi(\Sigma \cup \{\hat{0}\}; \mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \Psi_{\mathsf{pull}}(\mathcal{L}; \mathbf{a}, \mathbf{b})\,. \tag{1.4}$$

Note that this relates the evaluation of ${}_{\mathsf{ex}}\Psi(\Sigma \cup \{\hat{0}\}; y, \mathbf{a}, \mathbf{b})$ at $y = 0$ to the evaluation of ${}_{\mathsf{ex}}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ at $y = 1$. Equation (1.3) and thus also Equation (1.4) hold indeed in the more general context of oriented matroids.

Example 1.5. The pullback $\mathbf{ab}$-index of the poset from Example 1.2 is

$$\Psi_{\mathsf{pull}}(\mathcal{L}; \mathbf{a}, \mathbf{b}) = {}_{\mathsf{ex}}\Psi(\mathcal{L}; 1, \mathbf{a}, \mathbf{b}) = \mathbf{a}^2 + 5\mathbf{ba} + 5\mathbf{ab} + \mathbf{b}^2\,.$$

Consider the arrangement of three lines in the plane through a common intersection as shown below on the left in a way that emphasizes its face structure. Its lattice of flats is the poset $\mathcal{L}$ from Example 1.2. To the right, we draw its face poset Σ with $\hat{0}$ included.



The **ab**-index of $\Sigma \cup \{\hat{0}\}$ can be computed as

$$\mathbf{a}^3 + 5\mathbf{aba} + 5\mathbf{a}^2\mathbf{b} + \mathbf{ab}^2 = \mathbf{a}(\mathbf{a}^2 + 5\mathbf{ba} + 5\mathbf{ab} + \mathbf{b}^2) = \mathbf{a} \cdot \Psi_{\mathsf{pull}}(\mathcal{L}; \mathbf{a}, \mathbf{b})\,.$$

1.2 Main results

The main results of this paper concern *R-labeled posets*. These form a large family of posets including *distributive lattices*, *geometric lattices*, and *semimodular lattices*. In order to state Theorem 1.6, we introduce a combinatorial statistic on maximal chains of these posets and use this to describe the extended **ab**-index. In Section 2, we briefly discuss this combinatorial statistic for general edge labeled graded posets.

A function λ from the set of cover relations $X \lessdot Y$ in P into the positive integers is an R***-labeling*** of P if, for every interval $[X, Y]$ in P, there is a unique maximal chain $X = \mathcal{M}_i \lessdot \mathcal{M}_{i+1} \lessdot \cdots \lessdot \mathcal{M}_j = Y$ such that

$$\lambda(\mathcal{M}_i, \mathcal{M}_{i+1}) \leq \lambda(\mathcal{M}_{i+1}, \mathcal{M}_{i+2}) \leq \cdots \leq \lambda(\mathcal{M}_{j-1}, \mathcal{M}_j).$$

We say a poset P is R***-labeled*** if it is finite, graded, and admits an R-labeling. Throughout this section, we consider R-labeled posets with a fixed R-labeling λ.

The first result is a combinatorial statistic describing the coefficients of the extended **ab**-index which witnesses their nonnegativity. It generalizes [6, Corollary 7.2] and also reproves it using purely combinatorial arguments. For a maximal chain $\mathcal{M} = \{\mathcal{M}_0 \lessdot \mathcal{M}_1 \lessdot \cdots \lessdot \mathcal{M}_n\}$ in P, define the monomial $\mathsf{u}(\mathcal{M}) = u_1 \cdots u_n$ in $\mathbf{a}, \mathbf{b}$ given by $u_1 = \mathbf{a}$ and for $i \in \{2, \dots, n\}$ by

$$u_i = \begin{cases} \mathbf{a} & \text{if } \lambda(\mathcal{M}_{i-2}, \mathcal{M}_{i-1}) \leq \lambda(\mathcal{M}_{i-1}, \mathcal{M}_i)\,, \\ \mathbf{b} & \text{if } \lambda(\mathcal{M}_{i-2}, \mathcal{M}_{i-1}) > \lambda(\mathcal{M}_{i-1}, \mathcal{M}_i)\,. \end{cases} \tag{1.5}$$

Now, let $E \subseteq \{1, \dots, n\}$, viewed as a subset of the cover relations in the chain $\mathcal{M}$. Define the monomial $\mathsf{u}(\mathcal{M}, E) = v_1 \dots v_n$ in $\mathbf{a}, \mathbf{b}$ to be obtained from $\mathsf{u}(\mathcal{M})$ by

- replacing all variables $\mathbf{a}$ by $\mathbf{b}$ at positions $i \in \{1, \dots, n\}$ if $i \in E$ and

- replacing all variables $\mathbf{b}$ by $\mathbf{a}$ at positions $i \in \{2, \dots, n\}$ if $i - 1 \in E$.

In particular, we have $\mathsf{u}(\mathcal{M}, \emptyset) = \mathsf{u}(\mathcal{M})$, and $v_1 = \mathbf{b}$ if and only if $1 \in E$.

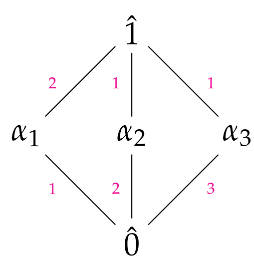
Theorem 1.6. *Let P be an R-labeled poset of rank n. Then*

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M},E)} y^{\#E} \cdot \mathsf{u}(\mathcal{M}, E)$$

where the sum ranges over all maximal chains $\mathcal{M}$ in P and all subsets $E \subseteq \{1, \dots, n\}$.

When P is a geometric lattice, setting $y = 0$ in Theorem 1.6 recovers [6, Corollary 7.2]. Specifically $\Psi(P; \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{M}} \mathsf{u}(\mathcal{M})$, where the sum ranges over all maximal chains $\mathcal{M} = \{\mathcal{M}_0 \lessdot \cdots \lessdot \mathcal{M}_n\}$.

Example 1.7. The poset from the previous examples admits the R-labeling given below on the left. On the right, we collect the relevant data to compute the combinatorial description of the extended **ab**-index.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

Then $_{\mathsf{ex}}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}$.

Corollary 1.8. *For an R-labeled poset P, we have*

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega\big(\Psi(P; \mathbf{a}, \mathbf{b})\big)$$

where the substitution ω replaces all occurrences of $\mathbf{ab}$ with $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{ba}$ and then simultaneously replaces all remaining occurrences of $\mathbf{a}$ with $\mathbf{a} + y\mathbf{b}$ and $\mathbf{b}$ with $\mathbf{b} + y\mathbf{a}$.

Using Corollary 1.8, the monomials $\mathsf{u}(\mathcal{M}, E)$ in Theorem 1.6 capture the same information as the *generalized descent sets* on *réseaux* as defined by Bergeron, Mykytiuk, Sottile, and van Willigenburg in [5, Section 7] in the context of quasisymmetric functions. The next corollary can be seen as a refinement of [27, Proposition 2.2] and of [5, Theorem 7.2], stated in terms of **ab**-indices rather than quasisymmetric functions. Both can be seen as the special case for the pullback **ab**-index: the first for *enriched P-partitions* and the second for general edge-labeled graded posets, compare with Section 2. We start by describing their relevant combinatorics in the present notation. Let $\mathcal{M}$ be a maximal chain with $\mathsf{u}(\mathcal{M}) = u_1 \dots u_n$, and let

$$\mathsf{Peak}(\mathcal{M}) = \big\{ i \in \{2, \dots, n\} \mid u_{i-1} = \mathbf{a}, u_i = \mathbf{b} \big\}$$

denote its ***peak set***. A set $S \subseteq \{1, \dots, n\}$ is then $\mathcal{M}$***-peak-covering*** if

$$\mathsf{Peak}(\mathcal{M}) \subseteq S \cup \{i+1 \mid i \in S\}\,.$$

For $\mathsf{u}(\mathcal{M}, S) = v_1 \cdots v_n$, let b-out$(\mathcal{M}, S)$ be the number of positions $i \in \{1, \dots, n\} \setminus S$ where $v_i = \mathbf{b}$.

Corollary 1.9. *For an R-labeled poset P of rank n, we have*

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M},S)} (1+y)^{\#S} \cdot y^{\text{b-out}(\mathcal{M},S)} \cdot \mathsf{wt}_S\,,$$

where the sum ranges over all maximal chains $\mathcal{M}$ and all $\mathcal{M}$-peak-covering subsets $S \subseteq \{1, \dots, n\}$ and where $\mathsf{wt}_S = w_1 \dots w_n$ as given in Equation (1.1).

Another consequence of Corollary 1.8 is that the Poincaré polynomial of P is in fact encoded in its **ab**-index. To see this, we define another substitution ι, which deletes the first letter from every **ab**-monomial, so $\iota(\mathbf{a}^3\mathbf{ba} + (1+y)\mathbf{ba}) = \mathbf{a}^2\mathbf{ba} + (1+y)\mathbf{a}$ for example. This gives us a way to obtain the Poincaré polynomial from the **ab**-index, a result which is similar in spirit to [6, Proposition 5.3].

Corollary 1.10. *For an R-labeled poset P of rank n, the Poincaré polynomial is the coeffcient of $\mathbf{a}^{n-1}$ in $\iota\big(\omega\big(\Psi(P; \mathbf{a}, \mathbf{b})\big)\big)$.*

Corollary 1.8 generalizes [6, Theorem 3.1] relating the **ab**-index of the lattice of flats of an oriented matroid with the **ab**-index of its face poset. As a consequence, we see that $_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b})$ is akin to a refinement of a **cd**-index. We make this observation precise in the following corollary.

Corollary 1.11. *For an R-labeled poset P, there exists a polynomial $\Phi(P; \mathbf{c}_1, \mathbf{c}_2, \mathbf{d})$ in noncommuting variables $\mathbf{c}_1, \mathbf{c}_2, \mathbf{d}$ such that*

$$_{\mathsf{ex}}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \Phi(P;\ \mathbf{a} + y\mathbf{b},\ \mathbf{b} + y\mathbf{a},\ \mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{ba}).$$

In particular, the pullback **ab***-index $\Psi_{\mathsf{pull}}(P; \mathbf{a}, \mathbf{b})$ is a polynomial in noncommuting variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $2\mathbf{d} = 2(\mathbf{ab} + \mathbf{ba})$.*

Remark 1.12 (The synthetic **cd**-index)**.** Recall that the **cd**-index of a poset exists if the **ab**-index can be written as a polynomial in $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = (\mathbf{ab} + \mathbf{ba})$. Bayer and Klapper proved a conjecture of Fine that a poset satisfies the *generalized Dehn-Sommerville relations* if and only if its **cd**-index exists and has integer coefficients [4, Theorem 4]. The **cd**-index of an Eulerian poset always exists (see [3, Theorem 2.1]) and has nonnegative coefficients when it comes from the face poset of a shellable regular *CW* sphere like the face poset of a convex polytope [26, Theorem 2.2] (or, more generally, from a Gorenstein* poset [17, Theorem 1.3]).

In [6], Billera, Ehrenborg, and Readdy give an elegant alternative proof of the non-negativity of the **cd**-index of the face poset of an oriented matroid. They use the support map from Equation (1.3) to relate the **ab**-index of the lattice of flats to the **ab**-index of the face poset. In our language, they interpret (using posets and polytopes) the extended **ab**-index of an oriented matroid at $y = 0$ and $y = 1$. Every matroid admits an extended **ab**-index, and the evaluation at $y = 0$ is the **ab**-index of its lattice of flats. This raises the natural question whether there is a geometric or poset-theoretic interpretation of the $y = 1$ evaluation of the extended **ab**-index. For this reason, we call the $y = 1$ evaluation of the extended **ab**-index rewritten in terms of **c** and **d** the ***synthetic* cd*-index***.

Example 1.13 (The Fano matroid)**.** Setting $y = 1$ and then $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ in the extended **ab**-index of the *Fano matroid* [8, Example 6.6.2(1)] gives the synthetic **cd**-index of the Fano matroid: $12\mathbf{cd} + 28\mathbf{dc} + \mathbf{c}^3$. A convex 3-polytope with this **cd**-index would have 30 vertices and 14 facets; see [21]. Thus its polar dual polytope would have 14 vertices and 30 facets, contradicting the the Upper Bound Theorem [20, p.180].

Example 1.14 (The Mac Lane matroid)**.** We compute the synthetic **cd**-index of the *Mac Lane matroid*; see [9, page 114] and [29, Section 2]. We get the synthetic **cd**-index $18\mathbf{cd} + 32\mathbf{dc} + \mathbf{c}^3$, which is the **cd**-index of a polytope!

Remark 1.15 (Oriented interval greedoids)**.** The argument used for oriented matroids and their lattices of flats also applies to *oriented interval greedoids*, where the analogue of Equation (1.3) is given in [24, Theorem 6.8]. Since the lattice of flats of an interval greedoid is a semimodular lattice, it admits an R-labeling; see [7, Theorem 3.7]. Applying Corollary 1.8 and setting $y = 1$ gives [24, Corollary 6.12].

Remark 1.16 (Distributive lattices & r-signed Birkhoff posets)**.** Ehrenborg discussed an ω-like substitution for arbitrary distributive lattices [11]. Remarkably, that substitution is equivalent to the substitution in Corollary 1.8 for $y = r - 1 \in \mathbb{N}$. In that case of distributive lattices, the parameter r is a fixed integer (rather than a variable) carrying information about the fiber sizes of a certain support map. For a (not necessarily graded) finite poset P, the *r-signed Birkhoff poset* $J_r(P)$ is the collection of pairs (F, f) where F is an *order ideal* in P and f is a map from the maximal elements in F to the set $\{1, \dots, r\}$, with order relation given by

$$(F, f) \leq (G, g) \quad \Longleftrightarrow \quad G \subseteq F \text{ and } f(x) = g(x) \text{ for all } x \in \max(F) \cap \max(G)\,.$$

These posets were defined in [15, 11] and studied in connection to the Birkhoff lattice $J(P) = J_1(P)$. The map $z : J_r(P) \to J(P)$ with $(F, f) \mapsto F$ is an order- and rank-preserving poset surjection for which the fiber size of a chain $\mathcal{C}$ in $J(P)$ can—in the notation from the previous sections—be computed by $\#z^{-1}(\mathcal{C}) = \mathsf{Poin}_{\mathcal{C}}(J(P); r - 1)$, see [11, Proposition 5.2]. Since distributive lattices are modular, they admit R-labelings; see [7, Theorem 3.7]. Thus, applying Corollary 1.8 for $y = r - 1$ gives the first part of [11, Theorem 4.2].

We next turn toward the coarse flag Hilbert–Poincaré series introduced and studied in [19]. The numerator of this rational function is defined in [19, Equation (1.13)], and we extend this definition to graded posets via

$$\mathsf{Num}(P;y,t) = \sum_{\mathcal{C}\text{ chain in } P\setminus\{\hat{0},\hat{1}\}} \mathsf{Poin}_{\{\hat{0}\}\cup\mathcal{C}}(P;y)\cdot t^{\#\mathcal{C}}(1-t)^{n-1-\#\mathcal{C}} \in \mathbb{Z}[y,t]\,.$$

By removing the first letter of every **ab** monomial and then specializing via $\mathbf{a}\mapsto 1$ and $\mathbf{b}\mapsto t$ we obtain a proof of [19, Conjecture E] and its generalization to R-labeled posets:

Corollary 1.17. *For an R-labeled poset P, the coefficients of $\mathsf{Num}(P;y,t)$ are nonnegative.*

Together with Corollary 1.10, we obtain $\mathsf{Poin}(P;y) = [t^0]\ \mathsf{Num}(P;y,t)$. The substitutions in the previous corollaries show that Theorem 1.6 also gives analogous combinatorial interpretations for the coefficients of $\iota\big({}_{\mathsf{ex}}\Psi(P;y,\mathbf{a},\mathbf{b})\big)$ and of $\mathsf{Num}(P;y,t)$.

Remark 1.18 (Geometric semilattices)**.** Note that [19, Conjecture E] concerns all hyperplane arrangements (central and affine). While the intersection posets of central hyperplane arrangements are geometric lattices and, thus, admit R-labelings [7, Example 3.8], the intersection posets of affine arrangements are part of a more general family called *geometric semilattices*, first explicitly studied by Wachs and Walker in [28]. A theorem of Ziegler shows that if $\mathcal{L}$ is a geometric semilattice, then $\mathcal{L}\cup\{\hat{1}\}$ admits an R-labeling [30, Theorem 2.2]. Thus Theorem 1.6 holds for intersection posets of affine arrangements.

Remark 1.19 (Implications for other zeta functions)**.** The coarse flag Hilbert–Poincaré polynomial of a poset P comes from a natural specialization of its flag Hilbert–Poincaré series. The flag Hilbert–Poincaré series is a rational function in $\mathbb{Q}[y](t_x \mid x\in P)$ given by

$$\mathsf{fHP}_P(y,\mathbf{t}) = \sum_{\mathcal{C}\text{ chain in } P\setminus\hat{0}} \mathsf{Poin}_{\mathcal{C}}(P;y)\prod_{x\in\mathcal{C}}\frac{t_x}{1-t_x}\,.$$

The coarse flag Hilbert–Poincaré polynomial $\mathsf{Num}(P;y,t)$ is obtained by setting all the t_x equal to t and considering $(1-t)^{\mathsf{rank}(P)}\mathsf{fHP}_P(y,t)$. Different specializations of $\mathsf{fHP}_P(y,\mathbf{t})$ yield other well-studied zeta functions like local Igusa zeta functions of hyperplane arrangements [10], motivic zeta functions of matroids from [16], and the conjugacy class counting zeta functions of certain group schemes defined in [23]. Moreover, each of these is obtained from $\mathsf{fHP}_P(y,\mathbf{t})$ by a monomial substitution of the form $y=-p^{-1}$ and $t_x = p^{\lambda_x}t^{\mu_x}$ for some integers λ_x and μ_x, where p is a prime and $t=p^{-s}$ for a complex variable s; see [19, Remark 1.3].

The specialization of $\mathsf{Num}(P;y,t)$ at $y=1$ was studied further for matroids and oriented matroids by the second author and Kühne in [18], who showed $\mathsf{Num}(P;1,t)$ is the sum of h-polynomials of simplicial complexes related to the chambers if P is the lattice of flats of a real central hyperplane arrangement. The following corollary proves a generalized version of the conjectured lower bound from [18, Conjecture 1.4].

Corollary 1.20. *Let P be an R-labeled poset of rank n. The coefficient of t^k in $\mathsf{Num}(P;1,t)$ is bounded below by $\binom{n-1}{k} \cdot \mathsf{Poin}(P;1)$.*

2 Connection to quasisymmetric functions

Theorem 1.6 shows that the extended **ab**-index of an R-labeled poset has nonnegative coefficients. Nonnegativity may fail, however, for posets that do not admit R-labelings. For example, the weak order for the symmetric group $\mathfrak{S}_3$ (the hexagon poset) does *not* admit an R-labeling and has extended **ab**-index

$$\begin{aligned}\mathbf{aaa} + (-1+2y)\mathbf{aab} + (1+2y)\mathbf{aab} + y(2+y^2)\mathbf{baa} + (2y^2-1)\mathbf{abb} \\ + (-y^3+2y^2)\mathbf{bab} + y^2(2+y)\mathbf{bba} + y(3y^2+2y-2)\mathbf{bbb}\,.\end{aligned}$$

Using the right-hand side in Theorem 1.6, we define the (***combinatorial***) ***extended ab-index*** of a finite edge-labeled graded poset P, which is *manifestly positive*, via

$$_{\mathsf{cx}}\Psi(P;y,\mathbf{a},\mathbf{b}) = \sum_{(\mathcal{M},E)} y^{\#E} \cdot \mathsf{u}(\mathcal{M},E) \ \in \mathbb{N}[y]\langle \mathbf{a},\mathbf{b}\rangle\,.$$

While $_{\mathsf{cx}}\Psi$ is in general not linked to the Poincaré polynomial, the proofs of Corollaries 1.8 and 1.9 still hold. In particular, $_{\mathsf{cx}}\Psi(P;y,\mathbf{a},\mathbf{b})$ is a polynomial in $\mathbf{c}_1 = \mathbf{a}+y\mathbf{b}, \mathbf{c}_2 = \mathbf{b}+y\mathbf{a}$ and $\mathbf{d} = \mathbf{ab}+y\mathbf{ba}+y\mathbf{ab}+y^2\mathbf{ba}$. This means that $2\cdot {}_{\mathsf{cx}}\Psi(P;1,\mathbf{a},\mathbf{b})$ is an **ab**-analogue of the *peak enumerator* from [5, Definition 7.1]. The remainder of this section is devoted to presenting a conjecture inspired by this specialization.

Let $S = \{s_1 < \cdots < s_k\}$ be a subset of $\{1,\ldots,n\}$. The ***monomial quasisymmetric function*** M_S is the power series

$$M_S = \sum_{i_1<i_2<\cdots<i_k<i_{k+1}} x_{i_1}^{s_1} x_{i_2}^{s_2-s_1} \cdots x_{i_k}^{s_k-s_{k-1}} x_{i_{k+1}}^{n+1-s_k} \quad \in \mathbb{Q}[[x_1,x_2,x_3,\ldots]]\,.$$

Note that M_S is homogeneous of degree $n+1$ and—although we surpress it in the notation—implicitly depends on n. The ring of ***quasisymmetric functions*** QSym is the (linear) span of $M_\bullet = 1$ and all M_S for $n \geq 0$. Following [12, Section 3], we define a vector space isomorphism $\Xi : \mathbb{Q}\langle \mathbf{a},\mathbf{b}\rangle \longrightarrow \mathsf{QSym}$ defined by sending wt_T to M_T. Using the isomorphism Ξ, we can view the map ω from Corollary 1.8 as a map from QSym to $\mathsf{QSym} \otimes \mathbb{Q}[y]$ given by $F_S \mapsto \omega(F_S) = \Xi\big(\omega(\mathsf{m}_S)\big)$, where F_S is given in [13, Equation 2]. In [27, Equation (1.8)], Stembridge shows how to obtain (skew) Schur functions as P-partition enumerators of certain posets given in [27, Section 1.3]. The following conjecture[1] concerning the Schur functions has been verified for all integer partitions of size at most 11 using `SageMath`.

[1]This conjecture was exhibited at the *90th Séminaire Lotharingien de Combinatoire* in Bad Boll, Germany in September 2023 in collaboration with Darij Grinberg.

Conjecture 2.1. *For any partition $\lambda \vdash n$, the quasisymmetric function $\omega(s_\lambda)$ is symmetric and Schur positive. Specifically, for each $\mu \vdash n$, there exist $c^\mu_\lambda(y) \in \mathbb{N}[y]$ such that*

$$\omega(s_\lambda) = \sum_{\mu \vdash n} c^\mu_\lambda(y) \cdot s_\mu \,.$$

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Colored Permutation Statistics by Conjugacy Class

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Abstract. We consider the moments of statistics on conjugacy classes of colored permutation groups $\mathfrak{S}_{n,r} = \mathbb{Z}_r \wr \mathfrak{S}_n$. We first show that any fixed moment of a statistic coincides on all conjugacy classes when all cycle lengths are sufficiently long. For permutation statistics that can be realized via a process called symmetric extension, we show that for fixed r, this moment on these conjugacy classes is a polynomial in n. Hamaker and Rhoades (arXiv, 2022) established analogous results for the symmetric group as part of their far-reaching representation-theoretic framework. Independently, Campion Loth, Levet, Liu, Stucky, Sundaram, and Yin (arXiv, 2023) arrived at independence and polynomiality results for the symmetric group using instead an elementary combinatorial framework. Our techniques in this paper build on this latter elementary approach. Finally, we extend the work of Fulman (*J. Comb. Theory Ser. A.*, 1998), to establish a central limit theorem for descents in conjugacy classes of the hyperoctahedral group with sufficiently long cycles.

Keywords: colored permutation, Coxeter group, hyperoctahedral group, moment, permutation constraint, permutation statistic

1 Introduction

For a finite group G, a *statistic* is a map $X : G \to \mathbb{R}$. The *distribution* of X is the function (x_k), where x_k is the number of elements $g \in G$ such that $X(g) = k$ (i.e., $x_k := |X^{-1}(k)|$). When G is the symmetric group $G = \mathfrak{S}_n$, we refer to the statistics as *permutation statistics*. The study of permutation statistics is a classical topic in algebraic combinatorics; Stanley's texts [16, 17] serve as a key reference in this area.

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In this paper, we build on the elementary methods in [4] to investigate the distribution of *colored* permutation statistics by conjugacy class. In contrast to the vast literature on permutation statistics in $\mathfrak{S}_n$, there has been considerably less work on statistics for arbitrary Coxeter groups or the colored permutation groups $\mathfrak{S}_{n,r}$, i.e., the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$. We are in particular not aware of work considering colored permutation statistics on individual conjugacy classes.

When $r = 2$, the colored permutation group $\mathfrak{S}_{n,2}$ coincides with the hyperoctahedral group B_n, which is the type B Coxeter group. A study of statistics over the entire Coxeter group for types B and D was initiated by Reiner, see e.g. [15], and carried further by Adin and Roichman, see e.g. [1], and Brenti and Carnevale [3]. There is also work on colored permutation statistics and their distribution, again over the whole group, by Steingrímsson [18], Fire [7], and Moustakas [13].

Recently, Hamaker and Rhoades [11] established a representation-theoretic framework for permutation statistics on $\mathfrak{S}_n$ by conjugacy class C_λ. They introduced so-called *local* permutation statistics; using representation-theoretic methods, they established that the moments of these statistics depend only on n and the number of short cycles in λ. In particular, these moments are independent of the conjugacy class when the cycles in λ are all sufficiently large.

Independently, and subsequent to the paper [11], Campion Loth, Levet, Liu, Stucky, Sundaram, and Yin [4] established similar independence and polynomiality results for conjugacy classes in $\mathfrak{S}_n$, using only elementary combinatorial techniques. The present paper builds on the framework in [4]. The full version of this paper appears in [5].

Main Results. Fix $r \geq 1$, and let $\boldsymbol{\lambda}$ be an r-partition of n. For a statistic X on $\mathfrak{S}_{n,r}$, denote by $\mathbb{E}_{\boldsymbol{\lambda}}[X]$ the expected value of X taken over the conjugacy class of $\mathfrak{S}_{n,r}$ indexed by $\boldsymbol{\lambda}$. Our main results are as follows:

- Theorem 12 in Section 3.2 shows that for any statistic X, its kth moment coincides on all conjugacy classes $C_{\boldsymbol{\lambda}}$ of $\mathfrak{S}_{n,r}$ that do not have "short" cycles. For each statistic X, making this notion of "short" precise is done through *colored permutation constraints* as given in Definition 4.

- Theorem 20 in Section 3.3 concerns sequences of statistics $(X_n)_{n\geq 1}$ on $(\mathfrak{S}_{n,r})_{n\geq 1}$ that can be constructed using *symmetric extensions*, as described in Definition 19. This theorem shows that a single polynomial in n gives $\mathbb{E}_{\boldsymbol{\lambda}_n}[X_n^k]$ on conjugacy classes $C_{\boldsymbol{\lambda}_n}$ of $\mathfrak{S}_{n,r}$ without "short" cycles. Note that this result applies to many statistics, including the inversion statistic on B_n defined in (2.2).

- Finally, Theorem 28 in Section 4 establishes asymptotic normality of the descent statistic on B_n for conjugacy classes with no "short" cycles. Our proof leverages a generating function of Reiner [15, Theorem 4.1] for the joint distribution of descent and major index by cycle type, an analogue of the corresponding generating function for the symmetric group [10]. The arguments then follow Fulman's analogous result for descents on conjugacy classes of $\mathfrak{S}_n$ [8, Theorem 1 and proof of Theorem 2], but the technical details are nontrivial and require care to execute.

Remark 1. One essential insight in our work was in developing the notion of *colored permutation constraints* (see Definition 4). It took considerable effort to arrive at this definition, and we discuss these technical difficulties in the full version [5, Remark 3.3]. The fact that Theorem 12 and Theorem 20 generalize analogous results on the symmetric group [11, 4] so cleanly suggests that Definition 4 might in fact be the right notion of colored permutation constraints.

2 Preliminaries

We recall preliminary notions of colored permutation groups. The *colored permutation group* $\mathfrak{S}_{n,r}$ is the wreath product [12, Chapter 4] $\mathbb{Z}_r \wr \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n elements and $\mathbb{Z}_r$ is the cyclic group on r elements. A *colored permutation* $(\omega, \tau) \in \mathfrak{S}_{n,r}$ can be expressed as an ordered pair consisting of a permutation $\omega \in \mathfrak{S}_n$ along with a function $\tau : [n] \to \mathbb{Z}_r$, where the representative elements of $\mathbb{Z}_r$ are taken in $\{0, \ldots, r-1\}$. The value $\tau(j)$ is called the *color* of the symbol j, and $\tau(j) + \tau'(j)$ is defined as a sum of elements in $\mathbb{Z}_r$.

The colored permutation group $\mathfrak{S}_{n,r}$ has a canonical embedding as a subgroup of the symmetric group $\mathfrak{S}_{rn}$, which we describe explicitly as follows. Writing $[n]^r$ for the set of rn elements $\{i^j | i \in [n], j \in \{0, 1, \ldots, r-1\}\}$ where the exponent indicates the color of an element in $[n]$, we can also think of the colored permutation (ω, τ) as a bijection $f : [n]^r \to [n]^r$ defined by $f(i^j) = \omega(i)^{\tau(\omega(i))+j}$ for all i, j, where $\tau(\omega(i)) + j$ is taken modulo r. In this sense, the coloring of the symbols τ and the underlying permutation ω are independently specified.

We now turn to discussing the conjugacy class structure of $\mathfrak{S}_{n,r}$. An *r-partition* of $n \in \mathbb{N}$ is an r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^j)_{j=0}^{r-1}$ where each λ^j is a partition of some n_j such that $\sum_{j=0}^{r-1} n_j = n$. When $r = 2$, we also call this a *bi-partition.* For a cycle in a permutation in $\mathfrak{S}_{n,r}$, the length of this cycle is the number of elements in it, and the color of this cycle is the sum of the colors in the cycle, taken modulo r. The *cycle type* of $(\omega, \tau) \in \mathfrak{S}_{n,r}$ is the r-partition $\boldsymbol{\lambda} = (\lambda^j)_{0 \le j \le r-1}$, where each λ^j consists of the cycles of color j. Then $m_i(\lambda^j)$ denotes the number of cycles in λ^j of length i, and $C_{\boldsymbol{\lambda}}$ denotes the elements in $\mathfrak{S}_{n,r}$ with cycle type $\boldsymbol{\lambda}$.

Example 2. Let $\omega \in \mathfrak{S}_5$ be the permutation specified by $\omega = [45132] = (143)(25)$ in one-line and cycle notation. Let $\tau = (3, 0, 1, 1, 3)$. The colored permutation $(\omega, \tau) \in \mathfrak{S}_{5,4}$ is completely specified by the function $f : [5]^4 \to [5]^4$ satisfying $f(i^0) = \omega(i)^{\tau(\omega(i))}$. Hence in two-line, one-line, and cycle notations we have:

$$(\omega,\tau) = \begin{pmatrix} 1^0 & 2^0 & 3^0 & 4^0 & 5^0 \\ 4^1 & 5^3 & 1^3 & 3^1 & 2^0 \end{pmatrix} = [4^1 5^3 1^3 3^1 2^0] = (1^3 4^1 3^1)(2^0 5^3).$$

It has a 3-cycle of color 1 and a 2-cycle of color 3. Its cycle type is thus $(\varnothing, (3), \varnothing, (2))$.

The conjugacy classes of $\mathfrak{S}_{n,r}$ are well understood in terms of cycle type.

Proposition 3. *[12, Theorem 4.2.8, Lemmas 4.2.9-4.2.10] The conjugacy classes of $\mathfrak{S}_{n,r}$ are given by $C_{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda}$ is an r-partition of n.*

In the special case $r = 2$, the hyperoctahedral group $\mathfrak{S}_{n,2} = B_n$ can be viewed as the group of signed permutations, i.e., bijections on $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ where positive and negative elements respectively correspond to colors 0 and 1. In this case, we will denote bipartitions as (λ, μ) and the corresponding conjugacy class as $C_{\lambda,\mu}$.

The type B descent statistic, whose distribution is the subject of Section 4, is then given by the following definition, with the convention that $\omega(0) = 0$. See [2, Proposition 8.1.2]:

$$\mathrm{des}_B(\omega) = |\{i \in \{0\} \cup [n-1] \mid \omega(i) > \omega(i+1)\}|. \tag{2.1}$$

Two other B_n-statistics that will be useful for illustrative purposes are inv and negsum, defined by (see [2, Equation 8.1 and page 308])

$$\mathrm{inv} = |\{(i,j) \in [n] \times [n] \mid i < j \text{ and } \omega(i) > \omega(j)\}|, \quad \mathrm{negsum}(\omega) = \sum_{i \in [n], \omega(i) < 0} \omega(i). \tag{2.2}$$

Also, the Coxeter length statistic inv_B is given by the formula [2, Proposition 8.1.1]

$$\mathrm{inv}_B(\omega) = \mathrm{inv}(\omega) - \mathrm{negsum}(\omega) \tag{2.3}$$

We will use the des, inv, and negsum statistics as running examples to illustrate our work. Results on inv and negsum naturally lead to statements about inv_B, illustrating the more general fact that our results behave nicely with statistics that are defined as linear combinations of other statistics.

Throughout this paper, we will use $\Pr_{\mathfrak{S}_{n,r}}$ and $\Pr_{\boldsymbol{\lambda}}$ to denote the probabilities in $\mathfrak{S}_{n,r}$ and $C_{\boldsymbol{\lambda}}$ (with respect to the uniform measure). We similarly use $\mathbb{E}_{\mathfrak{S}_{n,r}}$ and $\mathbb{E}_{\boldsymbol{\lambda}}$ for the expected values on the corresponding probability spaces.

3 Moments of colored permutation statistics

In this section, we will discuss the techniques involved in establishing the independence result, Theorem 12, and the polynomiality result, Theorem 20.

3.1 Colored permutation constraints

In this section, we will extend the notion of a permutation constraint from the setting of the symmetric group to the setting of colored permutations. We compare this to [4, Definition 7.1] as well as to the work of Hamaker and Rhoades [11], where permutation constraints are called *partial permutations*. A colored permutation constraint will have two components (K, κ). The first, K, will constrain a permutation ω by specifying a subset of its values. The second component, κ, will assign colors to these values.

Definition 4. Let $K = \{(i_1, j_1), \ldots, (i_m, j_m)\}$ consist of distinct ordered pairs, where $i_h, j_h \in [n]$. Let $\kappa : \{j_1, \ldots, j_m\} \to \mathbb{Z}_r$. We call the pair (K, κ) a *colored permutation constraint*, and we call m the size of the constraint. For $(\omega, \tau) \in \mathfrak{S}_{n,r}$, we say that ω satisfies K if $\omega(i_h) = j_h$ for all $h \in [m]$, and we say τ satisfies κ if $\tau(x) = \kappa(x)$ for all $x \in \{j_1, \ldots, j_m\}$. Finally we say that $(\omega, \tau) \in \mathfrak{S}_{n,r}$ satisfies (K, κ) if ω satisfies K and τ satisfies κ. We will sometimes denote a constraint as a set of ordered pairs

$$(K, \kappa) = \left\{ \left(i_h^0, j_h^{\kappa(j_h)} \right) \right\}_{h=1}^{m}$$

recording these conditions, and we sometimes omit set braces for brevity.

Recall from Section 2 that we view the hyperoctahedral group $\mathfrak{S}_{n,2} = B_n$ as the group of signed permutations. In this case, a constraint is of the form $(K, \kappa) = \{(i_h, \kappa(j_h) j_h)\}_{h=1}^{m}$, where $\kappa(j_h) = \pm 1$.

Definition 5. Let $\mathcal{C}$ be a set of colored permutation constraints. The *size* of $\mathcal{C}$ is defined as the maximum size over all constraints contained in $\mathcal{C}$, namely,

$$\mathrm{size}(\mathcal{C}) = \max_{(K,\kappa) \in \mathcal{C}} |K|.$$

Recall that a colored permutation statistic is simply a map $X : \mathfrak{S}_{n,r} \to \mathbb{R}$. We now introduce decompositions of colored permutation statistics as weighted sums of indicator functions corresponding to colored permutation constraints.

Definition 6. A colored permutation statistic X is *realizable* over a constraint set of size m if there exists a set of constraints $\mathcal{C}$ of size m and weights $\mathrm{wt}(K, \kappa) \in \mathbb{R}$ such that $X = \sum_{(K,\kappa) \in \mathcal{C}} \mathrm{wt}(K, \kappa) I_{(K,\kappa)}$, where $I_{(K,\kappa)}$ is the indicator function that a permutation satisfies the constraint (K, κ). Note that in general, the decomposition $\sum_{(K,\kappa) \in \mathcal{C}} \mathrm{wt}(K, \kappa) I_{(K,\kappa)}$ is not unique.

Example 7. Many statistics have a natural decomposition in terms of constraints. For the statistics defined on B_n given in Section 2, we have

$$\mathrm{des}_B = \sum_{j \in [n]} I_{(1,-j)} + \sum_{i \in [n-1]} \sum_{\substack{j_1, j_2 \in [\pm n] \\ j_1 < j_2}} I_{(i,j_2),(i+1,j_1)},$$

$$\text{inv} = \sum_{\substack{i_1,i_2\in[n]\\ i_1<i_2}} \sum_{\substack{j_1,j_2\in[\pm n]\\ j_1<j_2}} I_{(i_1,j_2),(i_2,j_1)},$$

$$\text{negsum} = \sum_{i\in[n]} \sum_{j\in[n]} (-j) I_{(i,-j)}.$$

This shows that des_B and inv are realizable over constraint sets of size 2, and negsum is realizable over a constraint set of size 1. Since inv_B is the difference of inv and negsum, we also see that inv_B is realizable over a constraint set of size 2.

Remark 8. We say that (K, κ) is *well-defined* if all of the $i_h \in [n]$ are distinct, and all of the $j_h \in [n]$ are distinct. Observe that if (K, κ) is not well-defined, then $I_{(K,\kappa)}$ is identically 0 on $\mathfrak{S}_{n,r}$, and hence can be omitted from any set realizing a given statistic. Consequently, we are only interested in well-defined constraints.

3.2 Independence of moments

In this section, we outline the steps leading to the proof of our independence result, Theorem 12. Our methods follow the strategy of [4, Section 7]. Proofs appear in [5].

Definition 9. A colored permutation constraint (K, κ) is *acyclic* if K is well-defined and the graph $G(K, \kappa)$, with vertex set $V = [n]$ and directed edge set K, does not contain any cycles. Observe that in this case, $G(K, \kappa)$ consists of a set of paths.

As a non-example, the size one constraint induced by $I_{(i,-i)}$ from Example 7 is *not* acyclic.

Lemma 10. *(Compare to $\mathfrak{S}_n$, cf. [4, Lemma 7.15]) Consider the group of all r-colored permutations $\mathfrak{S}_{n,r}$. Let $C_{\boldsymbol{\lambda}}$ be a conjugacy class of $\mathfrak{S}_{n,r}$. Let (K, κ) be a well-defined colored permutation constraint of size $m \leq n$, and suppose that each partition in $\boldsymbol{\lambda}$ has all parts of size at least $m+1$. If K is acyclic, then*

$$\Pr_{\boldsymbol{\lambda}}[(\omega, \tau) \textit{ satisfies } (K, \kappa)] = \tfrac{1}{(n-1)(n-2)\cdots(n-m)} \cdot \tfrac{1}{r^m}.$$

If K is not acyclic, then $\Pr_{\boldsymbol{\lambda}}[(\omega, \tau) \textit{ satisfies } (K, \kappa)] = 0$.

One essential observation in proving Lemma 10 is that the permutation and the coloring can be treated independently.

Lemma 10 can be used to analyze the first moment of a statistic $\mathbb{E}_{\boldsymbol{\lambda}}[X]$ by expressing X in terms of constraints. We need one final lemma to accommodate arbitrary moments $\mathbb{E}_{\boldsymbol{\lambda}}[X^k]$ in the main result of this section, Theorem 12.

Lemma 11. *Let $X_1, X_2 : \mathfrak{S}_{n,r} \to \mathbb{R}$ be realizable over constraint sets of size m_1, m_2 respectively. Then X_1X_2 is realizable over a constraint set of size $m_1 + m_2$. In particular, for any integer $k \geq 1$, we have that X_1^k is realizable over a constraint set of size km_1.*

This leads to the main theorem of this section.

Theorem 12. *Suppose $X : \mathfrak{S}_{n,r} \to \mathbb{R}$ is realizable over a constraint set of size m. For any $k \geq 1$, the kth moment $\mathbb{E}_{\boldsymbol{\lambda}}[X^k]$ coincides on all conjugacy classes $C_{\boldsymbol{\lambda}}$ with no cycles of length $1, 2, \ldots, mk$.*

Note that the above theorem makes precise the notion of "short" cycles. In particular, if we are considering the kth moment of a statistic X realizable over a constraint set of size m, then the "short" cycles are the ones of length at most mk.

Remark 13. Note that a colored permutation (ω, τ) is itself a colored permutation constraint of size n. Hence, we can express any statistic X using size n constraints. Additionally, one can show that if X is realizable over a constraint set of size m, then it is also realizable over a constraint set of size m' for $m \leq m' \leq n$. For the full strength of our results, we are primarily interested in minimizing m, and we call this minimum possible value the *size* of X.

Remark 14. The arguments leading to the proof of Theorem 12 have practical applications for computing moments of statistics on those conjugacy classes. For example, consider negsum on B_n, which can be expressed as $\text{negsum} = \sum_{i \in [n]} \sum_{j \in [n]} (-j) I_{(i,-j)}$. Note that here all constraints are acyclic except for $(i, -i)$. One can then show that for any bi-partition (λ, μ) of n where all the parts have size at least 2,

$$\begin{aligned}\mathbb{E}_{\lambda,\mu}[\text{negsum}] &= -\sum_{i \in [n]} i \cdot \mathbb{E}_{\lambda,\mu}[I_{(i,-i)}] - \sum_{i \in [n]} \sum_{j \in [n] \setminus i} j \cdot \mathbb{E}_{\lambda,\mu}[I_{(i,-j)}] \\ &= -\frac{1}{(n-1) \cdot 2} \cdot \sum_{i \in [n]} \sum_{j \in [n] \setminus i} j = -\frac{1}{2}\binom{n+1}{2}.\end{aligned}$$

More generally, one can use $\text{negsum} = \sum_{i \in [n]} \sum_{j \in [n]} (-j) I_{(i,-j)}$ to express negsum^k using constraints of size at most k. On conjugacy classes where all parts have size at least $k+1$, a similar approach as the one above can be used to calculate $\mathbb{E}_{\lambda,\mu}[\text{negsum}^k]$.

3.3 Symmetric colored permutation statistics

We now turn to extending the notion of a *symmetric* permutation statistic from [4] to the colored setting. We begin with some definitions.

Definition 15. The *support* of a colored permutation constraint $(K, \kappa) = \{(i_r^0, j_r^{\kappa(j_r)})\}_{r=1}^m$ is $\text{supp}(K, \kappa) = \{i_1, \ldots, i_m, j_1, \ldots, j_m\}$. We emphasize that $\text{supp}(K, \kappa)$ is a set and not a multiset.

Definition 16. Consider any colored permutation constraint (K, κ) with support given by $a_1 < \cdots < a_s$. For any order-preserving injection $f : \{a_1, \dots, a_s\} \to [n]$, define $f(K, \kappa)$ to be the constraint

$$f(K, \kappa) = (\, \{(f(i_1), f(j_1)), \dots, (f(i_m), f(j_m))\}, \{\kappa(f(j_1)) = k_1, \dots, \kappa(f(j_m)) = k_m\}\,).$$

Definition 17. A set of colored permutation constraints $\mathcal{C}$ is *symmetric* if for all $(K, \kappa) \in \mathcal{C}$ and any order-preserving injection $f : \operatorname{supp}(K, \kappa) \to [n]$, we have $f(K, \kappa) \in \mathcal{C}$. A statistic X is *symmetric* if it has the form $X = \sum_{(K,\kappa)\in\mathcal{C}} I_{(K,\kappa)}$ for some symmetric $\mathcal{C}$.

Many statistics naturally satisfy this condition.

Example 18. Consider the statistic inv on B_n that can be realized as

$$\text{inv} = \sum_{\substack{i,j\in[n] \\ i<j}} \sum_{\substack{k,\ell\in[\pm n] \\ k<\ell}} I_{\{(i,\ell),(j,k)\}}.$$

We denote the constraint set $\mathcal{C}$. If $k, \ell > 0$, then for any order preserving $f : \{i, j, k, \ell\} \to [n]$, we see that $\{(f(i), f(\ell)), (f(j), f(k))\} \in \mathcal{C}$. Note that the set $\{i, j, k, \ell\}$ need not consist of four distinct elements. If $k < 0$ and $\ell > 0$, we see that for any order-preserving $f : \{i, j, |k|, \ell\} \to [n]$, we have $\{(f(i), f(\ell)), (f(j), -f(|k|))\} \in \mathcal{C}$. The same argument holds for the case when $k, \ell < 0$.

Definition 19. Fix $n_0 \geq 2$. Let $X = \sum_{(K,\kappa)\in\mathcal{C}} I_{(K,\kappa)}$ be a symmetric statistic defined on $\mathfrak{S}_{n_0,r}$. Define the *r-colored symmetric extensions* of X to be the statistics $X_n = \sum_{(K,\kappa)\in\mathcal{C}_n} I_{(K,\kappa)}$ on $\mathfrak{S}_{n,r}$ with $\mathcal{C}_n$ defined as follows:

- If $n \leq n_0$, then $\mathcal{C}_n$ contains all $(K, \kappa) \in \mathcal{C}$ with support contained in $[n]$.
- If $n \geq n_0$, then $\mathcal{C}_n$ is the set of all $f(K, \kappa)$ where $(K, \kappa) \in \mathcal{C}$ and $f : [n_0] \to [n]$ is order-preserving.

Observe that by construction, each X_n is a symmetric statistic. We emphasize here that r is kept constant throughout this construction.

Many statistics can be constructed in this manner. For example, if $\mathcal{C}$ is the set of constraints for inv on B_4, then this results in the inv statistics on all B_n. In general, the moments of these statistics satisfy the following polynomial property.

Theorem 20. *Fix $r \geq 1$. Let (X_n) be the symmetric extensions of a symmetric statistic $X = X_{n_0}$ on $\mathfrak{S}_{n,r}$ induced by a constraint set $\mathcal{C}$ of size m. There exists a polynomial $p_X(n)$ of degree at most mk depending only on X such that $p_X(n) = \mathbb{E}_{\boldsymbol{\lambda}_n}[X_n^k]$ for any r-partition $\boldsymbol{\lambda}_n$ of n where all $\lambda_n^{(j)}$ have parts of size at least $mk + 1$.*

Note that one can show this polynomiality property for other statistics that are not symmetric extensions. The key requirement is that the weights for the various I_K behave in a way that allows us to divide by the denominators that result from applying Lemma 10.

4 Descents in conjugacy classes of hyperoctahedral groups

In this section we discuss the techniques involved in establishing our central limit theorem for descents in conjugacy classes of B_n that do not have short cycles. The descent statistic on B_n was defined in Eqn. (2.1). Let $(\lambda(\omega), \mu(\omega))$ denote the cycle type of $\omega \in B_n$, and let $m_i(\lambda)$ denote the number of parts of λ equal to i. While Reiner [15] uses a different notion of descents, the generating function [9, Theorem 5.3]

$$\sum_{\omega \in B_n} t^{\operatorname{des}_B(\omega)} \prod_i x_i^{m_i(\lambda(\omega))} y_i^{m_i(\mu(\omega))} \tag{4.1}$$

is unaffected.

Following Fulman [8], our approach involves examining the generating function given in (4.1), which allows us to analyze the generating function for des_B on a conjugacy class. We then relate this with the generating function for descents on all of B_n. In the case where there are no short cycles in $C_{\lambda,\mu}$, we will ultimately conclude that certain moments of des_B agree on $C_{\lambda,\mu}$ and B_n, and this in turn enables us to use the method of moments with a known central limit theorem of Chow and Mansour for des_B on B_n given below.

Proposition 21. *[6, Thm 3.4] Let X_n be des_B defined on B_n. Then X_n has mean $n/2$ and variance $(n+1)/12$, and as $n \to \infty$, the standardized random variable $(X_n - n/2)/\sqrt{(n+1)/12}$ converges to a standard normal distribution.*

We will need the well-known generating function of des_B over all of B_n.

Proposition 22. *[14, Eqn. (13.3)] Let $B_n(t) = \sum_{\omega \in B_n} t^{\operatorname{des}_B(\omega)+1}$. Then*

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 1} (2k-1)^n t^k.$$

We now analyze (4.1), which will allow us to derive an expression for the generating function of des_B on a conjugacy class $C_{\lambda,\mu}$. The following expression features prominently in our analysis.

Definition 23. [15] Let $\mu(d)$ be the number-theoretic Möbius function. Define, for nonnegative integers r and m,

$$N(r, 2m) = \frac{1}{2m} \sum_{\substack{d \mid m \\ d \text{ odd}}} \mu(d) \left(r^{m/d} - 1 \right).$$

Reiner [15, Theorem 4.1, Theorem 4.2] shows that $N(2k-1, 2m)$ must be a nonnegative integer for all $k, m \geq 1$.

For a fixed bi-partition (λ, μ) of n, we use the special case of [15, Theorem 4.1] appearing in [9, Theorem 5.3] to derive the following expressions for the generating function $B_{\lambda,\mu}(t) = \sum_{\omega \in C_{\lambda,\mu}} t^{\operatorname{des}_B(\omega)+1}$ of descents over the conjugacy class $C_{\lambda,\mu}$.

Proposition 24. *Let $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots)$ and $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)$. Then the following are equal to $B_{\lambda,\mu}(t)/(1-t)^{n+1}$:*

$$t\delta_{((1^n),\emptyset)} + \sum_{k\geq 2} t^k \left(\prod_{i\geq 1} \binom{N(2k-1,2i)}{m_i(\mu)} \prod_{i\geq 2} \binom{N(2k-1,2i)+m_i(\lambda)-1}{m_i(\lambda)}\right) \binom{N(2k-1,2)+m_1(\lambda)}{m_1(\lambda)}$$
$$= t\delta_{((1^n),\emptyset)} + \sum_{k\geq 2} t^k \frac{m_1(\lambda)+k-1}{k-1} \prod_{i\geq 1} \binom{N(2k-1,2i)-1+m_i(\lambda)}{m_i(\lambda)}\binom{N(2k-1,2i)}{m_i(\mu)}.$$

Here $\delta_{((1^n),\emptyset)}$ is the Kronecker delta which is 1 for the conjugacy class $\lambda, \mu = ((1^n), \emptyset)$, and zero otherwise.

By solving for $B_{\lambda,\mu}$ and extracting the coefficient of t^d, we also obtain the following corollary.

Corollary 25. *The number of permutations $\omega \in B_n$ that are of cycle type (λ, μ) and have $d-1$ descents is*

$$\sum_{k=1}^{d} (-1)^{d-k} \binom{n+1}{d-k} \binom{m_1(\lambda)+k-1}{m_1(\lambda)} \prod_{i\geq 2} \binom{N(2k-1,2i)+m_i(\lambda)-1}{m_i(\lambda)} \prod_{i\geq 1} \binom{N(2k-1,2i)}{m_i(\mu)}.$$

We now give an elegant analogue of a result of Fulman [8, Proof of Theorem 2], which will relate $B_{\lambda,\mu}(t)$ and $B_n(t)$.

Theorem 26. *Let $C_{\lambda,\mu}$ be the conjugacy class of B_n indexed by the bi-partition (λ, μ) of n, let $B_n(t) = \sum_{\omega \in B_n} t^{\mathrm{des}_B(\omega)+1}$, and let $B_{\lambda,\mu}(t) = \sum_{\omega \in C_{\lambda,\mu}} t^{\mathrm{des}_B(\omega)+1}$. Then*

$$\frac{B_{\lambda,\mu}(t)}{|C_{\lambda,\mu}|} = \frac{B_n(t)}{2^n n!} + \frac{1-t}{2n} \frac{B_{n-1}(t)}{2^{n-1}(n-1)!}[m_1(\lambda)^2 - m_1(\mu)^2] + (1-t)^2 g(t),$$

where $g(t)$ is some polynomial in t. Furthermore, when all cycles in $C_{\lambda,\mu}$ have length larger than $2k$,

$$\frac{B_{\lambda,\mu}(t)}{|C_{\lambda,\mu}|} = \frac{B_n(t)}{2^n n!} + (1-t)^{k+1} h(t),$$

where $h(t)$ is some polynomial in t.

The latter case allows us to obtain the following result involving moments of des_B on B_n and $C_{\lambda,\mu}$.

Corollary 27. *Let $C_{\lambda,\mu}$ be the conjugacy class of B_n indexed by the bi-partition (λ, μ) of n. The kth moment of des_B in $C_{\lambda,\mu}$ is equal to the kth moment of des_B in B_n if all cycles in $C_{\lambda,\mu}$ have length greater than $2k$.*

The main result of this section, Theorem 28, now follows by applying Corollary 27, the method of moments, and the asymptotic normality theorem for descents in B_n given in Proposition 21.

Theorem 28. *For every $n \geq 1$, pick a conjugacy class C_{λ_n,μ_n} in B_n indexed by the bi-partition (λ_n, μ_n) of n, where $\lambda_n = (1^{m_1(\lambda_n)}, 2^{m_2(\lambda_n)}, \ldots)$ and $\mu_n = (1^{m_1(\mu_n)}, 2^{m_2(\mu_n)}, \ldots)$. Define X_n to be des_B on C_{λ_n,μ_n}. Suppose that for all i, $m_i(\lambda_n) \to 0$ and $m_i(\mu_n) \to 0$ as $n \to \infty$. For sufficiently large n, X_n has mean $n/2$ and variance $(n+1)/12$. Furthermore, as $n \to \infty$, the random variable $(X_n - n/2)/\sqrt{(n+1)/12}$ converges to a standard normal distribution.*

5 Conclusion

In this paper, we have introduced a notion of constraints and size for any colored permutation statistic $X : \mathfrak{S}_{n,r} \to \mathbb{R}$, and we have used this framework to study the moments of X on conjugacy classes $C_{\boldsymbol{\lambda}}$. In particular, we have established that for a statistic of size m, the kth moment on $C_{\boldsymbol{\lambda}}$ is independent of conjugacy class $C_{\boldsymbol{\lambda}}$ when all parts of the partitions in $\boldsymbol{\lambda}$ have length at least $mk + 1$. For statistics on $\mathfrak{S}_{n,r}$ that can be expressed as symmetric extensions, these moments are polynomials in n. Our results directly generalize those in [4] on $\mathfrak{S}_n$. Given the numerous connections to [11], one natural problem is the following.

Problem 29. Use the representation theory of B_n and $\mathfrak{S}_{n,r}$ to establish analogues of the results in [11].

Finally, we note that $\mathfrak{S}_n$ and B_n are respectively the type A and type B Coxeter groups. The following is a natural problem to consider next.

Problem 30. Establish analogues of the results in this paper for the type D Coxeter groups.

It would also be of interest to establish analogous results for (irreducible) complex reflection groups.

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Upho lattices and their cores

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Abstract. A poset is called upper homogeneous, or "upho," if every principal order filter is isomorphic to the original poset. We study enumerative and structural properties of (finite type $\mathbb{N}$-graded) upho posets. The first important observation we make about upho posets is that their rank generating functions and characteristic generating functions are multiplicative inverses of one another. This means that each upho lattice has associated to it a finite graded lattice, called its core, which determines its rank generating function. We investigate which finite graded lattices arise as cores of upho lattices, providing both positive and negative results. On the one hand, we show that many well-studied finite lattices do arise as cores, and we present combinatorial and algebraic constructions of the upho lattices into which they embed. On the other hand, we show there are obstructions which prevent many finite lattices from being cores.

1 Introduction

Symmetry is a fundamental theme in mathematics. A close cousin of symmetry is self-similarity, where a part resembles the whole. Here we study certain partially ordered sets that are self-similar in a precise sense. Namely, a poset is called *upper homogeneous*, or "*upho*," if every principal order filter of the poset is isomorphic to the whole poset. In other words, a poset $\mathcal{P}$ is upho if, looking up from each element $p \in \mathcal{P}$, we see another copy of $\mathcal{P}$. Upho posets were introduced recently by Stanley [13, 14]. We believe they are a natural and rich class of posets which deserve further attention.

Upho posets are infinite. In order to be able to apply the tools of enumerative and algebraic combinatorics, we need to impose some finiteness condition on the posets we consider. Thus, we restrict our attention to finite type $\mathbb{N}$-graded posets. These are the infinite posets $\mathcal{P}$ that possess a rank function $\rho\colon \mathcal{P} \to \mathbb{N}$ for which we can form the *rank generating function*

$$F(\mathcal{P};x) := \sum_{p\in\mathcal{P}} x^{\rho(p)}.$$

Henceforth, upho posets are assumed finite type $\mathbb{N}$-graded unless otherwise specified.

The first important observation we make about (finite type $\mathbb{N}$-graded) upho posets is that their rank generating functions are related in a nice way to the values of their

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Möbius functions. Specifically, if we define the *characteristic generating function* of an upho poset $\mathcal{P}$ to be

$$\chi^*(\mathcal{P};x) := \sum_{p\in\mathcal{P}} \mu(\hat{0},p)x^{\rho(p)}$$

then

$$F(\mathcal{P};x) = \chi^*(\mathcal{P};x)^{-1} \tag{1.1}$$

i.e., $F(\mathcal{P};x)$ and $\chi^*(\mathcal{P};x)$ are multiplicative inverses as formal power series.

Gao et al. [7, §5] have shown that there are uncountably many different rank generating functions $F(\mathcal{P};x)$ of upho posets $\mathcal{P}$ (see also [6]). This prevents us from being able to say much more about the enumerative and structural properties of upho posets in general. However, the situation is different for upho *lattices*.

Let $\mathcal{L}$ be an upho lattice, and let $L := [\hat{0}, s_1 \vee \cdots \vee s_r]$ denote the interval in $\mathcal{L}$ from its minimum to the join of its atoms $s_1,\ldots,s_r$. We refer to the finite graded lattice L as the *core* of the upho lattice $\mathcal{L}$. Rota's cross-cut theorem implies that $\chi^*(\mathcal{L};x) = \chi^*(L;x)$, where $\chi^*(L;x) := \sum_{p\in L}\mu(\hat{0},p)x^{\rho(p)}$ is the (reciprocal) characteristic polynomial of L. Hence,

$$F(\mathcal{L};x) = \chi^*(L;x)^{-1} \tag{1.2}$$

Thus, the rank generating function of an upho lattice is the inverse of a polynomial with integer coefficients. We review (1.1) and (1.2) in Section 2.

Because of (1.2), the core of an upho lattice determines its rank generating function.[1] We caution that the core does not completely determine the upho lattice, i.e., there can be multiple upho lattices with the same core. Nevertheless, any complete understanding of upho lattices would have to start with an answer to the following question.

Question 1. *Which finite graded lattices arise as cores of upho lattices?*

Question 1 can be thought of as a kind of tiling problem: our goal is to tile an infinite, fractal lattice $\mathcal{L}$ using copies of some fixed finite lattice L, or show that no such tiling is possible. In addressing Question 1 here, we provide both positive and negative results.

We start with combinatorial constructions of upho lattices. In Section 3, we construct upho lattices as limits of sequences of finite lattices. We show that any member of a uniform sequence of supersolvable geometric lattices is the core of an upho lattice. Examples of uniform sequences of supersolvable geometric lattices include the Boolean lattices B_n, their q-analogues $B_n(q)$, and the partition lattices Π_n. Hence, these are all cores of upho lattices. Figure 1 depicts an upho lattice produced via this construction.

In addition to combinatorial constructions, we also explore algebraic constructions of upho lattices. In Section 4 we explain how monoids provide one algebraic source of upho lattices. A homogeneously finitely generated monoid M that is left cancellative is

[1] In fact, since the flag f-vector of any upho poset is determined by its rank generating function (see [14, §3]), the core of an upho lattice determines its entire flag f-vector.

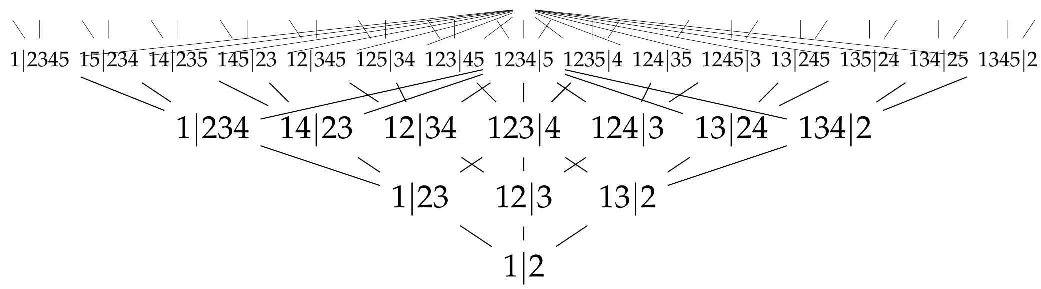

Figure 1: Partitions of sets of the form $\{1, 2, \ldots, n\}$ into 2 blocks, ordered by refinement. This is an upho lattice with core Π_3.

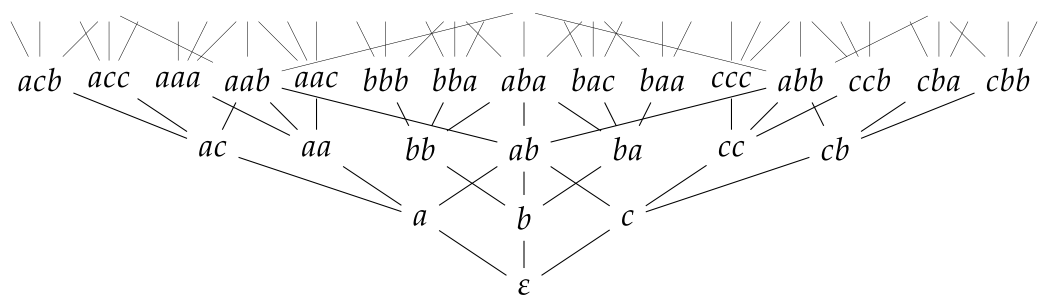

Figure 2: The dual braid monoid $\langle a, b, c \mid ab = bc = ca \rangle$ associated to the symmetric group S_3. This is an upho lattice with core the noncrossing partition lattice of S_3.

an upho poset under left division. Hence, if M is a lattice under left division, it is an upho lattice. An important example of such monoids are the Garside monoids coming from Coxeter theory. Thus, the weak order and noncrossing partition lattice of any finite Coxeter group are cores of upho lattices. Figure 2 depicts an upho lattice of this form.

On the negative side, in Section 5, we show that there are various obstructions which prevent arbitrary finite graded lattices from being realized as cores of upho lattices. There are constraints on the characteristic polynomial of the lattice coming from (1.2). There are also some structural obstructions, requiring the lattice to be partly self-similar. These obstructions allow us to show that the following plausible candidates cannot be realized as cores: the face lattices of the n-dimensional cross polytope and hypercube (for $n \geq 3$); the lattice of flats of the uniform matroid $U(k, n)$ (for $2 < k < n$).

The upshot is that Question 1 is quite subtle: it can be difficult to recognize when a given finite graded lattice is the core of an upho lattice. Many well-behaved finite lattices are cores of upho lattices, but many too are not. In Section 6 we briefly discuss future directions in the study of upho lattices that we intend to pursue.

This is just an extended abstract where we survey our recent results on upho lattices. The full articles containing these results, with proofs, are [8, 9].

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2 Upho poset basics

We use standard notation and terminology for posets, as laid out for instance in [12, §3]. Since we routinely work with both finite and infinite posets, we use the convention that finite posets are denoted by normal script letters (like P and L) and infinite posets are denoted by calligraphic letters (like $\mathcal{P}$ and $\mathcal{L}$).

Recall that a finite poset P is *graded (of rank n)* if we can write $P = \bigsqcup_{i=0}^{n} P_i$ as a disjoint union so that every maximal chain is of the form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ with $x_i \in P_i$. In this case, we define the *rank function* $\rho\colon P \to \mathbb{N}$ by $\rho(x) := i$ if $x \in P_i$. For such a P, we define its *rank generating polynomial* to be $F(P;x) := \sum_{p \in P} x^{\rho(p)}$. If P has a minimum $\hat{0}$, we define its *(reciprocal) characteristic polynomial* to be $\chi^*(P;x) := \sum_{p \in P} \mu(\hat{0}, p) x^{\rho(p)}$, where $\mu(\cdot,\cdot)$ is the Möbius function of P.

We say an infinite poset $\mathcal{P}$ is *$\mathbb{N}$-graded* if we can write $\mathcal{P} = \bigsqcup_{i=0}^{\infty} \mathcal{P}_i$ as a disjoint union so that every maximal chain is of the form $x_0 \lessdot x_1 \lessdot \cdots$ with $x_i \in \mathcal{P}_i$. In this case, we define the *rank function* $\rho\colon \mathcal{P} \to \mathbb{N}$ by $\rho(x) := i$ if $x \in \mathcal{P}_i$. We say that $\mathcal{P}$ is *finite type* $\mathbb{N}$-graded if $\#\mathcal{P}_i < \infty$ for all i. For such a $\mathcal{P}$, we define its *rank generating function* to be $F(\mathcal{P};x) := \sum_{p \in \mathcal{P}} x^{\rho(p)}$. If $\mathcal{P}$ has a minimum $\hat{0}$, we define its *characteristic generating function* to be $\chi^*(\mathcal{P};x) := \sum_{p \in \mathcal{P}} \mu(\hat{0}, p) x^{\rho(p)}$.

We say that a poset $\mathcal{P}$ is *upper homogeneous*, or "*upho*," if for every $p \in \mathcal{P}$, we have that $V_p \simeq \mathcal{P}$, where $V_p := \{q \in \mathcal{P} \colon q \geq p\}$ is the principal order filter generated by p. To avoid trivialities, let us assume the upho posets we consider have at least two elements; then they must be infinite. Examples of upho posets include:

- the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$, the nonnegative rational numbers $\mathbb{Q}_{\geq 0}$, and the nonnegative real numbers $\mathbb{R}_{\geq 0}$, all with their usual total orders;
- the poset of finite subsets of X ordered by inclusion, where X is any infinite set.

In order to be able to apply the tools of enumerative and algebraic combinatorics to study upho posets, we need to impose some finiteness conditions. Hence, from now on, **all upho posets are assumed finite type $\mathbb{N}$-graded**. Of the above examples, only $\mathbb{N}$ is finite type $\mathbb{N}$-graded. Here are more examples of (finite type $\mathbb{N}$-graded) upho posets:

Example 1. *Fix $r \geq 1$ and let $\mathcal{P}$ be the "infinite rooted r-ary tree" poset. The case $r = 2$ of this poset is depicted on the left in Figure 3. Note that this $\mathcal{P}$ is the "freest" upho poset with r atoms. It has $F(\mathcal{P};x) = \frac{1}{1-rx}$ and $\chi^*(\mathcal{P};x) = 1 - rx$.*

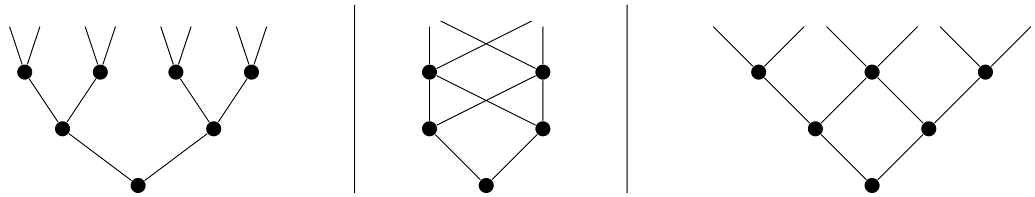

Figure 3: Various upho posets with two atoms.

Example 2. *Fix $r \geq 1$ and let $\mathcal{P} = \bigsqcup_{i=0}^{\infty} \mathcal{P}_i$ be the $\mathbb{N}$-graded poset with $\#\mathcal{P}_0 = 1$ and $\#\mathcal{P}_i = r$ for all $i \geq 1$, and with all cover relations between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ for all $i \geq 0$. The case $r = 2$ is depicted in the middle in Figure 3. Note that this $\mathcal{P}$ is the "least free" upho poset with r atoms. It has $F(\mathcal{P};x) = \frac{1+(r-1)x}{1-x}$ and $\chi^*(\mathcal{P};x) = \frac{1-x}{1+(r-1)x}$.*

Example 3. *Fix $r \geq 1$ and let $\mathcal{P} = \mathbb{N}^r$, i.e., the (Cartesian) product of r copies of $\mathbb{N}$. The case $r = 2$ is depicted on the right in Figure 3. It has $F(\mathcal{P};x) = \frac{1}{(1-x)^r}$ and $\chi^*(\mathcal{P};x) = (1-x)^r$.*

From the preceding examples, the reader might guess a relationship between the rank and characteristic generating functions of an upho poset. And indeed, the following result is proved by a straightforward application of Möbius inversion (see [12, §3.7]).

Theorem 1. *For an upho poset $\mathcal{P}$, we have $F(\mathcal{P};x) = \chi^*(\mathcal{P};x)^{-1}$.*

Lattices have well-behaved Möbius functions, so we can say even more about upho lattices. Let $\mathcal{L}$ be an upho lattice, and let $L := [\hat{0}, s_1 \vee \cdots \vee s_r]$ denote the interval in $\mathcal{L}$ from its minimum to the join of its atoms $s_1, \ldots, s_r$. We call L the *core* of $\mathcal{L}$. Rota's cross-cut theorem (see [12, Corollary 3.9.5]) and Theorem 1 together imply the following.

Corollary 1. *For an upho lattice $\mathcal{L}$ with core L, we have $F(\mathcal{L};x) = \chi^*(L;x)^{-1}$.*

Of the above examples of (finite type $\mathbb{N}$-graded) upho posets, only $\mathbb{N}^n$ is a lattice. The core of $\mathbb{N}^n$ is B_n, the rank n *Boolean lattice*, i.e., the lattice of subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion. And indeed, we have $\chi^*(B_n;x) = (1-x)^n = F(\mathbb{N}^n;x)^{-1}$. In what follows, we focus on Question 1, the question of which finite graded lattices are cores of upho lattices. For example, we just saw that the Boolean lattice B_n is a core for all $n \geq 1$.

3 Upho lattices from sequences of finite lattices

In this section we construct upho lattices as limits sequences of finite lattices that are appropriately embedded in one another. In order to make "appropriately embedded in one another" precise, we need two technical notions from the literature: the notion of a *supersolvable* geometric lattice, as introduced by Stanley in [11]; and the notion of a *uniform sequence* of geometric lattices, as introduced by Dowling in [5]. These two notions represent two different ways that a lattice can have a recursive structure.

Let L be a finite lattice. We say L is *atomic* if every element is the join of atoms. We say L is *(upper) semimodular* if it is graded and satisfies $\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y)$ for all $x, y \in L$. We say L is *geometric* if it is both atomic and semimodular. Geometric lattices are intensely studied because they are exactly the lattices of flats of matroids.

First we review supersolvability. So let L be a geometric lattice. We say $x \in L$ is *modular* if $\rho(x) + \rho(y) = \rho(x \vee y) + \rho(x \wedge y)$ for all $y \in L$. For instance, every element of a modular lattice is modular. The lattice L is called *supersolvable* if it has a maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$ of modular elements. Stanley proved the following remarkable factorization theorem for characteristic polynomials of supersolvable geometric lattices.

Theorem 2 (Stanley [11]). *Let L be a supersolvable geometric lattice with maximal chain of modular elements $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. Then $\chi^*(L; x) = (1 - a_1 x)(1 - a_2 x) \cdots (1 - a_n x)$ where $a_i := \#\{\text{atoms } s \in L \colon s \leq x_i, s \not\leq x_{i-1}\}$ for $i = 1, \ldots, n$.*

Next we review uniform sequences. A sequence $L_0, L_1, \ldots$ of geometric lattices is called *uniform* if each L_n is graded of rank n, and $[a, \hat{1}_{L_n}] \simeq L_{n-1}$ for every atom $a \in L_n$.

Now fix a uniform sequence of geometric lattices $L_0, L_1, \ldots$. We define their *Whitney numbers of the second and first kind*, denoted $V(i,j)$ and $v(i,j)$, respectively, to be the coefficients $F(L_i; x) = \sum_{j=0}^{i} V(i,j) x^{i-j}$ and $\chi^*(L_i, x) = \sum_{j=0}^{i} v(i,j) x^{i-j}$. By convention, we set $V(i,j) := 0$ and $v(i,j) := 0$ for $j > i$. Dowling showed that uniform sequences of geometric lattices have the following nice behavior for their Whitney numbers.

Theorem 3 (Dowling [5]). *The matrices $[V(i,j)]_{0 \leq i,j \leq \infty}$ and $[v(i,j)]_{0 \leq i,j \leq \infty}$ are inverses.*

In the case when the geometric lattices in our uniform sequence are supersolvable, we can combine Theorems 2 and 3 to yield a stronger result.

Corollary 2. *Suppose that the geometric lattices L_n in our uniform sequence are all supersolvable. Then their Whitney numbers are*

$$V(i,j) = h_{i-j}(a_1, \ldots, a_{j+1}) \qquad \text{and} \qquad v(i,j) = (-1)^{i-j} e_{i-j}(a_1, \ldots, a_i),$$

where h_k and e_k are the kth complete homogeneous and elementary symmetric polynomials, and $a_n := \#\{\text{atoms } s \in L_n\} - \#\{\text{atoms } s \in L_{n-1}\}$ for all $n \geq 1$.

The key to proving Corollary 2 is to show that each L_{n+1} has a modular coatom t for which $[\hat{0}_{L_{n+1}}, t] \simeq L_n$. In this way, we get rank-preserving embeddings $\iota_n \colon L_n \to L_{n+1}$. By abuse of terminology, we define a *uniform sequence of supersolvable geometric lattices* to be a uniform sequence of geometric lattices $L_0, L_1, \ldots$ for which we have *fixed* such embeddings $\iota_n \colon L_n \to L_{n+1}$, and for which these ι_n are *compatible* with the isomorphisms $[a, \hat{1}_{L_n}] \simeq L_{n-1}$ in the uniformity condition. See [9, §3] for the precise definition.

So now let $L_0, L_1, \ldots$ be a uniform sequence of *supersolvable* geometric lattices, and then define $\mathcal{L}_\infty := \bigcup_{n=0}^{\infty} L_n$, the direct limit of the L_n with respect to the $\iota_n \colon L_n \to L_{n+1}$. This $\mathcal{L}_\infty$ will almost be an upho lattice, except that it will not be finite type $\mathbb{N}$-graded: it will have infinitely many atoms. We need to "trim" $\mathcal{L}_\infty$ to produce an upho lattice. Hence, for each $k \geq 1$, define $\mathcal{L}_\infty^{(k)} := \{p \in \mathcal{L}_\infty \colon \nu(p) - \rho(p) < k\}$, where for $p \in \mathcal{L}_\infty$ we let $\nu(p) := \min\{n \colon p \in L_n\}$. Then we can prove the following.

Theorem 4. *For each $k \geq 1$, $\mathcal{L}_\infty^{(k)}$ is an upho lattice with core L_k.*

Theorem 4 would not be interesting if there were no interesting examples of uniform sequences of supersolvable geometric lattices. Fortunately, there are many interesting examples, which we now review.

Example 4. *Taking $L_n = B_n$, the rank n Boolean lattice, gives a uniform sequence of supersolvable geometric lattices. For this sequence, $\mathcal{L}_\infty$ is the lattice of all finite subsets of $\{1, 2, \ldots\}$ ordered by inclusion, and $\mathcal{L}_\infty^{(k)} = \{$finite $S \subseteq \{1, 2, \ldots\} \colon S \subseteq [\#S + k - 1]\}$. Note that, $\mathcal{L}^{(k)}$ is* not *isomorphic to $\mathbb{N}^k$ (for $k \geq 2$). This is the simplest example a finite graded lattice being the core of two different upho lattices. Of course, we have $F(\mathcal{L}_\infty^{(k)}; x)^{-1} = \chi^*(B_k; x) = (1 - x)^k$.*

Example 5. *Fix a prime power q. Recall that the* subspace lattice *$B_n(q)$ is the lattice of subspaces of $\mathbb{F}_q^n$ ordered by inclusion. Taking $L_n = B_n(q)$ gives a uniform sequence of supersolvable geometric lattices. For this sequence, $\mathcal{L}_\infty$ is the lattice of all finite-dimensional subspaces of $\mathbb{F}_q^\infty$, and $\mathcal{L}_\infty^{(k)} = \{$finite-dimensional $U \subseteq \mathbb{F}_q^\infty \colon U \subseteq \mathrm{Span}\{e_1, \ldots, e_{\dim(U)+k-1}\}\}$, with $e_1, e_2, \ldots$ an ordered basis of $\mathbb{F}_q^\infty$. We have $F(\mathcal{L}_\infty^{(k)}; x)^{-1} = \chi^*(B_k(q); x) = (1 - x)(1 - qx) \cdots (1 - q^{k-1}x)$.*

Example 6. *Recall that the* partition lattice *Π_n is the lattice of set partitions of $[n]$ ordered by refinement. Taking $L_n = \Pi_{n+1}$ gives a uniform sequence of supersolvable geometric lattices. For this sequence, $\mathcal{L}_\infty$ is the lattice of all set partitions of $\{1, 2, \ldots\}$ for which all but finitely many blocks are singletons (ordered by refinement). And $\mathcal{L}_\infty^{(k)}$ can be identified with the set partitions of a set of the form $[n]$ into k blocks, still ordered by refinement in the sense that $\pi_1 \leq \pi_2$ if each block of π_1 is a subset of a block of π_2. Figure 1 depicts $\mathcal{L}_\infty^{(2)}$ for this example. We have $F(\mathcal{L}_\infty^{(k)}; x)^{-1} = \chi^*(\Pi_{k+1}; x) = (1 - x)(1 - 2x) \cdots (1 - kx)$.*

Example 7. *Fix a finite group G, say with m elements. In [5], Dowling defined a lattice $Q_n(G)$, now called a* Dowling lattice*, consisting of certain "G-decorated" (partial) set partitions of $[n]$. When G is the trivial group, $Q_n(G) = \Pi_{n+1}$. And when $G = \mathbb{Z}/2\mathbb{Z}$, $Q_n(G)$ is the lattice of flats of the Type B_n Coxeter hyperplane arrangement. Dowling proved that $Q_n(G)$ is a uniform sequence of supersolvable geometric lattices, with $\chi^*(Q_n(G); x) = \prod_{i=1}^{n}(1 - (1 + m(i - 1))x)$. See [9, §3] for the description of the $\mathcal{L}_\infty$ and $\mathcal{L}_\infty^{(k)}$ for this example.*

4 Upho lattices from monoids

In this section, we explain how monoids give rise to upho lattices. The examples in this section are quite different from those in Section 3. For example, the characteristic polynomials in this section will not necessarily factor over the integers.

Recall that a *monoid* $M = (M, \cdot)$ is a set M with an associative binary product $\cdot$ that has an identity element. The *free monoid* on a set S is the monoid of words over the alphabet S with the product being concatenation and identity the empty word. A *presentation* of a monoid M is a way of writing $M = \langle S \mid R \rangle$ as a quotient of the free monoid over some generating set S by the relations R. A monoid M is *finitely generated* if it has a presentation $M = \langle S \mid R \rangle$ with S finite. Let us say that M is *homogeneously* finitely generated if it has a presentation $M = \langle S \mid R \rangle$ with S finite and R homogeneous. That is, we require that all relations in R are of the form $w_1 = w_2$ with $\ell(w_1) = \ell(w_2)$, where for a word w use $\ell(w)$ to denote its length.

Let M be a monoid. For $a, b \in M$, we say that a is a *left divisor* of b, and b is a *right multiple* of a, if $ax = b$ for some $x \in M$. We use $\leq_L$ for the preorder of left divisibility on M, which is actually a partial order if M is homogeneously finitely generated. The monoid M is called *left cancellative* if whenever $ab = ac$ then $b = c$, for all $a, b, c \in M$.

Lemma 1 (c.f. [7, Lemma 5.1] and [6]). *Let M be a homogeneously finitely generated monoid. If M is left cancellative, then $\mathcal{L} := (M, \leq_L)$ is an upho poset. If moreover every pair of elements in M has a least common right multiple, then $\mathcal{L}$ is an upho lattice.*

The significance of the Möbius function to enumeration in monoids, especially cancellative monoids, was already observed many years ago in the work of Cartier and Foata on free partially commutative monoids [3]. In practice, the left cancellative property of a monoid is not hard to check, but the lattice property is more difficult to establish. Nevertheless, some examples can be produced by hand:

Example 8. *Fix $n, r \geq 2$ and define $M := \langle x_1, \ldots, x_r \mid x_i x_1^{n-1} = x_1^n \text{ for all } i = 2, \ldots, r \rangle$. Then M satisfies the conditions of Lemma 1, so that $\mathcal{L} := (M, \leq_L)$ is an upho lattice. Its core is $L := \hat{0} \oplus r \cdot [n-1] \oplus \hat{1}$, i.e., the result of appending a minimum and maximum to the disjoint union of r $(n-1)$-element chains. We have $F(\mathcal{L}; x)^{-1} = \chi^*(L; x) = 1 - rx + (r-1)x^n$.*

Note in particular that taking $n = 2$ in Example 8 shows how every rank two finite graded lattice (with at least two atoms) can be realized as the core of an upho lattice. We will see in Section 5 that not all rank three lattices can be realized as cores.

To obtain more sophisticated examples of upho lattices from Lemma 1, we need some deeper theory. An important class of monoids satisfying the conditions of Lemma 1 are the (homogeneous) *Garside monoids*. We refer to [4] for a complete account of the theory of Garside monoids. Without giving the full definition, we note that a Garside monoid is not only left cancellative and a lattice under left divisibility, it is also right cancellative and a lattice under right divisibility.

The most significant examples of Garside monoids come from finite Coxeter groups. We give only a cursory account of the relevant Coxeter theory here; see [2, 1] for much more detail. Recall that W is a *finite Coxeter group* if W is a finite group with generating set $S = \{s_1, \ldots, s_r\} \subseteq W$ for which $W = \langle S \mid s_i^2 = 1 \text{ for all } i, (s_is_j)^{m_{i,j}} = 1 \text{ for } i < j\rangle$ for certain integers $m_{i,j} \geq 2$. We also say (W, S) is a *finite Coxeter system* in this case, and we say S are the *simple reflections* of W. For example, the symmetric group S_n of permutations of $[n]$ is a finite Coxeter group with simple reflections $\{s_1, \ldots, s_{n-1}\}$ the adjacent transpositions $s_i = (i, i+1)$; here we have $m_{i,j} = 2$ if $j - i \geq 2$ and $m_{i,i+1} = 3$.

Now fix a finite Coxeter system $(W, S = \{s_1, \ldots, s_r\})$. The *length* $\ell(w)$ of $w \in W$ is the minimum length of a way of writing $w = s_{i_1} \cdots s_{i_k}$ as a word in the s_i. The *weak order* of W is the partial order on W whose cover relations are $w \lessdot ws$ whenever $\ell(ws) = \ell(w) + 1$, for $w \in W$ and $s \in S$. It is well-known that the weak order is a finite graded lattice, with rank function ℓ. One Garside monoid associated to (W, S) is related to weak order:

Example 9. *Let* $\mathbf{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ *be a collection of letters corresponding to the simple reflections* $S = \{s_1, \ldots, s_r\}$. *For* $\mathbf{s}_i, \mathbf{s}_j \in \mathbf{S}$ *we write* $(\mathbf{s}_i, \mathbf{s}_j)^{[m]} := \mathbf{s}_i\mathbf{s}_j\mathbf{s}_i\mathbf{s}_j \cdots$, *a word with* m *letters. The* classical braid monoid *associated to* (W, S) *is* $M := \langle \mathbf{S} \mid (\mathbf{s}_i, \mathbf{s}_j)^{[m_{i,j}]} = (\mathbf{s}_j, \mathbf{s}_i)^{[m_{i,j}]} \text{ for } i < j\rangle$. *It is known that* M *is a Garside monoid (see [4, Chapter IX, §1]), which implies that* M *satisfies the conditions of Lemma 1, so* $\mathcal{L} := (M, \leq_L)$ *is an upho lattice. Its core is the weak order of* W.

Continue to fix the finite Coxeter system (W, S). There is another Garside monoid associated to (W, S) that is also very interesting. Let $T := \{w^{-1}sw \colon w \in W, s \in S\} \subseteq W$, which is called the set of *reflections* of W. The *absolute length* $\ell_T(w)$ of $w \in W$ is the minimum length of a way of writing $w = t_1 \cdots t_k$ with all $t_i \in T$. The *absolute order* of W is the partial order on W whose cover relations are $w \lessdot wt$ whenever $\ell_T(wt) = \ell_T(w) + 1$, for $w \in W$ and $t \in T$. Absolute order is a graded poset, with rank function ℓ_T, but it is not a lattice since it has multiple maximal elements. However, if $c \in W$ is a *Coxeter element* (a product $c = s_1 \cdots s_r$ of the simple reflections in some order), then the interval $[1, c]$ in absolute order *is* a lattice, whose isomorphism type does not depend on the choice of c. It is called the *noncrossing partition lattice* of W. When $W = S_n$, the noncrossing partition lattice is the restriction of Π_n to those partitions that are noncrossing when the numbers $1, 2, \ldots, n$ are arranged clockwise around a circle, and hence the name. The second Garside monoid attached to (W, S) is related to the noncrossing partition lattice:

Example 10. *Let* $\mathbf{T}$ *be a collection of letters corresponding to the reflections* T. *For* $\mathbf{s}, \mathbf{t} \in \mathbf{T}$, *we use the notation* $\mathbf{t}^{\mathbf{s}}$ *for the letter corresponding to the conjugate* $t^s := s^{-1}ts$. *The* dual braid monoid *associated to* (W, S) *is* $M := \langle \mathbf{T} \mid \mathbf{ts} = \mathbf{st}^{\mathbf{s}} \text{ for all } \mathbf{s} \neq \mathbf{t} \in \mathbf{T}\rangle$. *It is known that* M *is a Garside monoid (see [4, Chapter IX, §2]), which implies that* M *satisfies the conditions of Lemma 1, so* $\mathcal{L} := (M, \leq_L)$ *is an upho lattice. Its core is the noncrossing partition lattice of* W. *For example, Figure 2 depicts this upho lattice in the case when* $W = S_3$.

We note that the noncrossing partition lattice of S_3 happens to be isomorphic to Π_3, so Figures 1 and 2 are examples of non-isomorphic upho lattices with the same core.

5 Obstructions for cores of upho lattices

In this section we explore various methods for showing that a finite graded lattice *cannot* be realized as the core of an upho lattice. We first observe that there are constraints on the characteristic polynomial of a core coming from Corollary 1.

Lemma 2. *Let L be a finite graded lattice which is the core of an upho lattice. Then the formal power series $\chi^*(L;x)^{-1}$ has all positive coefficients.*

Already Lemma 2 can rule out some plausible candidate lattices from actually being realized as cores, as we now explain.

Let P be a convex polytope. The *face lattice* $L(P)$ of P is the poset of faces of P ordered by inclusion. It is always a finite graded lattice. For example, if P is an $(n-1)$-dimensional simplex, then $L(P) = B_n$, which we know is a core. It is reasonable to ask which other face lattices of convex polytopes are cores.

Example 11. *Let P be the octahedron. Then $\chi^*(L(P);x) = 1 - 6x + 12x^2 - 8x^3 + x^4$, and we can compute that $[x^{13}]\chi^*(L(P);x)^{-1} = -123704$, where $[x^n]F(x)$ means the coefficient of x^n in the power series $F(x)$. So by Lemma 2, $L(P)$ is not the core of any upho lattice.*

Let G be a connected, simple graph on vertex set $[n]$. The *bond lattice* $L(G)$ of G is the restriction of Π_n to the those set partitions π for which the induced subgraph of G on each block of π remains connected. It is always finite graded lattice; in fact, it is the lattice of flats of the graphic matroid of G. We have that $\chi^*(L(G);x) = x^n \cdot \chi(G;x^{-1})$ where $\chi(G;x)$ is the chromatic polynomial of G. For example, if $G = K_n$ is the complete graph, then $L(G) = \Pi_n$, which we know is a core. It is reasonable to ask which other bond lattices of graphs are cores.

Example 12. *Consider $G = C_4$, the 4-cycle graph. Then $\chi^*(L(C_4);x) = 1 - 4x + 6x^2 - 3x^3$ and we can compute that $[x^7]\chi^*(L(C_4);x)^{-1} = -80$. So by Lemma 2, $L(C_4)$ is not the core of any upho lattice.*

Beyond characteristic polynomial obstructions, there are also structural obstructions for cores. The following proposition follows trivially from the definition of the core of an upho lattice, but is still worth recording since it rules out many lattices as cores.

Proposition 1. *Let L be a finite graded lattice which is the core of an upho lattice. Then its maximum $\hat{1}$ is the join of its atoms.*

The previous propositions says something about the join of the elements covering $\hat{0}$. Looking at the join of the elements covering an arbitrary element x is a good idea, and leads to further, non-trivial obstructions for cores. The following lemma says that a core must already be "partly self-similar" in order to fit into an upho lattice.

Lemma 3. *Let L be a finite graded lattice which is the core of an upho lattice. Let $x \in L \setminus \{\hat{0}, \hat{1}\}$ and let $y_1, \ldots, y_k \in L$ be the elements covering x. Then there is a rank-preserving embedding of the interval $[x, y_1 \vee \cdots \vee y_k]$ into L.*

Lemma 3 lets us rule out many further plausible candidate cores, as we now explain.

Example 13. *Let $n \geq 1$ and let P be the n-dimensional cross polytope, i.e., the convex hull of all permutations of the vectors $(\pm 1, 0, \ldots, 0) \in \mathbb{R}^n$. Consider its face lattice $L := L(P)$. Concretely, the elements of L can be represented as length n words in the alphabet $\{0, +, -\}$, where $w \leq w'$ if w' is obtained from w by changing some 0's to $\pm$'s, together with a maximum element $\hat{1}$. Letting x be any atom of L, it can be seen that no embedding of the kind required by Lemma 3 exists when $n \geq 3$, so that L is not the core of any upho lattice.*

The 3-dimensional cross polytope is the octahedron, so Example 13 generalizes Example 11. We also remark that it can similarly be shown that the face lattice of the n-dimensional *hypercube* (the dual of the cross polytope) is not a core for $n \geq 3$.

Example 14. *Let $2 \leq k \leq n$ and let L be the lattice of flats of the uniform matroid $U(k, n)$. Concretely, L is obtained from the Boolean lattice B_n by removing all elements at rank k or higher, and then adding a maximum element $\hat{1}$. Letting x be any atom of L, it can be seen that no embedding of the kind required by Lemma 3 exists when $2 < k < n$, so that L is not the core of any upho lattice.*

The lattice of flats of the uniform matroid $U(n-1, n)$ is the same as the bond lattice $L(C_n)$ of the n-cycle graph C_n, so Example 14 generalizes Example 12.

6 Future directions

In this section we briefly discuss future directions in the study of upho lattices.

A question naturally suggested by our work here is the following:

Question 2. *For a finite graded lattice L, let $\kappa(L)$ denote the number of different upho lattices with core L. How does $\kappa(L)$ behave?*

In work in progress joint with Joel Lewis [10], we are pursuing Question 2. On the one hand, we can show that $\kappa(L)$ is finite for any lattice L which has no nontrivial automorphisms, suggesting that it may be finite for all L. On the other hand, we can show that $\kappa(L)$ is unbounded even when we restrict to lattices L of rank two.

Finally, if classifying all upho lattices is too difficult, we might instead hope to classify some subvarieties of upho lattices. Two of the most important subvarieties of lattices are the distributive lattices and the modular lattices. In planned future work, we will explore distributive and modular upho lattices. The only upho distributive lattices are $\mathbb{N}^n$, but upho modular lattices are much more interesting (c.f. [7, Conjecture 1.1]).

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Pattern heights and the minimal power of q in a Kazhdan–Lusztig polynomial

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Abstract. For w in the symmetric group, we use permutation patterns to provide an exact formula for the smallest positive power $q^{h(w)}$ appearing in the Kazhdan–Lusztig polynomial $P_{e,w}(q)$. We also use Weyl group patterns to provide a tight upper bound on $h(w)$ in simply-laced types, resolving a conjecture of Billey–Postnikov from 2002.

Keywords: Kazhdan–Lusztig polynomial, permutation pattern, Bruhat order

1 Introduction

Let G be a complex semisimple Lie group, with Borel subgroup B containing maximal torus T and corresponding Weyl group W. The Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$ gives rise to the *Schubert varieties* $X_w := \overline{BwB/B}$ inside the flag variety G/B, whose containments determine the Bruhat order on W: $y \leq w$ if $X_y \subset X_w$. The *Kazhdan–Lusztig polynomials* $P_{y,w}(q) \in \mathbb{Z}[q]$ have since their discovery [14] proven to underlie deep connections between canonical bases of Hecke algebras, singularities of Schubert varieties, and representations of Lie algebras.

Theorem 1 (Kazhdan and Lusztig [15])**.** *For $y \leq w$, let $IH^*(X_w)_y$ denote the local intersection cohomology of X_w at the T-fixed point yB, then*

$$P_{y,w}(q) = \sum_i \dim(IH^{2i}(X_w)_y)q^i.$$

Theorem 1 implies that $P_{y,w}(q)$ has nonnegative coefficients, a property which is completely obscured by their recursive definition (Definition 7). It is known that for all $y \leq w$ one has $P_{y,w}(0) = 1$.

Theorem 2 (Deodhar [11]; Peterson (see [9]))**.** *If G is simply-laced and $y \leq w$, then X_w is smooth at yB if and only if $P_{y,w}(q) = 1$. In particular, X_w is smooth if and only if $P_{e,w}(q) = 1$.*

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In light of Theorem 1, one wants to understand $P_{y,w}(q)$ explicitly enough to determine which coefficients vanish. Indeed, the view of the $P_{y,w}$ as a measure of the failure of local Poincaré duality in X_w was among the original motivations in [14]. Unfortunately, $P_{y,w}$ may be arbitrarily complicated [18] and the formulae [8] which exist involve cancellation, and are thus not well-suited to this problem. If X_w is singular (as is typically true) one can at least ask for the smallest nontrivial coefficient, the first degree in which Poincaré duality fails. Writing $[q^i]P_{y,w}$ for the coefficient of q^i in $P_{y,w}(q)$, define:

$$h(w) := \min\{i > 0 \mid [q^i]P_{e,w} \neq 0\} = \min_{y \leq w} \min\{i > 0 \mid [q^i]P_{y,w} \neq 0\}.$$

The second equality follows from the monotonicity property of the $P_{y,w}$ [7]. We make the convention that $h(w) = +\infty$ when X_w is smooth.

Conjecture 3 (Billey and Postnikov [2])**.** *Let G be simply-laced of rank r, and suppose X_w is singular. Then $h(w) \leq r$.*

Billey and Postnikov's conjecture is somewhat surprising, since $\deg(P_{y,w})$ may be as large as $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ which is of the order of r^2, where ℓ denotes length. An upper bound on $h(w)$ in certain special infinite Coxeter groups was given in [19].

The decomposition $X_w = \bigsqcup_{y \leq w} ByB/B$ is an affine paving, with the cell ByB/B having complex dimension $\ell(y)$. We thus have

$$L(w) := \sum_{y \leq w} q^{\ell(y)} = \sum_{j \geq 0} \dim(H^j(X_w)) q^{j/2},$$

the Poincaré polynomial of X_w. Björner–Ekedahl [6] gave a precise interpretation of $h(w)$ in terms of $L(w)$, as the smallest homological degree in which Poincaré duality fails.

Theorem 4 (Björner and Ekedahl [6])**.** *For $0 \leq i \leq \ell(w)/2$ we have $[q^i]L(w) \leq [q^{\ell(w)-i}]L(w)$, and*

$$h(w) = \min\{i > 0 \mid [q^i]L(w) < [q^{\ell(w)-i}]L(w)\}.$$

Theorem 4 will be a useful tool in this work, but cannot be directly used to resolve Conjecture 3 since it is difficult to compute $[q^i]L(w)$ in general.

Our first main theorem[1] is a refinement and proof of Conjecture 3.

Theorem 5. *Let G be simply-laced of rank r, and suppose X_w is singular. Then $h(w) \leq r - 2$.*

The bound of $r - 2$ is tight when G is a member of the infinite families SL_{r+1} or SO_{2r}. When G is one of the exceptional simply-laced groups of type E_6, E_7, or E_8, Theorem 5 follows from the computations made by Billey–Postnikov [2]. In the case $G = \mathrm{SL}_{n+1}$, the theorem can be derived from the classification of the singular locus of X_w [5, 17]. However, in this case we provide a new exact formula for $h(w)$ for any permutation w. This theorem is phrased in terms of *pattern containment* (see Section 2.5.2).

[1] A full version of this work is available at [13]

Theorem 6. *Let $G = \mathrm{SL}_{n+1}$, and suppose X_w is singular. Then*

$$h(w) = \begin{cases} 1 & \text{if } w \text{ contains } 4231, \\ \mathrm{mHeight}(w) & \text{otherwise,} \end{cases}$$

where $\mathrm{mHeight}(w)$ *denotes the minimum height of a* 3412 *pattern in* w.

In the case $P_{e,w}(1) = 2$, Theorem 6 follows from the work of Woo [21]. Our theorem adds to the deep [22] and ubiquitous [1] links between singularities of Schubert varieties and pattern containment.

2 Preliminaries

2.1 Bruhat order and Kazhdan–Lusztig polynomials

Let W be a Weyl group with simple reflections $S = \{s_1, s_2, \ldots\}$ and length function ℓ. Write $\mathcal{R}$ for the set of reflections (conjugates of simple reflections), then *Bruhat order* $\leq$ on W is defined as the transitive closure of the relation $y < yr$ if $r \in \mathcal{R}$ and $\ell(y) < \ell(yr)$.

The left (respectively, right) *descents* $D_L(w)$ (resp. $D_R(w)$) are those $s \in S$ such that $sw < w$ (resp. $ws < w$).

Definition 7 (Kazhdan and Lusztig [14]). Define polynomials $R_{y,w}(q) \in \mathbb{Z}[q]$ by setting $R_{y,w}(q) = 0$ if $y \not\leq w$, $R_{y,w}(q) = 1$ if $y = w$, and requiring:

$$R_{y,w}(q) = \begin{cases} R_{ys,ws}(q), & \text{if } s \in D_R(y) \cap D_R(w), \text{ and} \\ qR_{ys,ws}(q) + (q-1)R_{y,ws}, & \text{if } s \in D_R(w) \setminus D_R(y). \end{cases}$$

Then there is a unique family of polynomials $P_{y,w}(q) \in \mathbb{Z}[q]$, the *Kazhdan–Lusztig polynomials* satisfying $P_{y,w}(q) = 0$ if $y \not\leq w$, $P_{w,w}(q) = 1$, and such that if $y < w$ then $P_{y,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ and

$$q^{\ell(w)-\ell(y)} P_{y,w}(q^{-1}) = \sum_{a \in [y,w]} R_{y,a}(q) P_{a,w}(q).$$

Although not apparent from Definition 7, the $P_{y,w}$ satisfy an inversion symmetry:

Proposition 8. *Let $y, w \in W$, then $P_{y,w}(q) = P_{y^{-1},w^{-1}}(q)$. In particular, $h(w) = h(w^{-1})$.*

2.2 Fiber bundles of Schubert varieties

For $J \subset S$, we write W_J for the subgroup generated by J, P_J for the parabolic subgroup of G generated by B and J, and W^J for the set of minimal length representatives of the left cosets W/W_J. We have $W^J = \{w \in W \mid D_R(w) \cap J = \emptyset\}$. Each $w \in W$ decomposes uniquely as $w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$. Using right cosets instead gives decompositions $w = {}_J w\, {}^J w$ with ${}_J w \in W_J$ and ${}^J w \in {}^J W = (W^J)^{-1}$. Notice that $(w^{-1})_J = ({}_J w)^{-1}$.

We write $w_0(J)$ for the unique element of W_J of maximum length and write $[u,v]^J$ for the set $[u,v] \cap W^J$. Since parabolic decompositions are unique, we have an injection $[e,w^J]^J \times [e,w_J] \hookrightarrow [e,w]$ given by multiplication.

Schubert varieties $X^J_{w^J} := \overline{Bw^J P_J/P_J}$ in the partial flag variety G/P_J have an affine paving by ByP_J/P_J for $y \in W^J$ and $y \leq w^J$, and so

$$L^J(w^J) := \sum_{\substack{y \in W^J \\ y \leq w^J}} q^{\ell(y)} = \sum_{j \geq 0} \dim(H^j(X^J_{w^J}))q^{j/2}.$$

Definition 9 (Richmond and Slofstra [20])**.** The parabolic decomposition $w = w^J w_J$ is called a *Billey–Postnikov decomposition* or *BP-decomposition* of w if $\mathrm{supp}(w^J) \cap J \subset D_L(w_J)$.

Theorem 10 (Richmond and Slofstra [20])**.** *The map $X_w \twoheadrightarrow X^J_{w^J}$ induced by the map $G/B \to G/P_J$ is a bundle projection if and only if J is a BP-decomposition of w, and in this case the fiber is isomorphic to X_{w_J}. Taking Poincaré polynomials, we have $L^J(w^J)L(w_J) = L(w)$ in this case.*

2.3 Patterns in Weyl groups

Let Φ denote the root system for G, with positive roots Φ^+ and simple roots Δ. For $w \in W$, the *inversion set* is $\mathrm{Inv}(w) := \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}$.

A subgroup W' of W generated by reflections is called a *reflection subgroup*, and is itself a Coxeter group with reflections $\mathcal{R}' = \mathcal{R} \cap W'$. We write $\leq'$ for the intrinsic Bruhat order on W', Φ' for the root system, and Inv' for inversion sets.

Proposition 11 (Billey and Braden [4]; Billey and Postnikov [2])**.** *Let $W' \subset W$ be a reflection subgroup, there is a unique function* fl $: W \to W'$, *the* flattening map *satisfying:*

(1) fl *is W'-equivariant, and*

(2) *if* $\mathrm{fl}(x) \leq' \mathrm{fl}(wx)$ *for some $w \in W'$, then $x \leq wx$.*

Furthermore, fl *has the following explicit description:* $\mathrm{fl}(w)$ *is the unique element $w' \in W'$ with* $\mathrm{Inv}'(w') = \mathrm{Inv}(w) \cap \Phi'$. *If $W' = W_J$ is a parabolic subgroup, then* $\mathrm{fl}(w) = w_J$.

Definition 12. We say that $w \in W$ *contains the pattern $w'' \in W''$*, if W has some reflection subgroup W', with an isomorphism $W' \xrightarrow{\varphi} W''$ as Coxeter systems, such that $\varphi(\mathrm{fl}(w)) = w''$. Otherwise, w is said to *avoid* w''.

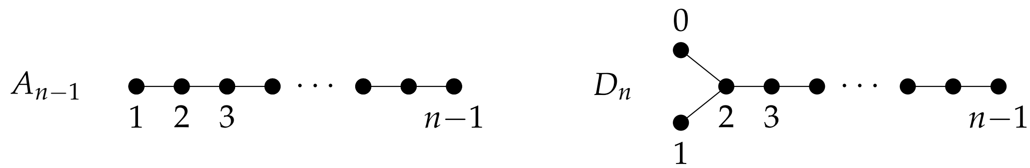

Figure 1: The Dynkin diagrams for Types A_{n-1} and D_n.

We will make use of the following result, which is proven using patterns.

Theorem 13 (Billey and Braden [4]). *Let $J \subset S$, then $h(w) \leq h(w_J)$.*

Billey and Postnikov gave the following characterization of smooth Schubert varieties, generalizing the work of Lakshmibai–Sandhya [16]. We write $W(Z)$ to denote the Weyl group of Type Z, where Z is one of the types in the Cartan–Killing classification.

Theorem 14 (Billey and Postnikov [2]). *Let G be simply-laced, then the Schubert variety $X_w \subset G/B$ is smooth if and only if w avoids the following patterns: $s_2s_1s_3s_2 \in W(A_3)$, $s_1s_2s_3s_2s_1 \in W(A_3)$, and $s_2s_0s_1s_3s_2 \in W(D_4)$.*

2.4 Conventions for simply-laced groups

2.4.1 $G = \mathrm{SL}_n$ (Type A_{n-1})

We let B be the set of lower triangular matrices in G, and $T \subset B$ the diagonal matrices in G. We have $\Phi(A_{n-1}) = \{e_j - e_i \mid 1 \leq i \neq j \leq n\}, \Phi^+(A_{n-1}) = \{e_j - e_i \mid 1 \leq i < j \leq n\}$, and $\Delta(A_{n-1}) = \{e_{i+1} - e_i \mid 1 \leq i \leq n-1\}$.

Under these conventions, the Weyl group $W(A_{n-1})$ acts on $\mathrm{Lie}_{\mathbb{R}}(T)^* = \mathbb{R}^n/(1,\ldots,1)$ by permutation of the coordinates, yielding an isomorphism $W(A_{n-1})$ with the symmetric group $\mathfrak{S}_n$. Letting $\alpha_i := e_{i+1} - e_i$, the corresponding simple reflection s_i is identified with the transposition $(i\, i+1)$. It will often be convenient for us to write permutations w in one-line notation as $w(1)\ldots w(n)$. The Dynkin diagram is shown in Figure 1.

2.4.2 $G = \mathrm{SO}_{2n}$ (Type D_n)

We let B be the set of lower triangular matrices in G, and $T \subset B$ the diagonal matrices in G. We have $\Phi(D_n) = \{e_j \pm e_i \mid 1 \leq i \neq j \leq n\}, \Phi^+(D_n) = \{e_j \pm e_i \mid 1 \leq i < j \leq n\}$, and $\Delta(D_n) = \{e_2 + e_1\} \cup \{e_{i+1} - e_i \mid 1 \leq i \leq n-1\}$.

Under these conventions, the Weyl group $W(D_n)$ acts on $\mathrm{Lie}_{\mathbb{R}}(T)^* = \mathbb{R}^n$ by permuting coordinates and negating pairs of coordinates. This identifies $W(D_n)$ with the permutations w of $\{-n,\ldots,-1,1,\ldots,n\}$ satisfying $w(i) = -w(-i)$ for all i, and such that $|\{w(1),\ldots,w(n)\} \cap \{-n,\ldots,-1\}|$ is even. We write $\mathfrak{D}_n$ for this realization of $W(D_n)$. Such a permutation can be uniquely specified by its *window notation* $[w(1)\ldots w(n)]$.

Write $\delta_0 = e_2 + e_1$ and $\delta_i = e_{i+1} - e_i$, $i = 1, 2, \ldots, n-1$ for the simple roots. It will often be convenient for us to write $\bar{i}$ for $-i$, and we use these interchangeably. We also make the convention that $e_{\bar{i}} = e_{-i} := -e_i$ for $i > 0$. We have simple reflections $s_0 = (1\,\bar{2})(\bar{1}\,2)$ and $s_i = (i\,i+1)(\bar{i}\,\overline{i+1})$ for $i = 1, \ldots, n-1$.

2.5 Reflection subgroups and diagram automorphisms

See Figure 1 for our labeling of the Dynkin diagrams. The following is clear:

Proposition 15. *The diagram of the Type A_{n-1} has an automorphism ε_A sending $\alpha_i \mapsto \alpha_{n-i}$ for $i = 1, \ldots, n-1$, and the diagram of Type D_n has an automorphism ε_D interchanging $\delta_0 \leftrightarrow \delta_1$. If $\varepsilon \in \{\varepsilon_A, \varepsilon_D\}$, then $h(w) = h(\varepsilon_D(w))$.*

2.5.1 Reflection subgroups

In light of Theorem 14, we will be concerned with reflection subgroups isomorphic to $W(A_3)$ and $W(D_4)$ inside $W(A_{n-1})$ and $W(D_n)$.

Proposition 16. *Reflection subgroups isomorphic to $W(A_3)$ and $W(D_4)$ inside $W(A_{n-1})$ and $W(D_n)$ are characterized as follows:*

- *(a) No reflection subgroup $W' \subset W(A_{n-1})$ is isomorphic to $W(D_4)$.*
- *(b) Reflection subgroups $W' \cong W(A_3)$ inside $W(A_{n-1})$ are conjugate to the parabolic subgroup $W(A_{n-1})_{\{1,2,3\}}$.*
- *(c) Reflection subgroups $W' \cong W(D_4)$ inside $W(D_n)$ are conjugate to the parabolic subgroup $W(D_n)_{\{0,1,2,3\}}$.*
- *(d) Reflection subgroups $W' \cong W(A_3)$ inside $W(D_n)$ come in two classes: those related to $W(D_n)_{\{1,2,3\}}$ by conjugacy and ε_D (Class I), and those conjugate to $W(D_n)_{\{0,1,2\}}$ (Class II).*

2.5.2 One line notation and patterns

We will be interested in occurrences of the patterns from Theorem 14 in elements $w \in W(A_{n-1})$ or $W(D_n)$. For $w \in W(D_n)$, it will sometimes be useful for us to distinguish between Class I and II patterns (see Proposition 16(d)). Realizing these Weyl groups as $\mathfrak{S}_n$ and $\mathfrak{D}_n$, respectively, allows for one-line interpretations of pattern containment (summarized in Figure 2). This approach to pattern containment is in some sense a hybrid between the approaches of Billey [3] using signed patterns and of Billey, Braden, and Postnikov [2, 4] using patterns in the sense of Definition 12. Our distinction between Class I and II patterns is seemingly novel and reflects the disparate effects that occurrences of these patterns can have on $h(w)$.

Type	Class	Pattern	One-line
A_3	I	$s_2s_1s_3s_2$	3412
A_3	II	$s_2s_1s_3s_2$	$\pm 12\bar{3}$
A_3	I	$s_1s_2s_3s_2s_1$	4231
A_3	II	$s_1s_2s_3s_2s_1$	$\pm 1\bar{3}\bar{2}$
D_4		$s_2s_0s_1s_3s_2$	$\pm 14\bar{3}2$

Figure 2: The patterns from Theorem 14 with their one-line notations, divided according to type and class.

Definition 17.

(i) For p a signed permutation of $[k]$, we say $w \in \mathfrak{D}_n$ contains p at positions $1 \le i_1 < \cdots < i_k \le n$ if $\mathrm{sign}(w(i_j)) = \mathrm{sign}(p(j))$ for $j = 1, \ldots, k$ and $|w(i_1)|, \ldots, |w(i_k)|$ are in the same relative order as $|p(1)|, \ldots, |p(k)|$.

(ii) For $p \in \mathfrak{S}_k$, we say $w \in \mathfrak{S}_n$ contains p at positions $1 \le i_1 < \cdots < i_k \le n$ if $w(i_1), \ldots, w(i_k)$ have the same relative order as $p(1), \ldots, p(k)$. We say $u \in \mathfrak{D}_n$ contains p at positions $i_1 < \cdots < i_k$, where each $i_j \in \pm[n]$ if $u(i_1), \ldots, u(i_k)$ have the same relative order as $p(1), \ldots, p(k)$ and $|i_1|, \ldots, |i_k|$ are distinct.

In each case, we say that the *values* of the occurrence are $w(i_1), \ldots, w(i_k)$.

The following is a translation of Theorem 14 in light of our conventions for patterns.

Proposition 18. *Let G be simply-laced; then $X_w \subset G/B$ is smooth if and only if w avoids the patterns* 3412, $\pm 12\bar{3}$, 4231, $\pm 1\bar{3}\bar{2}$, *and* $\pm 14\bar{3}2$ *(see Figure 2).*

The following statistic on occurrences of the pattern 3412 will be of special importance for us (see Theorem 6).

Definition 19 (See [10, 21])**.** We say an occurrence of 3412 in $w \in \mathfrak{S}_n$ or $\mathfrak{D}_n$ at positions $a < b < c < d$ has *height* equal to $w(a) - w(d)$. We let $\mathrm{mHeight}(w)$ denote the minimum height over all occurrences of 3412 in w.

3 Upper bounds on $h(w)$

3.1 Proof strategy

We will identify certain patterns p (among those from Proposition 18) such that if w contains p, then $h(w)$ can be computed using Theorem 4 and an analysis of the Bruhat covers of w. Then, for w avoiding these patterns and containing others, we will—by a combination of parabolic reduction (Theorem 13), inversion (Proposition 8), and diagram automorphisms (Proposition 15)—obtain a bound $h(w) \leq h(u)$ for u in some special family $\mathcal{S}$. Finally, we will show that elements $u \in \mathcal{S}$ have distinguished BP-decompositions such that the base and fiber in the bundle (Theorem 10) with total space X_u can be understood, allowing for the computation of $h(u)$. In the remainder, we refer primarily to the elements $w \in W$ rather than the Schubert varieties X_w that they index, although each of these steps has a geometric basis. We say w is *smooth* (resp. *singular*) if X_w is smooth (resp. singular).

We only have space to give a few representative proofs and proof ideas in this extended abstract.

Proposition 20. *Let $w \in \mathfrak{S}_n$ or $\mathfrak{D}_n$; we have $h(w) = 1$ if w contains:*

(i) 4231 *and $w \in \mathfrak{S}_n$,*

(ii) $\pm 12\bar{3}$,

(iii) $\pm 14\bar{3}2$, *or*

(iv) 3412 *of height one.*

Proof idea. The strategies for all cases are similar: containment of any of these patterns implies a relation $\tau_1 + \tau_2 = \tau_3 + \tau_4$ for $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathrm{Inv}(w)$. We show that this implies a relation between roots indexing lower Bruhat covers of w. By results of Dyer [12], this implies that $[q^{\ell(w)-1}]L(w) > [q]L(w)$, so that $h(w) = 1$ by Theorem 4. $\square$

3.2 Proof of Theorem 5 in Type A

In this section we obtain an upper bound on $h(w)$ for $w \in \mathfrak{S}_n$ in terms of $\mathrm{mHeight}(w)$; this establishes Theorem 5 for $W = \mathfrak{S}_n$ as well as one direction of Theorem 6.

Lemma 21. *For $n \geq 4$, consider $w \in \mathfrak{S}_n$ where $w(1) = n-1$, $w(2) = n$, $w(n-1) = 1$, $w(n) = 2$ and $w(i) = n-i+1$ for $3 \leq i \leq n-2$. Then $h(w) = n-3$.*

Proof. Let $J = \{2, 3, \ldots, n-2\}$ so that $w_J = w_0(J)$. The parabolic decomposition $w = w^J w_J$ is a Billey–Postnikov decomposition. Moreover, $L(w_J) = L(w_0(J))$ is palindromic, since $X_{w_0(J)}$ is a product of flag varieties and therefore smooth. Every $u \in W^J$ satisfies $u(2) < u(3) < \cdots < u(n-1)$ so by counting inversions with $u(1)$ and $u(n)$, we see $\ell(u) = (u(1)-1) + (n - u(n)) - \mathbf{1}_{u(1)>u(n)}$. Elements $u \in [e, w^J]^J$ are characterized by $u(1) \leq n-1$ and $u(n) \geq 2$ with $u(2) < \cdots < u(n-1)$. We are now able to count the rank sizes of $[e, w^J]^J$ to be $1, 2, 3, \ldots, n-4, n-3, n-2, n-1, n-3, n-4, \ldots, 2, 1$. Thus, $h(L^J(w^J)) = n-3$ and $h(w) = \min(h(L^J(w^J)), h(L(w_J))) = \min(n-3, \infty) = n-3$. $\square$

For an occurrence of a 3412 in w at indices $a < b < c < d$ with $w(c) < w(d) < w(a) < w(b)$ its *content* is $1 + |\{i \mid b < i < c, w(d) < w(i) < w(a)\}|$. Let $\mathrm{mCont}(w)$ be the minimum content of a 3412 pattern in w.

Lemma 22. *For $w \in \mathfrak{S}_n$ that contains* 3412*,* $\mathrm{mHeight}(w) = \mathrm{mCont}(w)$.

One advantage of working with content instead of height is that we evidently have $\mathrm{mCont}(w) = \mathrm{mCont}(w^{-1})$.

Lemma 23. *Suppose that $w \in \mathfrak{S}_n$ avoids* 4231 *and contains* 3412*. Then $h(w) \leq \mathrm{mHeight}(w)$.*

Proof. We use induction on n. The statement is true when $n = 4$, where $h(3412) = \mathrm{mHeight}(3412) = 1$. For a general n and $w \in \mathfrak{S}_n$, let $k = \mathrm{mHeight}(w) = \mathrm{mCont}(w)$. For $J = \{2, 3, \ldots, n-1\}$, if w_J has $\mathrm{mCont}(w_J) = k$, then we are done by the induction hypothesis and Theorem 13 which says $h(w) \leq h(w_J) \leq \mathrm{mCont}(w_J) = k$. We can thus assume without loss of generality that the index 1 appears in all 3412's of w with content k. Similarly, by considering $J = \{1, 2, \ldots, n-2\}$, we can also assume that the index n appears in all 3412's of w with content k. As $h(w) = h(w^{-1})$, with the same argument on w^{-1}, we can reduce to the case that w contains a unique 3412 of content k on indices $1 < w^{-1}(n) < w^{-1}(1) < n$ (see Figure 3). As we assume that w_J does not contain a

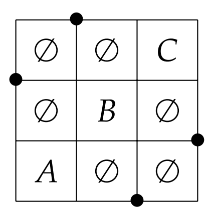

Figure 3: The permutation diagram for w with an occurrence of 3412 on the boundary. We draw permutation diagrams by putting $\bullet$'s at Cartesian coordinates $(i, w(i))$.

3412 of content k, there does not exist i such that $1 < i < w^{-1}(n)$ with $w(i) > w(n)$. By symmetry, we know six of the regions in Figure 3 are empty as shown, and label the other three regions as A, B, C. By definition, $|B| = k - 1$. If $k = 1$, then $h(w) = 1$ by Proposition 20. If $k > 1$, B is not empty; since w avoids 4231, A and C must be empty. Thus w is exactly the permutation in Lemma 21, which gives $h(w) = n - 3 = k$. $\square$

3.3 Extension to Type D

Proposition 24. *If $w \in \mathfrak{D}_n$ contains* 4231*, then $h(w) \leq 2$.*

Proof idea. We adapt the strategy for Proposition 20 to show that for most occurrences of 4231, we in fact have $h(w) = 1$. The few remaining cases are analyzed separately. $\square$

Definition 25. Define the *magnitude* $\mathrm{mag}(w)$ as the smallest $b > 0$ such that w has an occurrence of $\pm 1\bar{3}\bar{2}$ with values $a\bar{c}\bar{b}$.

Proposition 26. *Suppose $w \in \mathfrak{D}_n$ contains $\pm 1\bar{3}\bar{2}$ and avoids 4231, then $h(w) \leq \mathrm{mag}(w) - 1$.*

Proposition 27. *Let $W = \mathfrak{D}_n$ for $n \geq 5$, let $J = S \setminus \{1\}, J' = S \setminus \{0\}, K = S \setminus \{n-1\}$, and suppose $w \in \mathfrak{D}_n$ is singular, but satisfies:*

(i) w avoids 4231, $\pm 1\bar{3}\bar{2}, \pm 12\bar{3}, \pm 14\bar{3}2$, and neither w nor $\varepsilon_D(w)$ contains any occurrences of 3412 of height one,

(ii) $w_J, w_{J'}, w_K, {}_Jw, {}_{J'}w, {}_Kw$ are smooth.

Then $w = u := [n, 2, \bar{3}, \bar{4}, \ldots, \overline{n-1}, \pm 1]$ or $w = \varepsilon_D(u)$.

We are now ready to complete the proof of Theorem 5, resolving Conjecture 3.

Proof of Theorem 5. First suppose $G = \mathrm{SL}_{r+1}$, and let $w \in W(A_r) = \mathfrak{S}_{r+1}$ such that X_w is singular. By Theorem 14, w contains 4231 or 3412. If w contains 4231, then $h(w) = 1$ by Proposition 20. Otherwise w avoids 4231 and contains 3412, so $h(w) \leq \mathrm{mHeight}(w)$ by Lemma 23. It is clear by definition that $\mathrm{mHeight}(w) \leq r - 2$ for any w, so we are done.

Now suppose $G = \mathrm{SO}_{2r}$ for $r \geq 5$, and let $w \in W(D_r) = \mathfrak{D}_r$. Suppose by induction that the claim is true for $G = \mathrm{SO}_{2r'}$ for $r' < r$ (the base case $r' = 4$ is covered by the computations in [2]). If w contains 4231, then $h(w) \leq 2 \leq r - 2$ by Proposition 24, so we may assume that w avoids 4231. Then by Proposition 26, if w contains $\pm 1\bar{3}\bar{2}$ we have $h(w) \leq \mathrm{mag}(w) \leq r - 2$. If w contains any of the patterns from Proposition 20, then $h(w) = 1 \leq r - 2$. Let $J = S \setminus \{2\}, J' = S \setminus \{0\}, K = S \setminus \{r-1\}$; if any of $w_J, w_{J'}, w_K, {}_Jw, {}_{J'}w, {}_Kw$ is singular, then by the type A result, or by the induction hypothesis, we have $h(w) \leq r - 3$. Finally, if w does not fall into any of the above cases, then w satisfies the hypotheses (i) and (ii) of Proposition 27, so $w = u := [r, 2, \bar{3}, \bar{4}, \ldots, \overline{r-1}, \pm 1]$ or $w = \varepsilon_D(u)$.

We will now compute $h(u) = h(\varepsilon_D(u))$; suppose for convenience that r is even, the other case being exactly analogous. Let $I = \{1, 2 \ldots, r-2\}$, then we have $u_I = w_0(I)$ is the longest element of $\mathfrak{S}_{r-1}$, so $h(u_I) = \infty$. Thus we need to compute $h(L^I(u^I))$ with $u^I = [\overline{r-1}, \ldots, \bar{4}, \bar{3}, 2, r, \bar{1}]$. Notice $\ell(u^I) = N := \frac{1}{2}(r^2 - 3r + 4)$ with reduced word:

$$s_0(s_2s_0)(s_3s_2s_1) \cdots (s_{r-4}s_{r-5} \cdots s_3s_2s_0)(s_{r-3} \cdots s_3s_2s_1)(s_{r-2} \cdots s_3s_2s_0)s_{r-1}.$$

We claim that $L^I(u^I) = 1 + 2q + 3q^2 + \cdots + aq^{N-2} + 2q^{N-1} + q^N$, with $a \geq 4$, so that $h(u) = h(L^I(u^I)) = 2 < r - 2$. Indeed, the elements of length one in $[e, u^I]^I$ are $\{s_0, s_{r-1}\}$, the elements of length two are $\{s_0s_{r-1}, s_2s_0, s_{r-2}s_{r-1}\}$, and the elements of length $N - 1$ are $\{s_0u^I, s_2u^I\}$. Consider the four elements $z_1 = s_0s_2u^I, z_2 = s_2s_0u^I, z_3 = s_0u^Is_{r-1}, z_4 = s_3s_2u^I$. It is easy to check for $i = 1, 2, 3, 4$ that $\ell(z_i) = N - 2$, that $z_i \leq u^I$, and that $z_i \in W^I$; thus $a \geq 4$ as desired. $\square$

4 Exact formula when $G = \mathrm{SL}_n$

For $w \in \mathfrak{S}_n$, we have proved the upper bound in Theorem 6 in Section 3.2. The lower bound follows from Lemma 28 below.

Lemma 28. *Suppose that $w \in \mathfrak{S}_n$ avoids* 4231 *and contains* 3412. *Then* $h(w) \geq \mathrm{mHeight}(w)$.

Proof idea. This is an inductive argument using a diagram analysis, analogous to but more involved than the proof of Lemma 23. The relevant diagram is shown in Figure 4.

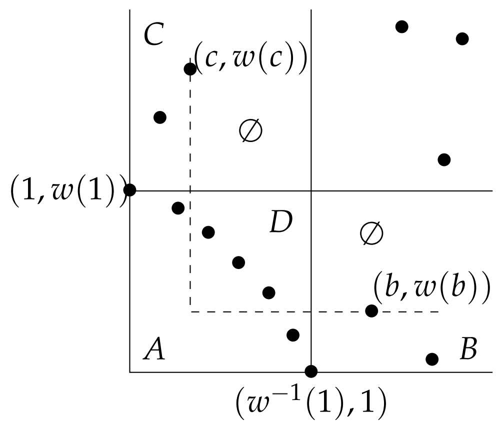

Figure 4: The permutation diagram of w used in the proof of Lemma 28.

□

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Rowmotion Markov Chains

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Abstract. *Rowmotion* is a certain well-studied bijective operator on the distributive lattice $J(P)$ of order ideals of a finite poset P. We introduce the *rowmotion Markov chain* $\mathbf{M}_{J(P)}$ by assigning a probability p_x to each $x \in P$ and using these probabilities to insert randomness into the original definition of rowmotion. More generally, we introduce a very broad family of *toggle Markov chains* inspired by Striker's notion of generalized toggling. We characterize when toggle Markov chains are irreducible, and we show that each toggle Markov chain has a remarkably simple stationary distribution.

We also provide a second generalization of rowmotion Markov chains to the context of semidistrim lattices. Given a semidistrim lattice L, we assign a probability p_j to each join-irreducible element j of L and use these probabilities to construct a rowmotion Markov chain $\mathbf{M}_L$. Under the assumption that each probability p_j is strictly between 0 and 1, we prove that $\mathbf{M}_L$ is irreducible. We also compute the stationary distribution of the rowmotion Markov chain of a lattice obtained by adding a minimal element and a maximal element to a disjoint union of two chains.

We bound the mixing time of $\mathbf{M}_L$ for an arbitrary semidistrim lattice L. In the special case when L is a Boolean lattice, we use spectral methods to obtain much stronger estimates on the mixing time, showing that rowmotion Markov chains of Boolean lattices exhibit the cutoff phenomenon.

Keywords: Toggle, rowmotion, Markov chain, stationary distribution, mixing time, lattice

1 Introduction

Let P be a finite poset, and let $J(P)$ denote the set of order ideals of P. For $S \subseteq P$, let

$$\Delta(S) = \{x \in P : x \le s \text{ for some } s \in S\} \quad \text{and} \quad \nabla(S) = \{x \in P : x \ge s \text{ for some } s \in S\},$$

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and let $\min(S)$ and $\max(S)$ denote the set of minimal elements and the set of maximal elements of S, respectively. *Rowmotion*, a well-studied operator in the growing field of dynamical algebraic combinatorics, is the bijection $\mathrm{Row}\colon J(P) \to J(P)$ defined by[1]

$$\mathrm{Row}(I) = P \setminus \nabla(\max(I)). \tag{1.1}$$

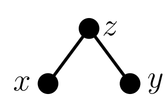
We refer the reader to [16, 17] for the history of rowmotion. The purpose of this extended abstract of the article [4] is to introduce randomness into the ongoing saga of rowmotion by defining certain Markov chains. We were inspired by the articles [1, 11, 14]; these articles define Markov chains based on the *promotion* operator, which is closely related to rowmotion in special cases [16] (though our Markov chains are fundamentally different from these promotion-based Markov chains).

For each $x \in P$, fix a probability $p_x \in [0,1]$. We define the *rowmotion Markov chain* $\mathbf{M}_{J(P)}$ with state space $J(P)$ as follows. Starting from a state $I \in J(P)$, select a random subset S of $\max(I)$ by adding each element $x \in \max(I)$ into S with probability p_x; then transition to the new state $P \setminus \nabla(S) = \mathrm{Row}(\Delta(S))$. Thus, for any $I, I' \in J(P)$, the transition probability from I to I' is

$$\mathbb{P}(I \to I') = \begin{cases} \left(\prod_{x \in \min(P \setminus I')} p_x\right)\left(\prod_{x' \in \max(I) \setminus \min(P \setminus I')} (1 - p_{x'})\right) & \text{if } \min(P \setminus I') \subseteq \max(I); \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if $p_x = 1$ for all $x \in P$, then $\mathbf{M}_{J(P)}$ is deterministic and agrees with the rowmotion operator. On the other hand, if $p_x = 0$ for all $x \in P$, then $\mathbf{M}_{J(P)}$ is deterministic and sends every order ideal of P to the order ideal P.

Example 1. Suppose P is the poset



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whose elements x, y, z are as indicated. Then $J(P)$ forms a distributive lattice with 5 elements. The transition diagram of $\mathbf{M}_{J(P)}$ is drawn over the Hasse diagram of $J(P)$ in Figure 1.

Suppose each probability p_x is strictly between 0 and 1. One of our main results will imply that $\mathbf{M}_{J(P)}$ is irreducible and that the probability of the state I in the stationary distribution of $\mathbf{M}_{J(P)}$ is

$$\frac{1}{Z(J(P))} \prod_{x \in I} p_x^{-1}, \tag{1.2}$$

[1] Many authors define rowmotion to be the inverse of the operator that we have defined. Our definition agrees with the conventions used in [2, 6, 17].

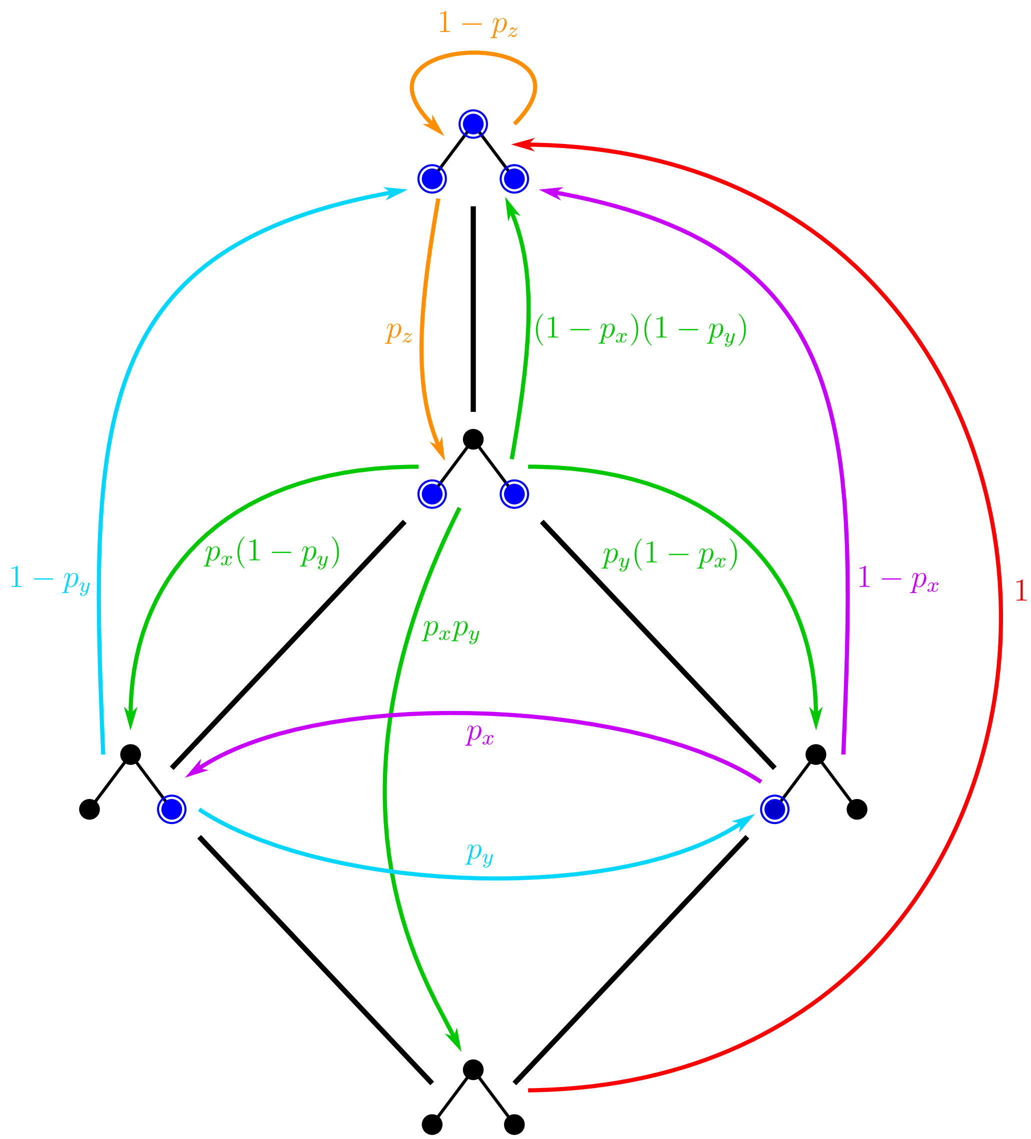


Figure 1: The transition diagram of $\mathbf{M}_{J(P)}$, where P is the 3-element poset from Example 1. The elements of each order ideal in $J(P)$ are circled and colored blue.

where $Z(J(P)) = \sum_{I' \in J(P)} \prod_{x' \in I'} p_{x'}^{-1}$.

It is surprising that there is such a clean formula for the stationary distribution in this level of generality. We will deduce this result from a more general result about a vastly broader family of Markov chains.

2 Toggle Markov Chains

Let P be a finite set of size n, and let $\mathcal{K}$ be a collection of subsets of P. For each $x \in P$, define the *toggle operator* $\tau_x \colon \mathcal{K} \to \mathcal{K}$ by

$$\tau_x(A) = \begin{cases} A \triangle \{x\} & \text{if } A \triangle \{x\} \in \mathcal{K} \\ A & \text{otherwise,} \end{cases}$$

where $\triangle$ denotes symmetric difference. Note that τ_x is an involution. Fix a tuple $\mathbf{x} = (x_1, \dots, x_n)$ that contains each element of P exactly once. In other words, $\mathbf{x}$ is an ordering of the elements of P. Given a set $Y \subseteq P$, let $\tau_Y = \tau_{y_r} \circ \cdots \circ \tau_{y_1}$, where $y_1, \dots, y_r$ is the list of elements of Y in the order that they appear within the list $x_1, \dots, x_n$.

Striker [15] viewed the map $\tau_P \colon \mathcal{K} \to \mathcal{K}$ as a generalization of rowmotion. Indeed, if P is a poset, $\mathbf{x} = (x_1, \dots, x_n)$ is a linear extension of P (meaning $i < j$ whenever $x_i < x_j$ in P), and $\mathcal{K} = J(P)$, then τ_P is equal to rowmotion. The recent article [7] studies the dynamical aspects of τ_P when P is a poset, $\mathbf{x}$ is a linear extension of P, and $\mathcal{K}$ is the collection of *interval-closed* (also called *convex*) subsets of P. The articles [3, 9, 10] consider τ_P when P is the vertex set of a particular graph, $\mathbf{x}$ is a special ordering of the vertices, and $\mathcal{K}$ is the collection of independent sets of the graph.

For each $x \in P$, fix a probability p_x. Define the *toggle Markov chain* $\mathbf{T} = \mathbf{T}(\mathcal{K}, \mathbf{x})$ as follows. The state space of $\mathbf{T}$ is $\mathcal{K}$. Suppose the Markov chain is in a state $A \in \mathcal{K}$. Choose a subset $T \subseteq A$ randomly so that each element $x \in A$ is included in T with probability p_x, and then transition from A to the new state $\tau_T(A)$.

To phrase this differently, define the *random toggle* $\widetilde{\tau}_x$ to be the stochastic operator that acts as follows on a set $A \in \mathcal{K}$. Let X be a Bernoulli random variable that takes the value 1 with probability p_x, and let

$$\widetilde{\tau}_x(A) = \begin{cases} \tau_x(A) & \text{if } x \notin A \text{ or } X = 1; \\ A & \text{if } x \in A \text{ and } X = 0. \end{cases}$$

Then the Markov chain transitions to the state obtained from A by applying the random toggles $\widetilde{\tau}_{x_1}, \dots, \widetilde{\tau}_{x_n}$ in this order. (Each time we apply a random toggle, we use a new Bernoulli random variable that is independent of those used before.)

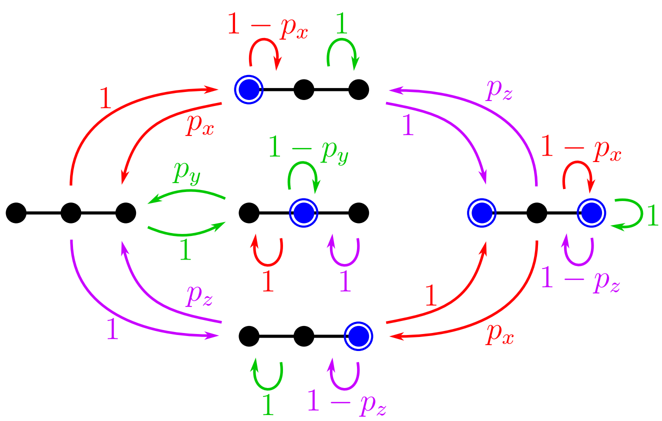
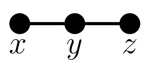


Figure 2: As in Example 2, we consider random toggles, where $\mathcal{K}$ is the collection of independent sets of a path graph with vertices x, y, z (from left to right). The elements of each independent set are circled and colored blue. To apply the random toggle $\widetilde{\tau}_x$ to an independent set A, we follow one of the red arrows starting at A; the probability that a particular arrow is used is written next to the arrow. Similarly, we follow a green arrow when we apply $\widetilde{\tau}_y$, and we follow a purple arrow when we apply $\widetilde{\tau}_z$.

Example 2. Suppose G is the graph  , whose vertices x, y, z are as indicated. Let $\mathcal{K}$ be the collection of independent sets of G. Figure 2 depicts the random toggles $\widetilde{\tau}_x, \widetilde{\tau}_y, \widetilde{\tau}_z$. If we let $\mathbf{x} = (x, y, z)$, then a transition of $\mathbf{T}(\mathcal{K}, \mathbf{x})$ consists of applying these random toggles in the order $\widetilde{\tau}_x, \widetilde{\tau}_y, \widetilde{\tau}_z$.

Given a set P, let $\mathcal{H}^P$ be the hypercube graph with vertex set 2^P (the power set of P) such that two sets $A, A' \subseteq P$ are adjacent if and only if $|A \triangle A'| = 1$. For $S \subseteq 2^P$, let $\mathcal{H}^P|_S$ be the induced subgraph of $\mathcal{H}^P$ with vertex set S.

Let us now state our main results about irreducibility and stationary distributions of toggle Markov chains. As before, we fix a finite set P, a collection $\mathcal{K}$ of subsets of P, an ordering $\mathbf{x}$ of the elements of P, and a probability p_x for each $x \in P$.

Theorem 1 ([4])**.** *Suppose $0 < p_x < 1$ for each $x \in P$. The toggle Markov chain $\mathbf{T}(\mathcal{K}, \mathbf{x})$ is irreducible if and only if the graph $\mathcal{H}^P|_{\mathcal{K}}$ is connected.*

If P is a finite poset and $\mathbf{x}$ is a linear extension of P, then one can show that $\mathbf{T}(\mathcal{J}(P), \mathbf{x})$ coincides with the rowmotion Markov chain $\mathbf{M}_{J(P)}$. In this case, every connected component of $\mathcal{H}^P|_{J(P)}$ contains the empty set as a vertex. Thus, it is immediate from Theorem 1 that the rowmotion Markov chain $\mathbf{M}_{J(P)}$ is irreducible whenever $0 < p_x < 1$ for every $x \in P$.

Theorem 2 ([4]). *Suppose the toggle Markov chain $\mathbf{T}(\mathcal{K}, \mathbf{x})$ is irreducible and $p_x > 0$ for every $x \in P$. For $A \in \mathcal{K}$, the probability of the state A in the stationary distribution of $\mathbf{T}(\mathcal{K}, \mathbf{x})$ is*

$$\frac{1}{Z(\mathcal{K})} \prod_{x \in A} p_x^{-1},$$

where $Z(\mathcal{K}) = \sum_{A' \in \mathcal{K}} \prod_{x' \in A'} p_{x'}^{-1}$.

Note that the stationary distribution in Theorem 2 is independent of the ordering $\mathbf{x}$ (though the Markov chain itself can certainly depend on $\mathbf{x}$).

3 Mixing Times

Suppose $\mathbf{M}$ is an irreducible finite Markov chain with state space Ω, transition matrix Q, and stationary distribution π. For $x \in \Omega$, let $Q^i(x, \cdot)$ denote the distribution on Ω in which the probability of a state x' is the probability of reaching x' by starting at x and applying i transitions (this probability is the entry in Q^i in the row indexed by x and the column indexed by x'). The *total variation distance* $d_{\mathrm{TV}} = d_{\mathrm{TV}}^{\Omega}$ is the metric on the space of distributions on Ω defined by

$$d_{\mathrm{TV}}(\mu, \nu) = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

For $\varepsilon > 0$, the *mixing time* of $\mathbf{M}$, denoted $t_{\mathbf{M}}^{\mathrm{mix}}(\varepsilon)$, is the smallest nonnegative integer i such that $d_{\mathrm{TV}}(Q^i(x, \cdot), \pi) < \varepsilon$ for all $x \in \Omega$.

The *width* of a finite poset P, denoted $\mathrm{width}(P)$, is the maximum size of an antichain in P. In [4], we use the method of coupling to prove the following bound on the mixing time of an arbitrary rowmotion Markov chain.

Theorem 3 ([4]). *Let P be a finite poset, and fix a probability $p_x \in (0, 1)$ for each $x \in P$. Let $\overline{p} = \max_{x \in P} p_x$. For each $\varepsilon > 0$, the mixing time of $\mathbf{M}_{J(P)}$ satisfies*

$$t_{\mathbf{M}_{J(P)}}^{\mathrm{mix}}(\varepsilon) \leq \left\lceil \frac{\log \varepsilon}{\log\left(1 - (1 - \overline{p})^{\mathrm{width}(P)}\right)} \right\rceil.$$

We can drastically improve the bound in Theorem 3 when P is an antichain (so $J(P)$ is a Boolean lattice). For simplicity, we assume that all probabilities p_x are equal to a single value p. In this setting, the Markov chain is reversible with respect to π; this allows us to give a spectral proof of the following result, which is an instance of the well-studied *cutoff phenomenon*.

Theorem 4 ([4]). *Let P be an n-element antichain, and fix a probability $p \in (0,1)$. Let $p_x = p$ for all $x \in P$. Let Q and π be the transition matrix and stationary distribution, respectively, of the Markov chain $\mathbf{M}_{J(P)}$.*

1. *For $c > \frac{1}{2}$ and $t = \frac{1}{2}\log_{1/p} n + c$, we have*

$$\max_{x \in J(P)} d_{\mathrm{TV}}(Q^t(x,\cdot),\pi) \leq \frac{1}{2}\left(e^{p^{2c-1}} - 1\right)^{1/2}.$$

2. *For $0 < c < \frac{1}{2}\log_{1/p} n$ and $t = \frac{1}{2}\log_{1/p} n - c$, we have*

$$\max_{x \in J(P)} d_{\mathrm{TV}}(Q^t(x,\cdot),\pi) \geq 1 - 4p^{2c+1} - 4p^{2c}.$$

It would be interesting to prove that other natural families of toggle Markov chains exhibit cutoff.

4 Semidistrim Lattices

If P is a finite poset, then we can order $J(P)$ by inclusion to obtain a distributive lattice. In fact, Birkhoff's Fundamental Theorem of Finite Distributive Lattices states that every finite distributive lattice is isomorphic to the lattice of order ideals of some finite poset. Thus, instead of viewing rowmotion as a bijective operator on the set of order ideals of a finite poset, one can equivalently view it as a bijective operator on the set of *elements* of a distributive lattice. This perspective has led to more general definitions of rowmotion in recent years. Barnard [2] showed how to extend the definition of rowmotion to the broader family of *semidistributive* lattices, while Thomas and Williams [17] discussed how to extend the definition to the family of *trim* lattices. (Every distributive lattice is semidistributive and trim, but there are semidistributive lattices that are not trim and trim lattices that are not semidistributive.)

One notable example motivating these extended definitions comes from Reading's *Cambrian lattices* [12]. Suppose c is a Coxeter element of a finite Coxeter group W. Reading [13] found a bijection from the c-Cambrian lattice to the c-noncrossing partition lattice of W; under this bijection, rowmotion on the c-Cambrian lattice corresponds to the well-studied *Kreweras complementation* operator on the c-noncrossing partition lattice of W [2, 17]. See [5, 8, 17] for other non-distributive lattices where rowmotion has been studied.

Recently, the first author and Williams [6] introduced the even broader family of *semidistrim* lattices and showed how to define a natural rowmotion operator on them; this is now the broadest family of lattices where rowmotion has been defined. It turns out that we can extend our definition of rowmotion Markov chains to semidistrim lattices; this provides a generalization of rowmotion Markov chains that is different from the toggle Markov chains discussed in Section 2. We sketch the details here, referring to [4] for the full definition of a semidistrim lattice and an explanation of why this definition specializes to the one given above when the lattice is distributive.

Let L be a semidistrim lattice, and let $\mathcal{J}_L$ and $\mathcal{M}_L$ be the set of join-irreducible elements of L and the set of meet-irreducible elements of L, respectively. There is a specific bijection $\kappa_L\colon \mathcal{J}_L \to \mathcal{M}_L$ satisfying certain properties. The *Galois graph* of L is the loopless directed graph G_L with vertex set $\mathcal{J}_L$ such that for all distinct $j, j' \in \mathcal{J}_L$, there is an arrow $j \to j'$ if and only if $j \not\leq \kappa_L(j')$. Let $\mathrm{Ind}(G_L)$ be the set of independent sets of G_L. There is a particular way to label the edges of the Hasse diagram of L with elements of $\mathcal{J}_L$; we write j_{uv} for the label of the edge $u \lessdot v$. For $w \in L$, let $\mathcal{D}_L(w)$ be the set of labels of the edges of the form $u \lessdot w$, and let $\mathcal{U}_L(w)$ be the set of labels of the edges of the form $w \lessdot v$. Then $\mathcal{D}_L(w)$ and $\mathcal{U}_L(w)$ are actually independent sets of G_L. Moreover, the maps $\mathcal{D}_L, \mathcal{U}_L\colon L \to \mathrm{Ind}(G_L)$ are bijections. The *rowmotion* operator $\mathrm{Row}\colon L \to L$ is defined by $\mathrm{Row} = \mathcal{U}_L^{-1} \circ \mathcal{D}_L$.

The *rowmotion Markov chain* $\mathbf{M}_L$ has L as its set of states. For each $j \in \mathcal{J}_L$, we fix a probability $p_j \in [0,1]$. Starting at a state $u \in L$, we choose a random subset S of $\mathcal{D}_L(u)$ by adding each element $j \in \mathcal{D}_L(u)$ into S with probability p_j and then transition to the new state $u' = \mathrm{Row}_L(\bigvee S)$.

When $p_j = 1$ for all $j \in \mathcal{J}_L$, the Markov chain $\mathbf{M}_L$ is deterministic and agrees with rowmotion; indeed, this follows from [6, Theorem 5.6], which tells us that $\bigvee \mathcal{D}_L(u) = u$ for all $u \in L$.

Our main result about rowmotion Markov chains of semidistrim lattices is as follows.

Theorem 5 ([4])**.** *Let L be a semidistrim lattice, and fix a probability $p_j \in (0,1)$ for each join-irreducible element $j \in \mathcal{J}_L$. The rowmotion Markov chain $\mathbf{M}_L$ is irreducible.*

Let us remark that this theorem is not at all obvious. Our proof uses a delicate induction that relies on some difficult results about semidistrim lattices proven in [6]. For example, we use the fact that intervals in semidistrim lattices are semidistrim.

We can also generalize Theorem 3 to the realm of semidistrim lattices in the following theorem. Given a semidistrim lattice L and an element $u \in L$, we write $\mathrm{ddeg}(u)$ for the *down-degree* of u, which is the number of elements of L covered by u. Let $\alpha(G_L)$ denote the independence number of the Galois graph G_L; that is, $\alpha(G_L) = \max\limits_{\mathcal{I} \in \mathrm{Ind}(G_L)} |\mathcal{I}|$. Equivalently, $\alpha(G_L) = \max\limits_{u \in L} \mathrm{ddeg}(u)$. If P is a finite poset, then $\alpha(G_{J(P)}) = \mathrm{width}(P)$.

Theorem 6 ([4])**.** *Let L be a semidistrim lattice, and fix a probability $p_j \in (0,1)$ for each $j \in \mathcal{J}_L$.*

Let $\overline{p} = \max_{j \in \mathcal{J}_L} p_j$*. For each* $\varepsilon > 0$*, the mixing time of* $\mathbf{M}_L$ *satisfies*

$$t^{\mathrm{mix}}_{\mathbf{M}_L}(\varepsilon) \leq \left\lceil \frac{\log \varepsilon}{\log\left(1 - (1-\overline{p})^{\alpha(G_L)}\right)} \right\rceil.$$

We were not able to find a formula for the stationary distribution of the rowmotion Markov chain of an arbitrary semidistrim (or even semidistributive or trim) lattice; this serves to underscore the anomalistic nature of the formula for distributive lattices in (1.2). However, there is one family of semidistrim (in fact, semidistributive) lattices where we were able to find such a formula. Given positive integers a and b, let $\hexagon_{a,b}$ be the lattice obtained by taking two disjoint chains $x_1 < \cdots < x_a$ and $y_1 < \cdots < y_b$ and adding a bottom element $\hat{0}$ and a top element $\hat{1}$. Let us remark that $\hexagon_{m-1,m-1}$ is isomorphic to the weak order of the dihedral group of order $2m$, whereas $\hexagon_{m-1,1}$ is isomorphic to the c-Cambrian lattice of that same dihedral group (for any Coxeter element c). We have $\mathcal{J}_{\hexagon_{a,b}} = \mathcal{M}_{\hexagon_{a,b}} = \{x_1, \ldots, x_a, y_1, \ldots, y_b\}$. For $2 \leq i \leq a$ and $2 \leq i' \leq b$, we have $\kappa_{\hexagon_{a,b}}(x_i) = x_{i-1}$ and $\kappa_{\hexagon_{a,b}}(y_{i'}) = y_{i'-1}$; moreover, $\kappa_{\hexagon_{a,b}}(x_1) = y_b$ and $\kappa_{\hexagon_{a,b}}(y_1) = x_a$. This is illustrated in Figure 3 when $a = 3$ and $b = 2$. Figure 4 shows the transition diagram of $\mathbf{M}_{\hexagon_{2,1}}$.

Theorem 7 ([4]). *Fix positive integers a and b, and let $\kappa = \kappa_{\hexagon_{a,b}}$. For each $j \in \mathcal{J}_{\hexagon_{a,b}}$, fix a probability $p_j \in (0,1)$. There is a constant $Z(\hexagon_{a,b})$ (depending only on a and b) such that in the stationary distribution of $\mathbf{M}_{\hexagon_{a,b}}$, we have*

$$\mathbb{P}(\hat{0}) = \frac{1}{Z(\hexagon_{a,b})} p_{x_1} p_{y_1} \left(1 - \prod_{j \in \mathcal{J}_{\hexagon_{a,b}}} p_j\right);$$

$$\mathbb{P}(\hat{1}) = \frac{1}{Z(\hexagon_{a,b})} \left(1 - \prod_{j \in \mathcal{J}_{\hexagon_{a,b}}} p_j\right);$$

$$\mathbb{P}(x_i) = \frac{1}{Z(\hexagon_{a,b})} \left((1-p_{x_1}) \prod_{\substack{j \in \mathcal{J}_{\hexagon_{a,b}} \\ \kappa(j) \geq x_i}} p_j + (1-p_{y_1}) \prod_{\substack{j \in \mathcal{J}_{\hexagon_{a,b}} \\ \kappa(j) \not\leq x_i}} p_j \right) \quad \text{for} \quad 1 \leq i \leq a;$$

$$\mathbb{P}(y_i) = \frac{1}{Z(\hexagon_{a,b})} \left((1-p_{y_1}) \prod_{\substack{j \in \mathcal{J}_{\hexagon_{a,b}} \\ \kappa(j) \geq y_i}} p_j + (1-p_{x_1}) \prod_{\substack{j \in \mathcal{J}_{\hexagon_{a,b}} \\ \kappa(j) \not\leq y_i}} p_j \right) \quad \text{for} \quad 1 \leq i \leq b.$$

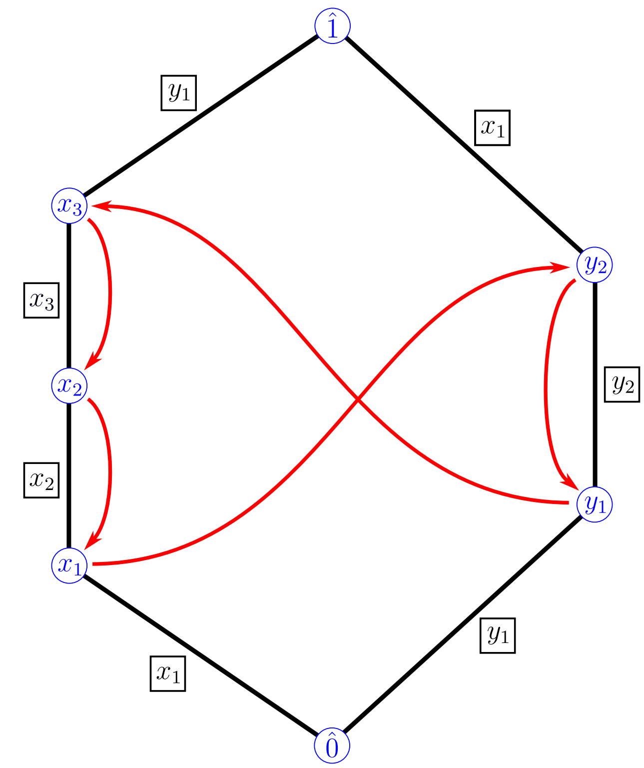


Figure 3: The lattice $\hexagon_{3,2}$. Next to each edge $u \lessdot v$ is a box containing the edge label j_{uv}. The red arrows represent the action of $\kappa_{\hexagon_{3,2}}$.

5 Future Directions

In Theorem 4, we saw that the rowmotion Markov chains of Boolean lattices exhibit the cutoff phenomenon. It would be very interesting to obtain similar results for other toggle Markov chains. Some particularly interesting toggle Markov chains $\mathbf{T}(\mathcal{K}, \mathbf{x})$ are as follows:

- Let P be the set of vertices of a graph G, let $\mathcal{K}$ be the collection of independent sets of G, and let $\mathbf{x}$ be some special ordering of P. For example, if G is a cycle graph, then $\mathbf{x}$ could be the ordering obtained by reading the vertices of G clockwise.
- Let P be an n-element set, and let $\mathbf{x}$ be an arbitrary ordering of the elements of P. For $0 \leq k \leq n$, let $\mathcal{K} = \{I \subseteq P : |I| \leq k\}$.
- Let P be an n-element set, and let $\mathbf{x}$ be an arbitrary ordering of the elements of P. For $0 \leq k \leq n$, let $\mathcal{K} = \{I \subseteq P : |I| \geq k\}$.

It would also be interesting to improve our estimates for the mixing times of rowmotion Markov chains for other families of semidistrim (or just distributive) lattices.

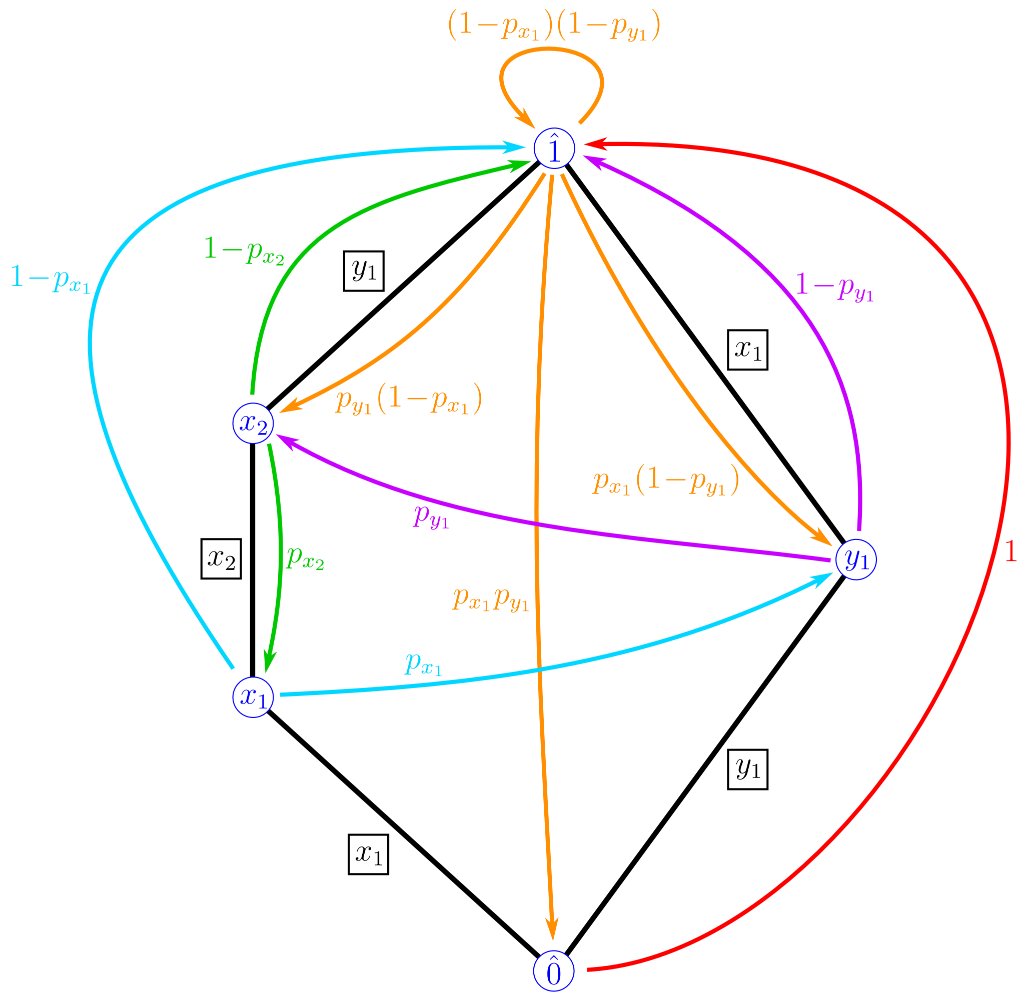


Figure 4: The transition diagram of $\mathbf{M}_{\mathcal{O}_{2,1}}$ drawn over the Hasse diagram of $\mathcal{O}_{2,1}$. Next to each edge $u \lessdot v$ is a box containing the edge label j_{uv}.

In Theorems 2 and 7, we computed the stationary distributions of rowmotion Markov chains of distributive lattices and the lattices $\mathcal{O}_{a,b}$. It would be quite interesting to find other special families of semidistrim lattices for which one can compute these stationary distributions.

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Enumerating the faces of split matroid polytopes

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Abstract. Computing f-vectors of polytopes is in general hard, and only little is known about their shape. We initiate the study of properties of f-vector of matroid base polytopes, by focusing on the class of split matroids, i.e., matroid polytopes arising from compatible splits of a hypersimplex. Unlike valuative invariants, the f-vector behaves in a much more unpredictable way, and the modular pairs of cyclic flats play a role in the face enumeration. We give a concise description of how the computation can be achieved without performing any convex hull or face lattice computation. As applications, we deduce formulas for sparse paving matroids and rank 2 matroids. These are two families that appear in other contexts within combinatorics.

Keywords: f-vectors, matroid polytopes, face numbers, split matroids, paving matroids

1 Introduction

A question that arises naturally in the study of a convex polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is how many faces of each dimension $\mathcal{P}$ has. The ***f*-vector** of $\mathcal{P}$ is defined by

$$f(\mathcal{P}) := (f_0, f_1, \ldots, f_{d-1}, f_d),$$

where $f_i := \#\{i\text{-dimensional faces of } \mathcal{P}\}$ for each $i \in \{0, \ldots, d\}$ and $d := \dim \mathcal{P}$. In particular, the number of vertices of $\mathcal{P}$ is just f_0, the number of facets of $\mathcal{P}$ is f_{d-1}, and $f_d = 1$.

The difficulty of calculating the f-vector may vary drastically depending on the polytope $\mathcal{P}$, on the properties it possesses, or on how it is described. For some concrete examples of the computation of f-vectors and certain related problems, see [21]. The family of possible vectors arising as the f-vector of a polytope is notoriously hard, and their classification is open in dimensions as low as four, see [23]. Even in the case of 0/1-polytopes of fixed dimension, although the set of possible f-vectors is finite, much remains to be discovered, see [22].

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In this article we will initiate the study of the explicit face enumeration of matroid polytopes, by focusing on the well-structured subclass of **(elementary) split matroids**. There are many equivalent ways of introducing these matroids. A matroid is *elementary split* whenever it does not contain a minor isomorphic to $\mathsf{U}_{0,1} \oplus \mathsf{U}_{1,2} \oplus \mathsf{U}_{1,1}$. Similarly, one may define the class of *split matroids* via five excluded minors [14, 7]. When the matroid is connected, these two notions agree. Geometrically, a connected matroid M is split whenever every pair of facet defining hyperplanes do not intersect in the interior of the hypersimplex containing the matroid polytope $\mathcal{P}(\mathsf{M})$.

The class of split matroids was introduced by Joswig and Schröter in [14] to study tropical linear spaces. They have received considerable attention in the past few years, including a forbidden minor characterization [7], hypergraphs descriptions [5], Tutte polynomial inequalities [11], subdivisions and computation of valuations [10], and conjectures about exchange properties on the bases [6] which are related to White's conjecture.

The face structure of some special classes as positroids and lattice path matroids appeared in previous work, however without an explicit enumeration. Even though the f-vector of the matroid base polytope constitutes an invariant of the matroid M under isomorphisms, it is not valuative; see Example 2.2 below. This makes its computation considerably subtler and difficult. In particular, for the case of split matroids we require a non-trivial modification of the machinery presented in [10].

One important reason why split matroids deserve to be studied is that they encompass the classes of paving and copaving matroids. A long-standing conjecture often attributed to Crapo and Rota, appearing in print in [16], predicts that asymptotically almost all matroids are sparse paving. There is some evidence supporting this assertion [18], but another intriguing conjecture affirms that even restricting to the enumeration of *non sparse paving* matroids, the class of split matroids will continue to be predominant [10, Conjecture 4.10].

As of today, the problem of face enumeration of matroid polytopes has not been approached systematically in the literature, and to the best of our knowledge there are no prior articles addressing their computation. Some articles such as [15, 19, 3, 12, 1] may be relevant, as they discuss other aspects indirectly related to the face enumeration for (some classes of) matroid polytopes.

In particular, perhaps as a reminiscence of the situation for polytopes in general (and even for 0/1-polytopes), questions about properties of f-vectors of matroid polytopes are widely open.

Summary of results

As mentioned before, the fact that the face numbers are not valuations makes the computation of the f-vector of matroid polytopes a delicate task. In the case of split matroids, we need more data than just the number of cyclic flats of each rank and size. Some information on their pairwise intersection is necessary.

In order to express the f-vector of a polytope $\mathcal{P}$ in a more compact fashion, we will often refer to the ***f*-polynomial**, which is defined via:

$$f_{\mathcal{P}}(t) := \sum_{i=0}^{d} f_i \cdot t^i.$$

Following the notation and terminology of [10], whenever we have a matroid M of rank k and cardinality n, we will denote by $\lambda_{r,h}$ the number of stressed subsets with non-empty cusp that M has. Although one of the main results of that article establishes that the numbers $\lambda_{r,h}$ are enough to compute any valuative invariant on M, we need further data to compute the f-vector.

For a matroid M as before, we will denote by $\mu_{\alpha,\beta,a,b}$ the number of **modular pairs** of cyclic flats $\{F_1, F_2\}$ such that $a = |F_1 \smallsetminus F_2|$, $b = |F_2 \smallsetminus F_1|$, $\alpha = \mathrm{rk}(F_1) - \mathrm{rk}(F_1 \cap F_2)$, and $\beta = \mathrm{rk}(F_2) - \mathrm{rk}(F_1 \cap F_2)$; see also equation ($\star$) below.

The following constitutes the main result of this article and is stated as Theorem 2.4 further below. It tells us that the numbers $\mu_{\alpha,\beta,a,b}$ are the precise additional datum needed to perform the computation of the f-vector of a split matroid polytope. Moreover, the statement tells us concretely how to calculate the number of faces of given dimension.

Theorem *Let M be a connected split matroid of rank k on n elements. The number of faces of its base polytope $\mathcal{P}(\mathsf{M})$ is given by the polynomial*

$$f_{\mathcal{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \sum_{r,h} \lambda_{r,h} \cdot u_{r,k,h,n}(t) - \sum_{\alpha,\beta,a,b} \mu_{\alpha,\beta,a,b} \cdot w_{\alpha,\beta,a,b}(t)$$

where the first sum ranges over all values with $0 < r < h < n$ and the second sum ranges over the values $0 < \alpha < a$, $0 < \beta < b$ for which either $a < b$ or $a = b$ and $\alpha \leq \beta$.

In the above theorem, the expressions $u_{r,k,h,n}(t)$ and $w_{\alpha,\beta,a,b}(t)$ are polynomials which depend only on their subindices. We present in Propositions 2.6 and 2.7 explicit (but complicated) formulas for them which can be used to calculate the face numbers effortlessly. A formula for the f-vector of the hypersimplex $\Delta_{k,n}$ is also given explicitly in Example 2.1. In particular, the entire calculation can be done bypassing the problem of building costly face lattices or computing convex hulls.

As two direct but interesting applications of our result, we particularize it to the classes of sparse paving and rank 2 matroids. The first is a class that made a prominent appearance in the theory of the extension complexity of independence polytopes [20]. The second bears a relevant connection with the theory of edge polytopes of graphs [17].

2 The number of faces of split matroids

2.1 The set up

Throughout this extended abstract we will assume that the reader is familiar with the usual terminology and notation in matroid theory. For the notions and machinery introduced very recently, in particular about **stressed subsets**, **relaxations**, and **cuspidal matroids** we refer the reader to our previous article [10, Sections 3–4]. Regarding **split matroids** and **elementary split matroids** the reader can consult the same article as well as [14, 5]. However, basic knowledge on polytopes should be enough to follow the arguments and methods in this manuscript.

For a d-dimensional polytope $\mathcal{P}$ we denote by $f(\mathcal{P}) := (f_0, \ldots, f_d)$ its *f-vector*, and by

$$f_{\mathcal{P}}(t) := \sum_{i=0}^{d} f_i \, t^i$$

its *f-polynomial*. In both cases, f_i denotes the number of i-dimensional faces of $\mathcal{P}$. Notice that we omit the inclusion of $f_{-1} := 1$ for the empty set in both the f-vector and the f-polynomial, but we do include $f_d = 1$ for the polytope itself.

Essential notation Following our prequel [10], whenever we have a matroid M, unless specified otherwise, the rank of M is denoted by k and the size of its ground set is denoted by n. We reserve the letters r and h for the rank and the size of stressed subsets that M may possess.

Note that under the assumption of being connected the classes of split matroids and elementary split matroids coincide [5, Theorem 11]. Since the base polytope of a direct sum of matroids $\mathsf{M}_1 \oplus \mathsf{M}_2$ is the cartesian product of $\mathcal{P}(\mathsf{M}_1)$ and $\mathcal{P}(\mathsf{M}_2)$, the f-vector of any disconnected split matroid can be recovered from the f-vector of the connected components, all of which are split as well.

The most basic example of a matroid polytope is the hypersimplex $\Delta_{k,n}$, the matroid base polytope of the uniform matroid $\mathsf{U}_{k,n}$ of rank k on n elements.

Example 2.1 The face enumeration of hypersimplices is encoded in the following f-polynomial:

$$f_{\mathcal{P}(\mathsf{U}_{k,n})}(t) = f_{\Delta_{k,n}}(t) = \binom{n}{k} + \sum_{i=1}^{n-1} \binom{n}{i+1} \sum_{j=1}^{i} \binom{n-i-1}{k-j} \cdot t^i \ .$$

For a detailed proof see for example [13, Corollary 1.4].

As we will see now, the assignment $\mathsf{M} \mapsto f_{\mathcal{P}(\mathsf{M})}(t)$ is an invariant of the matroid M that fails to be valuative. Hence its computation is a more delicate task, even for the case of paving or split matroids. In these cases, we cannot rely on the strength of [10, Theorem 6.6] — that result asserts that the evaluation of a valuative invariant on a split matroid M can be achieved by knowing relatively little about the matroid M, consisting of its rank k, its size n, and the parameters $\lambda_{r,h}$. If one is interested in knowing the f-vector of $\mathcal{P}(\mathsf{M})$, the first problem one faces is identifying what additional matroid data is required.

Example 2.2 Consider the four matroids $\mathsf{U}_{3,6}$, M, N_1 and N_2 with ground set $\{1,\dots,6\}$ and rank three, whose families of bases are given as follows:

$$\mathcal{B}(\mathsf{U}_{3,6}) := \binom{[6]}{3}, \qquad \mathcal{B}(\mathsf{N}_1) := \binom{[6]}{3} \smallsetminus \{\{1,2,3\},\{4,5,6\}\}$$
$$\mathcal{B}(\mathsf{M}) := \binom{[6]}{3} \smallsetminus \{\{1,2,3\}\}, \qquad \mathcal{B}(\mathsf{N}_2) := \binom{[6]}{3} \smallsetminus \{\{1,2,3\},\{3,4,5\}\}.$$

The f-vectors of their base polytopes are respectively:

$$f(\mathcal{P}(\mathsf{U}_{3,6})) = (20,90,120,60,12,1), \qquad f(\mathcal{P}(\mathsf{N}_1)) = (18,72,102,60,14,1),$$
$$f(\mathcal{P}(\mathsf{M})) = (19,81,111,60,13,1), \qquad f(\mathcal{P}(\mathsf{N}_2)) = (18,72,101,59,14,1).$$

All of these matroids are sparse paving. In particular, the two matroids N_1 and N_2 have, e.g., the same Tutte polynomial and the same Ehrhart polynomial — in fact, via [10, Corollary 6.7] any valuative invariant on these two matroids yields the same result. However, observe that their f-vectors differ in the third and the fourth entries.

2.2 Cuspidal matroids

By using [10, Corollary 6.2], we see that the intersection of the hypersimplex $\Delta_{k,n}$ with the half-space of a single split hyperplane leads to the polytope:

$$\mathcal{P}(\Lambda_{k-r,k,n-h,n}) = \left\{ x \in \Delta_{k,n} \ : \ \sum_{i=1}^{h} x_i \leq r \right\} \ . \tag{2.1}$$

for appropriate values r and h. This is the base polytope of the cuspidal matroid $\Lambda_{k-r,k,n-h,n}$, a matroid having exactly three cyclic flats: the empty set, the entire ground set, and one proper cyclic flat having size h and rank r. For the purposes of this paper, the reader may regard equation (2.1) as the definition of cuspidal matroids.

Let us introduce some notation that will help us formulate later our main results in a more compact way:

$$u_{r,k,h,n}(t) := f_{\Delta_{k,n}}(t) - f_{\mathcal{P}(\Lambda_{k-r,k,n-h,n})}(t). \tag{2.2}$$

A non-obvious property is that some of these coefficients may be negative while other are positive — moreover, the actual sign of each individual coefficient a priori depends on the four parameters r, k, h, n.

Before we go on, let us introduce a second polynomial, which will play an important role in the sequel. For fixed numbers $0 < \alpha < a$ and $0 < \beta < b$ let us define,

$$\begin{aligned} w_{\alpha,\beta,a,b}(t) &:= f_{\Delta_{\alpha+\beta,a+b}}(t) - f_{\Delta_{\alpha,a}}(t) \cdot f_{\Delta_{\beta,b}}(t) - u_{\alpha,\alpha+\beta,a,a+b}(t) - u_{\beta,\alpha+\beta,b,a+b}(t) \\ &= f_{\mathcal{P}(\Lambda_{\beta,\alpha+\beta,b,a+b})}(t) + f_{\mathcal{P}(\Lambda_{\alpha,\alpha+\beta,a,a+b})}(t) - f_{\Delta_{\alpha+\beta,a+b}}(t) - f_{\Delta_{\alpha,a}}(t) \cdot f_{\Delta_{\beta,b}}(t). \end{aligned}$$

Later, in Proposition 2.6, we provide a compact formula for the polynomials $w_{\alpha,\beta,a,b}(t)$ and a formula for the polynomials $u_{r,k,h,n}(t)$ in Proposition 2.7 both of which can be used to calculate these polynomials, bypassing the computation of f-vectors of cuspidal matroids using the polytopes themselves.

Remark 2.3 The intuition of why it is reasonable to consider and define the complicated expression above stems from [10, Example 6.5]. As follows from the explanation there, if the assignment $\mathsf{M} \mapsto f_{\mathcal{P}(\mathsf{M})}(t)$ were valuative, then the defining formula for $w_{\alpha,\beta,a,b}(t)$ would actually be identically zero. The polynomial $w_{\alpha,\beta,a,b}(t)$ quantifies (in a certain way) how far the map $\mathsf{M} \mapsto f_{\mathcal{P}(\mathsf{M})}(t)$ is from being valuative.

2.3 Face counting of split matroids

For a connected split matroid M, let us define the following numbers that we have already mentioned in the introduction. The number of stressed subsets with non-empty cusp having rank r and size h, denoted $\lambda_{r,h}$ — recall that by [10, Proposition 3.9], in a connected split matroid this is the same as the number of proper non-empty cyclic flats of rank r and size h. We also need the numbers $\mu_{\alpha,\beta,a,b}$ of (unordered) modular pairs $\{F_1, F_2\}$ of proper non-empty cyclic flats, i.e., F_1 and F_2 fulfilling the *modularity property*,

$$\operatorname{rk}(F_1) + \operatorname{rk}(F_2) = \operatorname{rk}(F_1 \cap F_2) + \operatorname{rk}(F_1 \cup F_2), \tag{$\star$}$$

where the indices denote the following quantities:

$$\begin{aligned} a &= |F_1 \smallsetminus F_2|, & \alpha &= \operatorname{rk} F_1 - \operatorname{rk}(F_1 \cap F_2) \\ b &= |F_2 \smallsetminus F_1|, & \beta &= \operatorname{rk} F_2 - \operatorname{rk}(F_1 \cap F_2)\ . \end{aligned}$$

Note that the set $F_1 \cap F_2 \subsetneq F_1 \subsetneq [n]$ can not contain a circuit if M is a connected split matroid, thus it is an independent set, i.e., $\operatorname{rk}(F_1 \cap F_2) = |F_1 \cap F_2|$.

Theorem 2.4 *Let M be a connected split matroid of rank k on n elements. The number of faces of its base polytope $\mathcal{P}(\mathsf{M})$ is given by the polynomials*

$$f_{\mathcal{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \sum_{r,h} \lambda_{r,h} \cdot u_{r,k,h,n}(t) - \sum_{\alpha,\beta,a,b} \mu_{\alpha,\beta,a,b} \cdot w_{\alpha,\beta,a,b}(t) \tag{2.3}$$

where the first sum ranges over all values with $0 < r < h < n$ *and the second sum ranges over the values* $0 < \alpha < a$, $0 < \beta < b$ *for which either* $a < b$ *or* $a = b$ *and* $\alpha \leq \beta$*.*

On one hand, note that the polynomials $f_{\Delta_{k,n}}(t)$, $u_{r,k,h,n}(t)$ and $w_{\alpha,\beta,a,b}(t)$ can be precomputed for all the occurring instances of the variables which appear as subindices. The first non-trivial fact that is deduced by our statement is that in addition to the parameters $\lambda_{r,h}$, which always appear in the computation of a valuative invariant, the precise additional matroidal datum needed to compute the f-vector consists of the numbers $\mu_{\alpha,\beta,a,b}$. Strikingly, the last sum in equation (2.3) does not take into consideration the rank nor the size of the matroid M itself, only the intersection data for the modular pairs of flats. The second non-trivial fact is that it explains how to put together this information in order to effectively computing the f-vector of $\mathcal{P}(\mathsf{M})$ for a split matroid, circumventing the necessity of constructing the polytope.

Example 2.5 Let us take a look again at Example 2.2. The matroids N_1 and N_2 are sparse paving, have rank $k = 3$ and size $n = 6$. In each case the proper non-empty cyclic flats are exactly the non-bases, yielding for both matroids $\lambda_{2,3,3,6} = 2$. One can compute the corresponding polynomial, $u_{2,3,3,6}(t) = 1 + 9t + 9t^2 - t^4$. In N_1, the intersection of the only pair of proper non-empty cyclic flats, $F_1 = \{1,2,3\}$ and $F_2 = \{4,5,6\}$, does not satisfy the property ($\star$), because $\operatorname{rk}(F_1 \cap F_2) + \operatorname{rk}(F_1 \cup F_2) = 0 + 3$, whereas $\operatorname{rk}(F_1) + \operatorname{rk}(F_2) = 2 + 2 = 4$.

For N_2, the situation is different, as $F_1 = \{1,2,3\}$ and $F_2 = \{3,4,5\}$ indeed satisfy ($\star$), and we have $a = |F_1 \smallsetminus F_2| = 2$, $b = |F_2 \smallsetminus F_1| = 2$, $\alpha = \operatorname{rk}(F_1) - |F_1 \cap F_2| = 2 - 1 = 1$, and $\beta = \operatorname{rk}(F_2) - |F_1 \cap F_2| = 2 - 1 = 1$, so that $\mu_{1,1,2,2} = 1$ and we need to subtract $w_{1,1,2,2}(t) = t^2 + t^3$ to obtain the correct f-polynomial, as we expected.

2.4 Explicit formulas

The polynomials $u_{r,k,h,n}(t)$ and $w_{\alpha,\beta,a,b}(t)$ in Theorem 2.4 are defined in terms of f-vectors of specific matroid polytopes. In this subsection we will present explicit descriptions for these polynomials, enabling us to do the face enumeration of a split matroid polytope, without any convex hull or face lattice computation. To express the formulas in a compact form, we will make use of multinomial coefficients. Let i, j, ℓ be non negative integers, then

$$\binom{i+j+\ell}{i,j} := \binom{i+j+\ell}{i,j,\ell} = \frac{(i+j+\ell)!}{i!j!\ell!} \quad .$$

We begin with an explicit formula for the polynomials $w_{\alpha,\beta,a,b}(t)$.

Proposition 2.6 *For any $0 < \alpha < a$ and $0 < \beta < b$, the following formula holds:*

$$w_{\alpha,\beta,a,b}(t) = \sum_{i=0}^{a-\alpha-1} \sum_{j=0}^{\alpha-1} \sum_{i'=0}^{b-\beta-1} \sum_{j'=0}^{\beta-1} \binom{a}{i,j}\binom{b}{i',j'} \cdot (1+t) \cdot t^{a+b-i-j-i'-j'-2} .$$

For the polynomials $u_{r,k,h,n}(t)$ we provide the following formula.

Proposition 2.7 *For any $0 < r < k < n$ and $r < h < n$ the following formula holds*

$$u_{r,k,h,n}(t) = p_{r,k,h,n}(t) - p'_{r,h}(t) \cdot p'_{k-r,n-h}(t) \cdot (1+t) + \sum_{i=r+1}^{k} \binom{h}{i}\binom{n-h}{k-i}$$

where $p'_{r,h}(t) = f_{\Delta_{r,h}}(t) - \binom{h}{r}$ and

$$p_{r,k,h,n}(t) = \sum_{j=0}^{h-r-1} \sum_{i=0}^{\min\{h-j,k-1\}} \sum_{\ell=0}^{\min\{k-i,k-r\}-1} \sum_{m=0}^{\min\{n-h-\ell,n-k-j-1\}} \binom{h}{i,j}\binom{n-h}{\ell,m} t^{n-1-s}.$$

where s denotes $i+j+\ell+m$ in the above sum.

Example 2.8 Let M be the projective geometry $\mathrm{PG}(2,3)$. This is a matroid on $n = 13$ elements of rank $k = 3$. It is split as it is in fact paving. This matroid has 13 stressed hyperplanes, i.e., rank $k-1 = 2$ flats, all of which have cardinality $h = 4$. In other words, we have $\lambda_{2,4} = 13$. In particular, to use the formula of Theorem 2.4, the polynomial

$$\begin{aligned} u_{2,3,4,13}(t) = &- t^{11} - 11\,t^{10} - 54\,t^9 - 156\,t^8 - 294\,t^7 - 378\,t^6 \\ &- 336\,t^5 - 195\,t^4 + t^3 + 166\,t^2 + 114\,t + 4 \end{aligned}$$

is required. Since projective geometries are modular matroids, any pair of distinct proper non-empty cyclic flats fulfills the property ($\star$). Also, every pair of them intersect in a single element. Moreover, for every pair of these cyclic flats we have $a = |F_1 \smallsetminus F_2| = 3$, and by symmetry $b = |F_1 \smallsetminus F_2| = 3$. Additionally, $\alpha = \mathrm{rk}(F_1) - |F_1 \cap F_2| = 2 - 1 = 1$ and again by symmetry $\beta = \mathrm{rk}(F_2) - |F_1 \cap F_2| = 1$. Therefore there is a single non-vanishing coefficient $\mu_{\alpha,\beta,a,b}$ which is

$$\mu_{1,1,3,3} = \binom{13}{2} = 78 .$$

It remains to compute:

$$w_{1,1,3,3}(t) = t^5 + 7t^4 + 15t^3 + 9t^2 .$$

Now applying Theorem 2.4, we obtain:

$$\begin{aligned} f_{\mathcal{P}(\mathrm{PG}(2,3))}(t) &= f_{\Delta_{3,13}}(t) - 13\,u_{2,3,4,13}(t) - 78\,w_{1,1,3,3}(t) \\ &= t^{12} + 39\,t^{11} + 455\,t^{10} + 2704\,t^9 + 9893\,t^8 + 24414\,t^7 + 42666\,t^6 + \\ &\quad 54054\,t^5 + 49608\,t^4 + 31707\,t^3 + 12870\,t^2 + 2808\,t + 234 . \end{aligned}$$

2.5 Face numbers of sparse paving matroids

As mentioned in the introduction, it is conjectured that almost all matroids are sparse paving; see [16] for the details. Furthermore, many famous examples of matroids fall into this class; notable examples are the Fano matroid, the Vámos matroid, the complete graph on four vertices, and the duals of each of them. Sparse paving and paving matroids are split, so we can make use of our main result.

Corollary 2.9 *Let* M *be a connected sparse paving matroid of rank* k *on* n *elements having exactly* λ *circuit-hyperplanes, and let* μ *count the pair of circuit-hyperplanes which have* $k-2$ *elements in common. Then*

$$f_{\mathcal{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \lambda \cdot u(t) - \mu \cdot (t^2 + t^3)$$

where $u(t)$ *is given by*

$$1 - k \cdot (n-k) \cdot (t+1) + \Big((n-k) \cdot (t+1)^{k+1} + k \cdot (t+1)^{n-k+1} - n \cdot (t+1) \Big) \cdot t^{-1} \\ + \Big((t+1)^k + (t+1)^{n-k} - (t+1)^n - 1 \Big) \cdot t^{-2} \ .$$

Remark 2.10 This formula can be used to prove that the number of facets of the base polytope of a matroid on n elements may be as large as $c2^n/n^{3/2}$ for an absolute constant c. However, for arbitrary $0/1$-polytopes in $\mathbb{R}^n$ it is known that the number of facets can be larger than $\left(\frac{cn}{\log n}\right)^{n/4}$, via a random construction [4].

Given a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$, an *extended formulation* of $\mathcal{P}$ is another lattice polytope $\mathcal{Q} \subseteq \mathbb{R}^m$ together with a projection map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ which projects $\mathcal{Q}$ onto $\mathcal{P}$. The *complexity* of an extended formulation is the number of facets of the polytope $\mathcal{Q}$. The *extension complexity* of $\mathcal{P}$, denoted $\mathrm{xc}(\mathcal{P})$, is the minimum complexity of an extended formulation of $\mathcal{P}$.

In a landmark paper [20, Corollary 6], Rothvoss proved [1] that for all n there exists a matroid M on n elements whose base polytope has extension complexity $\mathrm{xc}(\mathcal{P}(\mathsf{M})) \in \Omega\left(\frac{2^{n/2}}{n^{5/4}\sqrt{\log(2n)}}\right)$. Moreover, Rothvoss' proof is non-constructive and relies only on an enumerative result of matroids, that therefore guarantees that whatever these examples are, they must belong to the class of sparse paving matroids, and are therefore split matroids. It remains a notorious open problem to find an explicit family of matroids having exponential extension complexity. In fact, having one would yield an explicit infinite family of Boolean functions requiring superlogarithmic depth circuits, according

[1] To be precise, Rothvoss proved that the extension complexity of the *independence polytope* of some matroid is exponential, but an elementary reasoning shows that this is equivalent to an analogous statement for the base polytope. See for example the short explanation in [2, p. 1].

to an observation attributed to Göös in [2, Section 8]. We conjecture, however, that a certain class of "extremal" sparse paving matroids must already constitute such an example; for the details of the conjecture we refer to the extended version of the present paper [9].

2.6 Face numbers of rank two matroids

A loopless matroid of rank two is trivially paving, and hence a split matroid. This allows us to use the strength of Theorem 2.4 to compute their f-vectors. The hyperplanes, i.e., the flats of rank one, of a loopless matroid of rank two form a partition of the ground, and conversely, any partition of the ground set defines precisely a single rank two matroid having each part as a flat. The bases of the matroid are obtained by taking two elements of the ground set, not belonging to the same part.

Base polytopes of matroids of rank two have made prominent appearances throughout algebraic combinatorics, under various guises. Notably, as is pointed out in [8, Section 6.1], they coincide with edge polytopes of complete multipartite graphs — we refer to that paper for the precise definition of edge polytopes and a short overview of them. In this vein, the work of Ohsugi and Hibi [17] addresses the edge polytopes of complete multipartite graphs, motivated both from toric geometry and graph theory. In particular, the content of [17, Theorem 2.5] provides a formula for the f-vector of the edge polytope of an arbitrary complete multipartite graph, and thus for general rank two matroid polytopes. Let us point out that there appears to be an error in the formula as they stated it — in particular within the quantity they denote by α_i. As an application of Theorem 2.4 we can give another formula for the f-vector of these polytopes.

Corollary 2.11 *Let M be a loopless matroid of rank two having s hyperplanes with cardinalities $h_1, \dots, h_s$. Then, the number of i-dimensional faces of $\mathcal{P}(\mathsf{M})$ or, equivalently, the edge polytope of a complete multipartite graph with parts of sizes $h_1, \dots, h_s$ is given by:*

$$f_i(\mathcal{P}(\mathsf{M})) = \binom{n+1}{i+2} + (s-1)\binom{n}{i+2} - \sum_{j<\ell} \binom{h_j + h_\ell + 1}{i+2} \\ + (s-2)\sum_{j=1}^{s} \binom{h_j+1}{i+2} - \sum_{j=1}^{s} \binom{n-h_j}{i+2}.$$

2.7 Questions on the shape of f-vectors of matroids

A recent trend in matroid theory is that of proving unimodal and log-concave inequalities for various vectors of numbers associated to matroids. A finite sequence of numbers $(a_0, \dots, a_n)$ is said to be *unimodal* if there exists some index $0 \leq j \leq n$ with the property that

$$a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n.$$

If all the a_i's are positive, a stronger condition is that of *log-concavity*, which asserts that for each index $1 \leq j \leq n-1$ the inequalities $a_j^2 \geq a_{j-1}a_{j+1}$ hold.

It is quite inviting to ask the following question.

Question 2.12 Are the f-vectors of matroid base polytopes unimodal, or even log-concave?

It is known that there are simplicial polytopes having a non-unimodal f-vector; see [21, Chapter 8.6]. Within the existing literature we were not able to find any examples of non-unimodal f-vectors for the general class of 0/1-polytopes. We have been able to verify the log-concavity of the f-vectors of the following classes of matroids, in some cases relying critically on the results of this paper:

- All matroids on a ground set of size at most 9.
- Split matroids on a ground set of size at most 15.
- Sparse paving matroids on a ground set of size at most 40.
- Lattice path matroids on a ground set of size at most 13.
- Rank two matroids on a ground set of size at most 60.

Note: An extended version of this manuscript including all proofs can be found on the arXiv, see [9].

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From higher Bruhat orders to Steenrod cup-i coproducts

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Abstract. We show that the higher Bruhat orders of Manin and Schechtman provide a useful conceptual framework for understanding Steenrod's cup-i coproducts, which are used to define the cohomology operations known as Steenrod squares. Indeed, we show that the elements of the $(i+1)$-dimensional higher Bruhat order are in bijection with all possible cup-i coproducts on the chain complex of the simplex which give a homotopy between cup-$(i-1)$ and its opposite. The Steenrod cup-i coproduct and its opposite are then given by the maximal and minimal elements of the higher Bruhat order. This correspondence uses the geometric realisation of the higher Bruhat orders in terms of tilings of cyclic zonotopes, and enables us to give conceptual proofs of the fundamental properties of the cup-i coproducts.

Keywords: Higher Bruhat orders, zonotopal tilings, cubillages, cup-i coproducts, Steenrod squares

1 Introduction

In a classical article from 1947, N. E. Steenrod introduced the cup-i products on the cochains of a simplicial complex [12]. These can allow one to distinguish between non-homotopy-equivalent spaces with isomorphic cohomology rings, such as the suspensions of $\mathbb{C}P^2$ and $S^2 \vee S^4$. More recently, they were shown to be part of an E_∞-algebra structure on the cochains of a space X [11, 1], which faithfully encodes its homotopy type when X is of finite type and nilpotent [8, 9].

The cup-0 product is the usual cup product. This is commutative at the level of cohomology, but not at the level of cochains. The cup-1 product provides an explicit homotopy between cup-0 and its opposite which witnesses this fact. However, the cup-1

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product is itself not commutative, which gives rise to the cup-2 product, and so on. In this paper, we consider cup-i coproducts, which give rise to cup-i products via linear duality.

Over 40 years later, Yu. I. Manin and V. V. Schechtman introduced the higher Bruhat orders, with an entirely different purpose [10]. Their original motivation was to study hyperplane arrangements and higher braid groups, but the higher Bruhat orders have gone on to subsequently find connections with Soergel bimodules [4], the quantum Yang–Baxter equation [3], and many other areas of mathematics besides.

The 1-dimensional higher Bruhat order is the weak Bruhat order on the symmetric group. The elements of the 2-dimensional higher Bruhat order are then equivalence classes of maximal chains in the weak order, that is, reduced expressions for the longest element, up to swapping commuting simple reflections. The covering relations of the 2-dimensional order are then given by braid moves. This pattern repeats, with the $(i+1)$-dimensional higher Bruhat orders having as elements equivalence classes of maximal chains in the i-dimensional order.

Already, one can see a resemblance between the cup-i coproducts and the higher Bruhat orders insofar as in each case the objects in a given dimension give rise to the objects in one dimension higher. In fact, it is more than a resemblance, as we show. To state our main theorem, we need the following notation.

- $\mathcal{B}([0,n], i+1)$ is the $(i+1)$-dimensional higher Bruhat poset on the base set $[0,n] := \{0, 1, \ldots, n\}$.
- $\Delta_i \colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is the Steenrod cup-i coproduct on the chain complex of the n-simplex Δ^n, where we set $\Delta_{-1} = 0$.
- $T \colon C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is the isomorphism given by exchange of tensor factors, with the Koszul sign rule applied.

Theorem ([7, Theorem 3.6, Theorem 3.9])**.** *For all $i \geqslant 0$, there is a bijection between elements of $\mathcal{B}([0,n], i+1)$ and coproducts Δ'_i on $C_\bullet(\Delta^n)$ satisfying the homotopy formula*

$$\partial \circ \Delta'_i - (-1)^i \Delta'_i \circ \partial = (1 + (-1)^i T)\Delta_{i-1}, \tag{1.1}$$

and containing no redundant terms, with the additional assumption in the $i = 0$ case that $\Delta'_0(p) = p \otimes p$ for all vertices p of Δ^n.

Moreover, under this bijection, the minimal and maximal elements of the higher Bruhat orders correspond to Δ_i and $T\Delta_i$, up to sign.

The Steenrod cup-i coproduct Δ_i alternates between corresponding to the minimal element of the higher Bruhat poset and the maximum, according to the parity of i. This is due to differing conventions in the two theories.

The proof of this theorem uses the geometric realisation of the higher Bruhat orders $\mathcal{B}([0,n], i+1)$ in terms of zonotopal tilings of $Z([0,n], i+1)$, the $(i+1)$-dimensional cyclic zonotope with $n+1$ vertices. We call these tilings "cubillages" and refer to their tiles as "cubes". There is a natural bijection between cubes of cubillages and basis elements of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$, such that the terms of $\Delta'_i([0,n])$ in $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is described by a set of cubes of a cubillage of $Z([0,n], i+1)$. Simplifying slightly, this is used to derive the homotopy formula as follows. Taking the boundary ∂ corresponds to taking the boundary of the cubes of the cubillage. Facets of cubes which are shared with other cubes cancel out, so that the only remaining terms come from the boundary of the zonotope. These terms turn out to be precisely $(1 + (-1)^i T)\Delta_{i-1}$. Running the argument in reverse shows that all coproducts satisfying the homotopy formula must arise from cubillages, that is, elements of $\mathcal{B}([0,n], i+1)$.

This paper is an extended abstract of [7]. In Section 2, we give background on the Steenrod cup-i coproducts, followed by background on the higher Bruhat orders. We then outline our main results in Section 3, referring to [7] for complete proofs.

2 Background

In this section, we recall the definitions of Steenrod operations and higher Bruhat orders, and set up notation.

2.1 Steenrod cup-i coproducts

2.1.1 Chain complexes

By a (non-negatively graded) *chain complex* C we mean an $\mathbb{N}$-graded $\mathbb{Z}$-module with linear maps

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

satisfying $\partial_p \circ \partial_{p+1} = 0$ for each $p \in \mathbb{N}$. As usual, we refer to ∂_p as the p-th *boundary map* and suppress the subscript when convenient. A morphism of chain complexes, referred to as a *chain map*, is a morphism of $\mathbb{N}$-graded $\mathbb{Z}$-modules $f\colon C \to C'$ satisfying $\partial'_{p+1} f_{p+1} = f_p \partial_{p+1}$ for $p \in \mathbb{N}$.

The tensor product of two chain complexes X and Y is the chain complex $X \otimes Y$ whose degree r component is $(X \otimes Y)_r := \oplus_{p+q=r} X_p \otimes Y_q$, and whose differential is defined by $\partial(x \otimes y) := \partial(x) \otimes y + (-1)^{\deg(x)} x \otimes \partial(y)$. There is an isomorphism $T\colon X \otimes Y \to Y \otimes X$ defined by $T(x \otimes y) := (-1)^{\deg(x)\deg(y)} y \otimes x$.

2.1.2 Chain complex of the simplex

We denote the standard n-simplex $\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1,\, x_i \geqslant 0\}$. We refer to faces of the n-simplex using their vertex sets, where we use the notation $[p, q] := \{p, p+1, \dots, q\}$ and $(p, q) := [p, q] \setminus \{p, q\}$. When we give a set $\{v_0, v_1, \dots, v_q\} \subseteq [0, n]$, we mean that the elements are ordered $v_0 < v_1 < \cdots < v_q$.

We will consider the $\mathbb{Z}$-module given by the cellular chains $\mathrm{C}_\bullet(\Delta^n)$ on the standard n-simplex. This chain complex has as basis the faces of Δ^n, whose degree is given by the dimension; for example, the face $\{v_0, \dots, v_q\}$ has dimension q. The boundary map of this chain complex is given by

$$\partial(\{v_0, \dots, v_q\}) := \sum_{p=0}^{q} (-1)^p \{v_0, \dots, \hat{v}_p, \dots, v_q\}.$$

2.1.3 The Steenrod cup-i coproducts

An *overlapping partition* of $[0, n]$ is a family $\mathcal{L} = (L_0, L_1, \dots, L_{i+1})$ of intervals $L_p = [l_p, l_{p+1}]$ such that $l_0 = 0$, $l_{i+2} = n$, and for each $0 < p < i+1$ we have $l_p < l_{p+1}$. The *Steenrod cup-i-coproduct* is the coproduct $\Delta_i \colon \mathrm{C}_\bullet(\Delta^n) \to \mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ given by the formula

$$\Delta_i([0, n]) := \sum_{\mathcal{L}} (-1)^{\varepsilon(\mathcal{L})} (L_0 \cup L_2 \cup \cdots) \otimes (L_1 \cup L_3 \cup \cdots),$$

where the sum is taken over all overlapping partitions of $[0, n]$ into $i+2$ intervals. If $n \leqslant i-1$, there are no such overlapping partitions, and the coproduct is zero. Denoting by $w_{\mathcal{L}}$ the shuffle permutation putting $0, 1, \dots, n$ into the order

$$[0, l_1], [l_2, l_3], \dots, (l_1, l_2), (l_3, l_4), \dots,$$

the sign is given by $\varepsilon(\mathcal{L}) := \mathsf{sign}(w_{\mathcal{L}}) + in$. The coproduct Δ_i is then defined similarly on lower-dimensional faces, by summing over increasing overlapping partitions of their vertex sets. Steenrod [12] then shows that

$$\partial \Delta_i - (-1)^i \Delta_i \partial = (1 + (-1)^i T) \Delta_{i-1}, \tag{2.1}$$

where we set $\Delta_{-1} = 0$. We refer to this as the *homotopy formula*, since it is equivalent to saying that Δ_i gives a chain homotopy from $T\Delta_{i-1}$ to Δ_{i-1}.

2.2 Higher Bruhat orders

There are many ways of defining the higher Bruhat orders. For the purposes of this abstract, we only consider the geometric definition of the higher Bruhat orders in terms of cubillages of cyclic zonotopes due to [5, 13]. Other definitions can be found in [7].

Definition 2.1. Consider the *Veronese curve* $\xi \colon \mathbb{R} \to \mathbb{R}^{i+1}$, given by $\xi_t = (1, t, t^2, \ldots, t^i)$. The *cyclic zonotope* $Z([0,n], i+1)$ is defined to be the Minkowski sum of the line segments

$$\overline{\mathbf{0}\xi_0} + \cdots + \overline{\mathbf{0}\xi_n},$$

where $\mathbf{0}$ is the origin and $\overline{\mathbf{0}\xi_p}$ is the line segment from $\mathbf{0}$ to ξ_p. One similarly defines $Z(S, i+1)$ for $S \subseteq [0,n]$ a subset. There is a natural projection $\pi_{i+1} \colon Z([0,n], n+1) \to Z([0,n], i+1)$ given by forgetting the last $n-i$ coordinates.

Definition 2.2. A *cubillage* U of $Z([0,n], i+1)$ is a set of $(i+1)$-dimensional faces $\{F_p\}$ of $Z([0,n], n+1)$ such that π_{i+1} is a bijection when restricted to $\bigcup_p F_p$. Such faces F_p are necessarily $(i+1)$-dimensional, and we refer to them as the *cubes* of the cubillage.

A cubillage U of $Z([0,n], i+1)$ gives a subdivision of $Z([0,n], i+1)$ consisting of the images of the cubes under π_{i+1}. In the literature, cubillages are often called *fine zonotopal tilings*.

Recall that a *facet* of a polytope is a face of codimension one. The standard basis of $\mathbb{R}^{n+1}$ induces orientations of the faces of $Z([0,n], n+1)$, in the sense that the facets of a face can be partitioned into two sets, called upper facets and lower facets.

Definition 2.3. If F is a i-dimensional face of $Z([0,n], n+1)$, with G a facet of F, then G is a *lower* (resp. *upper*) facet of F if normal vectors to G which lie inside the affine span of F and point into F have positive (resp. negative) i-th coordinates.

One may similarly talk of lower and upper facets of $Z([0,n], i+1)$.

Definition 2.4 ([13, Thm. 2.1, Prop. 2.1], [5, Thm. 4.4])**.** The elements of $\mathcal{B}([0,n], i+1)$ consist of the cubillages of $Z([0,n], i+1)$. The covering relations of $\mathcal{B}([0,n], i+1)$ are given by pairs of cubillages $U \lessdot U'$ that differ by an *increasing flip*, that is when there is a $(i+2)$-face G of $Z([0,n], n+1)$ such that $(\bigcup_{F \in U} F) \setminus G = (\bigcup_{F' \in U'} F') \setminus G$ and such that U contains the lower facets of G, whereas U' contains the upper facets of G.

In Figure 1 we illustrate a pair of cubillages, where the right-hand cubillage is an increasing flip of the left. Here we have illustrated the cubillages using their images under π_2, but really they are sets of faces of the four-dimensional zonotope $Z([0,3], 4)$. We have labelled the vertices $\xi_A = \sum_{a \in A} \xi_a$ of $Z([0,3], 4)$ which lie in the cubillage by dropping the 'ξ' and only retaining the subscript 'A'; we will continue to do this.

The cyclic zonotope $Z([0,n], i+1)$ possesses two canonical cubillages. One is given by the set of faces $U_{\min}$ of $Z([0,n], n+1)$ that project to the lower facets of $Z([0,n], i+2)$ under the projection π_{i+2}. This is known as the *lower cubillage*. The other is given by the set of faces $U_{\max}$ that project to the upper facets of $Z([0,n], i+2)$, which we call the *upper cubillage*. The lower cubillage $U_{\min}$ of $Z([0,n], i+1)$ gives the unique minimum of the poset $\mathcal{B}([0,n], i+1)$, and the upper cubillage $U_{\max}$ gives the unique maximum. We have the following important theorem.

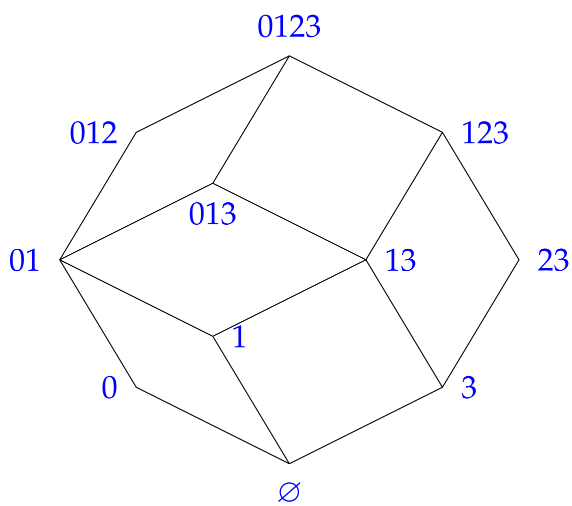

Figure 1: A pair of cubillages of $Z([0,3],2)$ such that the right is an increasing flip of the left.

Theorem 2.5 ([10, Thm. 2.3])**.** *There is a bijection between equivalence classes of maximal chains in $\mathcal{B}([0,n],i+1)$ and elements of $\mathcal{B}([0,n],i+2)$.*

The idea is that the $(i+2)$-dimensional faces which give covering relations in a maximal chain in $\mathcal{B}([0,n],i+2)$ give a cubillage of $Z([0,n],i+2)$. The equivalence relation mentioned in the theorem identifies maximal chains such that the set of these $(i+2)$-dimensional faces is the same.

Every $(i+1)$-dimensional face of $Z([0,n],n+1)$ is given by a Minkowski sum

$$\xi_A + \sum_{l\in L} \overline{\mathbf{0}\xi_l}$$

for some subset $L \in \binom{[0,n]}{i+1}$ and $A \subseteq [0,n] \setminus L$. We call L the set of *generating vectors* and A the *initial vertex*. In Figure 2, we illustrate the initial vertices and generating vectors of the cubillages from Figure 1. The initial vertices are labelled in blue, and the generating vectors are labelled in red in the centre of the cube. Given a cubillage $U \in \mathcal{B}([0,n],i+1)$, we write A_L^U for the initial vertex of the cube with generating vectors L in U. We also write $B_L^U := [0,n] \setminus (L \cup A_L^U)$ for the vectors which are neither generating vectors nor present in the initial vertex.

Given a cubillage U of $Z([0,n],i+1)$ and $k \in [0,n]$, we write U/k for the cubillage of $Z([0,n] \setminus k, i+1)$ given by the set of faces of $Z([0,n] \setminus k, n+1)$ which results from taking U and contracting all edges given by the vector ξ_k until they have length zero. For a

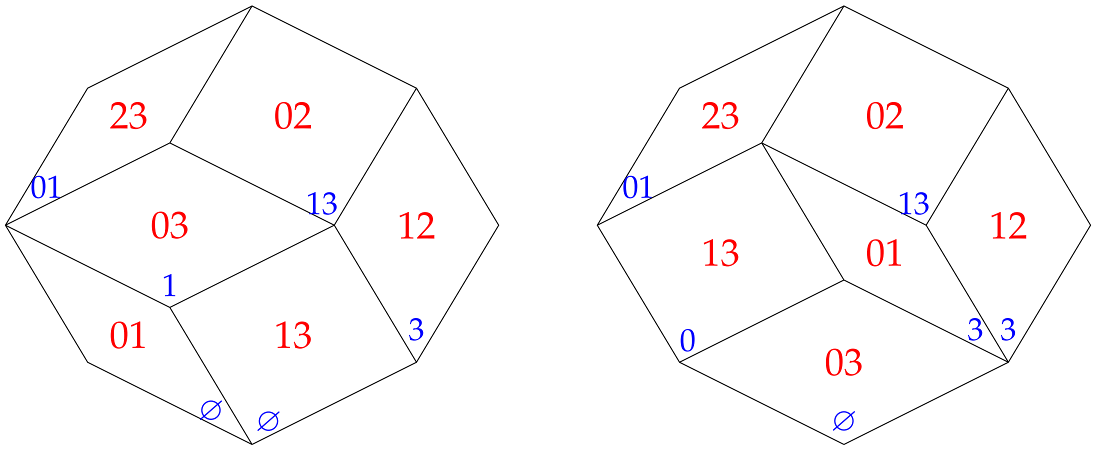

Figure 2: The pair of cubillages, with initial vertices and generating vectors indicated.

more precise construction, see [2, (4.3)], or for a construction using a different realisation of the higher Bruhat orders, see [7, Section 2.2.2].

3 Coproducts from cubillages

In this section, we show how one can construct a coproduct $\Delta_i^U\colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ from any cubillage $U \in \mathcal{B}([0,n], i+1)$. We show that all these coproducts give homotopies between Δ_{i-1} and $T\Delta_{i-1}$ and that all coproducts for which this is true arise from cubillages, provided they contain no redundant terms.

A central observation is as follows. Basis elements of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ are of the form $X \otimes Y$ for X and Y non-empty faces of Δ^n. Given a basis element $X \otimes Y \in C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$, we can always write $X = L \cup A$ and $Y = L \cup B$, where L, A, and B are pairwise disjoint. Given a subset $S \subseteq [0,n]$, we then say that $L \cup A \otimes L \cup B$ is *supported on* S if $L \cup A \cup B = S$.

Proposition 3.1. *There is a bijection between faces of $Z(S, |S|)$ excluding $\varnothing$ and S and basis elements of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ which are supported on S, given by sending a face with initial vertex A and generating vectors L to $L \cup A \otimes L \cup B$, where $B := S \setminus (L \cup A)$.*

Hence, we may identify basis elements of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ with the corresponding faces of $Z(S, |S|)$, in particular in the case $S = [0,n]$.

We illustrate in Figure 3 the elements of $C_\bullet(\Delta^3) \otimes C_\bullet(\Delta^3)$ assigned by Proposition 3.1 to the cubes of our running pair of cubillages. Compare this to the description of the generating vectors and initial points in Figure 2. The coproduct Δ_i^U associated to a cubillage U of $Z([0,n], i+1)$ is defined by setting $\Delta_i^U([0,n])$ as the sum of elements of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ assigned to the cubes of U.

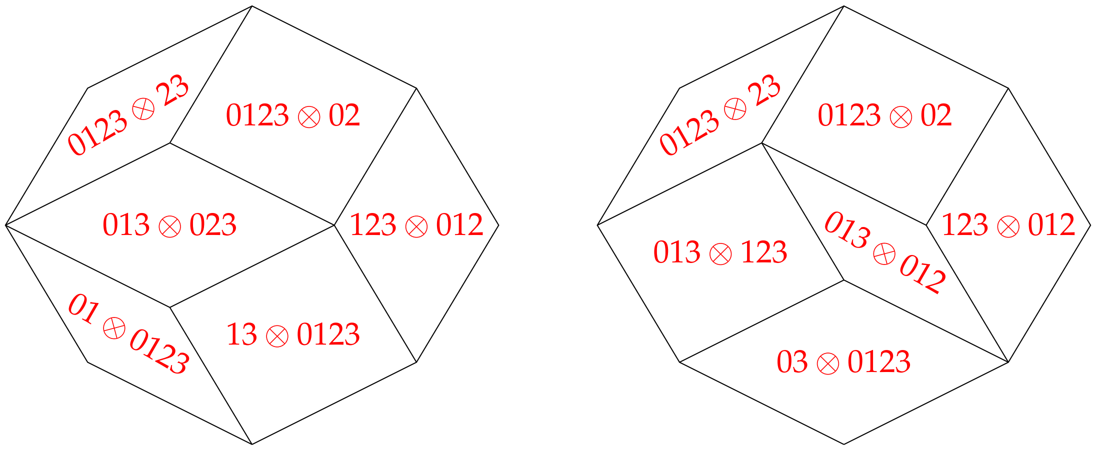

Figure 3: The terms of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ corresponding to the maximal cubes.

Construction 3.2. For any $U \in \mathcal{B}([0,n], i+1)$, where $n \geqslant i$, we now define the cup-i coproduct

$$\Delta_i^U \colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n).$$

We define Δ_i^U on the top face of Δ^n by the formula

$$\Delta_i^U([0,n]) := \sum_{L \in \binom{[0,n]}{i+1}} (-1)^{\varepsilon(L \cup A_L^U \otimes L \cup B_L^U)} L \cup A_L^U \otimes L \cup B_L^U,$$

where

$$\varepsilon(L \cup A_L^U \otimes L \cup B_L^U) := \sum_{b \in B_L^U} |A_L^U|_{<b} + \sum_{l \in L} |L|_{<l} + (n+1)|A_L^U|.$$

For codimension one faces, we define

$$\Delta_i^U([0,n] \setminus \{i\}) := \Delta_i^{U/i}([0,n] \setminus \{i\}).$$

In this way, we inductively extend the definition to lower-dimensional faces too. Once we reach a non-empty subset $S \subseteq [0,n]$ with $|S| \leqslant i$, we define $\Delta_i^U(S) := 0$, since in this case $\mathcal{B}(S, i+1)$ is empty.

The reader may wish to ignore the sign $\varepsilon(L \cup A_L^U \otimes L \cup B_L^U)$ for the purposes of this abstract. Many explicit sign calculations are carried out in [7, Appendix A].

3.1 Comparing with original Steenrod cup-i coproducts

Due to differing conventions, the Steenrod cup-i coproducts alternate between the minimal element of the higher Bruhat orders and the maximal element, according to the parity of i.

Theorem 3.3. *For i even, we have*

$$\Delta_i^{U_{\min}} = (-1)^{i/2}\Delta_i \quad \textit{and} \quad \Delta_i^{U_{\max}} = (-1)^{i/2}T\Delta_i,$$

whilst for i odd we have

$$\Delta_i^{U_{\min}} = (-1)^{\lceil i/2 \rceil}T\Delta_i \quad \textit{and} \quad \Delta_i^{U_{\max}} = (-1)^{\lfloor i/2 \rfloor}\Delta_i.$$

Sketch. Cubes of $U_{\min}$ or $U_{\max}$ with generating vectors L have initial points given by taking alternating parts of increasing overlapping paritions whose overlaps are given by L. This means that the terms of the coproducts coincide, up to applying T and adding a sign. □

3.2 Deriving the homotopy formula

The boundary of a term in the coproduct has a neat description in terms of the cubillage. Recalling Proposition 3.1, we may talk of upper and lower facets of a basis element F of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$, meaning the respective terms corresponding to the upper and lower facets of the face of $Z([0,n], n+1)$ corresponding to F. The following proposition follows from direct computation.

Proposition 3.4. *Let $F = L \cup A_L^U \otimes L \cup B_L^U$ be a basis element of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ supported on $[0,n]$. Adopting the notation $F/k := L \cup A_L^{U/k} \otimes L \cup B_L^{U/k}$, we have that*

$$\partial((-1)^{\varepsilon(F)}F) = \sum_{\substack{G \\ \textit{lower facet}}} (-1)^{\varepsilon(G)}G + \sum_{\substack{H \\ \textit{upper facet}}} (-1)^{\varepsilon(H)+1}H + \sum_{k \in [0,n]\setminus L} (-1)^{\varepsilon(F/k)+k+i+2}F/k.$$

Remark 3.5. Note that terms in $\partial((-1)^{\varepsilon(F)}F)$ are of one of the forms $L \setminus \{k\} \cup A_L^U \otimes L \cup B_L^U$, $L \cup A_L^U \otimes L \setminus \{k\} \cup B_L^U$, $L \cup A_L^U \setminus \{k\} \otimes L \cup B_L^U$, or $L \cup A_L^U \otimes L \cup B_L^U \setminus \{k\}$. The first and second of these may either be lower facets or upper facets, whilst the third and fourth give terms in the last sum in Proposition 3.4.

Showing that the coproduct Δ_i^U satisfies the homotopy formula is now straightforward.

Theorem 3.6 ([7, Theorem 3.6])**.** *For any $U \in \mathcal{B}([0,n], i+1)$, and for any $i \geqslant 0$, we have that*

$$\partial \circ \Delta_i^U - (-1)^i \Delta_i^U \circ \partial = (1 + (-1)^i T)\Delta_{i-1}^{U_{\min}}.$$

Sketch. Using Proposition 3.4, when we expand $\partial \circ \Delta_i^U([0,n])$ we see that terms from shared facets of cubes lying inside $Z([0,n], i+1)$ cancel, and that we are left with terms of the form $(-1)^{\varepsilon(F/k)+k+i+2}F/k$ and terms corresponding to the upper and lower facets of $Z([0,n], i+1)$. The former terms cancel with those from $(-1)^i\Delta_i^U \circ \partial$, and the latter terms give the right-hand side. □

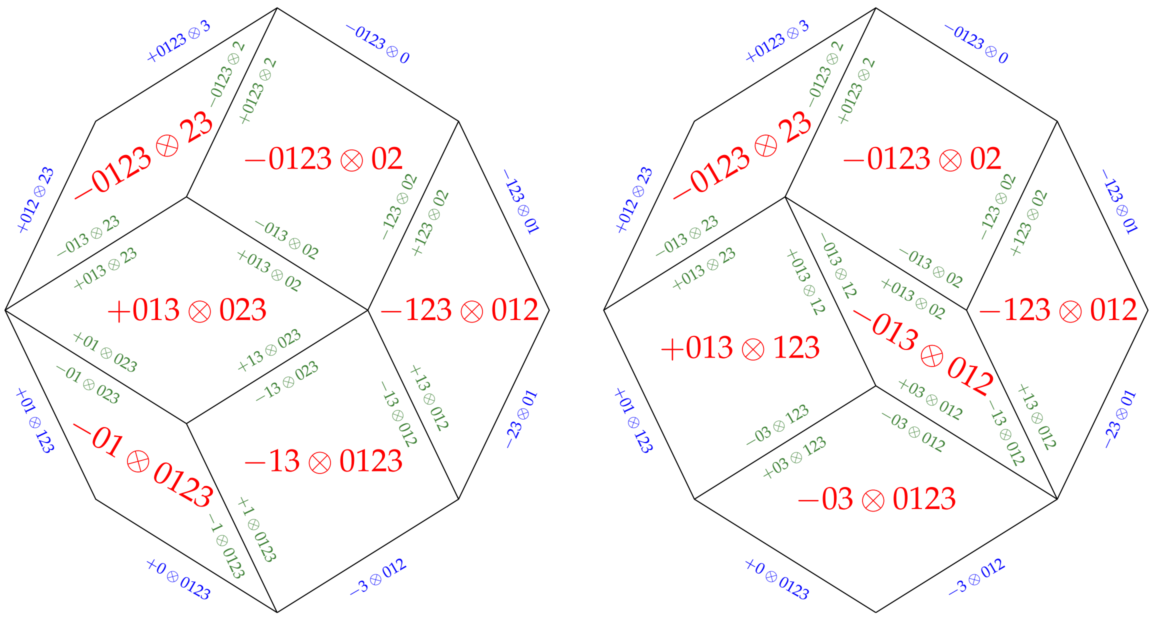

Figure 4: Illustrating why the homotopy formula holds.

Example 3.7. In Figure 4, we illustrate the proof of Theorem 3.6 for our two coproducts from Figure 3. The boundary of each red term from the cubes of the cubillage consists of terms coming from the facets of the cube F, along with terms of the form $(-1)^{\varepsilon(F/k)+k+i+2}F/k$. While we do not illustrate these latter terms, it can be seen that when we have two cubes sharing a facet inside the zonotope, the terms on the facet given by each cube have opposite sign, and so cancel. We are left with the terms on the boundary of the zonotope. The left-hand facets are the lower facets and correspond to terms of the usual Δ_0 cup-coproduct, whereas the terms on the right-hand facets are the terms of $-T\Delta_0$. Note that the terms on the boundary remain the same despite the different cubillages.

In fact, Construction 3.2 comprises *all* coproducts that satisfy the homotopy formula, up to redundancies. The idea is to run the proof of Theorem 3.6 in reverse, so that if a coproduct satisfies the homotopy formula, then the cubes corresponding to its terms must have come from a cubillage.

Theorem 3.8 ([7, Theorem 3.9]). *Suppose that we have a degree-i coproduct $\Delta'_i\colon \mathrm{C}_\bullet(\Delta^n) \to \mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ with $i \geqslant 0$, such that*

$$\partial \circ \Delta'_i - (-1)^i \Delta'_i \circ \partial = (1 + (-1)^i T)\Delta^{U_{\min}}_{i-1}. \tag{3.1}$$

1. *If, for $i > 0$, we have that for all non-empty $S \subseteq [0, n]$, $\Delta'_i(S)$ has a minimal number of*

terms amongst coproducts which satsify this formula, then we have that $\Delta_i' = \Delta_i^U$ for some $U \in \mathcal{B}([0,n], i+1)$.

2. *For $i = 0$, if we have that $\Delta_i'(p) = p \otimes p$ for all $p \in [0,n]$ and that $\Delta_i'(S)$ otherwise has a minimal number of terms for non-empty $S \subseteq [0,n]$, then we have $\Delta_i' = \Delta_i^U$ for some $U \in \mathcal{B}([0,n], 1)$.*

3.3 Extensions of the construction

There are several natural ways our construction can be extended.

1. One can consider cup-i coproducts on a whole simplicial complex, rather than a single simplex. In this case, for each maximal simplex in the simplicial complex, one can choose an element of the higher Bruhat orders. If these chosen elements agree on the intersections of the maximal simplices in an appropriate way, then they define a cup-i coproduct on the chain complex of the whole simplicial complex. The homotopy formula can then be verified simplex-by-simplex, as in Theorem 3.6.

   One interesting question is whether particular families of simplicial complexes give rise to other familiar objects on the combinatorial side.

2. It is natural to also consider cup-i coproducts on singular homology. Here, there are infinitely many singular simplices; the only feasible option is to assign the same element of the higher Bruhat orders to each of them. One can show that the only consistent way to do this is by assigning all of them the minimal elements, or all of them the maximal elements. Thus, the only cup-i coproducts that exist in the singular case are the Steenrod ones.

3. Instead of defining a homotopy from $T\Delta_{i-1}$ to Δ_{i-1}, one can instead consider homotopies from $T\Delta_{i-1}^U$ to Δ_{i-1}^U. Here the relevant posets are the "re-oriented higher Bruhat orders".

4. The cup-i coproducts give rise to the cohomology operations known as *Steenrod squares*. These operations are defined in mod 2 cohomology by $\mathrm{Sq}^i([\alpha]) = [\alpha \smile_i \alpha]$, where $\smile_i$ is the product which is the linear dual of the coproduct Δ_i. We show that different choices of elements of the higher Bruhat orders always produce the same Steenrod squares. There also exist mod p cohomology operations, which were described in terms of multi-arity versions of the cup-i products in [6].

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On the size of Bruhat intervals

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Abstract. For affine Weyl groups and elements associated to dominant coweights, we present a convex geometry formula for the size of the corresponding lower Bruhat intervals. Extensive computer calculations for these groups have led us to believe that a similar formula exists for all lower Bruhat intervals.

Keywords: Bruhat order, Affine Weyl groups, Polytopes, Lattice points

1 Introduction

In this extended abstract of the article [9], we study, for any affine Weyl group, the lower Bruhat interval for the element $\theta(\lambda)$ (see Definition 2.1) associated to a dominant coweight λ. These elements are intimately related to representation theory (character formulas for Lie groups, geometric Satake equivalence, quantum groups, among others). While calculating with indecomposable Soergel bimodules [12] and Kazhdan-Lusztig polynomials [4, 13], it became apparent that finding formulas for the cardinalities of lower Bruhat intervals played a crucial role. Surprisingly, little is known apart from length 2 (general) intervals [6, Lemma 2.7.3], lower intervals for smooth elements in Weyl groups [17, 14] and related results for affine Weyl groups [20, 7].

Our two main results relate the lower interval $\leq \theta(\lambda) := [\mathrm{id}, \theta(\lambda)]$, i.e. the elements below $\theta(\lambda)$ in the (strong) Bruhat order, with a certain convex polytope $\mathsf{P}(\lambda)$. We give a construction of $\leq \theta(\lambda)$ in terms of lattice points in $\mathsf{P}(\lambda)$. By using this construction, we then derive a formula which computes the cardinality of $\leq \theta(\lambda)$ as a linear combination of the volumes of the faces of $\mathsf{P}(\lambda)$. For the sake of clarity, we will first explain these results in a small example.

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Let us consider W the affine Weyl group of type $\widetilde{A}_2$, and the usual identification between elements in W and triangles (alcoves) in the tessellation of the plane by equilateral triangles. If x is an element of W, when we write $x \subset \mathbb{R}^2$, we mean the set of points in the closure of the alcove corresponding to x (the closed triangle). In Figure 1 we have the simple roots α_1 and α_2 in red and in blue, and the fundamental weights ϖ_1 and ϖ_2. The **id**-triangle is the fundamental alcove. For a dominant weight $\lambda \in X^+ := \mathbb{Z}_{\geq 0}\varpi_1 + \mathbb{Z}_{\geq 0}\varpi_2$ (depicted by a white dot in Figure 1), let $\theta(\lambda) \in W$ denote the λ-translate of the opposite of the fundamental alcove: those are the grey triangles.

Let also $P(\lambda)$ denote $\mathrm{Conv}(W_f \cdot \lambda)$, the convex hull of the orbit of λ under the finite Weyl group W_f. For $\lambda = 2\varpi_1 + \varpi_2$, it is the yellow hexagon in Figure 2. The faces of $P(\lambda)$ containing λ are

$$F_J := P(\lambda) \cap (\lambda + \sum_{i \in J} \mathbb{R}\alpha_i), \quad J \subset \{1, 2\}.$$

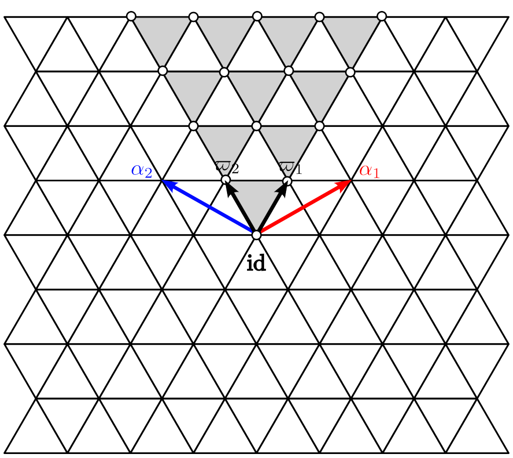

Figure 1

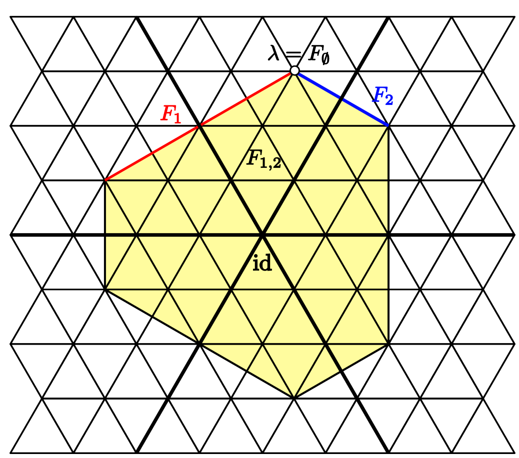

Figure 2

Consider the lattice $\mathcal{L} := \lambda + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$. Let $\lambda = 2\varpi_1 + \varpi_2$, as before. In Figure 3 the interval $\leq \theta(\lambda) := \{w \in W \mid w \leq \theta(\lambda)\}$ is colored in grey, and the green dots are the set $X_\lambda := P(\lambda) \cap \mathcal{L}$. Let $\mu \in X_\lambda$ and notice that there are six (grey) triangles adjacent to μ. Since the subgroup W_f of W corresponds to the six triangles adjacent to the origin (where the three thick lines meet), the triangles adjacent to μ are precisely the μ-translate of W_f. In fact, this describes all the grey triangles:

$$\leq \theta(\lambda) = \bigsqcup_{\mu \in X_\lambda} W_f + \mu. \tag{1.1}$$

In particular we get the following equation, which we call the *Lattice Formula*

$$|\leq \theta(\lambda)| = 6\,|P(\lambda) \cap \mathcal{L}|. \tag{1.2}$$

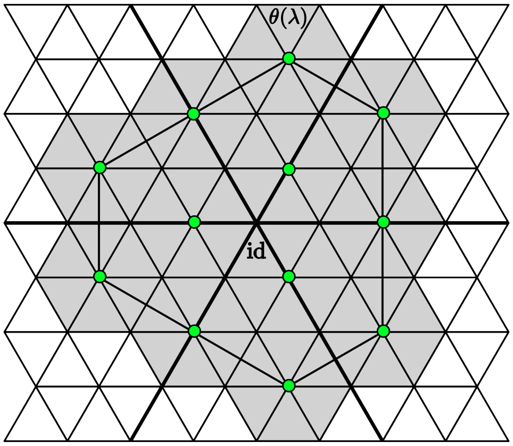

Figure 3: Lattice Formula

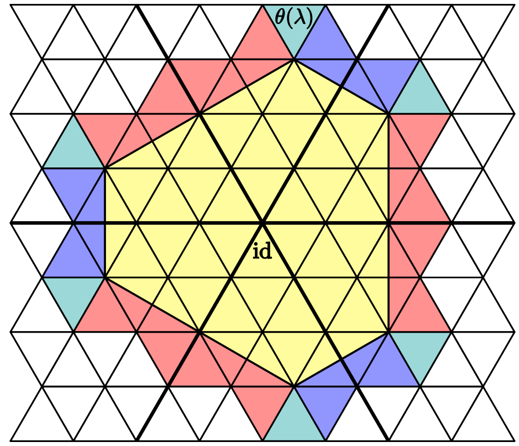

Figure 4: Geometric Formula

On the other hand, take the area of each colored part in Figure 4. By adding these areas and dividing by the area of any triangle, we get

$$|\leq \theta(\lambda)| = \mu_{1,2}\text{Area}(F_{1,2}) + \mu_1\text{Length}(F_1) + \mu_2\text{Length}(F_2) + \mu_\emptyset\text{Card}(F_\emptyset), \tag{1.3}$$

for some real numbers μ_J. That is, $\mu_{1,2}\text{Area}(F_{1,2})$ is the number of triangles in the yellow part, $\mu_1\text{Length}(F_1)$ corresponds to the red part, $\mu_2\text{Length}(F_3)$ to the blue part and the last term corresponds to the 6 turquoise triangles.

It is obvious that for a given λ, there are some μ's satisfying Equation (1.3). However it turns out that the coefficients μ's corresponding to the partition of Figure 4 do not depend on the choice of λ and that they are unique in this sense. We call this formula the *Geometric Formula*.

Remark 1.1. The reader may notice that the formula presented here bears strong similarities to Pick's theorem. For the proof of Theorem B, a generalization of the formula (1.3) applicable to any root system, we use a generalized version of Pick's theorem developed by Berline and Vergne. For more details see Section 3.2.

For any irreducible root system Φ one has an associated affine Weyl group W and one can define similar concepts as in the $\widetilde{A_2}$ case. For example, $\theta(\lambda)$ corresponds to the alcove touching λ in the direction of ρ (the sum of the fundamental weights). The following theorem, a generalization of Equation (1.2), builds the bridge between Coxeter combinatorics and convex geometry.

Theorem A (Lattice Formula)**.** *For every dominant coweight λ, we have*

$$|\leq \theta(\lambda)| = |W_f|\ |\text{Conv}(W_f \cdot \lambda) \cap (\lambda + \mathbb{Z}\Phi^\vee)|.$$

This formula is a key step to prove our main theorem below but it is also interesting in its own right, as we now explain. In [19] Postnikov studied permutohedra of general types. Among them, one of the most remarkable is the regular permutohedron of type A_n. The number of integer points of that polytope can be interpreted [21, §3] as the number of forests on $\{1, 2, \ldots, n\}$. There are other interpretations for the integer points of the regular permutohedron of type A_n, for instance, [1, Proposition 4.1.3] gives one as certain orientations of the complete graph. We remark that these interpretations are only for the regular permutohedron of type A_n. For non-regular permutohedra of any type, before the present paper, there was no interpretation of the integer points. Theorem A gives a first interpretation of this sort, and it is also of a different nature than the pre-existent ones in that it is not related to graph theory but to Coxeter theory.

This theorem also gives an interesting new insight. For a generic permutohedron (i.e. $\mathrm{Conv}(W_f \cdot \lambda)$ for some $\lambda \in \mathbb{Z}_{>0}\varpi_1 + \mathbb{Z}_{>0}\varpi_2$), the set of vertices is in bijection with the finite Weyl group $W_f = \{w \leq_R w_0\}$ where $\leq_R$ is the right weak Bruhat order on W_f and w_0 is the longest element. The Hasse diagram of $\leq_R$ on $\{w \leq_R w_0\}$ corresponds to the graph of the polytope.

Theorem A (or more precisely Proposition 2.4, a generalization of Equation (1.1)) says that if we consider the strong Bruhat order, the set $\leq \theta(\lambda)$ can be obtained from the lattice points inside the polytope. Heuristically, the weak Bruhat order gives the vertices of the polytope and the strong Bruhat order gives the lattice points inside the polytope.

Now we can present our main result. For $J \subseteq \{1, 2, \ldots, n\}$, one can define the face $F_J = \mathrm{Conv}(W_J \cdot \lambda)$ of $\mathrm{Conv}(W_f \cdot \lambda)$. See section 3.1 for more details.

Theorem B (Geometric Formula)**.** *For every rank n irreducible root system Φ, there are unique $\mu_J^{\Phi} \in \mathbb{R}$ such that for any dominant coweight λ,*

$$|\leq \theta(\lambda)| = \sum_{J \subset \{1,\ldots,n\}} \mu_J^{\Phi} \mathrm{Vol}(F_J),$$

Remark 1.2. This Theorem generalizes Equation (1.3). One should be careful with the intuition coming from type $\widetilde{A_2}$. In that small example, recall that the coefficients were determined by the partition in Figure 4. For a given λ one can always construct a partition $\mathcal{P}$ of (the alcoves of) $\leq \theta(\lambda)$ according to $\mathrm{Conv}(W_f \cdot \lambda)$, and then derive some coefficients μ's. It is fortuitous that in the $\widetilde{A_2}$ case, these coefficients coincide with the ones in the Geometric Formula. Already in $\widetilde{A_4}$ it is not true that $\mathrm{Conv}(W_f \cdot \lambda) \subset \leq \theta(\lambda)$, and in $\widetilde{A_{24}}$ there is a negative μ_J coefficient, so $\mu_J \mathrm{Vol}(F_J)$ is not the number of alcoves in some $p \in \mathcal{P}$.

Theorem B is proved by combining Theorem A with a particular formula for computing the number of lattice points developed by Berline-Vergne [5] and Pommersheim-Thomas [18]. The construction we use is part of a bigger family of formulae relating the number of lattice points of a polytope with the volumes of its faces, see [3, §6].

In [19], Postnikov gives several formulas for the volumes $\mathrm{Vol}(F_J)$ for any Φ. When Φ is the root system of type A_n, in Section 4 we give some geometric coefficients $\mu_J^{A_n}$.

The volumes are polynomials in the coordinates $m_1, \ldots, m_n$ of λ in the coweight basis. As a consequence of Theorem B we obtain that the size of the lower Bruhat intervals generated by $\theta(\lambda)$ is a polynomial function on the coordinates of λ.

2 Lattice Formula

We refer the reader to [11, 8] for more details about Weyl groups.

For the rest of this extended abstract, we fix an irreducible (reduced, crystallographic) root system Φ of rank n, and we denote by V be the ambient (real) Euclidean space spanned by Φ, with inner product $(-,-) : V \times V \to \mathbb{R}$.

Let $\alpha_1, \cdots, \alpha_n \in \Phi$ be a choice of simple roots. The *fundamental coweights* $\varpi_i^\vee$ are defined by the equations $(\varpi_i^\vee, \alpha_j) = \delta_{ij}$. They form a basis of V. A *coweight* is an integral linear combination of the fundamental coweights, and a *dominant coweight* is a coweight whose coordinates in this basis are non-negative. The set of coweights will be denoted by $\Lambda^\vee$.

For a root $\alpha \in \Phi$ and an integer $k \in \mathbb{Z}$, consider the hyperplane

$$H_{\alpha,k} = \{\lambda \in V \mid (\lambda, \alpha) = k\},$$

and the affine reflection $s_{\alpha,k}$ through this hyperplane. We write $s_i := s_{\alpha_i,0}$, for $1 \le i \le n$, and $s_0 := s_{\widetilde{\alpha},-1}$, where $\widetilde{\alpha}$ is the highest root. The *affine Weyl group* W is the group generated by $S := \{s_0, s_1, \ldots, s_n\}$. We have that (W, S) is a Coxeter system. We denote by $\le$ the (strong) Bruhat order on W: $u \le w$ if u can be obtained by deleting some letters of a reduced word for w. For $J \subset S$, the *parabolic subgroup* W_J is the subgroup of W generated by J. The *finite Weyl group* W_f is the parabolic subgroup of W generated by $S_f := \{s_1, \ldots, s_n\}$. It has a maximal element w_0 with respect to $\le$.

An alcove is a connected component of $V \setminus \cup_{\alpha,k} H_{\alpha,k}$. The closure of an alcove is a fundamental domain for the action of W on V. The *fundamental alcove* is the simplex

$$A_{\mathrm{id}} := \{\lambda \in V \mid -1 < (\lambda, \alpha) < 0, \ \forall \alpha = \alpha_1, \ldots, \alpha_n, \widetilde{\alpha}\}.$$

We have a bijection $w \mapsto A_w := wA_{\mathrm{id}}$ between W and the set of alcoves.

The *coroot* $\alpha^\vee$ corresponding to a root $\alpha \in \Phi$ is $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. The lattice $\Lambda^\vee$ contains $\mathbb{Z}\Phi^\vee$ as a subgroup of finite index. Consider the group $\Omega := \Lambda^\vee/\mathbb{Z}\Phi^\vee$. Define $v_i = -\varpi_i^\vee$ for $1 \le i \le n$ and let v_0 be the zero vector. Define $M := \{i \mid (\varpi_i^\vee, \widetilde{\alpha}) = 1\}$. The set $\{v_0, v_i \mid i \in M\}$ is a complete system of representatives of Ω. This group classifies all parabolic subgroups of W that are isomorphic to W_f. We will denote by W_σ the parabolic subgroup corresponding to $\sigma \in \Omega$. It is the subgroup generated by $S \setminus \{s_i\}$, where $\sigma = v_i$ in Ω. From now on, we will identify Ω with the representatives $\{v_0, v_i \mid i \in M\}$.

Definition 2.1. Let λ be a dominant coweight. Since $A_{w_0}+\lambda$ is an alcove, there exists a unique element $\theta(\lambda)\in W$ such that $A_{\theta(\lambda)}=A_{w_0}+\lambda$. See Figure 1 for an example.

For any $X\subset W$, let $A(X)$ be the union of alcoves corresponding to X. That is, $A(X)=\sqcup_{x\in X}A_x$. The following Lemma captures the geometric intuition needed to prove Theorem A.

Lemma 2.2. *Let λ be a dominant coweight and let $\sigma\in\Omega$ such that $\lambda\in\sigma+\mathbb{Z}\Phi^\vee$. Then,*

1. $A(W_\sigma)=A(W_f)+\sigma$.
2. $A(\theta(\lambda)W_\sigma)=A(W_f)+\lambda$.
3. *$\theta(\lambda)$ is maximal with respect to the Bruhat order in its double coset $W_f\theta(\lambda)W_\sigma$.*
4. *The maximal elements of the double cosets in $\bigsqcup_{\sigma\in\Omega}W_f\backslash W/W_\sigma$, are precisely the θ-elements.*

Definition 2.3. For any $\lambda\in V$, we define the *orbit polytope* $\mathsf{P}^\Phi(\lambda)$ as the convex polytope whose vertex set is the W_f-orbit of λ. See Figure 2 for an example.

As long as λ is not the zero vector, the orbit polytope is always full dimensional.

Using Lemma 2.2, we can derive the following Proposition, which describes the alcoves corresponding to $\leq\theta(\lambda)$ in terms of lattice points in $\mathsf{P}^\Phi(\lambda)$.

Proposition 2.4. *For every dominant coweight λ, we have*

$$A\Big(\leq\theta(\lambda)\Big)=\bigsqcup_{\mu\in X_\lambda}A(W_f)+\mu,\tag{2.1}$$

where $X_\lambda=\mathsf{P}^\Phi(\lambda)\cap(\lambda+\mathbb{Z}\Phi^\vee)$.

Then, by counting alcoves in Equation (2.1), we get the Lattice Formula.

Theorem 2.5 (Lattice Formula)**.** *For every dominant coweight λ, we have*

$$|\leq\theta(\lambda)|=|W_f|\,|\mathsf{P}^\Phi(\lambda)\cap(\lambda+\mathbb{Z}\Phi^\vee)|.\tag{2.2}$$

3 Geometric Formula

3.1 Faces of the orbit polytope and their volumes

For any $X\subset V$ we denote by $\mathsf{Conv}(X)$ the convex hull of X. Let λ be a dominant coweight. The faces of the orbit polytope $\mathsf{P}^\Phi(\lambda)$ are given by

$$\mathsf{F}(w,J)=w\mathsf{Conv}(W_J\cdot\lambda),$$

where $J\subset S_f$ and w ranges over any representatives of W/W_J. In particular, the facets of $\mathsf{P}^\Phi(\lambda)$ containing λ, are precisely $\mathsf{F}(\mathrm{id},S_f\setminus\{s_i\})$ for $1\leq i\leq n$.

Definition 3.1. For a subset $J \subset S_f$, we define $V_J^\Phi(\lambda)$ as the $|J|$-dimensional volume of the face $\mathsf{F}(\mathrm{id}, J)$ of $\mathsf{P}^\Phi(\lambda)$.

It will turn out that the volumes $V_J^\Phi(\lambda)$ can be seen as polynomials, as we now explain. For simplicity, suppose $J = S_f$ and that λ is *generic*[1], i.e. its coordinates (in the fundamental coweight basis) are strictly positive. We can decompose $\mathsf{P}^\Phi(\lambda)$ into pyramids having the facets of $\mathsf{P}^\Phi(\lambda)$ as their bases, and the zero vector as their apex. Thus we can compute the n-dimensional volume of $\mathsf{P}^\Phi(\lambda)$, i.e. $V_{S_f}^\Phi(\lambda)$, by adding up the volumes of these pyramids. After considering symmetries, we get the following equation.

$$V_{S_f}^\Phi(\lambda) = \frac{1}{n}\sum_{j=1}^{n}\left[W : W_{S_f\setminus\{s_j\}}\right]\frac{(\lambda, \varpi_j^\vee)}{\|\varpi_j^\vee\|}\, V_{S_f\setminus\{s_j\}}^\Phi(\lambda). \tag{3.1}$$

Now let $\mathbf{m} = (m_1, \ldots, m_n)$ be a n-tuple of positive integers. Define $V_{S_f}^\Phi(\mathbf{m}) := V_{S_f}^\Phi(m_1\varpi_1^\vee + \ldots + m_n\varpi_n^\vee)$. It is clear that the term $(\lambda, \varpi_j^\vee)$ (coming from the height of the pyramids) is a polynomial in $m_1, \ldots, m_n$. Since $V_\varnothing^\Phi(\lambda) = 1$, Equation (3.1) implies that $V_{S_f}^\Phi(\mathbf{m})$ is a homogeneous polynomial of degree n in $m_1, \ldots, m_n$, by induction.

For any $J \subset S_f$ and dominant coweight λ, a similar formula to Equation (3.1) allows us to see the volumes $V_J^\Phi(\lambda)$ as polynomials. Furthermore, we can deduce their linear independence. We collect this in the following Lemma (for more details, see [9, §4]).

Lemma 3.2. *Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a n-tuple of non-negative integers. For $J \subset S_f$, define $V_J^\Phi(\mathbf{m}) := V_J^\Phi(m_1\varpi_1^\vee + \ldots + m_n\varpi_n^\vee)$.*

- *$V_J^\Phi(\mathbf{m})$ is a homogeneous polynomial of degree $|J|$ in the variables m_j, for $j \in J$ (identifying J with a subset of $\{1, 2, \ldots, n\}$).*
- *The polynomials $V_J^\Phi(\mathbf{m})$ with $J \subset S_f$ are linearly independent.*

Remark 3.3. To compare our results to Potnikov's formulas for the volumes, suppose Φ has type A_n. In this case, $\mathsf{P}^\Phi(\lambda)$ is a permutohedron. Our variables $m_1, \ldots, m_n$ correspond to the variables $u_1, \ldots, u_n$ in [19, §16]. There is a missing scalar factor of $\sqrt{n+1}$, which is the Euclidean volume of the fundamental parallelepiped spanned by the simple roots, but his formulas are scaled so that its volume is 1.

[1] In the literature, a coweight is *regular* if it is not orthogonal to any root. Thus, a dominant coweight is generic if and only if it is regular.

3.2 Counting lattice points

For any (possibly non-pointed) cone C that includes the origin, we define its *polar* as

$$\mathsf{C}^\circ = \{v \in V \ : \ (v, w) \leq 0, \forall w \in \mathsf{C}\}.$$

Let $\Gamma \subset V$ be a lattice.

Definition 3.4. Let P be a full dimensional lattice polytope, that is, a convex polytope whose vertices lie in Γ. For a face $\mathsf{F} \subset \mathsf{P}$ let H be its affine span, L the corresponding linear subspace and $\pi : V \to L^\perp$ the orthogonal projection. We define four cones:

- The **normal cone** $\mathsf{n}(\mathsf{F}, \mathsf{P}) = \mathrm{cone}\{u_\mathsf{G} \ : \ \mathsf{G} \text{ is a facet such that } \mathsf{F} \subset \mathsf{G}\}$, where u_G is an outer normal for the facet $\mathsf{G} \subset \mathsf{P}$.
- The **feasible cone** $\mathsf{f}(\mathsf{F}, \mathsf{P})$ is the polar of the normal cone $\mathsf{n}(\mathsf{F}, \mathsf{P})$.
- The **supporting cone** $\mathsf{s}(\mathsf{F}, \mathsf{P}) := H + \mathsf{f}(\mathsf{F}, \mathsf{P})$. It is a translation of the feasible cone.
- The **transverse cone** $\mathsf{t}(\mathsf{F}, \mathsf{P}) = \pi(\mathsf{s}(\mathsf{F}, \mathsf{P}))$.

We say that a pointed cone C is rational if its vertex is a lattice point and every ray (1-dimensional face) contains a lattice point. The following is the *Euler-Maclaurin formula* developed by Berline and Vergne [5] (see also [2, Chapters 19-20] for an exposition). There exists a function ν on pointed rational cones such that the following is true for all lattice polytopes P.

$$|\mathsf{P} \cap \Gamma| = \sum_{\mathsf{F} \subseteq \mathsf{P}} \nu\left(\mathsf{t}(\mathsf{F}, \mathsf{P})\right) \mathrm{relVol}(\mathsf{F}), \tag{3.2}$$

where the sum is indexed over all nonempty faces of P. The relative volume $\mathrm{relVol}(\mathsf{F})$ of a face is the volume on its affine span H normalized with respect to the lattice $\Gamma \cap L$, where L is the linear subspace parallel to H. More precisely,

$$\mathrm{relVol}(\mathsf{F}) = \frac{\mathrm{Vol}(\mathsf{F})}{\det(\Gamma \cap L)}. \tag{3.3}$$

Remark 3.5. To be more precise, Berline and Vergne's main construction in [5] is a function μ that maps pointed rational cones to meromorphic functions [5, §4]. In this paper we only use the function ν which is μ evaluated at zero [5, Definition 25], and then Equation (3.2) is equivalent to [5, Theorem 26] when the function h is the constant function equal to 1.

We remark that for a single polytope P, it is obvious that there will be a formula resembling Equation (3.2). The interesting part of Berline-Vergne's theorem is that the ν function satisfies Equation (3.2) for all lattice polytopes simultaneously and has certain local properties. Namely, the following operations do not change the ν value of a transverse cone.

i Applying a lattice-preserving orthogonal transformation.

ii Translating by a lattice element.

We use these tools to prove Theorem B, which we restate for the reader's convenience.

Theorem 3.6 (Geometric Formula)**.** *For every irreducible root system Φ, there are unique $\mu_J^\Phi \in \mathbb{R}$ such that for any dominant coweight λ,*

$$|\leq \theta(\lambda)| = \sum_{J \subset S_f} \mu_J^\Phi V_J^\Phi(\lambda). \tag{3.4}$$

The sketch of the proof is as follows. We focus on proving the existence of the coefficients, since Lemma 3.2 implies uniqueness.

Let λ be a dominant coweight. The polytope $\mathsf{Q}^\Phi(\lambda) := \mathsf{P}^\Phi(\lambda) - \lambda$ is a lattice polytope with respect to the lattice $\mathbb{Z}\Phi^\vee$. Note that the Lattice Formula, Theorem 2.5, yields

$$|\leq \theta(\lambda)| = |W_f|\ |\mathsf{P}^\Phi(\lambda) \cap (\lambda + \mathbb{Z}\Phi^\vee)| = |W_f|\ |\mathsf{Q}^\Phi(\lambda) \cap \mathbb{Z}\Phi^\vee|. \tag{3.5}$$

Applying Berline-Vergne formula (3.2), we get

$$|\leq \theta(\lambda)| = |W_f| \sum_{\mathsf{F} \subseteq \mathsf{Q}^\Phi(\lambda)} \nu\left(\mathsf{t}(\mathsf{F}, \mathsf{Q}^\Phi(\lambda))\right) \mathrm{relVol}(\mathsf{F}). \tag{3.6}$$

The faces of the lattice polytope $\mathsf{Q}^\Phi(\lambda)$ are $\mathsf{G}_J(w,\lambda) := \mathsf{F}_J(w,\lambda) - \lambda$ for all pairs $w \in W_f$ and $J \subset S_f$. We define $\mathsf{G}_J(\lambda) := \mathsf{G}_J(\mathrm{id},\lambda)$. Recall that a generic dominant coweight is a positive integer linear combination of the fundamental coweights.

Lemma 3.7. *Let λ be a **generic** dominant coweight and $J \subset S_f$. Then*

1. *The ν value of the transverse cone of $\mathsf{G}_J(\lambda)$ in $\mathsf{Q}^\Phi(\lambda)$ is independent of λ.*
2. *The ν value of the transverse cones of $\mathsf{G}_J(\lambda)$ and $\mathsf{G}_J(\lambda, w)$ are equal for all $w \in W_f$.*
3. *For $w \in W_f$ we have that $\mathrm{Vol}(\mathsf{G}_J(\lambda)) = \mathrm{Vol}(\mathsf{G}_J(\lambda, w))$. Furthermore, $\mathrm{relVol}(\mathsf{G}_J(\lambda)) = \mathrm{relVol}(\mathsf{G}_J(\lambda, w))$.*

Combining Lemma 3.7 and Equation (3.6), we get the existence in the generic case.

Proposition 3.8 (Existence in the generic case)**.** *For every irreducible root system Φ, there exists $\mu_J^\Phi \in \mathbb{R}$ such that for any **generic** dominant coweight λ,*

$$|\leq \theta(\lambda)| = \sum_{J \subset S_f} \mu_J^\Phi V_J^\Phi(\lambda). \tag{3.7}$$

On the other hand, we can express $\mathsf{Q}^{\Phi}(\lambda)$ as a Minkowski sum $m_1\mathsf{Q}^{\Phi}(\varpi_1^\vee)+\cdots+m_n\mathsf{Q}^{\Phi}(\varpi_n^\vee)$, where $\lambda = m_1\varpi_1^\vee+\cdots+m_n\varpi_n^\vee$. Using Equation (3.5), we get the quasi-polynomiality of $|\leq\theta(\lambda)|$ (see [15, Theorem 7]).

Proposition 3.9 (Quasi-polynomiality)**.** *For every dominant coweight $\lambda=\sum_i m_i\varpi_i^\vee$ (generic or not), we have that $|\leq\theta(\lambda)|$ is a quasi-polynomial in $m_1,\ldots,m_n$.*

We now prove the Geometric Formula.

Proof of Theorem 3.6. Proposition 3.8 together with the fact that V_J^Φ are polynomials (by Lemma 3.2) imply that $|\leq\theta(\lambda)|=\sum\mu_J^\Phi V_J^\Phi(\lambda)$ is a polynomial in the coordinates $m_1,\ldots,m_n$ of λ (in the fundamental coweight basis) when they are positive integers. By Proposition 3.9 we know that $|\leq\theta(\lambda)|$ is in general a quasi-polynomial in the m_i's. Put $\mathbf{m}=(m_1,\ldots,m_n)$. We have a polynomial $\sum\mu_J^\Phi V_J^\Phi(\mathbf{m})$ agreeing with the quasi-polynomial $|\leq\theta(\mathbf{m})|$ on the set $\mathbb{Z}^n_{>0}$. Thus, they must agree on $\mathbb{Z}^n_{\geq 0}$. Therefore, formula (3.7) holds for every dominant coweight λ, generic or not, giving the existence in every case.

Finally, by Lemma 3.2, the volume polynomials are linearly independent hence the coefficients μ_J^Φ are unique. □

A direct consequence of the Geometric Formula 3.6, is that if Φ has rank n and $\lambda=(m_1,\ldots,m_n)$ in the fundamental coweight basis, then $|\leq\theta(\lambda)|$ is a polynomial of degree n in the $m_1,\ldots,m_n$. Taking the sum over a fixed rank $|J|=d$ gives the degree d part of the polynomial. We call the coefficients μ_J^Φ the *geometric coefficients*.

4 On the geometric coefficients μ_J^Φ

We finish by giving some values of the geometric coefficients. The coefficient corresponding to the empty set is easily determined. Using the Geometric Formula (3.4), we get

$$\mu_\emptyset^\Phi=\sum_{J\subseteq S_f}\mu_J^\Phi V_J^\Phi(\mathbf{0})=|\leq\theta(\mathbf{0})|=|\leq w_0|=|W_f|.$$

The coefficient corresponding to the set S_f also has a nice expression.

Lemma 4.1. *Let $Vol(A_{id})$ be the n-dimensional volume of the fundamental alcove. Then*

$$\mu_{S_f}^\Phi=\frac{1}{Vol(A_{id})}.$$

In Table 1, we show the values of $\mu_{S_f}^\Phi$, which were computed using [8, Plates I, ..., VI].

Type	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\mu^{\Phi}_{S_f}$	$\frac{(n+1)!}{\sqrt{n+1}}$	$n!2^{n-1}$	$n!2^n$	$n!2^{n-4}$	$24\sqrt{3}\cdot 6!$	$288\sqrt{2}\cdot 7!$	$17280\cdot 8!$	576	$12\sqrt{3}$

Table 1: Values of the geometric coefficient $\mu^{\Phi}_{S_f}$.

Now let Φ be the root system of type A_n and let D be the corresponding Dynkin diagram. We say that $J \subseteq S_f$ is connected if the subgraph of D corresponding to J is connected. For example, $\{s_1, s_2, \dots, s_l\} \subset S_f$ is connected for every $1 \le l \le n$.

In [9, §6.2], we compute the geometric coefficients $\mu^{A_n}_J$ for connected subsets $J \subseteq S_f$. To achieve this, we use the following Lemma.

Lemma 4.2. *For all $m \in \mathbb{Z}_{\ge 0}$, and for all $1 \le k \le n$,*

$$| \le \theta(m\varpi_k)| = (n+1)!\, E_{k,n+1}(m), \tag{4.1}$$

where $E_{k,n+1}$ is the Ehrhart polynomial of the hypersimplex

$$\Delta_{k,n+1} = \left\{ x \in [0,1]^{n+1} \mid x_1 + \cdots + x_{n+1} = k \right\}.$$

In [10], the author gave a polynomial expansion of $E_{k,d}(m)$. On the other hand, the polynomial expansion of $| \le \theta(m\varpi_k)| \in \mathbb{R}[m]$ via the Geometric Formula 3.6, depends on the polynomials $V^{A_n}_J(m\varpi_k)$. They are of the form $V^{A_n}_J(m\varpi_k) = c_{k,J} m^{|J|}$, for some number $c_{k,J}$ (depending on the Eulerian numbers [16, A008292]). The connectedness of J is necessary (but not sufficient) to assure that $c_{k,J} \neq 0$. After comparing coefficients in Equation (4.1), we get a system of linear equations which, upon solving, gives all the geometric coefficients of connected sets.

For example, for every $1 \le l \le n$,

$$\mu^{A_n}_{\{s_1, s_2, \dots, s_l\}} = \frac{l!}{\sqrt{l+1}}(n+1) \begin{bmatrix} n+1 \\ l+1 \end{bmatrix}, \tag{4.2}$$

where the brackets denote the (unsigned) Stirling numbers of the first kind [16, A008275].

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Vines and MAT-labeled graphs

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Abstract. The present note explores a connection between two concepts arising from different fields of mathematics. The first concept, called vine, is a graphical model for dependent random variables. This concept first appeared in a work of Joe (1994), and the formal definition was given later by Cooke (1997). Vines have nowadays become an active research area whose applications can be found in probability theory and uncertainty analysis. The second concept, called MAT-freeness, is a combinatorial property in the theory of freeness of logarithmic derivation module of hyperplane arrangements. This concept was first studied by Abe-Barakat-Cuntz-Hoge-Terao (2016), and soon afterwards investigated further by Cuntz-Mücksch (2020).

In the particular case of graphic arrangements, the last two authors (2023) recently proved that the MAT-freeness is completely characterized by the existence of certain edge-labeled graphs, called MAT-labeled graphs. In this paper, we first introduce a poset characterization of a vine. Then we show that, interestingly, there exists an explicit equivalence between the categories of locally regular vines and MAT-labeled graphs. In particular, we obtain an equivalence between the categories of regular vines and MAT-labeled complete graphs.

Several applications will be mentioned to illustrate the interaction between the two concepts. Notably, we give an affirmative answer to a question of Cuntz-Mücksch that MAT-freeness can be characterized by a generalization of the root poset in the case of graphic arrangements.

Keywords: MAT-labeling, graph, poset, vine copula, hyperplane arrangement, MAT-freeness

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1 Motivation

The starting point of our note is a question of Cuntz-Mücksch [8] (Question 1.3) in the theory of *free hyperplane arrangements*.

Let V be a finite dimensional vector space. A **hyperplane** in V is a 1-codimensional linear subspace of V. Let $\{x_1, \dots, x_\ell\}$ be a basis for the dual space V^*. Any hyperplane in V can be described by a linear equation of the form $a_1x_1 + \cdots + a_\ell x_\ell = 0$ where at least one of the a_i's is non-zero.

A **hyperplane arrangement** $\mathcal{A}$ is a finite set of hyperplanes in V. The **intersection lattice** of $\mathcal{A}$ is the set of all intersections of hyperplanes in $\mathcal{A}$, which is often referred to as the **combinatorics** of $\mathcal{A}$. An arrangement is said to be **free** if its *module of logarithmic derivations* is a free module. For basic definitions and properties of free arrangements, we refer the interested reader to [17, 14]. Freeness is an algebraic property of hyperplane arrangements which has been a major topic of research since the 1970s. A central question in the theory is to study the freeness of an arrangement by combinatorial structures, especially by the intersection lattice of the arrangement.

Among others, *MAT-freeness* is an important concept which was first used by Abe-Barakat-Cuntz-Hoge-Terao [1] to settle the conjecture of Sommers-Tymoczko [15] on the freeness of *ideal subarrangements of Weyl arrangements*. This concept is formally defined later by Cuntz-Mücksch [8] and we will use their definition throughout. For a hyperplane $H \in \mathcal{A}$, define the **restriction** $\mathcal{A}^H$ of $\mathcal{A}$ to H by

$$\mathcal{A}^H := \{K \cap H \mid K \in \mathcal{A} \setminus \{H\}\}.$$

Definition 1.1 (MAT-partition and MAT-free arrangement [8])**.** Let $\mathcal{A}$ be a nonempty arrangement. A partition (disjoint union of nonempty subsets) $\pi = (\pi_1, \dots, \pi_n)$ of $\mathcal{A}$ is called an **MAT-partition** if the following three conditions hold for every $1 \le k \le n$.

1. The hyperplanes in π_k are linearly independent.
2. There does not exist $H' \in \mathcal{A}_{k-1}$ such that $\bigcap_{H \in \pi_k} H \subseteq H'$, where $\mathcal{A}_{k-1} := \pi_1 \sqcup \cdots \sqcup \pi_{k-1}$ (disjoint union) and $\mathcal{A}_0 := \emptyset$ is the empty arrangement.
3. For each $H \in \pi_k$, $|\mathcal{A}_{k-1}| - |(\mathcal{A}_{k-1} \cup \{H\})^H| = k - 1$.

An arrangement is called **MAT-free** if it is empty or admits an MAT-partition.

As the name suggests, any MAT-free arrangement is a free arrangement. This follows from the remarkable Multiple Addition Theorem by Abe-Barakat-Cuntz-Hoge-Terao [1, Theorem 3.1] (justifying the abbreviation MAT). MAT-freeness is a helpful combinatorial tool (as it depends only on the intersection lattice) to examine the freeness of arrangements. One of its most famous applications we mentioned earlier is a proof that the ideal subarrangements of Weyl arrangements are free. The MAT-freeness has received increasing attention in recent years, see [2, 3, 13, 7] for some other applications.

Let $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot, \cdot)$. Let Φ be an irreducible (crystallographic) root system in V, with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta := \{\alpha_1, \dots, \alpha_\ell\}$. For $\alpha \in \Phi$, define $H_\alpha := \{x \in V \mid (\alpha, x) = 0\}$. For $\Sigma \subseteq \Phi^+$, the **Weyl subarrangement** $\mathcal{A}_\Sigma$ is defined by $\mathcal{A}_\Sigma := \{H_\alpha \mid \alpha \in \Sigma\}$. In particular, $\mathcal{A}_{\Phi^+}$ is called the **Weyl arrangement**.

We can make Φ^+ into a *poset* (partially ordered set) by defining a partial order $\leq$ on Φ^+ as follows: $\beta_1 \leq \beta_2$ if $\beta_2 - \beta_1 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i$. The poset $(\Phi^+, \leq)$ is called the **root poset** of Φ. For an *ideal* I (Definition 2.7) of the root poset Φ^+, the corresponding Weyl subarrangement $\mathcal{A}_I$ is called the **ideal subarrangement**.

Theorem 1.2 ([1, Theorem 1.1])**.** *Any ideal subarrangement $\mathcal{A}_I$ is MAT-free, hence free.*

The ideal subarrangements form a significant subclass of MAT-free arrangements. However, there are many MAT-free arrangements (or MAT-partitions of a given MAT-free arrangement) that do not arise from ideal subarrangements (Example 3.7). One may wonder if the hyperplanes in an arbitrary MAT-free arrangement satisfy some poset structure similar to the root poset? This question was asked by Cuntz-Mücksch [8] and is the main motivation of our work.

Question 1.3 ([8, Problem 47])**.** *Given an MAT-free arrangement $\mathcal{A}$, can we characterize all possible MAT-partitions of $\mathcal{A}$ by a poset structure generalizing the classical root poset?*

Cuntz-Mücksch's question is difficult in general as the number of different MAT-partitions of a given MAT-free arrangement might be very large. Also, the definition of an MAT-partition itself does not reveal a natural choice of the desirable partial order. In the present note, we pursue this question along *graphic arrangements*, a well-behaved class of arrangements in which both freeness and MAT-freeness are completely characterized by combinatorial properties of graphs.

Let G be a simple graph (i.e. no loops and no multiple edges) with vertex (or node) set $N_G = \{v_1, \dots, v_\ell\}$ and edge set E_G. The **graphic arrangement** $\mathcal{A}_G$ is an arrangement in an ℓ-dimensional vector space V defined by

$$\mathcal{A}_G := \{x_i - x_j = 0 \mid \{v_i, v_j\} \in E_G\}.$$

A graph is **chordal** if it does not contain an induced *cycle* of length greater than three. A chordal graph is **strongly chordal** if it does not contain a *sun graph* as an induced subgraph. Here an ***n*-sun** S_n ($n \geq 3$) is a graph with vertex set $N_{S_n} = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ and edge set

$$E_{S_n} = \{\{u_i, u_j\} \mid 1 \leq i < j \leq n\} \cup \{\{v_i, u_j\} \mid 1 \leq i \leq n, j \in \{i, i+1\}\},$$

where we let $u_{n+1} = u_1$.

Theorem 1.4 ([16], [9, Theorem 3.3])**.** *The graphic arrangement $\mathcal{A}_G$ is free if and only if G is chordal.*

Theorem 1.5 ([18, Theorem 2.10])**.** *The graphic arrangement $\mathcal{A}_G$ is MAT-free if and only if G is strongly chordal.*

While the definition of an MAT-free arrangement may seem technical at first glance, Theorem 1.5 enables us to view MAT-freeness as a rather natural property. Furthermore, the correspondence between MAT-freeness and strong chordality establishes a nice analog[1] of the classical correspondence between freeness and chordality.

The good thing about graphs is that MAT-partition of a graphic arrangement can be rephrased in terms of a special edge-labeling of graphs, the so-called *MAT-labeling* (Definition 2.1). A graph together with such a labeling is called an **MAT-labeled graph**. To approach Question 1.3 for graphic arrangements, the first question would be how many non-isomorphic MAT-labelings can a (strongly chordal) graph have? A computation aided by computer for complete graphs on up to 8 vertices gives us the sequence $1, 1, 1, 2, 6, 40, 560, 17024$. Surprisingly, we found out that this sequence coincides with the number of equivalence classes of *(graphical) regular vines* (or *R-vines*) in dimension up to 8 given in [12, §10.3]. This observation is indeed compelling as it leads us to the notion of the *node poset* of a *graphical vine* (Definitions 2.9 and 2.10), which is a perfect candidate for the poset structure we are looking for.

2 Definitions

2.1 MAT-labeled graphs

All graphs in this paper are undirected, finite and simple. Let $G = (N_G, E_G)$ be a graph with the set N_G of vertices (or nodes) and the set E_G of edges (unordered pairs of vertices). In this paper, a vertex and a node in a graph are synonyms. The former will be used more often for graphs, while the latter will be used for an element in a poset.

An **edge-labeled graph** is pair (G, λ) where G is a simple graph and $\lambda \colon E_G \longrightarrow \mathbb{Z}_{>0}$ is a map, called **(edge-)labeling**. The following definition of an MAT-labeling is equivalent to the original one in [18, Definition 4.2].

Definition 2.1 (MAT-labeling)**.** Let (G, λ) be an edge-labeled graph. For $k \in \mathbb{Z}_{>0}$, let $\pi_k := \lambda^{-1}(k) \subseteq E_G$ denote the set of edges of label k. Define $\pi_{\le k} := \pi_1 \sqcup \cdots \sqcup \pi_k$ and $\pi_{<1} := \varnothing$. The labeling λ is an **MAT-labeling** if the following two conditions hold for every $k \in \mathbb{Z}_{>0}$.

[1] Many important concepts in the classical theory such as *simplicial vertex* and *perfect elimination ordering* of chordal graphs have their analogs in MAT-labeled graphs (see [18] for more details).

1. Any edge $e \in \pi_{\leq k}$ does not form a cycle with edges in π_k.
2. Every edge $e \in \pi_k$ forms exactly $k-1$ triangles with edges in $\pi_{<k}$.

Given an edge $e \in \pi_k$, a **conditioning vertex** of e is a vertex that together with the endvertices of e forms two edges both of label $< k$. Condition (2) above can be rephrased as every edge e of label k has exactly $k-1$ conditioning vertices.

Definition 2.2 (MAT-labeled (complete) graph)**.** An edge-labeled graph (G, λ) is an **MAT-labeled graph** if λ is an MAT-labeling of G. In particular, an MAT-labeled graph (G, λ) is an **MAT-labeled complete graph** if G is a complete graph.

MAT-partition of a graphic arrangement is nothing but MAT-labeling of the underlying graph [18, Proposition 4.3]. Thus, MAT-free graphic arrangement and MAT-labeled graph are essentially the same object.

Recall that a **clique** of a graph is a subset of vertices such that every two distinct vertices in the clique are adjacent.

Lemma 2.3 (Principal clique)**.** *Let (G, λ) be an MAT-labeled graph. Let $e = \{i, j\} \in \pi_k$ be an edge in G of label k and $h_1, \ldots, h_{k-1}$ be the conditioning vertices of e. Then the set $K_e := \{i, j, h_1, \ldots, h_{k-1}\}$ is a clique of G. We call K_e the* ***principal clique generated by*** *e.*

Definition 2.4 (Label-preserving isomomorphism)**.** Let (G, λ) and (G', λ') be two edge-labeled graphs. A **label-preserving homomorphism** from (G, λ) to (G', λ'), written $\sigma \colon (G, \lambda) \longrightarrow (G', \lambda')$ is a map $\sigma \colon N_G \longrightarrow N_{G'}$ such that for all $u, v \in N_G$, $\{u, v\} \in E_G$ implies $\{\sigma(u), \sigma(v)\} \in E_{G'}$ and $\lambda(u, v) = \lambda'(\sigma(u), \sigma(v))$.

We call σ an **isomorphism** if σ is bijective and its inverse is a label-preserving homomorphism. The edge-labeled graphs (G, λ) and (G', λ') are said to be **isomorphic**, written $(G, \lambda) \simeq (G', \lambda')$ if there exists an isomorphism $\sigma \colon (G, \lambda) \longrightarrow (G', \lambda')$. If $(G, \lambda) \simeq (G, \lambda')$, we say that two labelings λ and λ' are the same (or isomorphic).

If $(G, \lambda) \simeq (G', \lambda')$ and (G, λ) is an MAT-labeled graph, then (G', λ') is also an MAT-labeled graph.

Definition 2.5 (Category of MAT-labeled (complete) graphs)**.** The **category** MG **of MAT-labeled graphs** is the category whose objects are the MAT-labeled graphs and whose morphisms are the label-preserving homomorphisms. The **category** MCG **of MAT-labeled complete graphs** is a full subcategory of MG whose objects are the MAT-labeled complete graphs.

2.2 Vines: graphical and poset definitions

All posets $\mathcal{P} = (\mathcal{P}, \leq_{\mathcal{P}})$ in this note are finite. Denote by $\max(\mathcal{P})$ (resp. $\min(\mathcal{P})$) the set of all maximal (resp. minimal) elements in a poset $\mathcal{P}$.

Definition 2.6 (Graded poset)**.** A finite poset $\mathcal{P}$ is **graded** if there exists a **rank function** $\mathrm{rk} = \mathrm{rk}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathbb{Z}_{\geq 0}$ satisfying the following three properties:

1. For any $x, y \in \mathcal{P}$, if $x < y$ then $\mathrm{rk}(x) < \mathrm{rk}(y)$.
2. If y covers x, then $\mathrm{rk}(x) = \mathrm{rk}(y) - 1$.
3. All minimal elements of $\mathcal{P}$ have the same rank. In this note, we assume[2] $\mathrm{rk}(x) = 1$ for all $x \in \min(\mathcal{P})$.

Equivalently, for every $x \in \mathcal{P}$, all maximal chains among those with x as greatest element have the same length.

The **dimension**[3] $\dim(\mathcal{P})$ of $\mathcal{P}$ is defined as $\dim(\mathcal{P}) := |\min(\mathcal{P})|$. The **rank** $\mathrm{rk}(\mathcal{P})$ of a graded poset $\mathcal{P}$ with rank function rk is defined as

$$\mathrm{rk}(\mathcal{P}) := \max\{\mathrm{rk}(x) \mid x \in \mathcal{P}\}.$$

Definition 2.7 (Ideal, principal ideal)**.** Let $\mathcal{P}$ be a poset. An (order) **ideal** $\mathcal{I}$ of $\mathcal{P}$ is a downward-closed subset, i.e. for every $x \in \mathcal{P}$ and $y \in \mathcal{I}$, $x \leq y$ implies that $x \in \mathcal{I}$. For $a \in \mathcal{P}$, the ideal

$$\mathcal{P}_{\leq a} := \{x \in \mathcal{P} \mid x \leq a\}$$

is called the **principal** ideal of $\mathcal{P}$ generated by a.

Definition 2.8 (Poset homomorphism)**.** Let $\mathcal{P}$ and $\mathcal{P}'$ be posets. A **(poset) homomorphism** $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ is an order-preserving map, i.e. $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ for all $x, y \in \mathcal{P}$. We call φ a **join-preserving** homomorphism if for any $x, y \in \mathcal{P}$ such that the join $x \vee y$ exists, then $\varphi(x) \vee \varphi(y)$ exists and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. We call φ an **isomorphism** if φ is bijective and its inverse is a homomorphism. The posets $\mathcal{P}$ and $\mathcal{P}'$ are said to be **isomorphic**, written $\mathcal{P} \simeq \mathcal{P}'$ if there exists an isomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$. When $\mathcal{P} = (\mathcal{P}, \mathrm{rk})$ and $\mathcal{P}' = (\mathcal{P}', \mathrm{rk}')$ are graded posets, a homomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ is called **rank-preserving** if $\mathrm{rk}'(\varphi(x)) = \mathrm{rk}(x)$ for all $x \in \mathcal{P}$.

Now we recall the graphical definition of a *vine* following [4, Definition 4.1].

Definition 2.9 (Graphical definition of vine)**.** Let $1 \leq n \leq \ell$ be positive integers. A **(graphical) vine** $\mathcal{V}$ on ℓ elements $[\ell] = \{1, \dots, \ell\}$ (or more generally, on an ℓ-element set called N_1) is an ordered n-tuple $\mathcal{V} = (F_1, F_2, \dots, F_n)$ such that

[2] A motivation for this assumption is the equivalence between D-vine and root poset of type A (Remark 3.4). The latter is graded by heights of positive roots, and all the minimal elements (simple roots) have rank (height) 1.

[3] The term "dimension" of a poset may have a different meaning in the other context. The present definition is to make a compatibility for dimensions of a vine (Remark 2.11) and the ambient space of graphic arrangements.

1. F_1 is a forest with nodes $N_1 = [\ell]$ and a set of edges denoted E_1,
2. for $2 \le i \le n$, F_i is a forest with nodes $N_i = E_{i-1}$ and edge set E_i.

We call F_i the **i-th associated forest** of $\mathcal{V}$. A graphical vine is uniquely determined by its associated forests. Denote by $N(\mathcal{V}) = N_1 \cup \cdots \cup N_n$ the set of nodes (of the associated forests) of $\mathcal{V}$. We call the numbers n and ℓ the **rank** and **dimension** of $\mathcal{V}$, respectively.

If node u is an element of node v, i.e. $u \in v$, we say that u is a **child** of v. If v is reachable from u via the membership relation: $u \in u_1 \in \cdots \in v$, we say that u is a **descendant** of v.

Definition 2.10 (Node poset)**.** Let $\mathcal{V}$ be a graphical vine with node set $N(\mathcal{V})$. The **node poset** $\mathcal{P} = \mathcal{P}(\mathcal{V})$ of $\mathcal{V}$ is the poset $(N(\mathcal{V}), \le)$ defined as follows: For any $u, v \in N(\mathcal{V})$,

$$u \le v \quad \text{if} \quad u \text{ is a descendant of } v.$$

Remark 2.11*.* We emphasize that a graphical vine is uniquely determined by its node poset. The terminology "rank" of a vine has motivation from poset theory. If a vine $\mathcal{V}$ is an ordered n-tuple, then $\mathcal{P} = \mathcal{P}(\mathcal{V})$ is a graded poset with rank function $\mathrm{rk}(v) = i$ for $v \in N_i$ $(1 \le i \le n)$. Thus this number n equals the rank of $\mathcal{P}$. In addition, the dimension of $\mathcal{V}$ equals the number of minimal elements in $\mathcal{P}$, or the dimension of $\mathcal{P}$.

Assumption & Notation 2.12. From now on, unless otherwise stated we assume that $\mathcal{P}$ is a finite graded poset with a rank function $\mathrm{rk} : \mathcal{P} \longrightarrow \mathbb{Z}_{>0}$. Denote $n := \mathrm{rk}(\mathcal{P})$ and $\ell := \dim(\mathcal{P})$. For $v \in \mathcal{P}$, denote by $\mathcal{E}(v)$ the set of elements covered by v. For $i \ge 0$, define $\mathcal{P}_i := \{v \in \mathcal{P} \mid \mathrm{rk}(v) = i\}$ and $\mathcal{E}(\mathcal{P}_i) := \{\mathcal{E}(v) \mid v \in \mathcal{P}_i\}$. If $\mathcal{P}$ is an ℓ-dimensional poset, we assume $\mathcal{P}_1 = \min(\mathcal{P}) = [\ell]$.

As noted earlier in Remark 2.11, we may think of a graphical vine and its node poset essentially as the same object. It is thus natural to look for a characterization of the node poset of a vine. We give below such a characterization obtained immediately from Definition 2.9.

Definition & Proposition 2.13 (Poset definition of vine)**.** A finite graded poset $\mathcal{P}$ is the node poset of a graphical vine if and only if $\mathcal{P}$ satisfies the following conditions:

1. Every non-minimal node covers exactly two other nodes, and any two distinct nodes of the same rank are covered by at most one node.
2. For each $1 \le i \le n = \mathrm{rk}(\mathcal{P})$, the graph $F_i = (N_i, E_i)$ with node set $N_i := \mathcal{P}_i$ and edge set $E_i := \mathcal{E}(\mathcal{P}_{i+1})$ is a forest.

Assumption & Notation 2.14. From now on, unless otherwise stated, by a vine $\mathcal{P}$ we mean a finite graded poset satisfying the two conditions in 2.13. We will also retain the notion i-th associated forest $F_i = (\mathcal{P}_i, \mathcal{E}(\mathcal{P}_{i+1}))$ $(1 \le i \le n)$ of $\mathcal{P}$. If v is a node in a vine $\mathcal{P}$ and $\mathcal{E}(v) = \{a, b\}$, we will often abuse notation and write $v = \{a, b\}$. This notation is compatible with the notation of node/edge in the graphical definition of a vine.

The main reason why we choose the poset definition of a vine is because many terms and properties of a (graphical) vine have natural meanings in the language of posets. Under this consideration, the following poset definition of a *regular vine* is equivalent to the well-known graphical definition of it in the literature, e.g. [4, Definition 4.1].

Definition 2.15 (R-vine)**.** A vine $\mathcal{P}$ is a **regular vine**, or an *R-vine* for short, if $\mathcal{P}$ satisfies the following conditions:

1. $\mathrm{rk}(\mathcal{P}) = \dim(\mathcal{P})$, i.e. $n = \ell$.
2. Each associated forest $F_i = (\mathcal{P}_i, \mathcal{E}(\mathcal{P}_{i+1}))$ is a tree ($1 \leq i \leq n$).
3. **Proximity**: For any distinct nodes $a, b \in \mathcal{P}_i$ for $i \geq 2$, if a and b are covered by a common node, then a and b cover a common node.

Next we introduce the notion of a *locally* regular vine.

Definition 2.16 (LR-vine)**.** A vine $\mathcal{P}$ is a **locally regular vine**, or an *LR-vine* for short, if every principal ideal of $\mathcal{P}$ is an R-vine.

Remark 2.17*.* Intuitively, an LR-vine is a vine that "locally" looks like an R-vine. In particular, any R-vine is an LR-vine. Any ideal of a vine (resp. an LR-vine) is itself a vine (resp. an LR-vine).

The following theorem indicates the equivalence between the ideals of an R-vine and LR-vines.

Theorem 2.18. *Let $\mathcal{P}$ be a vine. The following are equivalent:*

1. *$\mathcal{P}$ is an ideal of an R-vine.*
2. *$\mathcal{P}$ satisfies the proximity condition.*
3. *$\mathcal{P}$ is an LR-vine.*

Definition 2.19 (Category of (L)R-vines)**.** The **category** LRV **of LR-vines** is the category whose objects are the LR-vines and whose morphisms are the homomorphisms preserving rank and join. The **category** RV **of R-vines** is a full subcategory of LRV whose objects are the R-vines.

3 The main result

Having introduced the concepts, we are ready to state our main result.

Theorem 3.1. *The categories* MG *and* LRV *are equivalent. In particular, the categories* MCG *and* RV *are equivalent.*

To prove the equivalence between MG and LRV, we construct two functors $\Psi\colon \mathsf{MG} \longrightarrow \mathsf{LRV}$ and $\Omega\colon \mathsf{LRV} \longrightarrow \mathsf{MG}$. The former amounts to constructing an LR-vine from a given MAT-labeled graph which is presented in Theorem 3.2 below. The proof is direct and largely dependent upon the notion of *MAT-perfect elimination ordering* developed in an earlier work of the last two authors [18]. The argument on the functor Ω is however more complicated, and the details are omitted.

Theorem 3.2. *Let (G, λ) be an MAT-labeled graph with $N_G = [\ell]$. Define a finite graded poset $\mathcal{P} = (\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$ from (G, λ) as follows:*

1. *$\mathcal{P}$ consists of the sets $\{i\}$ for $1 \leq i \leq \ell$ and all the principal cliques in (G, λ) (Lemma 2.3).*
2. *For $u, v \in \mathcal{P}$, $u \leq_{\mathcal{P}} v$ if u is a subset of v.*
3. *$\mathrm{rk}_{\mathcal{P}}(v) = |v|$ for all $v \in \mathcal{P}$.*

Then the poset $\mathcal{P}$ is an LR-vine. In particular, if (G, λ) is an MAT-labeled complete graph, then $\mathcal{P}$ is an R-vine.

We give two examples to illustrate the construction in Theorem 3.2.

Definition 3.3 (D-vine)**.** An R-vine is called a **D-Vine** if each associated tree has the smallest possible number of vertices of degree 1. Equivalently, each associated tree is a *path graph*.

Remark 3.4. Let Φ be an irreducible root system in $\mathbb{R}^\ell$ with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Suppose that Φ is of type A_ℓ and the Dynkin diagram of Φ is the path graph $\alpha_1 - \alpha_2 - \cdots - \alpha_\ell$. Then the positive roots of Φ are given by

$$\Phi^+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \,\middle|\, 1 \leq i \leq j \leq m \right\}.$$

It is not hard to show that the D-vine $\mathcal{P}$ with the first associated tree $1 - 2 - \cdots - \ell$ is isomorphic to the root poset of type A_ℓ.

Example 3.5. Figure 1 depicts a 4-dimensional D-vine (middle) that can be constructed in three ways. First, it is the node poset of a graphical vine on [4] (left). Second, it is the poset defined an MAT-labeled complete graph (right) via Theorem 3.2. Third, it is the root poset of type A_4 by Remark 3.4. The elements in the poset are written without set symbol for simplicity. The conditioned set of a non-minimal node is given to the left of the "|" sign, while the conditioning set appears on the right. For example, the top node $\{1, 2, 3, 4\}$ (or the largest clique generated by $\{v_1, v_4\}$) is written by $14|23$.

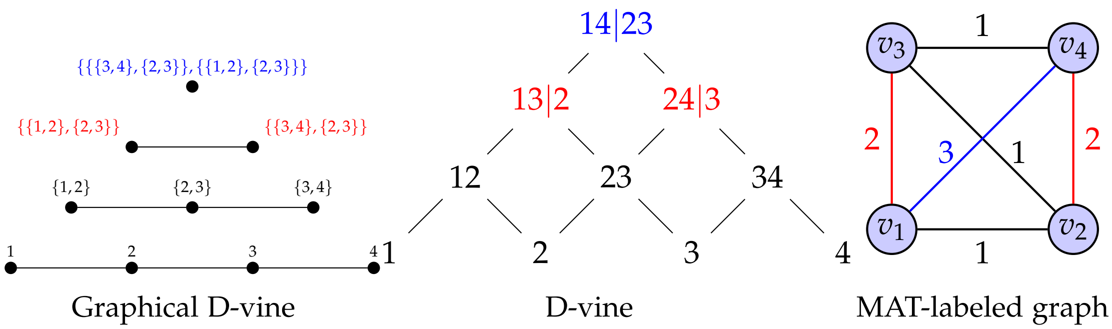

Figure 1: An MAT-labeled complete graph on 4 vertices (right), the D-vine (middle) (= type A root poset) defined by the graph via Theorem 3.2, and the corresponding graphical vine (left).

Definition 3.6 (C-vine)**.** An R-vine is called a **C-Vine** if each associated tree has the largest possible number of vertices of degree 1. Equivalently, each associated tree is a *star graph*.

D-vine and C-vine can be regarded as the "extreme" cases of R-vines.

Example 3.7. In dimension 4, there are exactly two non-isomorphic R-vine structures: D-vine and C-vine. Likewise, there are exactly two non-isomorphic MAT-labeled complete graphs on 4 vertices. Figure 2 depicts a graphical C-vine on $[4]$ (left), the corresponding node poset (middle), and the corresponding MAT-labeled complete graph (right) via Theorem 3.2. The C-vine in dimension ≥ 4 is not an ideal of any D-vine hence the corresponding MAT-partition is not obtained from an ideal of the type A root poset.

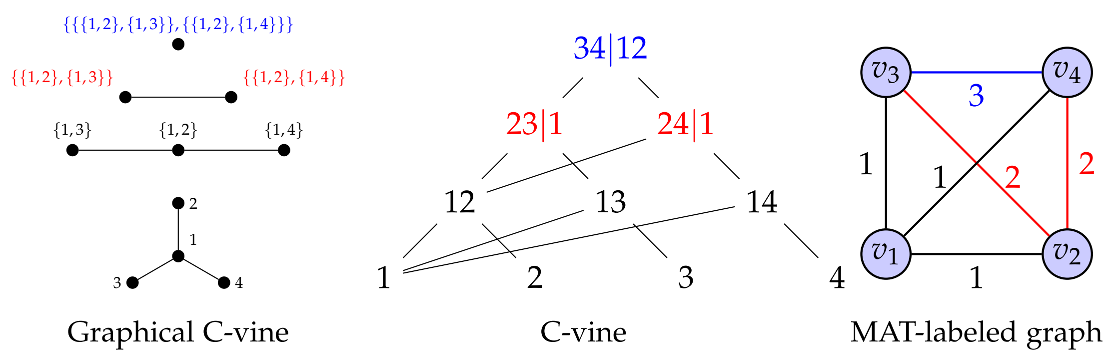

Figure 2: C-vine on 4 elements and the corresponding graphical version, MAT-labeled complete graph from Example 3.7.

4 Applications

From the view point of category theory, the equivalence establishes a strong similarity between the categories and allows many properties and structures to be translated from one to the other. We obtain two main applications from LR-vines to MAT-labeled graphs. First, LR-vine is an answer for Question 1.3 in the case of graphic arrangements. We find it interesting that although the class of MAT-free arrangements is strictly larger than that of ideal subarrangements in general, any MAT-free *graphic* arrangement is characterized by being an ideal of a poset structure (Theorem 2.18). Second, an explicit formula for the number of non-isomorphic MAT-labelings of complete graphs is obtained. This equals the number of equivalence classes of regular vines whose explicit formula is known [12, §10.3].

A vine is a graphical tool for representing the joint distribution of random variables. The first construction of a vine was given by Joe [10], and the formal definition was given and refined further by Cooke, Bedford and Kurowicka [5, 4, 11]. Vines have been studied extensively and proved to have various applications in probability theory and related areas. We refer the reader to [12] for a comprehensive handbook of vines. Our main result gives a new appearance and applications of vines in the arrangement theory. In the present note, we do not pursue the probabilistic or applied aspects of vines (neither does the proof of the main result) but emphasize and develop more on the theoretical aspects. In the full version of this extended abstract, we give several new combinatorial properties of vines, hoping that they will be useful for the future research on vines. For instance, we give an alternative way to associate an *m-vine* to a strongly chordal graph compared with the work of Zhu-Kurowicka [19], and an extension of the notion of *sampling order* [6] from R-vine to LR-vine.

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Configuration spaces and peak representations

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Abstract. *Eulerian* idempotents of types A and B generate representations with topological interpretations, as the cohomology of configuration spaces of types A and B. We provide an analogous cohomological interpretation for the representations generated by idempotents in the *peak algebra*, called the *peak representations*. We describe the peak representations as sums of *Thrall's higher Lie characters*, give Hilbert series and branching rule recursions for them, and discuss connections to Jordan algebras.

Keywords: Peak algebra, configuration spaces, Solomon's descent algebra, higher lie characters, hyperplane arrangements, Varchenko-Gelfand ring, Type A, Type B

1 Introduction

This abstract concerns the cohomology $H^*X = H^*(X, \mathbf{k})$ with coefficients in a field $\mathbf{k}$ for three different topological configuration spaces $X = X_n, Y_n, Z_n$ having large symmetry groups W. For each, the (ungraded) cohomology carries the regular representation of W, that is, $H^*X \cong \mathbf{k}\,W$. Our goal is to study and exploit the following surprising fact: for $\mathbf{k}$ of characteristic zero, the decomposition into H^iX matches a combinatorial direct sum decomposition for certain complete families $\{E_i\}$ of *orthogonal idempotents* in $\mathbf{k}\,W$:

$$H^*X = \bigoplus_i H^iX \qquad \cong \qquad \bigoplus_i (\mathbf{k}\,W)E_i = \mathbf{k}\,W. \tag{1.1}$$

The first two spaces X_n, Y_n are well-studied: X_n is the *ordered configuration space* of n points in $\mathbb{R}^3$ while Y_n is the $\mathbb{Z}_2$*-orbit configuration spaces* for the $\mathbb{Z}_2$-action via $\mathbf{x} \mapsto -\mathbf{x}$:

$$X_n := \mathrm{Conf}_n\,\mathbb{R}^3 = \{\mathbf{x} \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for } 1 \leq i < j \leq n\},$$
$$Y_n := \mathrm{Conf}_n^{\mathbb{Z}_2}\,\mathbb{R}^3 = \{\mathbf{x} \in (\mathbb{R}^3)^n : x_i \neq \pm x_j \text{ for } 1 \leq i < j \leq n, \text{ and } x_i \neq 0 \text{ for } 1 \leq i \leq n\}$$

Note that X_n has an action of the *symmetric group* $W = \mathfrak{S}_n$ permuting the coordinates of $\mathbf{x}$, while Y_n carries an action of the *hyperoctahedral group* $W = \mathfrak{S}_n^{\pm}$ by permuting and negating coordinates. Both spaces have cohomology concentrated only in even degrees and total cohomology carrying the regular representation $\mathbf{k}\,W$ for $W = \mathfrak{S}_n, \mathfrak{S}_n^{\pm}$.

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The idempotent decompositions of $\mathbf{k}\,\mathfrak{S}_n$ and $\mathbf{k}\,\mathfrak{S}_n^{\pm}$ will come from the *type A and B Eulerian idempotents* $\{E_k^{\mathfrak{S}_n}\}_{k=0,1,\ldots,n-1}$ in $\mathbf{k}\,\mathfrak{S}_n$ and $\{E_k^{\mathfrak{S}_n^{\pm}}\}_{k=0,1,\ldots,n}$ in $\mathbf{k}\,\mathfrak{S}_n^{\pm}$, defined in work of Reutenauer [13], Gerstenhaber–Schack [10], and F. Bergeron and N. Bergeron [4].

The Eulerian idempotents lie within the subalgebras of the group algebras $\mathbf{k}\,W$ known as *Solomon's descent algebra* $\mathrm{Sol}(W)$, meaning that when expressed as $\sum_{w\in W} c_w w$, their coefficients c_w depend only upon the Coxeter group *descent set* of w. Work of Hanlon [11], Sundaram-Welker [16] and Brauner [6] gives a correspondence between these objects:

$$H^{2k}X_n \cong (\mathbf{k}\,\mathfrak{S}_n)\, E_{n-1-k}^{\mathfrak{S}_n} \text{ for } k = 0, 1, \ldots, n-1, \tag{1.2}$$

$$H^{2k}Y_n \cong \left(\mathbf{k}\,\mathfrak{S}_n^{\pm}\right) E_{n-k}^{\mathfrak{S}_n^{\pm}} \quad \text{for } k = 0, 1, \ldots, n. \tag{1.3}$$

In this abstract, we use (1.2) and (1.3) as the starting point to give a third correspondence of the form (1.1) for the space $Z_n := Y_n / \mathbb{Z}_2^n \cong \mathrm{Conf}_n(\mathbb{RP}^2 \times (0,\infty))$, where $\mathbb{Z}_2^n$ is the normal subgroup of $\mathfrak{S}_n^{\pm}$ consisting of sign changes; thus $\mathfrak{S}_n \cong \mathfrak{S}_n^{\pm} / \mathbb{Z}_2^n$ acts on Z_n.

The idempotents $\{E_k^{\mathcal{P}_n}\}$ in this new correspondence lie inside the *peak algebra* $\mathcal{P}_n$, which is the further subalgebra of $\mathrm{Sol}(\mathfrak{S}_n)$ inside $\mathbf{k}\,\mathfrak{S}_n$ whose elements $\sum_{w\in W} c_w w$ have coefficients c_w depending only upon the *peak set* of $w = (w_0 := 0, w_1, \ldots, w_n)$

$$\mathrm{Peak}(w) := \{i : 1 \le i \le n-1 \text{ and } w_{i-1} < w_i > w_{i+1}\}.$$

Our main contribution is to relate the *peak representations* $(\mathbf{k}\,\mathfrak{S}_n)\, E_{n-k}^{\mathcal{P}_n}$ to the cohomology ring H^*Z_n, and to explicitly describe these families of representations in terms of Thrall's famed *higher Lie characters* Lie_λ for λ an integer partition of n.

Theorem 1.1. *Let* $\mathbf{k}$ *be a field of characteristic zero.*

(i) The peak idempotent $E_k^{\mathcal{P}_n}$ *in* $\mathbf{k}\,\mathfrak{S}_n$ *vanishes unless* $k \equiv n \bmod 2$.

(ii) The cohomology $H^iZ_n = H^i(Z_n, \mathbf{k})$ *vanishes unless* $i \equiv 0 \bmod 4$.

(iii) As a $\mathfrak{S}_n$*-representation, the total cohomology carries the regular representation:*

$$H^*Z_n \cong \mathbf{k}\,\mathfrak{S}_n.$$

(iv) For $0 \le k \le n$ *with* k *even, one has* $\mathfrak{S}_n$*-representation isomorphisms*

$$(\mathbf{k}\,\mathfrak{S}_n)\, E_{n-k}^{\mathcal{P}_n} \cong H^{2k}Z_n \cong \bigoplus_{\substack{\lambda \vdash n: \\ \mathrm{odd}(\lambda) = n-k}} \mathrm{Lie}_\lambda,$$

where $\mathrm{odd}(\lambda)$ *is the number of odd parts of* λ.

In fact, we refine Theorem 1.1 (see Theorems 4.4 and 4.6) by introducing several (compatible) decompositions of H^*Z_n and a family of primitive idempotents in $\mathcal{P}_n$.

Although $\mathcal{P}_n$ is a well-known subalgebra of $\mathrm{Sol}(\mathfrak{S}_n)$, it is in general difficult to directly relate the two algebras. Our work offers a step in this direction. The novelty of our approach is to avoid computations in the algebras themselves, and instead develop and utilize concrete combinatorial descriptions of the rings H^*X_n, H^*Y_n, and H^*Z_n.

The remainder of the abstract proceeds as follows. Section 2 gives necessary background on the Type A and B stories. We then develop properties of H^*Y_n in Section 3, which will be instrumental in proving our main results on the peak representations in Section 4. In Section 5 we provide generating function formulae and branching rule recursions for the peak representations, and relate this story to the free Jordan algebra.

2 Background

We review here in more detail the spaces X_n, Y_n, their cohomology rings, and their relationship to the Eulerian idempotents and Lie characters Lie_λ discussed in Section 1.

2.1 The (associated graded) Varchenko-Gelfand ring

The cohomology rings $\mathcal{X}_n := H^*X_n$ and $\mathcal{Y}_n := H^*Y_n$ are closely related to the *reflection hyperplane arrangements* $\mathcal{A}_W \subset V = \mathbb{R}^n$ associated to the groups $W = \mathfrak{S}_n, \mathfrak{S}_n^{\pm}$:

$$\mathcal{A}_{\mathfrak{S}_n} = \{x_i = x_j\}_{1 \le i < j \le n} \qquad \mathcal{A}_{\mathfrak{S}_n^{\pm}} = \{x_i = 0\}_{1 \le i \le n} \sqcup \{x_i = \pm x_j\}_{1 \le i < j \le n}.$$

In particular, Moseley [12] proved there are algebra isomorphisms

$$\mathcal{X}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n}) \qquad \mathcal{Y}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n^{\pm}}),$$

where $\mathcal{VG}(\mathcal{A})$ is the *(associated graded) Varchenko-Gelfand ring*, defined for any real hyperplane arrangement $\mathcal{A} \subset \mathbb{R}^n$ as the quotient of $\mathbf{k}[u_i]_{H_i \in \mathcal{A}}$ by an ideal[1]

$$\mathcal{J}_A = \langle u_i^2, \ \sum_{j=1}^{c} \epsilon(C, i_j) \cdot u_{i_1} u_{i_2} \cdots \widehat{u_{i_j}} \cdots u_{i_{c-1}} u_{i_c} \text{ for all } C \subset \mathcal{A} \rangle.$$

Here $C = (C_+, C_-)$ is an *oriented matroid* signed circuit of $\mathcal{A}$, with $\epsilon(C, i_j) = \pm 1$, depending on whether i_j lies in C_+ or C_-.

Example 2.1. When $\mathcal{A} = \mathcal{A}_{\mathfrak{S}_n}$, work of Arnol'd [2] and Cohen [8] shows that $\mathcal{X}_n$ has presentation given by

$$\mathcal{X}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n}) = \mathbf{k}[u_{ij}]_{1 \le i < j \le n} / \langle u_{ij}^2, \ \ u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk} \rangle.$$

Barcelo [3] constructed an elegant *non-broken circuit* monomial basis for $\mathcal{X}_n$, obtained by taking products with at most one element from each set U_i below:

$$U_1 = \{u_{12}\}, \ U_2 = \{u_{13}, \ u_{23}\}, \cdots, \ U_{n-1} = \{u_{1n}, \ u_{2n}, \ \cdots, \ u_{(n-1),n}\}.$$

[1] In fact, one can take coefficients in $\mathbb{Z}$ rather than $\mathbf{k}$. However, in what follows, we will want $\mathbf{k}$ to be a field with characteristic not dividing 2.

In [6], the second author showed that $\mathcal{VG}(\mathcal{A})$ admits a decomposition by intersection subspaces (i.e. flats) in $\mathcal{A}$. The component of $\mathcal{VG}(\mathcal{A})_X$ indexed by X is the $\mathbb{Z}$-span of all monomials $\{u_{i_1} \cdots u_{i_\ell}\}$ for which $H_{i_1} \cap \cdots \cap H_{i_\ell} = X$.

In the case of a reflection arrangement $\mathcal{A}_W$, we can group flats by their W-orbits $[X]$, which gives a coarser decomposition of $\mathcal{VG}(\mathcal{A}_W) = \bigoplus \mathcal{VG}(\mathcal{A}_W)_{[X]}$. The flats and flat orbits in $\mathcal{A}_{\mathfrak{S}_n}$ and $\mathcal{A}_{\mathfrak{S}_n^\pm}$ have elegant (and useful!) combinatorial descriptions.

Famously, the flats of $\mathcal{A}_{\mathfrak{S}_n}$ biject with set partitions of $[n]$. This isomorphism identifies a flat X with the set partition $\pi_X = \{B_1, \cdots, B_k\}$ where i and j are in the same block B_ℓ if and only if $x_i = x_j$ in X. The $\mathfrak{S}_n$-orbits of these flats biject with integer partitions of n: the orbit of π_X corresponds to the partition $\lambda_X = \{|B_1|, \cdots, |B_k|\}$.

Similarly, the flats in $\mathcal{A}_{\mathfrak{S}_n^\pm}$ can be identified with a set partition on a *subset* S of $[n]^\pm := \{\bar{1}, \bar{2}, \cdots, \bar{n}, 1, 2, \cdots n\}$, where S does not contain both i and $\bar{i}$. Given a flat X, identify $\bar{i}$ with $-x_i$ and let $\tau_X = \{C_1, \cdots C_k\}$ where for $i, j \in [n]$, indices i and j (resp. i and $\bar{j}$) appear in the same block C_ℓ if and only if $x_i = x_j \neq 0$ (resp. if and only if $x_i = -x_j \neq 0$) in X. Note that two set partitions related by $i \mapsto \bar{i}$ correspond to the same flat. The $\mathfrak{S}_n^\pm$ orbit of τ_X is indexed by a partition $\mu_X = \{|C_i|, \cdots, |C_k|\}$ of $0 \leq m \leq n$.

We write $\mathcal{X}_{\lambda_X}^{(n)} := \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n})_{[\pi_X]}$ and $\mathcal{Y}_{\mu_X}^{(n)} := \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n^\pm})_{[\tau_X]}$, giving the decompositions

$$\mathcal{X}_n = \bigoplus_{\lambda \vdash n} \mathcal{X}_\lambda^{(n)} \qquad\qquad \mathcal{Y}_n = \bigoplus_{\mu \vdash 0 \leq m \leq n} \mathcal{Y}_\mu^{(n)}.$$

2.2 The Eulerian idempotents and higher Lie characters

The idempotents $\{E_k^{\mathfrak{S}_n}\}$ and $\{E_k^{\mathfrak{S}_n^\pm}\}$ from Section 1 can be defined via the formula in [6]:

$$\sum_{k=0}^{r} t^k E_k^W = \frac{1}{|W|} \sum_{w \in W} \left(\prod_{i=1}^{\mathrm{des}(w)} (t - e_i) \prod_{i=1}^{r-\mathrm{des}(w)} (t + e_i) \right) \cdot w,$$

which recovers work of Garsia–Reutenauer [9] for $W = \mathfrak{S}_n$ and Bergeron–Bergeron [4] for $W = \mathfrak{S}_n^\pm$. Here, r is the *rank* of $\mathcal{A}_W$ ($r = n-1$ for $W = \mathfrak{S}_n$ and $r = n$ for $W = \mathfrak{S}_n^\pm$) and the e_i are the *exponents* of W ($e_i = i$ for $W = \mathfrak{S}_n$ and $e_i = 2i - 1$ for $W = \mathfrak{S}_n^\pm$). The *descent number*, $\mathrm{des}(w)$ is the number of simple reflections s of W with $\ell(ws) < \ell(w)$.

The E_k^W have a refinement due to Bergeron–Bergeron–Howlett–Taylor [5], who introduced families of complete, primitive orthogonal idempotents in $\mathrm{Sol}(W)$ for any finite Coxeter group W. These idempotents, which we will call the *BBHT idempotents*, are indexed by W-flat orbits. We omit the technical definitions, but note that by the discussion in §2.1, for $W = \mathfrak{S}_n, \mathfrak{S}_n^\pm$ they can be indexed as $\{E_\lambda^{\mathfrak{S}_n} : \lambda \vdash n\}$ and $\{E_\mu^{\mathfrak{S}_n^\pm} : \mu \vdash m, m \leq n\}$.

To recover the $\{E_k^{\mathfrak{S}_n}\}$ and $\{E_k^{\mathfrak{S}_n^\pm}\}$, group $\{E_\lambda^{\mathfrak{S}_n}\}$ and $\{E_\mu^{\mathfrak{S}_n^\pm}\}$ by partition *length* ℓ:

$$E_k^{\mathfrak{S}_n} = \sum_{\lambda :\ \ell(\lambda) = k} E_\lambda^{\mathfrak{S}_n} \qquad\qquad E_k^{\mathfrak{S}_n^\pm} = \sum_{\mu :\ \ell(\mu) = k} E_\mu^{\mathfrak{S}_n^\pm}. \tag{2.1}$$

We can also refine the isomorphisms in (1.2) and (1.3) using the BBHT idempotents:

Theorem 2.2 (Brauner, [6])**.** *There are $\mathfrak{S}_n$ and $\mathfrak{S}_n^{\pm}$ representation isomorphisms*

$$\mathcal{X}_\lambda^{(n)} \cong (\mathbf{k}\,\mathfrak{S}_n)\, E_\lambda^{\mathfrak{S}_n} \qquad\qquad \mathcal{Y}_\mu^{(n)} \cong \left(\mathbf{k}\,\mathfrak{S}_n^{\pm}\right) E_\mu^{\mathfrak{S}_n^{\pm}}.$$

In fact, there is more to say in the case of $W = \mathfrak{S}_n$, relating to the *higher Lie representations* $\{\mathrm{Lie}_\lambda\}$ of Thrall [17]. Let $\mathcal{C}_\lambda$ be the conjugacy class of $\mathfrak{S}_n$ indexed by the partition $\lambda = (1^{m_1}, 2^{m_2}, \cdots n^{m_n})$. The centralizer Z_λ of an element of $\mathcal{C}_\lambda$ has isomorphism type

$$Z_\lambda \cong \prod_{j=1}^{n} \mathfrak{S}_{m_j}[\mathbb{Z}_j],$$

where $\mathbb{Z}_j$ is the cyclic group of order j, and $\mathfrak{S}_{m_j}[\mathbb{Z}_j]$ is the wreath product. Specifically, the action of $\mathfrak{S}_{m_j}$ in this wreath product swaps the m_j blocks of λ of size j.

We will be interested in a linear character ω_λ on Z_λ obtained from extending faithful characters on each $\mathbb{Z}_j$ to Z_λ, where ω_λ restricts trivially on the wreath factors $\mathfrak{S}_{m_j}$ of Z_λ.

Write $\uparrow_H^G$ to be the representation induction from a subgroup H of G to G.

Definition 2.3. Give a partition $\lambda \vdash n$, define $\mathrm{Lie}_\lambda := \omega_\lambda \uparrow_{Z_\lambda}^{\mathfrak{S}_n}$.

Thrall proved that $\mathbf{k}\,\mathfrak{S}_n \cong \bigoplus_{\lambda\vdash n} \mathrm{Lie}_\lambda$. A beautiful result of Hanlon [11] then shows that $\mathrm{Lie}_\lambda \cong (\mathbf{k}\,\mathfrak{S}_n)E_\lambda^{\mathfrak{S}_n}$. Using (2.1), we can thus conclude

$$(\mathbf{k}\,\mathfrak{S}_n)\, E_{n-1-k}^{\mathfrak{S}_n} \cong \bigoplus_{\substack{\lambda\vdash n:\\ \ell(\lambda)=n-k}} \mathrm{Lie}_\lambda \cong H^{2k} X_n.$$

Example 2.4. When $\lambda = (n)$, the representation $\mathrm{Lie}_n := \mathrm{Lie}_{(n)}$ is isomorphic to the multilinear component of the free Lie algebra, defined and generalized in §5.1.

3 Presentations, Filtrations, and Decompositions of H^*Y_n

Our first task is to study the ring $\mathcal{Y}_n := H^*Y_n$ in greater detail. It will be important for the remainder of this section to assume that **the field k has characteristic larger than** n, so that $2 \in \mathbf{k}^\times$ and $\mathbf{k}[\mathfrak{S}_n^{\pm}]$ is semisimple. This allows us to make an invertible change-of-variables that diagonalizes the action of the normal subgroup $\mathbb{Z}_2^n$ within $\mathfrak{S}_n^{\pm}$.

The presentation of $\mathcal{Y}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n^{\pm}})$ was first given by Xicotencatl [18]; it is isomorphic to $\mathbf{k}[u_{ij}^+, u_{ij}^-, u_i]/J_{\mathfrak{S}_n^{\pm}}$ for $1 \le i < j \le n$, with generators corresponding to

$$u_{ij}^+ \longleftrightarrow \{x_i = x_j\} \qquad u_{ij}^- \longleftrightarrow \{x_i = -x_j\} \qquad u_i \longleftrightarrow \{x_i = 0\}$$

respectively. The generating relations for $\mathcal{J}_{\mathfrak{S}_n^{\pm}}$ are given in Table 1.

We will introduce a new basis for $\mathcal{Y}_n$, a filtration using that basis, and a corresponding associated graded ring. Along the way, we will see several useful decompositions of $\mathcal{Y}_n$.

Definition 3.1. For $1 \leq i < j \leq n$, define an isomorphism of graded $\mathbf{k}$-algebras $\mathcal{B}$ by

$$u_i \longmapsto u_i \qquad v_{ij} \longmapsto u_{ij}^+ + u_{ij}^- \qquad w_{ij} \longmapsto u_{ij}^+ - u_{ij}^-$$

with inverse given by $\mathcal{B}^{-1}(u_i) = u_i, \mathcal{B}^{-1}(u_{ij}^+) = \frac{1}{2}(v_{ij} + w_{ij}), \mathcal{B}^{-1}(u_{ij}^-) = \frac{1}{2}(v_{ij} - w_{ij})$.

We wish to rewrite the presentation $\mathcal{Y}_n := \mathbf{k}[u_{ij}^+, u_{ij}^-, u_i]/\mathcal{J}_{\mathfrak{S}_n^\pm}$ in terms of these new variables v_{ij}, w_{ij}, using a Gröbner basis argument. Introduce a lexicographic monomial ordering $\prec$ on $\mathbf{k}[v_{ij}, w_{ij}, u_i]$, in which the variables u_i, v_{ij}, w_{ij} are ordered as follows:

$$u_1 < u_2 < \cdots u_n < v_{12} < w_{12} < v_{13} < w_{13} < \cdots < v_{(n-1)1} < w_{(n-1)n}. \tag{3.1}$$

Theorem 3.2. *The isomorphism $\mathcal{B} : \mathbf{k}[v_{ij}, w_{ij}, u_i] \longrightarrow \mathbf{k}[u_{ij}^+, u_{ij}^-, u_i]$ induces a graded $\mathbf{k}$-algebra isomorphism, where $\mathcal{I}$ is generated by the relations $\mathcal{G}$ listed in Table 1 below:*

$$\mathbf{k}[v_{ij}, w_{ij}, u_i]/\mathcal{I} \xrightarrow{\sim} \mathbf{k}[u_{ij}^+, u_{ij}^-, u_i]/\mathcal{J}_{\mathfrak{S}_n^\pm} =: \mathcal{Y}_n,$$

Moreover, $\mathcal{G}$ gives a Gröbner basis for the ideal $\mathcal{I}$ with respect to $\prec$, in which the standard monomial $\mathbf{k}$-basis for the quotient $\mathbf{k}[v_{ij}, w_{ij}, u_i]/\mathcal{I}$ is the set of monomials $\mathcal{V}$ obtained from taking products with at most one element from each of these sets V_i:

$$V_1 = \{u_1\},\ V_2 = \{u_2,\ v_{12},\ w_{12}\},\ \cdots,\ V_n = \{u_n,\ v_{1n},\ w_{1n}, \cdots, v_{(n-1)n},\ w_{(n-1)n}\}.$$

We make two observations about the $\mathfrak{S}_n^\pm$ action on $\mathcal{Y}_n$. First, elements of $\mathbb{Z}_2^n \subset \mathfrak{S}_n^\pm$ scale all of u_i, v_{ij}, w_{ij} via ± 1; thus Theorem 3.2 will allow us to construct a monomial basis for $H^* Z_n \cong (\mathcal{Y}_n)^{\mathbb{Z}_2^n}$ in §4. Second, the generators segregate into two $\mathfrak{S}_n^\pm$-orbits: $\{u_i\}_{1 \leq i \leq n}$ and $\{v_{ij}, w_{ij}\}_{1 \leq i < j \leq n}$. This leads to a helpful *filtration*, as follows.

For $q \in \mathbf{k}[v_{ij}, w_{ij}, u_i]$, let $\deg(q)$ be the polynomial degree of q, $\deg_{\mathcal{V}}(q)$ to be the degree of q in the v_{ij} and w_{ij} variables, and $\deg_u(q)$ be the degree in the u_i variables. Our key insight is that $\mathcal{Y}_n$ admits a filtration by $\deg_u$. In particular, define the ideal

$$P^{(i)} := \{q \in \mathcal{Y}_n \subset \mathbf{k}[u_i, v_{ij}, w_{ij}] : \deg_u(q) \geq i\}.$$

For example, when $n = 2$ the ideal $P^{(1)}$ is the $\mathbf{k}$-span of $\{u_1,\ u_2,\ u_1 v_{12},\ u_1 w_{12},\ u_1 u_2\}$.

Proposition 3.3. *There are $\mathfrak{S}_n^\pm$-stable ascending filtrations on $\mathcal{Y}_n$ given by*

$$P^{(n)} \subset P^{(n-1)} \subset \cdots \subset P^{(1)} \subset P^{(0)}.$$

The associated graded ring $\overline{\mathcal{Y}_n} = \bigoplus_{i=0}^n P^{(i)}/P^{(i+1)}$ has presentation $\mathbf{k}[v_{ij}, w_{ij}, u_i]/\mathfrak{gr}(\mathcal{I})$ for $1 \leq i < j \leq n$, where the relations generating $\mathfrak{gr}(\mathcal{I})$ are given in Table 1.

The motivation for introducing and studying the associated graded ring $\overline{\mathcal{Y}_n}$ is that in our context (i.e. $\mathbf{k}\,\mathfrak{S}_n^\pm$ being a semisimple algebra), we have $\overline{\mathcal{Y}_n} \cong \mathcal{Y}_n$ as $\mathfrak{S}_n^\pm$-modules. Hence, it suffices to study the basis and representations on $\overline{\mathcal{Y}_n}$.

We will see that $\overline{\mathcal{Y}_n}$ has several useful decompositions that make studying the representations on $\mathcal{Y}_n$ (and eventually $H^* Z_n$) far more tractable.

Relations for $\mathcal{J}_{\mathfrak{S}_n^\pm}$	Relations for $\mathcal{I}$	Relations for $\mathfrak{gr}(\mathcal{I})$
u_i^2	u_i^2	u_i^2
$u_iu_{ij}^+ - u_iu_{ij}^- - u_{ij}^+u_{ij}^-$	$v_{ij}w_{ij}$	$v_{ij}w_{ij}$
$u_iu_j - u_iu_{ij}^- - u_ju_{ij}^-$	$u_iw_{ij} - u_jv_{ij}$	$u_iw_{ij} - u_jv_{ij}$
$(u_{ij}^+)^2$	$v_{ij}^2 - 2u_iw_{ij}$	v_{ij}^2
$(u_{ij}^-)^2$	$w_{ij}^2 + 2u_iw_{ij}$	w_{ij}^2
$u_iu_j - u_iu_{ij}^- - u_ju_{ij}^-$	$u_iv_{ij} - 2u_iu_j - u_jw_{ij}$	$u_iv_{ij} - u_jw_{ij}$
$u_{ij}^+u_{jk}^+ - u_{ij}^+u_{ik}^+ - u_{ik}^+u_{jk}^+$	$v_{ij}w_{jk} - w_{ij}w_{ik} - v_{ik}v_{jk}$	$v_{ij}w_{jk} - w_{ij}w_{ik} - v_{ik}v_{jk}$
$u_{ij}^-u_{jk}^+ - u_{ij}^-u_{ik}^- - u_{ik}^-u_{jk}^+$	$w_{ij}w_{jk} - v_{ij}w_{ik} - w_{ik}w_{jk}$	$w_{ij}w_{jk} - v_{ij}w_{ik} - w_{ik}w_{jk}$
$-u_{ij}^-u_{jk}^- + u_{ij}^-u_{ik}^+ - u_{ik}^+u_{jk}^-$	$v_{ij}v_{jk} - v_{ij}v_{ik} - v_{ik}w_{jk}$	$v_{ij}v_{jk} - v_{ij}v_{ik} - v_{ik}w_{jk}$
$-u_{ij}^+u_{jk}^- + u_{ij}^+u_{ik}^- - u_{ik}^-u_{jk}^-$	$w_{ij}v_{jk} - w_{ij}v_{ik} - w_{ik}v_{jk}$	$w_{ij}v_{jk} - w_{ij}v_{ik} - w_{ik}v_{jk}$

Table 1: Generating relations for the ideals $\mathcal{J}_{\mathfrak{S}_n}, \mathcal{I}$ and $\mathfrak{gr}(\mathcal{I})$.

First, one can show that the flat orbit decomposition from §2.1 persists in $\overline{\mathcal{Y}_n}$; we will abuse notation and write $\mathcal{Y}_\mu^{(n)}$ instead of $\overline{\mathcal{Y}_\mu}^{(n)}$ since they are isomorphic.

The second useful decomposition is the following bi-grading:

$$\mathcal{Y}_{k,\ell}^{(n)} := \mathrm{span}_{\mathbf{k}}\{q \in \overline{\mathcal{Y}_n} : \deg(q) = k \quad \deg_{\mathcal{V}}(q) = \ell\}.$$

In fact, this bi-grading can be refined to a third decomposition by signed partitions, which are pairs of partitions (λ^+, λ^-) such that $|\lambda^+| + |\lambda^-| = n$.

Definition 3.4. Given a monomial in $q \in \mathbb{Q}[u_i, v_{ij}, w_{ij}]$, associate to q a signed partition $(\lambda^+_{(q)}, \lambda^-_{(q)})$ as follows:

1. Construct a graph $\mathcal{G}(q)$ with vertex set $[n] = \{1, 2, \cdots, n\}$ by drawing an edge between i and j if v_{ij} or w_{ij} occurs in q, and drawing a loop at i if u_i occurs in q;

2. Let $\mathcal{G}_1 = (E_1, V_1), \cdots, \mathcal{G}_k = (E_k, V_k)$ be the connected components of $\mathcal{G}(q)$. Then

$$\lambda^+_{(q)} := \{|V_\ell| : \mathcal{G}_\ell \text{ has no loops}\} \quad \lambda^-_{(q)} := \{|V_\ell| : \mathcal{G}_\ell \text{ has loops}\}.$$

Proposition 3.5. *There is a decomposition of $\overline{\mathcal{Y}_n}$ by signed partitions $\overline{\mathcal{Y}_n} = \bigoplus_{(\lambda^+,\lambda^-)} \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)}$, where*

$$\mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)} := \mathrm{span}_{\mathbf{k}}\{\textit{monomials } q \in \overline{\mathcal{Y}_n} : (\lambda^+_{(q)}, \lambda^-_{(q)}) = (\lambda^+, \lambda^-)\}.$$

This decomposition is compatible with the other decompositions of $\overline{\mathcal{Y}_n}$, in the sense that:

$$\mathcal{Y}^{(n)}_{\mu} = \bigoplus_{(\lambda^+,\lambda^-):\ \lambda^+=\mu} \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)} \qquad \mathcal{Y}^{(n)}_{k,\ell} = \bigoplus_{\substack{(\lambda^+,\lambda^-):\ \ell(\lambda^+)=n-k \\ \ell(\lambda^+)+\ell(\lambda^-)=n-\ell}} \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)}$$

For example, suppose $n = 8$ and $q = w_{12} \cdot u_5 \cdot v_{56} \cdot u_7 \cdot v_{24}$. Then q is in the bi-graded piece $\mathcal{Y}^{(8)}_{5,3}$ and we have $\lambda^+_{(q)} = \{3,1,1\}$ and $\lambda^-_{(q)} = \{2,1\}$. Thus $q \in \mathcal{Y}^{(8)}_{((3,1,1),(2,1))} \subset \mathcal{Y}^{(8)}_{(3,1,1)}$.

Theorem 3.6. *There is a well-defined, $\mathfrak{S}_n$-equivariant surjection of* $\mathbf{k}$*-vector spaces*

$$\begin{aligned} \gamma : \overline{\mathcal{Y}_n} &\longrightarrow \mathcal{X}_n = \mathbf{k}[u_{ij}]_{1\le i<j\le n} / \langle u_{ij}^2,\ \ u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk} \rangle \\ \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)} &\longmapsto \mathcal{X}^{(n)}_{(\lambda^+\cup\lambda^-)}, \end{aligned}$$

defined by sending $\qquad \gamma(u_i) = 1, \qquad \gamma(w_{ij}) = u_{ij} \qquad \gamma(v_{ij}) = u_{ij}.$

Proof idea. The key observation is that the relations $u_iw_{ij} - u_jv_{ij}$ and $u_iv_{ij} - u_jw_{ij}$ in $\mathfrak{gr}(\mathcal{I})$ mean that one can give a presentation of $\overline{\mathcal{Y}_n}$ as a quotient of a subring of $\mathbf{k}[v_{ij}, w_{ij}, u_i]$, by an ideal $\tilde{\mathcal{I}} \subset \mathfrak{gr}(\mathcal{I})$ that omits the relation u_i^2. From this, one can define a surjection of *vector spaces*; note however that γ cannot be extended to a map of algebras. □

4 Main Results

At last, we are ready to analyze the peak representations. Our investigations began from an observation of Aguiar, Bergeron and Nyman [1] relating the descent algebras $\mathrm{Sol}(\mathfrak{S}_n)$ and $\mathrm{Sol}(\mathfrak{S}_n^{\pm})$ to the *peak algebra* $\mathcal{P}_n$.

Recall that one can express the hyperoctahedral group of all signed permutations as $\mathfrak{S}_n^{\pm} = \mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ where $\mathbb{Z}_2^n$ is the normal subgroup performing arbitrary sign changes in the coordinates. The quotient map $\mathfrak{S}_n^{\pm} \twoheadrightarrow \mathfrak{S}_n^{\pm} / \mathbb{Z}_2^n \cong \mathfrak{S}_n$ of groups, which forgets the signs in a signed permutation, gives rise to a surjective $\mathbf{k}$-algebra map $\varphi : \mathbf{k}\,\mathfrak{S}_n^{\pm} \twoheadrightarrow \mathbf{k}\,\mathfrak{S}_n$. In [1], it was shown that the peak subalgebra $\mathcal{P}_n$ is exactly the image under φ of $\mathrm{Sol}(\mathfrak{S}_n^{\pm})$, that is, φ restricts to an algebra surjection $\mathrm{Sol}(\mathfrak{S}_n^{\pm}) \overset{\varphi}{\twoheadrightarrow} \mathcal{P}_n$.

As a consequence, one can define a family of *peak idempotents* inside $\mathcal{P}_n \subset \mathbf{k}\,\mathfrak{S}_n$ via

$$E_k^{\mathcal{P}_n} := \varphi(E_k^{\mathfrak{S}_n^{\pm}}) \text{ for } k = 0, 1, \cdots, n \qquad E_\mu^{\mathcal{P}_n} := \varphi(E_\mu^{\mathfrak{S}_n^{\pm}}) \text{ for } \mu \vdash m \le n.$$

Both families inherit from $\{E_k^{\mathfrak{S}_n^{\pm}}\}$ and $\{E_\mu^{\mathfrak{S}_n^{\pm}}\}$ the property of being a complete system of orthogonal idempotents in $\mathbf{k}\,\mathfrak{S}_n$, and the $\{E_\mu^{\mathcal{P}_n}\}$ are also primitive if nonzero. Note that some of the $E_k^{\mathcal{P}_n}$ and $E_\mu^{\mathcal{P}_n}$ will be zero, which we characterize in Theorems 1.1 and 4.6. By construction, one recovers $E_k^{\mathcal{P}_n}$ from the $E_\mu^{\mathcal{P}_n}$ by summing over all μ of length k.

Our goal is to relate the peak idempotents to the ring $\mathcal{Z}_n := H^* Z_n$, where

$$Z_n := Y_n / \mathbb{Z}_2^n = \mathrm{Conf}_n(\left(\mathbb{R}^3 \setminus \{\mathbf{0}\}\right) / \mathbb{Z}_2) = \mathrm{Conf}_n(\mathbb{RP}^2 \times (0,\infty))$$

is the configuration space of n ordered points within the quotient $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ under the $\mathbb{Z}_2$-action via $\mathbf{x} \mapsto -\mathbf{x}$, so that $\left(\mathbb{R}^3 \setminus \{\mathbf{0}\}\right) / \mathbb{Z}_2 \;\cong\; \mathbb{RP}^2 \times (0,\infty)$.

Note that $(\mathcal{Y}_n)^{\mathbb{Z}_2^n} \cong \mathcal{Z}_n$. The filtration, bigrading, and finer decompositions (by flat orbits and signed partitions) on $\mathcal{Y}_n$ from Section 3 persist when one takes $\mathbb{Z}_2^n$-fixed spaces, giving a bigraded $\mathfrak{S}_n$-representation on an associated graded ring $\overline{\mathcal{Z}_n}$:

$$\mathcal{Z}_{k,\ell}^{(n)} := (\mathcal{Y}_{k,\ell}^{(n)})^{\mathbb{Z}_2^n}, \qquad \mathcal{Z}_{\mu}^{(n)} := (\mathcal{Y}_{\mu}^{(n)})^{\mathbb{Z}_2^n}, \qquad \mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)} := (\mathcal{Y}_{(\lambda^+,\lambda^-)}^{(n)})^{\mathbb{Z}_2^n}.$$

We first construct monomial a basis for $\mathcal{Z}_n$, using the fact that by Theorem 3.2, the basis $\mathcal{V}$ of $\mathcal{Y}_n$ diagonalizes the action of the normal subgroup $\mathbb{Z}_2^n \leq \mathfrak{S}_n^{\pm}$ on $\mathcal{Y}_n$.

Definition 4.1. For $1 \leq i < j < k \leq n$, let $\mathcal{I}_1 := \{u_i w_{ij}\}$, $\mathcal{I}_2 := \{w_{ij} w_{ik}\}$, $\mathcal{I}_3 := \{v_{ij} w_{jk}\}$. Let $\tilde{\mathcal{V}}$ be the monomials obtained from products in $\mathcal{I}_j$ for $j = 1,2,3$ that are also in $\mathcal{V}$.

Theorem 4.2. *The set $\tilde{\mathcal{V}}$ is a basis for $\mathcal{Z}_n$ and $\overline{\mathcal{Z}_n}$ that is compatible with the decomposition by signed partitions: $\overline{\mathcal{Z}_n} = \bigoplus \mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}$.*

Proof idea. We construct a bijection from $\tilde{\mathcal{V}}$ to the monomial basis of $\mathcal{X}_n$ from Example 2.1. This involves defining a "pairing lemma" to group quadratic terms appearing in $q \in \tilde{\mathcal{V}}$ and then mapping: $u_i w_{ij}$ to u_{ij}, $w_{ij} w_{ik}$ to $u_{ij} u_{ik}$, and $v_{ij} w_{jk}$ to $u_{ij} u_{jk}$. □

Example 4.3. The basis for $\mathcal{Z}_{4,2}^{(4)}$ is $\{(u_1 w_{12})(u_3 w_{34}),\ (u_1 w_{13})(u_2 w_{24}),\ (u_1 w_{14})(u_2 w_{23})\}$.

Given a partition λ of n, recall that $\ell(\lambda)$ is its number of parts and $|\lambda|$ is its size. Let $\mathrm{Odd}(\lambda)$ (resp. $\mathrm{Even}(\lambda)$) be the partition obtained by taking only the odd (resp. even) parts of λ. We call λ an *odd partition* if $\mathrm{Odd}(\lambda) = \lambda$ and an *even partition* if $\mathrm{Even}(\lambda) = \lambda$. Write $\mathrm{odd}(\lambda) = \ell(\mathrm{Odd}(\lambda))$ and $\mathrm{even}(\lambda) = \ell(\mathrm{Even}(\lambda))$.

Theorem 4.4. *The space $\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}$ vanishes unless λ^+ is an odd partition and λ^- is an even partition, while $\mathcal{Z}_{\mu}^{(n)}$ vanishes unless μ is an odd partition and $n - |\mu|$ is even.*
Moreover, the map γ restricts to an $\mathfrak{S}_n$-equivariant vector-space isomorphism $\gamma : \mathcal{Z}_n \longrightarrow \mathcal{X}_n$:

$$\gamma(\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}) = \mathcal{X}_{(\lambda^+ \cup \lambda^-)}^{(n)} \qquad \gamma^{-1}(\mathcal{X}_{\lambda}^{(n)}) = \mathcal{Z}_{(\mathrm{Odd}(\lambda),\mathrm{Even}(\lambda))}^{(n)}.$$

Thus, for non-vanishing $\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}$, $\mathcal{Z}_{\mu}^{(n)}$, and $\mathcal{Z}_{2k,\ell}^{(n)}$, there are $\mathfrak{S}_n$-representation isomorphisms

$$\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)} \cong \mathrm{Lie}_{(\lambda^+ \cup \lambda^-)}, \qquad \mathcal{Z}_{\mu}^{(n)} \cong \bigoplus_{\lambda:\ \mathrm{Odd}(\lambda) = \mu} \mathrm{Lie}_{\lambda}, \qquad \mathcal{Z}_{2k,\ell}^{(n)} \cong \bigoplus_{\substack{\lambda: \ell(\lambda) = n-\ell \\ \mathrm{odd}(\lambda) = n-2k}} \mathrm{Lie}_{\lambda}\,.$$

Example 4.5. When $n = 4$, the non-vanishing pieces $\mathcal{Z}_{\mu}^{(4)}$ are as follows:

$$\mathcal{Z}_{\varnothing}^{(4)} \cong \mathrm{Lie}_{(2,2)} \oplus \mathrm{Lie}_{(4)} \quad \mathcal{Z}_{(1,1)}^{(4)} \cong \mathrm{Lie}_{(2,1,1)} \quad \mathcal{Z}_{(3,1)}^{(4)} \cong \mathrm{Lie}_{(3,1)} \quad \mathcal{Z}_{(1,1,1,1)}^{(4)} \cong \mathrm{Lie}_{(1,1,1,1)}\,.$$

The non-vanishing bi-graded pieces $\mathcal{Z}^{(4)}_{2k,\ell}$ are

$$\mathcal{Z}^{(4)}_{0,0} \cong \mathrm{Lie}_{(1,1,1,1)} \quad \mathcal{Z}^{(4)}_{2,1} \cong \mathrm{Lie}_{(2,1,1)} \quad \mathcal{Z}^{(4)}_{2,2} \cong \mathrm{Lie}_{(3,1)} \quad \mathcal{Z}^{(4)}_{4,2} \cong \mathrm{Lie}_{(2,2)} \quad \mathcal{Z}^{(4)}_{4,3} \cong \mathrm{Lie}_{(4)} \,.$$

In fact, we now have all the tools necessary to provide a cohomological interpretation of the $\mathfrak{S}_n$-representations generated by the Peak idempotents, by analyzing the $\mathbb{Z}_2^n$ fixed spaces of Theorem 2.2 and applying Theorem 4.4.

Theorem 4.6. *The idempotent $E^{\mathcal{P}_n}_\mu$ does not vanish if and only if μ is an odd partition (including $\mu = \varnothing$) and $n - |\mu|$ is even. In this case, there are $\mathfrak{S}_n$-representation isomorphisms*

$$(\mathbf{k}\,\mathfrak{S}_n)E^{\mathcal{P}_n}_\mu \cong \mathcal{Z}^{(n)}_\mu \cong \bigoplus_{\lambda:\ \mathrm{Odd}(\lambda)=\mu} \mathrm{Lie}_\lambda \,.$$

Note that combining Proposition 3.5 with Theorems 4.4 and 4.6 implies Theorem 1.1.

5 Hilbert series and the free Jordan algebra

Having established the connection between the peak algebra and the ring $\mathcal{Z}_n$, we now develop enumerative and recursive properties of the latter.

Let Λ denote the *ring of symmetric functions* (of bounded degree, in infinitely many variables). It has a $\mathbb{Z}$-algebra isomorphism known as the *Frobenius characteristic map* $\mathrm{ch} : \oplus_{n\geq 0}\mathrm{Rep}(\mathfrak{S}_n) \to \Lambda$, where $\mathrm{Rep}(\mathfrak{S}_n)$ are the *virtual characters* of $\mathfrak{S}_n$. We will study the Frobenius characteristic of $\mathcal{Z}^{(n)}_{2k,\ell}$, using the fact that $\mathcal{Z}^{(n)}_{2k+1,\ell} = 0$ by Theorem 1.1.

Definition 5.1. Write $\Lambda_{\mathbb{Z}[t,q]}$ to be the ring Λ with coefficients in $\mathbb{Z}[t,q]$ and define

$$M_n(t,q) := \sum_{k,\ell} \dim\left(\mathcal{Z}^{(n)}_{2k,\ell}\right) t^k q^\ell \in \mathbb{Z}[t,q], \qquad \mathcal{M}^{(n)}(t,q) := \sum_{k,\ell} \mathrm{ch}\left(\mathcal{Z}^{(n)}_{2k,\ell}\right) t^k q^\ell \in \Lambda_{\mathbb{Z}[t,q]}.$$

For $w \in \mathfrak{S}_n$ let $\mathrm{even}(w), \mathrm{odd}(w)$ denote the number of even-sized and odd-sized cycles of w, and $\mathrm{cyc}(w)$ the number of cycles of w.

Theorem 5.2. *Write $L_\lambda := \mathrm{ch}(\mathrm{Lie}_\lambda)$. Then one can rewrite $M_n(t,q)$ and $\mathcal{M}^{(n)}(t,q)$ as follows:*

$$M_n(t,q) = \sum_{w\in\mathfrak{S}_n} t^{\frac{n-\mathrm{odd}(w)}{2}} q^{n-\mathrm{cyc}(w)}, \qquad \mathcal{M}^{(n)}(t,q) = \sum_{\lambda\vdash n} L_\lambda \cdot t^{\frac{|\lambda|-\mathrm{odd}(\lambda)}{2}} q^{|\lambda|-\ell(\lambda)}.$$

Using Theorem 5.2, we manipulate the symmetric functions in $\mathcal{M}^{(n)}(t,q)$ to give a branching rule recurrence for the bi-graded pieces $\mathcal{Z}^{(n)}_{2k,\ell}$. Let $\uparrow$ denote representation induction from $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$ and $\downarrow$ denote representation restriction from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$.

Theorem 5.3. *The restriction of $\mathcal{Z}^{(n)}_{2k,j}$ from an $\mathfrak{S}_n$ to an $\mathfrak{S}_{n-1}$-module is given by*

$$\mathcal{Z}^{(n)}_{2k,\ell}\downarrow = \mathcal{Z}^{(n-1)}_{2k,\ell} + \mathcal{Z}^{(n-2)}_{2(k-1),\ell-1}\uparrow + \left(\mathcal{Z}^{(n-2)}_{2(k-1),\ell-2}\uparrow\right) * \chi^{(n-2,1)},$$

where $*$ is the Kronecker product and $\chi^{(n-2,1)}$ is the irreducible reflection representation of $\mathfrak{S}_{n-1}$.

Theorem 5.3 implies a recursive formula for $M_n(t,q)$ with interesting specializations:

$$M_n(1,q) = (1+q)(1+2q)\cdots(1+(n-1)q), \tag{5.1}$$
$$M_n(t,1) = (1+(n-1)q)\cdot M_{n-1}(1,q), \tag{5.2}$$

where (5.1) is the generating function for the *Stirling numbers of the first kind*, and (5.2) describes the *Sheffer polynomials* [15] counting permutations w according to $\mathrm{odd}(w)$.

5.1 The space of simple Jordan elements

Finally, we mention an interesting connection between $\mathcal{Z}_n$ and the multilinear part of the *space of simple Jordan elements* within the free associative algebra $\mathbf{k}\langle \mathbf{x}\rangle = \mathbf{k}\langle x_1,\ldots,x_n\rangle$.

Consider a deformation of the Lie bracket on $\mathbf{k}\langle \mathbf{x}\rangle$ by $\alpha\in\mathbb{C}$: $[x,y]_\alpha := xy-\alpha yx$. Let J_α be the smallest $\mathbf{k}$-subspace of $\mathbf{k}\langle \mathbf{x}\rangle$ containing the generators $\mathbf{x}$ and closed under $[\cdot,\cdot]_\alpha$.

For example, $J_1\subset \mathbf{k}\langle \mathbf{x}\rangle$ is the free Lie algebra. Define $V_n(\alpha)\subset J_\alpha$ to be the $\mathbf{k}$-subspace spanned by these multilinear bracketings of homogeneous degree n for $w\in\mathfrak{S}_n$:

$$[[\cdots[x_{w(1)},x_{w(2)}]_\alpha,x_{w(3)}]_\alpha,\cdots]_\alpha,x_{w(n)}]_\alpha$$

Then $V_n(1)\cong \mathrm{Lie}_n$ is the multilinear component of the free Lie algebra, while $V_n(-1)$ is the multilinear part of the *space of simple Jordan elements*. The following was proved by Robbins in [14, §6, Thm. 7] and later in [7, Thm 2.1] by Calderbank–Hanlon–Sundaram:

$$V_n(-1)\cong \bigoplus_{\substack{\lambda\vdash n\\ \mathrm{odd}(\lambda)=\ell(\lambda)}} \mathrm{Lie}_\lambda\,. \tag{5.3}$$

We combine Theorem 4.4 and (5.3), to give a cohomological interpretation for $V_n(-1)$.

Corollary 5.4. *The space $V_n(-1)$ is isomorphic as an $\mathfrak{S}_n$-representation to $\bigoplus_k \mathcal{Z}^{(n)}_{2k,2k}$.*

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A determinantal point process approach to scaling and local limits of random Young tableaux

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Abstract. We obtain scaling and local limit results for large random multirectangular Young tableaux via the asymptotic analysis of a determinantal point process due to Gorin and Rahman (2019). In particular, we find an explicit description of the limiting surface, based on solving a complex-valued polynomial equation. As a consequence, we find a simple criterion to determine if the limiting surface is continuous in the whole domain, implying that, for multirectangular tableaux, the limiting surface is generically discontinuous.

1 Introduction

Random Young diagrams form a classical theme in probability theory, starting with the work of Logan–Shepp and Vershik–Kerov on the Plancherel measure [12, 19]. The topic is deeply connected with random permutations, random matrix theory and particle systems, and has known an increase of interest after the discovery of an underlying determinantal point process for a Poissonized version of the Plancherel measure [4]. It would be vain to do a complete review of the related literature, and we refer only to [8, 17] for books on the topic.

In comparison, random Young tableaux have a shorter history. Motivations to study random Young tableaux range from asymptotic representation theory to connections with other models of combinatorial probability, such as random permutations with short monotone subsequences [16] or most notably random sorting networks; see e.g. [1].

As in most of the literature, we are interested in the simple model where we fix a partition λ (or rather a sequence of growing partitions) and consider a uniform random tableau T of shape λ. In [14], Pittel and Romik derived a limiting surface result for uniform random Young tableaux of rectangular shapes, based on the hook length formula

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and analytic arguments. An earlier result of Biane in asymptotic representation theory [2] implies, in fact, the existence of such limiting surfaces for any underlying shape. However, getting explicit formulas for these limiting surfaces is difficult since their description involves the Markov–Krein correspondence and the free compression of probability measures. More recently, entropy optimization methods have been applied to prove the existence of limiting surfaces, extending the result to skew shapes [18]. These techniques lead to some natural gradient variational problems in $\mathbb{R}^2$ whose solutions are explicitly parameterized by κ-harmonic functions, as shown in [10].

Recently, in [5], a determinantal point process structure was discovered for a Poissonized version of random Young tableaux. This determinantal structure was used for a specific problem motivated by the aforementioned sorting networks, namely describing the local limit of uniform tableaux of staircase shape around their outer diagonal [5, 6].

The goal of the current paper is to exploit this determinantal point process structure in order to get limiting results for a large family of shapes. Namely, we consider shapes obtained as dilatations of any given Young diagram λ^0, i.e. multirectangular diagrams. Here is an informal description of our results.

- We obtain a new description of the limiting surface corresponding to the shape λ^0, based on solving a complex-valued polynomial equation (Theorem 4). This new description is more explicit compared to the one obtained through the existence approaches.

- This first result leads us to a surprising discontinuity phenomenon for the limiting surface corresponding to λ^0. More precisely, we establish a simple criterion – some equations involving the so-called *interlacing coordinates* of λ^0 – to determine whether the limiting surface is continuous (Theorem 6). This shows that such limiting surfaces are typically discontinuous for multirectangular tableaux.

- We also obtain a local limit result in the bulk of random Young tableaux. Due to space constraints, we do not present this result in this extended abstract and refer the interested reader to the long version of the article [3].

Remark 1. *In parallel to this work, explicit formulas for the limiting surfaces of random Young tableaux have also been obtained by Prause [15] through a different method (solving a variational problem obtained by the tangent plane method of Kenyon and Prause [10]).*

2 Results

2.1 Young tableaux and height function

Let us start by fixing terminology and notation. A *partition* of n is a non-increasing list $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers with $N = \sum_{i=1}^{l} \lambda_i$. We write $|\lambda| = N$ for the *size*

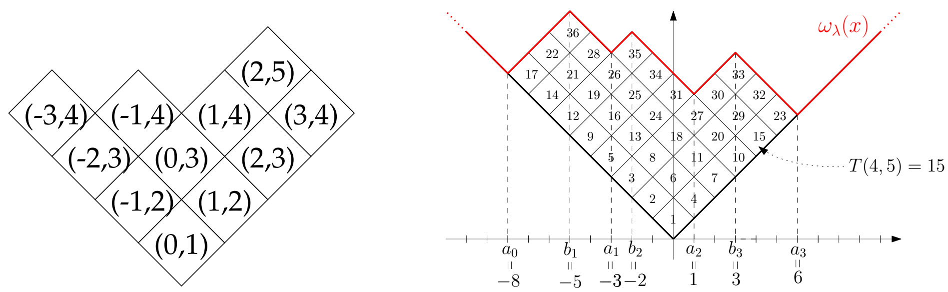


Figure 1: Left: The Young diagram of the partition $(4,4,2,1)$ drawn in Russian convention, with the coordinates of each box inside it. **Right:** A Young tableau $T : \lambda \to [N]$ of shape λ corresponding to the partition $(6,6,6,4,4,4,3,3)$ drawn according to the Russian convention; all the boxes are squares with area 2. We indicate the interlacing coordinates $a_0 < b_1 < a_1 < b_2 < \cdots < b_m < a_m$ below the x-axis.

of the partition and $\ell(\lambda) = l$ for the *length* of the partition and use the convention $\lambda_i = 0$ when $i > \ell(\lambda)$. We represent partitions graphically with the *Russian convention*, i.e. for each $i \leq \ell(\lambda)$ and $j \leq \lambda_i$ we have a square box whose sides are parallel to the lines $x = y$ and $x = -y$ and whose center has coordinates $(j - i, i + j - 1)$; see the left-hand side of Figure 1. This graphical representation is called *Young diagram* of shape λ.

When looking at a Young diagram λ, its upper boundary is the graph of a 1-Lipschitz function, denoted by $\omega_\lambda : \mathbb{R} \to \mathbb{R}$, and the diagram λ can be encoded using the local minima and maxima of the function ω_λ. Following Kerov [11], we denote them by

$$a_0 < b_1 < a_1 < b_2 < \cdots < b_m < a_m, \qquad a_i, b_i \in \mathbb{Z}, \tag{2.1}$$

and we call them *interlacing coordinates*. See the right-hand side of Figure 1 for an example. Note that $a_0 = -\ell(\lambda)$ and $a_m = \lambda_1$. Furthermore, interlacing coordinates satisfy $\sum_{i=0}^m a_i = \sum_{i=1}^m b_i$, see, e.g., [9, Proposition 2.4].

A *Young tableau* of shape λ is a filling of the boxes of λ with the numbers $1, 2, \dots, N$ such that the numbers along every row or column are increasing. We see Young tableaux as functions $T : \lambda \to [N] = \{1, 2, \dots, N\}$, where the Young diagram λ is identified with the set $\{(j - i, i + j - 1), i \leq \ell(\lambda), j \leq \lambda_i\}$; see again the right-hand side of Figure 1 for an example. The function $T : \lambda \to [N]$ can be thought of as the graph of a (non-continuous) surface above the set λ.

We also represent a tableau T of size N by its *height function* H_T (normalized in the second argument). It is a map from $([a_0, a_m] \cap \mathbb{Z}) \times [0,1]$ to $\mathbb{Z}_{\geq 0}$ defined by

$$H_T(x,t) = \#\left\{y : T(x,y) \leq Nt\right\}, \tag{2.2}$$

i.e. $H_T(x,t)$ is the number of entries smaller than Nt in the vertical line $\{x\} \times \mathbb{Z}_{\geq 0}$. Clearly, the tableau T is entirely determined by the height function H_T. Moreover, we have that

$$T(x,y) < Nt \quad \text{if and only if} \quad H(x,t) > \tfrac{1}{2}(y - |x|). \tag{2.3}$$

2.2 Previous results: existence of a limiting height function

We now look at growing Young diagrams, and the associated random tableaux. We fix a Young diagram λ^0, and take our growing sequence of diagrams as dilatations of λ^0. Namely, for $n > 0$, we define $N = N(n, \lambda^0) = n^2|\lambda^0|$ and consider the $(n \times n)$-dilated diagram λ_N obtained by replacing each box of λ^0 by a square of $n \times n$ boxes. Note that λ_N has size N. We set $\eta = 1/\sqrt{|\lambda^0|}$, so that scaling λ^0 in both directions by a factor η gives a diagram of area 2. Finally, we let T_N be a uniform random Young tableau of shape λ_N.

The following convergence result for the height function of T_N is proved in [18, Theorem 7.15]. It also follows indirectly from earlier concentration results on random Young diagrams by Biane [2]; see [13, Proposition 10.1].

Theorem 2 ([18, Theorem 7.15] and [13, Proposition 10.1])**.** *Let λ^0 be a fixed Young diagram and $a_0, \ldots, a_m$ be its interlacing coordinates as defined in* (2.1)*. We let T_N be a uniform random Young tableau of shape λ_N. Then there exists a deterministic function $H^\infty : [\eta\, a_0, \eta\, a_m] \times [0,1] \to \mathbb{R}$ such that the following convergence in probability holds:*

$$\frac{1}{\sqrt{N}}\, H_{T_N}\left(\lfloor x\sqrt{N} \rfloor, t\right) \xrightarrow[N \to +\infty]{} H^\infty(x,t), \tag{2.4}$$

uniformly for all (x,t) in $[\eta\, a_0, \eta\, a_m] \times [0,1]$.

In [18], the limiting function H^∞ is implicitly found to be the unique maximizer of a certain entropy functional subject to some boundary conditions depending on the diagram λ^0. Using the approach of [2, 13], for each $t \in [0,1]$, the section $H^\infty(\cdot, t)$ is described using the free cumulants of an associated probability measure. Both descriptions are difficult to manipulate. Our first result gives an alternative and more explicit description of H^∞ through the solution of a polynomial equation, called the *critical equation*.

2.3 First result: a compact description of the limiting height function

Let $a_0 < b_1 < a_1 < b_2 < \cdots < a_m$ be the interlacing coordinates of λ^0, introduced in (2.1). For (x,t) in $[\eta\, a_0, \eta\, a_m] \times [0,1]$, we consider the following polynomial equation, referred to throughout the paper as the *critical equation*:

$$U \prod_{i=1}^{m} (x - \eta\, b_i + U) = (1-t) \prod_{i=0}^{m} (x - \eta\, a_i + U). \tag{2.5}$$

This is a polynomial equation in the complex variable U of degree $m+1$. Using the fact that the a_i's and b_i's are alternating, one can easily prove that (2.5) has at least $m-1$ real solutions; see [3, Lemma 24] for details. Hence it has either 0 or 2 non-real solutions.

Definition 3. *We let L be the set of pairs (x,t) in $[\eta\, a_0, \eta\, a_m] \times [0,1]$ such that (2.5) has two non-real solutions and we call it* liquid region. *The complement of the liquid region in $[\eta\, a_0, \eta\, a_m] \times [0,1]$ will be referred to as the* frozen region.

For $(x,t) \in L$, we denote by $U_c = U_c(x,t)$ the unique solution with a positive imaginary part of the critical equation (2.5). We use the notation $\Re z$ and $\Im z$ for the real and imaginary parts of the complex number z. It turns out that the limiting height function H^∞ is expressed in terms of U_c using the following simple formula.

Theorem 4. *With the above notation, for $(x,t) \in [\eta\, a_0, \eta\, a_m] \times [0,1]$, we have*

$$H^\infty(x,t) = \frac{1}{\pi} \int_0^t \frac{\Im U_c(x,s)}{1-s} \mathbf{1}[(x,s) \in L]\, \mathrm{d}s\,.$$

Informally, the liquid region is the limit of the region where the height function H_{T_N} is strictly increasing in the t-direction, and the t-derivative of H_{T_N} in this region is roughly given by $\sqrt{N}\, \Im U_c(x,s)/(\pi(1-s))$.

2.4 Limiting surfaces and discontinuities

It is natural to try to translate the limiting result for the height function to a limit result for the tableau itself, seen as a discrete surface. Namely, we set

$$D_{\lambda^0} := \left\{ (x,y) \in \mathbb{R}^2 : |x| < y < \eta\, \omega_{\lambda^0}(x/\eta) \right\}, \tag{2.6}$$

which is the open domain of $\mathbb{R}^2$ corresponding to the diagram λ^0 (in Russian convention), normalized to have area 2. For (x,y) in D_{λ^0}, letting T_N be a uniform tableau of shape λ_N, we consider

$$\widetilde{T}_N(x,y) := \frac{1}{N} T_N\left(\lfloor x\sqrt{N} \rfloor, \lfloor y\sqrt{N} \rfloor + \delta \right), \tag{2.7}$$

where $\delta \in \{0,1\}$ is chosen so that the arguments of T_N have distinct parities. We want to find a scaling limit for the function $\widetilde{T}_N(x,y)$. To this end, we set for all $(x,y) \in D_{\lambda^0}$,

$$\begin{aligned} T_+^\infty = T_+^\infty(x,y) &:= \sup\left\{ t \in [0,1] \,:\, H^\infty(x,t) \le \tfrac{1}{2}(y - |x|) \right\}, \\ T_-^\infty = T_-^\infty(x,y) &:= \inf\left\{ t \in [0,1] \,:\, H^\infty(x,t) \ge \tfrac{1}{2}(y - |x|) \right\}. \end{aligned} \tag{2.8}$$

Comparing Equations (2.3) and (2.8), the following statement is an easy consequence of Theorem 2, see [3] for details.

Proposition 5. *For all $\varepsilon > 0$, the following limit holds uniformly for all $(x,y) \in D_{\lambda^0}$:*

$$\lim_{N\to+\infty} \mathbb{P}\big(\widetilde{T}_N(x,y) < T^\infty_- - \varepsilon\big) = \lim_{N\to+\infty} \mathbb{P}\big(\widetilde{T}_N(x,y) > T^\infty_+ + \varepsilon\big) = 0.$$

We let $D^{\mathrm{reg}}_{\lambda^0}$ be the set of coordinates $(x,y) \in D_{\lambda^0}$ such that $T^\infty_-(x,y) = T^\infty_+(x,y)$. For such points, we simply write $T^\infty(x,y)$ for this common value. Then Proposition 5 implies the following convergence in probability for $(x,y) \in D^{\mathrm{reg}}_{\lambda^0}$:

$$\lim_{N\to+\infty} \widetilde{T}_N(x,y) = T^\infty(x,y), \tag{2.9}$$

On the other hand, for (x,y) in $D_{\lambda^0} \setminus D^{\mathrm{reg}}_{\lambda^0}$, we do not know whether $\widetilde{T}_N(x,y)$ converges at all, and the limiting surface T^∞ is discontinuous at such points.

A natural question is whether such discontinuity points (x,y) exist at all in D_{λ^0}. Our second main result shows that such points indeed exist unless λ^0 is a rectangle, or unless its interlacing coordinates satisfy some specific equations.

Theorem 6. *For a Young diagram λ^0, the following assertions are equivalent:*

1. *The limiting surface T^∞ is continuous in the whole domain D_{λ^0};*
2. *The interlacing coordinates defined in (2.1) satisfy the system of equations:*

$$\sum_{\substack{i=0\\ i\neq i_0}}^{m} \frac{1}{a_{i_0} - a_i} = \sum_{i=1}^{m} \frac{1}{a_{i_0} - b_i}, \qquad \text{for all } i_0 = 1, \dots, m-1. \tag{2.10}$$

Note that when $m = 1$, i.e. when λ^0 has a rectangular shape, there are no equations in the second item. Indeed, the limiting surface T^∞ is always continuous in this case. For $m > 1$ however, the limiting surface is generically discontinuous.

3 Examples

In this section, we illustrate our results in the cases $m = 1$ (rectangular shapes) and $m = 2$ (L-shapes). Before starting, let us note that our model and all results are invariant when multiplying all interlacing coordinates of λ^0 by the same positive integers. We will therefore allow ourselves to work with diagrams λ^0 with rational (non-necessarily integer) interlacing coordinates.

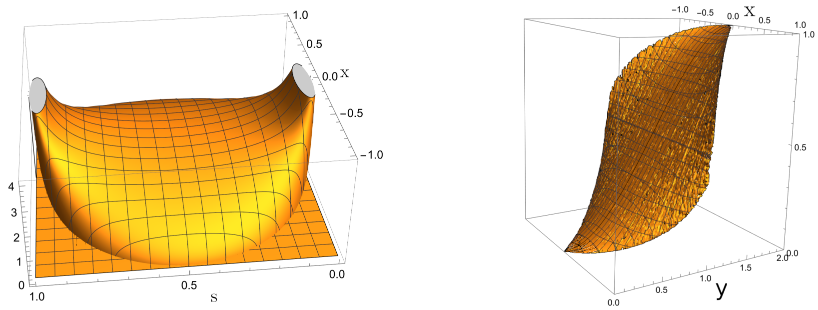


Figure 2: Left: The graphs of the function $\alpha(x,s) = \frac{\sqrt{4s-4s^2-x^2}}{2s-2s^2}$ from Remark 8. **Right:** The corresponding limiting surface $T_1^\infty(x,y)$ for squared diagrams. Note that we are using two different orientations of axes in order to improve the visualization.

3.1 An explicit formula for the rectangular case

In this section, we consider a rectangular diagram λ_0. Without loss of generality, we assume $a_0 = -1$ and write $r = a_1$. Necessarily, $b_1 = r - 1$. Solving explicitly the critical equation (2.5), which is in this case a degree 2 polynomial equation, we get:

Proposition 7. *The limiting height function corresponding to a $1 \times r$ rectangular shape λ^0 is given by*

$$H_r^\infty(x,t) = \frac{1}{\pi}\int_0^t \frac{\sqrt{s(4r-(1+r)^2 s) + 2(r-1)\sqrt{rsx} - rx^2}}{2\sqrt{r}(1-s)s}\,\mathrm{d}s \tag{3.1}$$

with the convention that $\sqrt{x} = 0$ if $x \le 0$. Furthermore, the limiting surface T_r^∞ is continuous on D_{λ^0} and is therefore implicitly determined by the equation

$$H_r^\infty(x, T_r^\infty(x,y)) = \tfrac{1}{2}(y - |x|). \tag{3.2}$$

Remark 8. *In the case $r = 1$ (square Young tableaux), we get*

$$H_1^\infty(x,t) = \frac{1}{\pi}\int_0^t \frac{\sqrt{4s-4s^2-x^2}}{2s-2s^2}\,\mathrm{d}s\,.$$

The graph of the function $\alpha(x,s) = \frac{\sqrt{4s-4s^2-x^2}}{2s-2s^2}$ is plotted on the left-hand side of Figure 2, while the corresponding limiting surface T_1^∞ is on the right. The above integral can be explicitly computed, recovering the formula found by Pittel and Romik from [14]. Pittel and Romik also give formulas for the general rectangular case, which should coincide with (3.1), though we could not verify directly the equivalence of both formulae.

One can also obtain explicit formulas for L-shaped diagrams; since the latter expressions are involved, we decided not to display them, and to discuss examples instead.

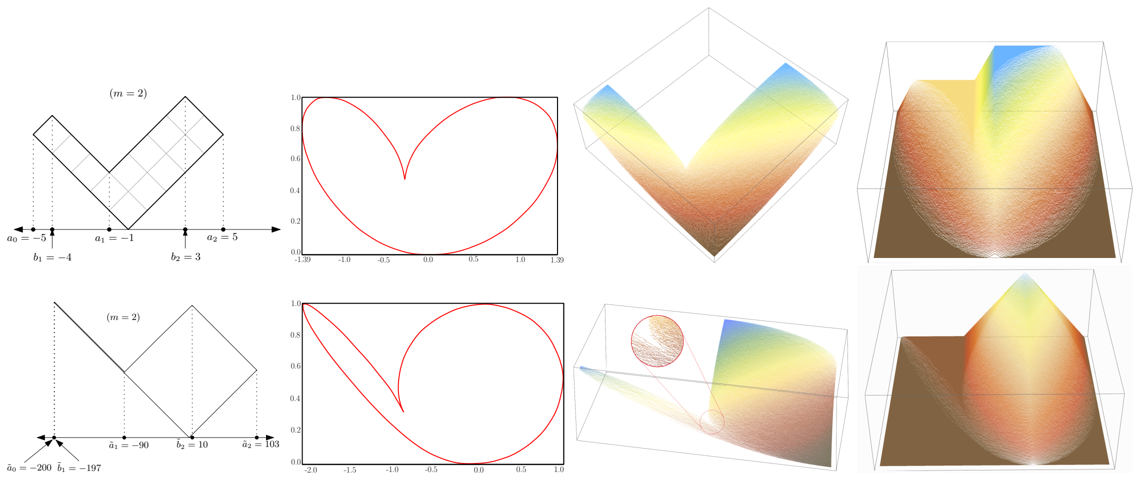


Figure 3: Figures for the heart example (top row) and for the pipe example (bottom row). **In each row, from left to right:** The Young diagram λ^0 or $\widetilde{\lambda}^0$ with its interlacing coordinates, the boundary curve of the corresponding liquid region, a uniform random tableau T_N of shape λ_N (with respectively $N = 130000$ and $N = 59400$ boxes) and the corresponding height function H_{T_N} (in 3D plots, the brown colour is used for small values of the surface and blue for large ones).

3.2 Two concrete examples of L-shape diagrams

We now consider two specific diagrams λ^0 and $\widetilde{\lambda}^0$ which are both L-shaped (i.e. $m = 2$). Due to the shape of the corresponding liquid regions (see the pictures in Figure 3), the first one is called the *heart example* and the second one the *pipe example*.

In the heart example, the Young diagram λ^0 has interlacing coordinates

$$a_0 = -5 \;<\; b_1 = -4 \;<\; a_1 = -1 \;<\; b_2 = 3 \;<\; a_2 = 5. \tag{3.3}$$

In this case we have $|\lambda^0| = 13$, so that $\eta = 1/\sqrt{|\lambda^0|} = 1/\sqrt{13}$ and $[\eta\, a_0, \eta\, a_m] = [-5/\sqrt{13}, 5/\sqrt{13}] \approx [-1.39, 1.39]$.

In the pipe example, the Young diagram $\widetilde{\lambda}^0$ has interlacing coordinates

$$\widetilde{a}_0 = -200 \;<\; \widetilde{b}_1 = -197 \;<\; \widetilde{a}_1 = -90 \;<\; \widetilde{b}_2 = 10 \;<\; \widetilde{a}_2 = 103. \tag{3.4}$$

In this case, we have $|\widetilde{\lambda}^0| = 9900$, so that $\widetilde{\eta} = \frac{1}{30\sqrt{11}}$ and $[\widetilde{\eta}\,\widetilde{a}_0, \widetilde{\eta}\,\widetilde{a}_m] = [-\frac{200}{30\sqrt{11}}, \frac{103}{30\sqrt{11}}] \approx [-2.01, 1.04]$.

For both examples, we have computed the boundary of the liquid region defined in Definition 3. Independently, we have also generated a uniform random tableau of shape λ_N for large N (using the Greene–Nijenhuis–Wilf hook walk algorithm [7]), and we present 3D plots both of the tableau T as a function from λ to $[0, 1]$ and of its height function H_T (which is a function from $[na_0, na_m] \times [0,1]$ to $\mathbb{Z}_{\geq 0}$). In both cases, the domain where the height function H_T is increasing in t fits very well with the liquid region, as predicted by Theorem 4.

An essential difference between the two examples is that the interlacing coordinates satisfy Condition (2.10) in the heart example, while this is not the case in the pipe example. From Theorem 6, the limiting surface is continuous in the heart example and not in the pipe example. This is indeed visible on the pictures, as we now explain.

In the heart example, the intersection of the liquid region with any vertical line is connected; in other terms, for every $x \in [\eta\, a_0, \eta\, a_m]$, the function $t \mapsto H^\infty(x,t)$ is first constant equal to 0, then strictly increasing, and then constant equal to its maximal value. Therefore, with the notation of (2.8), we have $T^\infty_-(x,y) = T^\infty_+(x,y)$ for all (x,y) in D_{λ^0} and the limiting surface T^∞ is defined and continuous on the whole set D_{λ^0}. Looking at the random tableau drawn as a discrete surface, it is indeed plausible that it converges to a continuous surface.

In the pipe example, however, we can find some x_0 just on the right of $\eta\, \widetilde{a}_1 = -\frac{3}{\sqrt{11}} \approx -0.9$ so that the liquid region intersects the line $x_0 \times [0,1]$ in two disjoint intervals. The function $t \mapsto H^\infty(x_0,t)$ is then constant, equal to some value y_0 between these two intervals. It follows that $T^\infty_-(x_0,y_0) < T^\infty_+(x_0,y_0)$ and the limiting surface T^∞ is discontinuous at (x_0,y_0). This discontinuity can be observed on the 3D plot of the tableau T_N (see the zoom inside the red circle on the left-hand side, where we observe a jump in the values of T_N).

4 Proof strategy

We now discuss the proof strategy of Theorems 4 and 6. Details can be found in [3].

4.1 Poissonized tableaux and determinantal point process

Following [5], we define a *Poissonized Young tableau* of shape λ as a function $\lambda \to [0,1]$ satisfying the same monotonicity constraints as standard tableaux. We encode such a tableau T by a set M_T of particles in $\mathbb{Z} \times [0,1]$ defined as

$$M_T = \big\{(x, T(x,y)), (x,y) \in \lambda\big\}.$$

A remarkable result of [5] states that, for any shape λ, if T is a uniform random Poissonized tableau of shape λ, then M_T is a determinantal point process with kernel

$$K_\lambda((x_1,t_1),(x_2,t_2)) = -\frac{1}{(2i\pi)^2} \oint_{\gamma_z} \oint_{\gamma_w} \frac{F_\lambda(z)}{F_\lambda(w)} \frac{\Gamma(w-x_1+1)}{\Gamma(z-x_2+1)} \frac{(1-t_2)^{z-x_2}\,(1-t_1)^{-w+x_1-1}}{z-w}\, \mathrm{d}w\, \mathrm{d}z\,, \quad (4.1)$$

where

- $F_\lambda(u) := \Gamma(u+1) \prod_{i=1}^{\infty} \frac{u+i}{u-\lambda_i+i} = \frac{\prod_{i=0}^{m} \Gamma(u-a_i+1)}{\prod_{i=1}^{m} \Gamma(u-b_i+1)}$;
- γ_w and γ_z are counterclockwise contours containing all the integers in $[a_0, x_1]$ and in $[x_2, a_m]$ respectively;
- γ_w and is inside (resp. outside) γ_z if $t_1 \geq t_2$ (resp. $t_1 < t_2$);
- the ratio $\frac{1}{z-w}$ remains uniformly bounded on the contours γ_w and γ_z.

4.2 Asymptotic behaviour of the kernel

To prove Theorem 4, we look for the asymptotic behaviour of the kernel in the regime

$$x_i = x_0\sqrt{N} + \xi_i, \quad t_i = t_0 + \tfrac{\tau_i}{\sqrt{N}} \qquad (i = 1, 2),$$

where (x_0, t_0) is fixed in $[\eta a_0, \eta a_m] \times [0, 1]$. In particular, the density of particles in M_T around $(x_0\sqrt{N}, t_0)$, normalized by $1/\sqrt{N}$, is given by $K_\lambda((x_0\sqrt{N}, t_0), (x_0\sqrt{N}, t_0))$, corresponding hence to $\xi_1 = \xi_2 = \tau_1 = \tau_2 = 0$. In this regime, a careful asymptotic analysis shows that the integrand in Equation (4.1) behaves as

$$\mathrm{Int}_N(W, Z) \simeq (\sqrt{N})^{\xi_2 - \xi_1} \mathrm{e}^{\sqrt{N}(S(W) - S(Z))}\, h(W, Z), \tag{4.2}$$

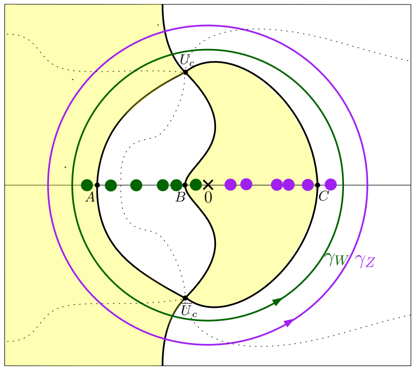
for some function S and h. The critical equation (2.5) corresponds to the equation $S'(U) = 0$, i.e. its solutions are critical points of S. The idea is then to move the integration contours so that $S(W) < S(Z)$ on the new contours, making the integrand and thus the integral tend to 0. Moving the integration contour may yield a residue term, which gives the non-trivial asymptotic of $K_\lambda((x_0\sqrt{N}, t_0), (x_0\sqrt{N}, t_0))$.

Let us explain briefly how this works when (x_0, t_0) is in the liquid region. By definition, in this case, S has two non-real critical points, which are necessarily conjugate, that we denote by U_c and $\bar{U}_c$. Comparing $\Re S(U)$ (for generic U) to $\Re S(U_c)$ divides the complex plane into regions as shown on Figure 4 (the shape of those regions is carefully justified in the long version of the paper [3]). We then move the integration contours so that $S(W) < S(U_c) < S(Z)$ almost everywhere on the new contours. Note that the new contour γ_W^{new} is not inside γ_Z^{new}, while γ_w is inside γ_z since $t_1 = t_2$ in the case of interest. Thus moving the contours yields a residue term associated with the pole $Z = W$ in Equation (4.1), which can be computed explicitly. We find

$$\lim_{N\to+\infty} K_\lambda((x_0\sqrt{N}, t_0), (x_0\sqrt{N}, t_0)) = \frac{\Im U_c(x_0, t_0)}{\pi(1-t_0)},$$

which implies after some work Theorem 4.

We note that the general proof strategy is standard in integrable probability but needs many careful estimations and arguments to justify the existence of the appropriate contours, and the asymptotic behaviour of the various terms (see [3]).



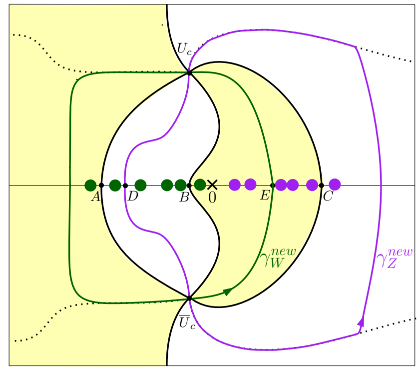


Figure 4: Left: The yellow regions correspond to $\{\Re S(U) < \Re S(U_c)\}$, while the white regions correspond to $\{\Re S(U) > \Re S(U_c)\}$. We plotted the original integration contours γ_W (in green) and γ_Z (in purple) appearing in Equation (4.1). The green and purple dots are respectively the W-poles and Z-poles of the integrand. **Right:** The new integration contours so that $S(W) < S(Z)$ almost everywhere on the contours.

4.3 Characterization of continuous limit shapes

We now discuss the proof of Theorem 6. Looking at the shape of the liquid regions in Figure 3 and at the discussions on the heart and pipe examples, we see that the limit shape is continuous if and only if the tangent vectors to the boundary of the liquid region at its cusp points are all vertical. The boundary of the liquid region is precisely the set of points (x, t) where the discriminant of the polynomial equation (2.5) vanishes, see [3, Proposition 27]. Using this description, we can obtain an explicit parametrization of this boundary curve, and compute the tangent vectors at its cusp points. Each cusp point gives one of the condition given in Equation (2.10), concluding the proof of the theorem.

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Generalized Heawood graphs and triangulations of tori

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Abstract. The Heawood graph is a remarkable graph that played a fundamental role in the development of the theory of graph colorings on surfaces in the 19th and 20th centuries. Based on permutahedral tilings, we introduce a generalization of the classical Heawood graph indexed by a sequence of positive integers. We show that the resulting generalized Heawood graphs are toroidal graphs, which are dual to higher dimensional triangulated tori. We also present explicit combinatorial formulas for their f-vectors and study their automorphism groups.

Keywords: Heawood graph, triangulations of tori, permutahedron, map coloring.

1 Introduction

The Heawood graph is a remarkable graph which played a fundamental role in the historical development of the theory of map colorings on surfaces. The four color theorem is an important result in this area, and perhaps one of the most well known results in mathematics in general. It states that for any map on a sphere, for example Europe, there is a coloring of that map with four colors, such that each region (or country) has one color and any two adjacent regions[1] have different colors. This problem has an interesting history dating back to 1852, but the theorem was only proved more than a hundred years later in 1976 by Kenneth Appel and Wolfgang Haken [1] after many false proofs and false counterexamples, and it is the first major result in mathematics that was proved using a computer.

One famous false proof of the four color theorem was given by Alfred Kempe in 1879 [4]. His proof was announced in *Nature* [5] and was regarded as an established fact for more than a decade. In 1890, Percy John Heawood found a gap in Kempe's proof, and modified his argument to show that five colors are sufficient to color a map on a sphere [3]. This became known as the five color theorem.

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[1]Two regions are adjacent if they share a common boundary curve segment, not just a point.

In the same paper [3], Heawood investigated coloring of maps on other surfaces. He showed that $N_p = \left\lfloor \frac{7+\sqrt{1+48p}}{2} \right\rfloor$ colors are sufficient to color a map on the oriented surface of genus $p \geq 1$, where $\lfloor x \rfloor$ is the largest integer not greater than x. For instance, it is possible to color any map on a torus (genus $p = 1$ surface) using seven colors. Heawood also showed that for $p = 1$ the number seven is tight, by showing a map of the torus where seven colors are necessary: a map consisting of seven regions for which any two regions are adjacent to each other.

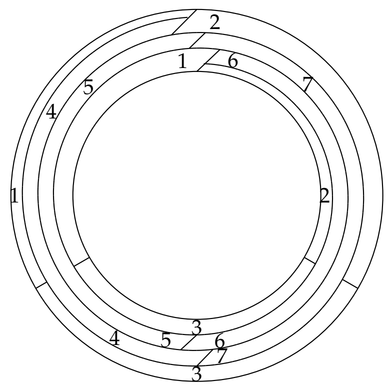


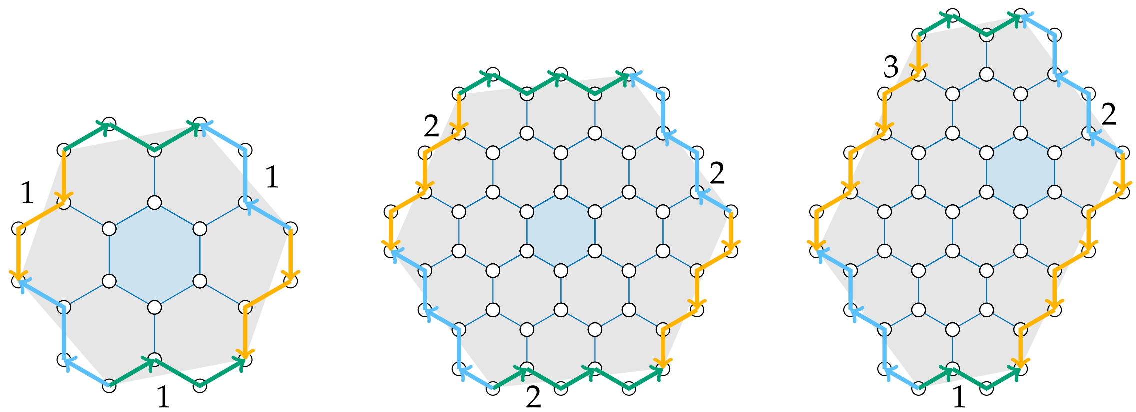
Figure 1: Reproduction of Heawood's map on a torus from 1890. The inner and outer circle are identified to produce a torus.

The fact that the number N_p is tight for a genus p orientable surface became known as Heawood's Conjecture, and was finally proved in 1968 [10]. The case $p = 1$ (the torus) is known as the seven color theorem, and has inspired beautiful math-art works.

The Heawood graph is defined as the graph of Heawood's map: its vertices are the common points of three pairwise adjacent regions, and the edges are the lines connecting these points. It is a toroidal and distance-transitive graph on 14 vertices and 21 edges. Our favorite representation of Heawood's graph is illustrated in Figure 2a, which is based on a highly symmetric representation due to Leech in [8, Figure 2]. Note that here, the graph is the graph induced by the edge graph of the seven hexagons, where the boundary is identified by gluing the opposite colored lines as illustrated.

The main purpose of this paper is to introduce a generalization $H_{\mathbf{k}}$ of Heawood's graph that extends Leech's representation. Our generalization is indexed by a sequence $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ of positive integers for some $d \geq 2$, and recovers the classical Heawood graph when $\mathbf{k} = (1,1,1)$. As in the classical case, we show that $H_{\mathbf{k}}$ is a toroidal graph which is naturally embedded in a d-dimensional torus.

When there are three parameters, the generalized Heawood graph $H_{(k_1,k_2,k_3)}$ is a 2-dimensional generalization of the classical Heawood graph. It is obtained by gluing together $\prod(k_i + 1) - \prod k_i$ regular hexagons: From a "central" hexagon one adds k_i



(a) $\mathbf{k} = (1, 1, 1)$ (b) $\mathbf{k} = (2, 2, 2)$ (c) $\mathbf{k} = (3, 1, 2)$

Figure 2: Examples of the Heawood graph $H_{\mathbf{k}}$ in dimension 2. The opposite sides (with the same color) are identified, making this graph a toroidal graph. The torus is the gray hexagon with opposite edges identified.

hexagons pointing in the direction at angle $(i-2)\frac{2\pi}{3}$ for $i = 1, 2, 3$; then fill the "big hexagon" that they generate with other small hexagons. Several examples are illustrated in Figure 2. We also provide three different choices of fundamental domain in Figure 3, where the torus can be visualized in its more common rectangular presentation.

The case $d = 3$ gives 3-dimensional generalizations of the Heawood graph. The smallest choice of parameters is $H_{(1,1,1,1)}$, which is obtained by gluing $15 = 2^4 - 1^4$ polytopes that are 3-dimensional permutahedra, see Figure 4. The boundary of the result is identified to itself to form the complex into a 3-dimensional torus (see Section 4.2).

One special object of interest is the dual triangulation of $H_{(1,1,\ldots,1)}$. This triangulation consists of $2^{d+1} - 1$ vertices and appeared in the work of Wolfgang Kühnel and Gunter Lassmann from the 1980's in [6, 7]. Interestingly, it is conjectured to be a minimal triangulation of the d-dimensional torus [9, Conjecture 21].

A longer version of this extended abstract with more details and proofs is available at [2].

2 The generalized Heawood graph

The generalized Heawood graph $H_{\mathbf{k}}$ is indexed by a sequence $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ of positive integers for some $d \geq 2$. It is obtained by making some identifications on an infinite graph $\widetilde{G}_d$, which is the graph of the d-dimensional permutahedral tiling. Before explaining this connection, we provide a direct definition in this section.

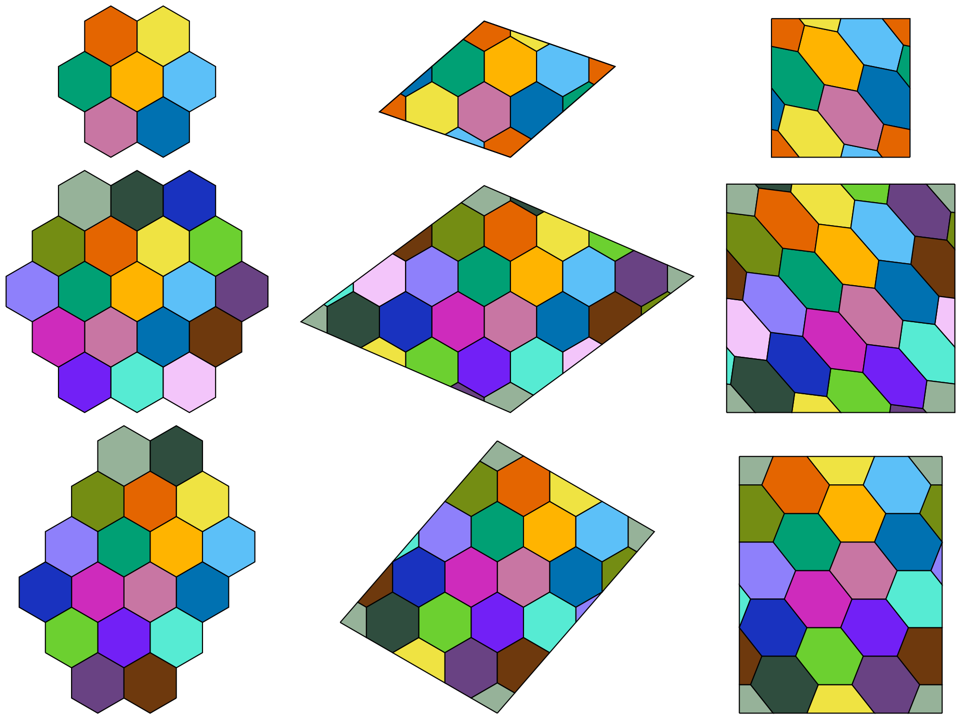


Figure 3: Different presentations of the fundamental domain for the Heawood graphs $H_{(1,1,1)}$, $H_{(2,2,2)}$ and $H_{(3,1,2)}$.

The vertices $\mathrm{Vert}(\widetilde{G}_d)$ of the *graph* $\widetilde{G}_d$ are the elements of the affine subspace

$$\{\mathbf{x} = (x_1, \ldots, x_{d+1}) : \; x_1 + \cdots + x_{d+1} = 1 + \cdots + (d+1)\} \subset \mathbb{R}^{d+1}$$

whose entries are integers containing all the numbers $1, 2, \ldots, d+1$ mod $(d+1)$. For instance, all permutations of $[d+1]$ satisfy this property. Two vertices $\mathbf{x}, \mathbf{y}$ of $\widetilde{G}_d$ are connected by an edge if $\mathbf{y} - \mathbf{x} = e_j - e_i$ for some $i \neq j$, where $e_1, \ldots, e_{d+1}$ denote the standard basis vectors in $\mathbb{R}^{d+1}$. Figure 5 shows a portion of the graph $\widetilde{G}_2$, where the blue hexagon is the convex hull of all permutations of [3].

For $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ we denote by $M_{\mathbf{k}}$ the matrix

$$M_{\mathbf{k}} = \begin{pmatrix} k_1+1 & -k_2 & & & & \\ & k_2+1 & -k_3 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & k_d+1 & -k_{d+1} \\ -k_1 & & & & & k_{d+1}+1 \end{pmatrix} \tag{2.1}$$

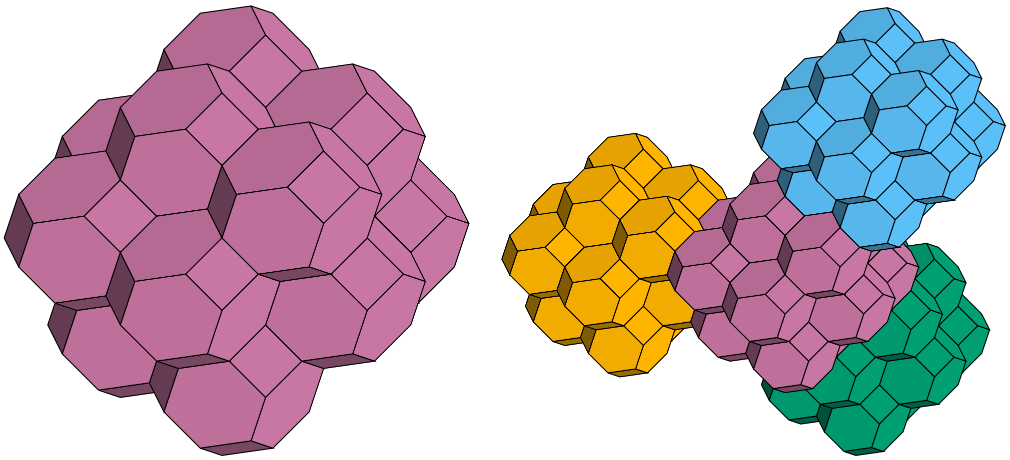


Figure 4: The Heawood graph $H_{(1,1,1,1)}$ is the edge graph of this portion of the 3-dimensional permutahedral tiling after properly identifying its boundary by translations (see Section 4.2), making it into a toroidal graph.

and let $w_1, \ldots, w_{d+1} \in \mathbb{Z}^{d+1}$ be the vectors

$$w_i = (d+1)e_i - \sum_{j=1}^{d+1} e_j. \tag{2.2}$$

Equivalently, w_i has ith coordinate equal to d and all other coordinates equal to -1.

Note that if $\mathbf{x} \in \mathrm{Vert}(\widetilde{G}_d)$ then $\mathbf{x} + w_i \in \mathrm{Vert}(\widetilde{G}_d)$. Moreover, if $\mathbf{x}, \mathbf{y} \in \mathrm{Vert}(\widetilde{G}_d)$ are connected by an edge then $\mathbf{x} + w_i$ and $\mathbf{y} + w_i$ are connected by an edge as well. In other words, the graph $\widetilde{G}_d$ is invariant under translations by the vectors $w_1, \ldots, w_{d+1}$.

We denote by $\mathcal{L}_d$ the lattice of integer linear combinations of the w_i

$$\mathcal{L}_d := \{a_1 w_1 + \cdots + a_{d+1} w_{d+1} : \ a_1, \ldots, a_{d+1} \in \mathbb{Z}^{d+1}\}, \tag{2.3}$$

and by $\mathcal{S}_\mathbf{k} \subset \mathcal{L}_d$ the sublattice

$$\mathcal{S}_\mathbf{k} := \left\{ a_1 w_1 + \cdots + a_{d+1} w_{d+1} : \begin{array}{l} (a_1, \ldots, a_{d+1}) = (b_1, \ldots, b_{d+1}) M_\mathbf{k} \\ \text{for some } b_1, \ldots, b_{d+1} \in \mathbb{Z} \end{array} \right\}. \tag{2.4}$$

That is, $\mathcal{S}_\mathbf{k}$ is the set of linear combinations of $w_1, \ldots, w_{d+1}$ whose coefficient vector $(a_1, \ldots, a_{d+1})$ is an integer linear combination of the rows of $M_\mathbf{k}$.

We say that $\mathbf{x}, \mathbf{y} \in \mathrm{Vert}(\widetilde{G}_d)$ are **k**-*equivalent*, in which case we write $\mathbf{x} \sim_\mathbf{k} \mathbf{y}$, if

$$\mathbf{y} = \mathbf{x} + v \text{ for some } v \in \mathcal{S}_\mathbf{k}. \tag{2.5}$$

Two edges of $\widetilde{G}_d$ are **k**-*equivalent* if one is a translation of the other by a vector in $\mathcal{S}_\mathbf{k}$.

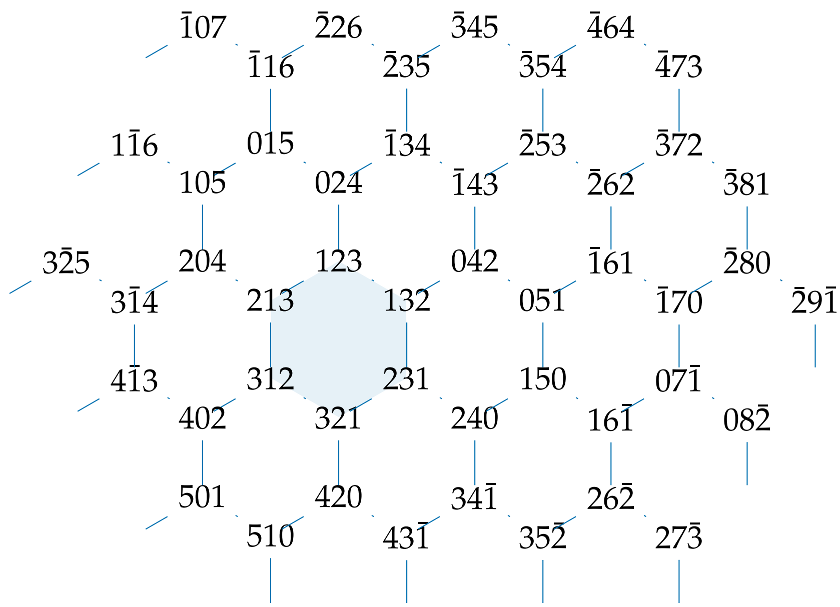


Figure 5: The graph $\widetilde{G}_2$ of the permutahedral tiling for $d = 2$. Commas and parenthesis are omited for simplicity. An overlined number $\bar{k}$ represent the negative number $-k$. For instance, $\bar{1}43$ represents the vertex $(-1, 4, 3)$.

Definition 2.1 (Generalized Heawood graph)**.** Let $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ be a sequence of positive integers for some $d \geq 2$. The *Heawood graph* $H_{\mathbf{k}}$ is the graph whose vertices and edges are the $\mathbf{k}$-equivalent classes of vertices and edges of $\widetilde{G}_d$, respectively. In other words, $H_{\mathbf{k}}$ is the graph obtained by identifying vertices and edges of $\widetilde{G}_d$ up to translation by vectors in $\mathcal{S}_{\mathbf{k}}$.

Example 2.2 (Classical Heawood graph)**.** The classical Heawood graph is obtained when $d = 2$ and $\mathbf{k} = (1, 1, 1)$, and is illustrated in Figure 6. The lattice $\mathcal{L}_2$ consists of integer linear combinations of the vectors $w_1 = (2, -1, -1)$, $w_2 = (-1, 2, -1)$, $w_3 = (-1, -1, 2)$.

The associated matrix is

$$M_{(1,1,1)} = \begin{pmatrix} 2 & -1 & \\ & 2 & -1 \\ -1 & & 2 \end{pmatrix}$$

The sublattice $\mathcal{S}_{(1,1,1)}$ consists of integer linear combinations of the rows of this matrix, when considered as vectors of coefficients of the w_i's, i.e. integer linear combinations of the vectors $2w_1 - w_2$, $2w_2 - w_3$, $2w_3 - w_1$.

Figure 6 shows a tiling of the plane, where each fundamental tile consists of seven hexagons: one hexagon in the center together with its six surrounding hexagons. The barycenters of the central hexagons correspond exactly to elements of the sublattice $\mathcal{S}_{(1,1,1)}$. The equivalence relation $\cong_{\mathbf{k}}$ then identifies vertices and edges via translations that transform one fundamental tile into another.

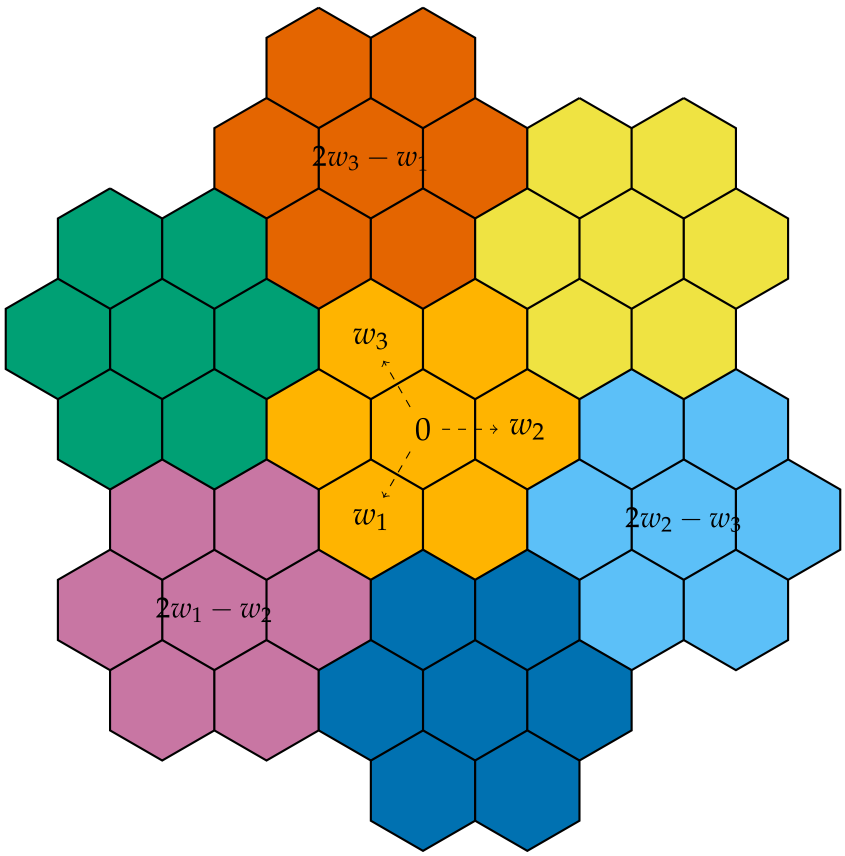


Figure 6: The classical Heawood graph $H_{(1,1,1)}$ as a quotient of the graph of the permutahedral tiling in dimension two.

Our aim is to prove some structural and enumerative properties of the generalized Heawood graph. Our first result is the following.

Theorem 2.3. *The generalized Heawood graph $H_{\mathbf{k}}$ is a vertex-transitive graph with $d!D_{\mathbf{k}}$ many vertices and $\frac{(d+1)!}{2}D_{\mathbf{k}}$ many edges, where*

$$D_{\mathbf{k}} = \det M_{\mathbf{k}} = \prod (k_i + 1) - \prod k_i. \tag{2.6}$$

Similarly to the classical case, the generalized Heawood graph is the dual graph of a triangulated torus, for which a simple combinatorial formula for its number of faces can be explicitly given.

We denote by $\left\{ {n \atop k} \right\}$ the Stirling number of the second kind, which counts the number of ways to partition a set of n objects into k non-empty subsets. These numbers can be explicitly calculated as

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n. \tag{2.7}$$

Theorem 2.4. *The generalized Heawood graph $H_{\mathbf{k}}$ is the dual graph of a triangulation of a d-dimensional torus with f-vector $(f_0, f_1, \dots, f_d)$ determined by*

$$f_i = i! \left\{ {d+1 \atop i+1} \right\} D_{\mathbf{k}}. \tag{2.8}$$

In particular,

$$f_0 = D_{\mathbf{k}}, \qquad f_d = d!\, D_{\mathbf{k}}, \qquad f_{d-1} = \frac{(d+1)!}{2} D_{\mathbf{k}}. \tag{2.9}$$

Table 1 shows the factor $c(i,d) := f_i / D_{\mathbf{k}}$ for some small values.

$d \backslash i$	0	1	2	3	4	5
2	1	3	2			
3	1	7	12	6		
4	1	15	50	60	24	
5	1	31	180	390	360	120

Table 1: The factor $c(i,d)$ for some small values of i and d.

Example 2.5 ($d = 2$)**.** We consider the classical Heawood graph, when $\mathbf{k} = (1,1,1)$. The factor $D_{(1,1,1)} = 2^3 - 1^3 = 7$ counts the number of hexagons in Figure 2a. The f-vector of its dual 2-dimensional triangulated torus is

$$(1 \cdot 7, 3 \cdot 7, 2 \cdot 7) = (7, 21, 14).$$

Interpreting this in the graph setting, we have 7 hexagons, 21 edges, and 14 vertices.

When $d = 2$, with a general $\mathbf{k}$, we have $D_{\mathbf{k}}$ many hexagons, $3D_{\mathbf{k}}$ many edges, and $2D_{\mathbf{k}}$ many vertices. Table 2 shows these numbers for all the examples in Figure 2.

3 The affine arrangement and the permutahedral tiling

In order to prove these results, it is useful to build on the connection with permutahedral tilings and their dual affine arrangements. We consider the collection of affine hyperplanes

$$H_{ij}^{k} = \{ \mathbf{x} \in \mathbb{R}^{d+1} : \; x_j - x_i = k \}$$

$\mathbf{k}$ \ i	0	1	2
$(1,1,1)$	$1\cdot 7$	$3\cdot 7$	$2\cdot 7$
$(2,2,2)$	$1\cdot 19$	$3\cdot 19$	$2\cdot 19$
$(3,1,2)$	$1\cdot 18$	$3\cdot 18$	$2\cdot 18$

Table 2: Number of hexagons, edges, and vertices for the Heawood graphs in Figure 2.

for $1 \leq i < j \leq d+1$ and $k \in \mathbb{Z}$.

The *affine Coxeter arrangement* $\widetilde{\mathcal{H}}_d$ of type $\widetilde{A}_d$ is the restriction of this arrangement to the hyperplane $V = \{\mathbf{x} \in \mathbb{R}^{d+1} :\ x_1 + \cdots + x_{d+1} = 0\}$. For $d = 2$, this is the arrangement of affine hyperplanes of a triangular lattice, which is illustrated on the left of Figure 7.

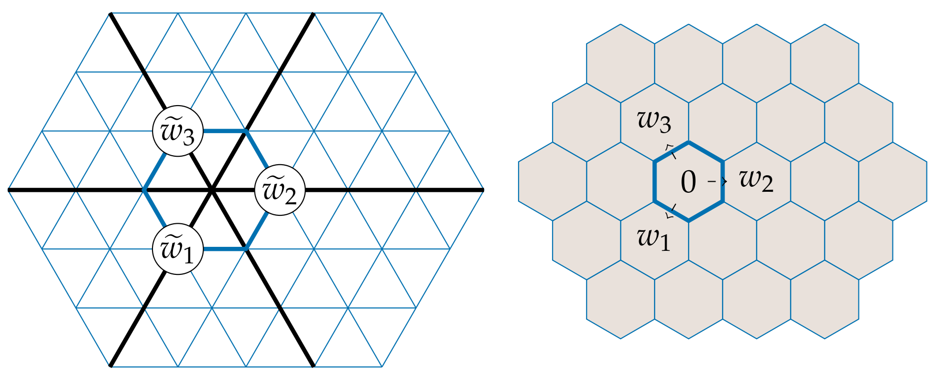


Figure 7: A finite piece of the simplicial complex $\widetilde{\mathcal{C}}_2$ of the affine Coxeter arrangement of type $\widetilde{A}_2$ (left). A finite piece of its dual tiling of space by permutahedra $\mathcal{PT}_2$ (right).

In general, the arrangement $\widetilde{\mathcal{H}}_d$ decomposes the space V into an infinite number of simplices, giving rise to an infinite simplicial complex that we denote by $\widetilde{\mathcal{C}}_d$. The vertices of this complex are the elements of

$$\widetilde{\mathcal{L}}_d := \{\mathbf{x} \in V :\ x_j - x_i \in \mathbb{Z},\ \text{for all } 1 \leq i < j \leq d+1\}. \tag{3.1}$$

This set is a lattice, which is known as the *weight lattice* of type A_d. It is generated by integer linear combinations of the vectors $\widetilde{w}_1, \ldots, \widetilde{w}_{d+1}$ determined by $(d+1)\widetilde{w}_i = w_i$, as defined in (2.2). They satisfy the relation

$$\widetilde{w}_1 + \cdots + \widetilde{w}_{d+1} = 0. \tag{3.2}$$

For $d = 2$, the dual of the triangular lattice is the hexagonal lattice, which is illustrated on the right of Figure 7. This generalizes to higher dimensions, where the dual of the complex $\widetilde{\mathcal{C}}_d$ is a combinatorial structure known as the permutahedral tiling.

The *permutahedron* Perm_d is the convex hull of all permutations of $[d+1]$:

$$\mathrm{Perm}_d = \mathrm{conv}\left\{(i_1,\dots,i_{d+1}) : \text{ for } \{i_1,\dots,i_{d+1}\} = [d+1]\right\} \subseteq \mathbb{R}^{d+1}. \tag{3.3}$$

The *permutahedral tiling* $\mathcal{PT}_d$ is the infinite tiling of the affine subspace

$$\{\mathbf{x} \in \mathbb{R}^{d+1} : \ x_1 + \cdots + x_{d+1} = 1 + \cdots + (d+1)\} \tag{3.4}$$

whose tiles are translates $\mathrm{Perm}_d + v$ of the permutahedron, for $v \in \mathcal{L}_d$. An example for $d = 2$ is shown on the right of Figure 7.

4 The triangulated torus and the Heawood complex

4.1 The triangulated torus

We consider the *sublattice* $\widetilde{\mathcal{S}}_{\mathbf{k}} \subset \widetilde{\mathcal{L}}_d$ defined by

$$\widetilde{\mathcal{S}}_{\mathbf{k}} := \left\{ a_1\widetilde{w}_1 + \cdots + a_{d+1}\widetilde{w}_{d+1} : \begin{array}{l} (a_1,\dots,a_{d+1}) = (b_1,\dots,b_{d+1})M_{\mathbf{k}} \\ \text{for some } b_1,\dots,b_{d+1} \in \mathbb{Z} \end{array} \right\}. \tag{4.1}$$

Its elements are integer linear combinations of the vectors $\widetilde{w}_1,\dots,\widetilde{w}_{d+1}$, whose vector of coefficients is an integer linear combination of the rows of the matrix $M_{\mathbf{k}}$.

We say that two faces $F, F' \in \widetilde{\mathcal{C}}_d$ are **k**-*equivalent*, and write $F \sim_{\mathbf{k}} F'$, if $F' = F + v$ for some $v \in \widetilde{\mathcal{S}}_{\mathbf{k}}$. That is, the face F' is a translation of F by $v \in \widetilde{\mathcal{S}}_d$.

Definition 4.1 (The torus). The *quotient complex* $\mathcal{T}_{\mathbf{k}} = \widetilde{\mathcal{C}}_d / \widetilde{\mathcal{S}}_{\mathbf{k}}$ is the simplicial complex of **k**-equivalent classes of faces of $\widetilde{\mathcal{C}}_d$. In other words, $\mathcal{T}_{\mathbf{k}}$ is the simplicial complex of faces of $\widetilde{\mathcal{C}}_d$ up to translation by vectors in $\widetilde{\mathcal{S}}_{\mathbf{k}}$.

We define the *fundamental vectors* with respect to **k** as the elements of the set

$$\widetilde{F}_{\mathbf{k}} = \left\{ a_1\widetilde{w}_1 + \cdots + a_{d+1}\widetilde{w}_{d+1} : \begin{array}{c} 0 \le a_i \le k_i \in \mathbb{Z}^{d+1} \\ \text{at least one } a_i = 0 \end{array} \right\}. \tag{4.2}$$

Lemma 4.2. *The quotient* $\widetilde{\mathcal{L}}_d / \widetilde{\mathcal{S}}_{\mathbf{k}}$ *is finite. Its cardinality is* $\det M_{\mathbf{k}} = \prod(k_i+1) - \prod k_i$. *The fundamental vectors in* $\widetilde{F}_{\mathbf{k}}$ *are element representatives of the classes of* $\widetilde{\mathcal{L}}_d / \widetilde{\mathcal{S}}_{\mathbf{k}}$.

Proposition 4.3. $\mathcal{T}_{\mathbf{k}}$ *is a triangulated d-dimensional torus on* $D_{\mathbf{k}} = \det M_{\mathbf{k}}$ *many vertices.*

The proof of this proposition is based on a parallelepiped domain of $\mathcal{T}_{\mathbf{k}}$, which we explain in the longer version of this manuscript [2], see the first illustration in Figure 8.

We also provide a permutahedral domain, see the second illustration in Figure 8, which leads to the following independent result.

Proposition 4.4. *The permutahedron* Perm_d *with opposite facets identified by translation is a topological d-dimensional torus.*

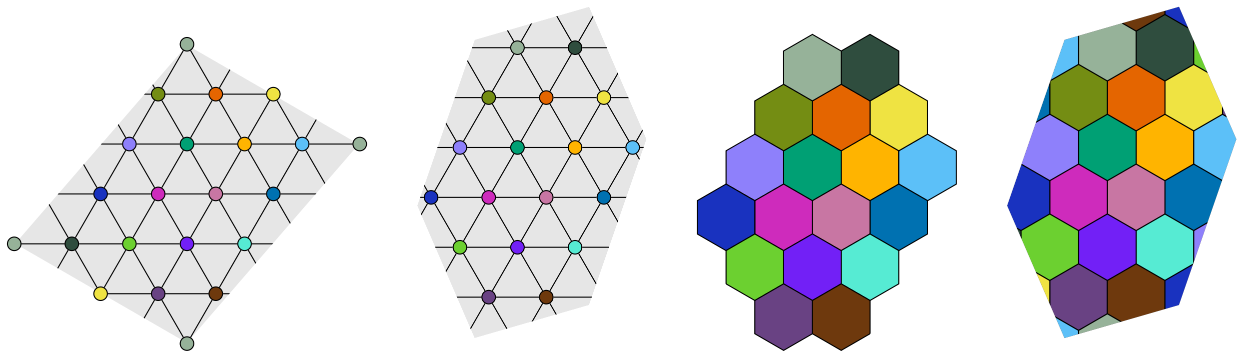


Figure 8: The parallelepiped domain and the permutahedron domain of $\mathcal{T}_{(3,1,2)}$, and the fundamental tile and the permutahedron domain of $\mathcal{HC}_{(3,1,2)}$.

4.2 The Heawood complex

We say that two faces $\mathbf{B}, \mathbf{B}'$ of the permutahedral tiling $\mathcal{PT}_d$ are **k**-*equivalent*, and write $\mathbf{B} \sim_{\mathbf{k}} \mathbf{B}'$, if $\mathbf{B}' = \mathbf{B} + v$ for some $v \in \mathcal{S}_{\mathbf{k}}$. That is, the face $\mathbf{B}'$ is a translation of $\mathbf{B}$ by a vector $v \in \mathcal{S}_{\mathbf{k}}$.

Definition 4.5 (The Heawood complex)**.** The *Heawood complex* $\mathcal{HC}_{\mathbf{k}} = \mathcal{PT}_d / \mathcal{S}_{\mathbf{k}}$ is the polytopal complex of **k**-equivalent classes of faces of $\mathcal{PT}_d$. That is, $\mathcal{HC}_{\mathbf{k}}$ is the polytopal complex of faces of $\mathcal{PT}_d$ up to translation by vectors in $\mathcal{S}_{\mathbf{k}}$.

We define

$$F_{\mathbf{k}} = \left\{ a_1 w_1 + \cdots + a_{d+1} w_{d+1} : \begin{array}{c} 0 \leq a_i \leq k_i \in \mathbb{Z}^{d+1} \\ \text{at least one } a_i = 0 \end{array} \right\}. \tag{4.3}$$

Since $w_i = (d+1)\widetilde{w}_i$, then the vectors in $F_{\mathbf{k}}$ are just the fundamental vectors in $\widetilde{F}_{\mathbf{k}}$, dilated by a factor of $d+1$. The *fundamental tile* $P_{\mathbf{k}}$ is the union of the permutahedra of the form $\mathrm{Perm}_d + v$ with $v \in F_{\mathbf{k}}$. An example of the fundamental tile $P_{(1,1,1)}$, including six translations of it, is illustrated in Figure 6.

In general, translations of the fundamental tile $P_{\mathbf{k}}$ by elements of the sublattice $\mathcal{S}_{\mathbf{k}}$ tile space. Thus, the Heawood complex is the complex of faces of this fundamental tile, where the boundary is identified according to how the translations glue together, see also Figure 4.

Proposition 4.6. *The following hold:*

1. *The Heawood graph* $H_{\mathbf{k}}$ *is the edge graph of the Heawood complex* $\mathcal{HC}_{\mathbf{k}}$.

2. *The Heawood complex and the torus are dual complexes:* $\mathcal{HC}_{\mathbf{k}} \cong \mathcal{T}^*_{\mathbf{k}}$.

3. *The Heawood graph $H_{\mathbf{k}}$ is the dual graph of the torus $\mathcal{T}_{\mathbf{k}}$.*

This, together with Proposition 4.3, finishes part of the proof of our main Theorem 2.4. The proof of the remaining enumerative part can be found in the longer version of this manuscript [2]. There, we also describe the automorphism groups of the triangulated torus $\mathcal{T}_{\mathbf{k}}$ and the generalized Heawood graph $H_{\mathbf{k}}$, and discuss about potential generalizations including the hyperbolic setting.

In view of Proposition 4.4, we finish with the following open question.

Question 4.7. What is the topology of other families of polytopes with opposite facets identified by translation?

Natural families that fit into this context are Permutahedra arising from finite Coxeter groups, Postnikov's generalized permutahedra obtained by removing some pairs of opposite facets of the classical permutahedron, and Zonotopes in general. A small example of the first and the last is an octagon. Identifying opposite edges of an octagon leads to a topological 2-hole torus.

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The somewhere-to-below shuffles in the symmetric group and Hecke algebras

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Abstract. We introduce and study the *somewhere-to-below shuffles*, which are elements of the group algebra of the symmetric group S_n defined as sums of cycles. We show that these elements are simultaneously triangularizable (in an easily-defined basis of $\mathbf{k}[S_n]$), and compute their joint eigenvalues with multiplicities. We furthermore discuss some identities between them, a card shuffling interpretation and its probabilistic properties, and a possible generalization to the Hecke algebra.

Keywords: symmetric group, permutations, card shuffling, top-to-random shuffle, group algebra, substitutional analysis, Fibonacci numbers, filtration, representation theory, Markov chain, Specht module, symmetric functions

1 Introduction

The group algebra $\mathbf{k}[S_n]$ of the symmetric group S_n is one of the most elementary, yet richest examples of an algebra in combinatorics. Over a characteristic-zero field, it is known (by the representation theory of the symmetric group) to be isomorphic to a direct product of matrix rings, a viewpoint that clarifies some of its features while obscuring others. The structure of $\mathbf{k}[S_n]$ becomes more interesting when $\mathbf{k}$ is less well-behaved (e.g., the ring $\mathbb{Z}$), but also when combinatorics is invited back onto the stage.

The latter can be done by defining a simple-looking family of elements of $\mathbf{k}[S_n]$ combinatorially and asking algebraic questions: Do its elements commute? Do they have integer eigenvalues (viewed as endomorphisms of $\mathbf{k}[S_n]$ by left multiplication)? What subalgebra do they generate? Such families often come with a rich provenance. Examples are the Young–Jucys–Murphy elements (originating from representation theory), the Eulerian idempotents (born in homological algebra) and the more recent Wronski–Purbhoo elements (inspired by mathematical physics).

A wide class of recent examples has come from probability theory, the most elementary example being perhaps the *top-to-random shuffle*

$$t_1 := \mathrm{cyc}_1 + \mathrm{cyc}_{1,2} + \mathrm{cyc}_{1,2,3} + \cdots + \mathrm{cyc}_{1,2,\ldots,n} \in \mathbf{k}[S_n],$$

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where $\mathrm{cyc}_{i_1,i_2,\ldots,i_m}$ denotes the m-cycle sending $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_m \mapsto i_1$. After this shuffle was fully analyzed in 1986 [6], several generalizations and extensions have come up and are still undergoing active research.

The work outlined in this abstract, and detailed in our papers [4] and [3] (and forthcoming work), concerns the perhaps simplest way to generalize the top-to-random shuffle: namely, by embedding it in the n-tuple $(t_1, t_2, \ldots, t_n)$ of the *somewhere-to-below shuffles*

$$t_i := \mathrm{cyc}_i + \mathrm{cyc}_{i,i+1} + \mathrm{cyc}_{i,i+1,i+2} + \cdots + \mathrm{cyc}_{i,i+1,\ldots,n} \in \mathbf{k}[S_n]$$

for all $i \in \{1, 2, \ldots, n\}$. These n shuffles have a simple probabilistic meaning (shuffling a deck of cards by picking the i-th card from the top and randomly moving it further down the deck), and are also related to the insertion sort algorithm and to subgroups (each t_i is a sum of coset representatives for a certain S_{n-i} subgroup inside S_{n-i+1}).

The somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$ do not commute, but they "commute up to nilpotent error terms". In rigorous language, this means that there exists a basis $(a_1, a_2, \ldots, a_{n!})$ of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ on which these elements act from the right as upper-triangular matrices (i.e., we have $a_k t_\ell \in \operatorname{span}(a_1, a_2, \ldots, a_k)$ for each k). This basis can be constructed explicitly over any ring $\mathbf{k}$, in contrast to the more classical diagonalizing bases that exist for various other known families but only over characteristic-0 fields. (A common diagonalizing basis is impossible for the $t_1, t_2, \ldots, t_n$, since some of their linear combinations fail to act semisimply.) A more conceptual but less catchy formulation of our main result is the existence of a filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$$

of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ that is preserved by the somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$ (acting from the right), and on whose quotients F_i / F_{i-1} these shuffles act as scalars. The length of this filtration is (rather unexpectedly) the $(n+1)$-st *Fibonacci number* f_{n+1}.

A consequence of all this is that each linear combination $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ of the somewhere-to-below shuffles has explicitly computable eigenvalues, which are all integers if the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are, and of which at most f_{n+1} are distinct. Their multiplicities (in the generic case) are certain divisors of $n!$, counting some kinds of permutations. We give the constructions and say a few words on the proofs below; details can be found in [4]. Some variants of these results (replacing right by left multiplication, and replacing the $t_1, t_2, \ldots, t_n$ by their "antipodes") are briefly outlined in Section 8.

The filtration above explains much but not everything. In particular, it shows that the commutators $[t_i, t_j]$ are nilpotent, but gives fairly bad (exponential) bounds on their nilpotency degrees. The actual nilpotency degrees, however, are much smaller (in fact, no larger than $n/2 + 1$). This is elaborated upon in Section 9, but the detailed proofs are too long to even hint at; they can be found in [3].

The motivation for studying the somewhere-to-below shuffles comes largely from probability theory. Card shuffling can be thought of applying a permutation at random, according to some probability, to a deck of cards. For us, it means acting on the right by an element of $\mathbf{k}[S_n]$ whose coefficients are nonnegative reals. A question of interest is thus, given the choice of an element of $\mathbf{k}[S_n]$, how many applications of it would suffice to shuffle the deck of cards properly. In Section 10, we give an optimal strong stationary time for linear combinations of the somewhere-to-below shuffles.

Anything about $\mathbf{k}[S_n]$ is, of course, connected to integer partitions and Young diagrams, since the irreducible representations of $\mathbf{k}[S_n]$ are the *Specht modules* S^λ assigned to the partitions λ of n. Thus, one can wonder how the somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$ act on a given Specht module S^λ. We answer this in Section 11; the proof will appear in forthcoming work.

In the last Section 12, we suggest a further potential generalization, replacing the symmetric group algebra $\mathbf{k}[S_n]$ by the Hecke algebra $\mathcal{H}_n(q)$. We have only just began the study of this setting, but it appears that many of our results extend to it. Research on this, as well as on our Specht module conjecture, is underway.

2 Definitions

2.1 Combinatorics

Let us first introduce some basic notations (more will be defined as needed). We set $\mathbb{N} := \{0, 1, 2, \ldots\}$. Furthermore, we set $[a, b] := \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ for any $a, b \in \mathbb{Z}$. For any $k \in \mathbb{Z}$, we set $[k] := [1, k] = \{1, 2, \ldots, k\}$.

We fix a positive integer n. We let S_n denote the n-th symmetric group; it consists of the $n!$ permutations of $[n]$, with multiplication given by composition: $(\alpha\beta)(i) = \alpha(\beta(i))$ for each $\alpha, \beta \in S_n$ and $i \in [n]$.

2.2 Algebra

We fix a commutative ring $\mathbf{k}$. (The cases $\mathbf{k} = \mathbb{Z}$ and $\mathbf{k} = \mathbb{Q}$ are fully sufficient.)

We let $\mathbf{k}[S_n]$ denote the group algebra of S_n over $\mathbf{k}$. This $\mathbf{k}$-algebra consists of all formal $\mathbf{k}$-linear combinations $\sum_{\sigma \in S_n} \lambda_\sigma \sigma$ of the permutations $\sigma \in S_n$, and its multiplication is the $\mathbf{k}$-linear extension of the multiplication on S_n. Its unity is $1 = \mathrm{id}_{[n]} \in S_n$.

For each $u \in \mathbf{k}[S_n]$, we define the two $\mathbf{k}$-linear maps $L(u) : \mathbf{k}[S_n] \to \mathbf{k}[S_n]$ and $R(u) : \mathbf{k}[S_n] \to \mathbf{k}[S_n]$ by

$$(L(u))(a) = ua \qquad \text{and} \qquad (R(u))(a) = au \qquad \text{for each } a \in \mathbf{k}[S_n].$$

These are just the left multiplication and the right multiplication by u. Being endomorphisms of the $\mathbf{k}$-module $\mathbf{k}[S_n]$, they can be represented as $n! \times n!$-matrices over $\mathbf{k}$ (since $\mathbf{k}[S_n]$ is a free $\mathbf{k}$-module of rank $n!$, with basis $(w)_{w \in S_n}$), and thus have characteristic polynomials, eigenvalues and eigenvectors (at least when $\mathbf{k}$ is a field).

2.3 Cycles, somewhere-to-below and other random-to-below shuffles

For any distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\operatorname{cyc}_{i_1,i_2,\ldots,i_k}$ be the permutation in S_n that cyclically permutes $i_1 \mapsto i_2 \mapsto i_3 \mapsto \cdots \mapsto i_k \mapsto i_1$ and leaves all other elements of $[n]$ unchanged. In particular, $\operatorname{cyc}_{i,j}$ is a transposition, while $\operatorname{cyc}_i = \operatorname{id} = 1$.

We are now ready for our main definition: For each $\ell \in [n]$, we define the element

$$t_\ell := \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell,\ell+1} + \operatorname{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \operatorname{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}[S_n].$$

These n elements $t_1, t_2, \ldots, t_n$ will be called the *somewhere-to-below shuffles*. The first of these elements, t_1, is also known as the *top-to-random shuffle* or the *Tsetlin library*, whereas the last is just the identity ($t_n = \operatorname{cyc}_n = 1$).

Linear combinations of the somewhere-to-below shuffles are also interesting. Assuming the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nonnegative reals, $\lambda_1 t_1 + \lambda_2 t_2 + \ldots + \lambda_n t_n$ represents the action of choosing the i-th somewhere-to-below shuffle with some probability dictated by λ_i. In particular, the *random-to-below shuffle* is the shuffle in which we pick i with uniform probability (among $[n]$), and then apply the i-th somewhere-to-below shuffle. In terms of card shuffling, this amounts to drawing a card (uniformly) at random and moving it weakly below. See [4, §3] for other interesting shuffles of this sort.

3 The descent-destroying basis

The n somewhere-to-below shuffles do not commute (e.g., we have $t_1 t_2 \neq t_2 t_1$ for $n = 3$). Nevertheless, they behave far better than a "random" family of elements of $\mathbf{k}[S_n]$. In particular, there exists a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ in which all of the endomorphisms $R(t_1), R(t_2), \ldots, R(t_n)$ are represented by upper-triangular matrices. We shall construct this basis now. This requires some more definitions.

For each $w \in S_n$, we define the *descent set* of w to be the set

$$\operatorname{Des} w := \{ i \in [n-1] \mid w(i) > w(i+1) \}.$$

For each $i \in [n-1]$, we define the *simple transposition* $s_i := \operatorname{cyc}_{i,i+1} \in S_n$.

For each $I \subseteq [n-1]$, we define the *Young subgroup* $G(I)$ to be the subgroup of S_n generated by the s_i for $i \in I$. This can be viewed as a product $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ with $n_1 + n_2 + \cdots + n_k = n$, embedded into S_n via the canonical homomorphism.

For each $w \in S_n$, we define

$$a_w := \sum_{\sigma \in G(\operatorname{Des} w)} w\sigma \in \mathbf{k}[S_n].$$

The following is easy to see by triangularity:

Proposition 1. *The family $(a_w)_{w \in S_n}$ is a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$.*

Example 1. *For $n = 3$, we have*

$$\begin{array}{ll|ll} a_{[123]} &= [123]; & a_{[231]} &= [231] + [213]; \\ a_{[132]} &= [132] + [123]; & a_{[312]} &= [312] + [132]; \\ a_{[213]} &= [213] + [123]; & a_{[321]} &= [321] + [312] + [231] + [213] + [132] + [123] \end{array}$$

(where we use one-line notation for permutations: $[i_1 i_2 \cdots i_n]$ means the permutation of $[n]$ that sends $1, 2, \ldots, n$ to $i_1, i_2, \ldots, i_n$).

Now, we claim that the endomorphisms $R(t_1), R(t_2), \ldots, R(t_n)$ are upper-triangular with respect to this basis (appropriately ordered). More concretely:

Theorem 1. *There is some partial order $\prec$ on S_n such that for any $w \in S_n$ and $\ell \in [n]$, we have*

$$a_w t_\ell = \mu_{w,\ell} a_w + \sum_{\substack{v \in S_n; \\ v \prec w}} \lambda_{w,\ell,v} a_v \qquad \text{for some } \mu_{w,\ell} \in \mathbb{N} \text{ and } \lambda_{w,\ell,v} \in \mathbb{Z}.$$

Example 2. *For $n = 4$, we have* $a_{[4312]} t_2 = a_{[4312]} + \underbrace{a_{[4321]} - a_{[4231]} - a_{[3241]} - a_{[2143]}}_{\text{subscripts are } \prec [4312]}.$

4 The invariant spaces $F(I)$

To prove Theorem 1 directly, we would need to understand how $R(t_\ell)$ acts on each single a_w. But this is not easy. Thus, we shall instead analyze the action of $R(t_\ell)$ on a certain filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$ of $\mathbf{k}[S_n]$ by left ideals (which are preserved by the $R(t_\ell)$). The basis $(a_w)_{w \in S_n}$ will then reveal itself to be compatible with this filtration (i.e., each F_i is spanned by some subfamily of this basis), and thus we will be able to draw conclusions about $a_w t_\ell$ from the action of $R(t_\ell)$ on the filtration. Essentially, the filtration will act as a "middleman" between the t_ℓ and the a_w.

In order to construct the filtration, we shall in turn need another middleman: some left ideals $F(I)$ defined for each $I \subseteq [n]$. These are easy to define:

For each subset I of $[n]$, we define the number

$$\operatorname{sum} I := \sum_{i \in I} i,$$

and the sets

$$\widehat{I} := \{0\} \cup I \cup \{n+1\} \qquad \text{and} \qquad I' := [n-1] \setminus (I \cup (I-1))$$

(where $I - 1 := \{i - 1 \mid i \in I\}$), and finally the left ideal

$$F(I) := \left\{ u \in \mathbf{k}[S_n] \mid us_i = u \text{ for all } i \in I' \right\} \subseteq \mathbf{k}[S_n]$$

(the "*invariant space*" corresponding to I).

Example 3. *Let $n = 9$ and $I = \{2,3,7\}$. Then, $\widehat{I} = \{0,2,3,7,10\}$ and $I' = [8] \setminus \{1,2,3,6,7\} = \{4,5,8\}$ and $F(I) = \{u \in \mathbf{k}[S_n] \mid us_4 = us_5 = us_8 = u\}$.*

The following is easy to see:

Proposition 2. *For each $I \subseteq [n]$, the family $(a_w)_{w \in S_n;\ I' \subseteq \operatorname{Des} w}$ is a basis of the $\mathbf{k}$-module $F(I)$.*

The main workhorse of our study of the somewhere-to-below shuffles is a lemma which, for each $I \subseteq [n]$ and $\ell \in [n]$ and $u \in F(I)$, expresses the product ut_ℓ as a scalar multiple of u plus a sum of "error terms" in "smaller" invariant spaces $F(J)$ (to be precise: invariant spaces $F(J)$ for subsets $J \subseteq [n]$ satisfying $\operatorname{sum} J < \operatorname{sum} I$). We can actually be more specific and characterize the scalar in front of the u as follows:

For any $\ell \in [n]$, we let $m_{I,\ell}$ be the distance from ℓ to the next-higher element of $\widehat{I}$. In other words,

$$m_{I,\ell} := \left(\text{smallest element of } \widehat{I} \text{ that is } \geq \ell\right) - \ell \in \{0,1,\ldots,n\}.$$

Example 4. *If $n = 9$ and $I = \{2,3,7\}$, then $\widehat{I} = \{0,2,3,7,10\}$ and*

$$(m_{I,1},\ m_{I,2},\ \ldots,\ m_{I,9}) = (1,0,0,3,2,1,0,2,1).$$

Lemma 1 (Workhorse lemma)**.** *Let $I \subseteq [n]$ and $\ell \in [n]$. Then,*

$$ut_\ell \in m_{I,\ell} u + \sum_{\substack{J \subseteq [n]; \\ \operatorname{sum} J < \operatorname{sum} I}} F(J) \qquad \text{for each } u \in F(I).$$

Proof idea. Expand ut_ℓ by the definition of t_ℓ, and break up the resulting sum into smaller bunches using the interval decomposition

$$[\ell, n] = [\ell, i_k - 1] \sqcup [i_k, i_{k+1} - 1] \sqcup [i_{k+1}, i_{k+2} - 1] \sqcup \cdots \sqcup [i_p, n]$$

(where $i_k < i_{k+1} < \cdots < i_p$ are the elements of I larger or equal to ℓ). The $[\ell, i_k - 1]$ bunch gives the $m_{I,\ell} u$ term; the others live in appropriate $F(J)$'s. See [4, Theorem 7.3] for details. $\square$

5 The Fibonacci filtration

The filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$ that we want to construct will consist of sums of certain invariant spaces $F(I)$. However, we do not need all $F(I)$, but only the ones that correspond to certain subsets I: namely, those that are *lacunar* (i.e., contain no two consecutive integers) and do not contain n. Arranging these lacunar subsets I in order of increasing sum, we will define F_i as the sum of the $F(I)$ corresponding to the first i many I's.

Let us elaborate on this. A set S of integers is called *lacunar* if it contains no two consecutive integers (i.e., we have $s+1 \notin S$ for all $s \in S$). The number of lacunar subsets of $[n-1]$ is known to be the Fibonacci number f_{n+1}. (Recall that the Fibonacci numbers $f_0, f_1, f_2, \ldots$ are defined by $f_0 = 0$ and $f_1 = 1$ and $f_k = f_{k-1} + f_{k-2}$ for each $k \geq 2$.)

The following lemma (essentially [4, Proposition 8.7]) is easy to check:

Lemma 2. *Let $J \subseteq [n]$ be a subset that fails to be lacunar or contains n. Then, there exists some subset $K \subseteq [n]$ such that $\operatorname{sum} K < \operatorname{sum} J$ and $K' \subseteq J'$ (so that $F(J) \subseteq F(K)$).*

Now, we let $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ be the f_{n+1} lacunar subsets of $[n-1]$, listed in such an order that $\operatorname{sum}(Q_1) \leq \operatorname{sum}(Q_2) \leq \cdots \leq \operatorname{sum}\left(Q_{f_{n+1}}\right)$. (We fix such an order once and for all.) Then, for each $i \in [0, f_{n+1}]$, define a left ideal

$$F_i := F(Q_1) + F(Q_2) + \cdots + F(Q_i) \qquad \text{of } \mathbf{k}[S_n]$$

(so that $F_0 = 0$). The resulting filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$$

satisfies the following crucial property:

Theorem 2. *For each $i \in [f_{n+1}]$ and $\ell \in [n]$, we have $F_i \cdot \left(t_\ell - m_{Q_i,\ell}\right) \subseteq F_{i-1}$ (so that $R(t_\ell)$ preserves F_i and F_{i-1}, and acts as multiplication by $m_{Q_i,\ell}$ on F_i / F_{i-1}).*

Proof idea. This follows from Lemmas 1 and 2. See [4, Theorem 8.1 **(c)**] for details. ☐

Now we claim that our basis $(a_w)_{w \in S_n}$ of $\mathbf{k}[S_n]$ respects the filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$. To make this precise, we introduce some more notation:

The *Q-index* $\operatorname{Qind} w$ of a permutation $w \in S_n$ is defined to be the **smallest** $i \in [f_{n+1}]$ such that $Q_i' \subseteq \operatorname{Des} w$. (Note that this depends on our ordering of $Q_1, Q_2, \ldots, Q_{f_{n+1}}$.)

The following facts ([4, §10]) are not hard to see:

Proposition 3. *Let $w \in S_n$ and $i \in [f_{n+1}]$. Then, $\operatorname{Qind} w = i$ if and only if $Q_i' \subseteq \operatorname{Des} w \subseteq [n-1] \setminus Q_i$.*

Theorem 3. *For each $i \in [0, f_{n+1}]$, the $\mathbf{k}$-module F_i is free with basis $(a_w)_{w \in S_n;\ \operatorname{Qind} w \leq i}$.*

Corollary 1. *For each $i \in [f_{n+1}]$, the $\mathbf{k}$-module F_i / F_{i-1} is free with basis $\left(\overline{a_w}\right)_{w \in S_n;\ \operatorname{Qind} w = i}$.*

6 Triangularizability

Combining Theorem 3 with Theorem 2, we easily obtain the following concretization of Theorem 1 ([4, Theorem 11.1]):

Theorem 4. *Let $w \in S_n$ and $\ell \in [n]$. Let $i = \operatorname{Qind} w$. Then,*

$$a_w t_\ell = m_{Q_i,\ell} a_w + \sum_{\substack{v \in S_n; \\ \operatorname{Qind} v < \operatorname{Qind} w}} \lambda_{w,\ell,v} a_v \qquad \text{for some integers } \lambda_{w,\ell,v}.$$

Thus, the endomorphisms $R(t_1), R(t_2), \ldots, R(t_n)$ are upper-triangular with respect to the basis $(a_w)_{w \in S_n}$, as long as the permutations $w \in S_n$ are ordered by increasing Q-index. Their diagonal entries are the numbers $m_{Q_{\operatorname{Qind} w},\ell} \in \mathbb{N}$.

Therefore, any $\mathbf{k}$-linear combination $R\left(\sum_{\ell=1}^{n} \lambda_\ell t_\ell\right) = \sum_{\ell=1}^{n} \lambda_\ell R(t_\ell)$ of these endomorphisms $R(t_1), R(t_2), \ldots, R(t_n)$ (with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$) is upper-triangular with respect to this basis as well, and its diagonal entries will be the appropriate $\mathbf{k}$-linear combinations $\sum_{\ell=1}^{n} \lambda_\ell m_{Q_{\operatorname{Qind} w},\ell}$. Hence, regarded as an $n! \times n!$-matrix, $R\left(\sum_{\ell=1}^{n} \lambda_\ell t_\ell\right)$ is triangularizable with eigenvalues $\sum_{\ell=1}^{n} \lambda_\ell m_{Q_{\operatorname{Qind} w},\ell}$ for $w \in S_n$.

This matrix is not always diagonalizable. A sufficient (but far from necessary) criterion can nevertheless be given:

Theorem 5. *Let $\mathbf{k}$ be a field, and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. Then, the eigenvalues of the operator $R(\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)$ are the linear combinations*

$$\lambda_1 m_{I,1} + \lambda_2 m_{I,2} + \cdots + \lambda_n m_{I,n} \qquad \text{for } I \subseteq [n-1] \text{ lacunar}$$

(with multiplicities discussed below). If all these f_{n+1} linear combinations are distinct, then $R(\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)$ is diagonalizable.

Proof idea. The first claim follows from the discussion above; the second uses Theorem 2 and some linear algebra. See [4, Corollary 12.2 and Theorem 12.3] for details. □

7 Multiplicities of the eigenvalues

We can also describe the multiplicities of the eigenvalues of $R(\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)$ ([4, Theorem 13.2]):

Theorem 6. *Assume that $\mathbf{k}$ is a field. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. For each $i \in [f_{n+1}]$, let δ_i be the number of all permutations $w \in S_n$ satisfying $\operatorname{Qind} w = i$, and let*

$$g_i := \sum_{\ell=1}^{n} \lambda_\ell m_{Q_i,\ell} \in \mathbf{k}.$$

Let $\kappa \in \mathbf{k}$. Then, the algebraic multiplicity of κ as an eigenvalue of $R(\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)$ equals the sum of the δ_i over all $i \in [f_{n+1}]$ satisfying $g_i = \kappa$.

Furthermore, these δ_i can be expressed by an explicit formula (similar to but simpler than the famous hook-length formula), and are divisors of $n!$ (just like in the hook-length formula); we refer to [4, Theorem 13.1] for details.

8 Variants

So far, we have directed our attention at the right multiplication maps $R(t_1), R(t_2), \ldots, R(t_n)$, while neglecting their left counterparts $L(t_1), L(t_2), \ldots, L(t_n)$. However, almost all our claims about the former can be extended to the latter using general properties of group algebras. In particular, there exists a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ in which all of the endomorphisms $L(t_1), L(t_2), \ldots, L(t_n)$ are represented by upper-triangular matrices. This basis is not the basis $(a_w)_{w \in S_n}$, but rather its dual basis with respect to a certain bilinear form (and its order is modified). Theorems 5 and 6 remain valid if "R" is replaced by "L" throughout them. For the proofs of all these claims, we refer to [4, §14]; all we shall say here is that they are derived from the analogous properties of R purely algebraically, with no further combinatorial input.

It is also natural to study the *below-to-somewhere shuffles* $t'_1, t'_2, \ldots, t'_n$, where

$$t'_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell+1,\ell} + \mathrm{cyc}_{\ell+2,\ell+1,\ell} + \cdots + \mathrm{cyc}_{n,n-1,\ldots,\ell} \in \mathbf{k}[S_n]$$

for each $\ell \in [n]$. Again, Theorems 5 and 6 remain valid if each t_ℓ is replaced by the corresponding t'_ℓ; but this is again not too surprising, since the t'_ℓ are the images of t_ℓ under a very simple $\mathbf{k}$-algebra anti-automorphism of $\mathbf{k}[S_n]$ called the *antipode* (sending each permutation $w \in S_n$ to its inverse w^{-1}). Thus, again, most properties can be transferred between the t_ℓ and the t'_ℓ by purely algebraic tools (see [4, §14] for details).

9 Nilpotent commutators

Since the endomorphisms $R(t_1), R(t_2), \ldots, R(t_n)$ are simultaneously triangularizable, their pairwise commutators are nilpotent. Hence, the pairwise commutators $[t_i, t_j]$ in $\mathbf{k}[S_n]$ are also nilpotent. A natural question is: How small is the required exponent?

As it turns out, it is much smaller than one might expect:

Theorem 7. *Let $1 \leq i \leq j \leq n$. Then,*

$$[t_i, t_j]^m = 0 \qquad \textit{holds for } m = \min\{j - i + 1,\ \lceil (n-j)/2 \rceil + 1\}.$$

We conjecture (and have verified for all $n \leq 12$) that this choice of m is optimal (i.e., that $[t_i, t_j]^{m-1} \neq 0$, at least for $\mathbf{k} = \mathbb{Z}$).

Actually, Theorem 7 can be generalized, replacing the m-th power of a single $[t_i, t_j]$ by a product of several $[t_i, t_j]$'s (with the same j but possibly different i's). The reader can find this generalization in [3, Theorems 8.15 and 9.10], where it is proved by long and tricky but completely elementary manipulations of permutations and sums.

Several other curious facts hold, such as the following ([3, Theorems 5.1 and 6.1, Corollaries 7.6 and 8.20]):

Proposition 4. *If $i \in [n-1]$, then $t_{i+1}t_i = (t_i - 1)\, t_i$.*
If $i \in [n-2]$, then $t_{i+2}(t_i - 1) = (t_i - 1)(t_{i+1} - 1)$.

Proposition 5. *Let $i, j \in [n]$. Then, $t_{n-1}[t_i, t_{n-1}] = 0$ and $[t_i, t_{n-1}][t_j, t_{n-1}] = 0$.*

These facts suggest that the $\mathbf{k}$-subalgebra $\mathbf{k}[t_1, t_2, \ldots, t_n]$ of $\mathbf{k}[S_n]$ has some interesting structure (apart from the "split-semisimple-by-nilpotent" decomposition following from Theorem 1). Yet it remains mysterious in many ways. For $\mathbf{k} = \mathbb{Q}$ and $n \in [8]$, here is its dimension as a $\mathbb{Q}$-vector space (the sequence is not in the OEIS as of 2023-11-07!):

n	1	2	3	4	5	6	7	8
$\dim(\mathbb{Q}[t_1, t_2, \ldots, t_n])$	1	2	4	9	23	66	212	761

. (9.1)

10 Probability theory

We shall now make a few comments on the probabilistic side of the one-sided cycle shuffles. Viewing them as shuffling operators, we are interested in the number of iterations needed to get a well-mixed deck of cards. We describe a strong stationary time for all one-sided cycle shuffles (see [4, §10]), imitating a similar result for the top-to-random shuffle ([1]). Once the strong stationary time is reached, the deck is perfectly mixed.

Theorem 8. *If $\lambda_1 \neq 0$, then the one-sided cycle shuffle $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ admits a stopping time τ obtained as follows: Place a bookmark right above the bottommost card of the deck. The bookmark itself does not move (but cards can move down past it). We let τ be the time it takes for the bookmark to reach the top of the deck.*

The distribution of the deck is uniform at time τ and any time afterwards; i.e., τ is a strong stationary time. Furthermore, this stopping time is optimal.

If $\lambda_1 = 0$, then the top card never moves, so the deck will never be uniformly mixed.

For the random-to-below shuffle, we can compute the waiting time explicitly:

Theorem 9. *Let H_n be the n-th harmonic number. The expected number of steps to get to the strong stationary time for the random-to-below shuffle is*

$$\mathbb{E}(\tau) = \sum_{i=2}^{n} \frac{n}{i\,(H_n - H_{i-1})} \leq n \log n + n \log(\log n) + n \log 2 + 1 \qquad \textit{if } n \geq 2.$$

We conjecture that the strong stationary time for the random-to-below shuffle satisfies $\mathbb{E}(\tau) = n(\log n + \log(\log n) + O(1))$, which makes the random-to-below shuffle slower than top-to-random, for which the strong stationary time approaches $n \log n$ ([1]).

11 Representation theory

Recall the maps $L(u)$ and $R(u)$ defined in Subsection 2.2 for any $u \in \mathbf{k}[S_n]$. Any representation theorist will recognize them as the actions of u on the left and the right regular representation of S_n. Similar maps can be defined for any other representation of S_n. It thus is natural to ask about analogues of Theorems 5 and 6 for arbitrary representations. We shall briefly summarize the answer (yet unpublished).

In this section, we assume that $\mathbf{k}$ is a field of characteristic 0. We shall use some basic notions from the representation theory of S_n and from symmetric functions; the reader can find all prerequisites in [2, Chapters 6 and 7]. For any partition λ of n, a Specht module S^λ is defined, which is an irreducible representation of S_n with a basis indexed by standard tableaux of shape λ. Each $u \in \mathbf{k}[S_n]$ acts (on the left) on this Specht module S^λ; we let $L_\lambda(u)$ denote this action (viewed as a $\mathbf{k}$-module endomorphism of S^λ).

We let $\mathcal{R}$ denote the representation ring of the symmetric groups (called R in [2, §7.3]), and Λ denote the ring of symmetric functions over $\mathbb{Z}$ (defined in [2, §6.2]). An isomorphism $\varphi : \Lambda \to \mathcal{R}$ (often called the *Frobenius characteristic map*) is defined in [2, §7.3], and the famous *Schur function* $s_\lambda \in \Lambda$ corresponding to a partition λ is the preimage of the Specht module S^λ under this isomorphism φ.

For each $m \in \mathbb{N}$, we let $h_m \in \Lambda$ denote the m-th complete homogeneous symmetric polynomial. For each $m > 0$, we let $z_m \in \Lambda$ denote the Schur function $s_{(m-1,1)} = h_{m-1}h_1 - h_m \in \Lambda$. (This is 0 for $m = 1$.)

For each subset I of $[n]$, we define a symmetric function $z_I := h_{i_1-1} \prod_{j=2}^{k} z_{i_j - i_{j-1}} \in \Lambda$, where $i_1, i_2, \ldots, i_k$ are the elements of $I \cup \{n+1\}$ in increasing order (so that $i_k = n+1$ and $I = \{i_1 < i_2 < \cdots < i_{k-1}\}$). When this symmetric function z_I is expanded in the basis $(s_\lambda)_{\lambda \text{ is a partition}}$ of Λ, the coefficient of a given Schur function s_λ shall be called c^I_λ. This coefficient c^I_λ is actually a Littlewood–Richardson coefficient (since z_I is a skew Schur function), hence $\in \mathbb{N}$.

We now claim the following:

Theorem 10. *Let ν be a partition. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. Then, the eigenvalues of the operator $L_\nu(\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)$ on the Specht module S^ν are the linear combinations*

$$\lambda_1 m_{I,1} + \lambda_2 m_{I,2} + \cdots + \lambda_n m_{I,n} \qquad \text{for } I \subseteq [n-1] \text{ lacunar satisfying } c^I_\nu \neq 0,$$

and their respective multiplicities are the c^I_ν in the generic case (i.e., if no two I's produce the same linear combination; otherwise the multiplicities of colliding eigenvalues should be added together). If all these linear combinations are distinct, then $L_\nu(\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)$ is diagonalizable.

Relatedly, (the isomorphism class of) the representation F_i / F_{i-1} of S_n is $\varphi(z_{Q_i})$.

12 Into the Hecke algebra

Like many objects originating in combinatorics, the symmetric group algebra $\mathbf{k}[S_n]$ has a q-deformation. This deformation is the *type-A Hecke algebra* (or *Iwahori-Hecke algebra*), defined in terms of a parameter $q \in \mathbf{k}$. It is commonly denoted by $\mathcal{H} = \mathcal{H}_q(S_n)$; it has a basis $(T_w)_{w \in S_n}$ indexed by the permutations $w \in S_n$, but a more intricate multiplication than $\mathbf{k}[S_n]$. We refer to [5] for the definition of this multiplication, and much more about $\mathcal{H}$. We can now define the *q-deformed somewhere-to-below shuffles* $t_1^{\mathcal{H}}, t_2^{\mathcal{H}}, \ldots, t_n^{\mathcal{H}}$ by

$$t_\ell^{\mathcal{H}} := T_{\mathrm{cyc}_\ell} + T_{\mathrm{cyc}_{\ell,\ell+1}} + T_{\mathrm{cyc}_{\ell,\ell+1,\ell+2}} + \cdots + T_{\mathrm{cyc}_{\ell,\ell+1,\ldots,n}} \in \mathcal{H}.$$

Surprisingly, these q-deformed shuffles appear to share many properties of the original $t_1, t_2, \ldots, t_n$. In particular, the analogues of Theorems 1 and 7 in $\mathcal{H}$ (where the t_ℓ are replaced by the $t_\ell^{\mathcal{H}}$) seem to hold. Even more surprisingly perhaps, the dimensions of $\mathbb{Q}[t_1, t_2, \ldots, t_n]$ tabulated in (9.1) (at least for $n \leq 6$) appear to be the same for the $\mathcal{H}$-analogue, which suggests that all algebraic relations between the $t_1, t_2, \ldots, t_n$ are "coming from" the Hecke algebra. Attempts to prove these conjectures are underway.

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Dyck combinatorics in p-Kazhdan–Lusztig theory

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Abstract. We survey some recent advances in combinatorial modular representation theory in type A through the lens of p-Kazhdan–Lusztig theory.

1 Introduction

The diagrammatic Hecke category has provided the intuition and tools necessary to cut through the most famous conjectures of Lie theory: the Lusztig and Kazhdan–Lusztig positivity conjectures. These conjectures place the Kazhdan–Lusztig polynomials (associated to parabolic Coxeter systems) centre-stage in the (modular) representation theory of Lie theoretic objects.

Kazhdan–Lusztig polynomials encode a great deal of character-theoretic and indeed cohomological information about cell modules. We further know that Kazhdan–Lusztig polynomials often carry information about the radical layers of indecomposable projective and cell modules. Given the almost ridiculous level of detail these polynomials encode, it is natural to ask *"what are the limits to what p-Kazhdan–Lusztig combinatorics can tell us about the structure of the Hecke category?"*

The family of ordinary Kazhdan–Lusztig polynomials which are combinatorially best understood are those for maximal parabolics of finite symmetric groups $\mathfrak{S}_m \times \mathfrak{S}_n \leqslant \mathfrak{S}_{m+n}$. These polynomials can be calculated in terms of the combinatorics of Dyck tilings [9]. The starting point of this project was to extend this to the modular case by proving that the p-Kazhdan–Lusztig polynomials of $\mathfrak{S}_m \times \mathfrak{S}_n \leqslant \mathfrak{S}_{m+n}$ are entirely independent of $p \geqslant 0$. We also find that there is a wealth of extra, richer combinatorial information which can be encoded into the Dyck tilings. Instead of looking only at the sets of Dyck tilings (which enumerate these Kazhdan–Lusztig polynomials) we look at the relationships for passing between these Dyck tilings. In fact, this "meta-Kazhdan–Lusztig

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combinatorics" is sufficiently rich as to completely determine the full structure of our Hecke categories. In this extended abstract, we discuss how this allows us to provide a complete combinatorial description of the submodule lattices of the cell modules for these categories.

We also proved in [2] that the Hecke categories of $\mathfrak{S}_m \times \mathfrak{S}_n \leqslant \mathfrak{S}_{m+n}$ control the structure of parabolic Verma modules for Lie algebras [4, 8, 9]; the representation category of the general linear supergroups [3]; arc algebras from categorified knot theory [5]; walled Brauer algebras [6]; and the combinatorics of attracting cells for torus fixed points in Springer fibers [11]. This makes the cell modules of these categories some of the most well-understood representations in all of non-semisimple Lie theory.

2 Kazhdan–Lusztig polynomials

Let (W, S_W) be a Coxeter system: W is the group generated by the finite set S_W subject to the relations $(\sigma\tau)^{m_{\sigma\tau}} = 1$ for $\sigma, \tau \in S_W$, $m_{\sigma\tau} \in \mathbb{N} \cup \{\infty\}$ satisfying $m_{\sigma\tau} = m_{\tau\sigma}$, and $m_{\sigma\tau} = 1$ if and only if $\sigma = \tau$. Let $\ell : W \to \mathbb{N}$ be the corresponding length function. Consider $S_P \subseteq S_W$ a subset and (P, S_P) its corresponding Coxeter system. We say that P is the parabolic subgroup corresponding to $S_P \subseteq S_W$. Let ${}^P W \subseteq W$ denote a set of minimal coset representatives in $P \backslash W$. For $\underline{w} = \sigma_1\sigma_2 \cdots \sigma_\ell$ an expression, we define a subword to be a sequence $\underline{t} = (t_1, t_2, \dots, t_\ell) \in \{0,1\}^\ell$ and set $\underline{w}^{\underline{t}} := \sigma_1^{t_1}\sigma_2^{t_2} \cdots \sigma_\ell^{t_\ell}$. We let $\leqslant$ denote the strong Bruhat order on ${}^P W$: namely $y \leqslant w$ if for some reduced expression $\underline{w}$ there exists a subword $\underline{t}$ and a reduced expression $\underline{y}$ such that $\underline{w}^{\underline{t}} = \underline{y}$. We denote the Hasse diagram of this poset by $\mathcal{G}_{(W,P)}$ and we refer to it as the *Bruhat graph* of the pair (W, P). Explicitly, the vertices of $\mathcal{G}_{(W,P)}$ are labelled by the elements of ${}^P W$ and for $\lambda \in {}^P W$ we have a directed edge $\lambda \to \lambda s_i$ if $\lambda < \lambda s_i \in {}^P W$ for some $s_i \in S_W$. We denote by $\varnothing$ (for the empty word in the generators) the minimal coset representative for the identity coset P.

We define the *extended Bruhat graph* $\widehat{\mathcal{G}}_{(W,P)}$ to be the directed graph having the same set of vertices as $\mathcal{G}_{(W,P)}$ but replacing each edge in $\mathcal{G}_{(W,P)}$ between λ and λs_i for $\lambda < \lambda s_i$ by four "up" and "down" directed edges

$$\lambda \xrightarrow{i} \lambda s_i, \quad \lambda \xrightarrow{i} \lambda, \quad \lambda s_i \xrightarrow{i} \lambda \quad \lambda s_i \xrightarrow{i} \lambda s_i, \tag{2.1}$$

which we denote U_i^1, U_i^0, D_i^1, D_i^0 respectively. We assign a degree to each edge in $\widehat{\mathcal{G}}_{(W,P)}$ by setting

$$\deg(\lambda \xrightarrow{i} \lambda s_i) = \deg(\lambda s_i \xrightarrow{i} \lambda) = 0 \qquad \deg(\lambda \xrightarrow{i} \lambda) = \begin{cases} 1 & \text{if } \lambda s_i > \lambda \\ -1 & \text{if } \lambda s_i < \lambda \end{cases}$$

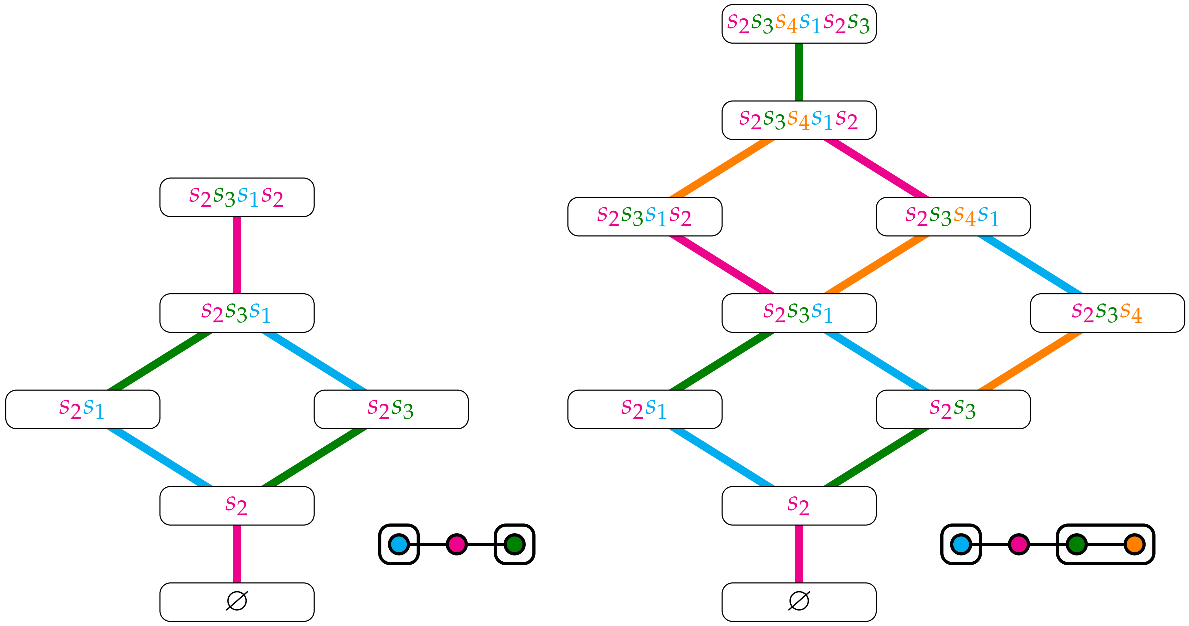


Figure 1: The graph $\mathcal{G}_{(W,P)}$ for $(W,P) = (\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ and $(\mathfrak{S}_5, \mathfrak{S}_2 \times \mathfrak{S}_3)$ respectively.

Given a path (or "Bruhat stroll") on $\widehat{\mathcal{G}}_{(W,P)}$

$$\mathsf{T} : \lambda_1 \xrightarrow{i_1} \lambda_2 \xrightarrow{i_2} \lambda_3 \xrightarrow{i_3} \dots \xrightarrow{i_{k-1}} \lambda_k,$$

we say that the *degree* $\deg(\mathsf{T})$ is the sum of the degrees of each edge in T. (The degree is also sometimes known as the "Deodhar defect".) We also define the weight of T, denoted by $w(\mathsf{T})$ to be the expression

$$w(\mathsf{T}) := s_{i_1}s_{i_2}s_{i_3}\dots s_{i_{k-1}}.$$

Given $\lambda \in {}^PW$, we let $\mathrm{Path}(\lambda)$ denote the set of all paths from $\varnothing$ and ending at λ in the extended Bruhat graph.

Definition 2.1. We say that a path $\mathsf{T} \in \mathrm{Path}(\mu)$ is *reduced* if it is a path of shortest possible length from $\varnothing$ to μ.

Throughout the paper we will fix one reduced path, $\mathsf{T}^\mu \in \mathrm{Path}(\mu)$, for each $\mu \in {}^PW$. For a fixed λ, we denote the set of all paths $\mathsf{T} \in \mathrm{Path}(\lambda)$ with $w(\mathsf{T}) = \mathsf{T}^\mu$ by $\mathrm{Path}(\lambda, \mathsf{T}^\mu)$.
Examples are given in Figure 2.

Definition 2.2. Given (W, P) a parabolic Coxeter system, we define the *matrix of light-leaves polynomials*

$$\Delta^{(W,P)} := (\Delta_{\lambda,\mu}(q))_{\lambda,\mu \in {}^P W} \qquad \Delta_{\lambda,\mu}(q) = \sum_{\mathsf{S} \in \mathrm{Path}(\lambda, \mathsf{T}^\mu)} q^{\deg(\mathsf{S})}$$

which is a (square) lower uni-triangular matrix. This matrix can be factorised *uniquely* as a product of lower uni-triangular matrices

$$N^{(W,P)} := (n_{\lambda,\nu}(q))_{\lambda,\nu \in {}^P W} \qquad B^{(W,P)} := (b_{\nu,\mu}(q))_{\nu,\mu \in {}^P W}$$

such that $n_{\lambda,\nu}(q) \in q\mathbb{Z}[q]$ for $\lambda \neq \nu$ and $b_{\nu,\mu}(q) \in \mathbb{Z}[q + q^{-1}]$. The polynomials $n_{\lambda,\nu}(q)$ are the anti-spherical Kazhdan–Lusztig polynomials of (W, P).

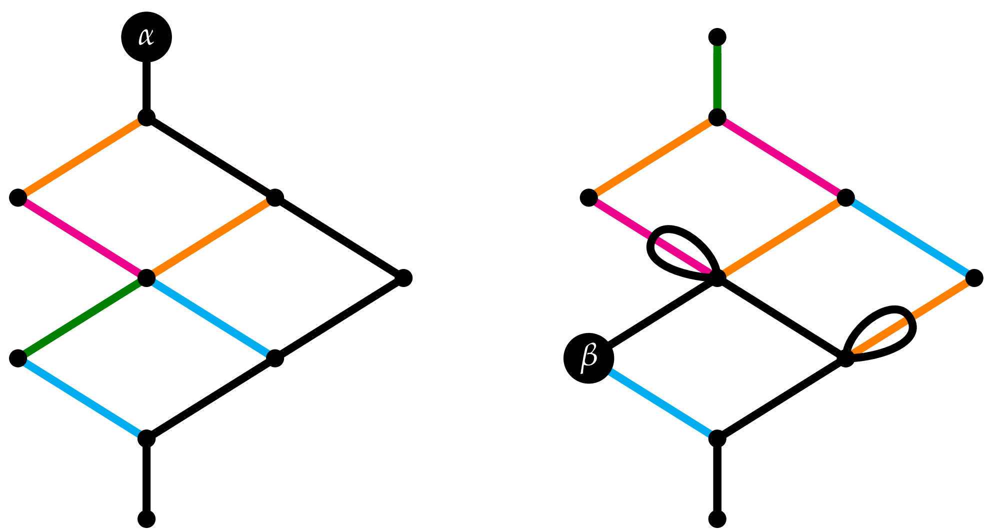


Figure 2: On the left we depict a path T^α and on the right we depict the unique element $\mathsf{S} \in \mathrm{Path}(\beta, \mathsf{T}^\alpha)$ for $\alpha = s_2 s_3 s_4 s_1 s_2 s_3$ and $\beta = s_2 s_1$. These are paths on $\widehat{\mathcal{G}}_{(\mathfrak{S}_5, \mathfrak{S}_2 \times \mathfrak{S}_3)}$ (also known as "Bruhat strolls") but we depict only the edges in $\mathcal{G}_{(\mathfrak{S}_5, \mathfrak{S}_2 \times \mathfrak{S}_3)}$ (for readability).

Example 2.3. The matrix $\Delta^{\Bbbk}$ in type $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ is depicted below.

$\Delta^{\Bbbk}$	$s_2s_1s_3s_2$	$s_2s_1s_3$	s_2s_1	s_2s_3	s_2	$\varnothing$
$s_2s_1s_3s_2$	1	·	·	·	·	·
$s_2s_1s_3$	q	1	·	·	·	·
s_2s_1	·	q	1	·	·	·
s_2s_3	·	q	·	1	·	·
s_2	q	q^2	q	q	1	·
$\varnothing$	q^2	·	·	·	q	1

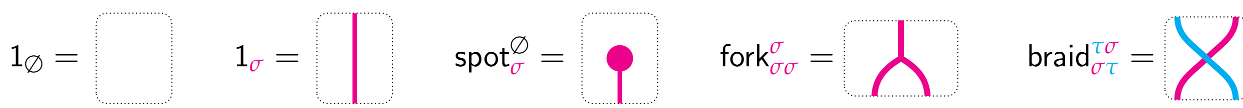
The factorisation of this matrix is trivial, with $N = \Delta^{\Bbbk}$ and $B = \mathrm{Id}_{6\times 6}$ the identity matrix.

The Hecke category (over the complex field) gives a categorification of this matrix factorisation.

3 Hecke categories and p-Kazhdan–Lusztig polynomials

Hecke categories provide the interface between Lie theory and Kazhdan–Lusztig theory. We begin by lifting the "folded paths" of the previous section to provide (what will be) a basis of the Hom-spaces of the Hecke category.

In this section, we will only explicitly discuss the generators and relations for $\mathscr{H}_{(W,P)}$, the category algebra of the Hecke category, when $W = \mathfrak{S}_{n+m}$ is a finite symmetric group and P is a maximal parabolic $P = \mathfrak{S}_m \times \mathfrak{S}_n$, as this simplifies the definitions considerably, whilst still illustrating the important points of the general case. We define the *Soergel generators* to be the framed graphs

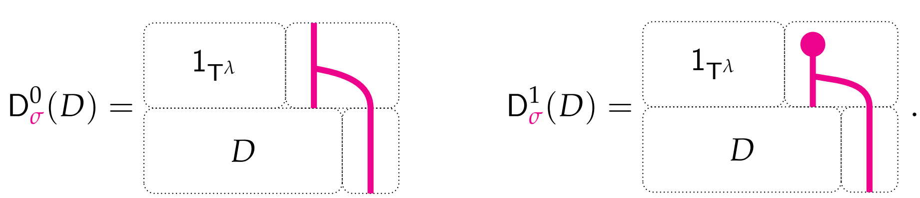


associated to any pair $\sigma, \tau \in S_W$ with $m_{\sigma\tau} = 2$. We define the northern/southern reading word of any diagram obtained from horizontal and vertical concatenation of Soergel generators to be the word in the alphabet S_W which records the colours along the northern/southern edge of the frame respectively. We let $\otimes$ to be horizontal concatenation of diagrams, the algebra multiplication $\circ$ will be given by vertical concatenation in the usual manner for diagram algebras. We let $*$ denote the anti-involution which flips a diagram through the horizontal axis.

Definition 3.1. We define up and down operators on diagrams as follows

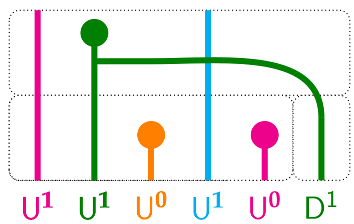
- Suppose that D has northern colour sequence T^{λ} with $\lambda\sigma > \lambda$. We define

- Now suppose that D has northern colour sequence $\mathsf{T}^{\lambda} \otimes \sigma$ with $\lambda\sigma > \lambda$. We define

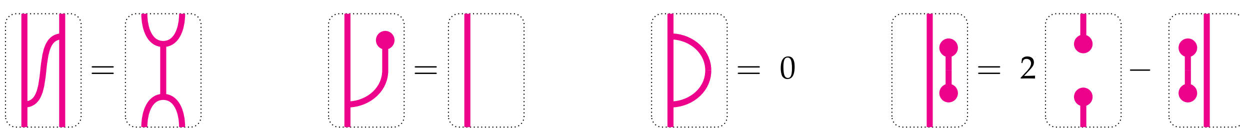


We do not emphasise the braids in our construction/notation since it will not matter if we pre- or post-multiply (at any stage of this construction) with a braid generator.

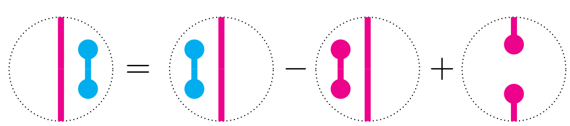
Definition 3.2. For $\mathsf{S} \in \mathrm{Path}(\lambda)$ we construct a Soergel diagram by performing the up and down operators of Definition 3.1 as we encounter any of the four up/down steps in the path. We denote the resulting diagram by c_{S}. The Soergel diagram corresponding to the path on the right hand side of Figure 2 is given below.

$c_{\mathsf{S}} =$ 

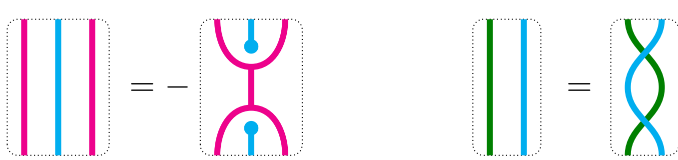
Definition 3.3. (B.-D.-H.-N. [1], Libedinsky–Williamson [10]) Let $(W, P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. The algebra $\mathscr{H}_{(W,P)}$ has a graded cellular basis given by $\{c^*_{\mathsf{S}} c_{\mathsf{T}} : \mathsf{S}, \mathsf{T} \in \mathrm{Path}(\lambda), \lambda \in \mathscr{P}_{m,n}\}$ with $\deg(c^*_{\mathsf{S}} c_{\mathsf{T}}) = \deg(\mathsf{S}) + \deg(\mathsf{T})$ with respect to the poset $(\mathscr{P}_{m,n}, \leqslant)$ and the anti-involution $*$. The multiplication is given by vertical concatenation subject to the following local relations together with their horizontal and vertical flips:



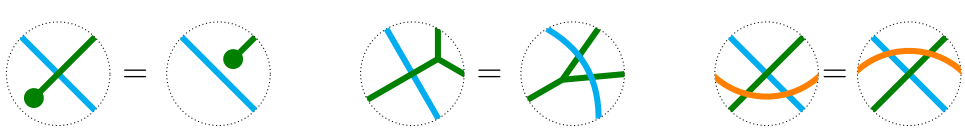
and for $m_{\sigma\tau} = 3$ we have the *2-colour barbell relation*,



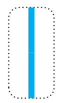
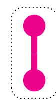
and for $m_{\sigma\tau} = 3$ and $m_{\tau\rho} = 2$ we have the *Temperley–Lieb relations*,



and for $m_{\tau\rho} = m_{\tau\pi} = m_{\rho\pi} = 2$ the *commutativity relations*,



Finally, we have the non-local *cyclotomic relations*,

 $\otimes 1_{\underline{w}} = 0$  $\otimes 1_{\underline{w}} = 0$

for $\sigma \in S_W$, $\tau \in S_P$, and $\underline{w}$ an arbitrary word for some $w \in W$.

The following theorems will hold true in the setting of arbitrary parabolic Coxeter systems (W, P). Thus we state them in that language (lifting the combinatorics from Section 2) despite the fact that we have only provided the (much simplified!) relations of the case $(W, P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$.

For $\lambda \in {}^P W$ we let $\mathscr{H}^{<\lambda}_{(W,P)}$ denote the span of all diagrams $c^*_{\mathsf{S}} c_{\mathsf{T}}$ with $\mathsf{S}, \mathsf{T} \in \mathrm{Path}(\mu)$ with $\mu < \lambda$.

Theorem 3.4 (The light leaves basis [10])**.** *For each $\lambda \in {}^P W$ the graded cell module $\Delta^{\Bbbk}(\lambda)$ has a basis given by*

$$\{c_{\mathsf{S}} + \mathscr{H}^{<\lambda}_{(W,P)} \mid \mathsf{S} \in \mathrm{Path}(\lambda)\}$$

This module has a unique proper maximal submodule, $\mathrm{rad}(\Delta^{\Bbbk}(\lambda))$*, with simple quotient*

$$L^{\Bbbk}(\lambda) = \Delta^{\Bbbk}(\lambda)/\mathrm{rad}(\Delta^{\Bbbk}(\lambda))$$

Moreover, the set $\{L^{\Bbbk}(\lambda)\langle k\rangle \mid \lambda \in {}^P W, k \in \mathbb{Z}\}$ provides a complete set of pairwise non-isomorphic graded simple modules for $\mathscr{H}_{(W,P)}$.

Theorem 3.5 (The Kazhdan–Lusztig positivity conjecture, Elias–Williamson [7])**.** *Let $\Bbbk$ be a field of characteristic $p \geqslant 0$. The p-Kazhdan–Lusztig polynomials are defined to be the graded composition factor multiplicities*

$${}^p n_{\lambda,\mu}(q) = \sum_{k \in \mathbb{Z}} [\Delta^{\Bbbk}(\lambda) : L^{\Bbbk}(\mu)\langle k\rangle] q^k.$$

For $p = 0$ we have that the ${}^p n_{\lambda,\mu}(q)$ specialise to the classical Kazhdan–Lusztig polynomials of Section 2 and thus the classical Kazhdan–Lusztig polynomials have non-negative coefficients.

4 Partitions and their Dyck combinatorics

Formally, a *partition* λ of ℓ is defined to be a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ which sum to ℓ. We call $\ell(\lambda) := \ell = \sum_i \lambda_i$ the length of the partition λ. We define the Young diagram of a partition to be the collection of tiles

$$[\lambda] = \{[r, c] \mid 1 \leqslant c \leqslant \lambda_r\}$$

depicted in Russian style with rows at 135° and columns at 45° (as in Figure 3). We identify a partition with its Young diagram and we write $\lambda \subseteq \mu$ if every box of λ is contained in μ (that is $\lambda_i \leqslant \mu_i$ for all $i \geqslant 1$). We let λ^t denote the transpose partition given by reflection of the Russian Young diagram through the vertical axis. Given $m, n \in \mathbb{N}$ we let $\mathscr{P}_{m,n}$ denote the set of all partitions which fit into an $m \times n$ rectangle, that is

$$\mathscr{P}_{m,n} = \{\lambda \mid \lambda_1 \leqslant m, \lambda_1^t \leqslant n\}.$$

For $\lambda \in \mathscr{P}_{m,n}$, the x-coordinate of a tile $[r,c] \in \lambda$ is equal to $r - c + m \in \{1, 2, \ldots, m+n\}$ and we define this x-coordinate to be the "colour" or "content" of the tile and we write $cont[r,c] = r - c + m$. It is well-known that a partition is uniquely determined by the contents of its boxes and this can be seen as the main ingredient in the following result:

Proposition 4.1. *For $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$ there is a poset isomorphism between $({}^P W, \leqslant)$ (the minimal coset representatives under the Bruhat ordering) and $(\mathscr{P}_{m,n}, \leqslant)$ (the partitions in an $(m \times n)$-rectangle ordered by inclusion), sending the identity coset to $\varnothing$ and the longest element to (m^n) (see Figures 1 and 3).*

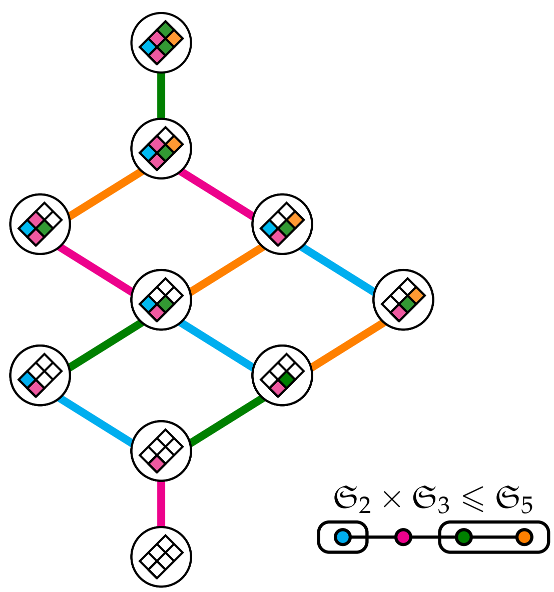


Figure 3: The partitions in a (2×3)-rectangle, ordered by inclusion. At the bottom we depict the empty partition inside a (2×3)-grid and at the top we depict the unique partition of maximal size, namely the rectangle (2^3). Compare this poset with the rightmost poset depicted in Figure 1.

Having encoded the Bruhat order in terms of partition combinatorics, we ask whether it is possible compute the Kazhdan–Lusztig polynomials in a similar fashion. The answer is yes, and makes use of the idea of Dyck paths. We define a path on λ to be a finite non-empty set P of tiles that are ordered $[r_1, c_1] \in \lambda, \dots, [r_s, c_s] \in \lambda$ for some $s \geqslant 1$ such that for each $1 \leqslant i \leqslant s-1$ we have $[r_{i+1}, c_{i+1}] = [r_i + 1, c_i]$ or $[r_i, c_i - 1]$. Note that the set $\mathsf{cont}(P)$ of contents of the tiles in a path P form an interval of integers. We say that P is a Dyck path if

$$\min\{r_i + c_i \,:\, 1 \leqslant i \leqslant s\} = r_1 + c_1 = r_s + c_s,$$

that is the minimal height of the path is achieved at the start and end of the path, see the leftmost diagram in Figure 4 for an example of a single Dyck path on a partition. We say that P and Q are *adjacent* if and only if the multiset given by the disjoint union $\mathsf{cont}(P) \sqcup \mathsf{cont}(Q)$ is an interval (see the central diagram in Figure 4 for an example).

Definition 4.2. Let $\lambda \subseteq \mu \in \mathscr{P}_{m,n}$. A Dyck tiling of the skew partition $\mu \setminus \lambda$ is a set $\{P^1, \dots, P^k\}$ of Dyck paths such that

$$\mu \setminus \lambda = \bigsqcup_{i=1}^{k} P^i$$

and for each $i \neq j$ we have P^i and P^j are not adjacent. If such a Dyck tiling exists, we call (λ, μ) a Dyck pair. Dyck tilings for a given $\mu \setminus \lambda$ are not unique. However, it can be shown that if we have two Dyck tilings $\mu \setminus \lambda = \sqcup_{i=1}^{k} P^i = \sqcup_{j=1}^{l} Q^j$ then we must have $k = l$ and there is a bijection $\{P^i\} \to \{Q^{j_i}\}$ satisfying $\mathsf{cont}(P^i) = \mathsf{cont}(Q^{j_i})$ for all $1 \leqslant i \leqslant k$. Thus it makes sense to define the degree of the Dyck pair (λ, μ) to be $\deg(\lambda, \mu) = k$.

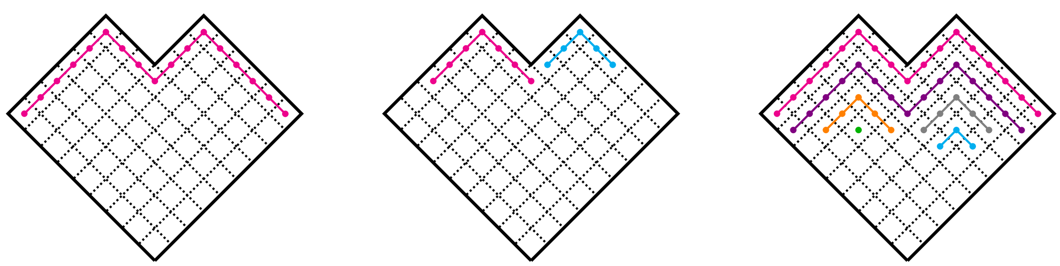


Figure 4: On the left we depict a Dyck path on $(9^6, 6^3)$. The centre diagram depicts two adjacent Dyck paths (and so $(9^6, 6^3) \setminus (9^2, 8^3, 5^3, 3)$ does not admit a Dyck tiling). On the right we depict a Dyck tiling of $(9^6, 6^3) \setminus (9, 7, 6, 5, 4, 2, 1^2)$ of degree 6.

We are now ready to provide a closed combinatorial interpretation for the p-Kazhdan–Lusztig polynomials of $(\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. This generalises existing results of Lascoux–Schutzenberger to arbitrary fields.

Theorem 4.3 (B.–D.–H.–Norton [1]). *Let $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$ and $p \geqslant 0$. We have that*

$$ {}^p n_{\lambda,\mu}(q) = \begin{cases} q^{\deg(\lambda,\mu)} & \text{if } (\lambda,\mu) \text{ is a Dyck pair;} \\ 0 & \text{otherwise.} \end{cases} $$

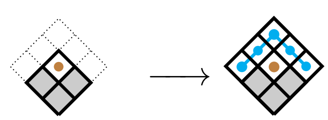
Proof. By Definition 3.3 we know that $\mathscr{H}_{(W,P)}$ has basis indexed by pairs of paths in the weak Bruhat graph of ${}^PW$. In [1] we provide a graded bijection between $\mathrm{Path}(\lambda, \mathsf{T}^\mu)$ and Dyck tilings of shape $\mu \setminus \lambda$. Any Dyck tiling $\mu \setminus \lambda$ is manifestly of *positive degree*, unless $\lambda = \mu$ in which case we obtain a unique (trivial) Dyck tableau of degree zero. Now, since any graded simple module is fixed by the anti-involution $*$ we deduce that it must have graded dimension belonging to $\mathbb{Z}_{\geqslant 0}[q+q^{-1}]$. Putting together the above facts, we deduce that the simple modules are 1-dimensional (concentrated in degree zero) regardless of the characteristic of the field and the result follows. □

5 Submodule lattices of cell modules

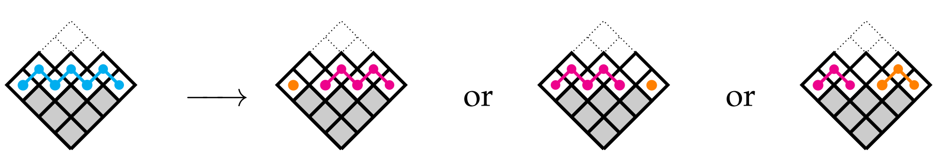
We are now ready to discuss one of the main results of [2]. Namely, we will provide the full submodule lattice of the cell modules for $\mathscr{H}_{(W,P)}$ when $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$ over any field $\Bbbk$. We prove in [2] that (the basic algebra of) $\mathscr{H}_{(W,P)}$ is generated in degrees 0 and 1 and hence the grading gives a submodule filtration of $\Delta^{\Bbbk}(\lambda)$. Thus to determine whether there is an extension between two composition factors $L^{\Bbbk}(\mu)$ and $L^{\Bbbk}(\nu)$ within $\Delta^{\Bbbk}(\lambda)$ (where (λ,μ) and (λ,ν) are Dyck pairs, by Theorem 4.3) it is enough to consider pairs of adjacent degree , that is where $\deg(\lambda,\nu) = \deg(\lambda,\mu)+1$. Using the presentations of [2, Theorem B] we are able to fully determine these extensions combinatorially as follows:

Definition 5.1. Let (λ,μ) and (λ,ν) be Dyck pairs of degree k and $k+1$ respectively. We write $(\lambda,\mu) \to (\lambda,\nu)$ if either:

- ν is obtained from μ by adding a Dyck path.



- ν is obtained from μ by removing a Dyck path, splitting some Dyck path in the tiling of $\mu \setminus \lambda$ into two distinct Dyck paths:



We extend this to a partial ordering, $\prec$, by taking the transitive closure of $\to$.

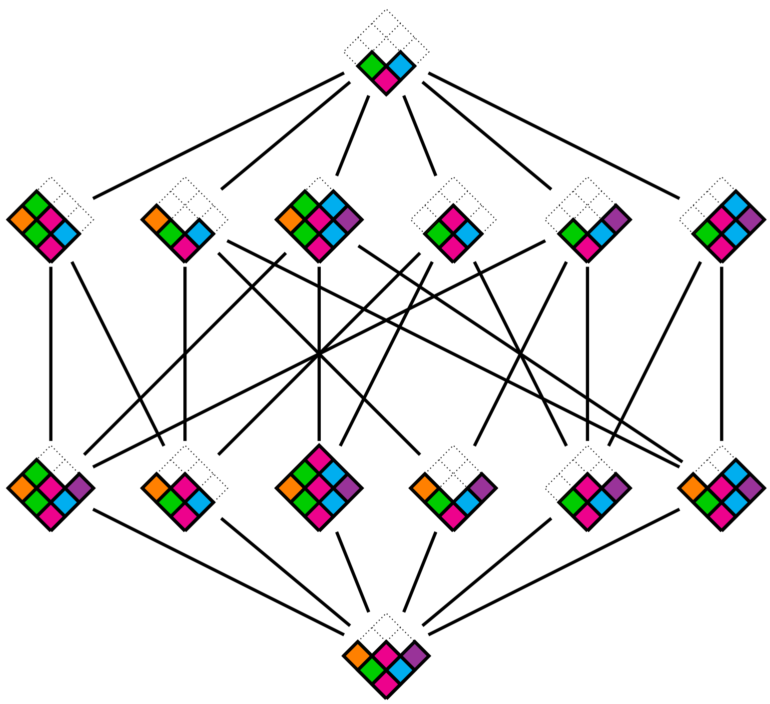
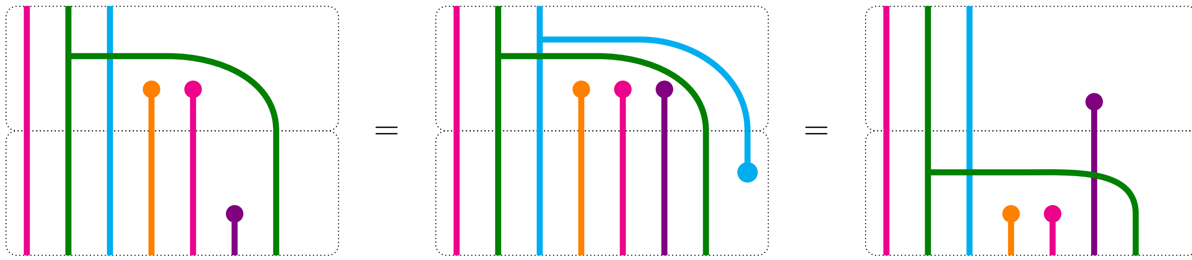


Figure 5: The submodule lattice of $\Delta^{\Bbbk}(2,1)$ for $m = n = 3$.

An example of the lattice on $\Delta^{\Bbbk}(\lambda)$ for $\lambda = (2,1)$ and $m = n = 3$ is depicted in Figure 5. With a little more work, one can prove that there is a unique Dyck pair (λ, α) of maximal degree (and that the submodule lattice is a bonafide lattice in the combinatorial sense!). Indeed we have the following:

Theorem 5.2 (B.-D.-H.-S. [2])**.** *Fix $\lambda \in {}^{P}W$ for $(W,P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. The module $\Delta^{\Bbbk}(\lambda)$ has a unique simple submodule, it is rigid (its socle and radical layers coincide) and the full submodule lattice of $\Delta(\lambda)$ is given by the partial ordering $\prec$.*

Proof. We first provide a full quiver and relations presentation of $\mathscr{H}_{(W,P)}$ and then use this to analyse the submodule structures of $\Delta^{\Bbbk}(\lambda)$. For example let $\mu = (3^2, 1)$ as in the leftmost vertex of the penultimate layer of the module of the module $\Delta^{\Bbbk}(2,1)$ depicted in Figure 5. The composition factor $L^{\Bbbk}(3^2, 1)$ has three distinct paths leading into it; these come from the simple modules labelled by (3^2), $(3^2, 2)$, and $(2, 1^2)$ respectively. These three paths can be seen to be equal using the fork-spot relations as follows:



The remaining cases also follow by fork-spot relations, but with a little more thought required. One must then show that these relations are exhaustive — this requires the full quiver and relations presentation of $\mathscr{H}_{(W,P)}$ alluded to above. □

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A framework unifying some bijections for graphs and its connection to Lawrence polytopes

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Abstract. Let G be a connected graph. The Jacobian group (also known as the Picard group or sandpile group) of G is a finite abelian group whose cardinality equals the number of spanning trees of G. The Jacobian group admits a canonical simply transitive action on the set $\mathcal{R}(G)$ of cycle-cocycle reversal classes of orientations of G. Hence one can construct combinatorial bijections between spanning trees of G and $\mathcal{R}(G)$ to build connections between spanning trees and the Jacobian group. The geometric bijections (defined by Backman, Baker, and Yuen) and the Bernardi bijections are two important examples. In this paper, we construct a new family of such bijections that includes both. Our bijections depend on a pair of atlases (different from the ones in manifold theory) that abstract and generalize certain common features of the two known bijections. The definitions of these atlases are derived from triangulations and dissections of the Lawrence polytopes associated to G. The acyclic cycle signatures and cocycle signatures used to define the geometric bijections correspond to regular triangulations. Our bijections can extend to subgraph-orientation correspondences. Most of our results hold for regular matroids. We present our work in the language of fourientations, which are a generalization of orientations.

Keywords: sandpile group, cycle-cocycle reversal class, Lawrence polytope, triangulation, dissection, fourientation

1 Overview

This paper is an extended abstract of our recent work [8] to be submitted to the conference FPSAC 2024. Most of this paper comes from [8, Section 1]. The major change we have made is that this paper is written in the setting of graphs rather than regular matroids. We hope this will benefit some readers who are not familiar with matroids.

Given a connected graph G, we build a new family of bijections between the set $\mathcal{T}(G)$ of spanning trees of G and the set $\mathcal{R}(G)$ of equivalence classes of orientations of G up to cycle and cocycle reversals. The new family of bijections includes the *BBY bijection* (also

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known as the geometric bijection) constructed by Backman, Baker, and Yuen [2], and the *Bernardi bijection*[1] in [6].

These bijections are closely related to the *Jacobian group* (also known as the *Picard group* or *sandpile group*) $\mathrm{Jac}(G)$ of G. The group $\mathrm{Jac}(G)$ and the set $\mathcal{T}(G)$ of spanning trees are equinumerous. Recently, many efforts have been devoted to making $\mathcal{T}(G)$ a *torsor* for $\mathrm{Jac}(G)$, i.e., defining a simply transitive action of $\mathrm{Jac}(G)$ on $\mathcal{T}(G)$. In [4], Baker and Wang interpreted the Bernardi bijection as a bijection between $\mathcal{T}(G)$ and *break divisors*. Since the set of break divisors is a canonical torsor for $\mathrm{Jac}(G)$ (see [1]), the Bernardi bijection induces the *Bernardi torsor*. In [14], Yuen defined the geometric bijection between $\mathcal{T}(G)$ and break divisors of G. Later, this work was generalized in [2] where Backman, Baker, and Yuen defined the BBY bijection between $\mathcal{T}(G)$ and the cycle-cocycle reversal classes $\mathcal{R}(G)$. The set $\mathcal{R}(G)$ was introduced by Gioan [10] and is known to be a canonical torsor for $\mathrm{Jac}(G)$ [2]. Hence any bijection between $\mathcal{T}(G)$ and $\mathcal{R}(G)$ makes $\mathcal{T}(G)$ a torsor. From the point of view in [2], replacing break divisors with $\mathcal{R}(G)$ provides a more general setting. In particular, we may also view the Bernardi bijection as a bijection between $\mathcal{T}(G)$ and $\mathcal{R}(G)$ and define the Bernardi torsor.

Our work puts all the above bijections in the same framework. It is surprising because the BBY bijection and the Bernardi bijection rely on totally different parameters. The main ingredients to define the BBY bijection are an *acyclic cycle signature* σ and an *acyclic cocycle signature* σ^* of G. The BBY bijection sends spanning trees to (σ, σ^*)-compatible orientations, which are representatives of $\mathcal{R}(G)$. The Bernardi bijection relies on a ribbon structure on the graph G together with a vertex and an edge as initial data. Although for planar graphs, the Bernardi bijection becomes a special case of the BBY bijection, they are different in general [14, 2]. The main ingredients to define our new bijections are a *triangulating atlas* and a *dissecting atlas* of G. These atlases (different from the ones in manifold theory) abstract and generalize certain common features of the two known bijections. They are derived from *triangulations* and *dissections* of the *Lawrence polytopes* associated to graphs. The acyclic cycle signatures and cocycle signatures used to define the BBY bijections correspond to *regular* triangulations.

Our bijections extend to subgraph-orientation correspondences. The construction is similar to the one that extends the BBY bijection in [9]. The extended bijections have nice specializations to forests and connected subgraphs.

Our results are also closely related to and motivated by Kálmán's work [11], Kálmán and Tóthmérész's work [12], and Postnikov's work [13] on *root polytopes* of *hypergraphs*, where the hypergraphs specialize to graphs, and the Lawrence polytopes generalize the root polytopes in the case of graphs. See [8, Section 1.8] for details.

We find it very efficient to present our theory in the language of *fourientations*, which are a generalization of orientations introduced by Backman and Hopkins [3].

[1] The Bernardi bijection in [6] is a subgraph-orientation correspondence. In this paper, by the Bernardi bijection we always mean its restriction to spanning trees.

Most of our results hold for *regular matroids* as in [2], although in this paper we focus on graphs. See [8] for the regular matroid version of this paper.

2 Notation and terminology

2.1 Cycles and cocycles of a graph

Let G be a connected finite graph with nonempty edge set E, where *loops* and *multiple edges* are allowed. For each edge $e \in E$, we may assign a direction to it and hence get an *arc*. Note that a loop also has two possible directions. An *orientation* of the graph G is an assignment of a direction to each edge, typically denoted by $\overrightarrow{O}$.

A subset C of E is called a *cycle* if there exist distinct vertices $v_1, v_2, \cdots, v_n$ such that $C = \{\text{edge } v_i v_{i+1} : i = 1, 2, \cdots, n\}$, where $v_{n+1} := v_1$. Note that a cycle may be a loop. If we direct every edge in C from v_i to v_{i+1} or direct every edge in C from v_{i+1} to v_i, then we get a *directed cycle*, typically denoted by $\overrightarrow{C}$. Given a subset W of vertices, the set of edges with one endpoint in W and the other one not in W is called a *cut*. A *cocycle* C^* is a cut which is minimal for inclusion. If we direct every edge in C^* from W to its complement (or in the other way), then we get a *directed cocycle*, typically denoted by $\overrightarrow{C^*}$.

When an arc $\overrightarrow{e}$, a directed cycle $\overrightarrow{C}$, or a directed cocycle $\overrightarrow{C^*}$ is specified, the corresponding underlying edge(s) will be denoted by e, C, or C^*, respectively. Viewing $\overrightarrow{O}$, $\overrightarrow{C}$, and $\overrightarrow{C^*}$ as sets of arcs, it makes sense to write $\overrightarrow{e} \in \overrightarrow{O}$, etc.

Now we define *cycle-cocycle reversal (equivalence) classes* of orientations of G introduced by Gioan [10]. If $\overrightarrow{C}$ is a directed cycle in an orientation $\overrightarrow{O}$ of G, then a *cycle reversal* replaces $\overrightarrow{C}$ with the opposite directed cycle in $\overrightarrow{O}$. The equivalence relation generated by cycle reversals defines the *cycle reversal classes* of orientations of G. Similarly, we may define the *cocycle reversal classes*. The equivalence relation generated by cycle and cocycle reversals defines the *cycle-cocycle reversal classes*. It is proved in [10] that the number of cycle-cocycle reversal classes of G equals the number of spanning trees of G.

Let T be a spanning tree of G and e be an edge. If $e \notin T$, then we call the unique cycle in $T \cup \{e\}$ the *fundamental cycle* of e (with respect to T); if $e \in T$, then we call the unique cocycle in $(E \backslash T) \cup \{e\}$ the *fundamental cocycle* of e (with respect to T).

2.2 Fourientations, potential cycles, and potential cocycles

It is convenient to introduce our theory in terms of *fourientations*. Fourientations of graphs are systematically studied by Backman and Hopkins [3]. We will only make use of the basic notions. A fourientation $\overrightarrow{F}$ of the graph G is a subset of the set of all the $2|E|$ arcs. Intuitively, a fourientation is a choice for each edge of G whether to make it *one-way* *oriented*, leave it *unoriented*, or *biorient* it. We denote by $-\overrightarrow{F}$ the fourientation obtained by reversing all the arcs in $\overrightarrow{F}$. In particular, the bioriented edges remain bioriented. We denote by $\overrightarrow{F}^c$ the set complement of $\overrightarrow{F}$, which is also a fourientation. A *potential cycle* of a fourientation $\overrightarrow{F}$ is a directed cycle $\overrightarrow{C}$ such that $\overrightarrow{C} \subseteq \overrightarrow{F}$. A *potential cocycle* of a fourientation $\overrightarrow{F}$ is a directed cocycle $\overrightarrow{C^*}$ such that $\overrightarrow{C^*} \subseteq -\overrightarrow{F}^c$. See Figure 1 for examples.

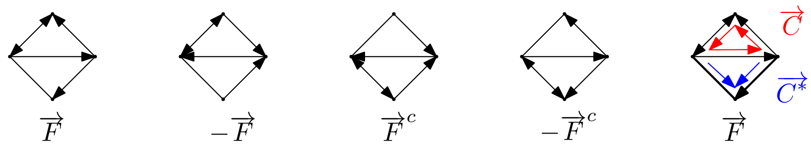

Figure 1: Examples of fourientation, potential cycle and potential cocycle

3 New framework: a pair of atlases and its induced map

The BBY bijection studied in [2] relies upon a pair consisting of an acyclic cycle signature and an acyclic cocycle signature. We will generalize this work by building a new framework where the signatures are replaced by *atlases* and the BBY bijection is replaced by a map $f_{\mathcal{A},\mathcal{A}^*}$. This section will introduce these new terminologies.

Definition 3.1. Let T be a tree of G (from now on, by trees we mean spanning trees).

(1) We call the edges in T *internal* and the edges not in T *external*.

(2) An *externally oriented tree* $\overrightarrow{T}$ is a fourientation where all the internal edges are bioriented and all the external edges are one-way oriented. Dually, an *internally oriented tree* $\overrightarrow{T^*}$ is a fourientation where all the external edges are bioriented and all the internal edges are one-way oriented.

(3) An *external atlas* $\mathcal{A}$ of G is a collection of externally oriented trees $\overrightarrow{T}$ such that each tree of G appears exactly once. Dually, an *internal atlas* $\mathcal{A}^*$ of G is a collection of internally oriented trees $\overrightarrow{T^*}$ such that each tree of G appears exactly once.

Given an external atlas $\mathcal{A}$ (resp. internal atlas $\mathcal{A}^*$) and a tree T, by $\overrightarrow{T}$ (resp. $\overrightarrow{T^*}$) we always mean the oriented tree in the atlas though the notation does not refer to the atlas.

Definition 3.2 (See Figure 2)**.** For a pair of atlases $(\mathcal{A}, \mathcal{A}^*)$, we define the following map

$$\begin{aligned} f_{\mathcal{A},\mathcal{A}^*} : \{\text{trees of } G\} &\to \{\text{orientations of } G\} \\ T &\mapsto \overrightarrow{T} \cap \overrightarrow{T^*} \text{ (where } \overrightarrow{T} \in \mathcal{A}, \overrightarrow{T^*} \in \mathcal{A}^*). \end{aligned}$$

In the forthcoming Example 3.4 and Example 3.5, we will put the BBY bijection and the Bernardi bijection in our framework. Before that, we recall the definitions of cycle (resp. cocycle) signatures and acyclic cycle (resp. cocycle) signatures in [2].

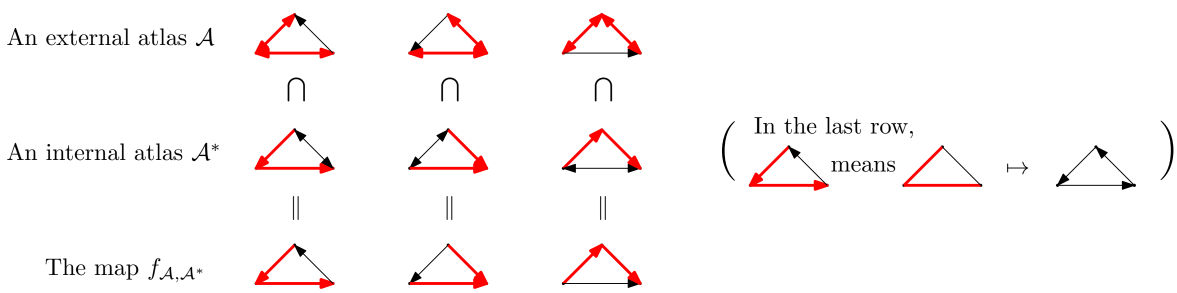

Figure 2: An example for Definition 3.1 and 3.2. The trees of the triangle graph are in red. For each tree T, the externally oriented tree $\overrightarrow{T} \in \mathcal{A}$ and the internally oriented tree $\overrightarrow{T^*} \in \mathcal{A}^*$ are in the same column. Their intersection $f_{\mathcal{A},\mathcal{A}^*} = \overrightarrow{T} \cap \overrightarrow{T^*}$ is displayed at the bottom.

Definition 3.3. Let G be a graph.

(1) A *cycle signature* σ of G is the choice of a direction for each cycle of G. For each cycle C, we denote by $\sigma(C)$ the directed cycle we choose for C. By abuse of notation, sometimes we also view σ as the set of the directed cycles of G chosen by σ.

(2) The cycle signature σ is said to be *acyclic* if whenever a_C are nonnegative reals with $\sum_C a_C\sigma(C) = 0$ in $\mathbb{R}^E$ we have $a_C = 0$ for all C, where the sum is over all cycles of G, and $\sigma(C)$ is viewed as a $\{0, \pm1\}$-vector in $\mathbb{R}^E$ w.r.t. a fixed reference orientation.

(3) *Cocycle signatures* σ^* and *acyclic cocycle signatures* are defined similarly.

Example 3.4 (Atlases $\mathcal{A}_\sigma, \mathcal{A}^*_{\sigma^*}$ and the BBY map (bijection))**.** Let σ be a cycle signature of G. We may construct an external atlas $\mathcal{A}_\sigma$ from σ such that for each externally oriented tree $\overrightarrow{T} \in \mathcal{A}_\sigma$, each external arc $\overrightarrow{e} \in \overrightarrow{T}$ is oriented according to the orientation of the fundamental cycle of e determined by σ. Similarly, we may construct an internal atlas $\mathcal{A}^*_{\sigma^*}$ from any cocycle signature σ^* such that for each internally oriented tree $\overrightarrow{T^*} \in \mathcal{A}^*_{\sigma^*}$, each internal arc $\overrightarrow{e} \in \overrightarrow{T^*}$ is oriented according to the orientation of the fundamental cocycle of e determined by σ^*. Then when the two signatures are acyclic, the map $f_{\mathcal{A}_\sigma,\mathcal{A}^*_{\sigma^*}}$ is exactly the BBY map defined in [2].

Example 3.5 (Atlases $\mathcal{A}_{\mathrm{B}}, \mathcal{A}^*_q$ and the Bernardi map (bijection))**.** The Bernardi bijection is defined for a connected graph G equipped with a *ribbon structure* and with initial data (q, e), where q is a vertex and e is an edge incident to the vertex; see [6, Section 3.2] for details. Here we use an example (Figure 3) to recall the construction of the bijection in the atlas language. The Bernardi bijection is a map from trees to certain orientations. The construction makes use of the *Bernardi tour* which starts with (q, e) and goes around a given tree T according to the ribbon structure. We may construct an external atlas $\mathcal{A}_{\mathrm{B}}$ of G as follows. Observe that the Bernardi tour cuts each external edge twice. We orient each external edge toward the first-cut endpoint, biorient all the internal edges of T, and hence get an externally oriented tree $\overrightarrow{T}$. All such externally oriented trees form the atlas $\mathcal{A}_{\mathrm{B}}$.

The internal atlas $\mathcal{A}_q^*$ of G is constructed as follows. For any tree T, we orient each internal edge away from q, biorient external edges, and hence get $\overrightarrow{T^*} \in \mathcal{A}_q^*$. We remark that $\mathcal{A}_q^*$ is a special case of $\mathcal{A}_{\sigma^*}^*$, where σ^* is an acyclic cocycle signature [2, Example 1.3.4].

The map $f_{\mathcal{A}_{\mathrm{B}},\mathcal{A}_q^*}$ is exactly the Bernardi map.

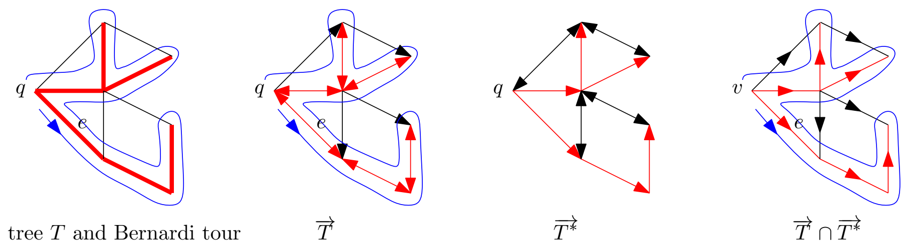

Figure 3: An example for the Bernardi map. The tree T is in red.

4 Bijections and the two atlases

We will see in this section that the map $f_{\mathcal{A},\mathcal{A}^*}$ induces a bijection between trees of G and cycle-cocycle reversal classes of G when the two atlases satisfy certain conditions which we call *dissecting* and *triangulating*. Furthermore, we will extend the bijection as in [9].

The following definitions play a central role in our paper. Although the definitions are combinatorial, they were derived from dissecting and triangulating Lawrence polytopes; see Section 6.

Definition 4.1. Let $\mathcal{A}$ be an external atlas and $\mathcal{A}^*$ be an internal atlas of G.

(1) We call $\mathcal{A}$ *dissecting* if for any two distinct trees T_1 and T_2, the fourientation $\overrightarrow{T_1} \cap (-\overrightarrow{T_2})$ has a potential cocycle. Dually, we call $\mathcal{A}^*$ *dissecting* if for any two distinct trees T_1 and T_2, the fourientation $(\overrightarrow{T_1^*} \cap (-\overrightarrow{T_2^*}))^c$ has a potential cycle.

(2) We call $\mathcal{A}$ *triangulating* if for any two distinct trees T_1 and T_2, the fourientation $\overrightarrow{T_1} \cap (-\overrightarrow{T_2})$ has no potential cycle. Dually, we call $\mathcal{A}^*$ *triangulating* if for any two distinct trees T_1 and T_2, the fourientation $(\overrightarrow{T_1^*} \cap (-\overrightarrow{T_2^*}))^c$ has no potential cocycle.

Remark 4.2. Being triangulating is stronger than being dissecting by [3, Proposition 2.6].

Now we are ready to present the first main result in this paper.

Theorem 4.3. Given a pair of dissecting atlases $(\mathcal{A}, \mathcal{A}^*)$ of a graph G, if at least one of the atlases is triangulating, then the map

$$\overline{f}_{\mathcal{A},\mathcal{A}^*} : \{\text{trees of } G\} \to \{\text{cycle-cocycle reversal classes of } G\}$$
$$T \mapsto [\overrightarrow{T} \cap \overrightarrow{T^*}]$$

is bijective, where $[\overrightarrow{T} \cap \overrightarrow{T^*}]$ denotes the cycle-cocycle reversal class containing $\overrightarrow{T} \cap \overrightarrow{T^*}$.

Example 4.4 (Example 3.4 continued)**.** One of the main results in [2] is that the BBY map induces a bijection between trees and cycle-cocycle reversal classes. Because $\mathcal{A}_\sigma$ and $\mathcal{A}^*_{\sigma^*}$ are triangulating ([8, Lemma 3.4]), Theorem 4.3 recovers this result.

Example 4.5 (Example 3.5 continued)**.** Theorem 4.3 also recovers the bijectivity of the Bernardi map for trees in [6]. In [6], it is proved that the Bernardi map is a bijection between trees and the *q-connected outdegree sequences*. Baker and Wang [4] observed that the q-connected outdegree sequences are essentially the same as the *break divisors*. Later in [2], the break divisors are equivalently replaced by cycle-cocycle reversal classes. The external atlas $\mathcal{A}_{\mathrm{B}}$ is dissecting ([8, Lemma 3.15]). The internal atlas $\mathcal{A}^*_q$ is triangulating because it equals $\mathcal{A}^*_{\sigma^*}$ for some acyclic signature σ^*. Hence our theorem applies.

In Theorem 4.3, if we do not further assume that one of the atlases is triangulating, then the map $\overline{f}_{\mathcal{A},\mathcal{A}^*}$ is not necessarily bijective; see [8, Example 1.11].

In [9], the BBY bijection is extended to a bijection between spanning subgraphs of G (i.e., subsets of E) and orientations of G in a canonical way. We also generalize this work by extending $f^{-1}_{\mathcal{A},\mathcal{A}^*}$ to $\varphi_{\mathcal{A},\mathcal{A}^*}$.

Definition 4.6 (The map $\varphi_{\mathcal{A},\mathcal{A}^*}$)**.** We will define a map $\varphi_{\mathcal{A},\mathcal{A}^*}$ from orientations to subgraphs such that $\varphi_{\mathcal{A},\mathcal{A}^*} \circ f_{\mathcal{A},\mathcal{A}^*}$ is the identity map, and hence $\varphi_{\mathcal{A},\mathcal{A}^*}$ extends $f^{-1}_{\mathcal{A},\mathcal{A}^*}$. We start with an orientation $\overrightarrow{O}$. By Theorem 4.3, we get a tree $T = \overline{f}^{-1}_{\mathcal{A},\mathcal{A}^*}([\overrightarrow{O}])$. Since $\overrightarrow{O}$ and $f_{\mathcal{A},\mathcal{A}^*}(T)$ are in the same cycle-cocycle reversal class, one can obtain one of them by reversing disjoint directed cycles $\{\overrightarrow{C_i}\}_{i\in I}$ and cocycles $\{\overrightarrow{C^*_j}\}_{j\in J}$ in the other ([8, Lemma 2.7]). Define $\varphi_{\mathcal{A},\mathcal{A}^*}(\overrightarrow{O}) = (T \cup \biguplus_{i\in I} C_i) \setminus \biguplus_{j\in J} C^*_j$, where the symbol $\uplus$ means disjoint union.

The amazing fact here is that $\varphi_{\mathcal{A},\mathcal{A}^*}$ is a bijection, and it has nice specializations.

Theorem 4.7. Fix a pair of dissecting atlases $(\mathcal{A}, \mathcal{A}^*)$ of G with ground set E. Suppose at least one of the atlases is triangulating.

(1) The map $\varphi_{\mathcal{A},\mathcal{A}^*}$ is a bijection from orientations of G to spanning subgraphs of G.

(2) The image of the spanning forests of G under the bijection $\varphi^{-1}_{\mathcal{A},\mathcal{A}^*}$ is a representative set of the cycle reversal classes of G.

(3) The image of the spanning connected subgraphs of G under the bijection $\varphi^{-1}_{\mathcal{A},\mathcal{A}^*}$ is a representative set of the cocycle reversal classes of G.

Remark 4.8. We can apply Theorem 4.7 to extend and generalize the Bernardi bijection; see [8, Corollary 3.16] for a formal statement.

5 Signatures and the two atlases

This section studies cycle signatures (resp. cocycle signatures) in terms of external atlases (resp. internal atlases). In particular, we will see Theorem 4.3 and Theorem 4.7 generalize the bijections in [2] and [9], respectively.

Recall in Example 3.4 that from signatures σ and σ^*, we may construct atlases $\mathcal{A}_\sigma$ and $\mathcal{A}^*_{\sigma^*}$. It is natural to ask: (1) Which signatures induce triangulating atlases? (2) Is any triangulating atlas induced by a signature?

The following definition and theorem answer these two questions.

Definition 5.1. A cycle signature σ is said to be *triangulating* if for any $\overrightarrow{T} \in \mathcal{A}_\sigma$ and any directed cycle $\overrightarrow{C} \subseteq \overrightarrow{T}$, $\overrightarrow{C}$ belongs to σ. Dually, a cocycle signature σ^* is said to be *triangulating* if for any $\overrightarrow{T^*} \in \mathcal{A}^*_{\sigma^*}$ and any directed cocycle $\overrightarrow{C^*} \subseteq \overrightarrow{T^*}$, $\overrightarrow{C^*}$ belongs to σ^*.

Theorem 5.2. The map $\alpha : \sigma \mapsto \mathcal{A}_\sigma$ is a bijection from the set of triangulating cycle signatures of G to the set of triangulating external atlases of G. Dually, the map $\alpha^* : \sigma^* \mapsto \mathcal{A}^*_{\sigma^*}$ is a bijection from the set of triangulating cocycle signatures of G to the set of triangulating internal atlases of G.

Remark 5.3. For a dissecting external atlas $\mathcal{A}$, it is possible for there to be no cycle signature σ such that $\mathcal{A}_\sigma = \mathcal{A}$; see [8, Remark 1.18].

Remark 5.4. Acyclic signatures are all triangulating; see [8, Lemma 3.4]. There exists a triangulating signature that is not acyclic; see [8, Proposition 3.14]. In Section 6, we will see acyclic signatures correspond to regular triangulations.

A nice thing about the acyclic signatures is that the associated compatible orientations (defined below) form representatives of orientation classes (proved in [2]). The triangulating signatures also have this property; see the proposition below.

Definition 5.5. Let G be a graph, σ be a cycle signature, σ^* be a cocycle signature, and $\overrightarrow{O}$ be an orientation of G.

(1) The orientation $\overrightarrow{O}$ is said to be *σ-compatible* if any directed cycle in $\overrightarrow{O}$ is in σ.

(2) The orientation $\overrightarrow{O}$ is said to be *σ^*-compatible* if any directed cocycle in $\overrightarrow{O}$ is in σ^*.

(3) The orientation $\overrightarrow{O}$ is said to be *(σ, σ^*)-compatible* if it is both σ-compatible and σ^*-compatible.

Proposition 5.6. Suppose σ and σ^* are *triangulating* signatures.

(1) The set of (σ, σ^*)-compatible orientations is a representative set of the cycle-cocycle reversal classes of G.

(2) The set of σ-compatible orientations (resp. σ^*-compatible orientations) is a representative set of the cycle (resp. cocycle) reversal classes of G.

To reformulate Theorem 4.3 and Theorem 4.7 in terms of signatures and compatible orientations, we write

$$\mathrm{BBY}_{\sigma,\sigma^*} = f_{\mathcal{A}_\sigma,\mathcal{A}^*_{\sigma^*}} \text{ and } \varphi_{\sigma,\sigma^*} = \varphi_{\mathcal{A}_\sigma,\mathcal{A}^*_{\sigma^*}}.$$

They are exactly the BBY bijection in [2] and the extended BBY bijection in [9] when the two signatures are acyclic. By the two theorems and a little extra work, we have the following theorems, which generalize the work in [2] and [9], respectively.

Theorem 5.7. Suppose σ and σ^* are *triangulating* signatures of a graph G. The map $\mathrm{BBY}_{\sigma,\sigma^*}$ is a bijection from trees of G to (σ,σ^*)-compatible orientations of G.

Theorem 5.8. Suppose σ and σ^* are *triangulating* signatures of a graph G.

(1) The map $\varphi_{\sigma,\sigma^*}$ is a bijection from orientations of G to spanning subgraphs of G.

(2) The map $\varphi_{\sigma,\sigma^*}$ specializes to a bijection between σ-compatible orientations and spanning forests of G.

(3)The map $\varphi_{\sigma,\sigma^*}$ specializes to a bijection between σ^*-compatible orientations and spanning connected subgraphs of G.

The definition of triangulating signatures is somewhat indirect. However, we have the following nice description for the triangulating *cycle* signatures, the proof of which is due to Gleb Nenashev. We do not know whether it holds for regular matroids.

Theorem 5.9. A *cycle* signature σ of a graph G is triangulating if and only if for any three directed cycles in σ, their sum (as vectors in $\mathbb{Z}^E$) is not zero.

6 Lawrence polytopes and the two atlases

In this section, we will introduce a pair of *Lawrence polytopes*[2] $\mathcal{P}$ and $\mathcal{P}^*$ associated to a graph G. We will see that dissections and triangulations of the Lawrence polytopes correspond to the dissecting atlases and triangulating atlases, respectively, which is actually how we derived Definition 4.1. We will also see that regular triangulations correspond to acyclic signatures.

By fixing a reference orientation of G, we get an *oriented incidence matrix* of G. The matrix is not of full rank. By deleting its last row, we get a matrix $M_{r\times n}$, where n equals the number of edges of G and r equals the number of edges of any tree of G. We can also construct another matrix $M^*_{(n-r)\times n}$ viewed as the dual of M. The construction is classic; see [9, Section 3.6]. For the readers who are familiar with matroids, we can simply say that M (resp. M^*) represents the graphic (resp. cographic) matroid associated to G.

[2]Readers can find some information on Lawrence polytopes in [5].

Definition 6.1. (1) We call

$$\begin{pmatrix} M_{r\times n} & \mathbf{0} \\ I_{n\times n} & I_{n\times n} \end{pmatrix}$$

the *Lawrence matrix*, where $I_{n\times n}$ is the identity matrix. The columns of the Lawrence matrix are denoted by $P_1, \cdots, P_n, P_{-1}, \cdots, P_{-n} \in \mathbb{R}^{n+r}$ in order.

(2) The *Lawrence polytope* $\mathcal{P} \subseteq \mathbb{R}^{n+r}$ of G is the convex hull of the points $P_1, \cdots, P_n$, $P_{-1}, \cdots, P_{-n}$.

(3) If we replace the matrix M in (1) with M^* , then we get the Lawrence polytope $\mathcal{P}^* \subseteq \mathbb{R}^{2n-r}$. We use the labels P_i^* for the points generating $\mathcal{P}^*$.

(4) We further assume that G is loopless when defining $\mathcal{P}$ and that G is coloopless when defining $\mathcal{P}^*$, to avoid duplicate columns of the Lawrence matrix.

We recall some basic notions in discrete geometry.

Definition 6.2. A *simplex* S is the convex hull of some affinely independent points. A *face* of S is a simplex generated by a subset of these points, which could be S or $\emptyset$.

Definition 6.3. Let $\mathcal{P}$ be a polytope of dimension d.

(1) If $d+1$ of the vertices of $\mathcal{P}$ form a d-dimensional simplex, we call such a simplex a *maximal simplex* of $\mathcal{P}$.

(2) A *dissection* of $\mathcal{P}$ is a collection of maximal simplices of $\mathcal{P}$ such that (I) the union is $\mathcal{P}$, and (II) the relative interiors of any two distinct maximal simplices in the collection are disjoint.

(3) If we replace the condition (II) in (2) with the condition (III) that any two distinct maximal simplices in the collection intersect in a common face (which could be empty), then we get a *triangulation*. (See Figure 4.)

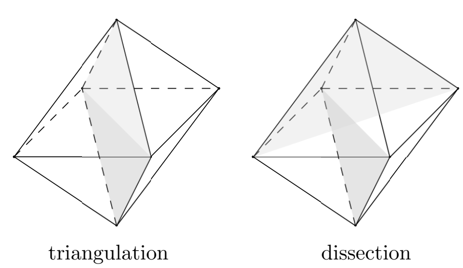

Figure 4: A triangulation and a dissection of an octahedron

The next two theorems build the connection between the geometry of the Lawrence polytopes and the combinatorics of the graph. To state them, we need to label the $2|E|$ arcs of G. Note that each column of M corresponds to the arcs of G in the reference orientation. We denote them by $\overrightarrow{e_1}, \cdots, \overrightarrow{e_n}$. For the rest of the arcs, we let $\overrightarrow{e_{-i}} = -\overrightarrow{e_i}$.

Theorem 6.4. We have the following threefold bijections, all of which are denoted by χ. (It should be clear from the context which one we are referring to when we use χ.)

(1) The Lawrence polytope $\mathcal{P} \subseteq \mathbb{R}^{n+r}$ is an $(n+r-1)$-dimensional polytope whose vertices are exactly the points $P_1, \cdots, P_n, P_{-1}, \cdots, P_{-n}$. Hence we may define a bijection

$$\begin{aligned} \chi : \{\text{vertices of } \mathcal{P}\} &\to \{\text{arcs of } G\} \\ P_i &\mapsto \overrightarrow{e_i} \end{aligned}$$

(2) The map χ in (1) induces a bijection

$$\begin{aligned} \chi : \{\text{maximal simplices of } \mathcal{P}\} &\to \{\text{externally oriented trees of } G\} \\ \begin{matrix}\text{a maximal simplex} \\ \text{with vertices } \{P_i : i \in I\}\end{matrix} &\mapsto \text{the fourientation } \{\chi(P_i) : i \in I\}. \end{aligned}$$

(3) The map χ in (2) induces two bijections

$$\begin{aligned} \chi : \{\text{triangulations of } \mathcal{P}\} &\to \{\text{triangulating external atlases of } G\} \\ \begin{matrix}\text{a triangulation with} \\ \text{maximal simplices } \{S_j : j \in J\}\end{matrix} &\mapsto \text{the external atlas } \{\chi(S_j) : i \in J\}, \end{aligned}$$

and

$$\begin{aligned} \chi : \{\text{dissections of } \mathcal{P}\} &\to \{\text{dissecting external atlases of } G\} \\ \begin{matrix}\text{a dissection with} \\ \text{maximal simplices } \{S_j : j \in J\}\end{matrix} &\mapsto \text{the external atlas } \{\chi(S_j) : j \in J\}. \end{aligned}$$

(4) The statements dual to (1), (2), and (3) hold for the Lawrence polytope $\mathcal{P}^*$.

Recall that the map $\alpha : \sigma \mapsto \mathcal{A}_\sigma$ is a bijection between triangulating cycle signatures and triangulating external atlases of G.

Theorem 6.5. The restriction of the bijection $\chi^{-1} \circ \alpha$ to the set of acyclic cycle signatures of G is bijective onto the set of *regular triangulations* of $\mathcal{P}$. The dual statement also holds. (See [7] for the definition of regular triangulations.)

We conclude this section with Table 1.

types of dissections of Lawrence polytope $\mathcal{P}$	dissection	triangulation	regular triangulation
types of external atlas $\mathcal{A}$	dissecting	triangulating	(no good description)
types of cycle signature σ	(may not exist)	triangulating	acyclic

Table 1: A summary of the correspondences among dissections of Lawrence polytopes, atlases, and signatures via α and χ. We omit the dual part.

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Shi arrangements and low elements in Coxeter groups

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Abstract. Given an arbitrary Coxeter system (W, S) and a nonnegative integer m, the m-Shi arrangement of (W, S) is a subarrangement of the Coxeter hyperplane arrangement of (W, S). The classical Shi arrangement ($m = 0$) was introduced in the case of affine Weyl groups by Shi to study Kazhdan-Lusztig cells for W. As two key results, Shi showed that each region of the Shi arrangement contains exactly one element of minimal length in W and that the union of their inverses form a convex subset of the Coxeter complex. The set of m-low elements in W were introduced to study the word problem of the corresponding Artin-Tits (braid) group and they turn out to produce automata to study the combinatorics of reduced words in W.

We generalize and extend Shi's results to any Coxeter system. First, for $m \in \mathbb{N}$ the set of minimal length elements of the regions in a m-Shi arrangement is precisely the set of m-low elements, settling a conjecture of the first and third authors in this case. Second, for $m = 0$ the union of the inverses of the (0-)low elements form a convex subset in the Coxeter complex, settling a conjecture by the third author, Nadeau and Williams.

Keywords: Coxeter groups, low elements, Shi arrangements, Garside shadows

1 Introduction

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and set of reflections $T = \cup_{w \in W} wSw^{-1} = \{s_\alpha \mid \alpha \in \Phi^+\}$, where Φ^+ is a set of positive roots in a root system Φ for (W, S). As a reflection group, W acts on the *Coxeter complex* $\mathcal{U}(W, S)$ that arises naturally from the Coxeter (hyperplane) arrangement $\mathcal{A}(W, S) = \{H_\alpha \mid \alpha \in \Phi^+\}$. The maximal simplices of $\mathcal{C}(W, S)$ are called *chambers* and they correspond to the connected

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components of the complement of $\mathcal{A}(W,S)$. The map $w \mapsto C_w$ is a bijection between W and the set of chambers; see for instance Figure 1 and Figure 2 below.

Let $m \in \mathbb{N}$. A positive root $\beta \in \Phi^+$ is *m-small* if there are at most m parallel, or ultraparallel, hyperplanes separating H_β from the fundamental chamber C_e (not counting H_β). Denote by Σ_m the set of m-small roots. Small roots were introduced by Brink and Howlett to prove that any finitely generated Coxeter system is automatic [2]; a key and remarkable result in their article was to prove that Σ_0 is a finite set. Later, Fu [6] proved that Σ_m is finite for all $m \in \mathbb{N}$. The sets of m-small roots are the building blocks of a family of regular automata that recognize the language of reduced words in (W,S).

The *m-Shi arrangement* $\mathrm{Shi}_m(W,S)$ *of* (W,S) is the hyperplane subarrangement of $\mathcal{A}(W,S)$:

$$\mathrm{Shi}_m(W,S) = \{H_\alpha \mid \alpha \in \Sigma_m\}.$$

The regions of $\mathrm{Shi}_m(W,S)$ are union of chambers and define therefore an equivalence relation $\sim_m$ on W. It was conjectured in [5, Conjecture 2] that each equivalence class under $\sim_m$ contains a unique minimal length element and that the set of these minimal length elements is the set of m-low elements. An element $w \in W$ is *m-low* if the inversion set $\Phi(w)$ of w is spanned by the m-small roots it contains. The set L_m of m-low elements turns out to be a finite Garside shadow [5, 3], that is, it shadows a finite Garside family in a corresponding Artin-Tits group.

The following two theorems are the main results of this abstract: the first theorem settles [5, Conjecture 2] and the second settles [7, Conjecture 3].

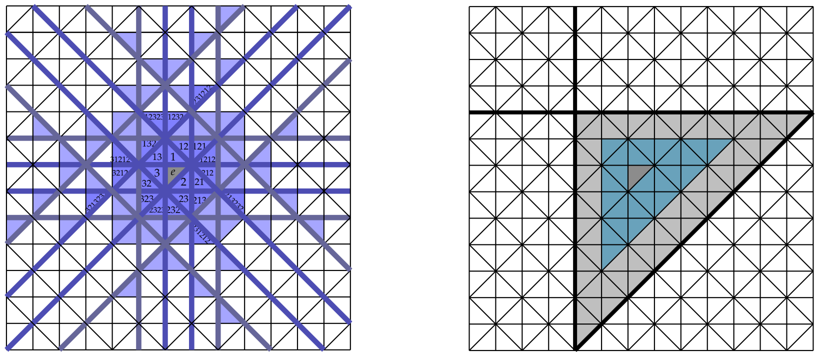
Theorem 1.1. *Let (W,S) be a Coxeter system and $m \in \mathbb{N}$.*

1. *Each region of* $\mathrm{Shi}_m(W,S)$ *contains a unique element of minimal length.*
2. *The set of the minimal length elements of* $\mathrm{Shi}_m(W,S)$ *is equal to the set L_m of m-low elements.*

A noteworthy consequence of the Theorem 1.1 and of the fact that L_m is a Garside shadow is that if the join z (under the right weak order) of two minimal elements of $\mathrm{Shi}_m(W,S)$ exists, then z is also the minimal element of a region of $\mathrm{Shi}_m(W,S)$.

Theorem 1.2. *Let (W,S) be a Coxeter system. The union of the chambers C_w for $w^{-1} \in L_0$ is a convex set.*

These theorems are illustrated in Figures 1 and 2. The proofs of these theorems depend on the *sandwich property* of *short inversion posets*, discussed in §3. The first author showed in 2019 that the *inversion set* $\Phi(w)$ *of* $w \in W$ is spanned by its *set of short inversions* $\Phi^1(w)$. We endow $\Phi^1(w)$ with a poset structure arising from the configuration of maximal dihedral reflection subgroups: $\alpha \dot{\prec}_w \beta$ if β is not in the simple system of the maximal dihedral reflection subgroup containing $\alpha, \beta \in \Phi^1(w)$, see §3.2. Then we prove that any short inversion $\beta \in \Phi^1(w)$ is *sandwiched* between a left-descent root and a



(a) The 1-Shi arrangement (with the 0-Shi arrangement in darker blue)

(b) The 1-Shi polyhedron (with the 0-Shi polyhedron inside)

Figure 1: The 1-Shi arrangement and the 1-Shi polyhedron for $\tilde{B}_2$.

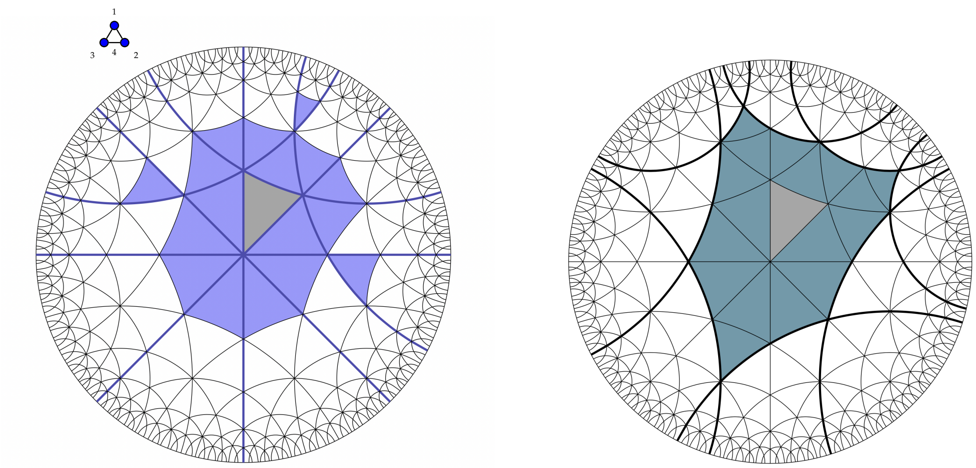
right-descent root, roots naturally defined from the left and right descent sets of w; this is Theorem 3.6, which is the core result of this abstract. We emphasize that these posets are new and have been very useful in analyzing elements of W.

In order to properly introduce m-Shi arrangements in as many realizations of the Coxeter arrangement as possible (e.g. Tits cones, Davis complexes, Euclidean and Hyperbolic spaces, etc.), the full paper uses the notion of *chambered sets*. Our discussion of chambered sets is omitted in the extended abstract.

Finally, in §5, we introduce extended Shi arrangements and we focus on Theorem 1.1 and Theorem 1.2. Combinatorics of roots and reduced words are surveyed in §2 while m-small roots and m-low elements are discussed in §4.

Let us give a bit of history about the m-Shi arrangement. For more details and references, see [4]. In 1986, Shi introduced the Shi arrangement $\mathrm{Shi}(W,S) = \mathrm{Shi}_0(W,S)$ in the case of irreducible affine Weyl groups to study Kazhdan-Lusztig cells for W. Surprising connections to Shi arrangements have been studied: to *ad*-nilpotent ideals of Borel subalgebras, and to Catalan arrangements, for example. In 1988, Shi proved a conjecture by Carter on the number of sign-types of an affine Weyl group. In order to prove that conjecture, Shi enumerated the number of regions in $\mathrm{Shi}_0(W,S)$. In particular, Shi proves that each region of the Shi arrangement contains a unique minimal element and that the union of the chambers corresponding to the inverses of those minimal elements is a convex subset of the Euclidean space. Theorems 1.1 and 1.2 are a generalization of both results to arbitrary Coxeter systems. Notice that in the case of affine Coxeter systems and for $m = 0$, Theorem 1.1 was proven by Chapelier-Laget and the second author, while for rank 3 and $m = 0$ it was proven by Charles. Osajda and Przytycki independently, in 2022, have a proof of Theorem 1.1(1) in the case $m = 0$,

As far as we know, the *m-(extended) Shi arrangements* were defined for affine Coxeter systems in Armstrong's thesis, but were implicit in Athanasiadis's work on *generalized*



(a) The 0-Shi arrangement. The low elements are shaded.

(b) The 0-Shi polyhedron

Figure 2: The 0-Shi arrangement and polyhedron of the Coxeter system with Coxeter graph given in the upper left.

Catalan numbers. In the extended case, the regions in $\mathrm{Shi}_m(W,S)$ were first enumerated by Yoshinaga using techniques from representation theory. In his thesis, Thiel gives a direct proof by extending Shi's result to any m in the case of affine Coxeter systems.

Theorem 1.1 shows that Thiel's minimal elements for $\mathrm{Shi}_m(W,S)$ are precisely the m-low elements. We recover Thiel's results as a direct consequence of the proof of Theorem 1.2.

Theorem 1.3. *If (W,S) is of affine type, then the union of the chambers C_w for $w^{-1} \in L_m$ is a convex set.*

Theorem 1.3 is not true for an indefinite Coxeter system, i.e., neither affine nor finite; for a counterexample see Figure 4. There are many new questions about the Shi arrangement in indefinite type; see [4] for a few of them.

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2 Preliminaries

Fix a Coxeter system (W,S) with length function $\ell : W \to \mathbb{N}$; the *rank of* (W,S) is the cardinality of S. We assume the reader familiar with the basics of the theory of Coxeter groups; see for instance [8, 1].

Combinatorics of reduced words We say that a word $s_1 \dots s_k$ ($s_i \in S$) is a *reduced word for* $w \in W$ if $w = s_1 \dots s_k$ and $k = \ell(w)$. For $u, v, w \in W$, we say u *is a prefix of* w if a reduced word for u can be obtained as a prefix of a reduced word for w; v *is a suffix of* w if a reduced word for u can be obtained as a suffix of a reduced word for w; and $w = uv$ is a *reduced product* if $\ell(w) = \ell(u) + \ell(v)$. More generally, we say that $w = u_1 \dots u_k$ is a *reduced product* if $\ell(w) = \ell(u_1) + \cdots + \ell(u_k)$, $u_i, w \in W$.

Weak and Bruhat orders This suffix/prefix terminology is best embodied by the *weak order*. The *right weak order* is the poset $(W, \leq_R)$ defined by $u \leq_R w$ if u is a prefix of w. The right weak order gives a natural orientation of the right Cayley graph of (W, S): for $w \in W$ and $s \in S$, we orient the edge $w \to ws$ if $w \leq_R ws$.

Recall that the *Bruhat order* is the poset $(W, \leq)$ defined as follows: $u \leq w$ if and only if a word for u can be obtained as a subword of a reduced word for w. We denote covering in the Bruhat order by $x \lhd y$.

Root system Please see [8] for information on geometric representations of (W, S), the symmetric bilinear form B, and root systems. We note that (1) if B is positive definite, then W is finite; if it is positive semi-definite but not positive definite, then W is affine; and otherwise W is indefinite; and (2) there is a bijection between the positive roots Φ^+ and the reflections T.

Depth of positive roots The *depth on* Φ^+ [2] is the function $\mathrm{dp} : \Phi^+ \to \mathbb{N}$ defined by:

$$\mathrm{dp}(\beta) = \min\{\ell(g) \mid g(\beta) \in \Delta\}.$$

There is a recursion for depth [1, Lemma 4.6.2] and $\mathrm{dp}(\alpha_s) = 0$ for all $s \in S$. The depth may be seen as measuring how far a positive root is from Δ in the orbit $\Phi = W(\Delta)$. There are many different depths and they are not equivalent. In this article we also consider the ∞-depth. For more on depths, lengths and weak orders on root systems, see [5, §5.1].

Inversion sets The *inversion set* $\Phi(w)$ *of* $w \in W$ is defined by:

$$\Phi(w) = \Phi^+ \cap w(\Phi^-) = \{\beta \in \Phi^+ \mid \ell(s_\beta w) < \ell(w)\}.$$

Its cardinality is $\ell(w)$ and is sometimes denoted in the literature by $N(w)$ or $\mathrm{inv}(w)$.

Reflection subgroups We end this section by recalling some useful facts about reflection subgroups and, in particular, about maximal dihedral reflection subgroups [5, §2.8]. A *reflection subgroup* W' of W is a subgroup $W' = \langle s_\beta \mid \beta \in A \rangle$ generated by the reflections associated to the roots in some $A \subseteq \Phi^+$. We set $\Phi_{W'} := \{\beta \in \Phi \mid s_\beta \in W'\}$ and $\Delta_{W'} := \{\alpha \in \Phi^+ \mid \Phi(s_\alpha) \cap \Phi_{W'} = \{\alpha\}\}$. The first author showed in 1990 that $\Phi_{W'}$ is a root system in (V, B) with simple root system $\Delta_{W'}$ and simple reflections $\chi(W') := \{s_\alpha \mid \alpha \in \Delta_{W'}\}$. There are corresponding positive roots: $\Phi^+_{W'} = \Phi_{W'} \cap \Phi^+$; both notions depend on (W, S) and not just W.

Maximal dihedral reflection subgroups A reflection subgroup W' of rank 2 is well-known to be isomorphic to a dihedral group and is so called a *dihedral reflection subgroup*. This following result gives a criterion for comparing depths of roots.

Proposition 2.1. *Let $\alpha, \beta \in \Phi^+$. Assume there is a dihedral reflection subgroup W' such that such that $\alpha \in \Delta_{W'}$ and $\beta \in \Phi^+_{W'} \setminus \Delta_{W'}$, then $\mathrm{dp}(\alpha) < \mathrm{dp}(\beta)$.*

A dihedral reflection subgroup W' is a *maximal dihedral reflection subgroup* if it is not contained in any other dihedral reflection subgroup but itself. Our partial order on the short inversions is based on maximal dihedral reflection subgroups. The following result is useful: it gives the form of inversion sets in maximal dihedral reflection subgroup.

Proposition 2.2. *Let W' be a maximal dihedral reflection subgroup. The inversion set of $u \in W'$, $u \neq e$, is of the form $\Phi_{W'}(u) = \mathrm{cone}_\Phi(\alpha, \beta)$ with $\alpha \in \Delta_{W'}$ and $\beta \in \Phi^+_{W'}$.*

Any dihedral reflection subgroup is contained in a unique maximal dihedral reflection subgroup. In particular, for $\alpha, \beta \in \Phi$ such that $\mathbb{R}\alpha \neq \mathbb{R}\beta$, the dihedral reflection subgroup $\langle s_\alpha, s_\beta \rangle$ is contained in the unique maximal dihedral reflection subgroup $\mathcal{M}_{\alpha,\beta}$, with root subsystem $\Phi_{\alpha,\beta} = (\mathbb{R}\alpha \oplus \mathbb{R}\beta) \cap \Phi$, and simple system $\Delta_{\mathcal{M}_{\alpha,\beta}}$. For simplicity, if $s = s_\alpha \in T$ and $t = s_\beta \in T$, we write $\mathcal{M}_{s,t} = \mathcal{M}_{\alpha,\beta}$.

Remark 2.3. The finite maximal dihedral reflection subgroups of (W, S) are precisely the finite parabolic subgroups of rank 2, that is, the conjugates of the standard parabolic subgroups $W_{s,t} = \langle s, t \rangle$ for $s, t \in S$ distinct such that the order $m_{s,t}$ of st is finite. Conversely, any conjugate of a rank 2 finite parabolic subgroup is maximal [3, Theorem 3.11(b)].

3 Short inversion posets

Among all inversions of an element of W, the short inversions span all the others. The key to proving Theorem 4.3 is to exhibit an order on the short inversions and to show that any short inversion is *sandwiched* between a *left descent-root* and a *right descent-root*.

3.1 Short inversions and descent roots

We think of $\Phi(w)$ as a polyhedral cone in $\Phi \subseteq V$ since $\Phi(w) = \mathrm{cone}_\Phi(\Phi(w))$. The *set of short inversions of* $\Phi(w)$ is the set

$$\Phi^1(w) = \{\beta \in \Phi^+ \mid \ell(s_\beta w) = \ell(w) - 1\} = \{\beta \in \Phi^+ \mid s_\beta w \lhd w\} \subseteq \Phi(w).$$

The first author showed in 1994 that $\Phi^1(w)$ is a basis of cone($\Phi(w)$): the set of extreme rays of $\mathrm{cone}_\Phi(\Phi(w))$ is indeed $\{\mathbb{R}_{\geq 0}\beta \mid \beta \in \Phi^1(w)\}$.

Proposition 3.1. *Let $w \in W$ and $\alpha, \beta \in \Phi^1(w)$ with $\alpha \neq \beta$. Then $\alpha \in \Delta_{\mathcal{M}_{\alpha,\beta}}$ or $\beta \in \Delta_{\mathcal{M}_{\alpha,\beta}}$. In particular: (1) if $\Delta_{\mathcal{M}_{\alpha,\beta}} = \{\alpha, \alpha'\}$ and $\beta \neq \alpha'$, then $\alpha' \notin \Phi(w)$; or (2) if $\Delta_{\mathcal{M}_{\alpha,\beta}} = \{\alpha, \beta\}$, then $\Phi^+_{\mathcal{M}_{\alpha,\beta}} \subseteq \Phi(w)$ and $\mathcal{M}_{\alpha,\beta}$ is finite.*

The well-known left and right descent sets of $w \in W$ have their natural counterparts in $\Phi^1(w)$. The *left descent set* $D_L(w) = \{s \in S \mid sw \lhd w\}$ is in bijection with the set of *left descent-roots:* $\Phi^L(w) = \Phi(w) \cap \Delta$. The *right descent set* $D_R(w) = \{s \in S \mid ws \lhd w\}$ is in bijection with the set of *right descent-roots:* $\Phi^R(w) = \{-w(\alpha_s) \mid s \in D_R(w)\}$.

3.2 Short inversion posets

Let $w \in W$. For $\alpha, \beta \in \Phi^1(w)$, we write $\alpha \dot{\prec}_w \beta$ if $\beta \notin \Delta_{\mathcal{M}_{\alpha,\beta}}$. By Proposition 3.1, this is equivalent to $\alpha \in \Delta_{\mathcal{M}_{\alpha,\beta}}$ and $\beta \notin \Delta_{\mathcal{M}_{\alpha,\beta}}$. Proposition 3.2 is a direct consequence of Proposition 2.1.

Proposition 3.2. *Let $w \in W$ and $\alpha, \beta \in \Phi^1(w)$. If $\alpha \dot{\prec} \beta$, then* $\mathrm{dp}(\alpha) < \mathrm{dp}(\beta)$.

For $w \in W$, we define the relation $\preceq_w$ to be the transitive and reflexive closure of $\dot{\prec}_w$, which turns out to be a partial order on $\Phi^1(w)$.

Proposition 3.3. *The relation $\preceq_w$ is a partial order on $\Phi^1(w)$. Moreover, for any reduced word $w = s_1 \ldots s_k$ consider the following total order $\leq$ on $\Phi(w)$: $\alpha_{s_1} < s_1(\alpha_{s_2}) < \cdots < s_1 \ldots s_{k-1}(\alpha_{s_k})$. Then $\alpha \preceq_w \beta$ implies $\alpha \leq \beta$ and* $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\beta)$ *for any $\alpha, \beta \in \Phi^1(w)$.*

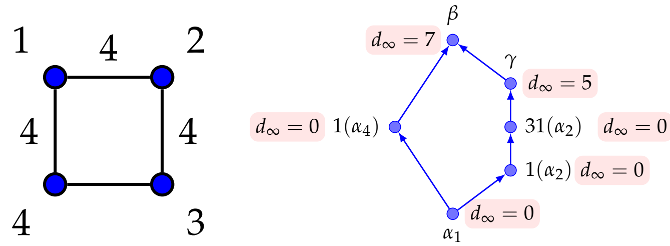
Remark 3.4. (1) The relation $\dot{\prec}_w$ is not the cover relation for $\preceq_w$. (2) The total order on $\Phi(w)$ in the statement of Proposition 3.3 is in fact the restriction of an *admissible order* on Φ^+ to $\Phi(w)$. Admissible orders on Φ^+ are in bijection with *reflection orders*, which plays a role in Kazhdan-Lusztig theory

Example 3.5. Consider (W, S) with $S = \{1, 2, 3, 4\}$ and the Coxeter graph in Figure 3. This is an indefinite Coxeter system. Let $w = 1234232314$, so that $\Phi^L(w) = \{\alpha_1\}$, $\Phi^R(w) = \{123432321(\alpha_4)\}$ and $\Phi^1(w) = \{\alpha_1, 1(\alpha_2), 31(\alpha_2), \gamma = 1234232(\alpha_1), 1(\alpha_4), \beta = 123432321(\alpha_4)\}$. The Hasse diagram of the short inversion poset of w is in Figure 3.

We state now the main result of this section, the sandwich theorem.

Theorem 3.6. *Let $w \in W$. For the poset $(\Phi^1(w), \preceq_w)$, the minimal elements are the left-descent roots in $\Phi^L(w)$ and the maximal elements are the right-descent roots in $\Phi^R(w)$. More precisely, for any $\beta \in \Phi^1(w)$ there is $\alpha \in \Phi^L(w)$ and $\gamma \in \Phi^R(w)$ such that $\alpha \preceq_w \beta \preceq_w \gamma$.*

The key to proving Theorem 3.6 is to explicitly construct, for $w \in W$ and for each $\beta \in \Phi^1(w) \setminus \Phi^L(w)$, a short inversion $\alpha \in \Phi^1(w)$ such that $\alpha \dot{\prec}_w \beta$. For such a $\beta \in \Phi^1(w) \setminus \Phi^L(w)$, we consider $g \in W$ such that $g(\beta) \in \Delta$ and $\ell(g) = \mathrm{dp}(\beta)$, which exists by definition of the depth.



(a) The Coxeter graph (b) The short root poset for w.

Figure 3: Observe that any short inversion is *sandwiched* between a left descent-root and a right descent-root in the short root poset. To the side of each root is its ∞-depth. See Example 3.5 and Section 4.1.

4 m-Small roots and m-low elements

Let (W, S) be a Coxeter system and $m \in \mathbb{N}$. In this section, we provide, as a consequence of Theorem 3.6, a key characterization of m-low elements: an element $w \in W$ is m-low if and only if $\Phi^R(w)$ consists of m-small roots, see Theorem 4.3 below.

4.1 Dominance order, dominance-depth, and m-small roots

Defined by Brink and Howlett [2], the *dominance order* is the partial order $\preceq_{\text{dom}}$ on Φ^+:

$$\alpha \preceq_{\text{dom}} \beta \iff (\forall w \in W,\ \beta \in \Phi(w) \implies \alpha \in \Phi(w)).$$

In the same paper, they introduced, in relation to the dominance order, another depth-statistic: the *dominance-depth* or ∞-*depth* $\text{dp}_\infty : \Phi^+ \to \mathbb{N}$ is defined by

$$\text{dp}_\infty(\beta) = |\{\alpha \in \Phi^+ \setminus \{\beta\} \mid \alpha \prec_{\text{dom}} \beta\}|.$$

In particular, $\text{dp}_\infty(\alpha_s) = 0$ for all $s \in S$ and there is a recurrence analogous to the recursion for depth. For $m \in \mathbb{N}$, the set Σ_m of *m-small roots* is the set of positive roots that dominate at most m distinct proper positive roots; that is, $\Sigma_m = \{\beta \in \Phi^+ \mid \text{dp}_\infty(\beta) \leq m\}$. The set Φ^+ is then $\bigcup_{m \in \mathbb{N}} \Sigma_m$. The m-small roots are defined in the introduction in relation with parallelism. Brink and Howlett [2] (for $m = 0$) and Fu [6] (for all m) proved that the set Σ_m is finite for all $m \in \mathbb{N}$ and finite S, which implies that the sets of m-small roots provides a decomposition of the positive roots into finite sets whenever S is finite.

4.2 m-small inversion sets and m-low elements

The *m-small inversion set of* $w \in W$ is the set:

$$\Sigma_m(w) = \Phi(w) \cap \Sigma_m.$$

The *set L_m of m-low elements* is, see [5] for more details:

$$L_m = \{w \in W \mid \Phi(w) = \mathrm{cone}_\Phi(\Sigma(w))\} = \{w \in W \mid \Phi^1(w) \subseteq \Sigma_m\}.$$

Example 4.1. (1) If W is finite, then $\Sigma_m = \Sigma_0 = \Phi^+$ for all $m \in \mathbb{N}$. Hence $L_m = L_0 = W$. (2) The elements of the set L_0 in affine type $\tilde{B}_2$ are the darker blue regions in Figure 1 (a), and the elements of L_1 are shaded a lighter blue. (3) The set L_0 of a non-affine Coxeter arrangement consists of the elements in the blue regions in Figure 2.

If S is finite, the set Σ_m is finite and therefore the set L_m is also finite. Actually, if S is finite, the set of m-low elements is a finite Garside shadow, that is, L_m contains S and is closed under taking suffixes and under taking join in the right weak order.

The key notion to prove that L_m is a Garside shadow is *bipodality*: a set $A \subseteq \Phi^+$ is *bipodal* if for any $\beta \in A$ and maximal dihedral reflection subgroup W' such that $\beta \in \Phi_{W'} \setminus \Delta_{W'}$ we have $\Delta_{W'} \subseteq A$; see [5, 3] for more information. Because L_m is bipodal and a Garside shadow, we have the following useful corollary.

Corollary 4.2. *Let $w \in W$, $\alpha, \beta \in \Phi^1(w)$ with $\alpha \preceq_w \beta$, then $\mathrm{dp}_\infty(\alpha) \leq \mathrm{dp}_\infty(\beta)$.*

As a direct consequence of Theorem 3.6 (the sandwich theorem) and Corollary 4.2, we obtain the following theorem. Together with Corollary 4.2, it establishes the relationship between our partial order $\preceq_w$ on Φ^1 and the ∞-depth.

Theorem 4.3. *Let $w \in W$ and set $d_w = \max\{\mathrm{dp}_\infty(\gamma) \mid \gamma \in \Phi^R(w)\}$. (1) The ∞-depth on $\Phi^1(w)$ is maximum on $\Phi^R(w)$: $\mathrm{dp}_\infty(\beta) \leq d_w$, for all $\beta \in \Phi^1(w)$. (2) The element w is a d_w-low element; (3) For $m \in \mathbb{N}$, $w \in L_m$ if and only if $m \geq d_w$.*

The following corollary proves [5, Conjecture 2], which is key to proving Theorem 1.1.

Corollary 4.4. *Let $m \in \mathbb{N}$. The map $\lambda_m : L_m \to \Lambda_m = \{\Sigma_m(w) \mid w \in W\}$, defined by $w \mapsto \Sigma_m(w)$, is a bijection.*

The next proposition is crucial to proving Theorem 1.2 and Theorem 1.3. For their proofs, we need the existence of a supporting hyperplane of C_{sw} which is not m-low and which separates C_{sw} from C_e.

Proposition 4.5. *Let $m \in \mathbb{N}$, $w \in L_m$ and $s \in S$. Then $sw \in L_{m+1}$. Moreover: (1) $sw \in L_{m+1} \setminus L_m$ if and only if $w < sw$ and there is $r \in D_R(w)$ such that $\mathrm{dp}_\infty(-sw(\alpha_r)) = m+1$. (2) Under the conditions above, $\alpha_s \prec_{\mathrm{dom}} -sw(\alpha_r)$ for any $r \in D_R(w)$ with $\mathrm{dp}_\infty(-sw(\alpha_r)) = m+1$.*

5 Extended Shi arrangements and low elements

Let (W, S) be a Coxeter system and $m \in \mathbb{N}$. In this section we first introduce extended Shi arrangements and discuss Theorem 1.1 and Theorem 1.2. We also discuss how we obtained, as a byproduct, a direct proof of Thiel's Theorem 1.3. Then we provide in a counterexample to the convexity of the inverses of L_m if $m > 0$ and (W, S) is indefinite.

5.1 Extended Shi arrangements and proof of Theorem 1.1

Let $m \in \mathbb{N}$. The *(extended) m-Shi arrangement* $\mathrm{Shi}_m(W,S)$ is the set of *m-small hyperplanes*:

$$\mathrm{Shi}_m(W,S) = \{H_\beta \mid \beta \in \Sigma_m\},$$

which consists of the hyperplanes in $\mathcal{A}$ that are separated from the fundamental chamber C by at most m parallel hyperplanes.

The closed regions for $\mathrm{Shi}_m(W,S)$ are called the *m-Shi regions*. The corresponding equivalence relation $\sim_{\Sigma_m}$ on W is abbreviated $\sim_m$ in this case. We have $u \sim_m v$ if and only if C_u and C_v are contained in the same m-Shi region.

Example 5.1. See Figures 1(a) and 2(a) where the blue chambers correspond to the m-low elements and are the unique minimal chamber of their corresponding m-Shi region. For $m = 0$, observe that the small hyperplanes (thick blue lines) do not have any other hyperplanes between them and C. In Figure 1, the 1-small hyperplanes consist of the small hyperplanes plus hyperplanes that have exactly one hyperplane between them and C.

Proposition 5.2. *For $m \in \mathbb{N}$ and $u, v \in W$, we have $u \sim_m v \iff \Sigma_m(u) = \Sigma_m(v)$. In other words, two chambers C_u and C_u are in the same m-Shi region if and only if u and v have the same m-small inversion set.*

In affine Weyl group and in the case $m = 0$, the map $w \mapsto \Sigma_0(w)$ from W to Λ_0 is the generalization of Shi's admissible sign type map. The following theorem proves in particular Theorem 1.1.

Theorem 5.3. *Let $m \in \mathbb{N}$. For any $w \in W$, there is a unique m-low element $u \in L_m$ such that $u \sim_m w$. Moreover $u \leq_R w$. In particular, each region of* $\mathrm{Shi}_m(W,S)$ *contains a unique element of minimal length, which is a low element.*

Remark 5.4. (1) The proof of Theorem 5.3 depends on the bijection between m-low elements and m-small short inversions given in Corollary 4.4. (2) In the terminology of Parkinson and Yau, Theorem 5.3 means that any m-Shi arrangement is *gated* and that L_m is the set of *gates* of $\mathrm{Shi}_m(W,S)$.

5.2 The m-Shi polyhedron and convexity

We now discuss the proofs of Theorem 1.2 and Theorem 1.3. Let $m \in \mathbb{N}$. Consider the set

$$\mathcal{B}_m = \{x^{-1}(\alpha_s) \mid x \in L_m,\ s \in S,\ sx \notin L_m\}.$$

Since the set L_m is a Garside shadow, it is stable under taking suffixes, so $s \in S \setminus D_L(x)$ in the definition above. The set $\mathcal{B}_m \subseteq \Phi^+$ and is finite if S is.

Definition 5.5. We define the *m-Shi polyhedron* to be the convex set:

$$\mathscr{S}_m = \bigcap_{\beta \in \mathcal{B}_m} H_\beta^+.$$

In the case of irreducible affine Weyl groups, Shi proved in in 1987 that $\mathscr{S}_0$ is a simplex with $|S|$ half-spaces in the above definition. See Figures 1 (b) and 2 (b) where the shaded regions correspond to the corresponding m-Shi polyhedron.

The following two theorems are Theorem 1.2 and Theorem 1.3.

Theorem 5.6. *The* 0*-Shi polyhedron is:*

$$\mathscr{S}_0 = \bigcup_{w \in L_0} C_{w^{-1}}.$$

Theorem 5.7. *Let (W, S) be an affine Coxeter system and let $m \in \mathbb{N}$. The m-Shi polyhedron is the union of $C_{w^{-1}}$ for $w \in L_m$.*

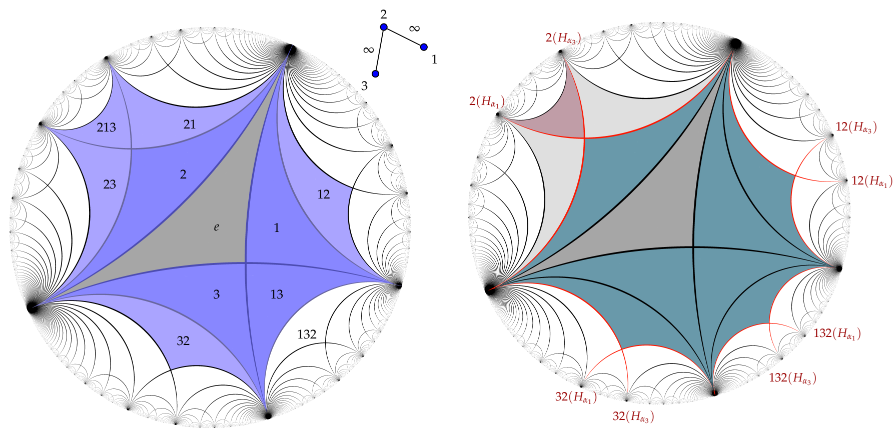
The proof that the m-Shi polyhedron is contained in the union of $C_{w^{-1}}$ for $w \in L_m$ is relatively straightforward and is a consequence of Lemma 5.8.

Lemma 5.8. *Let $m \in \mathbb{N}$ and $w \in W$ such that $\Phi(w^{-1}) \cap \mathscr{B}_m = \emptyset$. Then $w \in L_m$. In other words: $L_m \supseteq \{w \in W \mid \Phi(w^{-1}) \cap \mathscr{B}_m = \emptyset\}$.*

Proving that the union of $C_{w^{-1}}$ for $w \in L_m$ is contained in the Shi polyhedron is trickier and is not true in general for $m > 0$ in indefinite types–see Remark 5.9. It amounts to showing $L_m \subseteq \{w \in W \mid \Phi(w^{-1}) \cap \mathscr{B}_m = \emptyset\}$. The proof of this boils down to showing that if we have a $w \in W$ such that $\Phi(w^{-1}) \cap \mathscr{B}_m \neq \emptyset$, then w is not low. Here we need the existence of a supporting hyperplane which is not m-low (and some other conditions) and use Proposition 4.5 to obtain it.

Remark 5.9. In the proof of Theorem 5.7, we needed and proved the following property : if $\alpha, \beta, \gamma \in \Phi^+$ are such that $\alpha \preceq_{\text{dom}} \gamma$ and $\beta \preceq_{\text{dom}} \gamma$, then either $\alpha \preceq_{\text{dom}} \beta \preceq_{\text{dom}} \gamma$ or $\beta \preceq_{\text{dom}} \alpha \preceq_{\text{dom}} \gamma$. This property arises from the transitivity of the parallelism relation in Euclidean geometry. Unfortunately, it is not true in non-Euclidean space.

Convexity and extended Shi arrangements in indefinite Coxeter systems There can be no result analogous to Theorem 1.2 for all indefinite systems and $m > 0$. For instance, consider (W, S) be the indefinite system whose Coxeter graph is given in Figure 4. The red hyperplanes on the right do form a polyhedron, but C_{21} and C_{23} are not enclosed in it (light gray Figure 4(b)), although 12 and 32 are 1-low. The union of $C_{w^{-1}}$ for $w \in L_1$ is not even convex, since C_{213} (red) is not in the union as $312 = 132 \notin L_1$.



(a) The 0 and 1-Shi arrangements (b) The 0- and 1-polyhedron.

Figure 4: The 0 and 1-Shi arrangements and a counterexample of convexity for the indefinite system whose Coxeter graph is in the top middle of the picture. See Section 5.2.

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Universal Plücker coordinates for the Wronski map and positivity in real Schubert calculus

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Abstract. Given a d-dimensional vector space $V \subset \mathbb{C}[u]$ of polynomials, its Wronskian is the polynomial $(u+z_1)\cdots(u+z_n)$ whose zeros $-z_i$ are the points of $\mathbb{C}$ such that V contains a nonzero polynomial with a zero of order at least d at $-z_i$. Equivalently, V is a solution to the Schubert problem defined by osculating planes to the moment curve at $z_1, \dots, z_n$. The *inverse Wronski problem* involves finding all V with a given Wronskian $(u+z_1)\cdots(u+z_n)$. We solve this problem by providing explicit formulas for the Grassmann–Plücker coordinates of the general solution V, as commuting operators in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group. The Plücker coordinates of individual solutions over $\mathbb{C}$ are obtained by restricting to an eigenspace and replacing each operator by its eigenvalue. This generalizes work of Mukhin, Tarasov, and Varchenko (2013) and of Purbhoo (2022), which give formulas in $\mathbb{C}[\mathfrak{S}_n]$ for the differential equation satisfied by V. Moreover, if $z_1, \dots, z_n$ are real and nonnegative, then our operators are positive semidefinite, implying that the Plücker coordinates of V are all real and nonnegative. This verifies several outstanding conjectures in real Schubert calculus, including the positivity conjectures of Mukhin and Tarasov (2017) and of Karp (2021), the disconjugacy conjecture of Eremenko (2015), and the divisor form of the secant conjecture of Sottile (2003). The proofs involve the representation theory of $\mathfrak{S}_n$, symmetric functions, and τ-functions of the KP hierarchy.

Keywords: Wronskian, Schubert calculus, symmetric group, symmetric functions, KP hierarchy, total positivity

1 Introduction

For a system of real polynomial equations with finitely many solutions, we normally expect that some — but not all — of the solutions are real, while the remaining solutions come in complex-conjugate pairs. The precise number of real solutions usually depends in a complicated way on the coefficients of the equations. However, in some rare cases, it is possible to obtain a better understanding of the real solutions. A remarkable example occurs in the Schubert calculus of the Grassmannian $\mathrm{Gr}(d, m)$, for Schubert problems

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defined by flags osculating a rational normal curve. In 1993, Boris and Michael Shapiro conjectured that all such Schubert problems with real parameters have only real solutions. The corresponding systems of equations arise in various guises throughout mathematics, from algebraic curves [6, 20] to differential equations [30] to pole-placement problems [35, 7]. The conjecture was eventually proved by Mukhin, Tarasov, and Varchenko [31], using a reformulation in terms of Wronski maps, and machinery from quantum integrable systems and representation theory.

While the details of the Mukhin–Tarasov–Varchenko proof are rather intricate, the basic idea is relatively straightforward. They consider a family of commuting linear operators arising from the Gaudin model, and show that they satisfy algebraic equations defining a Schubert problem. Hence, by considering the spectra of these operators, they are able to infer some basic properties of the solutions to the Schubert problem. In this paper we extend these results, making the connection between the commuting operators and the corresponding solutions more explicit and concrete. Consequently, we obtain stronger results in real algebraic geometry, including several generalizations of the Shapiro–Shapiro conjecture. Namely, we resolve the divisor form of the secant conjecture of Sottile (2003), the disconjugacy conjecture of Eremenko [10], and the positivity conjectures of Mukhin–Tarasov (2017) and Karp [18]. Proofs and further details appear in the paper [19].

2 The Wronski map and the Bethe algebra

Let $\mathrm{Gr}(d,m)$ denote the Grassmannian of all d-dimensional linear subspaces of $\mathbb{C}^m$. We identify $\mathbb{C}^m$ with $\mathbb{C}_{m-1}[u]$, the m-dimensional vector space of univariate polynomials of degree at most $m-1$, via the isomorphism

$$(a_1,\ldots,a_m) \leftrightarrow \sum_{j=1}^{m} a_j \frac{u^{j-1}}{(j-1)!}\,. \tag{2.1}$$

In particular, we also view $\mathrm{Gr}(d,m)$ as the space of d-dimensional subspaces of $\mathbb{C}_{m-1}[u]$.

Now fix a nonnegative integer n, and let ν be a partition of n with at most d parts whose sizes are at most $m-d$. The ***Schubert cell*** $\mathcal{X}^\nu \subseteq \mathrm{Gr}(d,m)$ is the space of all d-dimensional linear subspaces of $\mathbb{C}[u]$ that have a basis $(f_1,\ldots,f_d)$, with $\deg(f_i) = \nu_i + d - i$. As a scheme, $\mathcal{X}^\nu$ is isomorphic to n-dimensional affine space.

Let $\mathcal{P}_n \subseteq \mathbb{C}[u]$ denote the n-dimensional affine space of monic polynomials of degree n. Given $V \in \mathcal{X}^\nu$, choose any basis $(f_1,\ldots,f_d)$ for V. We define $\mathrm{Wr}(V)$ to be the unique monic polynomial which is a scalar multiple of the Wronskian

$$\mathrm{Wr}(f_1,\ldots,f_d) := \begin{vmatrix} f_1 & f_1' & f_1'' & \cdots & f_1^{(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_d & f_d' & f_d'' & \cdots & f_d^{(d-1)} \end{vmatrix}\,.$$

It is not hard to see that $\mathrm{Wr}(V) \in \mathcal{P}_n$ is a polynomial of degree n, and is independent of the choice of basis. Thus we obtain a map $\mathrm{Wr} : \mathcal{X}^\nu \to \mathcal{P}_n$, called the ***Wronski map*** on $\mathcal{X}^\nu$. Abstractly, this is a finite morphism from n-dimensional affine space to itself.

Suppose $g(u) = (u + z_1) \cdots (u + z_n) \in \mathcal{P}_n$, where $z_1, \dots, z_n$ are complex numbers. The ***inverse Wronski problem*** is to compute the fibre $\mathrm{Wr}^{-1}(g) \subseteq \mathcal{X}^\nu$.

In their study of the Gaudin model for $\mathfrak{gl}_n$, Mukhin, Tarasov, and Varchenko [30, 32, 26, 31, 28] discovered a connection between the inverse Wronski problem, and the problem of diagonalizing the Gaudin Hamiltonians [16]. We will focus on the version of this story from [29], in which the Gaudin Hamiltonians generate the *Bethe algebra (of Gaudin type)* $\mathcal{B}_n(z_1, \dots, z_n) \subseteq \mathbb{C}[\mathfrak{S}_n]$, which is a commutative subalgebra of the group algebra of the symmetric group.

Let M^ν be the Specht module (i.e. irreducible $\mathfrak{S}_n$-representation) associated to the partition ν. Then $\mathcal{B}_n(z_1, \dots, z_n)$ acts on M^ν, and the image of this action defines a commutative subalgebra $\mathcal{B}_\nu(z_1, \dots, z_n) \subseteq \mathrm{End}(M^\nu)$. We have the following correspondence:

Theorem 2.1 (Mukhin, Tarasov, and Varchenko [29])**.** *The eigenspaces $E \subseteq M^\nu$ of the algebra $\mathcal{B}_\nu(z_1, \dots, z_n)$ are in one-to-one correspondence with the points $V_E \in \mathrm{Wr}^{-1}(g)$. The eigenvalues of the generators of $\mathcal{B}_\nu(z_1, \dots, z_n)$ are coordinates for V_E in some coordinate system.*

Unfortunately, Theorem 2.1 is poorly suited to studying certain properties of the Wronski map. This is because the generators of $\mathcal{B}_n(z_1, \dots, z_n)$ correspond to a somewhat unusual coordinate system for $\mathcal{X}^\nu$. Namely, given $V \in \mathcal{X}^\nu$, there is a unique ***fundamental differential operator*** $D_V = \partial_u^d + \psi_1(u)\partial_u^{d-1} + \cdots + \psi_d(u)$ with coefficients $\psi_j(u) \in \mathbb{C}(u)$, such that V is the space of solutions to the differential equation $D_V f(u) = 0$. The coefficients of D_V can be regarded as a coordinate system on $\mathcal{X}^\nu$. In the precise formulation of Theorem 2.1, the point $V_E \in \mathrm{Wr}^{-1}(g)$ is computed in these coordinates. In order to express V_E in standard coordinates, we need to solve a differential equation, resulting in highly non-linear formulas.

Our main result is Theorem 3.2 below, which is a new version of Theorem 2.1. Rather than using the fundamental differential operator coordinates, it computes $V_E \in \mathrm{Wr}^{-1}(g)$ in the ***Plücker coordinates***, which are the $d \times d$ minors of a $d \times m$ matrix whose rows form a basis for V_E. We introduce (by explicit formulas) a new set of generators β^λ for $\mathcal{B}_n(z_1, \dots, z_n)$, which are indexed by partitions λ. For any eigenspace $E \subseteq M^\nu$, the corresponding eigenvalues of the β^λ's are the Plücker coordinates of V_E.

There are three major advantages of this formulation. First, we obtain a more direct description of V_E which does not require solving a differential equation; the implicit part of our construction lies entirely in understanding the representation theory of $\mathfrak{S}_n$. Second, many natural objects of interest are given by *linear* functions of the Plücker coordinates. For example, we readily obtain explicit bases for V_E; the Wronskian and the fundamental differential operator coordinates are given as linear functions of the Plücker coordinates; and Schubert varieties and Schubert intersections are defined by linear equations in the

Plücker coordinates. Third, basic properties of the operators β^λ imply positivity results about the Plücker coordinates of V_E. This enables us to resolve several conjectures in real algebraic geometry, as we explain in Section 4.

3 Universal Plücker coordinates

We now state our main theorem. For every partition λ, define

$$\beta^\lambda(t) := \sum_{\substack{X\subseteq[n],\\ |X|=|\lambda|}} \sum_{\sigma\in\mathfrak{S}_X} \chi^\lambda(\sigma)\sigma \prod_{i\in[n]\setminus X}(z_i+t)\,. \tag{3.1}$$

Here $[n]=\{1,\dots,n\}$, $\mathfrak{S}_X \subseteq \mathfrak{S}_n$ is the group of permutations of X, and $\chi^\lambda : \mathfrak{S}_X \to \mathbb{C}$ is the character of the Specht module M^λ. We note that χ^λ is integer-valued, so $\beta^\lambda(t)$ is in fact defined over $\mathbb{Z}$. Also, $\beta^\lambda(t)$ is nonzero if and only if $|\lambda|\le n$. Set $\beta^\lambda := \beta^\lambda(0)$.

Example 3.1. If $\lambda=(1,1)$, then χ^λ is the sign character on $\mathfrak{S}_2$. When $n=3$, we get

$$\beta^{11} = (\mathbf{1}_{\mathfrak{S}_3}-\sigma_{1,2})z_3 + (\mathbf{1}_{\mathfrak{S}_3}-\sigma_{1,3})z_2 + (\mathbf{1}_{\mathfrak{S}_3}-\sigma_{2,3})z_1\,,$$

where $\mathbf{1}_{\mathfrak{S}_3}$ denotes the identity element of $\mathfrak{S}_3$, and $\sigma_{i,j} := (i\quad j)$ is the transposition swapping i and j.

Theorem 3.2. *Let $z_1,\dots,z_n\in\mathbb{C}$, and set $g(u) := (u+z_1)\cdots(u+z_n)\in\mathbb{C}[u]$. The operators $\beta^\lambda(t)\in\mathbb{C}[\mathfrak{S}_n]$ satisfy the following algebraic identities:*

(i) Commutativity relations:

$$\beta^\lambda(s)\beta^\mu(t) = \beta^\mu(t)\beta^\lambda(s) \qquad \text{for all partitions } \lambda \text{ and } \mu\,. \tag{3.2}$$

(ii) Translation identity:

$$\beta^\mu(s+t) = \sum_{\lambda\supseteq\mu} \frac{\mathsf{f}^{\lambda/\mu}}{|\lambda/\mu|!} t^{|\lambda/\mu|}\beta^\lambda(s) \qquad \text{for all partitions } \mu\,, \tag{3.3}$$

where $\mathsf{f}^{\lambda/\mu}$ denotes the number of standard Young tableaux of shape λ/μ.

(iii) The quadratic Plücker relations.

Furthermore:

(iv) For every partition λ and $t\in\mathbb{C}$, we have $\beta^\lambda(t)\in\mathcal{B}_n(z_1,\dots,z_n)$. The set $\{\beta^\lambda \mid |\lambda|\le n\}$ generates $\mathcal{B}_n(z_1,\dots,z_n)$ as an algebra.

(v) If $E\subseteq M^\nu$ is any eigenspace of $\mathcal{B}_\nu(z_1,\dots,z_n)$, then the corresponding eigenvalues of the operators β^λ are the Plücker coordinates of a point $V_E\in\mathcal{X}^\nu\subseteq \mathrm{Gr}(d,m)$ such that $\mathrm{Wr}(V_E)=g$. Every point of $\mathrm{Wr}^{-1}(g)$ corresponds to some eigenspace $E\subseteq M^\nu$ of $\mathcal{B}_\nu(z_1,\dots,z_n)$.

(vi) The multiplicity of V_E as a point of $\mathrm{Wr}^{-1}(g)$ is equal to $\dim \widehat{E}$, where $\widehat{E} \subseteq M^\nu$ is the generalized eigenspace of $\mathcal{B}_\nu(z_1, \ldots, z_n)$ containing E.

We note that while the translation identity in part (ii) is linear, parts (i) and (iii) both involve quadratic expressions in $\mathcal{B}_n(z_1, \ldots, z_n)$, making them intractable to prove directly. In both of these cases we proceed by reducing the problem to — and then proving — an easier identity, using a diverse set of algebraic tools. For part (i), we use properties of $\mathcal{B}_n(z_1, \ldots, z_n)$ and combinatorial ideas which appeared in [34]. For part (iii), we employ the translation identity, properties of the exterior algebra, new combinatorial identities of symmetric functions, and the theory of τ-functions of the KP hierarchy. Once identities (i)–(iii) are established, parts (iv)–(vi) are relatively straightforward consequences. See [19, Sections 3–4] for the details.

There is a precise scheme-theoretic formulation of Theorem 3.2(v); see [19, Section 5.1]. In [19, Section 5.2], we also use Theorem 3.2 to give two explicit bases for any element $V \in \mathrm{Wr}^{-1}(g)$, in terms of our operators $\beta^\lambda(t)$ acting on the associated eigenspace E.

Example 3.3. We illustrate Theorem 3.2 in the case $n = 2$, for the Grassmannian $\mathrm{Gr}(2,4)$. Writing $\mathfrak{S}_2 = \{\mathbf{1}_{\mathfrak{S}_2}, \sigma_{1,2}\}$, we have

$$\beta^0 = \mathbf{1}_{\mathfrak{S}_2} z_1 z_2\,, \qquad \beta^1 = \mathbf{1}_{\mathfrak{S}_2}(z_1 + z_2)\,, \qquad \beta^2 = \mathbf{1}_{\mathfrak{S}_2} + \sigma_{1,2}\,, \qquad \beta^{11} = \mathbf{1}_{\mathfrak{S}_2} - \sigma_{1,2}\,,$$

and $\beta^\lambda = 0$ for all other partitions λ. Note that the β^λ's satisfy the equation

$$-\beta^0\beta^{22} + \beta^1\beta^{21} - \beta^{11}\beta^2 = 0\,,$$

which is the first non-trivial Plücker relation.

There are two Specht modules for $\mathfrak{S}_2$, namely M^2 and M^{11}, which are both 1-dimensional. In M^2, both $\mathbf{1}_{\mathfrak{S}_2}$ and $\sigma_{1,2}$ act with eigenvalue 1, and so

$$\beta^0 \rightsquigarrow z_1 z_2\,, \qquad \beta^1 \rightsquigarrow z_1 + z_2\,, \qquad \beta^2 \rightsquigarrow 2\,, \qquad \beta^{1,1} \rightsquigarrow 0\,. \tag{3.4}$$

These are the Plücker coordinates of the element $V = \left\langle 1,\, z_1 z_2 u + \frac{z_1+z_2}{2} u^2 + \frac{1}{3} u^3 \right\rangle \in \mathcal{X}^2$. That is, when we represent V as the row span of the 2×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 z_2 & z_1 + z_2 & 2 \end{pmatrix}$$

(where vectors correspond to polynomials as in (2.1)), the maximal minors are precisely the β^λ's, where we read off the column set of a minor from λ as in Figure 1.

On the other hand, in M^{11}, the element $\mathbf{1}_{\mathfrak{S}_2}$ acts with eigenvalue 1 and $\sigma_{1,2}$ acts with eigenvalue -1, giving the solution $V = \left\langle \frac{z_1+z_2}{2} + u,\, -z_1 z_2 + u^2 \right\rangle \in \mathcal{X}^{1,1}$. We can check that both elements V of $\mathrm{Gr}(2,4)$ have Wronskian $g(u) = (u + z_1)(u + z_2)$.

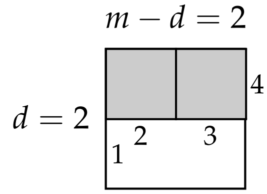

Figure 1: The partition $\lambda = (2)$ corresponds to the column set $\{1,4\}$, where $d=2$ and $m=4$. When we label the edges of the border of the diagram of λ by $1,\ldots,m$ from southwest to northeast, the elements of I are the labels of the vertical edges.

Example 3.4. We illustrate parts (i) and (iii) of Theorem 3.2 in the case $n=4$. Consider the 2-dimensional representation M^ν of $\mathfrak{S}_4$, $\nu=(2,2)$. Following the conventions used by `Sage` [37], the simple transpositions $\sigma_{1,2}$ and $\sigma_{3,4}$ both act as $\left(\begin{smallmatrix}1 & 0\\ 1 & -1\end{smallmatrix}\right)$, and $\sigma_{2,3}$ acts as $\left(\begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix}\right)$. Let $\beta^\lambda_\nu \in \mathrm{End}(M^\nu)$ denote the operator β^λ acting on M^ν, which we regard as a 2×2 matrix. Then

$$\beta^0_\nu = z_1z_2z_3z_4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \beta^1_\nu = (z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$\beta^2_\nu = \begin{pmatrix}2z_1z_2+z_1z_4+z_2z_3+2z_3z_4 & z_1z_3-z_1z_4-z_2z_3+z_2z_4\\ z_1z_2-z_1z_4-z_2z_3+z_3z_4 & 2z_1z_3+z_1z_4+z_2z_3+2z_2z_4\end{pmatrix},$$

$$\beta^{11}_\nu = \begin{pmatrix}2z_1z_3+z_1z_4+z_2z_3+2z_2z_4 & -z_1z_3+z_1z_4+z_2z_3-z_2z_4\\ -z_1z_2+z_1z_4+z_2z_3-z_3z_4 & 2z_1z_2+z_1z_4+z_2z_3+2z_3z_4\end{pmatrix},$$

$$\beta^{21}_\nu = 3(z_1+z_2+z_3+z_4)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \beta^{22}_\nu = 12\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and $\beta^\lambda_\nu = 0$ for all other partitions λ. We can see that the β^λ_ν's pairwise commute and satisfy the Plücker relation $-\beta^0_\nu\beta^{22}_\nu + \beta^1_\nu\beta^{21}_\nu - \beta^{11}_\nu\beta^2_\nu = 0$.

4 Conjectures in real algebraic geometry

We continue to work with the Schubert cell $\mathcal{X}^\nu \subseteq \mathrm{Gr}(d,m)$, where ν is a partition of n. The Schubert variety $\overline{\mathcal{X}}^\nu \subseteq \mathrm{Gr}(d,m)$ is the closure of $\mathcal{X}^\nu$. We write $\square$ for the rectangular partition $(m-d)^d = (m-d,\ldots,m-d)$. In this case, $\overline{\mathcal{X}}^{\square} = \mathrm{Gr}(d,m)$.

We will be mainly concerned with the following Schubert problem. Given $W_1,\ldots,W_n$ in $\mathrm{Gr}(m-d,m)$, determine all d-planes V such that

$$V \in \overline{\mathcal{X}}^\nu \qquad \text{and} \qquad V\cap W_i \neq \{0\} \;\text{ for all } i=1,\ldots,n\,. \tag{4.1}$$

When $W_1,\ldots,W_n$ are sufficiently general, the number of distinct solutions V to the Schubert problem (4.1) is exactly $\mathsf{f}^\nu = \dim M^\nu$.

We will be concerned with solving (4.1) over the real numbers when $W_1,\ldots,W_n$ are real, and especially with instances for which *all* the solutions are real. The interest in algebraic problems with only real solutions dates back at least to Fulton [13, Section 7.2], who wrote, "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems." Note that the property of having only real solutions is extremely rare; for example, for a 'random' Schubert problem on $\mathrm{Gr}(d,m)$ defined over $\mathbb{R}$, the number of real solutions is roughly the square root of the number of complex solutions [4]. We refer to [39] for a detailed survey of real enumerative geometry.

4.1 The Shapiro–Shapiro conjecture

The ***moment curve*** $\gamma : \mathbb{C} \to \mathbb{C}_{m-1}[u]$ is the parametric curve

$$\gamma(t) := \frac{(u+t)^{m-1}}{(m-1)!}. \tag{4.2}$$

The closure of the image of γ in $\mathbb{P}^{m-1}$ is a rational normal curve. A d-plane $V \in \mathrm{Gr}(d,m)$ ***osculates*** γ at $w \in \mathbb{C}$ if $(\gamma(w), \gamma'(w), \gamma''(w), \ldots, \gamma^{(d-1)}(w))$ is a basis for V. The ***Shapiro–Shapiro conjecture*** can be stated as follows:

Theorem 4.1 (Mukhin, Tarasov, and Varchenko [31])**.** *Let $z_1, \ldots, z_n$ be distinct real numbers. For $i = 1, \ldots, n$, let $W_i \in \mathrm{Gr}(m-d,m)$ be the osculating $(m-d)$-plane to γ at z_i. Then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real.*

Theorem 4.1 was conjectured by Boris and Michael Shapiro in 1993, and extensively tested and popularized by Sottile [38]. It was proved in the cases $d \leq 2$ and $m - d \leq 2$ by Eremenko and Gabrielov [8], and in general by Mukhin, Tarasov, and Varchenko [31]. Their proof was later restructured and simplified in [34]. A very different proof, based on geometric and topological arguments, is given in [23].

Using Theorem 3.2, we obtain a number of generalizations of Theorem 4.1:

4.2 The divisor form of the secant conjecture

Let $I \subseteq \mathbb{R}$ be an interval. An $(m-d)$-plane $W \in \mathrm{Gr}(m-d,m)$ is a ***secant*** to γ along I if there exist distinct points $w_1, \ldots, w_{m-d} \in I$ such that $(\gamma(w_1), \ldots, \gamma(w_{m-d}))$ is a basis for W. More generally, W is a ***generalized secant*** to γ along I if there exist distinct points $w_1, \ldots, w_k \in I$ and positive integers $m_1, \ldots, m_k$, such that $m_1 + \cdots + m_k = m - d$ and $\big(\gamma(w_1), \gamma'(w_1), \ldots, \gamma^{(m_1-1)}(w_1), \;\ldots\;, \gamma(w_k), \gamma'(w_k), \ldots, \gamma^{(m_k-1)}(w_k)\big)$ is a basis for W.

Around 2003, Frank Sottile formulated the ***secant conjecture***, which asserts in particular that Theorem 4.1 remains true when $W_1, \ldots, W_n$ are generalized secants to γ along disjoint intervals of $\mathbb{R}$. This statement is what we call the ***divisor form*** of the secant conjecture, since it arises from intersecting Schubert varieties of codimension one, i.e., *Schubert divisors*; the general form of the secant conjecture involves intersecting Schubert varieties of arbitrary codimension. Note that this case of the secant conjecture is a generalization of the Shapiro–Shapiro conjecture, since an osculating plane to γ is a special case of a generalized secant.

The secant conjecture appeared in [36] (cf. [39, Section 13.4]), and it was extensively tested experimentally in a project led by Sottile [15], as described in [17]. It has also been proved in special cases: Eremenko, Gabrielov, Shapiro, and Vainshtein [9, Section 3] established the case $m - d \leq 2$; and Mukhin, Tarasov, and Varchenko [27] (cf. [15, Section 3.1]) verified the case of the divisor form when there exists $r > 0$ such that every W_i is a (non-generalized) secant where $w_1, \ldots, w_{m-d} \in I_i$ are an arithmetic progression of step size r.

We show that the divisor form of the secant conjecture is true in general:

Theorem 4.2 (Secant conjecture, divisor form)**.** *Let $I_1, \ldots, I_n \subseteq \mathbb{R}$ be pairwise disjoint real intervals. For $i = 1, \ldots, n$, let $W_i \in \mathrm{Gr}(m-d, m)$ be a generalized secant to γ along I_i. Then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real.*

This verifies the secant conjecture in the first non-trivial case of interest for a Schubert problem on an arbitrary Grassmannian. We do not yet know how to address the general form of the secant conjecture with our methods.

4.3 The disconjugacy conjecture

Suppose that V is a d-dimensional vector space of real analytic functions, defined on an interval $I \subseteq \mathbb{R}$. Disconjugacy is concerned with the question of how many zeros a function in V can have. By linear algebra, there always exists a nonzero function $f \in V$ such that f has at least $d-1$ zeros on I. We say that V is ***disconjugate*** on I if every nonzero function in V has at most $d-1$ zeros on I (counted with multiplicities). Disconjugacy has long been studied because it is related to explicit solutions for linear differential equations; see [5], as well as [18, Section 4.1] and the references therein.

It is not always straightforward to decide if V is disconjugate on I. However, a necessary condition is that $\mathrm{Wr}(V)$ has no zeros on I. This is because $\mathrm{Wr}(V)$ has a zero at w if and only if there exists a nonzero $f \in V$ such that f has a zero at w of multiplicity at least d. In general, the converse is false; for example, $V = \langle \cos u, \sin u \rangle$ is not disconjugate on $I = \mathbb{R}$, and $\mathrm{Wr}(V) = 1$. Eremenko [10, 11] conjectured that the converse statement is actually correct under very special circumstances. This is known as the ***disconjugacy conjecture***, which we state now as a theorem:

Theorem 4.3 (Disconjugacy conjecture)**.** *Let $V \subseteq \mathbb{R}[u]$ be a finite-dimensional vector space of polynomials such that $\mathrm{Wr}(V)$ has only real zeros. Then V is disconjugate on every interval which avoids the zeros of $\mathrm{Wr}(V)$.*

The disconjugacy conjecture was previously verified in the case that $\dim(V) \leq 2$ [9] (cf. [10, p. 341]).

4.4 Positivity conjectures

A d-plane $V \in \mathrm{Gr}(d,m)$ is called ***totally nonnegative*** if all of its Plücker coordinates are real and nonnegative (up to rescaling). Similarly, V is called ***totally positive in*** $\mathcal{X}^\nu$ if $V \in \mathcal{X}^\nu$ and all of its Plücker coordinates which are not trivially zero on $\mathcal{X}^\nu$ are positive, i.e.,

$$\Delta^\lambda > 0 \ \text{ for all } \lambda \subseteq \nu \qquad \text{and} \qquad \Delta^\lambda = 0 \ \text{ for all } \lambda \not\subseteq \nu\,. \tag{4.3}$$

For example, each element $V \in \mathrm{Gr}(2,4)$ from Example 3.3 is totally nonnegative if and only if $z_1, z_2 \geq 0$, and is totally positive in its Schubert cell if and only if $z_1, z_2 > 0$.

The totally nonnegative part of $\mathrm{Gr}(d,m)$ is a totally nonnegative partial flag variety in the sense of Lusztig [24, 25] (see [3, Section 1] for further discussion), and was studied combinatorially by Postnikov [33]. Total positivity in Schubert cells was considered by Berenstein and Zelevinsky [2]. These and similar totally positive spaces have been extensively studied in the past few decades, with connections to representation theory [24], combinatorics [33], cluster algebras [12], soliton solutions to the KP equation [22], scattering amplitudes [1], Schubert calculus [21], topology [14], and many other topics.

Mukhin–Tarasov and Karp conjectured that the reality statements from Sections 4.1 and 4.2 have totally positive analogues. We verify this in slightly greater generality:

Theorem 4.4 (Positive Shapiro–Shapiro conjecture)**.** *Let $z_1, \ldots, z_n$ and $W_1, \ldots, W_n$ be as in Theorem 4.1.*

(i) If $z_1, \ldots, z_n \in [0, \infty)$, then all solutions to the Schubert problem (4.1) are real and totally nonnegative.

(ii) If $z_1, \ldots, z_n \in (0, \infty)$, then all solutions to the Schubert problem (4.1) are real and totally positive in $\mathcal{X}^\nu$.

Theorem 4.5 (Positive secant conjecture, divisor form)**.** *Let $I_1, \ldots, I_n$ and $W_1, \ldots, W_n$ be as in Theorem 4.2.*

(i) If $I_1, \ldots, I_n \subseteq [0, \infty)$, then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real and totally nonnegative.

(ii) If $I_1, \ldots, I_n \subseteq (0, \infty)$, then there are exactly f^ν distinct solutions to the Schubert problem (4.1), and all solutions are real and totally positive in $\mathcal{X}^\nu$.

In the special case $\nu = \square$, Theorem 4.4(i) was conjectured by Evgeny Mukhin and Vitaly Tarasov in 2017, and Theorems 4.4 and 4.5 were conjectured independently in [18].

4.5 Relationships between conjectures

We now explain how the conjectures stated in this section are related to each other, and why they follow from our main result Theorem 3.2. We have already noted that the divisor form of the secant conjecture (Theorem 4.2) implies the Shapiro–Shapiro conjecture

(Theorem 4.1). Eremenko showed that the disconjugacy conjecture (Theorem 4.3) implies the divisor form of the secant conjecture; in fact, his motivation was to generalize the argument used to prove the $m - d \leq 2$ case of the secant conjecture [9, Section 3]. Moreover, it was shown in [18] using topological arguments that the four statements in Theorems 4.4 and 4.5 in the case $\nu = \square$ are all pairwise equivalent, and that they are moreover equivalent to the disconjugacy conjecture. We can similarly show that Theorem 4.4 implies Theorem 4.5. Therefore to prove all of these statements, it suffices to establish Theorem 4.4. This is a direct consequence of Theorem 3.2; we briefly sketch the argument (see [19, Section 1] for the details).

If $W \in \mathrm{Gr}(m-d, m)$ osculates γ at $w \in \mathbb{C}$, then $V \cap W \neq \{0\}$ if and only if $-w$ is a zero of $\mathrm{Wr}(V)$. Hence in the setting of Theorem 4.4, the Schubert problem (4.1) is equivalent to $V \in \mathcal{X}^\nu$ and $\mathrm{Wr}(V) = g$, where $g(u) = (u + z_1) \cdots (u + z_n)$. By Theorem 3.2(v), we can write any such solution V as V_E for some eigenspace $E \subseteq M^\nu$ of $\mathcal{B}_\nu(z_1, \dots, z_n)$. This means that the Plücker coordinates $[\Delta^\lambda : \lambda \subseteq \square]$ of V are the eigenvalues of the operators β^λ on E. If $z_1, \dots, z_n \in [0, \infty)$, then one can show that each β^λ is positive semidefinite. Therefore the eigenvalues of β^λ are real and nonnegative, so V is totally nonnegative. This proves part (i) of Theorem 4.4. Similarly, if $z_1, \dots, z_n \in (0, \infty)$, then each β^λ with $\lambda \subseteq \nu$ is positive definite, and hence has positive eigenvalues. This implies that V is totally positive in $\mathcal{X}^\nu$, proving part (ii).

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Wilting Theory of Flow Polytopes

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Abstract. Many important polytopes and their canonical triangulations appear as DKK triangulations of a framed directed acyclic graph (DAG) Γ. These triangulations are combinatorially modelled by cliques of routes on the framed DAG. When Γ is amply framed, the dual graph of its DKK triangulation, or DKK graph, has a lattice structure called the DKK lattice. We study the clique complex of routes which avoid an arbitrary set of "wilted" edges. This leads to various decompositions of the DKK lattice into intervals, generalizing decompositions of the Tamari lattice into ν-Tamari intervals. We further classify the framed DAGs whose DKK graphs may be understood as an interval in the DKK lattice of an amply framed DAG. We realize ν-Tamari lattices and the s-weak order as DKK lattices of such "rooted" DAGs and we extend results about shellability and h^*-polynomials from the amply framed case to the rooted case.

Keywords: flow polytopes, triangulation, ν-Tamari lattice, gentle algebras

1 Introduction

Flow polytopes, which model the space of unit *flows* on a directed acyclic graph (DAG), are a fundamental object of combinatorial optimization and have relations to many fields such as representation theory and algebraic geometry. Danilov, Karzanov, and Koshevoy [5] introduced *framed DAGs* and defined a notion of pairwise compatibility on routes. The complex of *cliques*, or sets of pairwise compatible routes, of a framed DAG Γ serves as a combinatorial model for a (regular unimodular) *DKK triangulation* of the associated flow polytope. Many important classes of polytopes and their canonical triangulations appear in this way, such as associahedra, generalized permutahedra, s-permutahedra, and many order polytopes. We refer to the dual graph of the DKK triangulation as the *DKK Graph* $\mathcal{G}_\Gamma$. An *exceptional route* is one which is in every maximal clique, and Γ is *amply framed* if every edge is in an exceptional route. It was shown in [1] that the clique complex of an amply framed DAG Γ agrees with the support τ-tilting complex of a gentle algebra as described in [3, 7]; in particular, its dual graph has a lattice stucture which we call the *DKK Lattice* $\mathcal{L}_\Gamma$.

In this abstract, we mark a set W of edges of a framed DAG as *wilted* and we study the *lush subgraph* $\mathcal{G}_{(\Gamma,W)}$ of $\mathcal{G}_\Gamma$ of maximal cliques whose nonexceptional routes avoid

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all edges of W. We show that when Γ is amply framed, this gives an interval of the DKK lattice $\mathcal{L}_\Gamma$ which we call the *lush interval* $\mathcal{L}_{(\Gamma,W)}$. We call the wilted framed DAG (Γ, W), or the set W, *viable* if the lush subgraph is nonempty. Our first result provides a complete characterization of the viable edge sets W of a framed DAG G. By choosing a set S of exceptional routes and varying W across all ways to wilt exactly one edge from each route of S, we obtain the *wilted decomposition* of $\mathcal{G}_\Gamma$ by S into lush subgraphs. When Γ is amply framed, this is a decomposition of the DKK lattice into lush intervals. Polyhedrally, we are individually restricting the DKK triangulation of the flow polytope to all codimension-$|S|$ facets which avoid all vertices of exceptional routes in S; taking the cone of these triangulations with these exceptional vertices recovers the original DKK triangulation. As an application, we realize various decompositions of the Tamari lattice, which arises as the DKK lattice of a framed DAG $\mathrm{car}(1^n)$ [9, 2], into ν-Tamari intervals as wilted decompositions.

Next, we use wilting theory to define a new class of framed DAGs which we call *rooted*. Given a rooted DAG Γ, we construct an *ample envelope* (Γ', W') of Γ such that $\mathcal{L}_{(\Gamma',W')} \cong \mathcal{G}_\Gamma$. We thus prove that rooted DAGs are precisely the framed DAGs whose DKK graphs may be understood as intervals in the DKK lattice of an amply framed DAG. As a consequence, we induce a well-defined lattice structure on DKK graphs of rooted DAGs, we prove that clique complexes of rooted DAGs are shellable, and we get a formula for the h^*-vectors of rooted flow polytopes. Rooted DAGs thus inherit many nice properties of amply framed DAGs.

In recent years, the Hasse diagrams of many prominent lattices and their generalizations have been realized as DKK graphs of framed DAGs. In particular, the Hasse diagrams of the ν-Tamari lattice and the s-weak order have been realized as the DKK graphs of $\mathrm{car}(\nu)$ and $\mathrm{oru}(s)$ DAGs. In fact, these framed DAGs are rooted, and our lattice structure realizes their DKK lattices as the ν-Tamari lattice and s-weak order.

We remark that many of our results are phrased more generally for gentle algebras, though for brevity we do not treat this generality in this extended abstract.

2 Background on DAGs and Ample Framings

We start by recalling some background on flow polytopes and amply framed DAGs. Let $G = (V, E)$ be a finite directed acyclic graph (DAG) with vertex set V and edge set E. For each $v \in V$, let $\mathrm{in}(v)$ and $\mathrm{out}(v)$ denote the set incoming and outgoing edges of v, respectively. A vertex v is called a *source* if $\mathrm{in}(v) = \emptyset$, a *sink* if $\mathrm{out}(v) = \emptyset$, and *internal* otherwise. An edge $\alpha \in E$ is directed from its *tail* $t(\alpha)$ to its *head* $h(\alpha)$. The edge α is *internal* if it is between two internal vertices, and otherwise it is a *source edge* and/or a *sink edge*. A *route* of G is a maximal (directed) path in G.

Definition 2.1. A *flow* f on a DAG G is a function $f : E \to \mathbb{R}$ which preserves flow at each internal vertex, i.e., for every internal vertex v we have $\sum_{e\in\mathrm{in}(v)} f(e) = \sum_{e\in\mathrm{out}(v)} f(e)$. The *flow polytope* $\mathcal{F}_1(G)$ is the space of *unit flows* on G; i.e., flows satisfying $x_e \geq 0$ for all edges $e \in E$ and $\sum_{\substack{v \text{ is a source} \\ e\in\mathrm{out}(v)}} f(e) = 1$.

The dimension of $\mathcal{F}_1(G)$ is $\dim(\mathcal{F}_1) = |E| - \#\{v \in V \ : \ v \text{ is an inner vertex}\} - 1$. The vertices of $\mathcal{F}_1(G)$ are precisely the indicator vectors of routes of G.

Definition 2.2. Let $G = (V, E)$ be a DAG. For each internal vertex v of G, assign a linear order to the edges in $\mathrm{in}(v)$ and assign a linear order to the edges in $\mathrm{out}(v)$. This assignment is called a *framing* of G, which we denote by F. We use the symbol Γ to refer to a *framed DAG* (G, F). If e is less than f in the linear order for F on $\mathrm{in}(v)$, we write $e <_{F,\mathrm{in}(v)} f$ (and similarly for $\mathrm{out}(v)$). We may drop one or both subscripts when clear.

In the following, assume $\Gamma = (G, F)$ is a framed DAG. To denote a framing, we often label the half-edges or edges of a DAG with integers. See Figure 1 and Figure 2 for examples. An edge of a framed DAG Γ is *tail-highest* (respectively *tail-lowest*) if α is the greatest (respectively least) element in the partial order on $\mathrm{out}(t(\alpha))$. An edge which is neither tail-highest nor tail-lowest is *tail-middle*. Similarly, an edge may be *head-highest*, *head-lowest*, or *head-middle*. An edge which is both tail-highest and head-highest is called *highest*. We similarly may call edges *middle* or *lowest*. An edge is *steep* if it is head-highest and tail-lowest, or head-lowest and tail-highest.

Definition 2.3. A path p of Γ is *up-incompatible* to a path q if p contains $\alpha_1 R \alpha_2$ and q contains $\beta_1 R \beta_2$, for some path R and some edges α_i, β_i with $\alpha_1 >_{\mathrm{in}(v)} \beta_1$ and $\alpha_2 <_{\mathrm{out}(w)} \beta_2$. Two paths are *incompatible* if one is up-incompatible to the other. Otherwise, they are *compatible*. If a route p in Γ is compatible with every other route in Γ, we say that p is *exceptional*. A *clique* is a set of pairwise-compatible routes in Γ.

For example, in Figure 1, the route 121 and the route 211 are incompatible, as they share the first internal vertex but 121 enters this vertex with a higher edge and leaves with a lower edge compared to 211.

It follows that a route p of a framed DAG Γ is exceptional if and only if either every edge is highest, every edge is lowest, or p consists of a single edge. Note that an exceptional route is a route which is in every maximal clique. The *clique complex* $\mathcal{K}_\Gamma$ of Γ is the simplicial complex of cliques of Γ.

An edge α of Γ is an *idle edge* if $\mathrm{in}(h(\alpha)) = 1$ and $h(\alpha)$ is internal, or $\mathrm{out}(t(\alpha)) = 1$ and $t(\alpha)$ is internal. Idle edges may be contracted to obtain a new framed DAG whose clique complex and DKK graph agree with the original. Hence, we may safely assume that our DAGs have no idle edges. Γ is *amply framed* if every edge is contained in some exceptional route. In [1], it was shown that a framed DAG Γ with no idle edges is amply framed if and only if (1) Γ is full (i.e., for any internal vertex v of Γ, we have $|\mathrm{in}(v)| = 2 = |\mathrm{out}(v)|$), and (2) there is a map $\phi_F : E \to \{1, 2\}$ realizing the framing F (i.e., there are no steep edges in F).

2.1 Flow Polytopes and DKK Triangulations

Recall that vertices of the flow polytope $\mathcal{F}_1(\Gamma)$ are indicator vectors of routes of Γ. Through this correspondence, we may view maximal cliques of Γ as collections of vertices of $\mathcal{F}_1(G)$ which form a simplex of a regular unimodular triangulation:

Theorem 2.4 ([5])**.** *Let Γ be a framed DAG. The set of maximal cliques of Γ forms a regular unimodular triangulation of the flow polytope $\mathcal{F}_1(G)$.*

See Figure 1, where the top clique corresponds to the simplex whose vertices are given by (the indicator vectors of) the routes $\{111, 211, 221, 222\}$ appearing in the clique (the exceptional routes 111 and 222 are not drawn for readability). The triangulation from Theorem 2.4 is called the *DKK triangulation* of Γ. We will be particularly interested in the dual graph of a DKK triangulation (equivalently, the dual graph of the clique complex), which we refer to as the *DKK graph* $\mathcal{G}_\Gamma$.

When Γ is amply framed, $\mathcal{G}_\Gamma$ may be interpreted as the Hasse diagram of a lattice (whose lattice structure is inherited from the τ-tilting theory of an associated gentle algebra [1]), which we call the *DKK lattice* $\mathcal{L}_\Gamma$. This lattice structure may be described as follows using directional compatibility (see Figure 1 and Figure 5 for examples).

Definition 2.5 ([1, Definition 6.1])**.** Let $M_1 = M \cup \{p\}$ and $M_2 = M \cup \{q\}$ be adjacent maximal cliques in $\mathcal{G}_\Gamma$ for Γ amply framed. Then, without loss of generality, p is up-incompatible to q and q is not up-incompatible to p. In this case, $M_1 > M_2$ in $\mathcal{L}_\Gamma$.

3 Wilted Framed DAGs

We now *wilt* a set W of edges of a framed DAG and consider the maximal cliques whose nonexceptional routes avoid the wilted edges. In the amply framed case, these cliques form a *lush interval* in the DKK lattice $\mathcal{L}_\Gamma$. We characterize the sets of wilted edges giving nonempty lush intervals and we give a recipe to obtain canonical decompositions of $\mathcal{L}_\Gamma$ into lush intervals.

Definition 3.1. A *wilted framed DAG* (Γ, W) is a framed DAG $\Gamma = (G, F)$ along with a set W of edges of G considered as *wilted*. We say that a route of Γ is *wilted* if it contains an edge of W. Otherwise, it is *lush*. A clique is *wilted* if it contains a wilted nonexceptional route, and is otherwise *lush*. Let S be the set of exceptional routes containing an edge of W; then the *lush clique complex* $\mathcal{K}_{(\Gamma,W)}$ is the pure simplicial complex whose maximal simplices are of the form $M \backslash S$, for any lush maximal clique M. The *lush subgraph* $\mathcal{G}_{(\Gamma,W)}$ is the dual graph of $\mathcal{K}_{(\Gamma,W)}$. We say that (Γ, W) is *viable* if $\mathcal{G}_{(\Gamma,W)}$ is nonempty.

Remark 3.2. Let (Γ, W) be a viable wilted framed DAG. Let $\mathrm{del}(\Gamma, W)$ be the framed DAG obtained by deleting all edges of W from Γ. There is a natural bijection between lush routes of (Γ, W) and routes of $\mathrm{del}(\Gamma, W)$ which induces a bijection $\mathcal{K}_{(\Gamma,W)} \cong \mathcal{K}_{\mathrm{del}(\Gamma,W)}$. Hence, the lush subgraph $\mathcal{G}_{(\Gamma,W)}$ is isomorphic to the DKK graph $\mathcal{G}_{\mathrm{del}(\Gamma,W)}$.

The following result is proven representation-theoretically by considering the τ-tilting lattice of the associated gentle algebra of an amply framed DAG.

Proposition 3.3. *Let (Γ, W) be a wilted amply framed DAG. The lush subgraph $\mathcal{G}_{(\Gamma,W)}$ forms an interval in $\mathcal{L}_{(\Gamma,W)}$. We call this the* lush interval $\mathcal{L}_{(\Gamma,W)}$.

We now characterize the sets of edges which produce viable wilted framed DAGs.

Theorem 3.4. *Let (Γ, W) be a wilted framed DAG. Then (Γ, W) is viable if and only if*

1. *each edge of W is contained in an exceptional route, no exceptional route contains more than one edge of W, and*
2. *every internal vertex has an incoming and outgoing lush edge.*

3.1 Wilted Decompositions of Framed DAGs and Flow Polytopes

We define the wilted decomposition of a framed DAG with respect to a set of exceptional routes and provide an interpretation in terms of flow polytopes.

Definition 3.5. Let S be a subset of the set of exceptional routes of Γ. Define

$$\mathcal{W}_S := \{W \subseteq E \mid W \text{ consists of exactly one edge from each route of } S\}.$$

For each element W of $\mathcal{W}_S$, we obtain a wilted framed DAG (Γ, W) and a (possibly empty) lush subgraph $\mathcal{G}_{(\Gamma,W)}$. Each maximal clique M of Γ is contained in exactly one such lush subgraph $\mathcal{G}_{(\Gamma,W_M)}$. Hence, the set S gives a *wilted decomposition* of $\mathcal{G}_\Gamma$ into lush subgraphs. Conversely, given a viable wilted framed DAG (Γ, W), we may let S_W be the set of exceptional routes containing an edge of W; then $\mathcal{G}_{(\Gamma,W)}$ appears in the wilted decomposition of Γ by S_W.

When Γ is amply framed, $\mathcal{G}_\Gamma$ has a lattice structure $\mathcal{L}_\Lambda$ and Proposition 3.3 shows that each lush subgraph $\mathcal{G}_{(\Lambda,W)}$ is actually an interval $\mathcal{L}_{(\Lambda,W)} \subseteq \mathcal{L}_\Lambda$. In the future, we will see that this situation holds more generally for *rooted* DAGs. See Figure 1 or Figure 5 for an example of a decomposition of $\mathcal{L}_\Lambda$ into intervals for an amply framed DAG Γ.

Proposition 3.6. *Let Γ be a framed DAG and let S be a set of exceptional routes of Γ. The nonzero flow polytopes $\{\mathcal{F}_1(\mathrm{del}(\Gamma, W)) : W \in \mathcal{W}_S\}$ are precisely the codimension-$|S|$ faces of $\mathcal{F}_1(\Gamma)$ containing none of the vertices given by exceptional routes in S.*

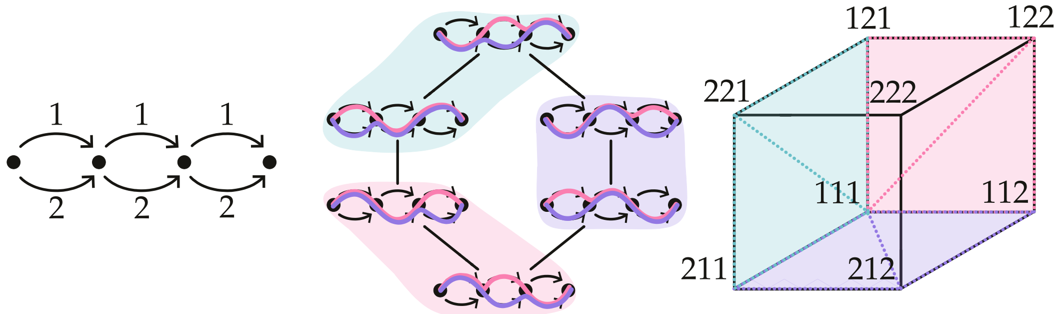

Figure 1: Shown is an amply framed DAG, the wilted decomposition of its DKK lattice by the route 222 (with no exceptional routes drawn for readability), and its flow polytope.

If S is a set of exceptional routes of Γ, then for any $W \in \mathcal{W}_S$, Proposition 3.6 shows that the lush subgraph $\mathcal{G}_{(\Gamma,W)} \subseteq \mathcal{G}_\Gamma$ is the dual graph the DKK triangulation of the codimension-$|S|$ face $\mathcal{F}_1(\mathrm{del}(\Gamma, W))$ of $\mathcal{F}_1(\Gamma)$. By taking the DKK triangulations $\mathcal{F}_1(\mathrm{del}(\Gamma, W))$ for all $W \in \mathcal{W}_S$ and adding the vertices corresponding to exceptional routes of S to all simplices, we recover the original DKK triangulation of $\mathcal{F}_1(\Gamma)$.

Example 3.7. Shown in Figure 1 is an amply framed DAG Γ and its flow polytope $\mathcal{F}_1(\Gamma)$, which is a cube. The vertex labelled 121, for example, corresponds to the route which first takes a 1-edge, then a 2-edge, then a 1-edge. Let $S = \{222\}$ consist only of the 2-route of Γ. The wilted decomposition of Γ by S separates $\mathcal{L}_\Gamma$ into three intervals, highlighted in different colors in the middle of Figure 1, based on which 2-edge is avoided by the nonexceptional routes. By Proposition 3.6, deleting any 2-edge of Γ restricts the triangulation to one of the three facets of $\mathcal{F}_1(\Gamma)$ not incident to the vertex corresponding to 222, which are highlighted in Figure 1. For example, wilting the sink 2-edge yields the lush interval highlighted in blue and deleting it yields the back face of the cube with vertices $\{111, 121, 211, 221\}$, highlighted in blue, with the dotted DKK triangulation. Taking the cone of these three separate DKK triangulations with the vertex corresponding to 222 gives the DKK triangulation of Γ.

4 Rooted Framed DAGs

In this section, we define a new class of framed DAGs which we call rooted. Given a rooted DAG, we obtain a wilted amply framed DAG (Γ', W') whose lush clique complex $\mathcal{K}_{(\Gamma',W')}$ is isomorphic to $\mathcal{K}_\Gamma$. As a consequence, we give a lattice structure to $\mathcal{G}_\Gamma$ and extend results about shellability and h^*-polynomials from the amply framed case.

Definition 4.1. An *exceptional segment* of a framed DAG Γ (with no idle edges) is a

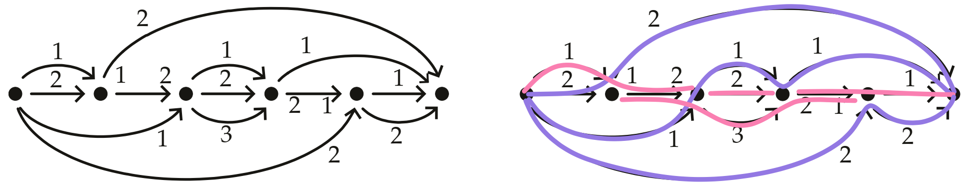

Figure 2: A framed DAG and its exceptional segments (exceptional routes in purple).

maximal path p of Γ which is compatible with every other path. An exceptional segment is *rooted* if it starts at a source vertex or ends at a sink vertex (or both). A framed DAG Γ is *rooted* if every exceptional segment of Γ is rooted.

An exceptional route is an exceptional segment which starts at a source vertex and ends at a sink vertex. Any middle edge makes up its own exceptional segment. When Γ has no idle edges, any steep edge is a part of exactly two exceptional segments, and any non-steep edge is a part of exactly one exceptional segment. See Figure 2.

Lemma 4.2. *Given a viable wilted amply framed DAG* (Γ, W)*, the framed DAG* $\mathrm{del}(\Gamma, W)$ *obtained by deleting all edges of* W *from* Γ *is rooted.*

Proof. Exceptional segments of $\mathrm{del}(\Gamma, W)$ correspond to maximal lush segments of exceptional routes of (Γ, W). Theorem 3.4 shows that any exceptional route of Γ contains at most one edge of W, ensuring that each exceptional segment of $\mathrm{del}(\Gamma, W)$ either starts at a source or ends at a sink. □

We now focus on showing the converse of Lemma 4.2. More concretely, given a rooted framed DAG Γ, we wish to obtain an amply framed DAG (Γ', W') such that $\mathcal{K}_\Gamma \cong \mathcal{K}_{(\Gamma', W')}$. If a framed DAG Γ is not amply framed, then either Γ is not full (i.e., there is an internal vertex of Γ with in-degree or out-degree greater than 2), or Γ has a steep edge. We will define operations which fix these issues while preserving the lush DKK graph. We first define an operation which pulls a framed DAG closer to being full.

Definition 4.3. Let α be a tail-middle edge of a viable wilted framed DAG (Γ, W). In particular, it is necessary that $h(\alpha)$ has an in-degree greater than 2. By Theorem 3.4, α is lush. The *wilted 2-decontraction of* (Γ, W) *with respect to* α is the wilted framed DAG (Γ', W') whose vertex set is given by $V' := \{v' \ : \ v \in V\} \cup \{v_\alpha\}$ and whose edges are described as follows. For any edge $\beta : i \to j$ of Γ, there is an edge $\beta' : i' \to j'$ (if $i \neq t(\alpha)$ or if $i = t(\alpha)$ and $\beta <_{\mathrm{out}(t(\alpha))} \alpha$) or $\beta' : v_\alpha \to j'$ (else). There is an additional *connecting edge* $\delta : v' \to v_\alpha$ and there is a wilted *dummy edge* $\epsilon : v_{\mathrm{source}} \to v_\alpha$. The framing of Γ' is inherited from the framing on Γ, with the stipulation that the connecting edge δ is highest and the dummy edge ϵ is lowest. Performing a wilted 2-decontraction to the left DAG of Figure 3 at its unique tail-middle edge results in the middle DAG of Figure 3.

Note that α' is tail-lowest in Γ' and that deleting the dummy edge ϵ and contracting Γ' along the connecting edge δ recovers (Γ, W). A *wilted 1-decontraction* with respect to an edge α which is tail-middle is obtained by reversing all partial orders of the framing F, performing a wilted 2-decontraction, and reversing the partial orders again. Dually, we may define *wilted 1-decontractions* and *wilted 2-decontractions* with respect to an edge which is head-middle. Given an edge α of (Γ, W) which is head-middle or tail-middle, any wilted decontraction of α which does not create a steep edge preserves the lush clique complex. This gives us the following lemma.

Lemma 4.4. *Let α be a head-middle or tail-middle edge of a viable wilted framed DAG (Γ, W). There exists a wilted decontraction (Γ', W') of (Γ, W) with respect to α such that (Γ', W') is viable and $\mathcal{K}_{(\Gamma', W')} \cong \mathcal{K}_{(\Gamma, W)}$.*

If Γ is a rooted framed DAG with no idle edges which is not full, then it must have an edge which is head-middle or tail-middle. Then we may repeatedly apply Lemma 4.4 to obtain a full wilted DAG (Γ', W') whose lush clique complex agrees with that of Γ. The framed DAG Γ' may not be amply framed, since it may have steep edges. We now define an operation to fix this.

Definition 4.5. Let (Γ, W) be a wilted framed full DAG. Let α be an edge of Γ which is steep. Without loss of generality, suppose α is tail-highest and head-lowest; the other case is similar. We define the *amplification of* (Γ, W) *with respect to* α as the wilted framed DAG (Γ', W') as follows. See the middle and right of Figure 3 for an example. The vertex set of Γ' consists of the vertices of Γ as well as an additional vertex v_α. For any edge β of Γ other than α, there is a corresponding edge β' of Γ'. Replacing α in Γ' is an edge α'_1 from $t(\alpha)$ to v_α which is highest in F, and an edge α'_2 from v_α to $h(\alpha)$ which is lowest in F. Additionally, there is a highest wilted edge γ from a source to v_α and a lowest wilted edge β from v_α to a sink.

Lemma 4.6. *If α is a steep edge of a full viable wilted framed DAG (Γ, W), then the amplification (Γ', W') of (Γ, W) with respect to α is viable and $\mathcal{K}_{(\Gamma', W')} \cong \mathcal{K}_{(\Gamma, W)}$.*

Theorem 4.7. *If Γ is a rooted framed DAG, then there is a wilted amply framed DAG (Γ', W') such that $\mathcal{K}_\Gamma \cong \mathcal{K}_{(\Gamma', W')}$.*

Proof. We may suppose that Γ is rooted and has no idle edges. First, we repeatedly apply Lemma 4.4 until we have reached (Γ'', W''), where Γ'' is full, and then we repeatedly apply Lemma 4.6 to fix the steep edges, resulting in an amply framed (Γ', W') with $\mathcal{K}_\Gamma \cong \mathcal{K}_{(\Gamma', W')}$. For any exceptional segment p of Λ, there is an exceptional route p' of Λ' that begins with a wilted edge if and only if p begins with an internal vertex and ends with a wilted edge if and only if p ends with an internal vertex. Hence, the condition that Γ is rooted corresponds to the condition that (Γ', W') is viable by Theorem 3.4. See Figure 3 for an example of this process. □

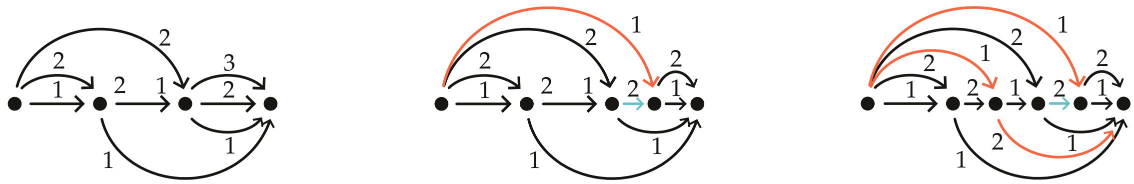

Figure 3: A framed DAG (left), a wilted 2-decontraction with respect to a its tail-middle edge (middle), and an amplification of the resulting DAG at its steep edge (right). Wilted edges are red and the connecting edge is blue.

We call (Γ', W') as in the statement of Theorem 4.7 an *ample envelope* of Γ. A consequence of the existence of ample envelopes is that the DKK graph of a rooted framed DAG has a lattice structure generalizing the amply framed case [1].

Definition 4.8. Let Γ be a rooted framed DAG. Let $M_1 = M \cup \{p\}$ and $M_2 = M \cup \{q\}$ be adjacent maximal cliques in $\mathcal{G}_\Gamma$. Then, without loss of generality, p is up-incompatible to q and q is not up-incompatible to p. In this case, we say that $M_1 > M_2$.

Corollary 4.9. *The transitive closure of the relations of Definition 4.8 gives $\mathcal{G}_\Gamma$ the structure of the Hasse diagram of a lattice, which we refer to as the* DKK lattice $\mathcal{L}_\Gamma$.

Corollary 4.9 is proven by inheriting the lattice structure of an ample envelope. Moreover, any lush subgraph of a rooted DAG may be considered as an interval in $\mathcal{L}_\Gamma$. Hence, the wilted decomposition of a rooted DAG by a set of exceptional routes (Definition 3.5) is a decomposition of $\mathcal{L}_\Gamma$ into lush intervals. The next corollary, which follows from Theorem 4.7 and Lemma 4.2, characterizes rooted DAGs as those whose DKK graphs may be understood as lush intervals of amply framed DAGs .

Corollary 4.10. *A nonempty lattice is of the form $\mathcal{L}_{(\Gamma,W)}$, where (Γ, W) is a wilted amply framed DAG, if and only if it is of the form $\mathcal{L}_{\Gamma'}$, where Γ' is a rooted framed DAG.*

It was shown in [1] that if Γ is amply framed then any linear extension of $\mathcal{L}_\Gamma$ is a shelling order for $\mathcal{K}_\Gamma$. By realizing the DKK graph of a rooted DAG as an interval in the DKK lattice of an ample envelope, we prove the following.

Theorem 4.11. *Let Γ be a rooted framed DAG. Then any linear extension of $\mathcal{L}_\Gamma$ gives a shelling order of the lush clique complex $\mathcal{K}_\Gamma$.*

Following [1, §6], we get a formula for the h^*-polynomials of flow polytopes arising from rooted framed DAGs.

Proposition 4.12. *Let Γ be a rooted DAG. The ith coefficient of the h^*-vector of $\mathcal{F}_1(\Gamma)$ is given by the number of elements in $\mathcal{L}_\Gamma$ covering exactly i elements.*

5 Motivating Example: The (ν-)Tamari Lattice

The Tamari lattice is a mathematical structure that captures the partial order of binary trees under a rotation operation. Préville-Ratelle and Viennot [8] introduced ν-Tamari lattices as a generalization of Tamari lattices and showed that the Tamari lattice has a decomposition into ν-Tamari intervals. In [4] the ν-Tamari lattice was realized as the one-skeleton of the polyhedral complex known as the ν-associahedron, and in [2] it was shown that ν-Tamari lattices arise as DKK graphs of a class of DAGs known as ν-caracol graphs. In this section, we interpret certain wilted decompositions on caracol graphs as decompositions of Tamari lattices into ν-Tamari intervals. Moreover, given a ν-Tamari lattice $\mathcal{L}_{\mathrm{car}(\nu)}$, we obtain a canonical Tamari lattice $\mathcal{L}_{\mathrm{car}(1^n)}$, which is the DKK lattice of a framed DAG $\mathrm{car}(1^n)$, and a set S of exceptional routes of $\mathrm{car}(1^n)$ such that the lattice $\mathcal{L}_{\mathrm{car}(\nu)}$ appears in the wilted decomposition of $\mathcal{L}_{\mathrm{car}(1^n)}$ by S.

Definition 5.1. Let a, b be nonnegative integers, and let $\nu := NE^{\nu_1}NE^{\nu_2}\dots NE^{\nu_a}$ be a lattice path from $(0,0)$ to (b,a) (with $\nu_i \geq 0$). The *ν-caracol graph* $\mathrm{car}(\nu)$ is the graph on the vertex set $\{0,1,\dots,a\}$, together with ν_i copies of the edge $(0,i)$ for $i = 1,\dots,a-1$, one copy of the edge (i,a) for $i = 1,\dots,a-1$, and the edges $(i,i+1)$ for $i = 0,\dots,a-1$. Give $\mathrm{car}(\nu)$ the framing such that the horizontal edges from i to $i+1$ are given the highest element in the vertex order on either side. See Figure 4 for an example. In the classical case, $\nu = 1^n = (1,\dots,1)$ for some n and the DKK lattice $\mathcal{L}_{\mathrm{car}(1^n)}$ is Tamari.

It was shown in [2, Theorem 1.2] that $\mathcal{G}_{\mathrm{car}(\nu)}$ is the Hasse diagram of the ν-Tamari lattice. In fact, the framed DAGs $\mathcal{G}_{\mathrm{car}(\nu)}$ are rooted, and our lattice structure realizes $\mathcal{L}_{\mathrm{car}(\nu)}$ as the ν-Tamari lattice. See Figure 5.

Theorem 5.2. *Let $V \subseteq [n]$. Then the wilted decomposition of* $\mathrm{car}(1^n)$ *by the set of 1-labelled routes whose internal vertices are in V is a decomposition of the Tamari lattice into ν-Tamari intervals. Any ν-Tamari lattice appears in such a decomposition, for some n and V.*

Proof. If W is a viable set of 1-edges of $\mathrm{car}(1^n)$, then deleting them and contracting yields some $\mathrm{car}(\nu)$. Conversely, it may be seen that any DAG $\mathrm{car}(\nu)$ has an ample envelope $(\mathrm{car}(1^n), W)$ where W consists of 1-edges. See the left and middle of Figure 5. □

Example 5.3. The left of Figure 5 shows the decomposition of the Tamari lattice $\mathrm{car}(T_3)$ into ν-Tamari intervals given by the first and last 1-labelled routes. The lush interval in red is the lush interval of the wilted framed DAG $\mathrm{car}(T_3)$ in the middle of Figure 4, which is the DKK graph of the $\mathrm{car}(\nu)$ DAG on the left of Figure 4 by Proposition 3.6.

The right of Figure 5 shows the wilted decomposition by the set of all 1-labelled routes, which recovers the partition introduced in [8]. This induces a wilted decomposition of $\mathcal{L}_{\mathrm{car}(\nu)}$ into two chains.

We end the extended abstract with some open questions:

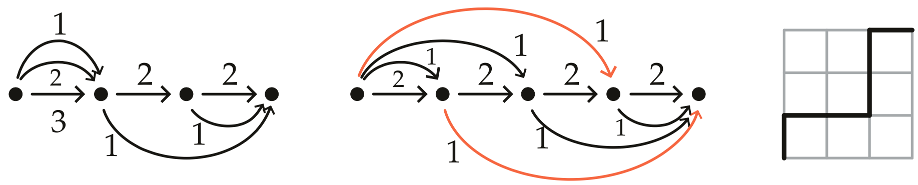

Figure 4: A ν-Caracol graph where $\nu = (0, 2, 0, 1)$, its realization as a wilted car(T_3) graph, and the corresponding lattice path.

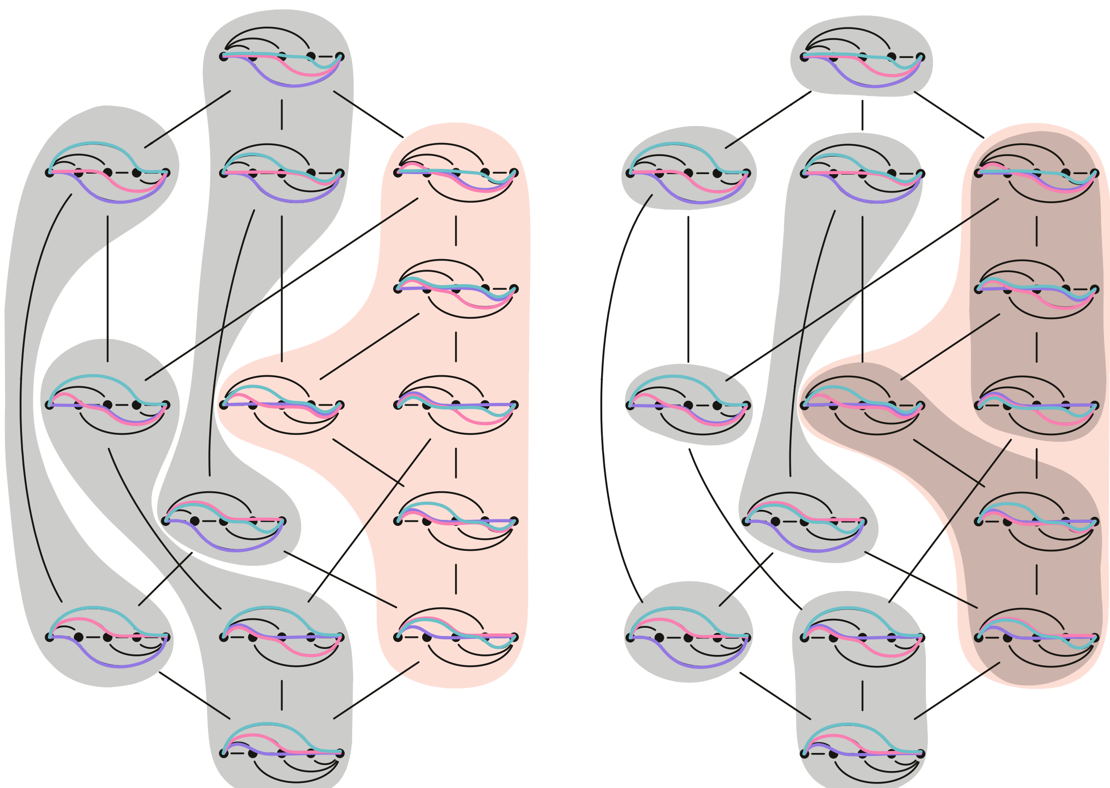

Figure 5: Shown is the wilted decomposition of the (Tamari) lattice $\mathcal{L}_{\mathrm{car}(1^n)}$, shown in the middle of Figure 4, induced by the first and third 1-routes (left) and by the set of all 1-routes (right). For readability, exceptional routes are not drawn.

1. Rooted framed DAGs inherit nice properties from amply framed DAGs. Examples of rooted framed DAGs include ν-caracol and s-oruga [6] graphs, whose DKK lattices are the ν-Tamari lattices and the s-weak order. What other lattices may be realized as DKK lattices of rooted framed DAGs?

2. The notable class of ν-Tamari lattices may be defined as the lush intervals of $\mathrm{car}(1^n)$ in its decomposition by some set of 1-routes. Can we realize other interesting lattice decompositions using wilting theory?

3. The DKK theory of rooted framed DAGs is in some sense equivalent to the wilting theory of viable wilted amply framed DAGs. What can be gained from studying amply framed DAGs with sets of wilted edges which are *not* viable? In particular, can we realize the DKK graph of an arbitrary framed DAG as an interval in the DKK lattice of an amply framed DAG?

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Character factorisations, z-asymmetric partitions, and plethysm

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Abstract. An old theorem of D. E. Littlewood asserts that the Schur function with variables "twisted" by a primitive t-th root of unity vanishes unless the t-core of the indexing partition is empty, in which case it factors as a product of Schur functions indexed by the t-quotient. Recently, Ayyer and Kumari generalised Littlewood's result to characters of the classical groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$. We show that Ayyer and Kumari's results may be lifted to the universal characters of the associated groups, and in doing so give a uniform extension involving a determinant of Bressoud and Wei which was later generalised by Hamel and King. What facilitates this extension is a new property of the Littlewood decomposition, extending results of Garvan, Kim and Stanton. We also explain the connection between Littlewood's original result and an instance of plethysm.

Keywords: Littlewood's decomposition, Schur functions, t-core, t-quotient, universal characters, z-asymmetric partitions

1 Introduction

In his classic 1940 book on group characters D. E. Littlewood gives a factorisation for the Schur function with variables "twisted" (not his term) by a primitive t-th root of unity ζ [13, §7.3]. More precisely, he proves that the Schur function s_λ with tn variables $\zeta^j x_i$ for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant t-1$ vanishes if the t-core of λ is nonempty. If the t-core is empty then, up to a sign, it factors as a product of Schur functions indexed by the partitions forming its t-quotient, each with variables $x_1^t, \dots, x_n^t$. His proof is based on a clever manipulation of the classical definition of s_λ as a ratio of alternants.

Inspired by a recent rediscovery of Littlewood's theorem, Ayyer and Kumari proved analogous factorisation theorems for the characters of the classical groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ using Littlewood's method [3]. As in the Schur case, when these twisted characters are nonzero they factor as a product of other group characters expressed in terms of the t-quotient of the indexing partition. However, the vanishing is now governed by the t-core having a particular form. Specifically, Ayyer and Kumari show that the twisted characters for $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ are nonzero

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if and only if t-core(λ) may be written in Frobenius notation as $(a+z \mid a)$ for $z = 1, -1$ and 0 respectively; see the next section for the relevant definitions.

In [2] we lifted the results of Ayyer and Kumari to the universal characters of the aforementioned groups as defined by Koike and Terada [11]. To describe how this works, let h_r denote the r-th complete homogeneous symmetric function, the set of which is algebraically independent over $\mathbb{Z}$ and generates Λ, the ring of symmetric functions. The notion of "twisting" by a root of unity is replaced by an endomorphism φ_t of Λ for each integer $t \geqslant 2$, defined by

$$\varphi_t h_r = \begin{cases} h_{r/t} & \text{if } t \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases}$$

This operator is occasionally referred to as the *t-th Verschiebung operator*, see for instance [7, §2.9] and references therein.[1] It is a quite a natural operator on symmetric functions, being the adjoint of plethysm by a power sum p_t with respect to the Hall scalar product; see Subsection 3.1. The main results of [2] are the action of φ_t on the universal characters.

In the present work we outline a new approach to proving these universal character factorisations. Following Ayyer and Kumari we call partitions which for $z \in \mathbb{Z}$ may be expressed in Frobenius notation as $(a+z \mid a)$ *z-asymmetric*. Bressoud and Wei [4] and Hamel and King [8] have defined a general symmetric function $\mathfrak{X}_\lambda(z)$ which essentially reduces to the above universal characters for $z = 1, -1$ or 0. They also show that $\mathfrak{X}_\lambda(z)$ may be expressed as a signed sum over skew Schur functions with inner shape a z-asymmetric partition. Using this expression and the action of φ_t on the skew Schur functions (Theorem 4) we can compute $\varphi_t \mathfrak{X}_\lambda(z)$. This is one of our main results, which, in order to keep things as simple as possible, we state only for $0 \leqslant z \leqslant t-1$ as Theorem 6 below. The cases $z \geqslant t$ and $z < 0$ require slightly cumbersome modifications, but no new techniques, so we defer these to future work [1]. The main advantage of our approach is that it produces a parameterised family of such factorisations which may be stated and proved uniformly. A key tool in our proof is the Littlewood decomposition, a bijection which maps a partition to its t-core and t-quotient. Our first main result, Theorem 2, is a characterisation of the Littlewood decomposition of z-symmetric partitions through restrictions on the core and quotient, which reduces to results of Garvan, Kim and Stanton for $z = 0, 1$ [6].

2 Littlewood's decomposition and z-asymmetric partitions

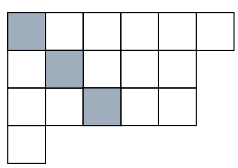
2.1 Preliminaries

A *partition* is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ such that the *size* $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ is finite. The nonzero λ_i are called *parts*

[1] *Verschiebung* is German for *shift*.

and the number of parts the *length*, denoted $l(\lambda)$. The set of all partitions is written $\mathscr{P}$ and the empty partition, the unique partition of 0, is denoted by $\varnothing$. We write (m^ℓ) for the partition with ℓ parts equal to m, and the difference $\lambda - (m^\ell)$ is then the partition obtained by subtracting m from the first ℓ parts of λ. We identify a partition with its *Young diagram*, which (in English notation) is the left-justified array of cells consisting of λ_i cells in row i with i increasing downward. An example is given in Figure 1. We define the *conjugate* partition λ' by reflecting the diagram of λ in the main diagonal, so that the conjugate of $(6,5,5,1)$ is $(4,3,3,3,3,1)$.

The *Frobenius rank* of a partition, $d(\lambda)$, is defined as the number of cells along its main diagonal. Another way to notate partitions is with *Frobenius notation*, which records the number of cells to the right of and below each cell on the main diagonal, which we write in terms of the partition λ as $\lambda = (\lambda_1 - 1, \ldots, \lambda_{d(\lambda)} - d(\lambda) \mid \lambda'_1 - 1, \ldots, \lambda'_{d(\lambda)} - d(\lambda))$; again, see Figure 1 for an example. Any two strictly decreasing nonnegative integer sequences a, b with the same number of elements, say k, thus give a unique partition $\lambda = (a \mid b)$ of Frobenius rank k. Clearly self-conjugate partitions are those of the form $(a \mid a)$. Now let $a + z := (a_1 + z, \ldots, a_k + z)$ for any $z \in \mathbb{Z}$. Ayyer and Kumari [3] define what they call *z-asymmetric partitions* to be those of the form $(a + z \mid a)$ for any sequence a (of any length) and fixed $z \in \mathbb{Z}$. The set of z-asymmetric partitions is denoted by $\mathscr{P}_z$ and $(6,5,5,1)$ in Figure 1 is 2-asymmetric.



9	7	6	5	4	1
7	5	4	3	2	
6	4	3	2	1	
1					

Figure 1: The partition $\lambda = (6,5,5,1) = (5,3,2 \mid 3,1,0)$ with its main diagonal shaded (left) and the same partition with hook length of each cell inscribed (right). We have $|\lambda| = 17$, $l(\lambda) = 4$ and $d(\lambda) = 3$.

Given a cell s in the Young diagram of λ its *hook length* is one more than the sum of the number of cells below and to the right of s; see Figure 1. For an integer $t \geqslant 2$ we say a partition is a *t-core* if it contains no cell with hook length t (or, equivalently, no cell with hook length divisible by t). For a pair of partitions λ, μ we say μ is *contained* in λ, written $\mu \subseteq \lambda$, if its Young diagram may be drawn inside the Young diagram of λ. The corresponding *skew shape* is the arrangement of cells formed by removing μ's diagram from λ's. A skew shape is a *ribbon* if it is edge-connected and contains no 2×2 square of cells, and a *t-ribbon* is a ribbon containing t cells.[2] The *height* of a ribbon R, $\mathrm{ht}(R)$, is one less than the number of rows it occupies; see Figure 2.

[2]Elsewhere in the literature ribbons are variously called *border strips*, *rim hooks* or *skew hooks*.

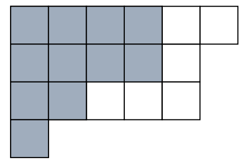


Figure 2: The pair of partitions $(4,4,2,1) \subseteq (6,5,5,1)$. The unshaded cells form a 6-ribbon of height 2.

We say a skew shape λ/μ is *t-tileable* if there exists a sequence of partitions

$$\mu =: \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(m-1)} \subseteq \nu^{(m)} := \lambda$$

such that the skew shapes $\nu^{(r)}/\nu^{(r-1)}$ are each t-ribbons for $1 \leqslant r \leqslant m$. It is a non-trivial fact, see, e.g. [17, Lemma 4.1], that the sign

$$\operatorname{sgn}_t(\lambda/\mu) := (-1)^{\sum_{r=1}^{m} \operatorname{ht}(\nu^{(r)}/\nu^{(r-1)})} \tag{2.1}$$

is constant over the set of all t-ribbon decompositions of λ/μ (so, indeed, the above is well-defined).

2.2 Littlewood's decomposition

The Littlewood decomposition is, for each integer $t \geqslant 2$, a bijection which decomposes a partition λ into a pair $(t\text{-core}(\lambda), \boldsymbol{\lambda})$, where t-core(λ) is the unique *t-core of* λ and $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is a t-tuple of partitions called the *t-quotient* [14]. Here we describe the Littlewood decomposition through the lens of Maya diagrams, which is essentially the abacus model of James and Kerber [9, §2.7]. A purely algebraic description may be found in [16, p. 12].

Given a partition λ its *Maya diagram* is the following subset of the set of half integers, sometimes called the *beta set*

$$\beta(\lambda) := \Big\{\lambda_i - i + \frac{1}{2} : i \geqslant 1\Big\}.$$

This is visualised as a configuration of "beads" on the real line placed at the positions indicated by $\beta(\lambda)$. The map from partitions to Maya diagrams is clearly a bijection, and one way to reconstruct λ from $\beta(\lambda)$ is to count the number of empty spaces to the left of each bead starting from the right. From the Maya diagram we extract t sub-diagrams formed by the beads at positions x such that $x - 1/2$ is r modulo t for $0 \leqslant r \leqslant t-1$, which we dub the *t-Maya diagram*. An example of this procedure is given in Figure 3. The corresponding partitions are denoted by $\lambda^{(r)}$ according to the residues modulo t of the original positions, and these precisely form Littlewood's *t-quotient*.

The key observation behind the definition of the t-core is that moving a bead one space to the left in the t-Maya diagram is equivalent to the removal of a t-ribbon from λ such that what remains is still a partition. Since such ribbons are in correspondence with hooks of length t in λ, pushing all beads to the left leaves a t-core. The t-Maya diagram shows that this is independent of the order in which such ribbons are removed, and so the resulting unique partition is denoted t-core(λ).

Theorem 1 (Littlewood's decomposition)**.** *For any integer $t \geqslant 2$ the above procedure encodes a bijection*

$$\begin{aligned} \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \dots, \lambda^{(t-1)})\big) \end{aligned}$$

such that $|\lambda| = |t\text{-core}(\lambda)| + t(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|)$.

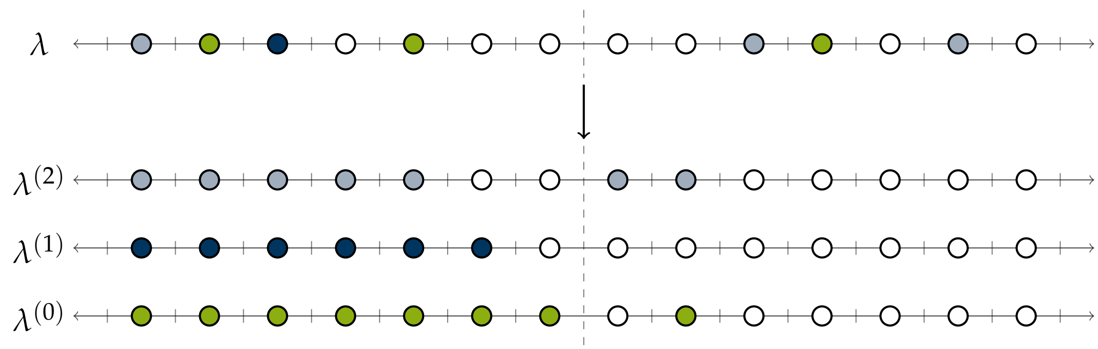


Figure 3: The Maya diagram of $\lambda = (6, 5, 5, 1)$ (top) and the 3-Maya diagram of the same partition (bottom). We have that 3-core$(\lambda) = (1, 1)$, $\kappa_3((1, 1)) = (1, -1, 0)$ and $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}) = ((1), \varnothing, (2, 2))$.

We will also need a different characterisation of t-cores. Call a Maya diagram *balanced* if it contains as many beads to the right of 0 as empty spaces to the left. The way we defined Maya diagrams ensures they are always balanced, but Figure 3 shows that the constituent diagrams of the quotient need not be. Let c_r^+ (resp. c_r^-) denote the number of beads to the right of 0 (resp. number of empty spaces to the left of 0) in row $\lambda^{(r)}$ of the t-Maya diagram. Now the sequence of integers $(c_0, \dots, c_{t-1})$ defined by $c_r := c_r^+ - c_r^-$ has total sum zero, and is invariant under valid bead movements. As observed by Garvan, Kim and Stanton, this encodes a bijection [6, Bijection 2]

$$\kappa_t : \mathscr{C}_t \longrightarrow \{(c_0, \dots, c_{t-1}) \in \mathbb{Z}^t : c_0 + \cdots + c_{t-1} = 0\} \tag{2.2}$$

such that for $\mu \in \mathscr{C}_t$

$$|\mu| = \sum_{r=0}^{t-1} \left(\frac{tc_r^2}{2} + rc_r \right).$$

The conjugate of a partition λ can be read off its Maya diagram by interchanging beads and empty spaces and then reflecting the picture about 0. Using this fact one may show that $t\text{-core}(\lambda') = t\text{-core}(\lambda)'$ which, if $\kappa_t(t\text{-core}(\lambda)) = (c_0, \dots, c_{t-1})$, translates to $\kappa_t(t\text{-core}(\lambda')) = (-c_{t-1}, \dots, -c_0)$ in terms of (2.2). Moreover, the quotient of λ' is given by $((\lambda^{(t-1)})', \dots, (\lambda^{(0)})')$. From these properties it is easy to see that the Littlewood decomposition of a self-conjugate partition must satisfy $t\text{-core}(\lambda) \in \mathscr{P}_0$, i.e., $c_r + c_{t-r-1} = 0$ for $0 \leqslant r \leqslant t-1$ and $\lambda^{(r)} = (\lambda^{(t-r-1)})'$ for r in the same range. Garvan, Kim and Stanton [6, §8] show that something similar holds for 1-asymmetric partitions. That is, if $\lambda \in \mathscr{P}_1$ then $t\text{-core}(\lambda), \lambda^{(0)} \in \mathscr{P}_1$ and the remaining entries in the quotient satisfy $\lambda^{(r)} = (\lambda^{(t-r)})'$ for $1 \leqslant r \leqslant t-1$.

Our first main result is a generalisation of the theorems of Garvan, Kim and Stanton to z-asymmetric partitions. To keep things simple we restrict to $0 \leqslant z \leqslant t-1$, with negative z being obtained by conjugation and larger values of z requiring only a minor, but slightly cumbersome, modification. To fix some notation, let $\mathcal{C}_{z;t} \subset \mathbb{Z}^t$ consist of those sequences for which $c_r + c_{z-r-1} = 0$ for $0 \leqslant r \leqslant z-1$ and $c_s + c_{t+z-s-1} = 0$ for $z \leqslant s \leqslant t-1$. Also let $d_c(\lambda)$ denote the Frobenius rank of the partition obtained by removing the first c rows of λ.

Theorem 2. *Let $t \geqslant 2$ and z be integers and λ a partition such that $0 \leqslant z \leqslant t-1$ and $\lambda \in \mathscr{P}_z$. Then $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{z;t}$ and the quotient $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is such that for $0 \leqslant r \leqslant z-1$ with $c_r \geqslant 0$ there exists a partition $\nu^{(r)}$ with*[3]

$$\lambda^{(r)} = \nu^{(r)} + (1^{c_r + d_{c_r}(\nu^{(r)})}) \quad \text{and} \quad \lambda^{(z-r-1)} = (\nu^{(r)})' + (1^{d_{c_r}(\nu^{(r)})}). \tag{2.3a}$$

Moreover, for $z \leqslant s \leqslant t-1$.

$$\lambda^{(s)} = (\lambda^{(t+z-s-1)})'. \tag{2.3b}$$

Before we comment on the proof of this characterisation an example is in order. Let $t = 5$, $z = 3$ and $\lambda = (21, 17, 16, 15, 12, 11, 6, 6, 5, 4, 4, 4, 3, 2, 1, 1, 1, 1)$, or, in Frobenius notation, $\lambda = (20\ 15\ 13\ 11\ 7\ 5 \mid 17\ 12\ 10\ 8\ 4\ 2)$. Then $5\text{-core}(\lambda) = (6, 5, 3, 2, 1, 1, 1, 1)$ which has associated integer vector $(2, 0, -2, 1, -1) \in \mathcal{C}_{3,5}$ and the quotient is given by

$$\big(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}\big) = \big((3,3,2), (3,1), (4,2), (2), (1,1)\big). \tag{2.4}$$

The reader may check the conditions (2.3) are satisfied by looking at Figure 4.

Note that if z is even then the $r = z/2$ case of (2.3a) just says that $\lambda^{(z/2)}$ is 1-asymmetric. This this it is clear that the 0- and 1-asymmetric cases are contained in the theorem. However, as the above example shows, not all cores with image in $\mathcal{C}_{z;t}$ are themselves z-asymmetric. The following corollary clarifies when the t-core of a z-asymmetric partition is again z-asymmetric, the first part of which is essentially contained in [3, Lemma 3.6].

[3] If $c_r = 0$ then $\nu^{(r)} = (\nu^{(z-r-1)})'$ is forced by (2.3a).

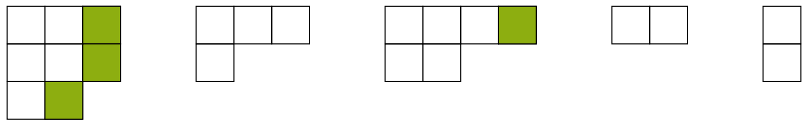


Figure 4: Young diagrams representing the 5-quotient (2.4). Note that $c_0 = 2$ so $d_{c_0}(\lambda^{(0)}) = 1$, $\nu^{(0)} = (2,2,1)$ and $\nu^{(1)} = (2,1)$. The highlighted cells in the first and third partitions denote those subtracted when verifying (2.3a).

Corollary 3. *A t-core μ is z-asymmetric if and only if $\kappa_t(\mu)$ satisfies $c_r = 0$ for $0 \leqslant r \leqslant z-1$. Moreover, for any sequence $\mathbf{c} \in \mathcal{C}_{z;t}$ the unique z-asymmetric partition $\mu_{\mathbf{c}}$ with $\kappa_t(t\text{-core}(\mu_{\mathbf{c}})) = \mathbf{c}$ and minimal $|\mu_{\mathbf{c}}|$ has quotient $\mu_{\mathbf{c}}^{(r)} = (1^{c_r})$ for those r with $0 \leqslant r \leqslant z-1$ and $c_r > 0$.*

Proof. By Theorem 2 a z-asymmetric partition μ must have $\kappa_t(t\text{-core}(\mu)) \in \mathcal{C}_{z;t}$ and $\lambda^{(r)} = \varnothing$ for all $0 \leqslant r \leqslant t-1$. However, the restrictions (2.3a) admit the empty partition as a solution if and only if $c_r = 0$. The second part of the corollary is then immediate. □

Theorem 2 may be proved by induction on z. For $z \geqslant 1$ there is an obvious bijection from $\mathscr{P}_{z-1}$ to $\mathscr{P}_z$ which adds one to the first $d(\lambda)$ parts of λ. We imagine the t-Maya diagram is wrapped around a cylinder, so that this bijection pushes the beads at positive positions up one row, and additionally moves the beads passing from row $t-1$ to row 0 one space to the right. This leads to an extension of Theorem 2 for $z \geqslant 0$, and to this end we say a pair of partitions satisfying (2.3a) are 1-*conjugate*. Then one writes $z = at + b$ for $a \geqslant 0$ and $0 \leqslant b \leqslant t-1$, so that the generalisation of Theorem 2 claims that $t\text{-core}(\lambda) \in \mathcal{C}_{b;t}$, and the partitions in (2.3a) will now be $(a+1)$-conjugate and those in (2.3b) a-conjugate [1].

3 Factorisations of universal characters

3.1 Symmetric functions and plethysm

As mentioned in the introduction, the ring of symmetric functions Λ has an algebraic basis given by the *complete homogeneous symmetric functions*, which for a countably infinite alphabet $X = (x_1, x_2, x_3, \dots)$ may be defined by the generating function

$$\prod_{i\geqslant 1} \frac{1}{1-ux_i} = \sum_{k\geqslant 0} u^k h_k(X).$$

The most important linear basis for Λ is given by the *Schur functions* s_λ, which we define at the generality of skew shapes by the Jacobi–Trudi determinant

$$s_{\lambda/\mu} := \det_{1\leqslant i,j\leqslant l(\lambda)} (h_{\lambda_i-\mu_j-i+j}), \tag{3.1}$$

where $h_{-k} := 0$ for $k \geqslant 1$. There is an inner product on Λ, the *Hall inner product*, under which the s_λ are orthonormal, so

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

with $\delta_{\lambda\mu}$ the usual Kronecker delta.

Plethysm is a composition of symmetric functions first introduced by Littlewood which we denote by $f \circ g$ for $f, g \in \Lambda$; see, e.g., [16, §1.9]. We only require the case where $g = p_t(X) := \sum_{i \geqslant 1} x_i^t$, the t-th power sum. This may most easily be defined by expanding f as a sum of monomials in X and then replacing each x_i by x_i^t. This particular plethysm satisfies $f \circ p_t = p_t \circ f$ and $p_s \circ p_t = p_{st}$ for $s, t \in \mathbb{N}$. Another way to define the operator φ_t is as the adjoint of the plethysm by a power sum with respect to the Hall inner product, i.e., for any $f, g \in \Lambda$,

$$\langle f \circ p_t, g \rangle = \langle f, \varphi_t g \rangle. \tag{3.2}$$

This may also be verified directly using, for instance, the orthonormality of the complete homogeneous and monomial symmetric functions.

As alluded to in the introduction, the content of [13, §7.3] is the computation of the action of φ_t on the Schur basis.[4] Later on, Farahat generalised this result to skew Schur functions of the form $s_{\lambda/t\text{-core}(\lambda)}$ [5], and the full skew Schur function case may be found in [16, p. 91]. Here the notion of "empty t-core" is replaced by the requirement that λ/μ is t-tileable. This is equivalent to $t\text{-core}(\lambda) = t\text{-core}(\mu)$ and $\lambda^{(r)} \supseteq \mu^{(r)}$ for each $0 \leqslant r \leqslant t-1$ [2, Lemma 2.1].

Theorem 4. *For any integer $t \geqslant 2$ and skew shape λ/μ we have that $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is t-tileable, in which case*

$$\varphi_t s_{\lambda/\mu} = \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}},$$

where the sign is defined in (2.1).

It is our opinion that this theorem has been somewhat neglected, and deserves to be better known. Although we will not give a full account its history here, the interested reader may find a few historical remarks in [2, §3] and its references. Using the Jacobi–Trudi formula (3.1) and the algebraic description of the t-core and t-quotient the proof is relatively straightforward, with the only difficulty being the identification of the sign.

One of the first applications of Littlewood's core and quotient construction is to the plethysm $s_\lambda \circ p_t$ which is now referred to as the *SXP rule* [14, p. 351].

[4] In his book [13] Littlewood does not use the language of cores and quotients, nor the map φ_t, and they appear only implicitly.

Theorem 5. *Let $c^{\lambda}_{\nu^{(0)},\dots,\nu^{(t-1)}}$ be the coefficient of s_λ in the Schur expansion of the product $s_{\nu^{(0)}}\cdots s_{\nu^{(t-1)}}$. Then for any $t \geqslant 2$,*

$$s_\lambda \circ p_t = \sum_{\substack{\nu \\ t\text{-core}(\nu)=\varnothing}} \operatorname{sgn}_t(\nu) c^{\lambda}_{\nu^{(0)},\dots,\nu^{(t-1)}} s_\nu.$$

By the adjoint relation (3.2) this is equivalent to Theorem 4 with $\mu = \varnothing$. Wildon has given a generalisation of the SXP rule for the expression $s_\tau(s_{\lambda/\mu} \circ p_t)$ [19], which may be derived from the full Theorem 4 in the same manner. Littlewood proved versions of the SXP rule for orthogonal and symplectic characters in the cases $t = 2, 3$ [15], and these were given lifts to the universal characters by Scharf and Thibon [18]. Lecouvey then greatly extended this by proving SXP rules for the universal symplectic and orthogonal characters for arbitrary $t \geqslant 2$ [12]. Using the adjoint relation (3.2), one can show that these expressions are equivalent to special cases of our Theorem 6 below.

3.2 Generalised universal characters

For a finite set of n variables the Schur function $s_\lambda(x_1, \dots, x_n)$ may be regarded as the character of the irreducible polynomial representation of $\mathrm{GL}(n, \mathbb{C})$ indexed by λ. The classical groups $\mathrm{O}(2n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$ also carry irreducible representations indexed by partitions. The characters of these representations are rather symmetric Laurent polynomials in n variables, however they may still be expressed as determinants in the complete homogeneous symmetric functions $h_r(x_1, 1/x_1, \dots, x_n, 1/x_n)$. Using these expressions, Koike and Terada defined the *universal characters* of the above groups, which are lifts of the characters to symmetric functions, and proved some expansions in terms of skew Schur functions [11, Theorem 2.3.1]. For example the universal character of $\mathrm{O}(2n, \mathbb{C})$ satisfies

$$\mathrm{o}_\lambda := \det_{1 \leqslant i,j \leqslant l(\lambda)} (h_{\lambda_i - i + j} - h_{\lambda_i - i - j}) = \sum_{\mu \in \mathscr{P}_1} (-1)^{|\mu|/2} s_{\lambda/\mu}. \tag{3.3}$$

with similar identities for the universal characters sp_λ and so_λ as sums over -1- and 0-asymmetric partitions respectively.

In [4] Bressoud and Wei proved an extension of (3.3) involving an integer $z \geqslant -1$ which reproduces the classical cases for $z \in \{-1, 0, 1\}$. This was generalised further by Hamel and King to an expression valid for all $z \in \mathbb{Z}$ and including an additional parameter q [8]. Let $[S]$ denote the *Iverson bracket*: $[S] = 1$ if the statement S is true and zero otherwise. Then the identity of Hamel and King is

$$\mathfrak{X}_\lambda(z; q) := \det_{1 \leqslant i,j \leqslant l(\lambda)} \big(h_{\lambda_i - i + j} + [j > -z] q h_{\lambda_i - i - j + 1 - z}\big) \tag{3.4a}$$

$$= \sum_{\mu \in \mathscr{P}_z} (-1)^{(|\mu| - d(\mu)(z+1))/2} q^{d(\mu)} s_{\lambda/\mu}. \tag{3.4b}$$

The odd and even orthogonal cases are recovered by setting $q = (-1)^z$ and then choosing $z = 0, 1$ respectively. The expression in terms of skew Schur functions immediately implies the following duality with respect to the involution ω on Λ which acts as $\omega s_{\lambda/\mu} = s_{\lambda'/\mu'}$:

$$\omega \mathfrak{X}_\lambda(z; q) = \mathfrak{X}_{\lambda'}(-z; (-1)^z q).$$

Again setting $q = (-1)^z$ and then $z = 1$ on the right then recovers the symplectic case of (3.3), thus extending $\omega \mathrm{o}_\lambda = \mathrm{sp}_{\lambda'}$ [11, Theorem 2.3.2]. Since our results require only minor modification to account for negative z we now restrict to $0 \leqslant z \leqslant t-1$.

Our main result is the action of φ_t on $\mathfrak{X}_\lambda(z; (-1)^z)$ (which was denoted simply $\mathfrak{X}_\lambda(z)$ in the introduction). To state this we need one more symmetric function, defined for $a, b, c \in \mathbb{N}$ by the following sum

$$\mathrm{rs}_{\lambda,\mu}(a, b; c) := \sum_\nu (-1)^{|\nu|} s_{\lambda/(\nu + (a^{c+d_c(\nu)}))} s_{\mu/(\nu' + (b^{d_c(\nu)}))},$$

where $d_c(\nu)$ is the modified Frobenius rank from Theorem 2. This symmetric function not only arises naturally in the proof of our main theorem, but also has Jacobi–Trudi-type determinantal expressions. For $c = 0$ this was also considered by Hamel and King, who also gave a Jacobi–Trudi-type expression [8]. If $a = b = c = 0$ it is essentially the universal character of the rational representation of $\mathrm{GL}(n, \mathbb{C})$ indexed by the pair (λ, μ) as defined by Koike [10]. (In fact, Koike's universal character was the inspiration for the extension of Hamel and King.) The function $\mathrm{rs}_{\lambda,\mu}(a, a; 0)$ is symmetric in λ and μ, however the same does not hold for $c \neq 0$. To make the statement below compact we adopt the convention that $\mathrm{rs}_{\lambda^{(r)},\lambda^{(z-r-1)}}(a, a; c_r) = \mathrm{rs}_{\lambda^{(z-r-1)},\lambda^{(r)}}(a, a; -c_r)$ if $c_r < 0$. Finally, recall that for $\mathbf{c} \in \mathcal{C}_{z;t}$, $\mu_\mathbf{c}$ is the unique smallest z-asymmetric partition with $\kappa_t(t\text{-core}(\mu_\mathbf{c})) = \mathbf{c}$ provided by Corollary 3. With this established, we are ready to state our second main result.

Theorem 6. *Let $t \geqslant 2$ and z be integers such that $0 \leqslant z \leqslant t-1$. Then $\varphi_t \mathfrak{X}_\lambda(z)$ vanishes unless $\kappa_t(t\text{-core}(\lambda)) := \mathbf{c} \in \mathcal{C}_{z;t}$ and $\lambda \supseteq \mu_\mathbf{c}$. If these conditions are satisfied, then*

$$\varphi_t \mathfrak{X}_\lambda(z; (-1)^z) = \varepsilon \prod_{r=0}^{\lfloor (z-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(z-r-1)}}(1, 1; c_r) \prod_{s=z}^{\lfloor (t+z-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(s)},\lambda^{(t+z-s-1)}}(0, 0; 0)$$
$$\times \begin{cases} 1 & \text{if } z \text{ even, } t \text{ even,} \\ \mathfrak{X}_{\lambda^{((z-1)/2)}}(1; -1) & \text{if } z \text{ odd, } t \text{ odd,} \\ \mathfrak{X}_{\lambda^{((t+z-1)/2)}}(0; 1) & \text{if } z \text{ even, } t \text{ odd,} \\ \mathfrak{X}_{\lambda^{((z-1)/2)}}(1; -1) \mathfrak{X}_{\lambda^{((t+z-1)/2)}}(0; -1) & \text{if } z \text{ odd, } t \text{ even,} \end{cases}$$

where the sign ε may be expressed as

$$\varepsilon = (-1)^{(|\mu_\mathbf{c}| + (z-1)d(\mu_\mathbf{c}))/2} \operatorname{sgn}_t(\lambda/\mu_\mathbf{c}).$$

For $z = 1$ the theorem states that $\varphi_t \mathrm{o}_\lambda$ vanishes unless t-core(λ) is 1-asymmetric, in which case

$$\varphi_t \mathrm{o}_\lambda = (-1)^{|t\text{-core}(\lambda)|/2} \operatorname{sgn}_t(\lambda/t\text{-core}(\lambda)) \mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-1)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & t \text{ even}, \\ 1 & t \text{ odd}, \end{cases}$$

where $\mathrm{so}^-_\nu := \mathfrak{X}_\nu(0, -1)$. This is precisely [2, Theorem 3.2], which generalises [3, Theorem 2.15]. One notable improvement on both our previous results and those of Ayyer and Kumari is that the sign in the above has a much nicer, combinatorial, expression. Another improvement is that Theorem 6 admits a uniform statement and uniform proof for $0 \leqslant z \leqslant t-1$. To obtain the symplectic case one must compute $\varphi_t \mathfrak{X}_\lambda(-z; (-1)^{z-1})$, which is completely analogous to the proof of the above, even though it is (unfortunately) not contained in Theorem 6. There is also a more general version of the theorem where $z \in \mathbb{N}$, as for Theorem 2, which we defer to future work [1].

Let us briefly sketch the proof of Theorem 6. The first step is to apply the map φ_t to each term in the skew Schur expansion (3.4b) using Theorem 4. Doing so gives the expression

$$\varphi_t \mathfrak{X}_\lambda(z; (-1)^z) = \sum_{\substack{\mu \in \mathscr{P}_z \\ \lambda/\mu \ t\text{-tileable}}} (-1)^{(|\mu| + (z-1)d(\mu))/2} \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$

From this vantage point the vanishing is already visible. Since λ/μ being t-tileable means that, in particular, t-core$(\lambda) = t$-core(μ) the first part of Theorem 2 implies that $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{z;t}$ for the sum to be non-vanishing. The second part of the same theorem combined with Corollary 3 tells us that $(1^{c_r}) \subseteq \mu^{(r)} \subseteq \lambda^{(r)}$ (since λ/μ is t-tileable) for those $0 \leqslant r \leqslant z-1$ such that $c_r > 0$, which is equivalent to the requirement that $l(\lambda^{(r)}) \geqslant c_r$ for $0 \leqslant r \leqslant z-1$. The next step is to show that the sum decouples as a product, and each factor in this product corresponds to the symmetric functions present in the factorisation. This is the meat of the proof, requiring a careful analysis of the sign and Frobenius rank as they relate to the Littlewood decomposition. Unfortunately, at this stage we are unable to include the parameter q present in (3.4) precisely because the Frobenius rank does not decompose nicely in terms of the Littlewood decomposition.

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Inhomogeneous particle process defined by canonical Grothendieck polynomials

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Abstract. We construct a time, particle, and position inhomogeneous discrete time particle process on the nonnegative integers that generalizes one of those studied in a Dieker and Warren. The particles move according to an inhomogeneous geometric distribution and stay in (weakly) decreasing order, where smaller particles block larger particles. We show that the transition probabilities for our particle process is given by a (refined) canonical Grothendieck function up to a simple overall factor.

Keywords: Grothendieck polynomial, particle process, transition probability

1 Introduction

The *totally asymmetric simple exclusion process* (TASEP) with sites on $\mathbb{Z}$ is a classical one dimensional model that has many interesting features and applications. In this stochastic process, there is at most one particle in each site and the particles move in one direction — say, to the right — according to some specified dynamic. For the continuous time process, particle p jumps one step with rate π_p, subject to an exclusion interaction, where a particle immediately to the right of p blocks it. For discrete time, then particles decide to move by flipping (biased) coins, where success rate π_p depends on particle. However, we need a rule to resolve when two particles move simultaneously that could interact. For the rule particles update right-to-left, this is Case B studied by Dieker and Warren [4]. On the other hand, if the particle keeps moving one step each time it flips the (biased) coin successfully until it fails, then it moves by the geometric distribution. This is [4, Case C] with instead updating the particles from left-to-right.

In a seemingly different area, the (refined) Grothendieck polynomials $G_{\lambda /\!/ \mu}(\mathbf{x}_n; \boldsymbol{\beta})$ originated from the (connective) K-theoretic Schubert calculus of the Grassmannian, and so they are a natural generalization of the Schur polynomials. They have been well-studied since their inception (for the unrefined case $\boldsymbol{\beta} = \beta$) in the work of Lascoux and

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Schützenberger [13], which includes explicit combinatorial descriptions [2, 3, 7]. However, this is related to the aforementioned particle processes as follows. When we additionally make the success probability of Case C TASEP depend on the time t according to $\pi_p x_t$, then the n-step transition probabilities can be easily seen to equal $G_{\lambda/\!/\mu}(\mathbf{x}_n; \boldsymbol{\beta})$ up to an explicit overall simple factor [8, Thm. 1.1]. Indeed, this can be seen in a number of different ways: Directly comparing the Jacobi–Trudi formula [9] with the natural symmetric function replacements in the determinants in [4], using extensions of the Schur operators to encode the dynamics [8, Sec. 4.2], or bijectively with set-valued tableaux [8, Sec. 5.3]. A similar statement holds for Case B with the weak Grothendieck polynomials.

A natural question is what particle process corresponds to the canonical Grothendieck polynomials [7, 16] (up to an analogous simple factor). However, it does not seem possible to build a particle process from naively combining the Case BC processes, which is similar to some of the combinatorial aspects of $G_{\lambda/\!/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$. Instead, we develop our stochastic model by using the Schur operators for $G_{\lambda/\!/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ developed in [8, Sec. 3] as they were shown to encode the particle movements when $\boldsymbol{\alpha} = 0$. This leads to a position inhomogeneous version of the Case BC process described above, which has been studied when $\boldsymbol{\beta} = 0$ in recent works [1, 11]. Our main result is that our new discrete time particle process has a transition kernel given by the canonical Grothendieck polynomials. Using this, we give a formula for the multi-point distribution for this process. All of our formulas can be described as determinants of contour integrals using [9].

This is an extended abstract based on [8, Sec. 8], and is organized as follows. In Section 2, we describe canonical Grothendieck polynomials. In Section 3, we give the necessary free fermion representations. In Section 4, we describe our particle process.

2 Grothendieck polynomials

Let $\mathcal{P}$ denote the set of all *partitions* $\lambda = (\lambda_1, \lambda_2, \dots)$, weakly decreasing sequences of nonnegative integers with finite sum. We draw our Young diagrams using English convention. Let $\ell(\lambda)$ denote the largest index ℓ such that $\lambda_\ell > 0$, called the *length* of λ. Let λ' denote the conjugate partition. A *hook* is a partition $a1^m$ with *arm* $a-1$ and *leg* m.

Let $\mathbf{x} = (x_1, x_2, \dots)$ denote a countably infinite sequence of indeterminates and denote $\mathbf{x}_n := (x_1, \dots, x_n, 0, 0, \dots)$. We make similar definitions for other such sequences. In particular, we take parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots)$.

A *hook-valued tableau* of skew shape λ/μ is a filling of the Young diagram by hook shaped tableau, fillings of a hook shape with entries weakly (resp. strictly) increasing along the arm (resp. leg), satisfying the local conditions

$$\begin{array}{|c|c|} \hline \mathsf{a} & \mathsf{b} \\ \hline \mathsf{c} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{ccc} \max(\mathsf{a}) & \leq & \min(\mathsf{b}) \\ \wedge & & \\ \min(\mathsf{c}) & & \end{array}$$

Work	[7]	[16]	[3]	[14]
Specialization	$G_\lambda(\mathbf{x}; -\boldsymbol{\alpha}, \boldsymbol{\beta})$	$G_\lambda(\mathbf{x}; -\alpha, -\beta)$	$G_\lambda(\mathbf{x}; 0, \beta)$	$G_\lambda(\mathbf{x}; 0, -\boldsymbol{\beta})$

Table 1: The relationship between our sign choices and some other papers.

(provided the requisite box exists). Note that this is a generalization of the semistandard conditions, which reduce to the usual ones when $\mathsf{a}, \mathsf{b}, \mathsf{c}$ all consist of a single entry.

For $\mu \subseteq \lambda$, the *canonical Grothendieck function* (we omit the word "refined" to simplify our nomenclature from [7]) is the generating function

$$G_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{T} \prod_{\mathsf{b} \in T} (-\alpha_i)^{a(\mathsf{b})} (-\beta_j)^{b(\mathsf{b})} \operatorname{wt}(\mathsf{b}),$$

where we sum over all hook-valued tableaux T of shape λ/μ, product over all entries b in T with $a(\mathsf{b})$ (resp. $b(\mathsf{b})$) the arm (resp. leg) of the shape of b and i (resp. j) the row (resp. column) of the entry. We indicate various specializations and relation with some of the literature in Table 1, which $G_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ also specializes those in [2, 12]. While technically we should work in a completion of the ring of symmetric functions, this does not affect our results, so we suppress this here. The set $\{G_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})\}_{\lambda \in \mathcal{P}}$ is a basis for (the completion of) symmetric functions (see, *e.g.*, [6, 7]).

The skew shape description is not natural from the algebraic perspective. Hence, refining [2, Eq. (6.4)] and [16, Prop. 8.8], we define [9, Sec. 4.1]

$$G_{\lambda /\!/ \mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \sum_{\nu \subseteq \mu} \prod_{(i,j) \in \mu/\nu} -(\alpha_i + \beta_j) G_{\lambda/\nu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{2.1}$$

where ν is formed by removing some corners of μ (boxes (i, μ_i) such that $\mu_i > \mu_{i+1}$).

Proposition 2.1 (Branching rules [9, Prop. 4.5])**.** *We have*

$$G_{\lambda/\mu}(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\nu \subseteq \lambda} G_{\lambda /\!/ \nu}(\mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) G_{\nu/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}),$$
$$G_{\lambda /\!/ \mu}(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\mu \subseteq \nu \subseteq \lambda} G_{\lambda /\!/ \nu}(\mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) G_{\nu /\!/ \mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}).$$

The *dual canonical Grothendieck functions* $\{g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})\}_{\lambda \in \mathcal{P}}$ are defined as the dual basis to the canonical Grothendieck functions under the Hall inner product, defined by $\{s_\lambda(\mathbf{x})\}_{\lambda \in \mathcal{P}}$, where $s_\lambda(\mathbf{x}) = G_\lambda(\mathbf{x}; 0, 0)$ are the Schur functions, is an orthonormal basis. A combinatorial definition of $g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ was given in [7], which is a refinement of the rim border tableaux description of [16].

We have the skew Cauchy formula [9, Thm. 4.6] (a non-skew version is in [7] or implied from [3, Rem. 3.9]). This is a refined version of [17, Thm. 1.1].

Theorem 2.2 (Skew Cauchy formula)**.** *We have*

$$\sum_{\lambda} G_{\lambda/\!/\mu}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) g_{\lambda/\nu}(\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{i,j} \frac{1}{1-x_i y_j} \sum_{\eta} G_{\nu/\!/\eta}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) g_{\mu/\eta}(\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\beta}).$$

3 Free fermions and Schur-type operators

We describe the free-fermion presentation of the (dual) canonical Grothendieck polynomials from [9]. For more details, we refer the reader to [10]. The unital associative Clifford algebra (over $\mathbb{C}$) is generated by $\{\psi_n, \psi_n^* \mid n \in \mathbb{Z}\}$ with relations

$$\psi_m \psi_n + \psi_n \psi_m = \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0, \qquad \psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m,n},$$

known as the canonical anti-commuting relations. The *current operators* are defined as $a_k := \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+k}^*$, (care is needed for $k = 0$, but we will not use this) and satisfy the Heisenberg algebra relations $[a_m, a_k] = m\delta_{m,-k}$, We will use the *Hamiltonian operators*

$$H(\mathbf{x}/\mathbf{y}) := \sum_{k>0} \frac{p_k(\mathbf{x}/\mathbf{y})}{k} a_k, \quad H^*(\mathbf{x}/\mathbf{y}) := \sum_{k>0} \frac{p_k(\mathbf{x}/\mathbf{y})}{k} a_{-k}, \quad \text{where } p_k(\mathbf{x}/\mathbf{y}) = \sum_{i=1}^{\infty} x_i^k - y_i^k,$$

and the corresponding *half vertex operators* $e^{H(\mathbf{x}/\mathbf{y})}$ and $e^{H^*(\mathbf{x}/\mathbf{y})}$. These satisfy the relations

$$e^{H(\mathbf{x}/\mathbf{y})} \psi_k e^{-H(\mathbf{x}/\mathbf{y})} = \sum_{i=0}^{\infty} h_i(\mathbf{x}/\mathbf{y}) \psi_{k-i}, \qquad e^{-H(\mathbf{x}/\mathbf{y})} \psi_k^* e^{H(\mathbf{x}/\mathbf{y})} = \sum_{i=0}^{\infty} h_i(\mathbf{x}/\mathbf{y}) \psi_{k+i}^*,$$

where $h_i(\mathbf{x}/\mathbf{y})$ is the homogeneous supersymmetric function.

Fermionic Fock space is the Clifford algebra representation $\mathcal{F}$ generated by the *shifted vacuum vectors* with relations

$$|m\rangle = \begin{cases} \psi_{m-1} \cdots \psi_0 |0\rangle & \text{if } m \geq 0, \\ \psi_m^* \cdots \psi_{-1}^* |0\rangle & \text{if } m < 0, \end{cases} \qquad \langle m| = \begin{cases} \langle 0| \psi_0^* \cdots \psi_{m-1}^* & \text{if } m \geq 0, \\ \langle 0| \psi_{-1} \cdots \psi_m & \text{if } m < 0. \end{cases}$$

Note that $e^{H(\mathbf{x}/\mathbf{y})}|m\rangle = |m\rangle$ and $\langle m| e^{H^*(\mathbf{x}/\mathbf{y})} = \langle m|$ for all m. We will use the vectors

$$|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} := \prod_{1 \leq i \leq \ell}^{\rightarrow} \left(e^{-H(A_{\lambda_i - 1})} \psi_{\lambda_i - i} e^{H(\beta_i)} e^{H(A_{\lambda_i - 1})} \right) |-\ell\rangle,$$
$$|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]} := \prod_{1 \leq i \leq \ell}^{\rightarrow} \left(e^{H^*(A_{\lambda_i})} \psi_{\lambda_i - i} e^{-H^*(\beta_i)} e^{-H^*(A_{\lambda_i})} \right) e^{H^*(A_{\lambda_\ell})} |-\ell\rangle,$$

here $A_k = -\boldsymbol{\alpha}_k = (-\alpha_1, \ldots, -\alpha_k)$ and the product is ordered $\Psi_1 \cdots \Psi_\ell$. We restrict ourselves to the subspace $\mathcal{F}^0$ and the bases [9, Thm. 3.10] $\{|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\}_{\lambda \in \mathcal{P}}$ and $\{|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\}_{\lambda \in \mathcal{P}}$.

There is also the dual representation $\mathcal{F}^*$, which has a canonical bilinear pairing called the *vacuum expectation value* that satisfies

$$\langle k|m\rangle = \delta_{km}, \qquad (\langle w|X)|v\rangle = \langle w|(X|v\rangle)$$

for all $k, m \in \mathbb{Z}$, operator X, $\langle w| \in \mathcal{F}^*$, and $|v\rangle \in \mathcal{F}$. Note that $|k\rangle^* = \langle k|$. Define by the anti-involution $\psi_i \leftrightarrow \psi_i^*$ the vectors ${}_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda| := (|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]})^*$ and ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda| := (|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]})^*$. We have the orthonormal bases [9, Thm. 3.10]

$$_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|\mu\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|\mu\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = \delta_{\lambda\mu}. \tag{3.1}$$

Moreover, there is the *boson-fermion correspondence* from $\mathcal{F}^0$ to symmetric functions defined by $|v\rangle \mapsto \langle 0|e^{H(\mathbf{x}/\mathbf{y})}|v\rangle$, which satisfies [9, Cor. 4.2, Eq. (4.1)]

$$G_{\lambda/\!/\mu}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu|e^{H(\mathbf{x})}|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}, \qquad g_{\lambda/\mu}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = {}_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu|e^{H(\mathbf{x})}|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}. \tag{3.2}$$

We denote $\kappa_i \colon \mathbf{k}[\mathcal{P}] \to \mathbf{k}[\mathcal{P}]$ the i-th (row) *Schur operator* that adds a box to the i-th row of a partition λ if $\lambda_i < \lambda_{i-1}$ (that is, we can add the box and obtain a partition) and is 0 otherwise. We define the linear operator $U_i^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ by

$$U_i^{(\boldsymbol{\alpha},\boldsymbol{\beta})} := \kappa_i + \Theta_i, \quad \text{where} \quad \Theta_i \cdot \lambda := \begin{cases} -\alpha_{\lambda_i}\lambda & \text{if } \lambda_i < \lambda_{i-1}, \\ \beta_{i-1}\lambda & \text{if } \lambda_i = \lambda_{i-1}, \end{cases}$$

for any $\lambda \in \mathcal{P}$. We consider $\lambda_0 = \infty$ and $\alpha_0 = 0$ (although our proofs could have α_0 be an arbitrary parameter). When there is no ambiguity, we will simply write $U_i := U_i^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

Lemma 3.1 ([8, Lemma 3.2])**.** *The operators $\mathbf{U} = \{U_i\}_{i=1}^{\infty}$ satisfy the weak Knuth relations.*

Lemma 3.1 implies we can use $\mathbf{U}$ with noncommutative symmetric functions [5].

Theorem 3.2 ([8, Thm. 3.3])**.** *We have ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|S_\mu(a_1, a_2, \dots) = {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle s_\mu(\mathbf{U}/\boldsymbol{\beta}) \cdot \lambda|$, where $S_\lambda(p_1(\mathbf{x}), p_2(\mathbf{x}), \dots) = s_\lambda(\mathbf{x})$.*

4 Particle Process

Now we describe a particle process whose transition kernel naturally uses the canonical Grothendieck polynomials. We start by explicitly defining the stochastic process, and then we will show how to interpret it using the noncommutative operators $\mathbf{U}$. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ be a sequence of parameters such that $0 \leq \pi_i x_j < 1$ for all i and j.

Let $G(j, i)$ denote the position of the j-th particle at time i, which is determined by

$$G(j,i) = \min\bigl(G(j, i-1) + w_{ji}, G(j-1, i-1)\bigr), \tag{4.1}$$

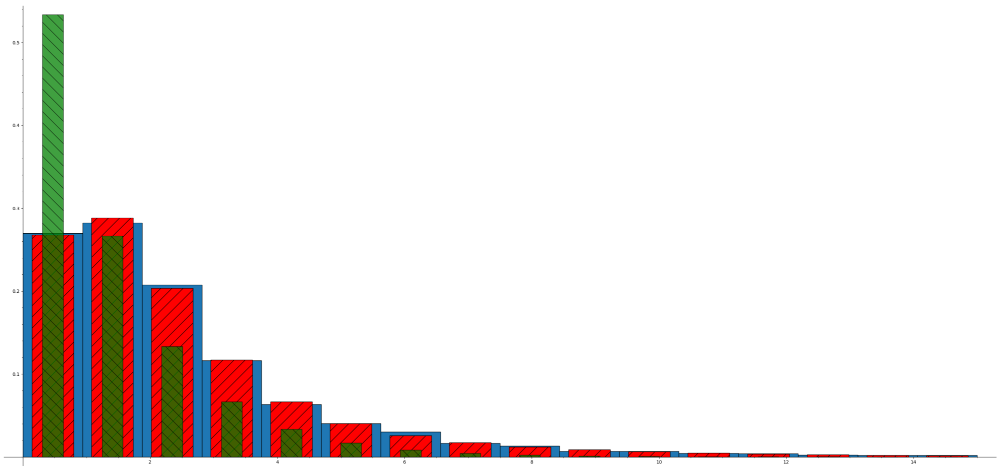

Figure 1: A sampling using 10000 samples of the inhomogeneous geometric distribution $\mathsf{P}_{\mathcal{G}}$ for $x_i = 1$, $\pi_j = .5$, and $\alpha_k = 1 - ke^{-k/2}$ (blue) under the exact distribution (red), which is under the geometric distribution with parameter $\pi_j x_i$ (green).

by convention $G(0, i-1) := \infty$, where the random variable w_{ij} (which depends on $G(j, i-1)$) is determined by the *inhomogeneous geometric distribution* defined by

$$\mathsf{P}_{\mathcal{G}}(w_{ji} = m' \mid G(j, i-1) = m) := \frac{1 - \pi_j x_i}{1 + \alpha_{m+m'} x_i} \prod_{k=m}^{m+m'-1} \frac{(\alpha_k + \pi_j) x_i}{1 + \alpha_k x_i}. \tag{4.2}$$

In other words, the j-th particle at time i attempts to jump w_{ji} steps, but can be blocked by the $(j-1)$-th particle, which updates its position after the j-th particle moves.

Let us digress slightly on why (4.2) is called an inhomogeneous geometric distribution. We can realize it as the waiting time for a failure in sequence of Bernoulli variables (*i.e.*, weighted coin flips), but the k-th trial given a probability of success $(\alpha_k + \pi_j) x_i (1 + \alpha_k x_i)^{-1}$. Indeed, we note that the probability of a failure is

$$1 - \frac{\alpha_k x_i + \pi_j x_i}{1 + \alpha_k x_i} = \frac{1 - \pi_j x_i}{1 + \alpha_k x_i}.$$

Hence, this gives us a sampling algorithm for the distribution $\mathsf{P}_{\mathcal{G}}$. We illustrate the effectiveness of this sampling in Figure 1. This perspective also allows us to easily see that we have a probability measure on $\mathbb{Z}_{\geq m}$ for any fixed m. The case $\boldsymbol{\pi} = 0$ can also be seen as a projection of the Warren–Windridge dynamics [15]; see also [1, Sec. 2.2].

We will give some remarks on the meaning of the $\boldsymbol{\alpha}$ parameters. From the behavior of the operators $\mathbf{U}$, it would be tempting to consider the $\boldsymbol{\alpha}$ parameters as a viscosity, but for $\boldsymbol{\alpha} > 0$, we have $\mathsf{P}_{\mathcal{G}}(w_{ji} = k) > \mathsf{P}_{Ge}(w_{ji} = k)$, where P_{Ge} denotes the usual geometric distribution with parameter $\pi_j x_i$. Thus, in this case, the $\boldsymbol{\alpha}$ parameters act as a current being applied to the system, the strength (and direction) of which can vary at each position. On the other hand, when $\boldsymbol{\alpha} < 0$, we have $\mathsf{P}_{\mathcal{G}}(w_{ji} = k) < \mathsf{P}_{Ge}(w_{ji} = k)$, and so indeed $\boldsymbol{\alpha}$ then acts as (position-based) viscosity. We can also introduce locations where certain particles must stop by having $-\alpha_k = \pi_j$ since this would have $\mathcal{P}_{\mathcal{G}}(w_{ij} = k') = 0$ for all k' that would move the j-th particle past position k.

To see how to obtain this process using the noncommutative operators $\mathbf{U}$, we initiate by taking the skew Cauchy formula (Theorem 2.2) with $\nu = \emptyset$ and with the specializations $\mathbf{y} = \boldsymbol{\pi}_1$ and $\beta_j = \pi_{j+1}$, yielding

$$\sum_{\lambda} G_{\lambda/\!/\mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}) g_{\lambda}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_i (1 - \pi_1 x_i)^{-1} g_{\mu}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4.3}$$

In particular, if we let $\widehat{\lambda}_i = \lambda_i - 1$ for all $1 \leq i \leq \ell(\lambda)$, then from the combinatorial description of [7, Thm. 7.2], we have $g_{\lambda}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\pi}^{1^{\ell(\lambda)}} \prod_{(i,j) \in \widehat{\lambda}} (\alpha_i + \pi_j)$. Hence, Equation (4.3) can be considered a Littlewood-type identity for canonical Grothendieck polynomials. Dividing this by the factor on the right hand side and taking the term corresponding to λ, we obtain a probability distribution for n step random growth process (since we must have $\mu \subseteq \lambda$ and currently the interpretation we have described is only on partitions) given by

$$\mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = \prod_{i=1}^{n} (1 - \pi_1 x_i) \boldsymbol{\pi}^{1^{\ell(\lambda)}/1^{\ell(\mu)}} \prod_{(i,j) \in \widehat{\lambda}/\widehat{\mu}} (\alpha_i + \pi_j) G_{\lambda/\!/\mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4.4}$$

Note that Equation (4.3) is equivalent to $\sum_{\lambda} \mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = 1$ for any fixed μ and n.

Rephrasing Equation (4.4) and adding an $\alpha_0 = 0$ parameter in order to simplify the product in $g_{\lambda}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta})$, what we have computed are coefficients

$$C_{\lambda\mu} = \prod_{i=1}^{n} (1 - \pi_1 x_i) (\vec{\boldsymbol{\alpha}} + \boldsymbol{\beta})^{\lambda/\mu}, \qquad \text{where } (\vec{\boldsymbol{\alpha}} + \boldsymbol{\beta})^{\lambda/\mu} := \prod_{(i,j) \in \lambda/\mu} (\alpha_{i-1} + \pi_j)$$

that is defined to be 0 if $\lambda \not\supseteq \mu$, such that

$$C_{\lambda\mu} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mu | e^{H(\mathbf{x}_n)} | \lambda \rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = \mathsf{P}_{\mathcal{C},n}(\lambda|\mu) \iff {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mu | e^{H(\mathbf{x}_n)} = \sum_{\lambda \supseteq \mu} \frac{\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)}{C_{\lambda\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \lambda |, \tag{4.5}$$

where the equivalence of the two formulas is given by the orthonormality (3.1).

We now restrict ourselves to a single timestep at time i in order to encode the growth process as a particle process by using the operators $\mathbf{U}$. This incurs no loss of generality as $\mathsf{P}_{\mathcal{C},n+n'}(\lambda|\mu) = \sum_{\nu} \mathsf{P}_{\mathcal{C},n}(\lambda|\nu) \mathsf{P}_{\mathcal{C},n'}(\nu|\mu)$ by the branching rules (Proposition 2.1) and we have a Markov process. Define the time evolution operator

$$\mathcal{T}_{\mathcal{C}} = \sum_{k=0}^{\infty} h_k(x_i \mathbf{U}) = \sum_{k=0}^{\infty} x_i^k h_k(\mathbf{U}).$$

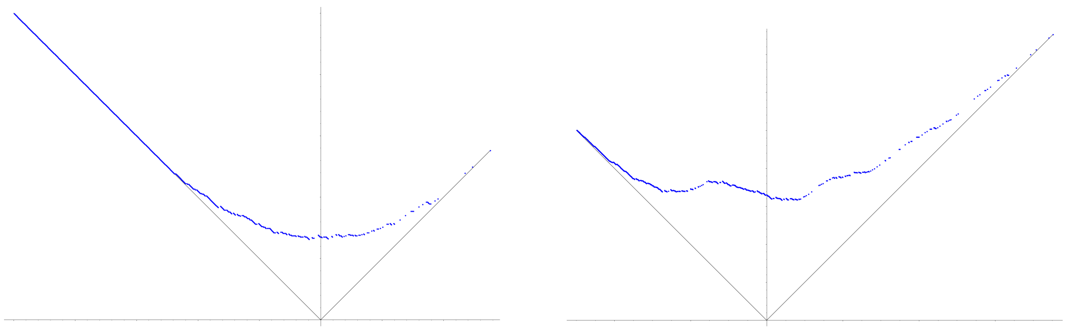

Figure 2: Samples of our process with $\ell = 500$ particles after $n = 50000$ time steps with (left) $\boldsymbol{\pi} = 1$, $\mathbf{x} = 0.01$, and $\boldsymbol{\alpha} = -0.5$; (right) $\boldsymbol{\pi} = 0.5$, $\mathbf{x} = .2$, and $\alpha_k = 0.5\sin(k/50)^6$.

By Theorem 3.2, by some algebraic and plethystic manipulations as in [8, Sec. 4.2]

$$ {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mu | e^{H(x_i)} = \prod_{j=2}^{\infty} \frac{1}{1-\pi_j x_i} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu |. $$

Thus, if we consider the expansion $\langle \mathcal{T}_{\mathcal{C}} \cdot \mu | = \sum_\lambda B_{\lambda\mu} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \lambda |$, and matching coefficients in (4.5) (equivalently, pairing with $|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}$), we obtain

$$ \mathsf{P}_{\mathcal{C}}(\lambda|\mu) = \frac{B_{\lambda\mu}}{(\vec{\boldsymbol{\alpha}} + \boldsymbol{\beta})^{\lambda/\mu}} \prod_{j=1}^{\infty} (1-\pi_j x_i)^{-1}. $$

Example 4.1. Consider $\mu = (1,1)$ and set $\pi_j = 0$ for all $j > 3$. Using

$$ h_1(\mathbf{u}_3) = u_1 + u_2 + u_3, \qquad h_2(\mathbf{u}_3) = u_1^2 + u_1u_2 + u_1u_3 + u_2^2 + u_2u_3 + u_3^2, $$
$$ h_3(\mathbf{u}_3) = u_1^3 + u_1^2u_2 + u_1^2u_3 + u_1u_2^2 + u_1u_2u_3 + u_1u_3^2 + u_2^3 + u_2^2u_3 + u_2u_3^2 + u_3^3, $$

and recalling we consider $\alpha_0 = 0$, we compute

$$ \begin{aligned} h_1(\mathbf{U}_3) \cdot \mu &= \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \pi_1 \ydiagram{1,1} + \ydiagram{1,1,1}, \\ h_2(\mathbf{U}_3) \cdot \mu &= \left(\alpha_1^2 \ydiagram{1,1} - (\alpha_1+\alpha_2) \ydiagram{2,1} + \ydiagram{3,1}\right) + \pi_1 \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) \\ &\quad + \pi_1^2 \ydiagram{1,1} + \pi_1 \ydiagram{1,1,1} + \pi_2 \ydiagram{1,1,1}, \\ h_3(\mathbf{U}_3) \cdot \mu &= \left(-\alpha_1^3 \ydiagram{1,1} + (\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2) \ydiagram{2,1} - (\alpha_1+\alpha_2+\alpha_3) \ydiagram{3,1} + \ydiagram{4,1}\right) \\ &\quad + \pi_1 \left(\alpha_1^2 \ydiagram{1,1} - (\alpha_1+\alpha_2) \ydiagram{2,1} + \ydiagram{3,1}\right) + \left(\alpha_1^2 \ydiagram{1,1,1} - (\alpha_1+\alpha_2) \ydiagram{2,1,1} + \ydiagram{3,1,1}\right) \\ &\quad + \pi_1^2 \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \pi_1 \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) + \pi_2 \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) \\ &\quad + \pi_1^3 \ydiagram{1,1} + \pi_1^2 \ydiagram{1,1,1} + \pi_1\pi_2 \ydiagram{1,1,1} + \pi_2^2 \ydiagram{1,1,1}. \end{aligned} $$

Recall that $A_k = -\boldsymbol{\alpha}_k$. Therefore, we have

$$\begin{aligned}
{}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu| &= (1 + h_1(\boldsymbol{\beta}_1 \sqcup A_1)x_i + h_2(\boldsymbol{\beta}_1 \sqcup A_1)x_i^2 + h_3(\boldsymbol{\beta}_1 \sqcup A_1)x_i^3 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1| \\
&\quad + x_i(1 + h_1(\boldsymbol{\beta}_1 \sqcup A_2)x_i + h_2(\boldsymbol{\beta}_1 \sqcup A_2)x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1| \\
&\quad + x_i(1 + h_1(\boldsymbol{\beta}_2 \sqcup A_1)x_i + h_2(\boldsymbol{\beta}_2 \sqcup A_1)x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1,1| \\
&\quad + x_i^2(1 + h_1(\boldsymbol{\beta}_1 \sqcup A_3)x_i + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 3,1| \\
&\quad + x_i^2(1 + h_1(\boldsymbol{\beta}_2 \sqcup A_2)x_i \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1,1| + \cdots \\
&= \frac{(1+\alpha_1 x_i)^{-1}}{1-\pi_2 x_i} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1| + \frac{(\alpha_1 x_i + \pi_1 x_i)(1+\alpha_1 x_i)^{-1}(1+\alpha_2 x_i)^{-1}}{(1-\pi_2 x_i)(\vec{\boldsymbol{\alpha}}+\boldsymbol{\pi})^{(2,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1| \\
&\quad + \frac{(\alpha_0 x_i + \pi_3 x_i)(1+\alpha_1 x_i)^{-1}}{(1-\pi_2 x_i)(1-\pi_3 x_i)(\vec{\boldsymbol{\alpha}}+\boldsymbol{\pi})^{(1,1,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1,1| \\
&\quad + \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_2 x_i + \pi_1 x_i)(1+\alpha_1 x_i)^{-1}(1+\alpha_2 x_i)^{-1}(1+\alpha_3 x_i)^{-1}}{(1-\pi_2 x_i)(\vec{\boldsymbol{\alpha}}+\boldsymbol{\pi})^{(3,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 3,1| \\
&\quad + \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_0 x_i + \pi_3 x_i)(1+\alpha_1 x_i)^{-1}(1+\alpha_2 x_i)^{-1}}{(1-\pi_2 x_i)(1-\pi_3 x_i)(\vec{\boldsymbol{\alpha}}+\boldsymbol{\pi})^{(2,1,1)/\mu}} {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1,1| + \cdots .
\end{aligned}$$

If we include α_0 in the $\mathbf{U}$ operators, then all terms will be multiplied by $(1+\alpha_0 x_i)^{-1}$ since the third particle can move from position 0. With this, some probabilities are

$$\begin{aligned}
\mathsf{P}_{\mathcal{C}}(1,1|\mu) &= \frac{(1-\pi_1 x_i)(1-\pi_3 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(2,1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(1-\pi_1 x_i)(1-\pi_3 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)(1+\alpha_2 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(1,1,1|\mu) &= \frac{(\alpha_0 x_i + \pi_3 x_i)(1-\pi_1 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(3,1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_2 x_i + \pi_1 x_i)(1-\pi_1 x_i)(1-\pi_3 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)(1+\alpha_2 x_i)(1+\alpha_3 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(2,1,1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_0 x_i + \pi_3 x_i)(1-\pi_1 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)(1+\alpha_2 x_i)}.
\end{aligned}$$

Any individual (free) particle motion is (up to changing $\pi_j \mapsto \pi_1$) equivalent to the first particle's motion. Thus, let us consider λ with $\ell(\lambda) = 1$, and a straightforward computation (say, at time i) using either the operators $\mathbf{U}$ or the combinatorial description of $G_{\lambda /\!/ \mu}(x_i; \boldsymbol{\alpha}, \boldsymbol{\beta})$ yields

$$\mathsf{P}_{\mathcal{C}}(m'|m) = \frac{1-\pi_j x_i}{1+\alpha_{m'+m} x_i} \prod_{k=m}^{m+m'-1} \frac{(\alpha_k + \pi_j) x_i}{1+\alpha_k x_i},$$

which is precisely the measure specified in (4.4). By (4.4), for any fixed m this is a probability measure for all $\alpha_k + \pi_j \geq 0$ with the natural assumptions $0 \leq \pi_j x_i < 1$ and

$\alpha_k x_i \geq -1$. This can also be extended to include $(\alpha_k)_{k\in\mathbb{Z}}$ by shifting the parameters $\alpha_k \mapsto \alpha_{k\pm 1}$. Hence, the same analysis as in [8, Sec. 4.2] yields the following.

Theorem 4.2. *Suppose $\ell(\lambda) \leq \ell$, $\pi_j x_i \in (0,1)$, $\alpha_k x_i > -1$, and $\alpha_k + \pi_j \geq 0$ for all i, j, k. Set $\beta_j = \pi_{j+1}$. Let $\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)$ denote the n-step transition probability for particle system using the distribution (4.2) for the jump probability of the particles with interactions as given by (4.1). Then the n-step transition probability is given by*

$$\mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = \prod_{i=1}^{n}(1-\pi_1 x_i)(\vec{\boldsymbol{\alpha}}+\boldsymbol{\pi})^{\lambda/\mu} G_{\lambda/\!/\mu}(\mathbf{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}).$$

Remark 4.3. Since the $\boldsymbol{\alpha}$ parameters used, and hence the probabilities, now depend on the positions of the particles, we can only work with the bosonic model, where multiple particles can occupy the same site. If we instead switch to a fermionic model by mapping the j-th particle at position λ_j to $\lambda_j - j$, then we are required to introduce additional parameters α_k for $k < 0$, in which case Theorem 4.2 no longer holds, or to account for the shifting of positions by replacing $\alpha_k \mapsto \alpha_{k+j}$ for the j-th particle distribution $\mathsf{P}_{\mathcal{G}}$.

We could also prove Theorem 4.2 by using the combinatorics of hook-valued tableaux as in [8, Sec. 5.3], where the positions of the particles is dictated by the smallest value in each entry of the hook-valued tableaux. The key observation is that we have a factor $x_i(1-\alpha_k x_i)^{-1}$ for every box in the k-th column that would normally contain an i in the set-valued tableaux (over all k), or where there is no arm. The leg (the column part except for the corner) corresponds to the choice between 1 and $-\pi_i x_j$ in the numerator of the normalization constant as in [8, Sec. 5.3]. The arm (the row part except for the corner) comes from waiting at that particular position and contributes an $-\alpha x_i$, which contributes a factor of $(1+\alpha x_i)^{-1}$ as in the Case B combinatorial proof [8, Sec. 5.4].

From [9, Thm. 4.1], we obtain determinant formulas for $\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)$, where we can write the entries of the matrix as contour integrals [9, Thm. 4.19]. We can also redo [8, Thm. 6.8] at this level of generality to obtain the multi-point distribution.

Theorem 4.4. *The multi-point distribution is given by*

$$\begin{aligned}\mathsf{P}_{\geq,n}(\nu|\mu) &:= \mathsf{P}(G(\ell,n)\geq \nu_\ell,\dots,G(1,n)\geq\nu_1 \mid G(\ell,0)=\mu_\ell,\dots,G(1,0)=\mu_1)\\ &= \prod_{j=2}^{\ell}\prod_{i=1}^{n}(1-\pi_j x_i)^{-1}\det\left[h_{\nu_i-\mu_j-i+j}\big(\mathbf{x}/\!/(A_{(\mu_j,\nu_i]}\sqcup\boldsymbol{\pi}_i/\boldsymbol{\beta}_j\big)\right]_{i,j=1}^{\ell}.\end{aligned}$$

We can give another simpler proof of Theorem 4.2 for the case when $\boldsymbol{\alpha} = \alpha$. This will follow from a straightforward generalization of the unrefined case [16, Prop. 3.4], noting our sign convention means we need to substitute $-\alpha$.

Proposition 4.5. $G_\lambda(\mathbf{x};\alpha,\boldsymbol{\beta}) = G_\lambda(\mathbf{x}/(1+\alpha\mathbf{x});0,\alpha+\boldsymbol{\beta})$ *by* $x_i \mapsto x_i/(1+\alpha x_i)$, $\beta_i \mapsto \alpha+\beta_i$.

Indeed, under this substitution, we have $\pi_j x_i \longmapsto \frac{(\alpha+\pi_j)x_i}{1+\alpha x_i}$. Hence, the geometric distribution P_{Ge} transforms to the distribution $\mathsf{P}_{\mathcal{G}}$ in (4.2) with $\boldsymbol{\alpha} = \alpha$. Moreover, in our formula in Case C of [8, Thm. 1.1], the total $\mathbf{x}$ degree and total $\boldsymbol{\pi}$ degree in each term of $\boldsymbol{\pi}^{\lambda/\mu} G_{\lambda/\!/\mu}(\mathbf{x};\boldsymbol{\beta})$ are equal, and so we can perform this substitution.

Remark 4.6. Let us discuss the relationship between this model and the doubly geometric inhomogeneous corner growth model defined in [11]. In their corresponding TASEP model, there is an additional set of position-dependent parameters $\boldsymbol{\nu}$ that are only involved after the initial movement of the particle (akin to static friction). Yet, if we set $\boldsymbol{\nu} = 0$, then the model in [11] is the fermionic realization of our model (*cf.* Remark 4.3) at $\boldsymbol{\beta} = 0$ with their parameters $(\mathbf{a}, \boldsymbol{\beta})$ equaling our parameters $(\boldsymbol{\alpha}, \mathbf{x})$. Hence, we end up with another TASEP version that is equivalent to Case B. It would be interesting to see if the model in [11] can be recovered from the free fermionic description.

We also remark that our model with $\boldsymbol{\pi} = 0$ was studied in [1], but using very different techniques based on Toeplitz matrices and Markov semigroups. Therefore, from the specialization of the canonical Grothendieck polynomials, it is essentially Case B as before, with a more probabilistic link being made by [1, Thm. 2.43].

We can similarly define a Bernoulli process with the position-dependent probability

$$\mathsf{P}_{\mathcal{B}}(w_{ji} = 1 \mid G(j, i-1) = m) := \frac{(\rho_j + \beta_m)x_i}{1 + \rho_j x_i}. \tag{4.6}$$

Analogously to Theorem 4.2 (including its proof), we have the following.

Theorem 4.7. *Suppose $\lambda_1 \leq \ell$, $\beta_k x_i \in (0,1)$, $\rho_j x_i > -1$, and $\rho_j + \beta_k \geq 0$ for all i, j, k. Set $\alpha_j = \rho_{j+1}$. The n-step transition probability for the particle system using Bernoulli jumps according to the distribution* (4.6) *is given by*

$$\mathsf{P}_{\mathcal{B},n}(\lambda|\mu) = \frac{(\vec{\beta} + \boldsymbol{\rho})^{\lambda/\mu}}{\prod_{i=1}^{n}(1+\rho_1 x_i)} G_{\lambda'/\!/\mu'}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}).$$

If we set $\boldsymbol{\alpha} = 0$ in this position-dependent version of [8, Case B], then we end up with a Bernoulli random variable version of [11] at $\boldsymbol{\nu} = 0$.

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Chow rings of matroids as permutation representations

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Abstract. Given a matroid and a group of its matroid automorphisms, we study the induced group action on the Chow ring of the matroid. This turns out to always be a permutation action. Work of Adiprasito, Huh and Katz showed that the Chow ring satisfies Poincaré duality and the Hard Lefschetz theorem. We lift these to statements about this permutation action, and suggest further conjectures in this vein.

Keywords: matroid, Chow ring, Koszul, log-concave, unimodal, Kahler package, Burnside ring, equivariant, Polya freqency, real-rooted

1 Introduction

A *matroid* $\mathcal{M}$ is a combinatorial abstraction of lists of vectors $v_1, v_2, \ldots, v_n$ in a vector space, recording only the information about which subsets of the vectors are linearly independent or dependent, forgetting their coordinates. In groundbreaking work, Adiprasito, Huh and Katz [1] affirmed long-standing conjectures of Rota–Heron–Welsh and Mason about vectors and matroids via a new methodology. Their work employed a certain graded $\mathbb{Z}$-algebra $A(\mathcal{M})$ called the *Chow ring* of $\mathcal{M}$, introduced by Feichtner and Yuzvinsky [4] as a generalization of the Chow ring of DeConcini and Procesi's *wonderful compactifications* for hyperplane arrangement complements. A remarkable integral Gröbner basis result proven by Feichtner and Yuzvinsky [4, Thm. 2] shows that for a matroid $\mathcal{M}$ of rank $r+1$ with Chow ring $A(\mathcal{M}) = \bigoplus_{k=0}^{r} A^k$, each homogeneous component is free abelian: $A^k \cong \mathbb{Z}^{a_k}$ for some Hilbert function $(a_0, a_1, \ldots, a_r)$. A key step in the work of Adiprasito, Huh and Katz shows not only *symmetry* and *unimodality* for the Hilbert function

$$a_k = a_{r-k} \text{ for } r \leq k/2 \tag{1.1}$$

$$a_0 \leq a_1 \leq \cdots \leq a_{\lfloor \frac{r}{2} \rfloor} = a_{\lceil \frac{r}{2} \rceil} \geq \cdots \geq a_{r-1} \geq a_r, \tag{1.2}$$

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but in fact proves that $A(\mathcal{M})$ enjoys a trio of properties referred to as the *Kähler package*, reviewed in Section 2.2 below. The first of these properties is *Poincaré duality*, proving (1.1) via a natural $\mathbb{Z}$-module isomorphism $A^{r-k} \cong \mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z})$. The second property, called the *Hard Lefschetz Theorem*, shows that after tensoring over $\mathbb{Z}$ with $\mathbb{R}$ to obtain $A(\mathcal{M})_{\mathbb{R}} = \bigoplus_{k=0} A^k_{\mathbb{R}}$, one can find *Lefschetz elements* ω in $A^1_{\mathbb{R}}$ such that multiplication by ω^{r-2k} give $\mathbb{R}$-linear isomorphisms $A^k_{\mathbb{R}} \to A^{r-k}_{\mathbb{R}}$ for $k \leq \frac{r}{2}$. In particular, multiplication by ω mapping $A^k_{\mathbb{R}} \to A^{k+1}_{\mathbb{R}}$ is *injective* for $k < \frac{r}{2}$, strengthening the unimodality (1.2).

We are interested in how these *Poincaré duality* and *Hard Lefschetz* properties interact with the group $G := \mathrm{Aut}(\mathcal{M})$ of symmetries of the matroid $\mathcal{M}$. It is not hard to check that G acts via graded $\mathbb{Z}$-algebra automorphisms on $A(\mathcal{M})$, giving $\mathbb{Z}G$-module structures on each A^k, and $\mathbb{R}G$-module structures on each $A^k_{\mathbb{R}}$. One can also check (see the proof of Corollary 6 below) that $A^r \cong \mathbb{Z}$ with trivial G-action. From this, the Poincaré duality pairing immediately gives rise to a $\mathbb{Z}G$-module isomorphism

$$A^{r-k} \cong \mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z}) \tag{1.3}$$

where g in G acts on φ in $\mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z})$ via $\varphi \mapsto \varphi \circ g^{-1}$; similarly $A^{r-k} \cong \mathrm{Hom}_{\mathbb{R}}(A^k, \mathbb{R})$ as $\mathbb{R}G$-modules. Furthermore, it is not hard to check (see Corollary 6 below) that one can pick an explicit Lefschetz element ω as in [1] which is G-fixed, giving $\mathbb{R}G$-module isomorphisms and injections

$$\begin{array}{ccc} A^k_{\mathbb{R}} \xrightarrow{\sim} A^{r-k}_{\mathbb{R}} \quad \text{for } r \leq \dfrac{k}{2} & \qquad & A^k_{\mathbb{R}} \hookrightarrow A^{k+1}_{\mathbb{R}} \quad \text{for } r < \dfrac{k}{2} \\ a \longmapsto a \cdot \omega^{r-2k} & & a \longmapsto a \cdot \omega. \end{array} \tag{1.4}$$

Our goal is to use Feichtner and Yuzvinsky's Gröbner basis result to prove a combinatorial strengthening of the isomorphisms and injections (1.3), (1.4). To this end, recall that the matroid $\mathcal{M}$ can be specified by its family $\mathbf{F}$ of *flats*; then the *Chow ring* $A(\mathcal{M})$ is presented as a quotient of the polynomial ring $S := \mathbb{Z}[x_F]$ having one variable x_F for each nonempty flat F in $\mathbf{F} \setminus \{\varnothing\}$. The presentation takes the form $A(\mathcal{M}) := S/(I+J)$ where I, J are certain ideals of S defined more precisely in Definition 1 below.

Feichtner and Yuzvinsky exhibited a Gröbner basis for $I + J$ that leads to the following standard monomial $\mathbb{Z}$-basis for $A(\mathcal{M})$, which we call the *FY-monomials* of $\mathcal{M}$:

$$\mathrm{FY} := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} : (\varnothing =: F_0) \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\}.$$

Here $\mathrm{rk}(F)$ denotes the matroid rank of the flat F. The subset FY^k of FY-monomials $x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell}$ of total degree $m_1 + \cdots + m_\ell = k$ then gives a $\mathbb{Z}$-basis for A^k. One can readily check (see Corollary 4 below) that the group $G = \mathrm{Aut}(\mathcal{M})$ permutes the $\mathbb{Z}$-basis FY^k for A^k, endowing A^k with the structure of a *permutation representation*, or *G-set*. Our main result is this strengthening of the isomorphisms and injections seen in (1.3), (1.4).

Theorem 1. *For every matroid $\mathcal{M}$ of rank $r+1$, there exist*

(i) G-equivariant bijections $\pi : \mathrm{FY}^k \xrightarrow{\sim} \mathrm{FY}^{r-k}$ *for* $k \leq \frac{r}{2}$*, and*

(ii) G-equivariant injections $\lambda : \mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$ *for* $k < \frac{r}{2}$.

2 Background

Among the many definitions of a matroid $\mathcal{M}$ on ground set E, the most useful here specifies its collection of *flats* $\mathbf{F} \subsetneq 2^E$, satisfying certain axioms. When ordered by inclusion the collection of flats $(\mathbf{F}, \subseteq)$ forms a geometric lattice; in an abuse of notation, we will use $\mathbf{F}$ to refer to both the lattice and the set. This lattice is ranked, with rank function denoted $\mathrm{rk}(F)$. The *rank* of the matroid $\mathcal{M}$ itself is defined to be $\mathrm{rk}(E)$, and we assume throughout that $\mathrm{rk}(E) = r+1$. An *automorphism* of the matroid $\mathcal{M}$ is any permutation $g : E \to E$ of the ground set E that carries flats to flats: for all F in $\mathbf{F}$ one has $g(F)$ in $\mathbf{F}$. Let $G = \mathrm{Aut}(\mathcal{M})$ denote the group of all automorphisms of $\mathcal{M}$. Since such automorphisms respect the partial order via inclusion on $\mathbf{F}$, they also preserve the rank function: $\mathrm{rk}(g(F)) = \mathrm{rk}(F)$ for all g in G and F in $\mathbf{F}$.

2.1 Chow Rings

As defined in the Introduction, Feichtner and Yuzvinsky [4] introduced the Chow ring $A(\mathcal{M})$ of a matroid $\mathcal{M}$.

Definition 1. The *Chow ring* $A(\mathcal{M})$ of a matroid $\mathcal{M}$ is the quotient Z-algebra

$$A(\mathcal{M}) := S/(I+J)$$

where $S = \mathbb{Z}[x_F]$ is a polynomial ring having one variable x_F for each nonempty flat $F \in \mathbf{F} \setminus \{\varnothing\}$, and where I, J are the following ideals of S:

- I is generated by products $x_F x_{F'}$ where F, F' are incomparable flats,
- J is generated by the linear elements $\sum_{a \in F \in \mathbf{F}} x_F$ for each atom a in the lattice $\mathbf{F}$.

The presentation of the Chow ring $A(\mathcal{M})$ only uses the information about the partial order on the lattice of flats $\mathbf{F}$ has some consequences. For one, the Chow ring depends only upon the associated *simple matroid* of $\mathcal{M}$ (one without loops and parallel edges); hence, we assume all matroids to be such. Another consequence is that any element g in $G = \mathrm{Aut}(\mathcal{M})$ will send the generators of the ideals I, J to other such generators. Thus $I+J$ is a G-stable ideal, and G acts on $A(\mathcal{M})$.

Note if one considers $S = \mathbb{Z}[x_F]$ as a graded $\mathbb{Z}$-algebra, then the ideals I, J are generated by homogeneous elements. Hence the quotient $A(\mathcal{M}) = S/(I+J)$ inherits the structure of a graded $\mathbb{Z}$-algebra $A(\mathcal{M}) = \bigoplus_{k=0}^{\infty} A^k$. Since the action of $G = \mathrm{Aut}(\mathcal{M})$ on the Chow ring preserves rank and hence degree, both $A(\mathcal{M})$ and each homogeneous component A^k become $\mathbb{Z}G$-modules.

The following crucial result appears as [4, Thm. 2]. To state it, define an *FY-monomial order* on $S = \mathbb{Z}[x_F]_{\varnothing \neq F \in \mathbf{F}}$ to be any monomial order based on a linear order of the variables with $x_F > x_{F'}$ if $F \subsetneq F'$.

Theorem 2. *Given a matroid $\mathcal{M}$ and any FY-monomial order on $S = \mathbb{Z}[x_F]_{\varnothing \neq F \in \mathbf{F}}$, the ideal $I + J$ presenting $A(\mathcal{M}) = S/(I+J)$ has a monic Gröbner basis $\{g_{F,F'}\}$ indexed by $F \neq F'$ in $\mathbf{F}$, with $g_{F,F'}$ and their initial terms $\mathrm{in}_{\prec}(g_{F,F'})$ as shown here:*

condition on $F \neq F'$ in $\mathbf{F}$	$g_{F,F'}$	$\mathrm{in}_{\prec}(g_{F,F'})$
F, F' non-nested	$x_F x_{F'}$	$x_F x_{F'}$
$\varnothing \neq F \subsetneq F'$	$x_F \left(\sum_{\substack{F'' \in \mathbf{F}: \\ F'' \supseteq F'}} x_{F''} \right)^{\mathrm{rk}(F') - \mathrm{rk}(F)}$	$x_F \cdot x_{F'}^{\mathrm{rk}(F') - \mathrm{rk}(F)}$
$\varnothing = F \subsetneq F'$	$\left(\sum_{\substack{F'' \in \mathbf{F}: \\ F'' \supseteq F'}} x_{F''} \right)^{\mathrm{rk}(F')}$	$x_{F'}^{\mathrm{rk}(F')}$

Corollary 3. *([4, Cor. 1]) For a matroid $\mathcal{M}$ of rank $r+1$, the Chow ring $A(\mathcal{M})$ has these properties:*

(i) $A(\mathcal{M})$ is free as a $\mathbb{Z}$-module, with $\mathbb{Z}$-basis given by the set of what we call FY-monomials

$$\mathrm{FY} := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} : F_1 \subsetneq \cdots \subsetneq F_\ell \in \mathbf{F}, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\}. \quad (2.1)$$

(ii) $A(\mathcal{M})$ vanishes in degrees strictly above r, that is, $A(\mathcal{M}) = \bigoplus_{k=0}^{r} A^k$.

(iii) A^r has $\mathbb{Z}$-basis $\{x_E^r\}$, and so a $\mathbb{Z}$-module isomorphism $\deg : A^r \longrightarrow \mathbb{Z}$ *sending $x_E^r \longmapsto 1$.*

Assertions (ii) and (iii) follow immediately from the first. To see this, note that the typical FY-monomial $x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell}$, has total degree

$$\sum_{i=1}^{\ell} m_i \leq \sum_{i=1}^{\ell} (\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1) = \mathrm{rk}(F_\ell) - \ell \leq (r+1) - 1 = r + 1.$$

Equality here occurs only if $\ell = 1$ and $F_\ell = E$, in which case the FY-monomial is x_E^r.

For any matroid automorphism g, the fact that $\mathrm{rk}(g(F)) = \mathrm{rk}(F)$ for every flat F in $\mathbf{F}$ implies that g sends any FY-monomial to another FY-monomial:

$$x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} \overset{g}{\longmapsto} x_{g(F_1)}^{m_1} x_{g(F_2)}^{m_2} \cdots x_{g(F_\ell)}^{m_\ell}.$$

This has a corollary, inspired by work of H.-C. Liao on Boolean matroids [8, Thm. 2.5].

Corollary 4. *For any matroid $\mathcal{M}$, the group $G = \mathrm{Aut}(\mathcal{M})$ permutes the set* FY*, as well as its subset of degree k monomials* $\mathrm{FY}^k \subset \mathrm{FY}$*. Consequently, the $\mathbb{Z}G$-modules on the Chow ring $A(\mathcal{M})$ and each of its homogeneous components A^k lift to G-permutation representations on* FY *and each* FY^k.

Example 1. Let $\mathcal{M} = U_{4,5}$ be the uniform matroid of rank 4 on $E = \{1,2,3,4,5\}$, associated to a list of 5 *generic* vectors v_1, v_2, v_3, v_4, v_5 in a 4-dimensional vector space, so that any quadruple v_i, v_j, v_k, v_ℓ is linearly independent. One has these flats of various ranks:

rank	flats $F \in \mathbf{F}$
0	$\varnothing$
1	$1, 2, 3, 4, 5$
2	$12, 13, 14, 15, 23, 24, 25, 34, 35, 45$
3	$123, 124, 125, 134, 135, 145, 234, 235, 245, 345$
4	$E = 12345$

The Chow ring $A(\mathcal{M}) = S/(I+J)$, where $S = \mathbb{Z}[x_i, x_{jk}, x_{\ell mn}, x_E]$ with $\{i\}, \{j,k\}, \{\ell, m, n\}$ running through all one, two and three-element subsets of $E = \{1,2,3,4,5\}$, and

$$I = \Big(x_F x_{F'} \Big)_{F \not\subset F', F' \not\subset F}, \qquad J = \Bigg(x_i + \sum_{\substack{1 \leq j < k \leq 5 \\ i \in \{j,k\}}} x_{jk} + \sum_{\substack{1 \leq \ell < m < n \leq 5 \\ i \in \{\ell, m, n\}}} x_{\ell mn} \; + x_E \Bigg)_{i=1,2,3,4,5}.$$

The FY-monomial bases for A^0, A^1, A^2, A^3 are shown here, together with the G-equivariant maps λ:

$$\begin{array}{ccccccc}
\mathbf{FY^0} & & \mathbf{FY^1} & & \mathbf{FY^2} & & \mathbf{FY^3} \\
1 & \overset{\lambda}{\longmapsto} & x_E & \overset{\lambda}{\longmapsto} & x_E^2 & & x_E^3 \\
 & & x_{ijk} & \overset{\lambda}{\longmapsto} & x_{ijk}^2 & & \\
 & & x_{ij} & \overset{\lambda}{\longmapsto} & x_{ij} \cdot x_E & &
\end{array}$$

Thus in this case, the ranks of the free $\mathbb{Z}$-modules (A^0, A^1, A^2, A^3) form the symmetric, unimodal sequence $(a_0, a_1, a_2, a_3) = (1, 21, 21, 1)$. Here the bijection $\pi : \mathrm{FY}^0 \to \mathrm{FY}^3$ necessarily maps $1 \longmapsto x_E^3$, and the bijection $\pi : \mathrm{FY}^1 \to \mathrm{FY}^2$ coincides with the map $\lambda : \mathrm{FY}^1 \to \mathrm{FY}^2$ above.

2.2 The Kähler package

The following theorem on the Kähler package for $A(\mathcal{M})$ compiles some of the main results of the work of Adiprasito, Huh and Katz [1].

Theorem 5. *For a matroid $\mathcal{M}$ of rank $r+1$, the Chow ring $A(\mathcal{M})$ satisfies the Kähler package:*

- *(Poincaré duality) For every $k \leq \frac{r}{2}$, one has a perfect $\mathbb{Z}$-bilinear pairing*

$$\begin{aligned} A^k \times A^{r-k} &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto \deg(a \cdot b) \end{aligned}$$

- *(Hard Lefschetz) Tensoring over $\mathbb{Z}$ with $\mathbb{R}$, the (real) Chow ring $A_{\mathbb{R}}(\mathcal{M}) = \sum_{k=0}^{r} A_{\mathbb{R}}^k$ contains **Lefschetz elements** ω in $A_{\mathbb{R}}^1$, meaning that $a \mapsto a \cdot \omega^{r-2k}$ is an $\mathbb{R}$-linear isomorphism $A_{\mathbb{R}}^k \to A_{\mathbb{R}}^{r-k}$ for $k \leq \frac{r}{2}$. In particular, multiplication by ω is an injection $A_{\mathbb{R}}^k \to A_{\mathbb{R}}^{k+1}$ for $k < \frac{r}{2}$.*

- *(Hodge-Riemann-Minkowski inequalities) The Lefschetz elements ω define quadratic forms $a \longmapsto (-1)^k \deg(a \cdot \omega^{r-2k} \cdot a)$ on $A_{\mathbb{R}}^k$ that become positive definite upon restriction to the kernel of the map $A_{\mathbb{R}}^k \longrightarrow A_{\mathbb{R}}^{r-k+1}$ that sends $a \longmapsto a \cdot \omega^{r-2k+1}$.*

In fact, they show that one obtains a Lefschetz element ω whenever $\omega = \sum_{\varnothing \neq F \in \mathbf{F}} c_F x_F$ has coefficients c_F coming from restricting to $\mathbf{F}$ any function $A \mapsto c_A$ that maps $2^E \to \mathbb{R}$ and satisfies these two properties:

(1) the *strict submodular inequality* $c_A + c_B > c_{A \cap B} + c_{A \cup B}$ for all $A \neq B$, and

(2) $c_\varnothing = c_E = 0$.

This has consequences for G acting on $A(\mathcal{M})$ and each A^k. Properties (1) and (2) above are refined by Theorem 1's parts (i) and (ii) respectively.

Corollary 6. *For any matroid $\mathcal{M}$, one has an isomorphism of $\mathbb{Z}G$-modules $A^{r-k} \to A^k$ for each $k \leq \frac{r}{2}$ and $\mathbb{R}G$-module maps $A_{\mathbb{R}}^k \to A_{\mathbb{R}}^{k+1}$ which are injective for $k < \frac{r}{2}$.*

3 Results

We recall the statement of the theorem, involving the FY-monomial $\mathbb{Z}$-basis for $A(\mathcal{M})$ in Corollary 3:

$$\mathrm{FY} := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} : \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell \text{ in } \mathbf{F}, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\}$$

This also means that the FY-monomials FY^k of degree k form a $\mathbb{Z}$-basis for A^k for each $k = 0, 1, 2, \ldots, r$.

Theorem 1 *For every matroid $\mathcal{M}$ of rank $r+1$, there exist*

(i) G-equivariant bijections $\pi : \mathrm{FY}^k \xrightarrow{\sim} \mathrm{FY}^{r-k}$ for $k \leq \frac{r}{2}$, and

(ii) G-equivariant injections $\lambda : \mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$ for $k < \frac{r}{2}$.

The prove this, we organize monomials according to the fibers of the following map.

Definition 2. Define the *extended support* $\mathrm{supp}_+(a) \subset \mathbf{F}$ of an FY-monomial $a = x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell}$ by

$$\mathrm{supp}_+(a) := \{F_1, \ldots, F_\ell\} \cup \{E\} = \begin{cases} \{F_1, \ldots, F_\ell\} \cup \{E\} & \text{if } F_\ell \subsetneq E, \\ \{F_1, \ldots, F_\ell\} & \text{if } F_\ell = E. \end{cases}$$

Define a partial order $<_+$ on the FY-monomials in which $a <_+ b$ if a divides b and $\mathrm{supp}_+(a) = \mathrm{supp}_+(b)$.

For integers $p < q$, let $[p, q]$ denote the integer linear order inclusively from p to q. Given a sequence of such pairs $p_i < q_i$ for $i = 1, 2, \ldots, m$, let

$$\prod_{i=1}^{n} [p_i, q_i] = [p_1, q_1] \times [p_2, q_2] \times \cdots \times [p_m, q_m] \tag{3.1}$$

denote their Cartesian product, partially ordered componentwise.

Proposition 7. *For any nested flag $\{F_1 \subsetneq \cdots \subsetneq F_\ell \subsetneq E\}$ in $\mathbf{F}$ containing E, with conventions $F_0 := \varnothing$ and $F_{\ell+1} := E$, the fiber $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}$ is the set of monomials $\{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} x_E^{m_{\ell+1}}\}$ satisfying these inequalities:*

$$\begin{aligned} &1 \leq m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i \leq \ell, \\ &0 \leq m_{\ell+1} \leq \mathrm{rk}(E) - \mathrm{rk}(F_\ell) - 1 = r - \mathrm{rk}(F_\ell). \end{aligned}$$

Consequently, the minimum and maximum degree of monomials in $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}$ are ℓ and $r - \ell$, and one has a poset isomorphism

$$\begin{array}{ccc} (\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}, <_+) & \longrightarrow & \displaystyle\prod_{i=1}^{\ell} [1, \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1] \ \times \ [0, r - \mathrm{rk}(F_\ell)] \\ x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_\ell}^{m_\ell} x_E^{m_{\ell+1}} & \longmapsto & (m_1, m_2, \ldots, m_\ell, m_{\ell+1}). \end{array}$$

Most assertions of the proposition are immediate from the definition of the order on FY-monomials $<_+$ and the map supp_+. The minimum and maximum degrees of monomials in $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\})$ are achieved by

$$\deg(x_{F_1}^1 x_{F_2}^1 \cdots x_{F_\ell}^1 x_E^0) = \ell \quad \text{and}$$

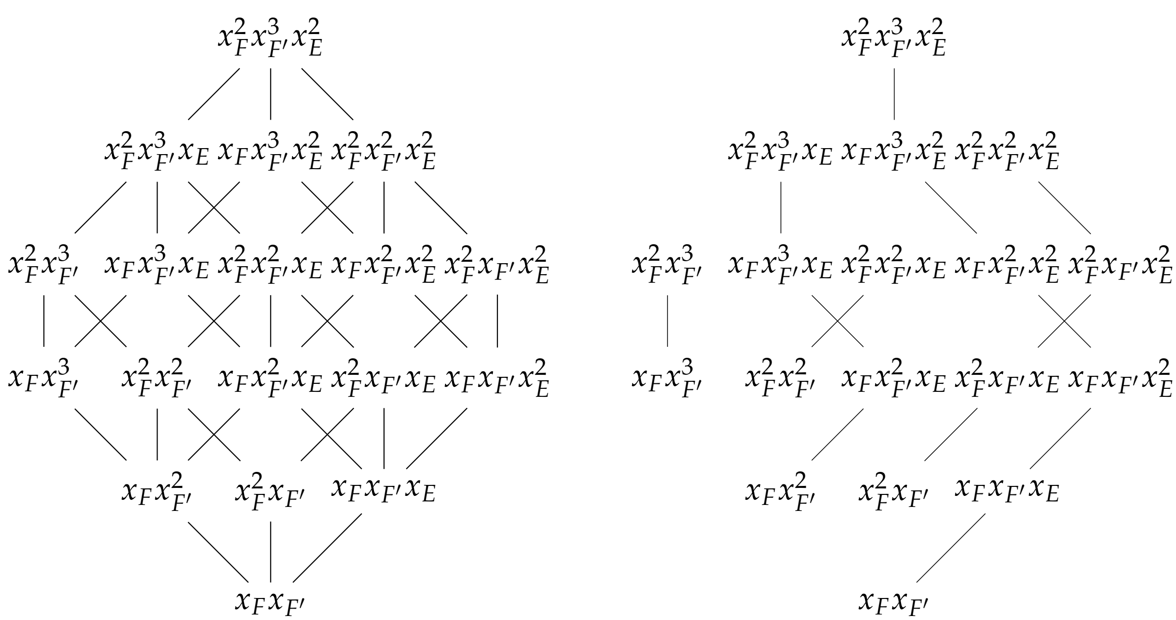
$$\begin{aligned}\deg\left(\prod_{i=1}^{\ell} x_{F_i}^{\mathrm{rk}(F_i)-\mathrm{rk}(F_i)-1} \cdot x_E^{\mathrm{rk}(E)-\mathrm{rk}(F_\ell)-1)}\right) &= \sum_{i=1}^{\ell+1} \big(\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\big) \\ &= \mathrm{rk}(E) - (\ell+1) \\ &= r - \ell.\end{aligned}$$

The idea behind the proof of Theorem 1 stems from the observation that all products of chains, as in (3.1), have *symmetric chain decompositions*, which can then be pulled back to each fiber $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}$.

Definition 3. A *symmetric chain decomposition (SCD)* of a finite ranked poset P of rank r is a disjoint decomposition $P = \bigsqcup_{i=1}^{t} C_i$ in which each C_i is a totally ordered subset containing one element of each rank $\{\rho_i, \rho_i+1, \ldots, r-\rho_i-1, r-\rho_i\}$ for some $\rho_i \leq \lfloor \frac{r}{2} \rfloor$.

It is not hard to check that when posets P_1, P_2 each have an SCD, then so does their Cartesian product. In particular, all products of chains have an SCD. *Fix one such SCD for each product poset in* (3.1), *once and for all,* and use the isomorphisms from Proposition 7 to induce an SCD on each fiber $\mathrm{supp}_+^{-1}\{F_1, \ldots, F_\ell, E\}$.

Example 2. Assume $\mathcal{M}$ has $\mathrm{rk}(E) = 10 = r+1$ with $r = 9$, and one has a pair of nested flats $F \subset F'$ with $\mathrm{rk}(F) = 3, \mathrm{rk}(F') = 7$. Then the poset $\mathrm{supp}_+^{-1}\{F, F', E\}$ and one choice of SCD for it look as follows:



4 Further questions and conjectures

So far, we have mentioned that the unimodality statement (1.2), asserting for $k < \frac{r}{2}$ that one has $a_k \leq a_{k+1}$, is weaker than the statement in Corollary 6 asserting that there are injective $\mathbb{R}G$-module maps $A^k_{\mathbb{R}} \to A^{k+1}_{\mathbb{R}}$, which is weaker than Theorem 1(ii) asserting that there are injective G-equivariant maps of the G-sets $\mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$. Here, we wish to consider not only unimodality for $(a_0, a_1, \ldots, a_r)$, but other properties like *log-concavity*, *the Pólya frequency property*, and how to similarly lift them to statements regarding $\mathbb{R}G$-modules and G-permutation representations. In phrasing this, it helps to consider the character and Burnside rings.

Definition 4. For a finite group G, its *virtual (complex) character ring* $R_{\mathbb{C}}(G)$ is the free $\mathbb{Z}$-submodule of the ring of (conjugacy) class functions $\{f : G \to \mathbb{C}\}$, having as a $\mathbb{Z}$-basis the irreducible complex characters of G. If a character χ can be written as a positive linear combination of irreducible characters of G, we say that χ is a *genuine character*, and write $\chi \geq_{R_{\mathbb{C}}(G)} 0$.

Similarly, one can define its *Burnside ring* $B(G)$ by now having as basis the isomorphism classes $[X]$ of finite G-sets X. Then $B(G)$ is the $\mathbb{Z}$-module that mods out by the span of all elements $[X \sqcup Y] - ([X] + [Y])$ and if $b \in B(G)$ can be written as a positive linear combination of isomorphism classes, then b is a *genuine permutation representation*, and $b \geq_{B(G)} 0$.

4.1 PF sequences and log-concavity

For a sequence of *positive* real numbers $(a_0, a_1, \ldots, a_r)$, the property of *unimodality* lies at the bottom of a hierarchy of concepts

$$\begin{array}{ccccccccc} \text{unimodal} & \Leftarrow & PF_2 & \Leftarrow & PF_3 & \Leftarrow & \cdots & \Leftarrow & PF_\infty \\ & & \| & & & & & & \| \\ & & \text{(strongly) log-concave} & & & & & & PF \end{array} \tag{4.1}$$

which we next review, along with their equivariant and Burnside ring extensions.

Definition 5. Say a sequence of positive reals $(a_0, a_1, \ldots, a_r)$ is *strongly log-concave* (or PF_2) if $0 \leq i \leq j \leq k \leq \ell \leq r$ and $i + \ell = j + k$ implies

$$a_i a_\ell \leq a_j a_k, \text{ or equivalently, } \det \begin{bmatrix} a_j & a_\ell \\ a_i & a_k \end{bmatrix} \geq 0.$$

For $\ell = 2, 3, 4, \ldots$, say that the sequence is *PF_ℓ* if the associated (infinite) *Toeplitz matrix*

$$T(a_0, \ldots, a_r) := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{r-1} & a_r & 0 & 0 & \cdots \\ 0 & a_0 & a_1 & \cdots & a_{r-2} & a_{r-1} & a_r & 0 & \cdots \\ 0 & 0 & a_0 & \cdots & a_{r-3} & a_{r-2} & a_{r-1} & a_r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has all *nonnegative* square minor subdeterminants of size $m \times m$ for $1 \leq m \leq \ell$. Say that the sequence is a *Pólya frequency sequence* (or PF_∞, or just *PF*) if it is PF_ℓ for all $\ell = 2, 3, \ldots$.

Definition 6. For a finite group G and (genuine, nonzero) $\mathbb{C}G$-modules $(A^0, A^1, \ldots, A^r)$, define the analogous notions of *equivariant unimodality*, *equivariant strong log-concavity*, *equivariant PF_r or PF_∞* by replacing the numerical inequalities in Definition 5 by inequalities in $R_{\mathbb{C}}(G)$, or, similarly, one can define all these concepts to be Burnside if these inequalities are in the Burnside ring $B(G)$.

We've seen for Chow rings $A(\mathcal{M})$ of rank $r+1$ matroids $\mathcal{M}$, and $G = \mathrm{Aut}(\mathcal{M})$, the sequence $(a_0, a_1, \ldots, a_r)$ with $a_k := \mathrm{rk}_{\mathbb{Z}} A_k$ is *unimodal*; after tensoring with $\mathbb{C}$, the sequence of $\mathbb{C}G$-modules $(A^0_{\mathbb{C}}, A^1_{\mathbb{C}}, \ldots, A^r_{\mathbb{C}})$ is *equivariantly unimodal*; and the sequence of G-sets $(\mathrm{FY}^0, \mathrm{FY}^1, \ldots, \mathrm{FY}^r)$ is *Burnside unimodal*.

Conjecture 1. *In the Chow ring of a rank $r+1$ matroid $\mathcal{M}$, one has that*

(i) (Ferroni-Schröter [6, Conj. 10.19]) $(a_0, \ldots, a_r)$ is PF_∞.

(ii) $(A^0_{\mathbb{C}}, \ldots, A^r_{\mathbb{C}})$ is equivariantly PF_∞.

(iii) $(\mathrm{FY}^0, \ldots, \mathrm{FY}^r)$ is Burnside PF_2 (Burnside log-concave).

Of course, in Conjecture 1, assertion (ii) implies assertion (i). However the same is not true of assertion (iii): it would only imply the weaker PF_2 part of the conjectural assertion (ii), and only imply the PF_2 part of Ferroni and Schröter's assertion (i), but not their PF_∞ assertions. We have some evidence for the following two further conjectures.

Conjecture 2. *For a Boolean matroid $\mathcal{M}$ of rank n and $i \leq j \leq k \leq \ell$ with $i + \ell = j + k$, not only is the element $[\mathrm{FY}^j][\mathrm{FY}^k] - [\mathrm{FY}^i][\mathrm{FY}^\ell] \geq_{B(\mathfrak{S}_n)} 0$, so that it is a genuine permutation representation, but furthermore one whose orbit-stabilizers are all Young subgroups $\mathfrak{S}_\lambda$.*

Conjecture 3. *For a matroid $\mathcal{M}$ of rank $r+1$ with Chow ring $A(\mathcal{M})$, and any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $m := \sum_i \alpha \leq r$, the analogous Toeplitz minors of G-sets have*

$$\det \begin{bmatrix} [\mathrm{FY}^{\alpha_1}] & [\mathrm{FY}^{\alpha_1+\alpha_2}] & [\mathrm{FY}^{\alpha_1+\alpha_2+\alpha_3}] & \cdots & [\mathrm{FY}^{m}] \\ [\mathrm{FY}^{0}] & [\mathrm{FY}^{\alpha_2}] & [\mathrm{FY}^{\alpha_2+\alpha_3}] & \cdots & [\mathrm{FY}^{m-\alpha_1}] \\ 0 & [\mathrm{FY}^{0}] & [\mathrm{FY}^{\alpha_3}] & \cdots & [\mathrm{FY}^{m-(\alpha_1+\alpha_2)}] \\ 0 & 0 & & & \vdots \\ \vdots & \vdots & & & [\mathrm{FY}^{\alpha_{\ell-1}+\alpha_\ell}] \\ 0 & 0 & \cdots & [\mathrm{FY}^{0}] & [\mathrm{FY}^{\alpha_\ell}] \end{bmatrix} \geq_{B(G)} 0.$$

4.2 Further Questions

So far, we have focused on the Chow ring of a matroid $\mathcal{M}$ using its *maximal* building set.

One relevant example of such a building set is the *minimal* building set, which is stable under the full automorphism group $\mathrm{Aut}(\mathcal{M})$, and which arises, for example, in the study of the moduli space $\overline{M}_{0,n}$ of genus 0 curves with n marked points; see, e.g., Dotsenko [3], Gibney and Maclagan [7].

Question 4. *Does the analogue of Theorem 1 hold for the Chow ring of a matroid $\mathcal{M}$ with respect to any G-stable building set? In particular, what about the minimal building set?*

In [9, Lem. 3.1], Stembridge provides a generating function for the symmetric group representations on each graded component of the Chow ring for all Boolean matroids; see also Liao [8]. Furthermore, Stembridge's expression exhibits them as *permutation representations*, whose orbit-stabilizers are all *Young subgroups* in the symmetric group.

Question 5. *Can one provide such explicit expressions as permutation representations for other families of matroids with symmetry?*

Hilbert functions $(a_0, a_1, \ldots, a_r)$ for Chow rings of rank $r+1$ matroids are not only symmetric and unimodal, but satisfy the stronger condition of *γ-positivity*, as shown by : one has *nonnegativity* for all coefficients $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{r}{2} \rfloor})$ appearing in the unique expansion

$$\sum_{i=0}^{r} a_i t^i = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \gamma_i \; t^i (1+t)^{r-2i}.$$

This has been shown, independently by Ferroni, Matherne, Stevens and Vecchi [5, Thm. 3.25] and by Wang (see [5, p. 29]), that the γ-positivity for Hilbert series of Chow rings of matroids follows from results of [2].

One also has the notion of *equivariant γ-positivity* for a sequence of G-representations $(A_0, A_1, \ldots, A_r)$: upon replacing each a_i with the element $[A_i]$ of $R_{\mathbb{C}}(G)$, one asks that the uniquely defined coefficients γ_i in $R_{\mathbb{C}}(G)$ have $\gamma_i \geq_{R_{\mathbb{C}}(G)} 0$.

Conjecture 6. *For any matroid $\mathcal{M}$ of rank $r+1$ and its Chow ring $A(\mathcal{M}) = \bigoplus_i A^i$, the sequence of G-representations $(A^0_{\mathbb{C}}, A^1_{\mathbb{C}}, \ldots, A^r_{\mathbb{C}})$ is equivariantly γ-positive.*

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Diagram model for the Okada algebra and monoid

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Abstract.

It is well known that the Young lattice is the Bratelli diagram of the symmetric groups expressing how irreducible representations restrict from $\mathfrak{S}_N$ to $\mathfrak{S}_{N-1}$. In 1988, Stanley discovered a similar lattice called the Young-Fibonacci lattice which was realized as the Bratelli diagram of a family of algebras by Okada in 1994.

In this paper, we realize the Okada algebra and its associated monoid using a labeled version of Temperley-Lieb arc-diagrams. We prove in full generality that the dimension of the Okada algebra is $n!$. In particular, we interpret a natural bijection between permutations and labeled arc-diagrams as an instance of Fomin's Robinson-Schensted correspondence for the Young-Fibonacci lattice. We prove that the Okada monoid is aperiodic and describe its Green relations. Lifting those results to the algebra allows us to construct a cellular basis of the Okada algebra.

Résumé. Il est bien connu que le treillis de Young peut s'interpréter comme le diagramme de Bratelli des groupes symétriques, décrivant, par exemple, comment les représentations irréductibles se restreignent de $\mathfrak{S}_n$ à $\mathfrak{S}_{n-1}$. En 1975, Stanley a découvert un treillis similaire appelée treillis de Young-Fibonacci qui a été interprété comme le diagramme de Bratelli d'une famille d'algèbres par Okada en 1994.

Dans cet article, nous réalisons l'algèbre d'Okada et le monoïde associé grâce à une version étiquetée des diagrammes d'arcs du monoïde de Jones et de l'algèbre de Tempeley-Lieb. Nous prouvons en toute généralité que l'algèbre d'Okada est de dimension $n!$. En particulier, nous interprétons la bijection naturelle entre les permutations et les diagrammes d'arcs comme une instance de la correspondance de Robinson-Schensted-Fomin associée au treillis de Young-Fibonacci. Nous prouvons que le monoïde est apériodique et décrivons ses relations de Green. En relevant, ces dernières à l'algèbre nous en construisant une base cellulaire.

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1 Introduction

The theory of 1-differential posets was developed by R. Stanley [7] as a framework for generalizing the Robinson-Schensted correspondence beyond the combinatorics of the Young lattice $\mathbb{Y}$ of integer partitions. A similar undertaking was made by S. Fomin in his work on dual graded graphs and growth processes, where the later technique was used to construct an explicit RS-correspondence for Stanley's *Young-Fibonacci* lattice $\mathbb{YF}$ [1, 7]. Both $\mathbb{Y}$ and $\mathbb{YF}$ are 1-differential and they are the only lattices having this property. Fomin's approach involves a Fibonacci version of standard tableaux; a notion later examined independently by T. Roby, K. Killpatrick, and J. Nzeutchap (whose formulation by-passes the growth construction altogether), see [5] and the references therein.

S. Okada [6] showed that the $\mathbb{YF}$-lattice supports a theory of *clone symmetric functions* with analogues of the classical bases (e.g. complete homogeneous, Schur, and power-sum symmetric functions) as well as a $\mathbb{YF}$-variant of the Littlewood-Richardson rule. The clone theory appears in Goodman-Kerov's determination of the *Martin boundary* of the $\mathbb{YF}$-lattice [2] and is related to various random processes.

The Okada algebras $\{\mathsf{O}_N(X,Y)\}_{N\geq 0}$ were introduced by S. Okada as a counterpart to the clone theory, and occupy a role similar to that played by the symmetric groups in the classical theory of symmetric functions. Okada algebras are finite dimensional, associative, and depend on parameters $X = (x_1, \ldots, x_{N-1})$ and $Y = (y_1, \ldots, y_{N-2})$. When those parameters are generic, they are semi-simple and their branching rule, which describes how irreducible representations restrict from $\mathsf{O}_N(X,Y)$ to $\mathsf{O}_{N-1}(X,Y)$, is expressed by the covering relations of the $\mathbb{YF}$-lattice.

In this paper we realize the Okada algebra $\mathsf{O}_N(X,Y)$ as a diagram algebra with a multiplicative/monoidal basis expressed in terms of certain arc-labeled, non-crossing perfect matchings (as appear in both the Temperley-Lieb and Martin-Saleur Blob algebras [4]). Like most diagram algebras, this basis is cellular and affords us with a novel, diagrammatic presentation of the irreducible representations of $\mathsf{O}_N(X,Y)$ (i.e. as *cell modules*). We interpret Fomin's RS-correspondence diagrammatically. This involves constructing a bijection between saturated chains in the $\mathbb{YF}$-lattice (presented in terms of sequences of *Fibonacci sets*) and *Okada half arc-diagrams*. In addition we examine the structure theory of the Okada algebra and monoid via a *dominance order* on Fibonacci sets.

2 Background

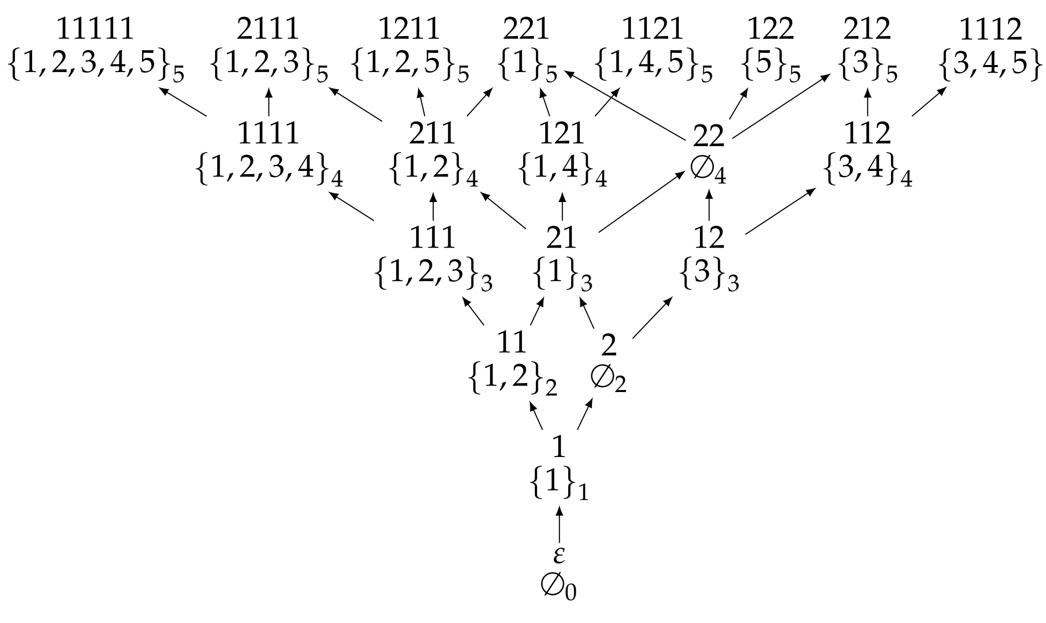
Throughout this paper N denotes a non-negative integer. We denote by $[N]$ the set $\{1, \ldots, N\}$. We often write negative numbers as overlined numbers such as $\overline{4}$. The cardinality of a set S is denoted $\# S$. For a non-negative integer N we endow $[N] \cup [\overline{N}]$ with the total order $\{1 < 2 < \cdots < N < \overline{N} < \cdots < \overline{2} < \overline{1}\}$. Overlining numbers which are negative should also help the reader remember this unusual ordering.

Stanley's original construction of the Young-Fibonacci lattice [7] involves endowing the set of *Fibonacci words*, i.e. binary words in the alphabet $\{1,2\}$, with a partial order. We present an alternative description using *Fibonacci sets*.

Definition 2.1. *A* Fibonacci set of rank N *is a subset* $S = \{s_1 < s_2 < \cdots < s_k\}$ *of* $[N]$ *whose size* k *has the same parity as* N *and such that* s_ℓ *have the same parity as* ℓ*. We write* $\mathbb{YFS}_N$ *for the collection of all rank* N *Fibonacci sets and* $\mathbb{YFS}$ *for the* disjoint *union of* $\mathbb{YFS}_N$ *as* N *varies.*

The entire interval $[N]$ itself is always a Fibonacci set of rank N, while $\emptyset$ is a Fibonacci set only when N is even. We emphasize on the fact that in $\mathbb{YFS}$ the set $\{1,2,5\}$ of rank 5 is not the same Fibonacci set as $\{1,2,5\}$ of rank 7. When they need to be distinguished we include N as a subscript, as in $\{1,2,5\}_5$ and $\{1,2,5\}_7$.

The covering relations which generate the lattice structure on $\mathbb{YFS}$ are defined by $S \lhd T$ if and only if $S \in \mathbb{YFS}_{N-1}$ and $T \in \mathbb{YFS}_N$ and one of these two sets can be obtained from the other one by removing its largest element. Stanley's description is equivalent to ours through the bijection sending a binary word w to the set of the sums of its suffixes whose first digit is 1. The Hasse diagram of $\mathbb{YFS}$ upto rank 5 is illustrated below.



Definition 2.2. *Fix a positive integer* N*. Given a field* $\mathbb{K}$*, fix also* $X = (x_1, \ldots, x_{N-1})$ *and* $Y = (y_1, \ldots, y_{N-2})$ *two sequences of parameters in* $\mathbb{K}$*. The* Okaka algebra $\mathsf{O}_N(X,Y)$ *is the algebra generated by* $\{\mathsf{E}_i \mid i = 1 \ldots N-1\}$ *with the relations*

$$\mathsf{E}_i^2 = x_i \mathsf{E}_i \qquad 1 \le i \le N-1, \tag{$I(X,Y)$}$$

$$\mathsf{E}_i \mathsf{E}_j = \mathsf{E}_j \mathsf{E}_i \qquad |i-j| \ge 2, \tag{C(X,Y)}$$

$$\mathsf{E}_{i+1}\mathsf{E}_i\mathsf{E}_{i+1} = y_i \mathsf{E}_{i+1} \qquad 1 \le i \le N-2, \tag{S(X,Y)}$$

If all the X's and the Y's are equal to 1, the Okada algebra is actually the algebra of a monoid; we call this the *Okaka Monoid* and denote it O_N. Recall that setting all $y_i := 1$ and all $x_i := q$ and adding the extra relation $\mathsf{E}_i\mathsf{E}_{i+1}\mathsf{E}_{i+1} = \mathsf{E}_i$ defines the Temperley-Lieb algebra which is also a deformation of the algebra of a monoid called the Jones monoid (obtained when $q = 1$).

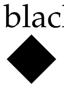
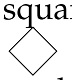
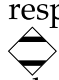
We now review some of Okada's results [6]: For generic values of the X and Y parameters $\mathsf{O}_N(X, Y)$ is semi-simple and its irreducible representations V_T correspond to rank N Fibonacci sets T. When V_T is restricted to the subalgebra $\mathsf{O}_{N-1}(X, Y) \subset \mathsf{O}_N(X, Y)$ it decomposes as a direct sum of irreducible representations V_S of $\mathsf{O}_{N-1}(X, Y)$ where $S \lhd T$ is a covering relation in $\mathbb{YFS}$.

The dimension of $\mathsf{O}_N(X, Y)$ is $N!$ and a basis for $\mathsf{O}_N(X, Y)$ can be constructed from permutations in the following way. Recall that the *code* of a permutation $\sigma \in \mathfrak{S}_N$ is $\mathsf{code}(\sigma) = (c_1, \dots c_N)$ where $c_i := \#\left\{j < i \mid \sigma^{-1}(j) > \sigma^{-1}(i)\right\}$. It is well known that the product $\prod_{i=1}^{n} \sigma_{i-1}\sigma_{i-2}\cdots\sigma_{i-c_i}$ taken from left to right, increasing with i, is the lexicographically minimal reduced factorization of σ into simple transpositions $\sigma_i = (i, i+1)$. Define $\mathsf{E}_\sigma := \prod_{i=1}^{n} \mathsf{E}_{i-1}\mathsf{E}_{i-2}\cdots\mathsf{E}_{i-c_i}$. Okada showed in [6] that the family $\{\mathsf{E}_\sigma \mid \sigma \in \mathfrak{S}_n\}$ is, generically, a basis of the Okada algebra. His proof, however, requires semi-simplicity and doesn't apply to degenerate specializations, such as the monoid case.

3 Diagram models for the Okada Monoid and Algebra

The goal of this section is to build a basis of the Okada algebra in full generality using rewriting techniques. Inspired by Viennot's theory of heaps of dimers [8], we use diagram rewriting rather than word rewriting.

A *diamond diagram* of rank-N is a trapezoidal arrangement of boxes with $N-1$ rows starting with a north-east diagonal and ending with south-east diagonal, where each box can be either black or white. The rows are indexed from bottom to top. The *reading* of such a diagram is the sequence $\underline{\mathbf{i}} = (i_1, \dots, i_\ell)$ obtained by recording the row index i_k of the k-th black box, starting on the left and reading each south-east diagonal from top to bottom. Associated to the reading $\underline{\mathbf{i}}$ is the monomial $\mathsf{E}_{\underline{\mathbf{i}}} := \mathsf{E}_{i_1}\cdots\mathsf{E}_{i_\ell}$ in the Okada algebra $\mathsf{O}_N(X, Y)$. We identify diagrams differing by empty south-east diagonals on the right. This identification is compatible with the reading and the associated monomial. See Figure 1 for some examples. Using rewriting techniques on such diagrams, one can show right away that the Okada algebra has dimension $N!$.

The relevant combinatorics becomes more transparent after we re-encode a diamond diagram as a *fully packed loop configuration* (FPLC). This is done by replacing black and white squares respectively with double U-turn and double horizontal squares:  $\to$  and  $\to$ . The paths fragments at the top and bottom of the trapezoid are completed by adding horizontal lines. The result is a set $\mathcal{C}$ of non-crossing planar loops and arcs. The endpoints of the arcs are situated on the left and right boundaries of the trapezoid and we number these endpoints, from bottom to top, with positive indices (on the left) and negative indices (overlined, on the right). See Figure 1 where we've colored some of the arcs in order to make the picture more legible.

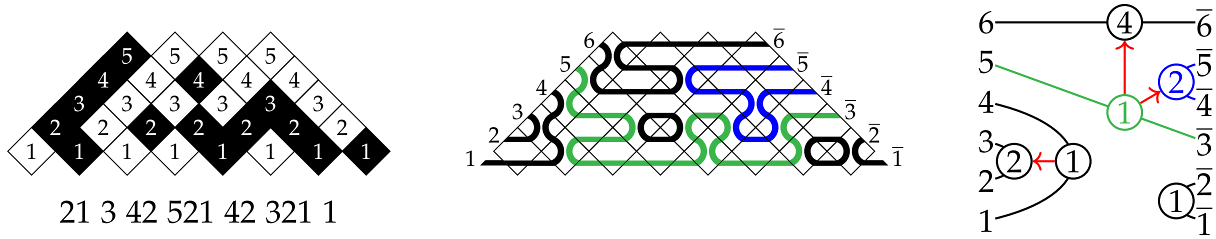
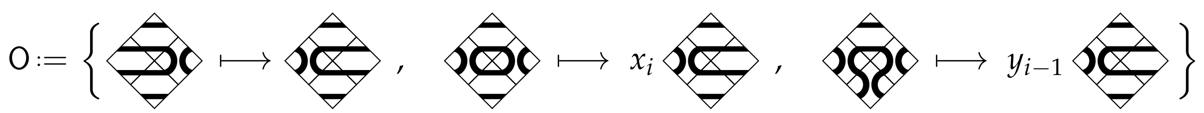


Figure 1: A diamond diagram, its reading together with the associated loop configuration and arc diagram

The horizontal arc segments in the -boxes occupy levels $1, \dots, N$ starting from the bottom of the trapezoid. The *height* of an arc/loop in an FPLC is the minimal level of the horizontal segments which form it. The height statistic of an arc/loop is invariant under the following local moves which implement the Okada relations:



The first and third moves can be viewed as restricted isotopies which transform arcs and loops horizontally and downward, while the second move erases loops. By repeatedly applying local moves, each FPLC can be brought to a *normal* form (*ie.* a configuration without any possible move). This normal form is independent of the sequence of moves used to obtain it and is therefore uniquely defined. It contains no arcs which go up and then down when followed in any direction; in particular, there are no loops. There is a bijection between permutations and normal forms which shows the following result:

Theorem 3.1. *For any N and for any specialization of the X, Y parameters, the map $\sigma \mapsto \mathsf{E}_\sigma$ is a bijection from $\mathfrak{S}_N$ to the monoid O_N and the family $(\mathsf{E}_\sigma)_{\sigma\in\mathfrak{S}_N}$ is a basis for the Okada algebra $\mathsf{O}_N(X, Y)$. In particular the dimension of the Okada algebra $\mathsf{O}_N(X, Y)$ is always $N!$.*

We abbreviate the structure of a FPLC $\mathcal{C}$ by removing its loops, labeling each arc by its respective height, and taking the isotopy class of what remains. We denote the result $[\mathcal{C}]$; an example is depicted in the third image of Figure 1. In view of the previous remarks $[\mathcal{C}] = [\mathcal{D}]$ whenever $\mathcal{C}$ and $\mathcal{D}$ are two FPLCs of rank N which are related by a sequences of moves. It turns out that this is actually an equivalence, providing us with a diagram model for the Okada algebra which we now examine.

Recall that a rank N *non-crossing arc-diagram* is a visualization of a perfect matching linking vertices $\{1, \dots, N\}$ and $\{\overline{1}, \dots, \overline{N}\}$ on the right and left boundaries of a rectangle by non-crossing arcs (drawn in the interior of the rectangle). A pair $\{a, b\}$ in the matching is depicted by an arc joining vertices $a, b \in [N] \cup [\overline{N}]$ and is denoted by $a \longmapsto b$. Either a, b are both positive, both negative, or else have different signs; in the later case we say the arc $a \longmapsto b$ is a *propagating*. Only the incidence relations of the arc-diagram are relevant, and so isotopic diagrams are considered equivalent.

An arc $a \longmapsto b$ is said to be *nested* in another arc $c \longmapsto d$ if $c < a < b < d$. Nesting defines a partial order on the arcs of a non-crossing arc diagram. The reader should be aware that any arc situated above a propagating arc is nested in the later. In particular, given any pair of propagating arcs, one arc must be nested in the other; consequently the nesting order is total when restricted to propagating arcs.

Definition 3.2. *A rank N Okada arc-diagram is a rank N non-crossing arc-diagram where each arc $a \longmapsto b$ is assigned an height-label $\mathbf{h}(a \longmapsto b)$ such that*

1. $\mathbf{h}(a \longmapsto b)$ *must be at least* 1 *and at most* $\min(|a|, |b|)$,
2. $\mathbf{h}(a \longmapsto b)$ *must have the same parity as* $\min(|a|, |b|)$,
3. $\mathbf{h}(a \longmapsto b) > \mathbf{h}(c \longmapsto d)$ *whenever $a \longmapsto b$ is nested in $c \longmapsto d$.*

The set of all Okada arc-diagrams of rank N is denoted $\mathfrak{A}_N$ and $\mathbb{C}\mathfrak{A}_N$ will denote the vector space spanned by all Okada arc-diagrams of rank N.

Definition 3.3. *Given $C, D \in \mathfrak{A}_N$ their composition $C \circ D$ is the diagram obtained by merging the right boundary nodes of C with the left boundary nodes of D and concatenating their respective arcs. The diagram $C \circ D$ may include loops, created from concatenated arc fragments of diagrams C and D. The height label of an arc/loop in $C \circ D$ is the minimum of the height labels of the arc fragments of C and D which comprise it. Let $[C \circ D]$ denoted the isotopy class of the height labeled, non-crossing arc-diagram obtained by removing all loops from the composition.*

Lemma 3.4. *If $C, D \in \mathfrak{A}_N$ then $[C \circ D] \in \mathfrak{A}_N$. Hence $\mathfrak{A}_N$ acquires the structure of a monoid denoted O_N whose unit is the identity Okada arc-diagram id_N which consists entirely of labeled propagating arcs $\mathbf{h}(a \longmapsto \overline{a}) = a$ for all $1 \leq a \leq N$.*

For simplicity we'll present the following results for arbitrary X-parameters together with the assumption that $y_k = 1$ for all $k \geq 1$. This is sufficiently generic to include the semi-simple case and all but the most degenerate specializations.

Definition 3.5. *Given $C, D \in \mathfrak{A}_N$ their product $C \cdot D$ is the element $X^{\underline{\ell}}[C \circ D]$ in $\mathbb{C}\mathfrak{A}_N$ where $X^{\underline{\ell}} = \prod_{k \geq 0} x_k^{\ell_k}$ and ℓ_k counts the number of loops γ in $C \circ D$ whose height label equals k.*

Lemma 3.6. *The product $C \cdot D$ endows $\mathbb{C}\mathfrak{A}_N$ with the structure of an associative, unital algebra, denoted $\widetilde{\mathsf{O}}_N(X, \mathbf{1})$. Using a rewriting rule, the Y-parameters can also be incorporated in the diagram product and $\widetilde{\mathsf{O}}_N(X, Y)$ will denote the corresponding diagram algebra. (See Figure 2 .)*

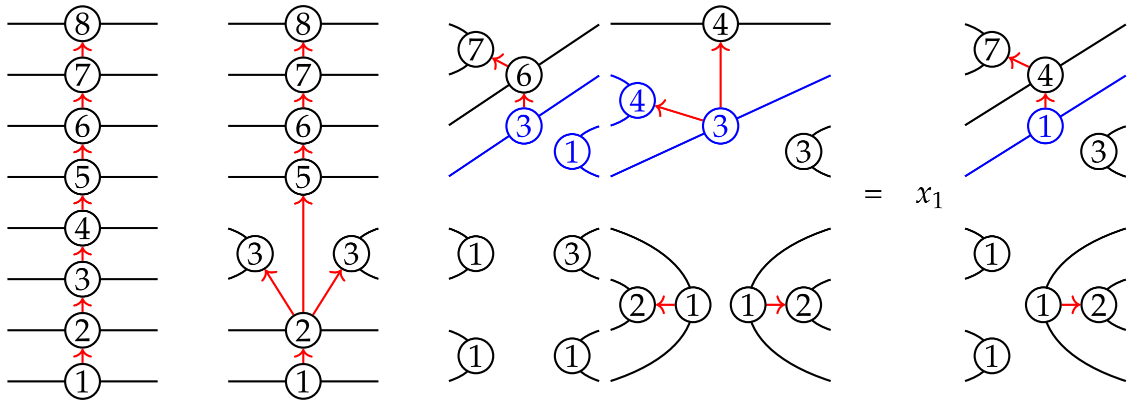


Figure 2: The identity, the generator G_3 and a composition of Okada arc-diagrams of rank 8. The red arrows indicate the Hasse diagram of the nesting order.

The *mirror* $D^\star$ of an Okada arc-diagram, obtained by reflecting D horizontally, is an Okada-arc diagram and the map $D \mapsto D^\star$ extends to an anti-isomorphism of $\widetilde{\mathsf{O}}_N(X,Y)$. Let ι_N denote the map from $\widetilde{\mathsf{O}}_N(X,Y)$ to $\widetilde{\mathsf{O}}_{N+1}(X,Y)$ adding the labeled, propagating arc $\mathbf{h}(N+1 \longmapsto \overline{N+1}) = N+1$ to each arc-diagram. Then ι_N is an injective algebra homomorphism with image the set of diagrams containing $\mathbf{h}(N+1 \longmapsto \overline{N+1}) = N+1$.

Definition 3.7. *For $1 \le i < N$, let G_i denote the elementary Okada arc-diagram containing the labeled arcs* $\mathbf{h}(j \longmapsto \bar{j}) = j$ *for* $j \neq i, i+1$, $\mathbf{h}(i \longmapsto i+1) = i$ *and* $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$.

The elementary Okada diagrams $\mathsf{G}_1, \ldots, \mathsf{G}_{N-1}$ satisfy Okada relations $I(X,Y)$, C(X,Y), and S(X,Y). To construct the isomorphism from $\mathsf{O}_N(X,Y)$ to $\widetilde{\mathsf{O}}_N(X,Y)$, we first need to show that the elements (G_i) generate $\widetilde{\mathsf{O}}_N(X,Y)$. It is clear from the definition that if a product ends with G_i, then the diagram contains the arc $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$. The converse is actually true: If an element $\mathbf{e} \in \mathfrak{A}_N$ contains the arc $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$ then it can be factored as $\mathbf{e} = \mathbf{f} \cdot \mathsf{G}_i$.

Proposition 3.8. *Suppose $D \in \mathfrak{A}_N$ doesn't contain the arc $\mathbf{h}(N \longmapsto \overline{N}) = N$. Then there exist an integer i such that D contains the arc $\mathbf{h}(\bar{i} \longmapsto \overline{i+1}) = i$. If I is the largest such integer, then there exists a unique arc diagram $D^\flat \in \mathfrak{A}_{N-1}$ such that D factorize as*

$$D = \iota_{N-1}(D^\flat)\mathsf{G}_{N-1}\mathsf{G}_{N-2}\cdots\mathsf{G}_I\,.$$

By induction, this proves the following theorem:

Theorem 3.9. *The dimension of $\widetilde{\mathsf{O}}_N(X,Y)$ is $N!$ and it is generated by the elementary diagrams $\mathsf{G}_1, \ldots, \mathsf{G}_{N-1}$. Furthermore the map sending E_i to G_i extends multiplicatively to a unique algebra isomorphism $\Theta : \mathsf{O}_N(X,Y) \to \widetilde{\mathsf{O}}_N(X,Y)$.*

We conclude this section by making explicit the relation between fully packed loop configurations and Okada arc-diagrams:

Proposition 3.10. *For simplicity assume $Y = \mathbf{1}$. Let $\mathcal{C}$ be a FPLC of rank N, let $\underline{\mathbf{i}}$ be its reading, and let $\mathsf{E}_{\underline{\mathbf{i}}}$ be the corresponding monomial in the Okada algebra $\mathsf{O}_N(X, \mathbf{1})$. Then $\Theta(\mathsf{E}_{\underline{\mathbf{i}}}) = X^{\underline{\ell}}\,[\mathcal{C}]$ where $X^{\underline{\ell}} = \prod_{k \geq 1} x_k^{\ell_k}$ and ℓ_k counts the number of loops in $\mathcal{C}$ with height k.*

4 Fomin correspondence and Okada arc-diagrams

We have a bijection between $\mathfrak{S}_N$ and the monoid $\widetilde{\mathsf{O}}_N$ of Okada arc-diagrams, however, it is circuitous: Starting from a permutation, first its code is computed, then the associated FPLC is drawn, from which an Okada arc-diagram is obtained. It is not obvious, for example, that the inverse of a permutation corresponds to the mirror of the associated Okada arc-diagram. The goal of this section is to better explicate this graphical bijection which turns out to be an incarnation of Fomin's Robinson-Schested correspondence for the Young-Fibonacci [7, 1] lattice. Recall that this is a bijection between permutations of $\mathfrak{S}_N$ and pairs of saturated chains in the Young-Fibonacci lattice starting at $\emptyset$ and sharing a common endpoint in $\mathbb{YF}_N$. The reader who is not familiar with Fomin's construction should refer to [1]. See Figure 3a for an example. We will see in this section that Okada arc-diagrams are also in natural bijection with the same pairs of chains.

Cutting a labeled arc-diagram D in the middle gives a natural notion of a *Okada half arc-diagram* containing either a labeled full arc $\mathbf{h}(a \longmapsto b)$ joining two nodes $a, b \in [N]$, or else a labeled half arc $\mathbf{h}(a \longmapsto)$ with a free end. Such a half arc is called *propagating*. The *bra* $\langle D|$ is the Okada half arc-diagram obtained by restricting D to its positive part. The *ket* $|D\rangle$ is defined to be the bra $\langle D^\star|$ of the mirror $D^\star$.

Definition 4.1. *The* propagating label set *of an Okada half arc-diagram H is the subset* $\mathrm{PLab}(H)$ *of $[N]$ consisting of the height labels of its propagating arcs.*

The propagating label set of an Okada half arc-diagram of rank N is always a Fibonacci set of rank N. The following trivial lemma-definition is of great importance:

Lemma 4.2 (Gluing lemma)**.** *For any Okada arc diagram D, the left and right half diagram $\langle D|$ and $|D\rangle$ are two Okada half arc diagrams which have the same propagating labels set. As consequence, it makes sense to define* $\mathrm{PLab}(D) := \mathrm{PLab}(\langle D|) = \mathrm{PLab}(|D\rangle)$.

Moreover if L and R are two Okada half arc-diagrams such that $\mathrm{PLab}(L) = \mathrm{PLab}(R)$, *there is a unique Okada arc-diagram $L \bowtie R$ such that $\langle L \bowtie R| = L$ and $|L \bowtie R\rangle = R$.*

To convert half arc diagrams to chains we need to restrict the former:

Definition 4.3. *For $r \leq N$, the* r-restriction *of an Okada half arc diagram H is the Okada half arc-diagram denoted by $H/[r]$ of rank r possessing*

- *a full arc* $\mathbf{h}(a \longmapsto\!\dashv b) = h$ *whenever* H *contains a full arc* $\mathbf{h}(a \longmapsto\!\dashv b) = h$ *with* $a, b \le r$
- *a half arc* $\mathbf{h}(a \longmapsto) = h$ *whenever* H *contains either a full arc* $\mathbf{h}(a \longmapsto\!\dashv b) = h$ *with* $a \le r < b$ *or a half arc* $\mathbf{h}(a \longmapsto) = h$ *with* $a \le r$.

If $r \le s$ then clearly the r-restriction of the s-restriction of any Okada half arc-diagram H coincides with the r-restriction of H.

Definition 4.4. *To any Okada half arc-diagram* H *of rank* N *we associate the sequence of Fibonacci sets* $\mathrm{Chain}(D) := (C_0, \dots, C_N)$ *defined by* $C_i := \mathrm{PLab}(H/[i])$.

Proposition 4.5. *The map* Chain *is a bijection between Okada half arc-diagrams and saturated chains of rank* N *in the* $\mathbb{YFS}$*-lattice. See Figure 3a for an example.*

Theorem 4.6. *Given a permutation* $\sigma \in \mathfrak{S}_N$, *let* L_σ *and* R_σ *denote the two Okada half arc-diagrams associated to the pair of saturated chains obtained from Fomin's RS-correspondence. Then* $\Theta(\mathsf{E}_\sigma) = L_\sigma \bowtie R_\sigma$. *Moreover* $\Theta(\mathsf{E}_\sigma)^\star = \Theta(\mathsf{E}_{\sigma^{-1}}) = R_\sigma \bowtie L_\sigma$.

5 Structure of the Okada algebra and monoid

From now on, we identify $\mathsf{O}_N(X,Y)$ and $\widetilde{\mathsf{O}}_N(X,Y)$ through the isomorphism Θ. The goal of this section is to understand the structure of the Okada algebra and its monoid via the $\mathbb{YFS}_N$ *dominance order*. In particular, we describe the stratification of the Okada algebra by two-sided ideals (generated by *free elements*) and the Green relations for the monoid (which allows us show that the monoid is aperiodic). This allows us to prove cellularity of the Okada algebra in the next section.

Definition 5.1. *Let* $S = \{s_1 < \cdots < s_k\}$ *and* $T = \{t_1 < \cdots < t_\ell\}$ *be two Fibonacci sets of the same rank* N. *We says that* S is dominated by T *and write* $S \preceq T$ *if* $k < \ell$ *and* $s_{k-i} \le t_{\ell-i}$ *for any* $0 \le i < k$. *We write* $S \prec T$ *if* $S \preceq T$ *but* $S \ne T$.

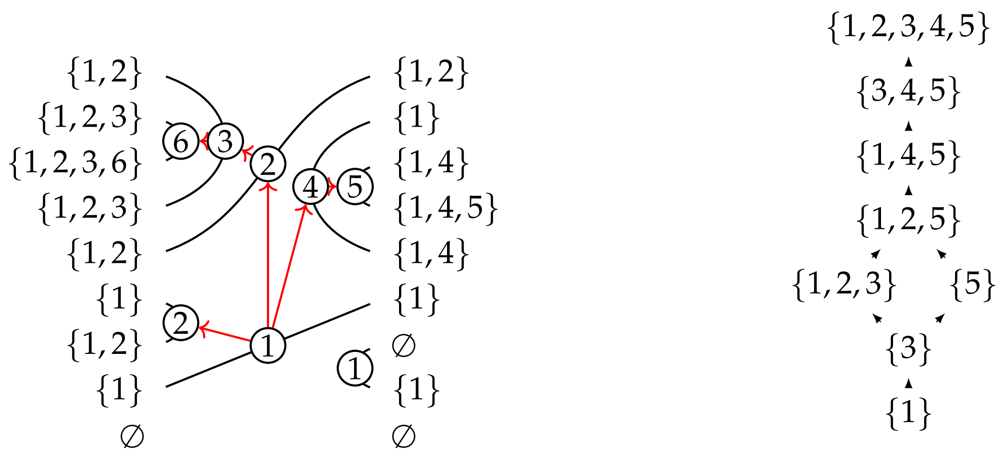
Proposition 5.2. $(\mathbb{YFS}_N, \preceq)$ *is a ranked lattice.*

Definition 5.3. *A* free set *of rank* N *is a subset of* $[N]$ *which does not contain both* i *and* $i+1$ *for all* $1 \le i < N$. *The map* $S \mapsto \mathfrak{F}(S) := \{i \mid i - \max\{s \in S \mid s < i\} \text{ is odd}\}$ *defines a bijection from the collection of rank* N *Fibonacci sets to the collection of rank* N *free sets.*

Definition 5.4. *For* $S \in \mathbb{YFS}_N$ *the associated* free element *in* $\mathsf{O}_N(X,Y)$ *is* $\mathsf{E}_S := \prod_{i \in \mathfrak{F}(S)} \mathsf{E}_i$.

Note that $\mathsf{E}_S = \mathsf{E}_{\sigma_S}$ where $\sigma_S := \prod_{i \in \mathfrak{F}(S)} \sigma_i$ is the associated *free involution* in $\mathfrak{S}_N$.

Remark 5.5. The half arc-diagrams $\langle \mathsf{E}_S|$ and $|\mathsf{E}_S\rangle$ coincide for any $S \in \mathbb{YFS}_N$. Furthermore $\langle \mathsf{E}_S|$ consists only of labeled propagating arcs $\mathbf{h}(s \longmapsto) = s$ for $s \in S$ and labeled full arcs $\mathbf{h}(i \longmapsto\!\dashv i+1) = i$ for $i \in \mathfrak{F}(S)$.



(a) An arc-diagram with its associated chains (b) The dominance order on $\mathbb{YFS}_5$

Proposition 5.6. *Let $\mathfrak{J}_S$ be the two-sided ideal in $\mathsf{O}_N(X,Y)$ generated a free element E_S for $S \in \mathbb{YFS}_N$, then $\mathfrak{J}_S \subseteq \mathfrak{J}_T$ if and only if $T \preceq S$ for any pair $S, T \in \mathbb{YFS}_N$.*

Theorem 5.7 (Triangular Factorization)**.** *For $\sigma \in \mathfrak{S}_N$ there exists a unique pair of permutations $\rho, \tau \in \mathfrak{S}_N$ such that $\mathsf{E}_\sigma = \mathsf{E}_\rho \cdot \mathsf{E}_S \cdot \mathsf{E}_\tau$ where $\ell(\sigma) = \#S + \ell(\rho) + \ell(\tau)$ and $S \preceq \inf(\mathrm{PLab}(\mathsf{E}_\rho), \mathrm{PLab}(\mathsf{E}_\tau))$ and where $S = \mathrm{PLab}(\mathsf{E}_\sigma)$.*

Returning to the Okada monoid, an element $\mathbf{e} \in \mathsf{O}_N$ is said to be *involutive* whenever it equals its mirror $\mathbf{e}^\star$. Involutive elements are always idempotents and thanks to the RS correspondence, these are precisely the basis monomials E_σ where $\sigma \in \mathfrak{S}_N$ is an involution (*i.e.* $\sigma^2 = 1$).

Remark 5.8. The set of idempotents in O_N is not exhausted by the involutive elements. For example in O_3 all element are idempotents, while $\mathsf{E}_1\mathsf{E}_2\mathsf{E}_3$ and $\mathsf{E}_3\mathsf{E}_2\mathsf{E}_1$ are the only non-idempotents in O_4. Computer calculations show that the number of idempotents for $N \leq 10$ are: 1, 1, 2, 6, 22, 108, 594, 4116, 30500, 274006, 2560400.

Proposition 5.9. *Let $\mathbf{e}, \mathbf{f} \in \mathsf{O}_N$. Then either $\langle \mathbf{ef}| = \langle \mathbf{e}|$ and thus $\mathrm{PLab}(\mathbf{ef}) = \mathrm{PLab}(\mathbf{e})$ or $\mathrm{PLab}(\mathbf{ef}) \prec \mathrm{PLab}(\mathbf{e})$. As a consequence, $\mathrm{PLab}(\mathbf{ef}) \preceq \inf(\mathrm{PLab}(\mathbf{e}), \mathrm{PLab}(\mathbf{f}))$.*

The previous proposition is the main ingredient of the following theorem which describe the structure of the Okada monoid:

Theorem 5.10. *The monoid O_N is aperiodic, i.e. there exists an integer K such that $\mathbf{e}^K = \mathbf{e}^{K+1}$ for all $\mathbf{e} \in \mathsf{O}_N$. Equivalently, all the groups in O_N are trivial.*

Recall that $\mathcal{R}$ (resp. $\mathcal{J}$) is the equivalence relation on O_N such that $\mathbf{e}\,\mathcal{R}\,\mathbf{f}$ if $\mathbf{e}$ and $\mathbf{f}$ generate the same right (resp. two-sided) ideals.

Theorem 5.11. *Each $\mathcal{R}$-class of O_N contains a unique involutive element. Each $\mathcal{J}$-class of O_N contains a unique free element. Moreover, the free representative of $\mathbf{e} \in \mathsf{O}_N$ is the free element having the same propagating set as $\mathbf{e}$.*

6 Cellular structure of the Okada algebra

Recall that a cellular algebra A is a finite dimensional algebra with distinguished cellular basis which is particularly well-adapted to studying the representation theory of A, especially as the ground ring/field varies. For brevity, we skip a general discussion about cellular algebras and point the reader to [3] for definitions and context.

Definition 6.1. *Let $\mathfrak{H}_N$ and $\mathbb{C}\mathfrak{H}_N$ denote respectively the set and the vector space spanned by all Okada half arc-diagrams of rank N. Likewise $\mathfrak{H}_N^S$ and $\mathbb{C}\mathfrak{H}_N^S$ will denote the set and the vector space spanned by all half diagrams $H \in \mathfrak{H}_N$ for which $\mathrm{PLab}(H) = S$ where $S \in \mathbb{YFS}_N$. We extend the bra map $D \mapsto \langle D|$ by linearity to obtain a map from $\mathsf{O}_N(X,Y)$ to $\mathbb{C}\mathfrak{H}_N$.*

The following result is a consequence of the factorization given in Proposition 5.7:

Theorem 6.2. *The Okada algebra $\mathsf{O}_N(X,Y)$ is cellular with the following data*

1. *A cell-poset is $\Lambda_N = (\mathbb{YFS}_N, \preceq)$.*
2. *An index set $M_S = \mathfrak{H}_N^S$ for each $S \in \mathbb{YFS}_N$*
3. *A cellular basis element $C_{L,R}^S := L \bowtie R$ associated to $L, R \in \mathfrak{H}_N^S$*
4. *An involutive anti-isomorphism given by the mirror map $\star : L \bowtie R \mapsto R \bowtie L$.*

Remark 6.3. The left $\mathsf{O}_N(X,Y)$ *cell module* associated to $S \in \mathbb{YFS}_N$ can be realized by the vector space $\mathbb{C}\mathfrak{H}_N^S$ equipped with the left action defined by

$$D \bullet H \ := \begin{cases} \langle D \cdot (H \bowtie H)| & \text{if } \mathrm{PLab}\left(D \circ (H \bowtie H)\right) = S \\ 0 & \text{otherwise} \end{cases}$$

where $D \in \mathsf{O}_N(X,Y)$ and $H \in \mathfrak{H}_N^S$. For generic values of X and Y, $\mathbb{C}\mathfrak{H}_N^S$ is irreducible.[1]

7 Prospectives

For a fixed choice of a threshold $k \geq 1$, we *truncate* any Okada arc-diagram D, replacing its height labels h by $\min(h,k)$. The k-truncated Okada arc-diagrams form a multiplicative basis for a *higher Blob algebra* $\mathrm{Blob}_N^{(k)}$, which can be realized as a quotient of the Okada algebra $\mathsf{O}_N(X,Y)$ after specializing the X, Y parameters appropriately. In particular the Temperely-Lieb and Martin-Saleur Blob algebras [4] are recovered for $k = 1, 2$ respectively. It seems that the corresponding Bratelli diagram $\mathbb{YF}^{(k)}$ naturally embeds

[1] The cell module $\mathfrak{H}_N^S$ carries an invariant bilinear form φ_S. We conjecture an explicit value for the determinant of the associated Gram matrix G_S, which we express in terms of clone Schur functions.

into the $\mathbb{YF}$-lattice and can be seen as a Fibonacci counterpart of the sublattice of integer partitions with at most k parts. Both the Temperely-Lieb and the Blob algebras are *intertwiner algebras* which raises the question of whether the higher Blob algebras have such a description for $k \geq 3$. If so, this would be indicative of a Fibonacci version of *Schur-Weyl duality*, and would entail, on a combinatorial level, a well-behaved version of the RSK-correspondence.

It should be possible to incorporate height labels into other diagram algebras such as the partition and Brauer algebras. Can one, for example, define a suitable notion of height labeled *braids* together with *skein relations* consistent with these labels? A satisfactory answer might shed light onto the problem of identifying appropriate *Jucys-Murphy* elements for the Okada algebras.

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Stanley chromatic functions and a conjecture in the representation theory of unipotent groups

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Abstract. This extended abstract is an introduction to a conjecture attempting to relate the representation theory of finite unipotent groups to the representation theory of symmetric groups via combinatorial Hopf algebras. Chromatic symmetric functions arise naturally through the representation theory of unipotent groups, and a positive answer to the conjecture should have useful things to say about the e-positivity of these functions.

Keywords: set partitions, symmetric functions, graph coloring, combinatorial Hopf algebras, categorification

1 Introduction

Stanley chromatic symmetric functions have seen increased attention in recent years with attempts to construct S_n-modules via Hessenberg varieties [9], and connections to the representation theory of the finite general linear groups via induced characters from unipotent groups [6]. This paper explores a seemingly more direct relationship between the representation theory of the finite groups of unipotent upper-triangular matrices and the representation theory of symmetric groups that has chromatic functions at its core. A framework developed by Aguiar–Bergeron-Sottile [1] for canonical maps on combinatorial Hopf algebras gives the mechanism underlying this connection. In particular, for a cocommutative Hopf algebra $\mathcal{H}$, we get Hopf algebra morphisms

$$\mathrm{ch} : \mathcal{H} \to \mathrm{Sym} \cong \mathrm{cf}(S_\bullet),$$

where Sym is the Hopf algebra of symmetric functions and

$$\mathrm{cf}(S_\bullet) = \bigoplus_{n \geq 0} \mathrm{cf}(S_n)$$

is the Hopf algebra of class functions of the finite symmetric groups S_n with product given by induction from Young subgroups and coproduct given by restriction to Young

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subgroups (e.g. [8]). While the function ch can be given quite explicitly, it unfortunately does not obviously lend itself to representation theoretic interpretations (vis-a-vis S_n).

The paper [2] established a Hopf algebra structure on

$$\mathrm{cf}(\mathrm{UT}_\bullet) = \bigoplus_{n\geq 0} \mathrm{cf}(\mathrm{UT}_n),$$

where $\mathrm{cf}(\mathrm{UT}_n)$ is the set of class functions on the finite group of upper-triangular matrices UT_n (the product comes from inflation and the coproduct from restriction). In fact, this paper lifts the Hopf structure from a subHopf algebra

$$\mathrm{scf}(\mathrm{UT}_\bullet) = \bigoplus_{n\geq 0} \mathrm{scf}(\mathrm{UT}_n)$$

defined in [3]. While the latter paper also shows this Hopf algebra is isomorphic to the symmetric functions in non-commuting variables NCSym, this point of view will not be the focus of this abstract.

In summary, we have the following Hopf algebras of interest:

$$\begin{tikzcd}
\mathrm{cf}(\mathrm{UT}_\bullet) \arrow[dr, two heads, "\mathrm{ch}_{\mathbb{1}_\bullet\rangle}"] & & \mathrm{cf}(\mathrm{UT}_\bullet)^* \arrow[dd, two heads] \\
& \mathrm{Sym} \cong \mathrm{cf}(S_\bullet) \cong \mathrm{Sym}^* \arrow[ur, hook, "\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}"] \arrow[dr, hook, "\tilde{\mathrm{ch}}^*_{\mathbb{1}_\bullet\rangle}"] & \\
\mathrm{scf}(\mathrm{UT}_\bullet) \arrow[uu, hook] \arrow[ur, two heads, "\tilde{\mathrm{ch}}_{\mathbb{1}_\bullet\rangle}"] & & \mathrm{scf}(\mathrm{UT}_\bullet)^*
\end{tikzcd}$$

It is worth noting that while $\tilde{\mathrm{ch}}_{\mathbb{1}_\bullet\rangle}$ is the restriction of $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}$, the functions $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}$ and $\tilde{\mathrm{ch}}^*_{\mathbb{1}_\bullet\rangle}$ are fundamentally different, and only $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}$ seems to be functorial. The main conjecture of this paper is as follows.

Conjecture 1. *The functions* $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}$ *and* $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}$ *come from adjoint functors* UT_n***-mod*** $\longrightarrow$ S_n***-mod*** *and* S_n***-mod*** $\longrightarrow$ UT_n***-mod****.*

In particular, if we apply either function to a character we should obtain a character. By construction, it will be clear that both functions are in fact virtual characters (send a character to a $\mathbb{Z}$-linear combination of characters), but all evidence seems to indicate that the signs all cancel.

2 Setting the stage

This section reviews the Aguiar–Bergeron–Sottile framework for combinatorial Hopf algebras, and introduces the main Hopf algebra of interest on characters of the unipotent upper-triangular matrices.

2.1 The Aguiar–Bergeron–Sottile framework

The framework developed by Aguiar–Bergeron–Sottile [1] takes a pair $(\mathcal{H}, \zeta)$ —where $\mathcal{H}$ is a cocommutative, graded, connected Hopf algebra and $\zeta : \mathcal{H} \to \mathbb{C}$ is an algebra morphism— and constructs a canonical Hopf algebra homomorphism $\mathrm{ch}_\zeta : \mathcal{H} \to \mathrm{Sym}$ given explicitly on graded components by

$$\begin{array}{rccc} \mathrm{ch}_\zeta : & \mathcal{H}_n & \longrightarrow & \mathrm{Sym}_n \\ & h & \mapsto & \displaystyle\sum_{\mu \vdash n} \zeta(\Delta^\mu(h)) m_\mu, \end{array}$$

where m_μ is the monomial symmetric functions corresponding to the integer partition μ, and if $\ell(\mu) = \ell$, then Δ^μ is the composition of Δ^ℓ with the standard projection $\mathcal{H}^{\otimes \ell} \to \mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_\ell}$; in this case, ζ is applied diagonally.

While often applied to other situations, the framework can in fact be applied to the classical situation of

$$\begin{array}{rccc} \mathrm{ch} : & \mathrm{cf}(S_\bullet) & \longrightarrow & \mathrm{Sym} \\ & \psi^\lambda & \mapsto & s_\lambda, \end{array} \tag{2.1}$$

where λ is an integer partition, ψ^λ is the corresponding irreducible character of $S_{|\lambda|}$, and s_λ is the corresponding Schur function. Let $\mathbb{1}_n$ denote the trivial character of S_n, and $\langle \cdot, \cdot \rangle$ the usual inner product on class functions. If $\mathbb{1}_\bullet\rangle : \mathrm{cf}(S_\bullet) \to \mathbb{C}$ is the algebra morphism on graded components given by

$$\begin{array}{rccc} \mathbb{1}_\bullet\rangle : & \mathrm{cf}(S_n) & \longrightarrow & \mathbb{C} \\ & \gamma & \mapsto & \langle \gamma, \mathbb{1}_n \rangle, \end{array}$$

then $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}$ is the same as the standard function (2.1).

2.2 The Hopf algebra $\mathrm{cf}(\mathrm{UT}_\bullet)$

Fix a power of a prime q, and for $n \in \mathbb{Z}_{\geq 0}$, let

$$\mathrm{UT}_n = \{g \in \mathrm{GL}_n(\mathbb{F}_q) \mid (g - \mathrm{Id}_n)_{ij} \neq 0 \text{ implies } i < j\}$$

be the subgroup of unipotent upper-triangular matrices with entries in the finite field $\mathbb{F}_q$. The representation theory of these groups is well-known to be wild, but we won't let that deter us. In particular, the space of class functions $\mathrm{cf}(\mathrm{UT}_n)$ has a canonical basis given by the irreducible characters $\mathrm{Irr}(G)$.

We form a graded vector space,

$$\mathrm{cf}(\mathrm{UT}_\bullet) = \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{UT}_n),$$

which has an inner product

$$\langle \gamma, \psi \rangle = \begin{cases} \dfrac{1}{|\mathrm{UT}_n|} \displaystyle\sum_{u \in \mathrm{UT}_n} \gamma(u)\overline{\psi(u)} & \text{if } \gamma, \psi \in \mathrm{cf}(\mathrm{UT}_n), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The basis of irreducible characters forms an orthonormal basis of this space. We upgrade to a graded Hopf algebra with the graded product

$$\begin{array}{rccc} \cdot : \mathrm{cf}(\mathrm{UT}_m) \otimes \mathrm{cf}(\mathrm{UT}_n) & \longrightarrow & \mathrm{cf}(\mathrm{UT}_{m+n}) \\ \psi \otimes \gamma & \mapsto & \mathrm{Inf}_{\mathrm{UT}_m \oplus \mathrm{UT}_n}^{\mathrm{UT}_{m+n}}(\psi \otimes \gamma), \end{array}$$

where $\mathrm{UT}_m \oplus \mathrm{UT}_n$ is the block diagonal quotient (and inflation Inf lifts functions up from that quotient), and coproduct

$$\begin{array}{rccc} \Delta : \mathrm{cf}(\mathrm{UT}_n) & \longrightarrow & \displaystyle\bigoplus_{j=0}^{n} \mathrm{cf}(\mathrm{UT}_j) \otimes \mathrm{cf}(\mathrm{UT}_{n-j}) \\ \psi & \mapsto & \displaystyle\sum_{A \subseteq \{1,2,\ldots,n\}} \mathrm{Res}_{\mathrm{UT}_A \times \mathrm{UT}_{\bar{A}}}^{\mathrm{UT}_n}(\psi), \end{array}$$

where $\bar{A}$ is the complement of A and UT_A ($\cong \mathrm{UT}_{|A|}$) is the subset of matrices whose nonzero entries above the diagonal only occur in the rows and columns in A.

We obtain the dual Hopf algebra $\mathrm{cf}(\mathrm{UT}_\bullet)^*$ by dualizing using the inner product (2.2). The underlying space is the same, but uses the adjoint functor induction for the product and deflation for the coproduct.

Returning to the ABS framework, we have an algebra morphism suggested by the symmetric group case given by

$$\begin{array}{rccc} \mathbb{1}_\bullet\rangle : & \mathrm{cf}(\mathrm{UT}_n) & \longrightarrow & \mathbb{C} \\ & \gamma & \mapsto & \langle \gamma, \mathbb{1}_n \rangle, \end{array}$$

which gives a corresponding canonical map $\mathrm{ch}_{\mathbb{1}_\bullet\rangle} : \mathrm{cf}(\mathrm{UT}_\bullet) \longrightarrow \mathrm{Sym}$. In particular, for $\gamma \in \mathrm{cf}(\mathrm{UT}_n)$,

$$\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\gamma) = \sum_{\substack{\underline{A} \vDash \{1,2,\ldots,n\} \\ \mathrm{bl}(\underline{A}) \vDash n}} \langle \mathrm{Res}_{\mathrm{UT}_{\underline{A}}}^{\mathrm{UT}_n}(\gamma), \mathbb{1} \rangle m_{\mathrm{bl}(\underline{A})},$$

where $\underline{A} = (A_1, \ldots, A_\ell) \vDash \{1, 2, \ldots, n\}$ is a set composition (an ordered list of nonempty subsets that partition $\{1, 2, \ldots, n\}$) and $\mathrm{bl}(\underline{A}) = (|A_1|, |A_2|, \ldots, |A_\ell|)$ is a composition of n. In particular, since the transition matrix from monomial to symmetric functions is integral, we see that the image of a character will be a virtual character.

3 Evidence for the conjecture

In this section we gather some evidence for the conjecture (though we omit complete computations of smaller examples). We begin by examining some natural UT_n characters that are more understandable than the basis $\mathrm{Irr}(\mathrm{UT}_n)$. Then we examine the two functions $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}$ and $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}$, individually.

3.1 More combinatorial spaces of characters

An $\mathbb{F}_q^\times$*-set partition* of $\{1,2,\ldots,n\}$ is a subset

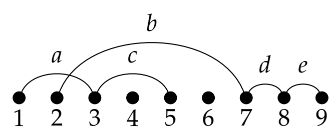
$$\lambda \subseteq \{(i,j;a) \mid 1 \leq i < j \leq n, a \in \mathbb{F}_q^\times\}$$

such that if $(i,k;a),(j,l;b) \in \lambda$, then $i=j$ or $k=l$ implies $(i,k;a)=(j,l;b)$. Let

$$\mathcal{P}_n(q) = \{\mathbb{F}_q^\times\text{-set partitions of } \{1,2,\ldots,n\}\}.$$

We typically view λ as an edge labeled graph Γ_λ on vertices $\{1,2,\ldots,n\}$ with an edge (called an *arc*) labeled by a from i to j if $(i,j;a) \in \lambda$. For example,

$$\left\{\begin{matrix}(1,3;a),(2,7;b),(3,5;c),\\(7,8;d),(8,9;e)\end{matrix}\right\} \leftrightarrow$$



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In practice, the labels are not particularly important for our purposes, so we will usually omit the edge labels in the graph Γ_λ, and we obtain a more standard interpretation of set partition if we let the blocks of the set partition be the connected components of Γ_λ.

We say an element $\lambda \in \mathcal{P}_n(q)$ is

- *non-nesting* if the set of nestings $\mathrm{NST}_\lambda = \emptyset$, where

$$\mathrm{NST}_\lambda = \{((i,l;a),(j,k;b)) \in \lambda \times \lambda \mid i<j<k<l\}.$$

- *non-crossing* if the set of crossings $\mathrm{CRS}_\lambda = \emptyset$, where

$$\mathrm{CRS}_\lambda = \{((i,k;a),(j,l;b)) \in \lambda \times \lambda \mid i<j<k<l\}.$$

In either case, we can evaluate in the graph Γ_λ whether the edges have any nestings or crossings.

Using this combinatorics we construct two families of characters by inducing from families of subgroups. Fix a non-trivial homomorphism $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$. For $\lambda \in \mathcal{P}_n(q)$, define

$$\begin{array}{rccc}\vartheta_\lambda : & \mathrm{UT}_n & \longrightarrow & \mathbb{C}\\ & u & \mapsto & \displaystyle\prod_{(i,j;a)\in\lambda} \vartheta(au_{ij}),\end{array}$$

which restricts to a linear character of the subgroup

$$\mathrm{UT}_\lambda = \{u \in \mathrm{UT}_n \mid u_{ij} = 0 \text{ if } (i,k;a) \in \lambda,\, i < j < k\}.$$

This gives us an induced character

$$\chi^\lambda = \mathrm{Ind}_{\mathrm{UT}_\lambda}^{\mathrm{UT}_n}(\vartheta_\lambda).$$

For $\lambda, \mu \in \mathcal{P}_n(q)$, these characters are orthogonal

$$\langle \chi^\lambda, \chi^\mu \rangle = q^{|\mathrm{CRS}_\lambda|}\delta_{\lambda\mu}, \tag{3.1}$$

and every irreducible character in $\mathrm{Irr}(G)$ is a constituent of exactly one such character [5]. Here, $\chi^\emptyset = \mathbb{1}_{\mathrm{UT}_n}$ is the trivial character.

If we take the space spanned by these characters we get a subspace

$$\mathrm{scf}(\mathrm{UT}_\bullet) = \bigoplus_{n\geq 0} \mathrm{scf}(\mathrm{UT}_n), \quad \text{where} \quad \mathrm{scf}(\mathrm{UT}_n) = \mathbb{C}\text{-span}\{\chi^\lambda \mid \lambda \in \mathcal{P}_n(q)\},$$

that forms a subHopf algebra of $\mathrm{cf}(\mathrm{UT}_\bullet)$ [3].

Another family of characters comes from $\lambda \in \mathcal{P}_n(2)$ non-nesting. Define

$$\bar{\chi}^\lambda = \mathrm{Ind}_{\overline{\mathrm{UT}}_\lambda}^{\mathrm{UT}_n}(\mathbb{1}) \quad \text{where} \quad \overline{\mathrm{UT}}_\lambda = \{u \in \mathrm{UT}_n \mid u_{jk} = 0, \text{ if } i \leq j < k \leq l,\, (i,l;a) \in \lambda\}.$$

For example, if $\lambda = \{(1,4;1),(3,5;1),(5,6;1)\} \in \mathcal{P}_6(2)$, then

$$\overline{\mathrm{UT}}_\lambda = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & \mathbf{0} & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \subseteq \mathrm{UT}_\lambda = \begin{bmatrix} 1 & 0 & 0 & \circledast & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & 0 & \circledast & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & \circledast \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where the coordinates of λ are indicated by bold $\mathbf{0}$ or circled $\circledast$. Both the regular character $\bar{\chi}^{\{(1,n;1)\}}$ and the trivial character $\bar{\chi}^\emptyset$ of UT_n are of this form.

While these characters are no longer pairwise orthogonal, we still have that

$$\overline{\mathrm{scf}}(\mathrm{UT}_\bullet) = \bigoplus_{n\geq 0} \overline{\mathrm{scf}}(\mathrm{UT}_n), \quad \text{where} \quad \overline{\mathrm{scf}}(\mathrm{UT}_n) = \mathbb{C}\text{-span}\{\bar{\chi}^\lambda \mid \lambda \in \mathcal{P}_n(2),\ \text{non-nesting}\},$$

is a subHopf algebra of $\mathrm{scf}(\mathrm{UT}_\bullet)$ [4].

3.2 The function $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}$

We begin by considering the image of the characters χ^λ for $\lambda \in \mathcal{P}_n(q)$. Our most complete answer is for those characters χ^λ corresponding to elements $\lambda \in \mathcal{P}_n(q)$ that are both non-nesting and non-crossing. By (3.1), these are also irreducible characters.

Chromatic symmetric functions arise naturally in the image. Recall, a proper coloring of a graph $\Gamma = (V, E)$ is a function $c : V \to \mathbb{Z}_{\geq 1}$ such that if $(a, b) \in E$ then $c(a) \neq c(b)$. Stanley [10] defined the chromatic symmetric function

$$X_\Gamma = \sum_{\substack{c:V\to\mathbb{Z}_{\geq 1}\\ \text{a proper coloring}}} X_c, \quad \text{where} \quad X_c = X_1^{|c^{-1}(1)|} X_2^{|c^{-1}(2)|} \cdots .$$

For $\lambda \in \mathcal{P}_n(q)$, let

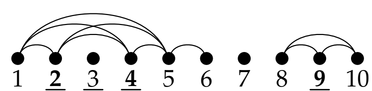
$$\mathcal{N}_\lambda = \{1 \leq j \leq n \mid i < j < k, (i, k, a) \in \lambda\},$$

and for any subset $M \subseteq \mathcal{N}_\lambda$ define a graph Γ_λ^M with vertices $\{1, 2, \ldots, n\}$ and edges

$$\{\{j, k\} \mid i \leq j < k \leq l, (i, l, a) \in \lambda, j, k \in M \cup \{i, l\}\}.$$

For example, if $\lambda = \{(1, 5; a), (5, 6; b), (8, 10; c)\}$, then

$$\mathcal{N}_\lambda = \{2, 3, 4, 9\} \quad \text{and} \quad \Gamma_\lambda^{\{2,4,9\}} =$$



.

Note that Γ_λ is a subgraph of Γ_λ^M for every subset M. We now get the image of some of the irreducible characters of UT_n.

Theorem 1. *Let $t = q - 1$. For $\lambda \in \mathcal{P}_n(q)$ non-nesting and non-crossing,*

$$\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\chi^\lambda) = \sum_{M \subseteq \mathcal{N}_\lambda} t^{|M|} X_{\Gamma_\lambda^M}.$$

The following lemma gives an essential outline for how to compute the restriction of characters with an eye towards finding a copy of the trivial character. Heuristically, we can think of restriction as picking a subset of vertices in our graph Γ_λ. If an edge has endpoints in the subset, that edge remains. If an edge is missing one or two endpoints, we either re-attach the unattached endpoints in all possible ways such that the new edge weakly nests in the original edge or remove the edge.

Lemma 1 ([11])**. (a) Factorization.** *For $\lambda \in \mathcal{P}_n(q)$ and $\underline{A} = (A_1, A_2, \ldots, A_\ell) \vDash n$,*

$$\mathrm{Res}^{\mathrm{UT}_n}_{\mathrm{UT}_{\underline{A}}}\Big(\frac{\chi^\lambda}{\chi^\lambda(1)}\Big) = \bigodot_{\substack{1\leq j\leq \ell\\ (i,l;a)\in\lambda}} \mathrm{Res}^{\mathrm{UT}_n}_{\mathrm{UT}_{A_j}}\Big(\frac{\chi^{(i,l;a)}}{\chi^{(i,l;a)}(1)}\Big),$$

where $\odot$ denotes the pointwise product on functions.

(b) Local restriction. *For $(i,l,a)\in\lambda$ and $A\subseteq\{1,2,\ldots,n\}$,*

$$\mathrm{Res}^{\mathrm{UT}_n}_{\mathrm{UT}_A}(\chi^{(i,l;a)})=\begin{cases} q^{\#\{1\leq j\leq l\mid j\notin A\}}\chi^{(i,l;a)} & \text{if } i,l\in A,\\ q^{\#\{1\leq j\leq l\mid j\notin A\}}\Big(\mathbb{1}+\sum\limits_{\substack{i<k<l,\\ k\in A, b\in\mathbb{F}_q^\times}}\chi^{(i,k;b)}\Big) & \text{if } i\in A,\, l\notin A,\\ q^{\#\{1\leq j\leq l\mid j\notin A\}}\Big(\mathbb{1}+\sum\limits_{\substack{i<k<l\\ k\in A, b\in\mathbb{F}_q^\times}}\chi^{(k,l;b)}\Big) & \text{if } i\notin A,\, l\in A,\\ q^{\#\{1\leq j\leq l\mid j\notin A\}}\Big((|A\cap[i,l]|t+1)\mathbb{1}+\sum\limits_{\substack{i<j<k<l\\ j,k\in A, b\in\mathbb{F}_q^\times}}\chi^{(j,k;b)}\Big) & \text{if } i,l\notin A.\end{cases}$$

(c) Conflict resolution. *For $i\leq j<k\leq l$,*

$$\chi^{(i,k;a)}\odot\chi^{(j,l,b)}=\begin{cases}\chi^{\{(i,k;a),(j,l;b)\}} & \text{if } i\neq j \text{ and } k\neq l,\\ \chi^{(i,l;b)}+\sum\limits_{\substack{i<i'<k\\ c\in\mathbb{F}_q^\times}}\chi^{\{(i',k;c),(i,l;b)\}} & \text{if } i=j \text{ and } k\neq l,\\ \chi^{(i,k;a)}+\sum\limits_{\substack{j<l'<l\\ c\in\mathbb{F}_q^\times}}\chi^{\{(i,k;a),(i,l';c)\}} & \text{if } i\neq j \text{ and } k=l.\end{cases}$$

Using this lemma we see that to get the trivial character (corresponding to the graph with no edges) when $\lambda\in\mathcal{P}_n(q)$ is non-nesting, we must detach an endpoint of every arc.

Lemma 2. *Suppose $\lambda\in\mathcal{P}_n(q)$, and $\underline{A}\vDash n$. Then*

(a) If $(i,j,a)\in\lambda$ implies i and j are in different blocks of $\underline{A}$, then

$$\langle\mathrm{Res}^{\mathrm{UT}_n}_{\mathrm{UT}_{\underline{A}}}(\chi^\lambda),\mathbb{1}\rangle\neq 0.$$

(b) If λ is non-nesting and

$$\langle\mathrm{Res}^{\mathrm{UT}_n}_{\mathrm{UT}_{\underline{A}}}(\chi^\lambda),\mathbb{1}\rangle\neq 0,$$

then $(i,j,a)\in\lambda$ implies i and j are in different blocks of $\underline{A}$.

Note that if $\lambda\in\mathcal{P}_n(q)$, then every $\underline{A}$ specifies a function

$$\begin{array}{rccc} c_{\underline{A}}: & \{1,2,\ldots,n\} & \longrightarrow & \{1,2,\ldots,\ell(\underline{A})\}\\ & j & \mapsto & i, \text{ where } j\in A_i\end{array}$$

By the Lemma 2, when λ is also non-nesting, this function is a proper coloring of the graph Γ_λ if and only if

$$\langle\mathrm{Res}^{\mathrm{UT}_n}_{\mathrm{UT}_{\underline{A}}}(\chi^\lambda),\mathbb{1}\rangle\neq 0.$$

Lemma 3. *If $\lambda \in \mathcal{P}_n(q)$ is a non-nesting and non-crossing, then*

$$\mathrm{ch}_{\mathbb{1}\rangle}(\chi^{\lambda}) = \sum_{\substack{c:V_\lambda \to \mathbb{Z}_{\geq 1} \\ \text{a proper coloring} \\ \text{of } \Gamma_\lambda}} \prod_{\substack{(i,k;a)\in\lambda \\ d\notin\{c(i),c(k)\}}} \Big(\#\{i<j<k \mid c(j)=d\}t+1\Big) X_c.$$

Since the graphs in question are unit interval graphs, from Gasharov [7] we obtain the following corollary to Theorem 1.

Corollary 1. *For $\lambda \in \mathcal{P}_n(q)$ non-nesting and non-crossing, $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\chi^{\lambda})$ is a non-negative linear combination of schur functions with coefficients in $\mathbb{Z}_{\geq 0}[t]$.*

Examples 1. Some easy examples include:

(a) Since the trivial character $\mathbb{1}_n \in \mathrm{cf}(\mathrm{UT}_n)$ is in fact $\mathrm{Inf}_{\{1\}}^{\mathrm{UT}_n}(\mathbb{1})$, by the multiplicativity of the canonical map,

$$\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\mathbb{1}_{\mathrm{UT}_n}) = \mathrm{Ind}_{S_1\times S_1\times\cdots\times S_1}^{S_n}(\mathbb{1}_1\otimes\mathbb{1}_1\otimes\cdots\otimes\mathbb{1}_1),$$

or the regular character of S_n.

(b) The linear characters of UT_n are all obtained from $\mathbb{F}_q^\times$-set partitions and they correspond to $\lambda \in \mathcal{P}_n(q)$ such that $(i,j;a)\in\lambda$ implies $j-i=1$. At $q=2$ these are in bijective correspondence with integer compositions and give a subHopf algebra of $\mathrm{cf}(\mathrm{UT}_n)$ isomorphic to the Hopf algebra of noncommutative symmetric functions NSym. For the single block partition σ_n, we have

$$\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\chi^{\sigma_n}) = X_{P_n},$$

where P_n is the path graph. In general, we get a product of path graphs corresponding to the composition part lengths.

(c) The minimal n such that $\mathrm{scf}(\mathrm{UT}_n) \neq \mathrm{cf}(\mathrm{UT}_n)$ is $n=4$. In particular, for $a,b\in\mathbb{F}_q^\times$,

$$\chi^{\{(1,3;a),(2,4;b)\}} = \sum_{c\in\mathbb{F}_q} \chi_c^{\{(1,3;a),(2,4;b)\}},$$

is a decomposition into irreducible characters, where

$$\chi_c^{\{(1,3;a),(2,4;b)\}}(u) = \begin{cases} q\vartheta_c(u_{12})\vartheta_a(u_{13})\vartheta_b(u_{24}) & \text{if } u_{23}=0 \text{ and } u_{12}a=u_{34}b, \\ 0, & \text{otherwise.}\end{cases}$$

Then direct computation gives

$$\begin{aligned}\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\chi_c^{\{(1,3;a),(2,4;b)\}}) &= 2(1+\delta_{c,0})m_{(2,2)} + (6+2q)m_{(2,1,1)} + 24qm_{(1^4)} \\ &= \begin{cases} 4s_{(2,2)} + 2(2+t)s_{(2,1,1)} + (18t+4)s_{(1^4)} & \text{if } c=0, \\ 2s_{(2,2)} + 2(3+t)s_{(2,1,1)} + (18t+2)s_{(1^4)} & \text{if } c\neq 0.\end{cases}\end{aligned}$$

For the permutation modules $\bar{\chi}^\lambda$ we again get a sum of chromatic symmetric functions with coefficients powers of $t = q - 1$.

Theorem 2. *For $\lambda \in \mathcal{P}_n$ non-nesting,*

$$\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\bar{\chi}^\lambda) = \sum_{E \subseteq E_\lambda^{\mathcal{N}_\lambda}} t^{|E|} X_{(\{1,2,\ldots,n\},E)},$$

where $E_\lambda^{\mathcal{N}_\lambda}$ is the edge set of $\Gamma_\lambda^{\mathcal{N}_\lambda}$.

The key step to this theorem is the following lemma that writes the image of $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}$ in terms of monomials. Given a coloring $c : \{1,2,\ldots,n\} \to \mathbb{Z}_{\geq 1}$ of a graph $\Gamma_\lambda^{\mathcal{N}_\lambda}$ (not necessarily proper), we define

$$M_c(\lambda) = \max\{E \mid c \text{ is a proper coloring of } (\{1,2,\ldots,n\},E)\},$$

where maximality is with respect to containment.

Lemma 4. *For $\lambda \in \mathcal{P}_n$ non-nesting,*

$$\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\bar{\chi}^\lambda) = \sum_{c:\{1,2,\ldots,n\}\to\mathbb{Z}_{\geq 1}} q^{|M_c(\lambda)|} X_c.$$

Note that Theorem 2 hardly seems like evidence, since we get plenty of graphs showing up that are not unit-interval graphs. In fact, in the case of $\lambda = \{(1,n;a)\}$, we get all possible graphs appearing in the sum, since $\Gamma_\lambda^{\mathcal{N}_\lambda}$ is the complete graph. However, it appears that we still get positive sums, as the bad graphs get corrected by good ones. For example,

$$\begin{aligned}\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\bar{\chi}^{\{(1,4;a)\}}) =& t^0(s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} + s_{(1,1,1,1)}) \\ &+ t^1 12(s_{(3,1)} + s_{(2,2)} + 2s_{(2,1,1)} + 1s_{(1,1,1,1)}) \\ &+ t^2 12(s_{(3,1)} + 2s_{(2,2)} + 6s_{(2,1,1)} + 5s_{(1,1,1,1)}) \\ &+ t^3 4(s_{(3,1)} + 5s_{(2,2)} + 23s_{(2,1,1)} + 38s_{(1,1,1,1)}) \\ &+ t^4 6(s_{(2,2)} + 9s_{(2,1,1)} + 31s_{(1,1,1,1)}) \\ &+ t^5 12(s_{(2,1,1)} + 9s_{(1,1,1,1)}) + t^6 24s_{(1,1,1,1)}.\end{aligned}$$

In fact, the constant term is a familiar module.

Proposition 1. *For $\lambda \in \mathcal{P}_n(2)$ non-nesting,*

$$\lim_{t\to 0} \mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\bar{\chi}^\lambda) = h_{(1^n)},$$

or the symmetric function corresponding to the regular character of S_n.

3.3 The function $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}$

In this section, we investigate the dual map $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle} : \mathrm{Sym} \to \mathrm{cf}(\mathrm{UT}_\bullet)^*$, given by

$$\langle \mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(f(\underline{X})), \gamma\rangle = \langle f(\underline{X}), \mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\gamma)\rangle.$$

The previous section allows us to quickly compute the image of h_n by duality.

Proposition 2. *For $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(h_n) = \mathbb{1}_{\mathrm{UT}_n}.$$

Note that the codomain is in fact the dual to $\mathrm{cf}(\mathrm{UT}_\bullet)$, so has product given by

$$\gamma_m \cdot \varphi_n = \sum_{\substack{\underline{A}=(A_1,A_2)\vDash m+n \\ |A_1|=m, |A_2|=n}} \mathrm{Ind}_{\mathrm{UT}_{\underline{A}}}^{\mathrm{UT}_{m+n}}(\gamma_m \otimes \varphi_n) \quad \text{for } \gamma_m \in \mathrm{cf}(\mathrm{UT}_m)^*,\ \varphi_n \in \mathrm{cf}(\mathrm{UT}_n)^*.$$

Remark 1. Here we note that while $\mathrm{scf}(\mathrm{UT}_\bullet) \subseteq \mathrm{cf}(\mathrm{UT}_\bullet)$ as Hopf algebras, the same does not hold for the dual spaces, as we instead obtain quotient Hopf algebras. While the coproduct is defined in the same way for both dual spaces, the product will use different adjoint functors to restriction in each case. In this paper we are therefore using the dual of $\mathrm{cf}(\mathrm{UT}_\bullet)$, since it preserves modules unlike the variant of induction used in the dual to $\mathrm{scf}(\mathrm{UT}_\bullet)$. However, it is worth noting that Proposition 2 holds for the dual of $\mathrm{scf}(\mathrm{UT}_\bullet)$ as well.

By Proposition 2 and because $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}$ is a Hopf algebra morphism, for any integer partition $\lambda \vdash n$ of length k,

$$\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(h_\lambda) = \sum_{\substack{\underline{A}=(A_1,\ldots,A_k)\vDash\{1,2,\ldots,n\} \\ |A_j|=\lambda_j}} \mathrm{Ind}_{\mathrm{UT}_{\underline{A}}}^{\mathrm{UT}_n}(\mathbb{1}).$$

In particular, we get that the permutation module $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{1})$ gets sent to a UT_n-module.

If we add the Jacobi–Trudi formula we obtain that for an integer partition $\lambda \vdash n$ of length k,

$$\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(s_\lambda) = \sum_{w\in S_k} (-1)^{\ell(w)} \sum_{\substack{\underline{A}=(A_1,A_2,\ldots,A_k)\vDash\{1,2,\ldots,n\} \\ |A_j|=\lambda_{w(j)}-w(j)+j}} \mathrm{Ind}_{\mathrm{UT}_{\underline{A}}}^{\mathrm{UT}_n}(\mathbb{1}). \tag{3.2}$$

In particular, it is evident that $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(s_\lambda)$ will be a virtual character.

A particular case of interest is the sign character or e_n.

Lemma 5. *For $n \in \mathbb{Z}_{\geq 0}$*

$$e_n = \sum_{\mu\vDash n} (-1)^{n-\ell(\mu)} h_\mu.$$

We conclude with the following intriguing consequence concerning the antipode S^* of the dual Hopf algebra $\mathrm{cf}(\mathrm{UT}_\bullet)^*$.

Corollary 2. *For $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(e_n) = (-1)^n S^*(\mathbb{1}_n).$$

Remark 2. Note that if one could show that $\mathrm{ch}^*_{\mathbb{1}_\bullet\rangle}(e_n)$ is a character, then this would imply that $\mathrm{ch}_{\mathbb{1}_\bullet\rangle}(\chi)$ is e-positive for all $\chi \in \mathrm{Irr}(\mathrm{UT}_n)$.

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Regular Schur labeled skew shape posets and their 0-Hecke modules

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Abstract. Assuming Stanley's P-partitions conjecture holds, the regular Schur labeled skew shape posets are precisely the finite posets P with underlying set $\{1, 2, \ldots, |P|\}$ such that the P-partition generating function is symmetric and the set of linear extensions of P, denoted $\Sigma_L(P)$, is a left weak Bruhat interval in the symmetric group $\mathfrak{S}_{|P|}$. We describe the permutations in $\Sigma_L(P)$ in terms of reading words of standard Young tableaux when P is a regular Schur labeled skew shape poset, and classify $\Sigma_L(P)$'s up to descent-preserving isomorphism as P ranges over regular Schur labeled skew shape posets. The results obtained are then applied to classify the 0-Hecke modules M_P associated with regular Schur labeled skew shape posets P up to isomorphism. Then we characterize regular Schur labeled skew shape posets as the finite posets P whose linear extensions form a dual plactic-closed subset of $\mathfrak{S}_{|P|}$. Using this characterization, we construct distinguished filtrations of M_P with respect to the Schur basis when P is a regular Schur labeled skew shape poset.

Keywords: labeled poset, P-partition, weak Bruhat order, 0-Hecke algebra, representation, skew Schur function

1 Introduction

Schur labeled skew shape posets naturally appear in the context of the celebrated Stanley's P-partition conjecture. Let P_n be the set of posets on $[n] := \{1, 2, \ldots, n\}$. To each poset $P \in \mathsf{P}_n$, one can associate a quasisymmetric function K_P, called the *P-partition generating function*. In 1972, Stanley [15, p. 81] proposed a conjecture stating that for $P \in \mathsf{P}_n$, K_P is

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a symmetric function if and only if P is a Schur labeled skew shape poset. While this conjecture has been verified to be true for all posets P with $|P| \leq 8$, it remains open in the general case (see [12]). For the definition of Schur labeled skew shape posets, see Subsection 2.3. We denote by SP_n the set of all Schur labeled skew shape posets in P_n.

On the other hand, *regular posets* were introduced by Björner–Wachs [4] during their investigation of the convex subsets of the symmetric group $\mathfrak{S}_n$ on $[n]$ under the right weak Bruhat order. For $P \in \mathsf{P}_n$ with the partial order $\preceq$, let $\Sigma_R(P)$ be the set of permutations $\pi \in \mathfrak{S}_n$ satisfying that if $x \preceq y$, then $\pi^{-1}(x) \leq \pi^{-1}(y)$. They observed that every convex subset of $\mathfrak{S}_n$ under the right weak Bruhat order appears as $\Sigma_R(P)$ for some $P \in \mathsf{P}_n$, and every right weak Bruhat interval in $\mathfrak{S}_n$ is convex. This observation led them to characterize the posets $P \in \mathsf{P}_n$ satisfying that $\Sigma_R(P)$ is a right weak Bruhat interval. They introduced the notion of regular posets, and proved that $P \in \mathsf{P}_n$ is a regular poset if and only if $\Sigma_R(P)$ is a right weak Bruhat interval in $\mathfrak{S}_n$. For the definition of regular posets, refer to Definition 2.1. We denote by RP_n the set of all regular posets in P_n. Let $\Sigma_L(P) := \{\gamma^{-1} \mid \gamma \in \Sigma_R(P)\}$. By considering the left Bruhat order and $\Sigma_L(P)$ instead of the right Bruhat order and $\Sigma_R(P)$, we can establish a similar characterization. However, we prefer the former over the latter as it is better suited for handling left $H_n(0)$-modules.

Let $\mathsf{RSP}_n := \mathsf{RP}_n \cap \mathsf{SP}_n$. In the following, we explain the reason why we study regular Schur labeled skew shape posets from the perspective of the representation theory of the 0-Hecke algebra.

In 1996, Duchamp–Krob–Leclerc–Thibon [7] introduced a ring isomorphism, called the *quasisymmetric characteristic*, from the Grothendieck ring $\mathcal{G}_0(H_\bullet(0))$ of the tower of 0-Hecke algebras to the ring QSym of quasisymmetric functions. For the definition of the quasisymmetric characteristic, see Subsection 2.4. In 2002, Duchamp–Hivert–Thibon [6] associated a right $H_n(0)$-module M_P with each poset $P \in \mathsf{P}_n$, such that the image of M_P under the quasisymmetric characteristic is K_P. This was achieved by defining a suitable right $H_n(0)$-action on $\Sigma_R(P)$. For technical reasons, we use a slightly different 0-Hecke module, denoted as M_P, instead of Duchamp–Hivert–Thibon's module M_P. Our M_P is a left $H_n(0)$-module with the basis $\Sigma_L(P)$. For the precise definition of M_P, refer to Definition 2.4.

Since the middle of 2010, various left 0-Hecke modules, each equipped with a tableau basis and yielding an important quasisymmetric characteristic image, have been constructed ([1, 3, 14, 16, 17]). In order to handle these modules in a uniform manner, Jung–Kim–Lee–Oh [9] introduced a left $H_n(0)$-module $\mathsf{B}(I)$, referred to as *the weak Bruhat interval module associated with* I, for each left weak Bruhat interval I in $\mathfrak{S}_n$. Furthermore, they showed that the Grothendieck ring $\bigoplus_{n\geq 0} \mathcal{G}_0(\mathscr{B}_n)$ is isomorphic to QSym as Hopf algebras, where $\mathscr{B}_n$ is the category direct sums of finitely many isomorphic copies of weak Bruhat interval modules of $H_n(0)$. Recently, Choi–Kim–Oh [5] clarified the exact relationship between the weak Bruhat interval modules and the 0-Hecke modules M_P, using Björner–Wachs' characterization.

The aim of this paper is to give a comprehensive investigation of regular Schur labeled skew shape posets and their associated 0-Hecke modules. Firstly, we provide an explicit description of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$. Next, we study the classification of left weak Bruhat intervals in $\mathfrak{S}_n$ up to descent-preserving poset isomorphism. Using the classification, we classify the $H_n(0)$-modules M_P up to isomorphism as P ranges over RSP_n. Then, we characterize regular Schur labeled skew shape posets as the posets such that $\Sigma_L(P)$ is a dual plactic-closed subset of $\mathfrak{S}_n$. This characterization is applied to show that for $P \in \mathsf{RSP}_n$, M_P admits a distinguished filtration with respect to the Schur basis. A tableau description of M_P for $P \in \mathsf{RSP}_n$ is also provided. Lastly, we discuss further issues concerned with the classification of the $H_n(0)$-modules M_P.

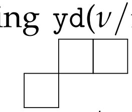
For details and more results, we refer the reader to [10].

2 Preliminaries

Throughout this paper, n will denote a nonnegative integer unless otherwise stated.

2.1 Compositions, Young diagrams, and bijective tableaux

A *composition* α of n, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^{k} \alpha_i = n$. We call $k =: \ell(\alpha)$ the *length* of α and $n =: |\alpha|$ the *size* of α. Given $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}) \models n$, we define $\mathrm{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \sum_{i=1}^{\ell(\alpha)-1} \alpha_i\}$. The *reverse composition* α^{r} *of* α is the composition $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$ and the *complement* α^{c} *of* α is the unique composition satisfying $\mathrm{set}(\alpha^c) = [n-1] \setminus \mathrm{set}(\alpha)$. If a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \models n$ satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, then it is called a *partition* of n and denoted as $\lambda \vdash n$. Given two partitions λ and μ with $\ell(\lambda) \geq \ell(\mu)$, we write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $1 \leq i \leq \ell(\mu)$. A *skew partition* λ/μ is a pair (λ, μ) of partitions with $\lambda \supseteq \mu$. We call $|\lambda/\mu| := |\lambda| - |\mu|$ the *size* of λ/μ.

Given a partition λ, we define the *Young diagram* $\mathtt{yd}(\lambda)$ *of* λ to be the left-justified array of n boxes, where the ith row from the top has λ_i boxes for $1 \leq i \leq k$. Similarly, given a skew partition λ/μ, we define the *Young diagram* $\mathtt{yd}(\lambda/\mu)$ *of* λ/μ to be the Young diagram $\mathtt{yd}(\lambda)$ with all boxes belonging to $\mathtt{yd}(\mu)$ removed. A skew partition is called *basic* if the corresponding Young diagram contains neither empty rows nor empty columns. In this paper, every skew partition is assumed to be basic unless otherwise stated. For skew partitions λ/μ and ν/κ, $\lambda/\mu \oplus \nu/\kappa$ is the skew partition whose Young diagram is obtained by taking a rectangle of empty squares with the same number of rows as $\mathtt{yd}(\lambda/\mu)$ and the same number of columns as $\mathtt{yd}(\nu/\kappa)$, and putting $\mathtt{yd}(\nu/\kappa)$ below and $\mathtt{yd}(\lambda/\mu)$ to the right of this rectangle. For instance, $\mathtt{yd}((2) \oplus (1)) =$  .

Given a skew partition λ/μ of size n, a *bijective tableau* of shape λ/μ is a filling of

$\mathtt{yd}(\lambda/\mu)$ with distinct entries in $[n]$. For later use, we denote by $\tau_0^{\lambda/\mu}$ the bijective tableau of shape λ/μ obtained by filling $1, 2, \ldots, n$ from right to left starting with the top row. A bijective tableau is referred to as a *standard Young tableau* if the elements in each row are arranged in increasing order from left to right, and the elements in each column are arranged in increasing order from top to bottom. We denote by $\mathrm{SYT}(\lambda/\mu)$ the set of all standard Young tableaux of shape λ/μ. And, we let $\mathrm{SYT}_n := \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda)$.

2.2 Weak Bruhat orders on the symmetric group

Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. For $1 \leq i \leq n-1$, let s_i be the simple transposition $(i, i+1)$. For $\sigma \in \mathfrak{S}_n$, let

$$\mathrm{Des}_L(\sigma) := \{i \in [n-1] \mid \ell(s_i\sigma) < \ell(\sigma)\} \text{ and } \mathrm{Des}_R(\sigma) := \{i \in [n-1] \mid \ell(\sigma s_i) < \ell(\sigma)\},$$

where $\ell(\sigma)$ is the length of σ. The *left weak Bruhat order* $\preceq_L$ (resp., *right weak Bruhat order* $\preceq_R$) on $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ whose covering relation $\preceq_L^c$(resp., $\preceq_R^c$) is given as follows: $\sigma \preceq_L^c s_i\sigma$ if and only if $i \notin \mathrm{Des}_L(\sigma)$ (resp, $\sigma \preceq_R^c \sigma s_i$ if and only if $i \notin \mathrm{Des}_R(\sigma)$). Given $\sigma, \rho \in \mathfrak{S}_n$, the *left weak Bruhat interval* $[\sigma, \rho]_L$ (resp., *right weak Bruhat interval* $[\sigma, \rho]_R$) denotes the closed interval $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$ (resp., $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_R \gamma \preceq_R \rho\}$). Let $\mathrm{Int}(n)$ be the set of nonempty left weak Bruhat intervals in $\mathfrak{S}_n$.

2.3 Regular posets and Schur labeled skew shape posets

Let P_n be the set of posets on $[n]$. Given $P \in \mathsf{P}_n$, we write the partial order of P as $\preceq_P$.

Definition 2.1. A poset $P \in \mathsf{P}_n$ is said to be *regular* if the following holds: for all $x, y, z \in [n]$ with $x \preceq_P z$, if $x < y < z$ or $z < y < x$, then $x \preceq_P y$ or $y \preceq_P z$.

We denote by RP_n the set of all regular posets in P_n. From the result of Björner–Wachs [4, Theorem 6.8], it follows that

(i) for $P \in \mathsf{P}_n$, P is regular if and only if $\Sigma_L(P)$ is a left weak Bruhat interval, and

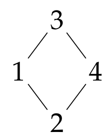
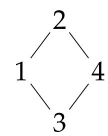
(ii) the map $\mathsf{RP}_n \to \mathrm{Int}(n)$ sending P to $\Sigma_L(P)$ is a one-to-one correspondence.

Here, $\Sigma_L(P) := \{\sigma \in \mathfrak{S}_n \mid \sigma(i) \leq \sigma(j) \text{ for all } i, j \in [n] \text{ with } i \preceq_P j\}$.

Next, let us introduce Schur labeled skew shape posets. Let λ/μ be a skew partition of size n. Given a bijective tableau τ of shape λ/μ, we define $\mathsf{poset}(\tau)$ to be the poset $([n], \preceq_\tau)$, where $i \preceq_\tau j$ if and only if i lies weakly northeast of j in τ. A *Schur labeling of shape* λ/μ is a bijective tableau of shape λ/μ such that the entries in each row decrease from left to right and the entries in each column increase from top to bottom. Let $\mathsf{S}(\lambda/\mu)$ be the set of all Schur labelings of shape λ/μ. Since $\tau_0^{\lambda/\mu}$ is a Schur labeling, $\mathsf{S}(\lambda/\mu)$ is nonempty. Set $\mathsf{SP}(\lambda/\mu) := \{\mathsf{poset}(\tau) \mid \tau \in \mathsf{S}(\lambda/\mu)\}$ and $\mathsf{SP}_n := \bigcup_{|\lambda/\mu|=n} \mathsf{SP}(\lambda/\mu)$.

Definition 2.2. A poset in P_n is said to be a *Schur labeled skew shape poset* if it is contained in SP_n.

Example 2.3. When $\lambda/\mu = (2,2)$, we have $\mathsf{S}(\lambda/\mu) = \left\{ \tau_1 := \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array}\ , \tau_2 := \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & 2 \\ \hline \end{array} \right\}$. Therefore, $\mathsf{SP}(\lambda/\mu)$ consists of the following posets:

$\mathsf{poset}(\tau_1) =$  and $\mathsf{poset}(\tau_2) =$  .

For simplicity, we set $\mathsf{RSP}_n := \mathsf{RP}_n \cap \mathsf{SP}_n$.

2.4 The 0-Hecke algebra and the quasisymmetric characteristic

The 0-Hecke algebra $H_n(0)$ is the associative $\mathbb{C}$-algebra with 1 generated by $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the following relations: (1) $\pi_i^2 = \pi_i$ for $1 \le i \le n-1$, (2) $\pi_i \pi_{i+1} \pi_i = \pi_{i+1}\pi_i\pi_{i+1}$ for $1 \le i \le n-2$, and (3) $\pi_i \pi_j = \pi_j \pi_i$ if $|i-j| \ge 2$. According to [13], there are 2^{n-1} pairwise nonisomorphic irreducible $H_n(0)$-modules which are naturally parametrized by compositions of n. For each $\alpha \models n$, the irreducible module $\mathbf{F}_\alpha$ corresponding to α is the 1-dimensional $H_n(0)$-module spanned by a vector v_α, which is annihilated by π_i if $i \in \mathrm{set}(\alpha)$ or fixed by π_i otherwise for all $1 \le i \le n-1$.

Let $\mathcal{G}_0(H_n(0))$ be the *Grothendieck group* of the category of finitely generated left $H_n(0)$-modules and $\mathcal{G}_0(H_\bullet(0)) := \bigoplus_{n\ge 0} \mathcal{G}_0(H_n(0))$ the ring equipped with the induction product. In [7], Duchamp–Krob–Leclerc–Thibon showed that the linear map

$$\mathrm{ch} : \mathcal{G}_0(H_\bullet(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha,$$

called *quasisymmetric characteristic*, is a ring isomorphism. Here, QSym is the ring of quasisymmetric functions and F_α is the *fundamental quasisymmetric function*.

2.5 Modules arising from posets and weak Bruhat interval modules

Definition 2.4. (cf. [6, Definition 3.18]) Let $P \in \mathsf{P}_n$. Define M_P to be the left $H_n(0)$-module with $\mathbb{C}\Sigma_L(P)$ as the underlying space and with the $H_n(0)$-action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma), \\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin \Sigma_L(P), \\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in \Sigma_L(P). \end{cases}$$

For $P \in \mathsf{P}_n$, a map $f : [n] \to \mathbb{Z}_{\ge 0}$ is called a *P-partition* if (i) $f(i) \le f(j)$ for all $i \preceq_P j$ and (ii) $f(i) < f(j)$ for all $i \preceq_P j$ with $i > j$. The *P-partition generating function* is defined by $K_P := \sum_{f:P\text{-partition}} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots$.

Theorem 2.5. ([6, Theorem 3.21(i)]) *For $P \in \mathsf{P}_n$, we have* $\mathrm{ch}([\mathsf{M}_P]) = \psi(K_P)$*, where ψ is the involution of* QSym *defined by $\psi(F_\alpha) = F_{\alpha^c}$.*

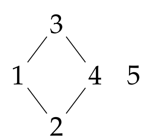
In order to provide a unified method for dealing with $H_n(0)$-modules constructed using tableaux in [1, 3, 14, 16, 17], Jung–Kim–Lee–Oh [9] introduced the *weak Bruhat interval module* $\mathsf{B}(I)$ associated with a left weak Bruhat interval I in $\mathfrak{S}_n$. For $I \in \mathrm{Int}(n)$, $\mathsf{B}(I)$ can be defined as M_P, where P is the unique poset in RP_n such that $\Sigma_L(P) = I$.

3 The weak Bruhat interval structure of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$

First, we introduce a specific Schur labeling associated with a Schur labeled skew shape poset. For $P \in \mathsf{SP}_n$, we define τ_P to be the unique Schur labeling such that

$$\mathrm{sh}(\tau_P) \text{ is basic}, \quad \mathsf{poset}(\tau_P) = P, \quad \text{and} \quad \min_i(\tau_P) < \min_j(\tau_P) \ \text{ for } 1 \le i < j \le k. \tag{3.1}$$

Here, $\min_i(\tau_P)$ is the smallest entry in the ith connected component of τ_P from the top for all $1 \le i \le k$ and k is the number of connected components of P.

Example 3.1. Let $P =$  . There are two Schur labelings τ such that $\mathsf{poset}(\tau) = P$, more precisely,

$$\tau_1 := \begin{array}{|c|c|c|} \hline & 2 & 1 \\ \hline & 4 & 3 \\ \hline 5 & & \\ \hline \end{array} \quad \text{and} \quad \tau_2 := \begin{array}{|c|c|c|} \hline & & 5 \\ \hline 2 & 1 & \\ \hline 4 & 3 & \\ \hline \end{array}\ .$$

Since τ_1 is a Schur labeling and satisfies (3.1), $\tau_P = \tau_1$.

Definition 3.2. Let $P \in \mathsf{SP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. The *$\tau_P$-reading* is the map

$$\mathsf{read}_{\tau_P} : \mathrm{SYT}(\lambda/\mu) \to \mathfrak{S}_n, \quad T \mapsto \mathsf{read}_{\tau_P}(T),$$

where $\mathsf{read}_{\tau_P}(T)$ is the permutation in $\mathfrak{S}_n$ given by $\mathsf{read}_{\tau_P}(T)(k) = T_{\tau_P{}^{-1}(k)}$, the entry of T in the box $\tau^{-1}(k)$, for $1 \le k \le n$.

With these notions, we state the following theorem.

Theorem 3.3. *Let $P \in \mathsf{SP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. Then, $\Sigma_L(P) = \mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$. In particular, if $P \in \mathsf{RSP}_n$, then*

$$\Sigma_L(P) = [\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})]_L.$$

Here, $T_{\lambda/\mu}$(resp. $T'_{\lambda/\mu}$) is the standard Young tableau obtained by filling $\mathtt{yd}(\lambda/\mu)$ by $1, 2, \ldots, n$ from left to right starting with the top row (resp. from top to bottom starting with leftmost column).

Example 3.4. Let P be the poset given in Example 3.1 and $\lambda/\mu = (3,3,1)/(1,1)$. Note that

$$T_{\lambda/\mu} = \begin{array}{ccc} & \boxed{1} & \boxed{2} \\ & \boxed{3} & \boxed{4} \\ \boxed{5} & & \end{array}, \quad T'_{\lambda/\mu} = \begin{array}{ccc} & \boxed{2} & \boxed{4} \\ & \boxed{3} & \boxed{5} \\ \boxed{1} & & \end{array}, \quad \text{and} \quad \tau_P = \begin{array}{ccc} & \boxed{2} & \boxed{1} \\ & \boxed{4} & \boxed{3} \\ \boxed{5} & & \end{array}.$$

Since $\mathsf{read}_{\tau_P}(T_{\lambda/\mu}) = 21435$ and $\mathsf{read}_{\tau_P}(T'_{\lambda/\mu}) = 42531$, we have $\Sigma_L(P) = [21435, 42531]_L$.

4 An equivalence relation on $\mathrm{Int}(n)$

For $I_1, I_2 \in \mathrm{Int}(n)$, a poset isomorphism $f : (I_1, \preceq_L) \to (I_2, \preceq_L)$ is called *descent-preserving* if $\mathrm{Des}_L(\gamma) = \mathrm{Des}_L(f(\gamma))$ for all $\gamma \in I_1$. We define an equivalence relation $\overset{D}{\simeq}$ on $\mathrm{Int}(n)$ by $I_1 \overset{D}{\simeq} I_2$ if there is a descent-preserving poset isomorphism between I_1 and I_2. In [10, Section 4], we show that $\mathsf{B}(I_1) \cong \mathsf{B}(I_2)$ for all $I_1, I_2 \in \mathrm{Int}(n)$ with $I_1 \overset{D}{\simeq} I_2$. This leads us to study the equivalence classes under $\overset{D}{\simeq}$. The following theorem provides significant information regarding equivalence classes under $\overset{D}{\simeq}$.

Theorem 4.1. *Let C be an equivalence class under $\overset{D}{\simeq}$. Then, $\{\sigma \mid [\sigma, \rho]_L \in C\}$ is a right weak Bruhat interval in $\mathfrak{S}_n$.*

According to Theorem 4.1, every equivalence class C can be expressed as follows:

$$C = \{[\gamma, \xi_C \gamma]_L \mid \gamma \in [\sigma_0, \sigma_1]_R\},$$

where $\xi_C := \rho\sigma^{-1}$ for any $[\sigma, \rho]_L \in C$ and $\sigma_0, \sigma_1 \in \mathfrak{S}_n$ with $[\sigma_0, \sigma_1]_R = \{\sigma \mid [\sigma, \rho]_L \in C\}$. When $P \in \mathsf{RSP}_n$, we explicitly describe the equivalence class of $\Sigma_L(P)$ in the following theorem.

Theorem 4.2. *Let $P \in \mathsf{RSP}_n$ and C the equivalence class of $\Sigma_L(P)$ under $\overset{D}{\simeq}$. Then,*

$$C = \{\Sigma_L(Q) \mid Q \in \mathsf{RSP}_n \textit{ with } \mathrm{sh}(\tau_Q) = \mathrm{sh}(\tau_P)\}.$$

Theorem 4.2 tells us that $\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\}$ is closed under $\overset{D}{\simeq}$ and the equivalence classes in this set are parametrized by skew partitions of size n. To be precise, for any skew partition λ/μ of size n, let

$$C_{\lambda/\mu} = \{\Sigma_L(P) \mid P \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_P) = \lambda/\mu\}.$$

This set is nonempty since $\mathsf{poset}(\tau_0^{\lambda/\mu}) \in C_{\lambda/\mu}$, and therefore it is an equivalence class by Theorem 4.1. To summarize, $\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\} = \bigsqcup_{|\lambda/\mu|=n} C_{\lambda/\mu}$ (disjoint union).

5 The classification of M_P's for $P \in \mathsf{RSP}_n$

In [15], Stanley proposed the following conjecture, called *Stanley's P-partitions conjecture*.

Conjecture 5.1. ([15, p. 81]) *For $P \in \mathsf{P}_n$, if K_P is symmetric, then $P \in \mathsf{SP}_n$.*

Assuming Stanley's P-partitions conjecture holds, Theorem 2.5 implies that for any $P \in \mathsf{RSP}_n$ and $Q \in \mathsf{RP}_n \setminus \mathsf{SP}_n$, $\mathrm{ch}([\mathsf{M}_P])$ is symmetric but $\mathrm{ch}([\mathsf{M}_Q])$ is not symmetric, thus $\mathsf{M}_P \not\cong \mathsf{M}_Q$. In addition, by the correspondence between RP_n and $\mathrm{Int}(n)$ in Subsection 2.3,

$$\{\mathsf{M}_P \mid P \in \mathsf{RSP}_n\} = \{\mathsf{B}(I) \mid I \in \mathrm{Int}(n) \text{ and } \mathrm{ch}([\mathsf{B}(I)]) \in \mathrm{Sym}\}.$$

This leads us to consider the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{RSP}_n\}$. We solve this problem by determining the projective cover and injective hull of M_P ($P \in \mathsf{RSP}_n$) up to isomorphism.

It is well known that there is a one-to-one correspondence between the set of irreducible $H_n(0)$-modules and that of projective indecomposable $H_n(0)$-modules. For $\alpha \models n$, let $\mathbf{P}_\alpha$ be the projective indecomposable module corresponding to $\mathbf{F}_\alpha$, that is, $\mathbf{P}_\alpha/\mathrm{rad}(\mathbf{P}_\alpha) \cong \mathbf{F}_\alpha$. In [6, Propsition 4.1], it was shown that $H_n(0)$ is a Frobenius algebra. Thus, an $H_n(0)$-module M is projective if and only if it is injective (see [2, Proposition 1.6.2]).

A *generalized composition* $\boldsymbol{\alpha}$ *of* n is a formal sum $\alpha^{(1)} \oplus \alpha^{(2)} \oplus \cdots \oplus \alpha^{(k)}$, where $\alpha^{(i)} \models n_i$ for positive integers n_i's with $n_1 + n_2 + \cdots + n_k = n$. For a generalized composition $\boldsymbol{\alpha} = \alpha^{(1)} \oplus \alpha^{(2)} \oplus \cdots \oplus \alpha^{(k)}$ of n, set $\boldsymbol{\alpha}^{\mathrm{c}} := (\alpha^{(1)})^{\mathrm{c}} \oplus (\alpha^{(2)})^{\mathrm{c}} \oplus \cdots \oplus (\alpha^{(k)})^{\mathrm{c}}$ and $\boldsymbol{\alpha}^{\mathrm{r}} := (\alpha^{(k)})^{\mathrm{r}} \oplus (\alpha^{(k-1)})^{\mathrm{r}} \oplus \cdots \oplus (\alpha^{(1)})^{\mathrm{r}}$. And, define $\mathbf{P}_{\boldsymbol{\alpha}} := \mathbf{P}_{\alpha^{(1)}} \otimes \mathbf{P}_{\alpha^{(2)}} \otimes \cdots \otimes \mathbf{P}_{\alpha^{(k)}} \uparrow_{H_{n_1}(0) \otimes H_{n_2}(0) \otimes \cdots \otimes H_{n_k}(0)}^{H_n(0)}$, where $n_i := |\alpha_i|$ for $1 \le i \le k$. This module is projective and its decomposition into projective indecomposable modules was provided in [8, Theorem 3.3].

For a connected skew partition λ/μ of size n, define

$$\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) := (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_{\ell(\lambda)} - \mu_{\ell(\lambda)}).$$

And, for a disconnected skew partition λ/μ of size n, define

$$\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu) := \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(1)}/\mu^{(1)}) \oplus \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(2)}/\mu^{(2)}) \oplus \cdots \oplus \boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{(k)}/\mu^{(k)}),$$

where $\lambda/\mu = \lambda^{(1)}/\mu^{(1)} \oplus \lambda^{(2)}/\mu^{(2)} \oplus \cdots \oplus \lambda^{(k)}/\mu^{(k)}$ with connected $\lambda^{(i)}/\mu^{(i)}$'s ($1 \le i \le k$). Set

$$\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu) := (\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda^{\mathrm{t}}/\mu^{\mathrm{t}})^{\mathrm{c}})^{\mathrm{r}},$$

where λ^{t} and μ^{t} denote the transpose of λ and μ, respectively.

Lemma 5.2. *For $P \in \mathsf{RSP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$, $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{proj}}(\lambda/\mu)}$ (resp. $\mathbf{P}_{\boldsymbol{\alpha}_{\mathrm{inj}}(\lambda/\mu)}$) is the projective cover (resp. the injective hull) of M_P.*

Using this lemma, we establish the following classification theorem of M_P's for $P \in \mathsf{RSP}_n$ up to $H_n(0)$-module isomorphism.

Theorem 5.3. *Let $P, Q \in \mathsf{RSP}_n$. Then $\mathsf{M}_P \cong \mathsf{M}_Q$ if and only if $\mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q)$.*

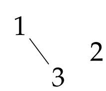
The "if" part can be derived from Theorem 4.2. Let us briefly explain how we prove the "only if" part. Considering Lemma 5.2 together with Huang's decomposition of $\mathbf{P}_\alpha$ in [8, Theorem 3.3], we prove that for $P, Q \in \mathsf{RSP}_n$, M_P and M_Q have either nonisomorphic projective covers or nonisomorphic injective hulls if τ_P and τ_Q have different shapes.

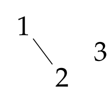
6 A characterization of regular Schur labeled skew shape posets P and distinguished filtrations of M_P

We first characterize regular Schur labeled skew shape posets from the viewpoint of dual plactic congruence. The *Robinson–Schensted correspondence* is a one-to-one correspondence between $\mathfrak{S}_n$ and $\bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$. For $\sigma \in \mathfrak{S}_n$, we use the notation $(\mathtt{ins}(\sigma), \mathtt{rec}(\sigma))$ to represent the image of σ under this bijection. The *dual plactic congruence* is an equivalence relation $\overset{K^*}{\cong}$ on $\mathfrak{S}_n$ defined by $\sigma \overset{K^*}{\cong} \rho$ if $\mathtt{rec}(\sigma) = \mathtt{rec}(\rho)$. A subset S of $\mathfrak{S}_n$ is called *dual plactic-closed* if S is a union of equivalence classes under the dual plactic congruence.

In [11, Theorem 1], Malvenuto proved that if $\Sigma_L(P)$ is dual plactic-closed, then $P \in \mathsf{SP}_n$. We improve Malvenuto's result by providing the following characterization of regular Schur labeled skew shape posets.

Theorem 6.1. *For $P \in \mathsf{P}_n$, P is a regular Schur labeled skew shape poset if and only if $\Sigma_L(P)$ is dual plactic-closed.*

Example 6.2. Consider the posets $P =$ 

and $Q =$ 

in SP_3. One sees that (i) P is non-regular and Q is regular and that (ii) $\Sigma_L(P) = \{231, 312, 321\}$ is not dual plactic-closed and $\Sigma_L(Q) = \{213, 312, 321\}$ is dual plactic-closed.

Using the characterization given in Theorem 6.1, we construct a filtration of M_P ($P \in \mathsf{RSP}_n$) which provides a representation theoretic interpretation of $s_{\lambda/\mu} = \sum_{\nu \vdash n} c^\lambda_{\mu,\nu} s_\nu$, the expansion of the skew Schur function $s_{\lambda/\mu}$ in the Schur basis $\{s_\nu \mid \nu \vdash n\}$. Here, $\lambda/\mu = \mathrm{sh}(\tau_P)$ and $c^\lambda_{\mu,\nu}$ is the Littlewood–Richardson coefficient. To handle such filtrations in a uniform manner, we introduce the notion of distinguished filtrations.

Definition 6.3. Let $\mathcal{B} = \{\mathcal{B}_\alpha \mid \alpha \in I\}$ be a linearly independent subset of QSym_n with the property that $\mathcal{B}_\alpha$ is F-positive for all $\alpha \in I$, where I is an index set. Given a finite dimensional $H_n(0)$-module M, a *distinguished filtration of M with respect to $\mathcal{B}$* is an $H_n(0)$-submodule series of M

$$0 =: M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l := M$$

such that for all $1 \le k \le l$, $\mathrm{ch}([M_k/M_{k-1}]) = \mathcal{B}_\alpha$ for some $\alpha \in I$.

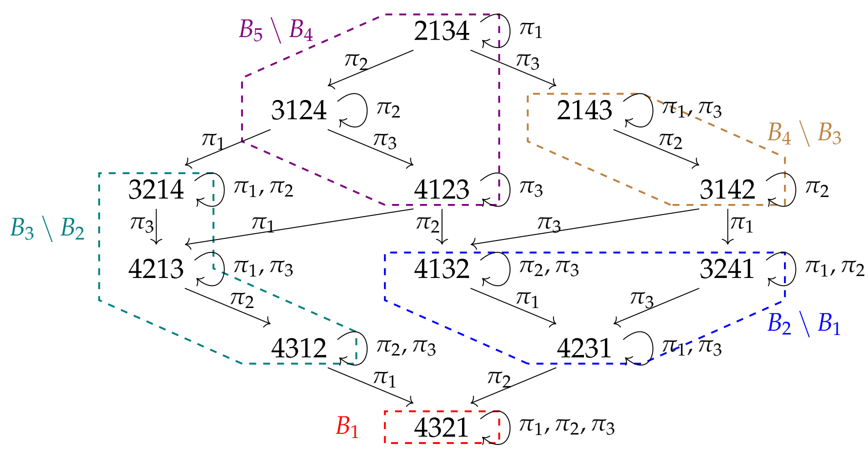
It should be remarked that a distinguished filtration of M with respect to $\mathcal{B}$ may not exist even if $\mathrm{ch}([M])$ expands positively in $\mathcal{B}$. For instance, see [10, Example 6.6]. If such a filtration exists, we have a representation theoretic interpretation of the expansion of $\mathrm{ch}([M])$ in $\mathcal{B}$.

Theorem 6.4. *For every $P \in \mathsf{RSP}_n$, M_P admits a distinguished filtration with respect to the Schur basis $\{s_\lambda \mid \lambda \vdash n\}$.*

Example 6.5. Let $P = \mathsf{poset}(\tau_0^{(4,2,1)/(2,1)})$. The set $\{\mathtt{rec}(\gamma) \mid \gamma \in \Sigma_L(P)\}$ is equal to

$$\left\{ Q_1 := \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}\,,\ Q_2 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}\,,\ Q_3 := \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}\,,\ Q_4 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,,\ Q_5 := \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}.$$

For $0 \le k \le 5$, let $B_k := \{\gamma \in \mathfrak{S}_4 \mid \mathtt{rec}(\gamma) = Q_l \text{ for some } 1 \le l \le k\}$. Then, $0 = \mathbb{C}B_0 \subset \mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \mathbb{C}B_3 \subset \mathbb{C}B_4 \subset \mathbb{C}B_5 = \mathsf{M}_P$ is a filtration of M_P, as seen in the figure:



Moreover, since $\mathrm{ch}([\mathbb{C}B_k/\mathbb{C}B_{k-1}]) = s_{\mathrm{sh}(Q_k)^{\mathrm{t}}}$ for all $1 \le k \le 5$, it is a distinguished filtration with respect to $\{s_\lambda \mid \lambda \vdash 4\}$.

7 A tableau description of M_P for $P \in \mathsf{RSP}_n$

Let λ/μ be a skew partition of size n. Define $X_{\lambda/\mu}$ to be the $H_n(0)$-module with the underlying space $\mathbb{C}\mathrm{SYT}(\lambda/\mu)$ and with the $H_n(0)$-action given by

$$\pi_i \cdot T = \begin{cases} T & \text{if } i \text{ is strictly left of } i+1 \text{ in } T, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same column of } T, \\ s_i \cdot T & \text{if } i \text{ is strictly right of } i+1 \text{ in } T \end{cases}$$

for $1 \leq i \leq n-1$ and $T \in \mathrm{SYT}(\lambda/\mu)$. Here, $s_i \cdot T$ is the tableau obtained from T by swapping i and $i+1$. One can see that this $H_n(0)$-action is well defined and $X_{\lambda/\mu} \cong \mathsf{M}_{\mathsf{poset}(\tau_0^{\lambda/\mu})}$. Theorem 5.3 says that $\mathsf{M}_P \cong X_{\mathrm{sh}(\tau_P)}$ for $P \in \mathsf{RSP}_n$, and $X_{\lambda/\mu} \not\cong X_{\nu/\kappa}$ for distinct skew partitions λ/μ, ν/κ of size n.

Proposition 7.1. *We have the following isomorphisms.*

(1) *For skew partitions λ/μ of size n and ν/κ of size m,*

$$X_{\lambda/\mu} \otimes X_{\nu/\kappa} \uparrow_{H_n(0)\otimes H_m(0)}^{H_{n+m}(0)} \cong X_{\lambda/\mu \oplus \nu/\kappa} \quad \text{as } H_{n+m}(0)\text{-modules.}$$

(2) *For a skew partition λ/μ of size n and $1 \leq k \leq n-1$,*

$$X_{\lambda/\mu} \downarrow_{H_k(0)\otimes H_{n-k}(0)} \cong \bigoplus_{\substack{|\nu/\mu|=k \\ \mu\subset\nu\subset\lambda}} X_{\overline{\nu/\mu}} \otimes X_{\overline{\lambda/\nu}} \quad \text{as } H_k(0) \otimes H_{n-k}(0)\text{-modules.}$$

Here, $\overline{\nu/\mu}$ and $\overline{\lambda/\nu}$ denote the basic skew partitions whose Young diagrams are obtained from $\mathtt{yd}(\nu/\mu)$ and $\mathtt{yd}(\lambda/\nu)$, respectively, by removing empty rows and empty columns.

8 Final remarks

In Theorem 5.3, we show that for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q). \tag{8.1}$$

Since $\mathsf{RSP}_n = \mathsf{RP}_n \cap \mathsf{SP}_n$, it would be natural to consider the classification problem for $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$ and $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$.

(1) Although the notion 'the shape of τ_P' is available for $P \in \mathsf{SP}_n$, (8.1) does not hold for $P, Q \in \mathsf{SP}_n$ in general (see [10, Section 7.1.1]).

(2) Unlike (i), the notion 'the shape of τ_P' is not available for $P \in \mathsf{RP}_n$ in general. For this reason, we modify (8.1) in the following form: for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \text{if and only if} \quad \Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q), \tag{8.2}$$

which can be obtained by combining Theorem 4.2 and Theorem 5.3. Since the equivalence relation $\overset{D}{\simeq}$ is defined on $\mathrm{Int}(n) = \{\Sigma_L(P) \mid P \in \mathsf{RP}_n\}$, we expect that this classification can be extended to RP_n in its current form. The validity of this expectation has been checked for values of n up to 6 with the aid of the computer program SageMath. Also, we show that (8.2) holds when $P \in \mathsf{RSP}_n$, $Q \in \mathsf{RP}_n$, and $\mathrm{ch}([\mathsf{M}_P])$ is a Schur function. For more detail, see [10, Section 7.1.2].

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Acyclonestohedra

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Abstract. Given a building set $\mathcal{B}$ and an oriented matroid $\mathcal{M}$ on the same ground set, we define the acyclic nested complex as the simplicial complex of nested sets on $\mathcal{B}$ which are in some sense acyclic with respect to $\mathcal{M}$. We prove that this complex is always the face lattice of an oriented matroid, obtained as a stellar subdivision of the positive tope of the oriented matroid $\mathcal{M}$. When the oriented matroid $\mathcal{M}$ is the oriented matroid of a vector configuration $\boldsymbol{A}$, we moreover prove that this complex is the boundary complex of an acyclonestohedron, a polytope obtained as the section of a nestohedron for $\mathcal{B}$ by the evaluation space of $\boldsymbol{A}$. Our work specializes to explicit polytopal realizations of the poset associahedra and affine poset cyclohedra of Galashin.

Résumé. Étant donné un ensemble de construction $\mathcal{B}$ et un matroïde orienté $\mathcal{M}$ sur le même ensemble, nous définissons le complexe imbriqué acyclique comme le complexe simplicial des ensembles imbriqués de $\mathcal{B}$ qui sont acycliques pour $\mathcal{M}$ en un certain sens. Nous montrons que ce complexe est toujours le treillis des faces d'un matroïde orienté, obtenu par subdivisions stellaires du tope positif du matroïde orienté $\mathcal{M}$. Quand le matroïde orienté $\mathcal{M}$ est le matroïde orienté d'une configuration de vecteurs $\boldsymbol{A}$, nous montrons que ce complexe est le complexe de bord d'un acyclonestoèdre, un polytope obtenu comme la section du nestoèdre de $\mathcal{B}$ par l'espace vectoriel des évaluations linéaires sur $\boldsymbol{A}$. Notre travail se spécialise à des réalisations polytopales explicites des associaèdres d'ordres et des cycloèdres d'ordres affines de Galashin.

Keywords: building sets, nested complexes, oriented matroids, poset associahedra

Introduction

Motivated by the recent work of Galashin on poset associahedra and affine poset cyclohedra [8], we introduce the acyclic nested complexes and the acyclonestohedra, some simplicial complexes and polytopes at the interface between nestohedra [10, 4, 6, 12] (Section 1.1) and oriented matroids [1] (Section 1.2).

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The input data is an oriented building set $(\mathcal{B}, \mathcal{M})$ (Section 2.1), that is, a building set $\mathcal{B}$ and an oriented matroid $\mathcal{M}$ on the same ground set so that any circuit of $\mathcal{M}$ is a block of $\mathcal{B}$. The acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is the simplicial complex of nested sets on $\mathcal{B}$ which are in some sense acyclic with respect to $\mathcal{M}$ (Section 2.2).

Prototypical examples are graphical oriented building sets. The graphical oriented building set of a directed graph D is formed by the graphical building set of the line graph $L(D)$ together with the graphical oriented matroid of D. The graphical acyclic nested complex is then given by all tubings T on $L(D)$ such that for each tube $t \in T$, the contraction in the restriction $D_{|t}$ of all arcs contained in some tube $s \in T$ with $s \subsetneq t$ yields an acyclic directed graph. It is not difficult to see that this definition actually only depends upon the transitive closure of D and coincides with the poset associahedron of [8]. A similar (but slightly more intricate) construction shows that the affine poset cyclohedra of [8] are also acyclic nested complexes of specific oriented building sets.

Our main results are geometric realizations of acyclic nested complexes (Sections 2.3 and 2.4). We show that the acyclic nested complex of an oriented building set $(\mathcal{B}, \mathcal{M})$ is

(i) the face lattice of an oriented matroid obtained by stellar subdivisions of $\mathcal{M}$,
(ii) the boundary complex of a convex polytope, obtained by stellar subdivisions of the positive tope of $\mathcal{M}$ when the latter is realizable,
(iii) the boundary complex of the polar of the acyclonestohedron, a polytope obtained as the section of a nestohedron for $\mathcal{B}$ with the evaluation space of $\boldsymbol{A}$, when $\mathcal{M}$ is realized by the vector configuration $\boldsymbol{A}$.

Note that (i) is valid for all oriented matroids (realizable or not), while (ii) and (iii) only apply to realizable oriented matroids. The advantage of (iii) over (ii) is that it leads to explicit realizations with controlled integer coordinates. For poset associahedra and affine poset cyclohedra, (ii) recovers the construction of [8] using stellar subdivisions of order polytopes, and (iii) answers a question left open in [8], and independently settled in [11].

In fact, the oriented building sets and their acyclic nested complexes are closely related to the lattice building sets and their lattice nested complexes of [4, 6]. Namely, we show that the building sets on the Las Vergnas face lattice of $\mathcal{M}$ are obtained from the oriented building sets $(\mathcal{B}, \mathcal{M})$ by keeping only the blocks of $\mathcal{B}$ which are faces of $\mathcal{M}$, and that the two notions of nested complexes coincide (Section 3). We exploit this correspondence in both directions: we recover our results on stellar subdivisions as reformulations of [5, 4], and we use our acyclonestohedra to get explicit polytopal realizations with integer coordinates for the all nested complexes over face lattices of realizable matroids.

Finally, Galashin's main motivation for poset associahedra was that they model compactifications of the space of order preserving maps $P \to \mathbb{R}$, which can be identified with the interior of an order polytope. We observe that results of [7] imply that acyclonestohedra are associated to nice compactifications of interiors of polytopes (Section 4).

Many details and all proofs are omitted in this extended abstract due to space limitations. We refer to the long version of this work which should soon become available.

1 Preliminaries

1.1 Building sets, nested complexes, and nestohedra

We start with the classical definitions of building sets, nested sets, nested complexes, and nestohedra from [10, 4, 6, 12] and their specializations to the graphical case [2].

Definition 1.1 ([10, 4, 6])**.** A *building set* on S is a set $\mathcal{B}$ of non-empty subsets of S such that

- $\mathcal{B}$ contains all singletons $\{s\}$ for $s \in S$, and
- if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \varnothing$, then $B \cup B' \in \mathcal{B}$.

We denote by $\kappa(\mathcal{B})$ its set of *$\mathcal{B}$-connected components*, *i.e.,* its inclusion maximal elements.

Example 1.2 ([2])**.** Consider a graph G on S. A *tube* of G is a non-empty subset of S which induces a connected subgraph of G. The set $\mathcal{B}(G)$ of all tubes of G is a *graphical building set*. Moreover, the blocks of $\kappa(\mathcal{B}(G))$ are the connected components of G.

Remark 1.3 ([3])**.** More generally, a hypergraph H on S defines a building set $\mathcal{B}(H)$ on S given by all non-empty subsets of S which induce connected subhypergraphs of H. Any building set $\mathcal{B}$ on S is the building set of a hypergraph, but not always of a graph.

Definition 1.4 ([10, 4, 6])**.** A *nested set* is a subset $\mathcal{N}$ of $\mathcal{B}$ containing $\kappa(\mathcal{B})$ such that

- for any $B, B' \in \mathcal{N}$, either $B \subseteq B'$ or $B' \subseteq B$ or $B \cap B' = \varnothing$,
- for any $k \geq 2$ pairwise disjoint $B_1, \dots, B_k \in \mathcal{N}$, the union $B_1 \cup \dots \cup B_k$ is not in $\mathcal{B}$.

The *nested complex* of $\mathcal{B}$ is the simplicial complex $\mathfrak{N}(\mathcal{B})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all nested sets $\mathcal{N}$ on $\mathcal{B}$.

Example 1.5 ([2])**.** Consider a graph G on S. Two tubes t, t' of G are *compatible* if they are either nested (*i.e.,* $t \subseteq t'$ or $t' \subseteq t$), or disjoint and non-adjacent (*i.e.,* $t \cup t' \notin \mathcal{B}(G)$). A *tubing* on G is a set T of pairwise compatible tubes of G containing all connected components $\kappa(G)$. Tubings are precisely the nested sets of the graphical building set $\mathcal{B}(G)$. The nested complex $\mathfrak{N}(\mathcal{B}(G))$ is a *graphical nested complex*.

We now introduce restrictions and contractions of building sets. These operations are used to describe links of nested complexes [12, Prop. 3.2], and will be crucial here to define acyclic nested complexes.

Definition 1.6. For any $R \subseteq S$, define

- the *restriction* of $\mathcal{B}$ to R as the building set $\mathcal{B}_{|R} := \{B \in \mathcal{B} \mid B \subseteq R\}$ on R,
- the *contraction* of R in $\mathcal{B}$ as the building set $\mathcal{B}_{/R} := \{B \smallsetminus R \mid B \in \mathcal{B},\ B \not\subseteq R\}$ on $S \smallsetminus R$.

Example 1.7. For a graph G on S and $R \subseteq S$,

- $\mathcal{B}(G)_{|R} = \mathcal{B}(G_{|R})$ where $G_{|R}$ is the subgraph of G induced by R,
- $\mathcal{B}(G)_{/R} = \mathcal{B}(G_{/R})$ where $G_{/R}$ is the *reconnected complement* of R in G, *i.e.,* the graph on $S \smallsetminus R$ with an edge $\{r, s\}$ if there is a path between r and s in G with vertices in $R \cup \{r, s\}$, see [2].

Finally, we recall the definition of the nestohedron which realizes the nested complex. See for instance Figure 1. We denote by $(\boldsymbol{e}_s)_{s\in S}$ the standard basis of $\mathbb{R}^S$. For a building set $\mathcal{B}$, we denote by $\mathbb{R}^{\mathcal{B}}_+ := \{\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}} \mid \lambda_B > 0 \text{ for all } B \in \mathcal{B} \text{ with } |B| \ge 2\}$.

Definition 1.8 ([10, 6]). For a building set $\mathcal{B}$ and a positive vector $\boldsymbol{\lambda} = (\lambda_B)_{B\in\mathcal{B}} \in \mathbb{R}^{\mathcal{B}}_+$, the *nestohedron* $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$ is the Minkowski sum $\sum_{B\in\mathcal{B}} \lambda_B \triangle_B$, where $\triangle_B := \operatorname{conv}\{\boldsymbol{e}_b \mid b \in B\}$ denotes the face of the standard simplex $\triangle_S$ corresponding to B.

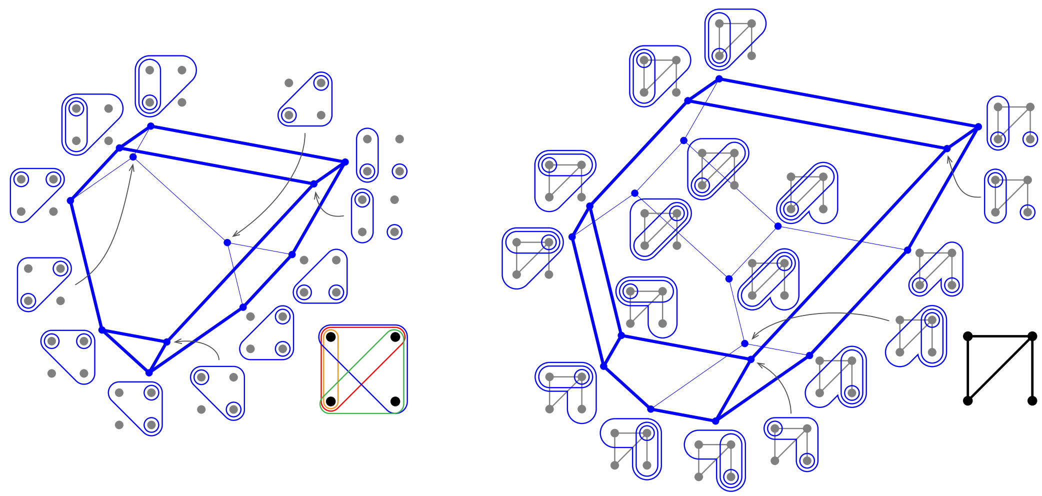

Figure 1: A nestohedron whose vertices are labeled by the corresponding maximal nested sets (left), and a graph associahedron whose vertices are labeled by the corresponding maximal tubings (right). The maximal block or tubing is always omitted.

Theorem 1.9 ([10, 6, 12]). *For a building set $\mathcal{B}$ and any $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}}_+$, the nested complex $\mathfrak{N}(\mathcal{B})$ is isomorphic to the boundary complex of the polar of the nestohedron* $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$.

Proposition 1.10. *For $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}}_+$, the vertex of the nestohedron $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$ corresponding to a maximal nested set $\mathcal{N}$ is*

$$\boldsymbol{v}(\mathcal{N}, \boldsymbol{\lambda}) = \sum_{s\in S} \sum_{B\in\mathcal{B},\ s\in B\subseteq B(v,\mathcal{N})} \lambda_B \boldsymbol{e}_s,$$

where $B(s, \mathcal{N})$ denotes the inclusion minimal block of $\mathcal{N}$ containing s.

Proposition 1.11. *For $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{B}}_+$, the nestohedron $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\lambda})$ is given by the equalities $g_B(\boldsymbol{x}) = 0$ for all $B \in \kappa(\mathcal{B})$ and the inequalities $g_B(\boldsymbol{x}) \ge 0$ for all $B \in \mathcal{B}$, where*

$$g_B(\boldsymbol{x}) := \Big\langle \sum_{b\in B} \boldsymbol{e}_b \;\Big|\; \boldsymbol{x} \Big\rangle - \sum_{B'\in\mathcal{B},\ B'\subseteq B} \lambda_{B'}.$$

Example 1.12. For a graph G on S, the nestohedra of $\mathcal{B}(G)$ are the *graph associahedra* of G, introduced in [2]. For instance, the associahedron of the complete graph is a permutahedron, the associahedron of a path graph is an associahedron, and the associahedron of a cycle graph is a cyclohedron.

1.2 Vector configurations and oriented matroids

We now recall some aspects of oriented matroids. We only give the precise definition for those associated to vector configurations and refer to [1] for the general definition.

Definition 1.13. For a finite vector configuration $\boldsymbol{A} := (\boldsymbol{a}_s)_{s\in S} \in (\mathbb{R}^d)^S$, we denote by
- $\mathcal{D}(\boldsymbol{A}) := \{\boldsymbol{\delta} \in \mathbb{R}^S \mid \sum_{s\in S} \delta_s \boldsymbol{a}_s = \boldsymbol{0}\}$ the space of linear *dependences* on $\boldsymbol{A}$,
- $\mathcal{D}^*(\boldsymbol{A}) := \{(f(\boldsymbol{a}_s))_{s\in S} \in \mathbb{R}^S \mid f \in (\mathbb{R}^d)^*\}$ the space of *evaluations* of linear forms on $\boldsymbol{A}$.

Note that, $\mathcal{D}^*(\boldsymbol{A})$ and $\mathcal{D}(\boldsymbol{A})$ are orthogonal spaces whose dimensions are the *rank* $\mathrm{rk}(\boldsymbol{A})$ and the he *corank* $\mathrm{rk}^*(\boldsymbol{A}) := |S| - \mathrm{rk}(\boldsymbol{A})$ of $\boldsymbol{A}$ respectively.

Notation 1.14. Define $\sigma(S) := \{(x_+, x_-) \mid x_+, x_- \subseteq S \text{ and } x_+ \cap x_- = \varnothing\}$. The *signature* of $\boldsymbol{\delta} \in \mathbb{R}^S$ is $\sigma(\boldsymbol{\delta}) := (\{s \in S \mid \delta_s > 0\}, \{s \in S \mid \delta_s < 0\})$ in $\sigma(S)$. For $x = (x_+, x_-) \in \sigma(S)$, we define the *support* of x by $\underline{x} := x_+ \cup x_-$, and the *opposite* of x by $-x := (x_-, x_+)$.

Definition 1.15. The *oriented matroid* $\mathcal{M}(\boldsymbol{A})$ of a finite vector configuration $\boldsymbol{A} \subset \mathbb{R}^d$ is the combinatorial data given equivalently by
- the *vectors* $\mathcal{V}(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.*, signatures of linear dependences of $\boldsymbol{A}$,
- the *covectors* $\mathcal{V}^*(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.*, signatures of linear evaluations on $\boldsymbol{A}$,
- the *circuits* $\mathcal{C}(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.*, support minimal signatures of linear dependences of $\boldsymbol{A}$,
- the *cocircuits* $\mathcal{C}^*(\boldsymbol{A})$ of $\boldsymbol{A}$, *i.e.*, support minimal signatures of linear evaluations on $\boldsymbol{A}$.

Example 1.16. Consider a directed graph D with vertex set V and arc set S (loops and multiple arcs are allowed). Let $(\boldsymbol{b}_v)_{v\in V}$ denote the standard basis of $\mathbb{R}^V$. The *incidence configuration* $\boldsymbol{A}_D$ of D has a vector $\boldsymbol{a}_{(u,v)} := \boldsymbol{b}_u - \boldsymbol{b}_v \in \mathbb{R}^V$ for each arc (u, v) of D. Its oriented matroid, whose ground set is the set S of arcs of D, is the *graphical oriented matroid* $\mathcal{M}(D)$ of D. See [9, Prop. 1.1.7 & Chap. 5] and [1, Sect. 1.1].

In this paper, we will consider abstract oriented matroids, which are combinatorial abstractions for the dependences and evaluations of vector configurations considered in Definitions 1.13 and 1.15. We rest on [1] to avoid the detailed axioms.

Definition 1.17. An *oriented matroid* on S is the combinatorial data $\mathcal{M}$ given by four subsets of $\sigma(S)$, the *vectors* $\mathcal{V}(\mathcal{M})$, *covectors* $\mathcal{V}^*(\mathcal{M})$, *circuits* $\mathcal{C}(\mathcal{M})$ and *cocircuits* $\mathcal{C}^*(\mathcal{M})$, which satisfy the axioms of [1, Sect. 3].

Definition 1.18. An oriented matroid $\mathcal{M}$ is *realizable* if there exists a vector configuration $\boldsymbol{A} := (\boldsymbol{a}_s)_{s\in S} \in (\mathbb{R}^d)^S$ such that $\mathcal{V}(\mathcal{M}) = \mathcal{V}(\boldsymbol{A})$, $\mathcal{V}^*(\mathcal{M}) = \mathcal{V}^*(\boldsymbol{A})$, $\mathcal{C}(\mathcal{M}) = \mathcal{C}(\boldsymbol{A})$, and $\mathcal{C}^*(\mathcal{M}) = \mathcal{C}^*(\boldsymbol{A})$ (these four conditions are actually equivalent).

Definition 1.19. An oriented matroid $\mathcal{M}$ is acyclic if it has no positive circuit.

Example 1.20. A realizable oriented matroid $\mathcal{M}(\boldsymbol{A})$ is acyclic if and only if $\boldsymbol{A}$ has no positive dependence, *i.e.*, if and only if $\boldsymbol{A}$ is contained in a positive linear half-space of $\mathbb{R}^d$. A graphical oriented matroid $\mathcal{M}(D)$ is acyclic if and only if D is acyclic (no directed cycle).

Definition 1.21. Let $\mathcal{M}$ be an acyclic oriented matroid. A set $F \subseteq S$ is a *face* of $\mathcal{M}$ if it is the complement of a non-negative covector, *i.e.*, $(S \smallsetminus F, \varnothing) \in \mathcal{V}^*(\mathcal{M})$. The *Las Vergnas face lattice* $\mathcal{F}(\mathcal{M})$ is the poset of faces of $\mathcal{M}$ ordered by inclusion.

We conclude with restrictions and contractions in oriented matroids.

Definition 1.22. For any $R \subseteq S$, define

- the *restriction* $\mathcal{M}_{|R}$ as the oriented matroid on R with circuits $\{c \in \mathcal{C}(\mathcal{M}) \mid \underline{c} \subseteq R\}$,
- the *contraction* $\mathcal{M}_{/R}$ as the oriented matroid on $S \smallsetminus R$ with vectors $\{v \smallsetminus R \mid v \in \mathcal{V}(\mathcal{M})\}$, where $v \smallsetminus R := (v_+ \smallsetminus R, v_- \smallsetminus R)$.

Example 1.23. For a vector configuration $\boldsymbol{A} := (\boldsymbol{a}_s)_s \in S$ and $R \subseteq S$,

- $\mathcal{M}(\boldsymbol{A})_{|R} = \mathcal{M}(\boldsymbol{A}_{|R})$ where $\boldsymbol{A}_{|R}$ is the vector subconfiguration $(\boldsymbol{a}_r)_{r \in R}$,
- $\mathcal{M}(\boldsymbol{A})_{/R} = \mathcal{M}(\boldsymbol{A}_{/R})$ where $\boldsymbol{A}_{/R}$ is the vector configuration obtained by projecting the vectors $\boldsymbol{a}_s$ with $s \notin R$ on the space orthogonal to all vectors $\boldsymbol{a}_r$ with $r \in R$.

For a directed graph D and a subset R of arcs of D,

- $\mathcal{M}(D)_{|R} = \mathcal{M}(D_{|R})$ where $D_{|R}$ is the subgraph of D formed by the arcs in R,
- $\mathcal{M}(D)_{/R} = \mathcal{M}(D_{/R})$ where $D_{/R}$ is the contraction of the arcs of R in D.

2 Acyclic nested complexes and acyclonestohedra

2.1 Oriented building sets

Definition 2.1. An *oriented building set* is a pair $(\mathcal{B}, \mathcal{M})$ where $\mathcal{B}$ is a building set and $\mathcal{M}$ is an oriented matroid on the same ground set S such that $\underline{c} \in \mathcal{B}$ for any $c \in \mathcal{C}(\mathcal{M})$. We say that $(\mathcal{B}, \mathcal{M})$ is *realizable* if $\mathcal{M}$ is realizable.

Example 2.2. Consider a directed graph D with vertex set V and arc set S. The *line graph* of D is the graph $L(D)$ on S with an edge between two arcs of D if and only if they share an endpoint. The *graphical oriented building set* of D is the pair $(\mathcal{B}(L(D)), \mathcal{M}(D))$. Note that it is indeed an oriented building set: S is the ground set of both $\mathcal{B}(L(D))$ and $\mathcal{M}_D$, and the circuits in $\mathcal{M}(D)$ are cycles in D, hence of $L(D)$, thus belong to $\mathcal{B}(L(D))$.

Lemma 2.3. *If $(\mathcal{B}, \mathcal{M})$ is an oriented building set on S and $R \subseteq S$, then both $(\mathcal{B}_{|R}, \mathcal{M}_{|R})$ and $(\mathcal{B}_{/R}, \mathcal{M}_{/R})$ are oriented building sets on R and $S \smallsetminus R$ respectively.*

Definition 2.4. Given an oriented building set $(\mathcal{B}, \mathcal{M})$, a nested set $\mathcal{N}$ on $\mathcal{B}$ and $B \in \mathcal{N}$, we consider the oriented buiding set $(\mathcal{B}, \mathcal{M})_{B \in \mathcal{N}} := (\mathcal{B}_{B \in \mathcal{N}}, \mathcal{M}_{B \in \mathcal{N}})$ on $S_{B \in \mathcal{N}} := B \smallsetminus R$ defined by $\mathcal{B}_{B \in \mathcal{N}} := (\mathcal{B}_{|B})_{/R}$ and $\mathcal{M}_{B \in \mathcal{N}} := (\mathcal{M}_{|B})_{/R}$, where $R = R_{B \in \mathcal{N}} := \bigcup_{B' \in \mathcal{N},\, B' \subsetneq B} C$.

Example 2.5. Consider the graphical oriented building set of a directed graph D of Example 2.2, and a tube t in a tubing T of $L(D)$. The oriented building set $(\mathcal{B}(L(D)), \mathcal{M}(D))_{t \in T}$ is the graphical oriented building set of the directed graph obtained as the contraction in the restriction $D_{|t}$ of all arcs contained in some tube $s \in T$ with $s \subsetneq t$.

2.2 Acyclic nested complexes

Definition 2.6. The *acyclic nested complex* of an oriented building set $(\mathcal{B}, \mathcal{M})$ is the simplicial complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all nested sets $\mathcal{N}$ of $\mathcal{B}$ such that $\mathcal{M}_{B \in \mathcal{N}}$ is acyclic for every $B \in \mathcal{N}$.

Remark 2.7. Observe that:

- For any building set $\mathcal{B}$ on S, the nested complex $\mathfrak{N}(\mathcal{B})$ is the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{I})$, where $\mathcal{I}$ is the independent (*i.e.*, no circuit) oriented matroid on S.
- If $\mathcal{M}$ is not acyclic, then the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is empty.
- If $\mathcal{M}$ contains a circuit $c = (c_+, c_-)$ with $|c_-| = 1$, then $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is isomorphic to $\mathfrak{A}(\mathcal{B}_{|S \smallsetminus \{s\}}, \mathcal{M}_{|S \smallsetminus \{s\}})$.

Example 2.8. From Example 2.2, consider a directed graph D and its graphical oriented building set $(\mathcal{B}(L(D)), \mathcal{M}(D))$. The *graphical acyclic nested complex* $\mathfrak{A}(\mathcal{B}(L(D)), \mathcal{M}(D))$ is then given by all tubings T on $L(D)$ such that for each tube $t \in T$, the contraction in the restriction $D_{|t}$ of all arcs contained in some tube $s \in T$ with $s \subsetneq t$ yields an acyclic directed graph. Figure 2 illustrates two graphical acyclic nested complexes. Note that these two directed graphs have the same line graph, but distinct graphical oriented matroids, and thus distinct graphical acyclic nested complexes.

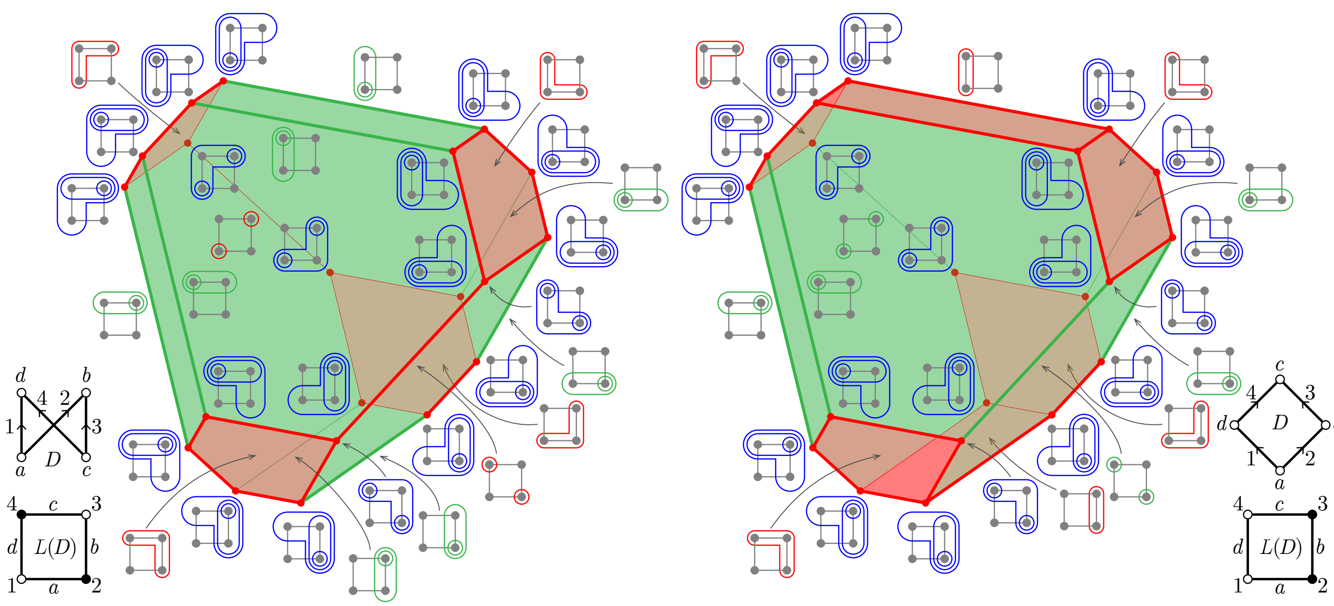

Figure 2: Two graphical acyclic nested complexes. For each one, we have drawn the directed graph D, its line graph $L(D)$ with vertices colored black and white according to the sign of the corresponding arcs in the only circuit of D, and all tubings of $L(D)$ labeling the faces of the graph associahedron, colored green if acyclic and red if cyclic.

Remark 2.9. It follows from Remark 2.7 that the graphical acyclic nested complex of D is

- isomorphic to the classical nested complex of the line graph $L(D)$ when D is an oriented forest (for instance, it is isomorphic to the simplicial permutahedron if D is a star, and to the simplicial associahedron if D is a path),
- empty if D is cyclic (*i.e.,* has an oriented cycle),
- isomorphic to the graphical acyclic nested complex of the Hasse diagram of the transitive closure of D if D is acyclic.

Hence, graphical acyclic nested complexes are in fact intrinsically associated to posets. The graphical case of Examples 2.2 and 2.8 actually motivated Definitions 2.1 and 2.6, and was inspired from the poset associahedra defined in [8]. The following statement can serve as definition of poset associahedra, which we omit here for space reason.

Proposition 2.10. *The poset associahedron of a finite poset P defined in [8] is isomorphic to the graphical acyclic nested set of the Hasse diagram of P.*

We note that affine poset associahedra of [8] are also acyclic nested complexes of certain specific oriented building sets, although their definition is slightly more intricate.

2.3 Stellar subdivisions

We now show that the acyclic nested complex of any oriented building set (realizable or not) is always the face lattice of an oriented matroid, hence a topological sphere [1, Thm. 4.3.5]. The main tool here is that of stellar subdivisions.

Definition 2.11. For a cell σ in a regular cell complex Δ, the *stellar subdivision* $\mathrm{sd}(\Delta, \sigma)$ is the cell complex obtained by gluing the cone $s * (\overline{\mathrm{star}}(\sigma, \Delta) \smallsetminus \mathrm{star}(\sigma, \Delta))$ to $\Delta \smallsetminus \mathrm{star}(\sigma, \Delta)$ along $\overline{\mathrm{star}}(\sigma, \Delta) \smallsetminus \mathrm{star}(\sigma, \Delta)$, where s is a new vertex, and $\mathrm{star}(\sigma, \Delta) := \{\tau \in \Delta \mid \sigma \subseteq \tau\}$ is the star of σ and $\overline{\mathrm{star}}(\sigma, \Delta) := \{\rho \in \Delta \mid \rho \subseteq \tau \text{ for some } \tau \in \mathrm{star}(\sigma, \Delta)\}$ is its closure.

Proposition 2.12 ([1, Prop. 9.2.3 & Sect. 7.2])**.** *Let $\mathcal{M}$ be an acyclic oriented matroid with ground set S, and F be one of its proper faces. Then the face lattice of the stellar subdivision $\mathrm{sd}(\Delta(\mathcal{M}), F)$ is isomorphic to the face lattice of an oriented matroid on $S \cup \{F\}$ (this oriented matroid is not unique, but its face lattice is). Moreover, this oriented matroid can be chosen to be realizable when $\mathcal{M}$ is realizable.*

Theorem 2.13. *For any oriented building set $(\mathcal{B}, \mathcal{M})$ (realizable or not), the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is the face lattice of an oriented matroid, obtained by stellar subdivisions of $\mathcal{M}$.*

Corollary 2.14. *For any realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$, the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ is isomorphic to the boundary complex of a convex polytope, obtained by stellar subdivisions of the positive tope of $\boldsymbol{A}$.*

Example 2.15. In the graphical situation discussed in Examples 2.2 and 2.8, Remark 2.9, and Proposition 2.10, we obtain that the poset associahedron of a poset P can be realized as a stellar subdivision of the order polytope of P, thus recovering the construction of [8].

2.4 Acyclonestohedra

We now consider a realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$. From Corollary 2.14, we know that the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ is realizable as a polytope by stellar subdivisions of the positive tope of $\boldsymbol{A}$. However, this non-explicit approach does not allow any control on the coordinates of the realizations. In this section, we obtain explicit polytopal realizations with controlled integer coordinates, using sections of nestohedra.

Definition 2.16. As each $c \in \mathcal{C}(\boldsymbol{A})$ is the signature of a unique (up to rescaling) linear dependence $\delta \in \mathcal{D}(\boldsymbol{A})$, we define $r_c := \max \delta^{\neq 0} / \min \delta^{\neq 0}$ where $\delta^{\neq 0} := \{|\delta_s| \mid s \in S\} \smallsetminus \{0\}$ and $R := |\mathcal{B}| \cdot \max_{c \in \mathcal{C}(\boldsymbol{A})} r_c$. We then define $\boldsymbol{\rho} := (\rho_B)_{B \in \mathcal{B}} \in \mathbb{R}_+^{\mathcal{B}}$ by $\rho_B := 0$ if $|B| = 1$ and $\rho_B := R^{|B|}$ if $|B| \geq 2$.

We use these coefficients $\boldsymbol{\rho} \in \mathbb{R}_+^{\mathcal{B}}$ to define two polytopes $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ and $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$ that we both call *acyclonestohedra*. While these two polytopes are affinely equivalent, the first is more natural for our construction, but the second has the advantage to live in the right dimensional space.

Definition 2.17. The *acyclonestohedron* $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ is the polytope in $\mathbb{R}^S$ defined as the intersection of the nestohedron $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\rho})$ with the evaluation space $\mathcal{D}^*(\boldsymbol{A})$ of $\boldsymbol{A}$.

Definition 2.18. The *acyclonestohedron* $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$ is the polytope of $\mathbb{R}\boldsymbol{A}$ defined by the equalities $\overline{g}_B(\boldsymbol{y}) = 0$ for all $B \in \kappa(\mathcal{B})$ and the inequalities $\overline{g}_B(\boldsymbol{y}) \geq 0$ for all $B \in \mathcal{B}$, where

$$\overline{g}_B(\boldsymbol{y}) := \Big\langle \sum_{b \in B} \boldsymbol{a}_b \;\Big|\; \boldsymbol{y} \Big\rangle - \sum_{B' \in \mathcal{B},\, B' \subseteq B} \rho_{B'}.$$

Proposition 2.19. *The acyclonestohedron* $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A}) \subset \mathbb{R}^S$ *of Definition 2.17 and the acyclonestohedron* $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A}) \subset \mathbb{R}\boldsymbol{A}$ *of Definition 2.18 are affinely equivalent.*

Theorem 2.20. *For any realizable oriented building set* $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$*, the acyclic nested complex* $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ *is isomorphic to the boundary complex of the polar of the acyclonestohedron* $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ *(or equivalently of* $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$*).*

Remark 2.21. Following Remark 2.7, note that if $\boldsymbol{A}$ is linearly independent, then its evaluation space $\mathcal{D}^*(\boldsymbol{A})$ is $\mathbb{R}^S$, and the acyclonestohedra $\mathsf{Acyc}(\mathcal{B}, \boldsymbol{A})$ and $\mathsf{A}\overline{\mathsf{cyc}}(\mathcal{B}, \boldsymbol{A})$ both coincide with the classical nestohedron $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\rho})$. For instance, the acyclonestohedron of the graphical oriented building set of an oriented forest D is the graph associahedron of $L(D)$ (for instance, a permutahedron if D is a star, and an associahedron if D is a path).

Example 2.22. Specializing Definitions 2.17 and 2.18 and Theorem 2.20 to the graphical situation discussed in Examples 2.2 and 2.8, Remark 2.9, and Proposition 2.10, we obtain that the poset associahedron of a poset P with Hasse diagram D is explicitly realized as

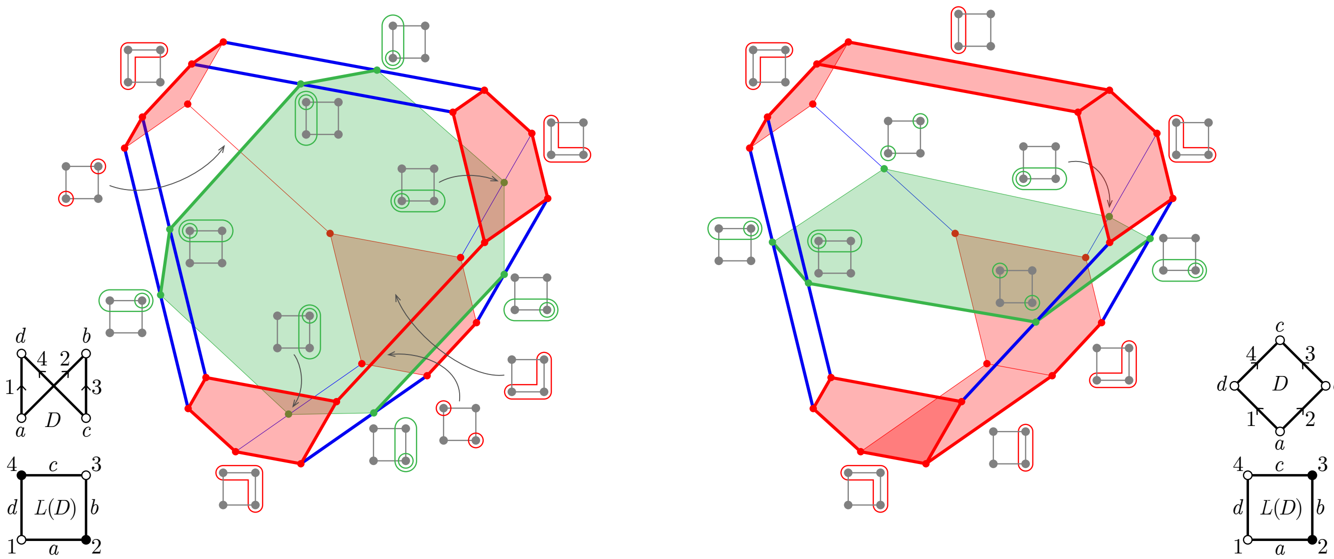

Figure 3: The graphical acyclonestohedra (green polygons) realizing the graphical acyclic nested complexes of Figure 2, obtained as the section of the line graph of $L(D)$ by the evaluation space of the graphical oriented matroid of D.

- the section of a graph associahedron of the line graph of D with the linear hyperplanes normal to $\mathbb{1}_{c_+} - \mathbb{1}_{c_-}$ for all circuits $c = (c_+, c_-)$ of D, see Figure 3,
- the polytope in $\mathbb{R}^P$ defined by the equality $\overline{g}_P(\boldsymbol{y}) = 0$ and the inequalities $\overline{g}_t(\boldsymbol{y}) \geq 0$ for all $t \in \mathcal{B}(\mathrm{P})$, where

$$\overline{g}_t(\boldsymbol{y}) := \Big\langle \sum_{\substack{p,q \in t \\ p \prec q}} \boldsymbol{b}_p - \boldsymbol{b}_q \;\Big|\; \boldsymbol{y} \Big\rangle - \sum_{\substack{t' \in \mathcal{B}(\mathrm{P}) \\ t' \subseteq t}} |\mathcal{B}(\mathrm{P})|^{|t|}.$$

This answers an open question of [8]. During the completion of this paper, we became aware that this question was independently solved in [11]. The approach of [11] is quite different but leads essentially to the same realization of poset associahedra. We actually want to acknowledge that we originally only worked with the acyclonestohedron of Definition 2.17, and that the affinely equivalent acyclonestohedron of Definition 2.18 was motivated by the approach of [11].

Remark 2.23. To conclude this section, we want to give a vague idea of the proof of Theorem 2.20. As illustrated in Figures 2 and 3, the main point is that our choice of coefficients $\boldsymbol{\rho}$ guaranties that a face of $\mathsf{Nest}(\mathcal{B}, \boldsymbol{\rho})$ intersects the evaluation space $\mathcal{D}^*(\boldsymbol{A})$ if and only if the corresponding nested set on $\mathcal{B}$ is acyclic for $\mathcal{M}(\boldsymbol{A})$. Note that the coefficients could sometimes be chosen smaller, our exponential choice is just a convenient hammer to kill all small contributions. For instance, for graphical oriented building sets, the coefficient ρ_t of a tube t can in fact be chosen of order $4^{|t|}$.

3 Nested complexes of building sets of the face lattice

Although less popular in the combinatorics community, the original definition of [4] for building sets and their nested complexes depends upon an underlying lattice.

Definition 3.1 ([4, 6])**.** A subset $\mathcal{B}$ of a finite lattice $\mathcal{L}$ is a *$\mathcal{L}$-building set* if the lower interval of any element $x \in \mathcal{L}$ is the direct product of the lower intervals of the maximal elements of $\mathcal{B}$ below x. We denote by $\kappa(\mathcal{B}) := \max(\mathcal{B})$ the set of *$\mathcal{B}$-connected components*.

Definition 3.2 ([4, 6])**.** Let $\mathcal{B}$ be an $\mathcal{L}$-building set. An *$\mathcal{L}$-nested set* $\mathcal{N}$ on $\mathcal{B}$ is a subset of $\mathcal{B}$ containing $\kappa(\mathcal{B})$ and such that for any $k \geq 2$ pairwise incomparable elements $B_1, \dots, B_k \in \mathcal{N}$, the join $B_1 \vee \dots \vee B_k$ does not belong to $\mathcal{B}$. The *$\mathcal{L}$-nested complex* of $\mathcal{B}$ is the simplicial complex $\mathfrak{N}_{\mathcal{L}}(\mathcal{B})$ whose faces are $\mathcal{N} \smallsetminus \kappa(\mathcal{B})$ for all $\mathcal{L}$-nested sets $\mathcal{N}$ on $\mathcal{B}$.

Example 3.3. If $\mathcal{L}$ is the boolean lattice on S, then the $\mathcal{L}$-building sets are the building sets on S of Definition 1.1 and the $\mathcal{L}$-nested sets are the nested sets of Definition 1.4.

We will use these definitions over the Las Vergnas face lattice $\mathcal{F}(\mathcal{M})$ of the oriented matroid $\mathcal{M}$, see Definition 1.21. We first select the facial part of an oriented building set.

Definition 3.4. The *facial building set* $\widehat{\mathcal{B}}$ of an oriented building set $(\mathcal{B}, \mathcal{M})$ is the set of blocks $B \in \mathcal{B}$ that are also faces of $\mathcal{M}$.

Theorem 3.5. *The facial building sets of $\mathcal{M}$ coincide with the $\mathcal{F}(\mathcal{M})$-building sets.*

Definition 3.6. The *facial nested complex* $\mathfrak{N}_{\mathcal{F}(\mathcal{M})}(\widehat{\mathcal{B}})$ is the $\mathcal{F}(\mathcal{M})$-nested complex of $\widehat{\mathcal{B}}$.

Theorem 3.7. *Let $\widehat{\mathcal{B}}$ be the facial building set of an oriented building set $(\mathcal{B}, \mathcal{M})$. Then the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ and the facial nested complex $\mathfrak{N}_{\mathcal{F}(\mathcal{M})}(\widehat{\mathcal{B}})$ coincide.*

Example 3.8. If $\mathcal{M}$ is independent (*i.e.,* no circuit), then its positive tope is a simplex, its Las Vergnas face lattice is boolean, so that we are in the classical situation of Example 3.3.

We conclude with a few remarks in light of Theorems 3.5 and 3.7. First, we observe that this interpretation actually recovers the results of Section 2.3. Namely,

- [5, Cor. 4.3] proved that the nested complex of a finite atomic meet-semilattice is homeomorphic to its order complex. Since the face lattices of oriented matroids encode face lattices of regular cell decompositions of spheres [1, Thm. 4.3.5], their order complexes are the face lattices of the barycentric subdivisions of these spheres.
- The stellar subdivisions of Theorem 2.13 are actually oriented matroid realizations of the combinatorial blowups of [4, Thm. 3.4] on face lattices of oriented matroids.

In turn, our acyclonestohedra of Section 2.4 provide explicit polytopal realizations with integer coordinates for the $\mathcal{F}(\mathcal{M})$-nested complexes over realizable matroids. To sum up:

Corollary 3.9. *Any $\mathcal{F}(\mathcal{M})$-nested complex of any $\mathcal{F}(\mathcal{M})$-building set over the face lattice $\mathcal{F}(\mathcal{M})$ of an acyclic oriented matroid $\mathcal{M}$ is the face lattice of an oriented matroid obtained by stellar subdivisions of the positive tope of $\mathcal{M}$. When $\mathcal{M}$ is realizable, it can be realized as a polytope either by realizing these stellar subdivisions polytopaly, or as the polar of a section of a nestohedron.*

4 Compactifications

Galashin's main motivation for defining poset associahedra was that they model compactifications of the space of order preserving maps $P \to \mathbb{R}$, which can be identified with the interior of an order polytope. In fact, the connection above reveals that all acyclonestohedra are associated to nice compactifications of interiors of polytopes, via [7].

Theorem 4.1 ([7])**.** *Consider a realizable oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$, and let P be the polytope associated to the positive tope. Then there is a compactification $P^{\mathcal{B}}$ of the interior of P that is a stratified C^∞ manifold with corners such that*

(i) except for the open dense stratum, all the strata lie in the boundary,

(ii) the codimension 1 strata are in correspondence with the facial blocs of $\widehat{\mathcal{B}}$,

(iii) the intersection of the closures of the strata indexed by a subset $\mathcal{N} \subseteq \widehat{\mathcal{B}}$ is non empty if and only if $\mathcal{N}$ is a $\mathcal{F}(\mathcal{M})$-nested set,

(iv) the strata of $P^{\mathcal{B}}$ can be indexed by the faces of the acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$.

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Determinant of the distance matrix of a tree

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Abstract. We present a combinatorial proof of the Graham–Pollak formula for the determinant of the distance matrix of a tree, via sign-reversing involutions and the Lindström–Gessel–Viennot lemma.

Résumé. Nous présentons une preuve combinatoire de la formule de Graham et Pollak pour le déterminant de la matrice des distances d'un arbre, en utilisant des involutions et le lemme de Lindström–Gessel–Viennot.

Keywords: Distance matrix of a tree. Lindström–Gessel–Viennot's Lemma. Sign-reversing involutions. Bijective Combinatorics.

1 Introduction

Consider a tree T with vertices labeled from one to n, and edge set E. Define the *distance* between vertices i and j, denoted by $d(i,j)$, as number of edges in the unique path of T connecting i and j. Define the *distance matrix* of T as $M(T) = (d(i,j))_{1\leq i,j\leq n}$.

In their influential 1971 paper [6], Graham and Pollak established that the determinant of the distance matrix of T obeys the *Graham–Pollak formula*:

$$\det M(T) = (-1)^{n-1}(n-1)2^{n-2} \tag{1.1}$$

Observe that this implies that the determinant of the distance matrix of T is solely dependent on its number of vertices, and not on its tree structure.

Multiple techniques drawn from linear algebra, ranging from Gauss elimination to Charles Dodgson's condensation formula, have been used to prove the Graham–Pollak formula [4, 6, 8, 9, 10]. However, the expression $(-1)^{n-1}(n-1)2^{n-2}$ suggests the existence of a signed enumeration problem solved by $\det M(T)$.

Pursuing this trail has led us to a novel combinatorial proof of the Graham–Pollak formula that relies on the existence of sign-reversing involutions, and on the celebrated

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Lindström–Gessel–Viennot lemma. Our journey concludes by establishing that our combinatorial proof provides a solid framework for many of the existing generalizations and q-analogues of the Graham–Pollak formula, and facilitates the derivation of new ones.

2 Catalysts

Where we introduce the idea of a catalyst *for a tree, and demonstrate how* $\det(T)$ *does a signed enumeration of all catalysts of a fixed tree* $T = ([n], E)$.

Fix a tree $T = ([n], E)$. Let $E^{\pm} = \{(i, j) : \{i, j\} \in E\}$ denote the set of arcs supported on T. Given a permutation σ in $\mathfrak{S}_n$ and a map $f : [n] \to E^{\pm}$, the ordered pair (σ, f) is a *catalyst for* T if for each vertex i, $f(i) = (v_i, v_{i+1})$ is a pair of successive vertices in the path $P(i, \sigma(i))$ (i.e., an arrow). The *sign of a catalyst* is the sign of its underlying permutation.

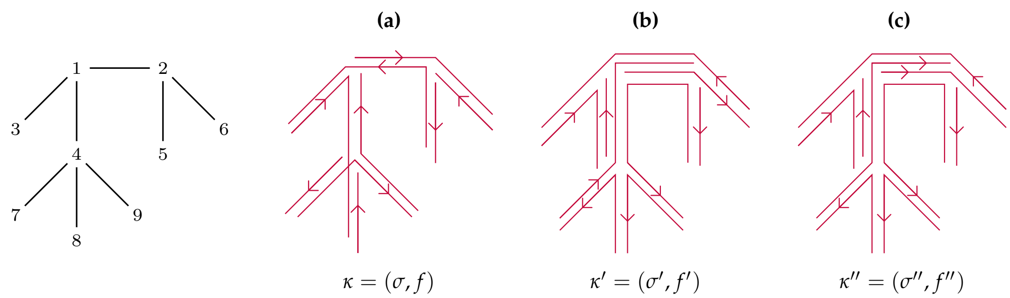

Figure 1: A tree T and the diagrams of three of its catalysts. We can recover a catalyst from its diagram. E.g., from the first diagram we see that $\sigma(1) = 6$ and $f(1) = (1, 2)$, that $\sigma(2) = 5$ and $f(2) = (2, 5)$, that $\sigma(3) = 8$ and $f(3) = (3, 1)$, and so on.

The determinant $\det M(T)$ does a signed enumeration of all catalysts for T. This is so because $d(i, \sigma(i))$ counts the number of edges in the unique path $P(i, \sigma(i))$ in T. Indeed,

$$\det M(T) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, d(1, \sigma(1)) d(2, \sigma(2)) \ldots d(n, \sigma(n)) = \sum_{\kappa \in K} \operatorname{sgn} \kappa, \tag{2.1}$$

where we are summing over K, the set of all catalysts for T. It is worth noting that the definition of catalyst implies that its underlying permutation must be a *derangement*, that is, a permutation without fixed points.

Partitioning catalysts by their underlying permutations proves ineffective in our search of a combinatorial proof of the Graham–Pollak formula, as in general, there are no cancellations between resulting summands.

3 Arrowflows and the Graham–Pollak formula

Where we present the definition of the arrowflow induced on T by a catalyst, and show how the Graham–Pollak formula becomes transparent when catalysts are partitioned according to them.

An *arrowflow on T* is a directed multigraph with vertex set $[n]$, with exactly n arcs when counted with multiplicity, and whose underlying simple graph is a subgraph of T. By definition, given any catalyst $\kappa = (\sigma, f)$, the image of f, considered as a multiset, is always an arrowflow on T. We refer to it as *arrowflow induced* by κ.

We say that an arrowflow A is connected when its underlying simple graph is. If A is a connected arrowflow, there exist precisely two vertices that belong to more than one arrow of A. It turns out that these two vertices always belong to precisely two arrows, that we call the *repeated arrows*. We say that the repeated arrows of a connected arrowflow are *parallel* when they point in the same direction, and *anti-parallel* when they point in opposite directions.

An arrowflow is said to be *unital* when it is connected and its repeated arrows are anti-parallel, as illustrated in Figure 2 (a). Otherwise, it is said to be *zero-sum*. There are two possible causes for an arrowflow to be zero-sum. Either the arrowflow is disconnected, as illustrated in Figure 2 (b), or the arrowflow is connected, but the repeated arrows are parallel, as in Figure 2 (c).

Example 3.1. Figure 2 shows the three arrowflows induced by the three catalysts of Figure 1. The first arrowflow (a) is unital. The second arrowflow (b) is zero-sum because it is disconnected. The last one (c) is zero-sum because arc $(1, 2)$ appears twice.

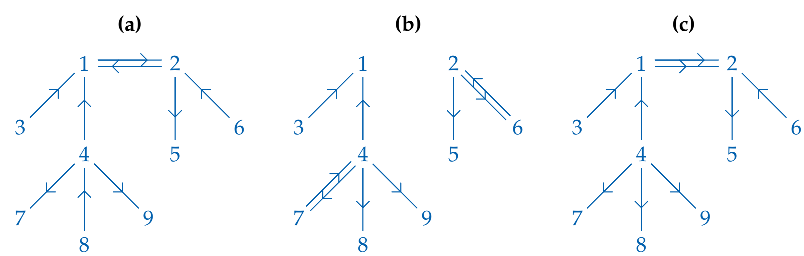

Figure 2: (a) Unital, **(b)** disconnected zero-sum, **(c)** connected zero-sum arrowflows.

It is crucial to observe that different catalysts on T can result on the same arrowflow. On the other hand, it is worth pointing out that there exist arrowflows on T that are not induced by any catalyst for T. We leave it to the reader to come up with such examples.

We define the *arrowflow class of A*, denoted by $C(A)$, as the set of catalysts inducing A on T. An arrowflow class $C(A)$ is unital or zero-sum according to whether A is unital or zero-sum. Nonempty arrowflow classes define a partition $K = \bigsqcup_A C(A)$ of the set

K of all catalysts for T, that we call the *arrowflow partition of* K. It allows us to rewrite Equation (2.1) as

$$\det M(T) = \sum_{\substack{A \\ \text{arrowflow}}} \sum_{\kappa \in C(A)} \operatorname{sgn}(\kappa), \tag{3.1}$$

where the first sum is taken over all arrowflows on T, and the second one over all catalysts κ in the arrowflow class $C(A)$.

Theorem 3.2. *The arrowflow partition defines an optimal way of partitioning the set of catalysts for T. More precisely, if $C(A)$ is an arrowflow class, then*

$$\sum_{\kappa \in C(A)} \operatorname{sgn}(\kappa) = \begin{cases} (-1)^{n-1} & \textit{if } A \textit{ is a unital arrowflow,} \\ 0 & \textit{if } A \textit{ is a zero-sum arrowflow.} \end{cases} \tag{3.2}$$

The proof of this result will unfold in the following two sections.

We close this section by showing how a combinatorial proof of the Graham–Pollak formula can be obtained by gathering all the elements of our reasoning. First observe that Theorem 3.2 implies that there can be no cancellations between the different summands in Equation (3.1). Therefore, it suffices to show that

$$\sum_{\substack{A \text{ unital} \\ \text{arrowflow}}} (-1)^{n-1} = (-1)^{n-1}(n-1)\, 2^{n-2}.$$

Or equivalently, that there exists $(n-1)\, 2^{n-2}$ unital arrowflows on T. This is immediate as the factor $(n-1)$ counts the number of ways of selecting the edge of T that gives rise to the anti-parallel repeated arrows, whereas factor 2^{n-2} counts the number of ways in which the remaining $n-2$ edges can be oriented. Finally, to show that the sign of a unital arrowflow class is $(-1)^{n-1}$, we show that the underlying permutation of the unique catalyst that survives the involution process is always an n-cycle.

4 Zero-sum arrowflows

Where we present a sign-reversing involution without fixed points on each zero-sum arrowflow class, and conclude that the signed sum of catalyst in such a class is always zero.

This is achieved in Lemma 4.1, which implies that the signed sum of catalysts in a zero-sum arrowflow class $C(A)$ is zero. This constitutes one half of Lemma 3.2.

Lemma 4.1. *Let A be a zero-sum arrowflow on T. If A is connected, let i and j be the two preimages of the repeated arrow (a, b) of A. On the other hand, if A is disconnected, we let $\{i, j\}$*

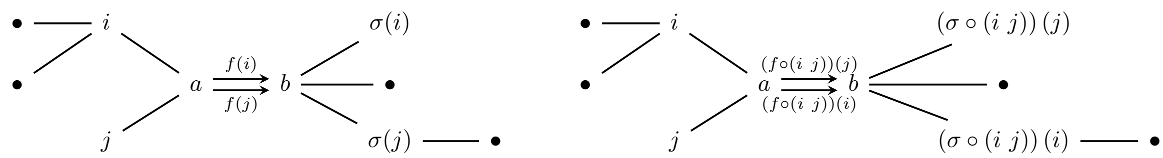

Figure 3: Involution φ acting on a zero-sum connected arrowflow A.

be an edge of T such that neither (i,j) nor (j,i) is in A. Then, the map $\varphi : \mathrm{C}(A) \to \mathrm{C}(A)$ that sends the catalyst (σ, f) to the catalyst $(\sigma \circ (i\ j), f \circ (i\ j))$ is a sign-reversing involution without fixed points.

Sketch of proof. It is enough to show that $f(i)$ is an arc in both $P(j,\sigma(i))$ and $P(i,\sigma(i))$. Observe that if A is connected, then i, j and a lie in one connected component of the graph obtained by deleting from T edge $\{a, b\}$, while $\sigma(i), \sigma(j)$ and b lie in the other one. See Figure 3.

On the other hand, when A is disconnected, then $P(j,\sigma(i))$ and $P(i,\sigma(i))$ differ in exactly one arc, either (i, j) or (j,i). Therefore, since $f(i)$ is neither of them, we conclude that $f(i)$ belongs to both paths. See Figure 4.

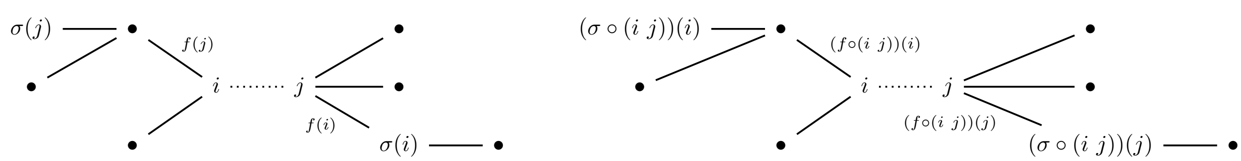

Figure 4: Involution φ acting on zero-sum disconnected arrowflow A.

$\square$

5 Unital arrowflows

Where we rely on the Lindström–Gessel–Viennot Lemma to compute the signed sum of all catalysts in a unital arrowflow class.

We rely on the following version of the Lindström–Gessel–Viennot lemma here.

Lemma 5.1 (Lindström [7], Gessel–Viennot [5]). *Let $\mathcal{R}$ be an acyclic directed graph. Distinguish two sequences of nodes $(v(1), \ldots, v(n))$ and $(v'(1), \ldots, v'(n))$ with no repeated nodes in either of them. Let $\mathcal{P}$ be the set of all sequences of paths $(P_1, ..., P_n)$ for which there exists a permutation $\sigma_P \in \mathbb{S}_n$ such that, for each i in $[n]$, the path P_i stars at $v(i)$ and finishes at $v'(\sigma(i))$.*

Then,

$$\sum_{(P_1,\ldots,P_n)\in\mathcal{P}} \operatorname{sgn}(\sigma_P) = \sum_{\substack{(P_1,\ldots,P_n)\in\mathcal{P} \\ \textit{non-intersecting}}} \operatorname{sgn}(\sigma_P).$$

We break down our argument into three parts. First, we prepare for the application of the Lindström–Gessel–Viennot Lemma and create an acyclic directed graph, the *route map* $\mathcal{R}_A$, from any unital arrowflow A on T. Subsequently, we establish a sign-preserving bijection that sends each catalyst in $C(A)$ to a family of n paths on $\mathcal{R}_A$, which we refer to as *n-paths*. The application of the Lindström–Gessel–Viennot Lemma reduces our original quest to the problem of finding a description of the non-intersecting n-paths. Finally, we establish that in each unital arrowflow class, there exists exactly one catalyst κ that generates a non-intersecting n-path, and that its underlying permutation of κ is always an n-cycle. Therefore, the sole surviving catalyst, has sign $(-1)^{n-1}$.

5.1 The route map $\mathcal{R}_A$. To construct $\mathcal{R}_A$, the route map of T, we proceed in several steps. First, we use arrowflow A to define a plane rooted directed tree T_0. Then, we define the *Southern hemisphere,* an acyclic directed graph; and its anti-isomorphic counterpart, the *Northern hemisphere*. Finally, we add bridges connecting both hemispheres and pointing from South to North.

Step 1. *Construct a rooted directed tree A_0 from A.*

Let $e = \{a,b\}$ be the edge of T connecting the two vertices appearing in the repeated edge of A. We construct a *rooted directed tree* A_0 from A by adding a new vertex r as a root and substituting the arcs (a,b) and (b,a) by (r,b) and (r,a) respectively. This construction induces a bijection between arcs of A and A_0. An arc of A_0 will be said to be *ascending* if it points to the root, and *descending* if it points away from the root. A child node u of a parent node v is termed *ascending* when the associated arc for the edge $\{u,v\}$ is ascending, and *descending* when it is descending. See Figures 5 (a) and (b).

We denote the underlying undirected rooted tree of A_0 by T_0.

Step 2. *Give a compatible plane structure to the rooted directed tree A_0.*

A plane structure for A_0 is said to be *compatible* if for each node v with children $u_1,\ldots,u_k$, every ascending child of v lies *to the left* of every descending child. In general there exist multiple compatible plane structures on A_0. We just choose one of them. See Figure 5 (c).

The underlying rooted tree T_0 inherits a plane rooted tree structure. The neighbors of a vertex i of T_0 are ordered starting with the children of i in increasing order (as in the plane structure of T_0), and ending with the parent.

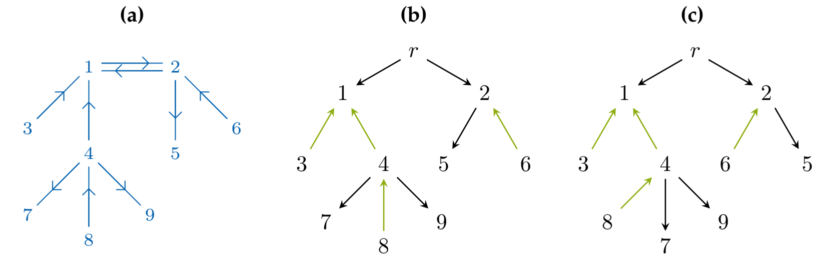

Figure 5: **(a)** A unital arrowflow A with repeated edge $\{1, 2\}$. **(b)** The rooted directed tree A_0 with root r. **(c)** A compatible plane structure for A_0.

Step 3. *Construct the Southern hemisphere of T_0.*

The *Southern hemisphere* $\mathcal{S}(T_0)$ of an undirected plane rooted tree T_0 is a directed multigraph whose vertex set is composed of three types of nodes (v-node, e-node, and s-node). Each node i of T_0, including the root, contributes with a node $v(i)$. Each edge $\{i, j\}$ of T_0, contributes two nodes $e(i, j)$ and $e(j, i)$. Finally, we add two nodes $s_i(j_{k-1}, j_k)$ and $s_i(j_k, j_{k-1})$ for each vertex i of T_0 and each pair of consecutive neighbors j_{k-1}, j_k of i. See Figure 6.

The arcs of the route map connect these nodes in a natural way, as to allow one to understand the paths of T_0 as paths in the route map. The explicit construction of the set of arcs can be daunting, but these arcs do not need to be included in the graphical representations of $\mathcal{S}(T_0)$ as they can be inferred from the set of nodes.

To construct the set of arcs of $\mathcal{S}(T_0)$ we add, for each i, two arcs between s-nodes for each three consecutive neighbors j_{k-1}, j_k, j_{k+1}, an arc $(v(i), s_i(j_1, j_2))$ for each node i that is not a leaf, and an arc $(v(i), e(i, j_1))$ for each node i. Additionally, from each e-node $e(j, i)$ there is an arc to (at most) two s-nodes around i, and conversely from each s-node $s_i(j_k, j_{k+1})$ to its corresponding $e(i, j_{k+1})$ node.

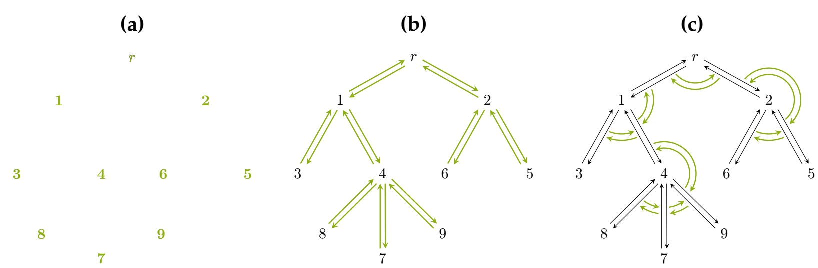

Figure 6: Highlighted, the sets of **(a)** v-nodes, **(b)** e-nodes, and **(c)** s-nodes of $\mathcal{S}(T_0)$.

Step 4. *Construct the Northern hemisphere, an anti-isomorphic copy of the Southern hemisphere.*

Let T_0 be an undirected plane rooted tree, let T'_0 be its *mirror image*; a copy of T_0 in which local orders are inverted. The *Northern hemisphere* $\mathcal{N}(T_0)$ of T_0 is constructed from $\mathcal{S}(T'_0)$ by replacing each arc $(v(i), e(i, j_1))$ by the arc $(e(j_1, i), v(i))$, and, whenever i is not a leaf, replacing arc $(v(i), s_i(j_1, j_2))$ by the arc $(s_i(j_2, j_1), v(i))$. We denote nodes of $\mathcal{N}(T_0)$ using primed letters.

Step 5. *Construct the route map $\mathcal{R}_A$.*

The *route map* $\mathcal{R}_A$ of the unital arrowflow A is the directed multigraph obtained by adding to $\mathcal{S}(T_0) \cup \mathcal{N}(T_0)$ an arc $(e(u, v), e'(u, v))$ for each arc (u, v) of A_0. These arcs are referred to as the *bridges* between hemispheres of $\mathcal{R}_A$.

The key property of the route map $\mathcal{R}_A$ is that it is always acyclic. The Southern hemisphere $\mathcal{S}(T_0)$ is acyclic because any cycle in $\mathcal{S}(T_0)$ would induce a cycle in the rooted plane tree T_0. The Northern hemisphere $\mathcal{N}(T_0)$ is acyclic because it is an anti-isomorphic copy of $\mathcal{S}(T_0)$. Finally, the route map is acyclic because all the bridges point from South to North.

5.2 Catalyst and n-paths. Let A be a unital arrowflow, and (σ, f) be a catalyst in $C(A)$. Let Λ_i be the unique path of $\mathcal{R}_A$ going from $v(i)$ to $v'(\sigma(i))$ and passing through the bridge $(e(u_i, v_i), e'(u_i, v_i))$, where (u_i, v_i) is the arc of A_0 defined by $f(i)$. See Figure 7.

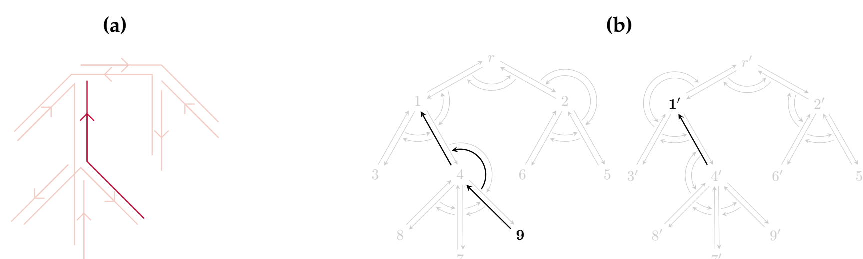

Figure 7: **(a)** A path $P(9, \sigma(9) = 1)$ marked at $f(9) = (4, 1)$. **(b)** The path Λ_9 of $\mathcal{R}_A$.

We define the *n-path induced by catalyst $\kappa = (\sigma, f)$* on the route map $\mathcal{R}_A$ as $\Lambda(\kappa) = \{\Lambda_1, \ldots, \Lambda_n\}$, and say that κ has been lifted to the n-path $\Lambda(\kappa)$. One can recover the permutation σ from $\Lambda(\kappa)$. Thus we define $\operatorname{sgn}(\Lambda(\kappa))$ as $\operatorname{sgn}(\sigma)$.

Example 5.2. Let κ be the catalyst of Figure 1 (a). Figure 8 (b) illustrates the n-path induced by κ, where we mark path Λ_i with subscript i. Moreover, since each node in the route map belongs to at most one path, it is an example of a non-intersecting n-path.

We say that an n-path is *full* when every bridge $(e(u, v), e'(u, v))$ belongs to exactly one of its paths. Since any n-path that is not full must contain an intersection at some bridge, non-intersecting n-paths are always full. Moreover, the lifting of any catalyst belonging to a unital arrowclass is always full.

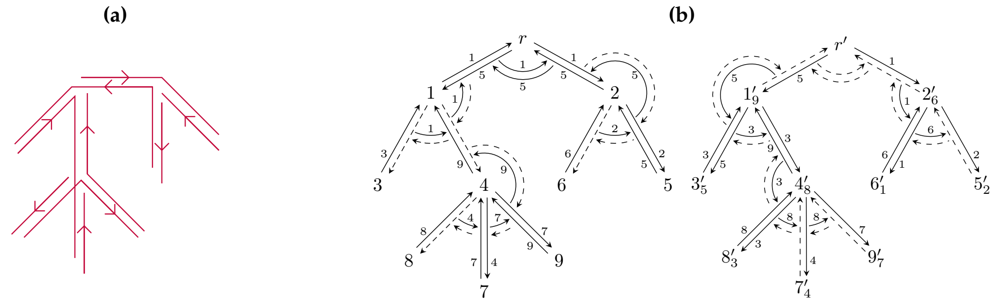

Figure 8: **(a)** A catalyst and **(b)** its induced n-path $\Lambda(\kappa) = \{\Lambda_1, \ldots, \Lambda_n\}$.

Lemma 5.3. *The operation of lifting defines a permutation-preserving bijection between the set of catalysts with unital arrowflow A and the set of full n-paths of $\mathcal{R}_A$.*

Sketch of proof. To prove that the lifting map is a bijection, we define its inverse. Any n-path P defines a permutation σ, where P_i is a path from $v(i)$ to $v'(\sigma(i))$. On the other hand, we use the bridges of P to define a map $f : [n] \to E^{\pm}$. If the arc e defining the bridge of P_i does not include the root, we define $f(i) = e$. Otherwise, we let $f(i)$ be the repeated edge with the appopiate orientation. It can be shown that $(\sigma, f) \in C(A)$ and, that the map we just defined is the inverse of the lifting map. Since the underlying permutation of a catalyst is the permutation induced by its lifting, we conclude that the lifting map is a permutation-preserving bijection. □

Lemma 5.3 allows us to rewrite Equation (3.2) as $\sum_{\kappa \in C(A)} \operatorname{sgn}(\kappa) = \sum \operatorname{sgn}(\Lambda)$, where the second sum is taken over the set of full n-paths on $\mathcal{R}_A$.

5.3 There is an unique catalyst inducing a non-intersecting n-path. Moreover, its underlying permutation is always an n-cycle.

Proposition 5.4. *Let A be a unital arrowflow on T. Consider all n-paths in the route map $\mathcal{R}_A$ induced by catalysts in $C(A)$. There exists precisely one catalyst inducing a non-intersecting n-path in $\mathcal{R}_A$. Its underlying permutation is an n-cycle. Therefore, its sign is $(-1)^{n-1}$.*

Sketch of proof. Fix a plane rooted tree A_0 and let Λ be an n-path. Assume Λ is the lift of catalyst (σ, f). Consider the set $\mathcal{E}$ consisting of the e-nodes appearing in the paths of Λ. We can show that $\mathcal{E}$ uniquely determines the catalyst (σ, f). On the other hand, for each arc (i, j) of A_0, the set $\mathcal{E}$ contains both $e(i, j)$ and $e'(i, j)$. Furthermore, if Λ is non-intersecting, a counting argument allows us to show that (i, j) is ascending if and only if $\mathcal{E}$ contains $e'(j, i)$, and descending if and only if it contains $e(j, i)$. This gives uniqueness.

The argument concludes by noting that the underlying permutation of the sole catalyst inducing a non-intersecting n-path is always an n-cycle, and that the depth-first search algorithm allows us to explicitly describe this cycle, as illustrated in Figure 9. □

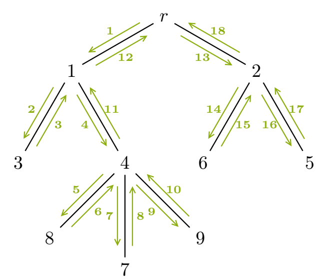

Figure 9: Applying the depth-first search algorithm to this rooted tree results in the word $r\,1\,3\,1\,4\,8\,4\,7\,4\,9\,4\,1\,r\,2\,6\,2\,5\,2\,r$, which we identify with the cycle $(3\,8\,4\,7\,9\,1\,6\,2\,5)$.

Example 5.5. Figure 9 illustrates how the depth-first search algorithm describes the non-intersecting path in the route map $\mathcal{R}_A$ of our running example.

The Lindström–Gessel–Viennot Lemma [5, 7] allows us to conclude that when we perform the signed sum of all catalysts in a unital arrowflow class, catalysts that induce intersecting n-paths on $\mathcal{R}_A$ cancel each other out. Finally, since the unique non-intersecting n-path has as its underlying permutation an n-cycle, the signed sum of catalysts in a unital arrowflow class is equal to $(-1)^{n-1}$, which concludes the proof of Lemma 3.2.

6 Beyond the Graham–Pollak formula

Where we show that our combinatorial proof of the Graham–Pollak formula not only establishes a solid framework for the understanding of the existing generalizations but also paves the way for the creation of new ones.

Various generalizations of the Graham–Pollak formula exist in the literature. In [1], a version of this formula is presented for simple trees with weighted edges, while the situation of arc-weighted trees is treated in [2]. In both cases, the weight of a path is defined as the sum of the weights of its edges. In contrast, using q-integers to define the vertex distance results in q-analogues of the results. Simple trees obey the q-Graham–Pollak formula $(-1)^{n-1}(n-1)(1+q)^{n-2}$ [9, Cor. 2.3], and a q-analogue for trees weighted with integers is given in [9, Thm. 2.4]. There also exist q-analogues for the Graham–Pollak formula when T is arc-weighted with integers [2, Thm. 3.1], or over a commutative ring [10, Thm. 4], or with matrices over a commutative ring [10, Thm. 7].

We present a new generalization of the Graham–Pollak formula. Towards this end, we define a *q-sum*, denoted by ⓠ, as $a \,ⓠ\, b = a + b + (q-1)ab$. This operation allows us to simplify [9, Thm. 2.4], by noting that $[1]_q = 1$ and $[a+b]_q = [a]_q \,ⓠ\, [b]_q$. More crucially, the q-sum operation is well-defined over any commutative ring, and not just integers. This makes our setting more general.

Let e^+ and e^- be the arcs originating from edge e, and let $E^\pm$ be the set of arcs of T. Let R be a commutative ring, and $\alpha : E^\pm \to R[q]$ be a weight function. Let the (i_k, i_{k+1})'s be the arcs in the unique path from $i = i_0$ to $j = i_{d(i,j)}$. Define the *q-distance* between vertices i and j as $d_q(i,j) = \alpha(i, i_1) \circledast \alpha(i_1, i_2) \circledast \cdots \circledast \alpha(i_{d(i,j)-1}, j)$. For any arc a of $E^\pm$, we write α_a for $\alpha(a)$. Then, the following theorem holds.

Theorem 6.1. *The determinant of the d_q-distance matrix of a tree T is*

$$(-1)^{n-1} \sum_{e \in E} \Big(\alpha_{e^+} \alpha_{e^-} \prod_{\substack{f \in E \\ f \neq e}} (\alpha_{f^+} \circledast \alpha_{f^-}) \Big).$$

Before discussing this result's proof, it's worth noting that the same argument provides a combinatorial proof for the very general Choudhury–Khare formula [3, Thm. A]. It is interesting to note that while the Choudhury–Khare's generalization is, in some precise sense, the most general possible [3, Example 1.13], Theorem 6.1 stands independently from this framework. It represents the most natural simultaneous generalization of [10, Thm. 4] and [9, Thm. 2.4], as depicted in the diagram appearing in Figure 10.

$$\begin{tikzcd}
\text{\shortstack{[6, p. 2511]\\ Graham–Pollak}} & \text{\shortstack{[1, Cor. 2.5]\\ weighted edges}} \arrow[l, "\alpha_{ij}=1"'] & \text{\shortstack{[10, Thm. 4]\\ weighted arcs}} \arrow[l, "\alpha_{ij}=\alpha_{ji}"'] \\
\text{\shortstack{[9, Cor. 2.3]\\ q-Graham–Pollak}} \arrow[u, "q=1"] & \text{\shortstack{[9, Thm. 2.4]\\ weighted edges, q-sum}} \arrow[l, "\alpha_{ij}=1"'] \arrow[u, "q=1"] & \text{\shortstack{Thm. 6.1\\ weighted arcs, q-sum}} \arrow[l, "\alpha_{ij}=\alpha_{ji}"'] \arrow[u, "q=1"]
\end{tikzcd}$$

Figure 10: Relationship between the formulas found in the literature.

Both Theorem 6.1 and the Choudhury–Khare formula [3, Thm. A] readily follow from our combinatorial construction. In both situations, we want to compute the determinant of an appropriate matrix $M'(T)$. Towards this end, we define a weight function on the catalyst set of T, in such a way that the determinant of $M'(T)$ does the weighted (q-) sum of all catalysts, as in Equation 3.1.

To show that the weighted (q-) sum of all catalysts in a zero-sum arrowflow class is zero, we show that the involution φ defined in Section 4 is weight-preserving. On the other hand, we use the constructions presented in Section 5 to compute the weighted (q-) sum of all catalysts in a unital arrowflow class. For this, we assign weights to the edges of the route map $\mathcal{R}_A$, and show that the lifting map is weight-preserving. A weighted version of the Lindström–Gessel–Viennot Lemma allows us to conclude that $\det M'(T)$ does a weighted (q-) sum of non-intersecting n-paths within $\mathcal{R}_A$. Finally, we use the characterization of the sole catalyst inducing a non-intersecting n-path on $\mathcal{R}_A$ obtained in Proposition 5.4 to deduce the desired formula for the determinant of $M'(T)$.

Acknowledgements

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Pop-Stack for Cambrian Lattices

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Abstract. The *pop-stack operator* of a finite lattice L is the map that sends each x in L to the meet of x with the set of elements covered by x. Using tools from representation theory, we provide simple Coxeter-theoretic and lattice-theoretic descriptions of the image of the pop-stack operator of a Cambrian lattice of a finite irreducible Coxeter group. When specialized to a bipartite Cambrian lattice of type A, this result settles a conjecture of Choi and Sun. We also settle a related enumerative conjecture of Defant and Williams. When L is an arbitrary lattice quotient of the weak order on W, we prove that the maximum size of a forward orbit under the pop-stack operator of L is at most the Coxeter number of W; when L is a Cambrian lattice, we provide an explicit construction to show that this maximum forward orbit size is actually equal to the Coxeter number.

Keywords: torsion classes, Cambrian lattice, weak order, pop-stack operator

1 Introduction

Let L be a finite lattice with meet operation $\wedge$ and join operation $\vee$. The *pop-stack operator* $\mathrm{pop}_L^{\downarrow}\colon L \to L$ and the *dual pop-stack operator* $\mathrm{pop}_L^{\uparrow}\colon L \to L$ are defined by

$$\mathrm{pop}_L^{\downarrow}(x) = x \wedge \left(\bigwedge\{y \mid y \lessdot x\}\right) \quad \text{and} \quad \mathrm{pop}_L^{\uparrow}(x) = x \vee \left(\bigvee\{y \mid x \lessdot y\}\right),$$

where we write $u \lessdot v$ to mean that u is covered by v in L. These operators have appeared in various contexts; they serve as both useful tools and objects of interest in their own right. When the lattice L is understood, we will omit subscripts and simply denote these operators by $\mathrm{pop}^{\downarrow}$ and $\mathrm{pop}^{\uparrow}$.

Given an element $x \in L$, the *forward orbit* of x under $\mathrm{pop}_L^{\downarrow}$ is the set

$$\mathcal{O}_L(x) = \left\{x, \mathrm{pop}_L^{\downarrow}(x), (\mathrm{pop}_L^{\downarrow})^2(x), \ldots\right\},$$

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where $(\mathrm{pop}_L^{\downarrow})^t$ is the map obtained by composing $\mathrm{pop}_L^{\downarrow}$ with itself t times. If t is sufficiently large, then $(\mathrm{pop}_L^{\downarrow})^t(x)$ is equal to the minimal element $\hat{0}$ of L (which is the unique fixed point of $\mathrm{pop}_L^{\downarrow}$). Thus, $|\mathcal{O}_L(x)| - 1$ is equal to the number of iterations of $\mathrm{pop}_L^{\downarrow}$ needed to send x to $\hat{0}$.

Given an interesting lattice L, one of the primary problems one can consider about its pop-stack operator is that of maximizing $\mathcal{O}_L(x)$. When L is the weak order on a finite irreducible Coxeter group W, Defant [7] proved that $\max_{x \in L} |\mathcal{O}_L(x)|$ is the Coxeter number h of W; in type A, this result was originally proven much earlier by Ungar [17]. Defant also studied this problem for ν-Tamari lattices in [6].

Defant and Williams [8] found that it is fruitful to study the image of the pop-stack operator when L is a semidistributive (or more generally, a semidistrim) lattice; this is because the image of $\mathrm{pop}^{\downarrow}$ has numerous interesting properties, some of which relate to a certain bijective *rowmotion operator* $\mathrm{row}\colon L \to L$. For example, $|\mathrm{pop}^{\downarrow}(L)|$ and $|\mathrm{pop}^{\uparrow}(L)|$ are both equal to the number of elements $x \in L$ such that $\mathrm{row}(x) \leq x$. The images of $\mathrm{pop}^{\downarrow}$ and $\mathrm{pop}^{\uparrow}$ are also naturally in bijection with the set of facets of a certain simplicial complex called the *canonical join complex* of L.

In our full article [3], we take a representation-theoretic perspective and consider a finite-dimensional basic algebra Λ over a field K. The set of torsion classes of finitely-generated (right) Λ-modules forms a lattice, denoted $\mathsf{tors}\,\Lambda$ [13]. While the pop-stack operator of $\mathsf{tors}\,\Lambda$ has already appeared (sometimes under different names) in the theory of lattices of torsion classes (see e.g. [1, 4, 9]; a longer list can be found in introduction of our full article [3]), it has primarily been used as a tool rather than a dynamical operator worthy of its own investigation. Our full article, on the other hand, studies the image and dynamical properties of the pop-stack operator of $\mathsf{tors}\,\Lambda$ in the case when $\mathsf{tors}\,\Lambda$ is finite. We show that applying the pop-stack operator and its dual to a torsion class corresponds to performing certain mutations on associated *2-term simple-minded collections*. We characterize the preimages of a prescribed torsion class under $\mathrm{pop}^{\downarrow}_{\mathsf{tors}\,\Lambda}$ and $\mathrm{pop}^{\uparrow}_{\mathsf{tors}\,\Lambda}$. As corollaries, we obtain descriptions of the elements of $\mathsf{tors}\,\Lambda$ that require exactly 1 or exactly 2 iterations of $\mathrm{pop}^{\downarrow}$ to reach $\hat{0}$.

When Λ is a Dynkin quiver (or more generally, a Dynkin species), the lattice $\mathsf{tors}\,\Lambda$ is isomorphic to a Cambrian lattice [12]. For the sake of remaining explicit and combinatorial, we will devote this extended abstract to the pop-stack operators of Cambrian lattices; we will also consider Cambrian lattices of arbitrary finite irreducible Coxeter groups (not just crystallographic). That said, some of our proofs, which we omit in this extended abstract, are heavily representation-theoretic.

Let c be a (standard) Coxeter element of a finite irreducible Coxeter group W. Let $\mathrm{Weak}(W)$ denote the (right) weak order on W. The set of c-sortable elements of W (see Section 2 for definitions) forms a sublattice Camb_c of $\mathrm{Weak}(W)$ called the *c-Cambrian lattice*. Hong [11] found a description of the image of the pop-stack operator on a Tamari lattice (a particular Cambrian lattice of type A), and Choi and Sun [5] found a similar description for the image of the pop-stack operator on a type B analogue of the Tamari lattice. Choi and Sun also conjectured a description of the image of the pop-stack operator on a type A Cambrian lattice associated to a bipartite Coxeter element.

In Section 3, we provide an explicit description of the image of the pop-stack operator on an arbitrary Cambrian lattice; we were only able to discover this description by thinking representation-theoretically (it involves *projective modules*), but we can state it in purely Coxeter-theoretic and lattice-theoretic terms. This characterization allows us to obtain a surprising dynamical result (see Theorem 2). When W is of type A and $c = c^{\times}$ is a bipartite Coxeter element, our description of the image of the pop-stack operator allows us to resolve the aforementioned conjecture of Choi and Sun [5]. We then construct a bijection from the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}$ to a certain set of Motzkin paths (Theorem 3); this allows us to resolve an enumerative conjecture of Defant and Williams [8]. This result provides an enumeration of the facets of the canonical join complex of a bipartite type A Cambrian lattice.

When $W_{\equiv}$ is a lattice quotient of the weak order on a finite irreducible Coxeter group W, we show that $\max_{x \in W_{\equiv}} |\mathcal{O}_{W_{\equiv}}(x)| \leq h$, where h is the Coxeter number of W. We prove that this inequality is actually an equality when $W_{\equiv}$ is the c-Cambrian lattice associated to a Coxeter element c of W.

Section 2 provides background on posets, lattices, Coxeter groups, and Cambrian lattices. Section 3 is devoted to the images of the pop-stack operators of Cambrian lattices, and Section 4 is devoted to studying maximum-sized orbits. In Section 5, we collect several ideas for future work.

2 Background

2.1 Posets and Lattices

Let P be a poset. For $x, y \in P$, we say y *covers* x and write $x \lessdot y$ if $x < y$ and there does not exist $z \in P$ such that $x < z < y$. The *dual* of P is the poset P^* with the same underlying set as P defined so that $x \leq y$ in P^* if and only if $y \leq x$ in P. A *lattice* is a poset L such that any two elements $x, y \in L$ have a greatest lower bound, which is called their *meet* and denoted by $x \wedge y$, and a least upper bound, which is called their *join* and denoted by $x \vee y$. We write $\bigwedge X$ and $\bigvee X$ for the meet and join, respectively, of a finite subset X of a lattice. Given lattices L and L', a *lattice homomorphism* is a map $\phi\colon L \to L'$ such that $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for all $x, y \in L$. We say L' is a *lattice quotient* if there is a surjective lattice homomorphism from L to L'.

Assume L is a finite lattice. Then L has a unique minimal element $\hat{0} = \bigwedge L$ and a unique maximal element $\hat{1} = \bigvee L$. An element $j \in L$ is called *join-irreducible* if it covers exactly one element of L. Dually, an element $m \in L$ is *meet-irreducible* if it is covered by exactly one element of L. A set $A \subseteq L$ is *join-irredundant* (resp. *meet-irredundant*) if $\bigvee A' < \bigvee A$ (resp. $\bigwedge A' > \bigwedge A$) for every proper subset A' of A. Let JIrr_L (resp. MIrr_L) be the set of join-irredundant (resp. meet-irredundant) subsets of L. The *canonical join representation* of an element $x \in L$ (if it exists) is the unique set $A \in \mathrm{JIrr}_L$ satisfying $x = \bigvee A$ with the property that for every $B \in \mathrm{JIrr}_L$ satisfying $x = \bigvee B$, there exist $a \in A$ and $b \in B$ such that $a \leq b$. Dually, the *canonical meet representation* of x (if it exists) is the unique set $A \in \mathrm{MIrr}_L$ satisfying $x = \bigwedge A$ with the property that for every $B \in \mathrm{MIrr}_L$ satisfying $x = \bigwedge B$, there exist $a \in A$ and $b \in B$ such that $a \geq b$.

We say L is *semidistributive* if for all $x, y, z \in L$, we have

$$x \wedge y = x \wedge z \implies x \wedge y = x \wedge (y \vee z) \quad \text{and} \quad x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z).$$

Suppose L is finite and semidistributive. It is known that every element v of L has a canonical join representation $\mathcal{D}(v)$ and a canonical meet representation $\mathcal{U}(v)$; in fact, the existence of both representations for every $v \in L$ is equivalent to semidistributivity. Moreover, the collection of canonical join representations (resp. canonical meet representations) of elements of L forms a simplicial complex called the *canonical join complex* (resp. *canonical meet complex*) of L. The canonical join complex and canonical meet complex of L are isomorphic simplicial complexes by [2, Corollary 5]. Moreover, the number of facets in each of these simplicial complexes is equal to both $|\mathrm{pop}_L^{\downarrow}(L)|$ and $|\mathrm{pop}_L^{\uparrow}(L)|$ by [8, Theorem 9.13]. Indeed, the facets of the canonical join complex (resp. canonical meet complex) of L are precisely the canonical meet representations (resp. canonical join representations) of the elements of $\mathrm{pop}_L^{\downarrow}(L)$ (resp. $\mathrm{pop}_L^{\uparrow}(L)$). Let $\mathbf{P}_L(q)$ be the generating function that counts the facets of the canonical join complex (equivalently, the canonical meet complex) according to their sizes. Then

$$\mathbf{P}_L(q) = \sum_{v \in \mathrm{pop}_L^{\downarrow}(L)} q^{|\mathcal{U}(v)|} = \sum_{v \in \mathrm{pop}_L^{\uparrow}(L)} q^{|\mathcal{D}(v)|}. \tag{2.1}$$

2.2 Coxeter groups

Let (W, S) be a finite Coxeter system. This means that S is a finite set and that W is a finite group with a presentation of the form $\langle S \mid (ss')^{m(s,s')} = e \text{ for all } s, s' \in S \rangle$, where e is the identity element of W and we have $m(s,s) = 1$ and $m(s,s') = m(s',s) \in \{2, 3, \ldots\}$ for all distinct $s, s' \in S$. (We often refer to just the Coxeter group W, tacitly assuming that this refers to the Coxeter system (W, S).)

The elements of S are called the *simple reflections*. A *reflection* is an element of W of the form wsw^{-1} for $s \in S$ and $w \in W$. The *Coxeter graph* of W is the graph Γ_W with vertex set S in which two simple reflections s and s' are connected by an edge whenever $m(s,s') \geq 3$; this edge is labeled with the number $m(s,s')$ if $m(s,s') \geq 4$. We will assume that W is *irreducible*, which means that Γ_W is connected.

A *reduced word* for an element $w \in W$ is a word over S that represents w and is as short as possible. The number of letters in a reduced word for w is called the *length* of w and is denoted $\ell(w)$. A *left inversion* of w is a reflection t such that $\ell(tw) < \ell(w)$. The *(right) weak order* is the partial order $\leq$ on W defined so that $u \leq v$ if and only if there exists a reduced word for v that has a reduced word for u as a prefix. Let $\mathrm{Weak}(W)$ denote the poset $(W, \leq)$. It is well known that $\mathrm{Weak}(W)$ is a lattice. A *descent* of an element $w \in W$ is a simple reflection $s \in S$ such that $ws < w$ in $\mathrm{Weak}(W)$. The *long element* of W, denoted $w_\circ$, is the unique maximal element of $\mathrm{Weak}(W)$.

A *(standard) Coxeter element* of W is an element c obtained by multiplying the simple reflections in some order (with each appearing once in the product). Thus, a reduced word for c is a word in which each simple reflection appears exactly once.

Fix a reduced word c for a Coxeter element c, and consider the infinite word $\mathsf{c}^\infty = \mathsf{c}^{(1)}\mathsf{c}^{(2)}\cdots$, where each $\mathsf{c}^{(k)}$ is a copy of c. Following Reading [15], we define the *c-sorting word* of an element $w \in W$ to be the reduced word $\mathsf{sort_c}(w)$ for w that is lexicographically first as a subword of c^∞. Let $\mathbf{I}_c^{(k)}(w)$ be the set of simple reflections that are taken from $\mathsf{c}^{(k)}$ when we form $\mathsf{sort_c}(w)$ as the lexicographically first subword of c^∞. Although $\mathbf{I}_c^{(k)}(w)$ depends on the Coxeter element c, it does not depend on the choice of the reduced word c. The element w is called *c-sortable* if $\mathbf{I}_c^{(1)}(w) \supseteq \mathbf{I}_c^{(2)}(w) \supseteq \cdots$. The set of c-sortable elements of W forms a sublattice of $\mathrm{Weak}(W)$ called the *c-Cambrian lattice*, which we denote by Camb_c.

For each $w \in W$, the set $\mathrm{Camb}_c \cap \{v \in W \mid v \leq w\}$ has a unique maximal element in the weak order; we denote this element by $\pi_\downarrow^c(w)$. The map $\pi_\downarrow^c$ is a surjective lattice homomorphism from $\mathrm{Weak}(W)$ to Camb_c, so Camb_c is a lattice quotient of $\mathrm{Weak}(W)$ [15]. According to [6, Theorem 3.2], we have $\mathrm{pop}^\downarrow_{\mathrm{Camb}_c} = \pi_\downarrow^c \circ \mathrm{pop}^\downarrow_{\mathrm{Weak}(W)}$.

3 The Image of Pop-Stack on a Cambrian Lattice

Let c be a Coxeter element of a finite irreducible Coxeter group W. Let $s_1, \ldots, s_n$ be the simple reflections of W; these are the elements that cover $\hat{0}$ in Camb_c. For $1 \leq i \leq n$, let

$$p_i = \bigvee \{w \in \mathrm{Camb}_c \mid s_i \leq w \text{ and } s_j \not\leq w \text{ for all } s_j \in S \setminus \{s_i\}\}.$$

Our main result describing the image of $\mathrm{pop}^\downarrow_{\mathrm{Camb}_c}$ is as follows.

Theorem 1 ([3]). *For $w \in \mathrm{Camb}_c$, the following are equivalent:*

1. *w is in the image of $\mathrm{pop}^\downarrow_{\mathrm{Camb}_c}$.*
2. *The descents of w all commute, and w has no left inversions in common with c^{-1}.*
3. *The interval $[\mathrm{pop}^\downarrow_{\mathrm{Camb}_c}(w), w]$ is Boolean, and $p_i \not\leq w$ for all $i \in [n]$.*

In [3], we apply Theorem 1 (together with further representation-theoretic arguments) to deduce the following result.

Theorem 2 ([3])**.** *Let c be a Coxeter element of a finite irreducible crystallographic Coxeter group W. If $w \in \mathrm{Camb}_c$ and $t \geq 0$, then*

$$(\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(W)})^t(\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w)) = (\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c})^{t+1}(w).$$

The Coxeter group A_n is the same as the symmetric group whose elements are permutations of the set $[n+1] = \{1, \ldots, n+1\}$. We will frequently represent a permutation $w \in A_n$ in *one-line notation* as the word $w(1) \cdots w(n+1)$. The simple reflections of A_n are $s_1, \ldots, s_n$, where s_i is the transposition that swaps i and $i+1$. The Coxeter graph Γ_{A_n} is a path that contains an (unlabeled) edge $\{s_i, s_{i+1}\}$ for each $i \in [n]$. Let

$$c^{\times}_{(n)} = c_1 c_2, \quad \text{where} \quad c_1 = \prod_{\substack{i \in [n] \\ i \text{ odd}}} s_i \quad \text{and} \quad c_2 = \prod_{\substack{i \in [n] \\ i \text{ even}}} s_i. \tag{3.1}$$

We refer to the Coxeter element $c^{\times}_{(n)}$ as a *bipartite Coxeter element*.

3.1 Arc diagrams

Let c denote an arbitrary Coxeter element of A_n. Define $\nu_c \colon \{2, \ldots, n\} \to \{\mathbf{A}, \mathbf{B}\}$ by

$$\nu_c(i) = \begin{cases} \mathbf{A} & \text{if } s_i \text{ precedes } s_{i-1} \text{ in every reduced word for } c; \\ \mathbf{B} & \text{if } s_{i-1} \text{ precedes } s_i \text{ in every reduced word for } c. \end{cases} \tag{3.2}$$

The map $c \mapsto \nu_c$ is a bijection from the set of Coxeter elements of A_n to the set of functions from $\{2, \ldots, n\}$ to $\{\mathbf{A}, \mathbf{B}\}$. Reading [14, Example 4.9] showed that a permutation $w \in A_n$ is c-sortable if and only if for all $i \in [n+1]$ and $j \in [n]$ such that $w(j+1) < w(i) < w(j)$, we have $\nu_c(w(i)) = \mathbf{A}$ if and only if $j < i$. Arrange $n+1$ points along a horizontal line, and identify them with the numbers $1, \ldots, n+1$ from left to right. An *arc* is a curve that moves monotonically rightward from a point i to another point j (for some $i < j$), passing above or below each of the points $i+1, \ldots, j-1$. Two arcs are considered to be the same if they have the same endpoints and they pass above the same collection of numbered points. A *noncrossing arc diagram* (of type A_n) is a collection of arcs that can be drawn so that no two arcs have the same left endpoint or have the same right endpoint or cross in their interiors. We write $|\delta|$ for the number of arcs in a noncrossing arc diagram δ. Let AD_n be the set of noncrossing arc diagrams of type A_n.

Given a permutation $w \in A_n$, form a noncrossing arc diagram $\Delta(w) \in \mathrm{AD}_n$ as follows. For each index i such that $w(i) > w(i+1)$, draw an arc from $w(i+1)$ to $w(i)$ such that for each integer k satisfying $w(i+1) < k < w(i)$, the arc passes above (resp.

below) the point k if $i+1 < w^{-1}(k)$ (resp. $w^{-1}(k) < i$). This defines a map $\Delta\colon A_n \to \mathrm{AD}_n$, and it is straightforward to check that Δ is a bijection.

Given a Coxeter element c of A_n, say an arc $\mathfrak{a}$ with left endpoint i and right endpoint j is *c-sortable* if for every $k \in \{i+1, \dots, j-1\}$, $\mathfrak{a}$ passes above (resp. below) k if $\nu_c(k) = \mathbf{A}$ (resp. $\nu_c(k) = \mathbf{B}$). Note that for all $1 \le i < j \le n+1$, there is a unique c-sortable arc from i to j. Let $\mathrm{AD}(c) = \Delta(\mathrm{Camb}_c)$ be the set of noncrossing arc diagrams of c-sortable elements of A_n. It is immediate from Reading's characterization of c-sortable elements that a noncrossing arc diagram is in $\mathrm{AD}(c)$ if and only if all of its arcs are c-sortable. Hence, $\mathrm{AD}(c)$ is a simplicial complex whose vertices are the c-sortable arcs.

Cambrian lattices are semidistributive, so we can consider the canonical join complex and the canonical meet complex of Camb_c (and we know these simplicial complexes are isomorphic by [2, Corollary 5]). An element $v \in \mathrm{Camb}_c$ is join-irreducible if and only if it has exactly one descent, and this occurs if and only if $\Delta(v)$ contains a single arc. Therefore, Δ establishes a one-to-one correspondence between the join-irreducible elements of Camb_c and the c-sortable arcs. Then for each $w \in \mathrm{Camb}_c$, the noncrossing arc diagram $\Delta(w)$ corresponds to the canonical join representation of w. It follows that the simplicial complex $\mathrm{AD}(c)$ is isomorphic to the canonical join complex of Camb_c. Say a noncrossing arc diagram in $\mathrm{AD}(c)$ is *maximal* if it is a facet of $\mathrm{AD}(c)$. In other words, a noncrossing arc diagram in $\mathrm{AD}(c)$ is maximal if it is not properly contained in another noncrossing arc diagram in $\mathrm{AD}(c)$. Let $\mathrm{MAD}(c)$ denote the set of maximal noncrossing arc diagrams in $\mathrm{AD}(c)$.

The preceding discussion yields the identity

$$\mathbf{P}_{\mathrm{Camb}_c}(q) = \sum_{\delta \in \mathrm{MAD}(c)} q^{|\delta|}, \tag{3.3}$$

where $\mathbf{P}_{\mathrm{Camb}_c}(q)$ is the generating function defined in Equation (2.1). Defant and Williams conjectured [8, Conjecture 11.2] that

$$\sum_{n\ge 1} \mathbf{P}_{\mathrm{Camb}_{c^{\times}_{(n)}}}(q) z^n = \frac{1}{qz}\left(\frac{2}{1 - qz(1-2z) + \sqrt{1 + q^2z^2 - 2qz(1+2z)}} - 1\right) - 1. \tag{3.4}$$

The remainder of this section is devoted to stating the bijection that we use in [3] to prove this conjecture.

3.2 Motzkin paths

A *Motzkin path* is a lattice path in the plane that consists of up (i.e., $(1,1)$) steps, down (i.e., $(1,-1)$) steps, and horizontal (i.e., $(1,0)$) steps, starts at the origin, never passes below the horizontal axis, and ends on the horizontal axis. Let `U`, `D`, and `H` denote up, down, and horizontal steps, respectively. Given a word P over the alphabet $\{\mathtt{U}, \mathtt{D}, \mathtt{H}\}$, let

$\#_{\mathtt{U}}(P)$, $\#_{\mathtt{D}}(P)$, and $\#_{\mathtt{H}}(P)$ denote the number of $\mathtt{U}'s$, the number of D's, and the number of H's in P, respectively. We can think of a Moztkin path as a word M over the alphabet $\{\mathtt{U},\mathtt{D},\mathtt{H}\}$ such that $\#_{\mathtt{U}}(M)=\#_{\mathtt{D}}(M)$ and $\#_{\mathtt{U}}(P)\geq\#_{\mathtt{D}}(P)$ for every prefix P of M.

A *peak* of a Motzkin path M is a point (j,k) where an up step in M ends and a down step in M begins; the *height* of this peak is the number k. If we view M as a word over $\{\mathtt{U},\mathtt{D},\mathtt{H}\}$, then a peak corresponds to a consecutive occurrence of UD, and the height of the peak is $\#_{\mathtt{U}}(P)-\#_{\mathtt{D}}(P)$, where P is the prefix of M that ends with the up step involved in the peak. The only peak of the Motzkin path at the bottom of Figure 1 is $(5,2)$.

Let $\overline{\mathcal{M}}_n$ be the set of Motzkin paths of length n that have no peaks of height 1. Let $\overline{\mathbf{M}}(q,z)=\sum_{n\geq 0}\sum_{M\in\overline{\mathcal{M}}_n}q^{\#_{\mathtt{U}}(M)}z^n$. In [3], we use straightforward enumerative techniques to show that

$$\overline{\mathbf{M}}(q,z)=\frac{2}{1-z+2qz^2+\sqrt{1-2z+(1-4q)z^2}}. \tag{3.5}$$

Using Equation (3.5), one can readily check that the expression on the right-hand side of Equation (3.4) is

$$\frac{1}{qz}\left(\overline{\mathbf{M}}(1/q,qz)-1\right)-1=\sum_{n\geq 1}\sum_{M\in\overline{\mathcal{M}}_{n+1}}q^{n-\#_{\mathtt{U}}(M)}z^n.$$

Therefore, in order to prove Equation (3.4), it suffices (by Equation (3.3)) to exhibit a bijection $\Psi\colon \mathrm{MAD}(c^{\times}_{(n)})\to\overline{\mathcal{M}}_{n+1}$ such that $|\delta|=n-\#_{\mathtt{U}}(\Psi(\delta))$ for every $\delta\in\mathrm{MAD}(c^{\times}_{(n)})$.

3.3 The bijection

Throughout the remainder of this section, fix a positive integer n, and write $c^{\times}=c^{\times}_{(n)}$. The map $\nu_{c^{\times}}\colon\{2,\ldots,n\}\to\{\mathbf{A},\mathbf{B}\}$ is such that $\nu_{c^{\times}}(i)=\mathbf{A}$ if i is odd and $\nu_{c^{\times}}(i)=\mathbf{B}$ if i is even.

Suppose $\delta\in\mathrm{MAD}(c^{\times})$. Let $\Psi(\delta)$ be the word $\mathtt{M}_1\cdots\mathtt{M}_{n+1}$, where for $1\leq i\leq n+1$, we define

$$\mathtt{M}_i=\begin{cases}\mathtt{U} & \text{if } i\leq n \text{ and } i+1 \text{ is not the right endpoint of an arc in } \delta;\\ \mathtt{D} & \text{if } i\geq 2 \text{ and } i-1 \text{ is not the left endpoint of an arc in } \delta;\\ \mathtt{H} & \text{otherwise.}\end{cases} \tag{3.6}$$

In [3], we prove that $\Psi(\delta)$ is well defined in the sense that no letter in $\Psi(\delta)$ can be both U and D. See Figure 1 for an illustration of Ψ.

We can now state the main theorem of this section; as mentioned at the end of Section 3.2, this theorem implies the identity Equation (3.4), thereby settling the conjecture of Defant and Williams.

Theorem 3 ([3])**.** *The map Ψ is a bijection from $\mathrm{MAD}(c^{\times})$ to $\overline{\mathcal{M}}_{n+1}$. For each $\delta\in\mathrm{MAD}(c^{\times})$, we have $\Psi(\delta)\in\overline{\mathcal{M}}_{n+1}$ and $|\delta|=n-\#_{\mathtt{U}}(\Psi(\delta))$.*

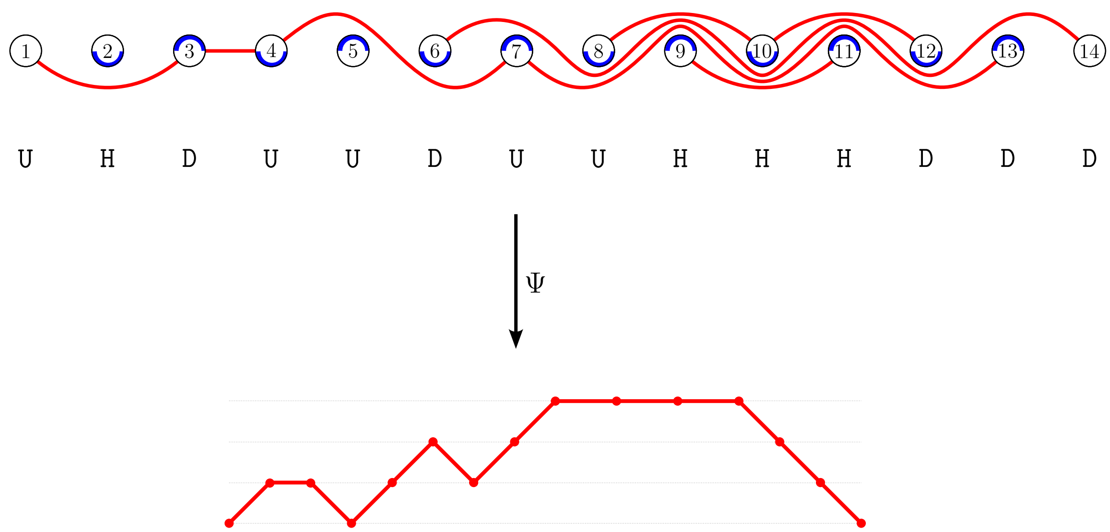

Figure 1: When $n = 13$, the map Ψ sends a noncrossing arc diagram of type A_{13} to a Motzkin path of length 14 with no peaks of height 1. For each $2 \leq i \leq 13$, a blue semicircle appears on the top (resp. bottom) of the circle containing i if $\nu_{c^\times}(i) = \mathbf{A}$ (resp. if $\nu_{c^\times}(i) = \mathbf{B}$). (The letters drawn below the noncrossing arc diagram represent the Moztkin path; they are not part of the noncrossing arc diagram.)

4 Maximum-Size Pop-Stack Orbits

As above, let (W, S) be a finite irreducible Coxeter system. The *Coxeter number* of W is the quantity $h = 2|T|/|S|$, where T is the set of reflections in W.

Theorem 4 ([3])**.** *If $W_\equiv$ is a lattice quotient of* $\mathrm{Weak}(W)$*, then*

$$\max_{x \in W_\equiv} |\mathcal{O}_{W_\equiv}(x)| \leq h.$$

The next theorem states that the inequality in Theorem 4 is tight for Cambrian lattices.

Theorem 5 ([3])**.** *For each Coxeter element c of W, we have*

$$\max_{x \in W_\equiv} |\mathcal{O}_{\mathrm{Camb}_c}(x)| = h.$$

The *spine* of Camb_c, denoted $\mathrm{spine}(\mathrm{Camb}_c)$, is the union of the maximum-length chains of Camb_c. Hohlweg, Lange, and Thomas [10] proved that $\mathrm{spine}(\mathrm{Camb}_c)$ is a distributive sublattice of Camb_c. Let us define $\mathbf{z}_c = (\mathrm{pop}^{\uparrow}_{\mathrm{spine}(\mathrm{Camb}_c)})^{h-1}(e)$ (where $e = \hat{0}$ is the identity element). In our full article [3], we prove Theorem 5 by showing that $|\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)| = h$. To do so, we make use of *combinatorial AR quivers* and the combinatorial aspects of the *c-sorting word* for the long element of W (we omit this proof here).

Example 1. Let W be the hyperoctahedral group B_3. Then $S = \{s_0, s_1, s_2\}$, and we have $m(s_0, s_1) = 4$, $m(s_1, s_2) = 3$, and $m(s_0, s_2) = 2$. Let $c = s_0s_2s_1$ and $c' = s_0s_1s_2$. The lattices Camb_c and $\mathrm{Camb}_{c'}$ are shown on the left and right, respectively, in Figure 2. The spine of each lattice has been colored in red. The Coxeter number of B_3 is $h = 6$. In the lattice on the left, we obtain the element $\mathbf{z}_c$, which is marked by a blue circle, by starting at the bottom element and applying the dual pop-stack operator *in the spine* $h - 1 = 5$ times. This amounts to traveling up the blue dotted curves. If we start at $\mathbf{z}_c$ and iteratively apply $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$, then we just travel down the same blue dotted curves (the fact that this happens for arbitrary Cambrian lattices is not obvious). This shows that $\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)$ is contained in the spine of Camb_c and that $|\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)| = h$. Similarly, $\mathbf{z}_{c'}$ is obtained by traveling up the blue dotted curves in the lattice on the right, and $\mathcal{O}_{\mathrm{Camb}_{c'}}(\mathbf{z}_{c'})$ has size h and is contained in the spine of $\mathrm{Camb}_{c'}$.

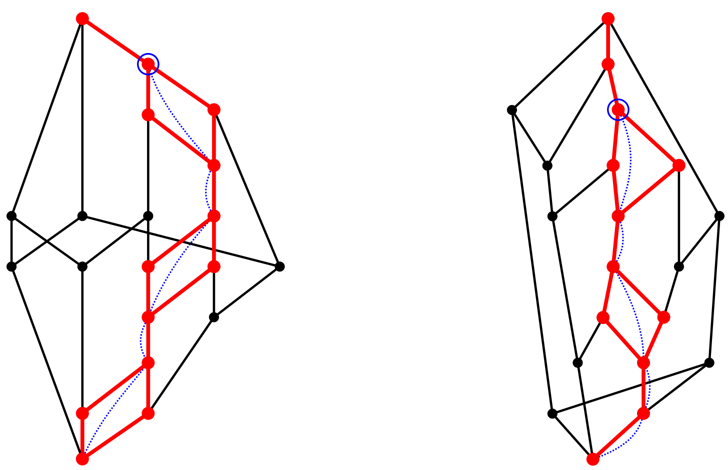

Figure 2: Two Cambrian lattices of type B_3. The spine of each lattice is in thick red. In each lattice, we have circled in blue an element whose forward orbit under the pop-stack operator has size $h = 6$.

5 Future Directions

Consider the *linear Coxeter element* $c^{\rightarrow} = s_1s_2\cdots s_n$ of A_n. The Cambrian lattice $\mathrm{Camb}_{c^{\rightarrow}}$ is the $(n+1)$-st *Tamari lattice*. Hong [11] proved that the size of the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}$ is the n-th *Motzkin number* (i.e., the number of Motzkin paths of length n). In Section 3, we determined the size of the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}$, where $c^{\times} = c^{\times}_{(n)}$ is the bipartite Coxeter element of A_n defined in Equation (3.1). Using these formulas, one can verify that $|\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}(\mathrm{Camb}_{c^{\rightarrow}})| \leq |\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}(\mathrm{Camb}_{c^{\times}})|$. Numerical evidence has led us to conjecture that the linear and bipartite Coxeter elements are, in some sense, extremal with regard to the sizes of the images of pop-stack operators.

Conjecture 1. *For every Coxeter element c of A_n, we have*

$$|\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}(\mathrm{Camb}_{c^{\rightarrow}})| \leq |\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c}}(\mathrm{Camb}_{c})| \leq |\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}}}(\mathrm{Camb}_{c^{\times}})|.$$

Suppose L is a finite lattice, and let $v_L = \max_{x \in L} |\mathcal{O}_L(x)|$. Let

$$\mathrm{Y}_L = \{x \in L \mid |\mathcal{O}_L(x)| = v_L\}$$

be the set of elements of L whose forward orbits under $\mathrm{pop}^{\downarrow}_L$ attain the maximum possible size.

Let c be a Coxeter element of a finite irreducible Coxeter group W. We saw in Theorem 5 that the maximum possible size of the forward orbit of an element of Camb_c under $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$ is the Coxeter number h; however, we said nothing about the number of elements that actually attain this maximum. In the case when $W = A_n$ and c is the linear Coxeter element $c^{\rightarrow}$ (i.e., Camb_c is the $(n+1)$-st Tamari lattice), it is known that $|\mathrm{Y}_{\mathrm{Camb}_c}|$ is the $(n-1)$-st Catalan number [6]. It would be interesting to understand $|\mathrm{Y}_{\mathrm{Camb}_c}|$ for other Cambrian lattices Camb_c. In particular, we have the following conjecture.

Conjecture 2. *The number of elements of* $\mathrm{Camb}_{c^{\times}}$ *whose forward orbits under the pop-stack operator have size h is* 1 *if n is even and is* 2 *if n is odd.*

The original use of the term *pop-stack* comes from the setting where L is the weak order on A_n; in this case, Ungar proved that $\max_{x \in \mathrm{Weak}(A_n)} |\mathcal{O}_{\mathrm{Weak}(A_n)}(x)|$ is $n+1$ (which is the Coxeter number of A_n).

Question 1. *What can be said about $|\mathrm{Y}_{\mathrm{Weak}(A_n)}|$?*

Defant [7] proved that if W is a finite irreducible Coxeter group with Coxeter number h, then $\max_{x \in W} |\mathcal{O}_{\mathrm{Weak}(W)}(x)| = h$. In Theorem 4, we found that $\max_{x \in L} |\mathcal{O}_L(x)| \leq h$ whenever L is a lattice quotient of $\mathrm{Weak}(W)$, and we saw in Theorem 5 that this inequality is an equality whenever L is a Cambrian lattice. We are naturally led to ask the following questions.

Question 2. *Let W be a finite irreducible Coxeter group with Coxeter number h. For which lattice quotients L of* $\mathrm{Weak}(W)$ *is it the case that* $\max_{x \in L} |\mathcal{O}_L(x)| = h$*?*

Question 3. *Let L' be a lattice quotient of a finite lattice L. Is it necessarily the case that*

$$\max_{x' \in L'} |\mathcal{O}_{L'}(x')| \leq \max_{x \in L} |\mathcal{O}_L(x)|?$$

It would be interesting to see how much of our work on Cambrian lattices can be extended to more general families of lattices. For example, it could be interesting to study the pop-stack operators on *m-Cambrian lattices*, which were introduced by Stump, Thomas, and Williams [16].

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The characteristic quasi-polynomials for exceptional well-generated complex reflection groups

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Abstract. Kamiya, Takemura, and Terao initiated the theory of the characteristic quasi-polynomial of an integral arrangement, which is a function counting the elements in the complement of the arrangement modulo positive integers. The characteristic quasi-polynomials of crystallographic root systems exhibit many interesting properties. Recently, the authors extended the concept of the characteristic quasi-polynomials for arrangements over a Dedekind domain, where every residue ring with respect to nonzero ideal is finite. In this article, we investigate the characteristic quasi-polynomials for exceptional well-generated complex reflection groups, using the root systems over the rings of definition introduced by Lehrer and Taylor. We demonstrate that a specific relation between the Coxeter numbers and the LCM-periods of the characteristic quasi-polynomials is generalized in this context.

Résumé. Kamiya, Takemura et Terao ont initié la théorie du quasi-polynôme caractéristique d'un agencement intégral, qui est une fonction comptant les éléments dans le complément de l'agencement modulo les entiers positifs. Les quasi-polynômes caractéristiques des systèmes de racines cristallographiques présentent de nombreuses propriétés intéressantes. Récemment, les auteurs ont étendu le concept des quasi-polynômes caractéristiques aux agencements sur un domaine de Dedekind, où chaque anneau résiduel par rapport à un idéal non nul est fini. Dans cet article, nous examinons les quasi-polynômes caractéristiques pour les groupes de réflexion complexes exceptionnellement bien générés, en utilisant les systèmes de racines sur les anneaux définition introduits par Lehrer et Taylor. Nous démontrons qu'une relation spécifique entre les nombres de Coxeter et les LCM-périodes des quasi-polynômes caractéristiques est généralisée dans ce contexte.

Keywords: hyperplane arrangement, complex reflection group, root system, characteristic quasi-polynomial

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1 Introduction

1.1 Characteristic quasi-polynomials

For a positive integer ℓ, let $\mathcal{A} = \{c_1, \ldots, c_n\} \subseteq \mathbb{Z}^\ell$ be a finite subset consisting of nonzero integral column vectors. Define the hyperplane arrangement $\mathcal{A}(\mathbb{R})$ in the vector space $\mathbb{R}^\ell$ by $\mathcal{A}(\mathbb{R}) := \{H_1, \ldots, H_n\}$, where

$$H_j := \left\{ \boldsymbol{x} := (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell \;\middle|\; \boldsymbol{x}c_j = 0 \right\} \quad (j \in [n] := \{1, \ldots, n\}).$$

Let $L(\mathcal{A}(\mathbb{R})) := \{ H_J \mid J \subseteq [n] \}$ be the set of intersections $H_J := \bigcup_{j \in J} H_j$. The set $L(\mathcal{A}(\mathbb{R}))$ equipped with the order defined by $X \leq Y \Leftrightarrow X \supseteq Y$ is called the **intersection lattice**. The **characteristic polynomial** $\chi_{\mathcal{A}(\mathbb{R})}$ is defined by

$$\chi_{\mathcal{A}(\mathbb{R})}(t) := \sum_{Z \in L(\mathcal{A}(\mathbb{R}))} \mu(Z) t^{\dim Z},$$

where μ denotes the **Möbius function** on $L(\mathcal{A}(\mathbb{R}))$, which is defined recursively by

$$\mu(\mathbb{R}^\ell) := 1 \qquad \text{and} \qquad \mu(Z) := -\sum_{Y < Z} \mu(Y) \text{ for } Z \neq \mathbb{R}^\ell.$$

The **complement** of an arrangement is the complement of the union of the members of the arrangement in the ambient space. Each connected component of the complement of $\mathcal{A}(\mathbb{R})$ is called a chamber. Zaslavsky [20] proved that the numbers of chambers and bounded chambers coincide with $|\chi_{\mathcal{A}(\mathbb{R})}(-1)|$ and $|\chi_{\mathcal{A}(\mathbb{R})}(1)|$. Orlik and Solomon [13] proved that $\chi_{\mathcal{A}(\mathbb{R})}(t)$ is equivalent to the Poincaré polynomial of the complement of the complexification of $\mathcal{A}(\mathbb{R})$.

Next, for any positive integer q, we define the **q-reduced arrangement** $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ in $(\mathbb{Z}/q\mathbb{Z})^\ell$ by $\mathcal{A}(\mathbb{Z}/q\mathbb{Z}) := \{H_{1,q}, \ldots, H_{n,q}\}$, where

$$H_{j,q} := \left\{ [\boldsymbol{x}]_q \in (\mathbb{Z}/q\mathbb{Z})^\ell \;\middle|\; \boldsymbol{x}c_j \equiv 0 \pmod{q} \right\} \quad (j \in [n])$$

and $[\boldsymbol{x}]_q$ denotes the equivalence class of $\boldsymbol{x}$.

Athanasiadis [1, Theorem 2.2] provided a method to compute the characteristic polynomial of an integral arrangement by counting the points of the complement of $\mathcal{A}(\mathbb{Z}/p\mathbb{Z})$ for large enough prime numbers p. Athanasiadis [2, Theorem 2.1] also proved that the characteristic polynomial can be computed by counting the points of the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ for large enough integers q relatively prime a constant which depends only on $\mathcal{A}$.

Kamiya, Takemura, and Terao developed Athanasiadis' method by considering the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ for all positive integers q as follows. For a nonempty subset

$J = \{j_1, \dots, j_k\} \subseteq [n]$, suppose that the matrix $C_J := (c_{j_1} \ \cdots \ c_{j_k})$ has the Smith normal form

$$\begin{pmatrix} d_{J,1} & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & d_{J,2} & & \vdots & & & \vdots \\ \vdots & & \ddots & 0 & & & \\ 0 & \cdots & 0 & d_{J,r(J)} & & & \\ \vdots & & & & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 \end{pmatrix},$$

where $d_{J,i}$ is a positive integer such that $d_{J,i}$ divides $d_{J,i+1}$. Define $\rho_{\mathcal{A}} \in \mathbb{Z}_{>0}$ by

$$\rho_{\mathcal{A}} := \operatorname{lcm}\left\{ d_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \right\}.$$

Theorem 1.1 (Kamiya–Takemura–Terao [6])**.** *Let $M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z})) := (\mathbb{Z}/q\mathbb{Z})^{\ell} \setminus \bigcup_{J\subseteq[n]} H_{J,q}$ denote the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$. Then the function $|M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$ is a monic integral quasi-polynomial in $q \in \mathbb{Z}_{>0}$ with a period $\rho_{\mathcal{A}}$. Namely, there exist monic polynomials $f^k_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ $(1 \le k \le \rho_{\mathcal{A}})$ such that $f^k_{\mathcal{A}}(q) = |M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$ if $q \equiv k \pmod{\rho_{\mathcal{A}}}$. Furthermore, the quasi-polynomial has the **GCD-property**, that is, $f^k_{\mathcal{A}}(t) = f^{k'}_{\mathcal{A}}(t)$ when $\gcd(k, \rho_{\mathcal{A}}) = \gcd(k', \rho_{\mathcal{A}})$.*

Definition 1.2. We call the quasi-polynomial

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) := |M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$$

the **characteristic quasi-polynomial** of $\mathcal{A}$. The period $\rho_{\mathcal{A}}$ is called the **LCM-period**. The polynomial $f^k_{\mathcal{A}}(t)$ is said to be the k**-constituent** of $\chi^{\text{quasi}}_{\mathcal{A}}(q)$.

Interestingly enough, each constituent of the characteristic quasi-polynomial has a combinatorial interpretation (See [12, 17] for details). In particular, the following holds.

Theorem 1.3 (Kamiya–Takemura–Terao [6, Theorem 2.5])**.** *The 1-constituent of the characteristic quasi-polynomial of $\mathcal{A}$ is the characteristic polynomial of the hyperplane arrangement $\mathcal{A}(\mathbb{R})$. Namely, $f^1_{\mathcal{A}}(t) = \chi_{\mathcal{A}(\mathbb{R})}(t)$.*

For a decade, it was an open problem whether the LCM-period is minimum or not. Recently Higashitani, Tran, and Yoshinaga gave an affirmative answer for central arrangements.

Theorem 1.4 (Higashitani–Tran–Yoshinaga [4, Theorem 1.2])**.** *The LCM-period $\rho_{\mathcal{A}}$ is the minimum period of the characteristic quasi-polynomial $\chi^{\text{quasi}}_{\mathcal{A}}(q)$.*

Remark 1.5. The characteristic quasi-polynomial and its LCM-period can be considered for non-central arrangements [8]. Higashitani, Tran, and Yoshinaga [4] also studied non-central arrangements such that the LCM-periods are not minimum.

1.2 Crystallographic root systems

Let Φ be an irreducible crystallographic root system and Φ^+ a positive system of Φ. Every positive root is expressed as a linear combination of the simple roots with integral coefficients. Gathering the coefficient column vectors, we obtain the set $\mathcal{A}_\Phi$ consisting of integral column vectors. Kamiya, Takemura, and Terao [5, 7] computed the characteristic quasi-polynomial $\chi_\Phi^{\text{quasi}}(q)$ of $\mathcal{A}_\Phi$ and its LCM-period explicitly by using the classification of root systems. Note that Suter [16] gave essentially the same calculation in terms of the number of lattice points in the fundamental alcoves (the Ehrhart quasi-polynomials).

Kamiya, Takemura, and Terao [7, Theorem 3.1] gave an explicit formula of the generating function $\Gamma_\Phi := \sum_{q=1}^{\infty} \chi_\Phi^{\text{quasi}}(q)t^q$ for an irreducible crystallographic root system Φ in terms of the coefficient of the highest root and the Coxeter number. We obtain the following corollaries.

Corollary 1.6 (Kamiya–Takemura–Terao [7, Corollary 3.2])**.** *Let $n_1, \dots, n_\ell$ be the coefficient of the highest root of Φ with respect to the simple roots. Then $\operatorname{lcm}(n_1, \dots, n_\ell)$ coincides with the LCM-period of $\chi_\Phi^{\text{quasi}}(q)$.*

Corollary 1.7 (Kamiya–Takemura–Terao [7, Corollary 3.4])**.** *Let h be the Coxeter number of Φ. Then $\chi_\Phi^{\text{quasi}}(q) > 0$ if and only if $q \geq h$.*

The characteristic quasi-polynomial of an irreducible crystallographic root system also has duality with respect to the Coxeter number. The duality can be shown from the explicit expressions given by Kamiya, Takemura, and Terao [5], or Suter [16]. Yoshinaga [19] gave a classification-free proof.

Theorem 1.8 (Yoshinaga [19, Corollary 3.8])**.** *Let Φ be an irreducible crystallographic root system of rank ℓ and h its Coxeter number. Then $\chi_\Phi^{\text{quasi}}(q) = (-1)^\ell \chi_\Phi^{\text{quasi}}(h-q)$.*

Note that the duality holds as quasi-polynomials but not the level of the constituents. Yoshinaga [18] studied the condition for the constituents to hold the duality in detail.

Combining Theorem 1.8, Theorem 1.3, and the following theorem, we can deduce that the characteristic polynomial $\chi_\Phi(t)$ of the arrangement $\mathcal{A}_\Phi$ satisfies the duality (Corollary 1.10).

Theorem 1.9 (Kamiya–Takemura–Terao [7], Suter [16])**.** *The radical of the LCM period of $\chi_\Phi^{\text{quasi}}(q)$ divides the Coxeter number h.*

Corollary 1.10. *Let Φ be an irreducible crystallographic root system of rank ℓ and h its Coxeter number. Then $\chi_\Phi(q) = (-1)^\ell \chi_\Phi(h-q)$.*

1.3 Characteristic quasi-polynomials over residually finite Dedekind domains

Let $\mathcal{O}$ be a Dedekind domain such that the residue ring $\mathcal{O}/\mathfrak{a}$ is finite for every nonzero ideal $\mathfrak{a}$. Such a ring $\mathcal{O}$ is called a **residually finite Dedekind domain** or a Dedekind domain with the finite norm property. The ring $\mathbb{Z}$ is an example of a residually finite Dedekind domain. More generally, the ring of integers of an algebraic number field is a residually finite Dedekind domain. The authors generalized the notion of characteristic quasi-polynomials for $\mathcal{O}$ as follows.

Let $\mathcal{A} = \{c_1, \dots, c_n\} \subseteq \mathcal{O}^\ell$ and $\mathfrak{a} \in I(\mathcal{O})$, where $I(\mathcal{O})$ denotes the set of nonzero ideals of $\mathcal{O}$. Define the **$\mathfrak{a}$-reduced arrangement** $\mathcal{A}(\mathcal{O}/\mathfrak{a})$ by $\mathcal{A}(\mathcal{O}/\mathfrak{a}) := \{\, H_{j,\mathfrak{a}} \mid j \in [n] \,\}$, where

$$H_{j,\mathfrak{a}} := \left\{ [x]_\mathfrak{a} \in (\mathcal{O}/\mathfrak{a})^\ell \,\middle|\, xc_j \equiv 0 \pmod{\mathfrak{a}} \right\}.$$

Let $M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))$ denote the complement of $\mathcal{A}(\mathcal{O}/\mathfrak{a})$. Namely

$$M(\mathcal{A}(\mathcal{O}/\mathfrak{a})) := (\mathcal{O}/\mathfrak{a})^\ell \setminus \bigcup_{j=1}^{n} H_{j,\mathfrak{a}}.$$

Definition 1.11. The function $\chi^{\mathrm{quasi}}_{\mathcal{A}} \colon I(\mathcal{O}) \to \mathbb{Z}$ determined by $\chi^{\mathrm{quasi}}_{\mathcal{A}}(\mathfrak{a}) := |M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ is called the **characteristic quasi-polynomial** of $\mathcal{A}$.

The function $\chi^{\mathrm{quasi}}_{\mathcal{A}}$ is described by using finitely many polynomials periodically as ordinary quasi-polynomials.

Theorem 1.12 ([9, Theorem 3.1])**.** *There exists an ideal $\rho \in I(\mathcal{O})$ such that the following statement holds: For any divisor $\kappa \mid \rho$ there exists a monic polynomial $f^\kappa_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ such that*

$$\mathfrak{a} + \rho = \kappa \Longrightarrow \chi^{\mathrm{quasi}}_{\mathcal{A}}(\mathfrak{a}) = f^\kappa_{\mathcal{A}}(N(\mathfrak{a})),$$

where $N(\mathfrak{a}) := |\mathcal{O}/\mathfrak{a}|$, the ***absolute norm*** *of $\mathfrak{a}$.*

The ideal ρ above is called a **period**. We can construct a period $\rho_{\mathcal{A}}$ (called the **LCM-period**) for $\chi^{\mathrm{quasi}}_{\mathcal{A}}(\mathfrak{a})$ using the structure theorem for finitely generated modules over Dedekind domains and the authors proved that the LCM-period $\rho_{\mathcal{A}}$ is minimum (See [9, Theorem 5.1] for details). If $\mathcal{O}$ is a Euclidean domain, then we can compute the LCM-period algorithmically by computing the Smith normal forms and elementary divisors.

2 Characteristic quasi-polynomials for exceptional well-generated complex reflection groups

Let V be a finite-dimensional complex vector space. A map $r \in \mathrm{GL}(V)$ is called a **reflection** if $\ker(r - \mathrm{id}_V)$ has codimension 1. A finite subgroup $G \subseteq \mathrm{GL}(V)$ is called a

complex reflection group is G is generated by reflections. We say that G is **irreducible** if there are no nontrivial G-invariant subspaces. In this case, the dimension of the ambient space is called the **rank** of G. An irreducible complex reflection group G of rank ℓ is **well-generated** if G is generated by ℓ reflections. Irreducible complex reflection groups are classified by Shephard and Todd [14]. There are an infinite family $G(m, p, \ell)$ and 34 exceptional cases labeled by $G_4, \ldots, G_{37}$. Among the exceptional groups, we have 26 well-generated ones, which are listed in Table 1.

Definition 2.1. Let G be an irreducible reflection group. Define the **field of definition** $K(G)$ by

$$K(G) \coloneqq \mathbb{Q}(\operatorname{tr}(\sigma) \mid \sigma \in G).$$

Define the **ring of definition** $\mathcal{O}(G)$ as the ring of integers of $K(G)$.

It is shown that G can be representable a vector space U over $K(G)$. Note that since $K(G)/\mathbb{Q}$ is a finite extension, the ring of definition $\mathcal{O}(G)$ is a residually finite Dedekind domain.

Let $(-,-)$ denote the Hermitian inner product of $V = \mathbb{C} \otimes_{K(G)} U$. Let $\boldsymbol{\mu}(\mathcal{O}(G))$ denote the group of roots of unity in $\mathcal{O}(G)$. For every $a \in U \setminus \{0\}$ and $\lambda \in \boldsymbol{\mu}(\mathcal{O}(G))$, we define a reflection $r_{a,\lambda}$ by

$$r_{a,\lambda}(v) \coloneqq v - (1-\lambda)\frac{(v,a)}{(a,a)}a.$$

Lehrer and Taylor [10] defined a generalization of root systems for algebraic integers and showed that every finite complex reflection group admits a "root system". Namely there exists a pair (Σ, f) satisfying the following.

- Σ is a finite subset of $U \setminus \{0\}$ and Σ spans U.
- $f \colon \Sigma \to \boldsymbol{\mu}(\mathcal{O}(G))$.
- G is generated by the reflections $\left\{ r_{a,f(a)} \;\middle|\; a \in \Sigma \right\}$.
- For all $a \in \Sigma$ and all $\lambda \in K(G)$ we have $\lambda a \in \Sigma \Leftrightarrow \lambda \in \boldsymbol{\mu}(\mathcal{O}(G))$.
- For all $a \in \Sigma$ and $\lambda \in \boldsymbol{\mu}(\mathcal{O}(G))$ we have $f(\lambda a) = f(a) \neq 1$.
- For all $a, b \in \Sigma$ we have $(1 - f(b))(a,b)/(b,b) \in \mathcal{O}(G)$.
- For all $a, b \in \Sigma$ we have $r_{a,f(a)}(b) \in \Sigma$ and $f(r_{a,f(a)}(b)) = f(b)$.

We call (Σ, f) a $\mathcal{O}(G)$**-root system** for G. If $\mathcal{O}(G) = \mathbb{Z}$, then the above definition coincides with the definition of crystallographic root system. Namely a $\mathbb{Z}$-root system is a crystallographic system.

When G is well-generated, there exist roots $a_1, \ldots, a_\ell \in \Sigma$ such that every root in Σ is reperesented by a linear combination of $a_1, \ldots, a_\ell$ over $\mathcal{O}(G)$. Hence we can obtain a finite coefficient column vectors $\mathcal{A}_{(\Sigma,f)} \subseteq \mathcal{O}(G)^\ell$ from the root system (Σ, f) over $\mathcal{O}(G)$. Note that the characteristic quasi-polynomial determined by (Σ, f) does not depend on the choice of roots $a_1, \ldots, a_\ell$.

Lehrer and Taylor listed the Cartan matrices of $\mathcal{O}(G)$-root systems for exceptional irreducible complex reflection groups. We can recover the root system from the corresponding Cartan matrix.

Example 2.2. Consider the group G_4. The rank of G_4 is two and its ring of definition is $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$. The matrix

$$C_4 = \begin{pmatrix} 1-\omega & 1 \\ -\omega & 1-\omega \end{pmatrix}$$

is a Cartan matrix for G_4. Let $\{a_1, a_2\}$ a basis for $\mathbb{C}^2$ and r_1, r_2 the correspondence reflections. The Cartan matrix C_4 tells us $r_j(a_i) = a_i - c_{ij}a_j$, where c_{ij} denotes the (i,j) entry of C_4. Thus we have

$$\begin{aligned} r_1(a_1) &= a_1 - (1-\omega)a_1 = \omega a_1, & r_2(a_1) &= a_1 - 1 \cdot a_2 = a_1 - a_2, \\ r_1(a_2) &= a_2 - (-\omega)a_1 = \omega a_1 + a_2, & r_2(a_2) &= a_2 - (1-\omega)a_2 = \omega a_2. \end{aligned}$$

Hence we obtain the matrix representations of r_1, r_2 with respect to the basis $\{a_1, a_2\}$ as follows.

$$r_1 = \begin{pmatrix} \omega & \omega \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 \\ -1 & \omega \end{pmatrix}.$$

Therefore $G_4 = \langle r_1, r_2 \rangle \subseteq \mathrm{GL}_2(\mathbb{C})$ and Σ can be recovered as $\Sigma = \{\, r(a_i) \mid r \in G_4,\ i = 1, 2 \,\}$ with $a_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $a_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. As a result, Σ consists of the following 24 vectors.

$$\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda \begin{pmatrix} \omega \\ 1 \end{pmatrix}, \quad \lambda \in \boldsymbol{\mu}(\mathbb{Z}[\omega]) = \{\pm 1, \pm\omega, \pm\omega^2\}.$$

Setting $f(a_1) = f(a_2) = \omega$, we obtain the $\mathbb{Z}[\omega]$-root system (Σ, f) and

$$\mathcal{A}_{(\Sigma,f)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} \omega \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{Z}[\omega]^2.$$

Since $\mathbb{Z}[\omega]$ is a Euclidean domain, we can compute the LCM-period by finding the elementary divisors. The LCM-period is the unit ideal $\langle 1 \rangle$ and hence the characteristic quasi-polynomial has only one constituent (the characteristic polynomial)

$$f^{\langle 1 \rangle}(t) = t^2 - 4t + 3 = (t-1)(t-3).$$

Example 2.3. Consider G_{33} and the Cartan matrix

$$C_{33} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & -\omega & 0 \\ 0 & -1 & -\omega^2 & 2 & -\omega^2 \\ 0 & 0 & 0 & -\omega & 2 \end{pmatrix}.$$

The ring of definition is $\mathbb{Z}[\omega]$ and the LCM-period is $\langle 2\sqrt{-3} \rangle$. The characteristic quasi-polynomial consists of the following constituents:

$$\begin{aligned} f^{\langle 1 \rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 17169t - 12285. \\ &= (t-1)(t-7)(t-9)(t-13)(t-15). \\ f^{\langle 2 \rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 17574t - 18360. \\ &= (t-4)(t-15)(t^3 - 26t^2 + 196t - 306). \\ f^{\langle \sqrt{-3} \rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 18129t - 20925. \\ &= (t-3)(t-9)(t^3 - 33t^2 + 327t - 775). \\ f^{\langle 2\sqrt{-3} \rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 18534t - 27000. \end{aligned}$$

The authors calculated the LCM-period for the root systems determined by the Cartan matrices in Table 2. Note that C_{20} is modified from the one in [10] so that it recovers the root system correctly. According to [11] and [15], the rings of definition for exceptional irreducible complex reflection groups are Euclidean domains except for $\mathcal{O}(G_{21})$. Although the authors are not sure whether $\mathcal{O}(G_{21})$ is Euclidean or not, since $\mathcal{O}(G_{21})$ is a principal ideal domain (See [3]), there exist the Smith normal forms. Fortunately the authors could find the Smith normal forms and hence the LCM-period for G_{21}. We summarize the results in Table 1. From this computational result, we have the following theorem, which is a generalization of Theorem 1.9.

Theorem 2.4. *Every exceptional well-generated irreducible complex reflection group G admits an $\mathcal{O}(G)$-root system such that the radical of the LCM-period divides the Coxeter number.*

Remark 2.5. We anticipated phenomenon analogous to Corollary 1.6, Corollary 1.7, Theorem 1.8, and Theorem 1.9. However, only Theorem 1.9 has been observed.

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Table 1: LCM-periods and Coxeter numbers

G	$\mathcal{O}(G)$	LCM-period	h	Coexponents
G_4	$\mathbb{Z}[\omega]$	$\langle 1 \rangle$	6	$1, 3$
G_5	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	12	$1, 7$
G_6	$\mathbb{Z}[i, \omega]$	$\langle 1 + i \rangle$	12	$1, 9$
G_8	$\mathbb{Z}[i]$	$\langle 1 + i \rangle$	12	$1, 5$
G_9	$\mathbb{Z}[\zeta_8]$	$\langle 6 \rangle$	24	$1, 17$
G_{10}	$\mathbb{Z}[i, \omega]$	$\langle (1 + i)\sqrt{-3} \rangle$	24	$1, 13$
G_{14}	$\mathbb{Z}[\omega, \sqrt{-2}]$	$\langle 6 \rangle$	24	$1, 19$
G_{16}	$\mathbb{Z}[\zeta_5]$	$\langle 1 - \zeta_5 \rangle$	30	$1, 11$
G_{17}	$\mathbb{Z}[i, \zeta_5]$	$\langle 6\sqrt{5} \rangle$	60	$1, 41$
G_{18}	$\mathbb{Z}[\omega, \zeta_5]$	$\langle 2\sqrt{-3}(1 - \zeta_{15}^3) \rangle$	60	$1, 31$
G_{20}	$\mathbb{Z}[\omega, \tau]$	$\langle 2\sqrt{-3} \rangle$	30	$1, 19$
G_{21}	$\mathbb{Z}[i, \omega, \tau]$	$\langle 6\sqrt{5} \rangle$	60	$1, 49$
$G_{23} = \mathrm{H}_3$	$\mathbb{Z}[\tau]$	$\langle 2 \rangle$	10	$1, 5, 9$
G_{24}	$\mathbb{Z}[\lambda]$	$\langle 4 \rangle$	14	$1, 9, 11$
G_{25}	$\mathbb{Z}[\omega]$	$\langle \sqrt{-3} \rangle$	12	$1, 4, 7$
G_{26}	$\mathbb{Z}[\omega]$	$\langle 6 \rangle$	18	$1, 7, 13$
G_{27}	$\mathbb{Z}[\omega, \tau]$	$\langle 4\sqrt{-3} \rangle$	30	$1, 19, 25$
$G_{28} = \mathrm{F}_4$	$\mathbb{Z}$	$\langle 12 \rangle$	12	$1, 5, 7, 11$
G_{29}	$\mathbb{Z}[i]$	$\langle 10(1 + i) \rangle$	20	$1, 9, 13, 17$
$G_{30} = \mathrm{H}_4$	$\mathbb{Z}[\tau]$	$\langle 6\sqrt{5} \rangle$	30	$1, 11, 19, 29$
G_{32}	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	30	$1, 7, 13, 19$
G_{33}	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	18	$1, 7, 9, 13, 15$
G_{34}	$\mathbb{Z}[\omega]$	$\langle 84 \rangle$	42	$1, 13, 19, 25, 31, 37$
$G_{35} = \mathrm{E}_6$	$\mathbb{Z}$	$\langle 6 \rangle$	12	$1, 4, 5, 7, 8, 11$
$G_{36} = \mathrm{E}_7$	$\mathbb{Z}$	$\langle 12 \rangle$	18	$1, 5, 7, 9, 11, 13, 17$
$G_{37} = \mathrm{E}_8$	$\mathbb{Z}$	$\langle 60 \rangle$	30	$1, 7, 11, 13, 17, 19, 23, 29$

$$i = \sqrt{-1},\ \omega = \frac{-1 + \sqrt{-3}}{2},\ \tau = \frac{1 + \sqrt{5}}{2},\ \lambda = \frac{-1 + \sqrt{-7}}{2},\ \zeta_k = e^{2\pi i/k}.$$

Table 2: Cartan matrices (C_{20} is modified)

$$C_4 = \begin{pmatrix} 1-\omega & 1 \\ -\omega & 1-\omega \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1-\omega & 1 \\ -2\omega & 1-\omega \end{pmatrix}, \quad C_6 = \begin{pmatrix} 2 & 1 \\ 1-\omega+i\omega^2 & 1-\omega \end{pmatrix},$$

$$C_8 = \begin{pmatrix} 1-i & 1 \\ -i & 1-i \end{pmatrix}, \quad C_9 = \begin{pmatrix} 2 & 1 \\ (1+\sqrt{2})\zeta_8 & 1+i \end{pmatrix}, \quad C_{10} = \begin{pmatrix} 1-\omega & 1 \\ -i-\omega & 1-i \end{pmatrix},$$

$$C_{14} = \begin{pmatrix} 1-\omega & 1 \\ 1-\omega+i\omega^2\sqrt{2} & 2 \end{pmatrix}, \quad C_{16} = \begin{pmatrix} 1-\zeta_5 & 1 \\ -\zeta_5 & 1-\zeta_5 \end{pmatrix}, \quad C_{17} = \begin{pmatrix} 2 & 1 \\ 1-\zeta_5-i\zeta_5^3 & 1-\zeta_5 \end{pmatrix},$$

$$C_{18} = \begin{pmatrix} 1-\omega & 1 \\ -\omega-\zeta_5 & 1-\zeta_5 \end{pmatrix}, \quad C_{20} = \begin{pmatrix} 1-\omega & \tau-1 \\ \omega(1-\tau) & 1-\omega \end{pmatrix}, \quad C_{21} = \begin{pmatrix} 2 & 1 \\ 1-\omega-i\omega^2\tau & 1-\omega \end{pmatrix},$$

$$C_{23} = \begin{pmatrix} 2 & -\tau & 0 \\ \tau & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C_{24} = \begin{pmatrix} 2 & -1 & -\lambda \\ -1 & 2 & -1 \\ 1+\lambda & -1 & 2 \end{pmatrix}, \quad C_{25} = \begin{pmatrix} 1-\omega^2 & \omega^2 & 0 \\ -\omega^2 & 1-\omega & -\omega^2 \\ 0 & \omega^2 & 1-\omega \end{pmatrix},$$

$$C_{26} = \begin{pmatrix} 1-\omega & -\omega^2 & 0 \\ \omega^2 & 1-\omega & -1 \\ 0 & -1+\omega & 2 \end{pmatrix}, \quad C_{27} = \begin{pmatrix} 2 & -\tau & -\omega \\ -\tau & 2 & -\omega^2 \\ -\omega^2 & -\omega & 2 \end{pmatrix},$$

$$C_{28} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad C_{29} = \begin{pmatrix} 2 & -1 & i+1 & 0 \\ -1 & 2 & -i & 0 \\ -i+1 & i & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

$$C_{30} = \begin{pmatrix} 2 & -\tau & 0 & 0 \\ -\tau & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad C_{32} = \begin{pmatrix} 1-\omega & \omega^2 & 0 & 0 \\ -\omega^2 & 1-\omega & -\omega^2 & 0 \\ 0 & \omega^2 & 1-\omega & \omega^2 \\ 0 & 0 & -\omega^2 & 1-\omega \end{pmatrix},$$

$$C_{33} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & -\omega & 0 \\ 0 & -1 & -\omega^2 & 2 & -\omega^2 \\ 0 & 0 & 0 & -\omega & 2 \end{pmatrix}, \quad C_{34} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -\omega & 0 \\ 0 & 0 & -1 & -\omega^2 & 2 & -\omega^2 \\ 0 & 0 & 0 & 0 & -\omega & 2 \end{pmatrix}.$$

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Macdonald characters from a new formula for Macdonald polynomials

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Abstract. We introduce a new operator $\mathbf{\Gamma}$ on symmetric functions, which enables us to obtain a creation formula for Macdonald polynomials. This formula makes a connection between the theory of Macdonald operators initiated by Bergeron, Garsia, Haiman and Tesler, and shifted Macdonald polynomials introduced by Knop, Okounkov and Sahi.

We use this formula to introduce a two-parameter generalization of Jack characters. Finally, we provide a change of variables in order to formulate several positivity conjectures related to these generalized characters. Our conjectures extend some important problems on Jack polynomials, including some famous conjectures of Goulden and Jackson.

Keywords: Macdonald polynomials, Macdonald characters, Matchings-Jack conjecture

1 Introduction

1.1 Jack and Macdonald polynomials

Jack polynomials are symmetric functions depending on one parameter α which have been introduced by Jack [13]. The combinatorial analysis of Jack polynomials has been initiated by Stanley [20] and a first combinatorial interpretation has been given by Knop and Sahi in terms of tableaux [14]. A second family of combinatorial objects related to Jack polynomials is given by *maps*, which are roughly graphs embedded in surfaces. This connection has first been observed in the conjectures of Goulden and Jackson [11] and important progress has recently been made in this direction [3, 2] with a first "topological expansion" of Jack polynomials in terms of maps.

Macdonald polynomials are symmetric polynomials introduced by Macdonald in 1989, which depend on two parameters q and t. Jack polynomials can be obtained from

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Macdonald polynomials by taking an appropriate limit. Several combinatorial results on Jack polynomials have been generalized to the Macdonald case, in particular, an interpretation in terms of tableaux was established in [12]. However, no connection between Macdonald polynomials and maps is known, even conjecturally. As a first step towards a Macdonald generalization of maps, we introduce in this paper some new tools that make the parallel between the Jack and Macdonald stories more compelling.

First, we prove a creation formula (Equations (1.1) and (1.2)) for Macdonald polynomials inspired from the one used in [2] to connect Jack polynomials to maps. Second, we use this formula to introduce a Macdonald analog of Jack characters (Section 1.4). Finally, we formulate a Macdonald version of some Jack conjectures, including Goulden and Jackson's Matchings-Jack and b-conjectures.

1.2 Preliminaries

For the results of this section we refer to [7, 17]. A *partition* $\lambda = [\lambda_1, ..., \lambda_\ell]$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq ... \geq \lambda_\ell > 0$. We denote by $\mathbb{Y}$ the set of integer partitions. The integer ℓ is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + ... + \lambda_\ell$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers λ_1,...,λ_ℓ are called the *parts* of λ. For $i \geq 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ. We then set $z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}$. We identify a partition λ with its *Young diagram*, defined by $\lambda := \{(i,j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. Fix a box $\Box := (i,j) \in \lambda$. Its *arm*, *leg*, *co-arm* and *co-leg* are respectively given by

$$a_\lambda(\Box) := |\{(i,c) \in \lambda, c > j\}|, \qquad \ell_\lambda(\Box) := |\{(r,j) \in \lambda, r > i\}|,$$

$$a'_\lambda(\Box) := |\{(i,c) \in \lambda, c < j\}|, \text{ and } \ell'_\lambda(\Box) := |\{(r,j) \in \lambda, r < i\}|.$$

Finally, n and n' denote respectively the two statistics on Young diagram given by

$$n(\lambda) := \sum_{\Box \in \lambda} \ell_\lambda(\Box) \quad \text{and} \quad n'(\lambda) := \sum_{\Box \in \lambda} a_\lambda(\Box).$$

We consider the graded algebra $\Lambda = \oplus_{r\geq 0} \Lambda^{(r)}$ of symmetric functions in the alphabet $(x_1, x_2, \dots)$ with coefficients in $\mathbb{Q}(q,t)$. Let p_λ and h_λ denote the power-sum and the complete symmetric functions in $(x_i)_{i\geq 1}$, respectively. We use here a variable u to keep track of the degree of the functions, and an extra variable v; all the functions considered are in $\Lambda[v][\![u]\!]$. Consider the *Hall scalar product* defined by $\langle p_\mu, p_\nu \rangle = \delta_{\mu,\nu} z_\mu$. Let $f^\perp$ denote the adjoint of multiplication by $f \in \Lambda$ with respect to $\langle , \rangle$.

We will use the *plethystic notation*: if $E(q,t,u,v,x_1,x_2,\dots) \in \Lambda[v][\![u]\!]$ and $f \in \Lambda$ then $f[E]$ is the image of f under the algebra morphism defined by

$$\begin{aligned} \Lambda[v][\![u]\!] &\longrightarrow \Lambda[v][\![u]\!] \\ p_k &\longmapsto E(t^k, q^k, u^k, v^k, x_1^k, \dots). \end{aligned}$$

Set $X := x_1 + x_2 + \dots$. Notice that $f[X] = f(x_1, x_2, \dots)$ for any f. Moreover,

$$p_k\left[X\frac{1-q}{1-t}\right] = \frac{1-q^k}{1-t^k}p_k(x_1, x_2, \dots) \text{ and } p_\lambda[-X] = (-1)^{\ell(\lambda)}p_\lambda(x_1, x_2, \dots).$$

We consider the scalar product $\langle , \rangle_{q,t}$ on Λ defined by

$$\langle p_\mu[X], p_\nu[X]\rangle_{q,t} = \delta_{\mu,\nu}z_\mu(q,t) := \delta_{\mu,\nu}z_\mu p_\mu\left[\frac{1-q}{1-t}\right].$$

The *integral form of Macdonald polynomials* $J_\lambda^{(q,t)}$ can be defined as the unique family of polynomials satisfying a triangularity property in the monomial basis, and orthogonal with respect to $\langle , \rangle_{q,t}$. More precisely $\left\langle J_\lambda^{(q,t)}, J_\rho^{(q,t)} \right\rangle_{q,t} = \delta_{\lambda,\rho} j_\lambda^{(q,t)}$ with

$$j_\lambda^{(q,t)} := \prod_{\square\in\lambda}\left(1 - q^{a_\lambda(\square)+1}t^{\ell_\lambda(\square)}\right)\left(1 - q^{a_\lambda(\square)}t^{\ell_\lambda(\square)+1}\right).$$

For every $r \in \mathbb{N}$, the set $\{J_\lambda^{(q,t)} \mid \lambda \vdash r\}$ is a basis of $\Lambda^{(r)}$.

Finally, let $\mathcal{P}_Z$ be the operator such that $\mathcal{P}_Z \cdot f[X] = \mathrm{Exp}[ZX]f[X]$, i.e. the multiplication by the *plethystic exponential* $\mathrm{Exp}[ZX] := \sum_{n\geq 0} h_n[ZX]$, and let $\mathcal{T}_Z := \sum_{\mu\in\mathbb{Y}} z_\mu^{-1}p_\mu[Z]p_\mu^\perp$ be the translation operator, so that $\mathcal{T}_Z \cdot f[X] = f[X+Z]$. Note that $\mathcal{P}_{Z+Z'} = \mathcal{P}_Z \cdot \mathcal{P}_{Z'}$.

1.3 A new formula for Macdonald polynomials

Consider the operators[1] $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}_v$ on symmetric functions defined by

$$\boldsymbol{\nabla} \cdot J_\lambda^{(q,t)} = (-1)^{|\lambda|}\left(\prod_{\square\in\lambda} q^{a'(\square)}t^{-\ell'(\square)}\right)J_\lambda^{(q,t)}, \quad \boldsymbol{\Delta}_v \cdot J_\lambda^{(q,t)} = \prod_{\square\in\lambda}\left(1 - v\cdot q^{a'(\square)}t^{-\ell'(\square)}\right)J_\lambda^{(q,t)}.$$

These are integral versions of some known operators on modified Macdonald polynomials (see Section 2.1).

We finally introduce the following operator[2] on $\Lambda[v][\![u]\!]$

$$\boldsymbol{\Gamma}(u,v) := \boldsymbol{\Delta}_{1/v}\mathcal{P}_{\frac{uv(1-t)}{1-q}}\boldsymbol{\Delta}_{1/v}^{-1}.$$

The fact that $\boldsymbol{\Gamma}(u,v)$ is a polynomial in v is a consequence of the Pieri rule.

We can state our new formula for Macdonald polynomials.

Theorem 1.1. *For any partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$, we have*

$$\boldsymbol{\Gamma}_{\lambda_1}^{(+)}\boldsymbol{\Gamma}_{\lambda_2}^{(+)}\cdots\boldsymbol{\Gamma}_{\lambda_k}^{(+)}\cdot 1 = J_\lambda^{(q,t)} \quad \textit{where} \quad \boldsymbol{\Gamma}_m^{(+)} := [u^m]\boldsymbol{\nabla}^{-1}\boldsymbol{\Gamma}(u,q^m)\boldsymbol{\nabla}. \tag{1.1}$$

[1] We use boldface symbols to distinguish these operators from their relatives from Section 2.1.

[2] This operator is a close relative of the Theta operator in [7], first introduced in [5].

It turns out that Theorem 1.1 is an easy consequence of the following *creation formula*.

Theorem 1.2. *For any partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$, we have*

$$\mathbf{\Gamma}(u, q^{\lambda_1})\mathbf{\Gamma}(t^{-1}u, q^{\lambda_2}) \cdots \mathbf{\Gamma}(t^{-(k-1)}u, q^{\lambda_k}) \cdot 1 = t^{-n(\lambda)} \boldsymbol{\nabla} \mathcal{T}_{\frac{1}{u(1-t)}} J_\lambda^{(q,t)}[uX]. \tag{1.2}$$

In Section 2.3, we prove an analogous result for modified Macdonald polynomials Theorem 2.1 from which we deduce Theorem 1.2.

In addition to giving a direct construction of Macdonald polynomials, Theorem 1.2 provides a dual approach to study the structure of these polynomials. Indeed, Equation (1.2) allows to think of $J_\lambda^{(q,t)}$ as a function in the partition λ described by the alphabet $(q^{\lambda_1}, q^{\lambda_2}, \dots)$. This dual approach plays a key role in this paper and is used in Section 3 to introduce a q,t-deformation of Jack characters.

1.4 Macdonald characters

Jack characters, introduced by Lassalle in [16], can be seen as a one parameter deformation of the symmetric group characters, and are directly related to the coefficients of Jack polynomials in the power-sum basis. Jack characters have been useful to understand asymptotic behavior of large Young diagram sampled with respect to a Jack deformed Plancherel measure [4, 6]. Moreover, Goulden and Jackson's Mathings-Jack conjecture has a natural interpretation in terms of structure coefficients of Jack characters. Recently, a combinatorial interpretation of Jack characters in terms of maps on non orientable surfaces has been proved in [2], answering a positivity conjecture of Lassalle.

Using the operator $\mathbf{\Gamma}$, we introduce a two parameter deformation $\boldsymbol{\theta}_\mu^{(q,t)}$ of Jack characters. These characters have a structure of shifted symmetric functions and are related to shifted Macdonald polynomials, see [15, 18]. Macdonald characters $\boldsymbol{\theta}_\mu^{(q,t)}$ can be thought of as a natural generalization of the coefficients of Macdonald polynomials in the power-sum basis (see Equation (3.5)) which are hard to guess without the new operator Γ.

In Section 4, we make several positivity conjectures related to the new characters $\boldsymbol{\theta}_\mu^{(q,t)}$. These conjectures suggest that the characters $\boldsymbol{\theta}_\mu^{(q,t)}$ have a combinatorial structure which generalizes the one given by maps and that we hope to investigate in future works.

2 A new creation formula for Macdonald polynomials

2.1 Modified Macdonald polynomials

In [8], Garsia and Haiman introduced a *modified* version of *Macdonald polynomials*

$$\widetilde{H}_\lambda^{(q,t)} = t^{n(\lambda)} J_\lambda^{(q,1/t)} \left[\frac{X}{1-1/t}\right].$$

The operators ∇ and Δ_v are defined by

$$\nabla \widetilde{H}_\lambda^{(q,t)} := (-1)^{|\lambda|} \prod_{\square \in \lambda} q^{a'_\lambda(\square)} t^{\ell'_\lambda(\square)} \widetilde{H}_\lambda^{(q,t)}, \qquad \Delta_v \widetilde{H}_\lambda^{(q,t)} := \prod_{\square \in \lambda} \left(1 - v q^{a'_\lambda(\square)} t^{\ell'_\lambda(\square)}\right) \widetilde{H}_\lambda^{(q,t)}.$$

These operators are related by the *five-term relation* of Garsia and Mellit [10]

$$\nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{M}} = \Delta_{1/v} \mathcal{P}_{\frac{uv}{M}} \Delta_{1/v}^{-1}, \tag{2.1}$$

where $M := (1-q)(1-t)$. Let $B_\lambda := \sum_{\square \in \lambda} q^{a'_\lambda(\square)} t^{\ell'_\lambda(\square)} = \sum_{1 \le i \le \ell(\lambda)} t^{i-1} \frac{1-q^{\lambda_i}}{1-q}$, and $D_\lambda := MB_\lambda - 1$. We state another fundamental identity for Macdonald polynomials, due to Garsia, Haiman and Tesler [9]: for any partition λ

$$\nabla \mathcal{P}_{-\frac{u}{M}} \mathcal{T}_{\frac{1}{u}} \widetilde{H}_\lambda[uX] = \operatorname{Exp}\left[-\frac{uXD_\lambda}{M}\right]. \tag{2.2}$$

2.2 Creation formula for modified Macdonald polynomials

We start by proving a modified version of Theorem 1.2. Set $\Gamma(u,v) := \Delta_{1/v} \mathcal{P}_{\frac{uv}{1-q}} \Delta_{1/v}^{-1}$.

Theorem 2.1. *For $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ partition we have*

$$\Gamma(u, q^{\lambda_1}) \Gamma(tu, q^{\lambda_2}) \cdots \Gamma(t^{\ell-1}u, q^{\lambda_\ell}) \cdot 1 = \nabla \mathcal{T}_{\frac{1}{u}} \widetilde{H}_\lambda[uX] = \nabla \widetilde{H}_\lambda[uX+1]. \tag{2.3}$$

Lemma 2.2. *We have*

$$\Gamma(u,v) = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{1-q}} \nabla \mathcal{P}_{\frac{-tu}{M}} \nabla^{-1}.$$

Proof. The operator Γ can be rewritten as follows

$$\Gamma(u,v) = \left(\Delta_{1/v} \mathcal{P}_{\frac{uv}{M}} \Delta_{1/v}^{-1}\right) \left(\Delta_{1/v} \mathcal{P}_{\frac{-tuv}{M}} \Delta_{1/v}^{-1}\right) = \left(\Delta_{1/v} \mathcal{P}_{\frac{uv}{M}} \Delta_{1/v}^{-1}\right) \left(\Delta_{1/v} \mathcal{P}_{\frac{tuv}{M}} \Delta_{1/v}^{-1}\right)^{-1}.$$

Using the five-term relation (2.1) on each one of the two factors, we obtain

$$\Gamma(u,v) = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{M}} \mathcal{P}_{\frac{-utv}{M}} \nabla \mathcal{P}_{\frac{-tu}{M}} \nabla^{-1} = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv}{1-q}} \nabla \mathcal{P}_{\frac{-tu}{M}} \nabla^{-1}. \quad \square$$

We now prove Theorem 2.1.

Proof of Theorem 2.1. It follows, using Lemma 2.2, that

$$\Gamma(u,v_1)\Gamma(tu,v_2)\cdots\Gamma(t^{k-1}u,v_k) \cdot 1 = \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv_1}{1-q}} \mathcal{P}_{\frac{utv_2}{1-q}} \cdots \mathcal{P}_{\frac{ut^{k-1}v_k}{1-q}} \nabla \mathcal{P}_{\frac{-ut^k}{M}} \nabla^{-1} \cdot 1 \ .$$

Using $\nabla^{-1} \cdot 1 = 1$ and $\nabla \mathcal{P}_{-\frac{z}{M}} \cdot 1 = \mathcal{P}_{\frac{z}{M}} \cdot 1$ (see e.g. [7, Eq. (1.47)] with $k = n$), we get

$$\begin{aligned}
\Gamma(u, v_1)\Gamma(tu, v_2) \cdots \Gamma(t^{k-1}u, v_k) \cdot 1 &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \mathcal{P}_{\frac{uv_1}{1-q}} \mathcal{P}_{\frac{utv_2}{1-q}} \cdots \mathcal{P}_{\frac{ut^{k-1}v_k}{1-q}} \mathcal{P}_{\frac{ut^k}{M}} \cdot 1 \\
&= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[\frac{ut^k X}{M} + \frac{uX}{1-q} \sum_{1 \le i \le k} t^{i-1} v_i \right] \\
&= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[\frac{uX}{M} - \frac{uX}{M}(1-t) \sum_{1 \le i \le k} t^{i-1}(1 - v_i) \right].
\end{aligned}$$

Fix now a partition λ. Applying the previous equation, we get

$$\begin{aligned}
\Gamma(u, q^{\lambda_1})\Gamma(tu, q^{\lambda_2}) \cdots \Gamma(t^{\ell-1}u, q^{\lambda_\ell}) \cdot 1 &= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[\frac{uX}{M} - \frac{uX}{M}(1-t) \sum_{i \ge 1} t^{i-1}(1 - q^{\lambda_i}) \right] \\
&= \nabla \mathcal{P}_{\frac{u}{M}} \nabla^{-1} \operatorname{Exp}\left[-\frac{uXD_\lambda}{M} \right],
\end{aligned}$$

Now applying Equation (2.2) concludes the proof of the theorem. □

2.3 Proof of Theorem 1.2

In this subsection we deduce Theorem 1.2 from Theorem 2.1.

Consider the transformation ϕ on Λ defined by

$$f = \sum_\mu d^f_\mu(q, t) p_\mu[X] \longmapsto \phi(f) := \sum_\mu d^f_\mu(q, 1/t) p_\mu \left[\frac{X}{1 - 1/t} \right],$$

where d^f_μ are the coefficients of f in the power-sum basis. Notice that ϕ is invertible and

$$\phi^{-1}(f) = \sum_\mu d^f_\mu(q, 1/t) p_\mu \left[X(1-t) \right] \text{ for any } f.$$

With this definition, one has $\widetilde{H}^{(q,t)}_\lambda = t^{n(\lambda)} \phi(J^{(q,t)}_\lambda)$. Moreover, $\boldsymbol{\nabla} = \phi^{-1} \cdot \nabla \cdot \phi$ and $\boldsymbol{\Delta}_v = \phi^{-1} \cdot \Delta_v \cdot \phi$. Finally $\phi^{-1} \cdot \mathcal{P}_{\frac{ut^i}{1-q}} \cdot \phi = \mathcal{P}_{\frac{t^{-i}(1-t)u}{1-q}}$, for any $i \ge 0$. We deduce that $\phi^{-1} \cdot \Gamma(t^i u, v) \cdot \phi = \boldsymbol{\Gamma}(t^{-i}u, v)$, for any $i \ge 0$.

On the other hand, one can check that $\phi^{-1} \cdot \mathcal{T}_{\frac{1}{u}} \cdot \phi = \mathcal{T}_{\frac{1}{u(1-t)}}$. Hence, applying ϕ^{-1} on Equation (2.3), we obtain Equation (1.2).

3 A two-parameter generalization of Jack characters

The results of this section will be proved in the long version of the paper.

3.1 Shifted symmetric functions

Definition 3.1. We say that a *polynomial* in k variables $f(u_1,\dots,u_k)$ is *shifted symmetric* if it is symmetric in the variables $u_1, u_2t^{-1},\dots,u_kt^{1-k}$. A *shifted symmetric function* $f(u_1,u_2,\dots)$ is a sequence of polynomials $(f_k)_{k\geq 1}$ of bounded degrees, such that f_k is a shifted symmetric polynomial in k variables for each k, and $f_{k+1}(v_1,\dots,v_k,1) = f_k(v_1,\dots,v_k)$.

If f is a shifted symmetric function, we consider its evaluation on a Young diagram $\lambda = [\lambda_1,\dots,\lambda_k]$ defined by $f(\lambda) := f(q^{\lambda_1}, q^{\lambda_2},\dots,q^{\lambda_k},1,1,\dots)$. It is well known that the *space of shifted symmetric functions* Λ^* can be identified with a subspace of the space of functions on Young diagrams through the map $f \longmapsto (f(\lambda))_{\lambda\in\mathbb{Y}}$.

Theorem 3.2 (Shifted Macdonald polynomials)**.** *[18] Let μ be a partition. There exists a unique function $J^*_\mu(v_1,v_2,\dots)$ such that*

1. *J^*_μ is shifted symmetric of degree $|\mu|$.*
2. *$J^*_\mu(\mu) = (-1)^{|\mu|}q^{n'(\mu)}t^{-2n(\mu)}j^{(q,t)}_\mu$ (normalization property).*
3. *for any partition $\lambda \not\supset \mu$ one has $J^*_\mu(\lambda) = 0$ (vanishing property).*

Moreover, the top homogeneous part of J^*_μ is $J^{(q,t)}_\mu(v_1, t^{-1}v_2, t^{-2}v_3,\dots)$.

Since Macdonald polynomials form a basis of Λ, using a triangularity argument it can be deduced that shifted Macdonald polynomials form a basis of Λ^*. As a consequence we can extend the map $J_\mu \longmapsto J^*_\mu$ to a linear isomorphism

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda^* \\ f & \longmapsto & f^*. \end{array} \tag{3.1}$$

3.2 An explicit isomorphism between the spaces of symmetric and shifted-symmetric functions

The main purpose of this subsection is to give two explicit formulas for the isomorphism (3.1). The first one is Equation (3.2), which gives the image of a function f^* as a shifted symmetric function. The second formula is Equation (3.3), which gives the image as a function on Young diagrams. The proof, that we omit, is based on Equation (1.2) and the Pieri rule.

Theorem 3.3. *For any symmetric function f, the following holds*

$$f^*(v_1,\dots,v_k) = \left\langle f, \mathbf{\Gamma}(1,v_1)\mathbf{\Gamma}(t^{-1},v_2)\cdots\mathbf{\Gamma}(t^{-(k-1)},v_k)\cdot 1\right\rangle_{q,t}. \tag{3.2}$$

Equivalently, for any Young diagram λ,

$$f^*(\lambda) = \left\langle \mathcal{P}_{\frac{1}{1-q}} \boldsymbol{\nabla} \cdot f, t^{-n(\lambda)} J_\lambda \right\rangle_{q,t}. \tag{3.3}$$

Remark 1. The isomorphism given in Equation (3.3) has been implicitly described by Lassalle, see [15, Definition 1]. However, the formula of Equation (3.2) seems to be new.

3.3 Macdonald characters

Definition 3.4. The *Macdonald character* $\boldsymbol{\theta}_\mu^{(q,t)}$ is the function defined by

$$\boldsymbol{\theta}_\mu^{(q,t)}(v_1, v_2, \dots) := (p_\mu)^*(v_1, v_2, \dots) = \left\langle p_\mu, \boldsymbol{\Gamma}(1, v_1)\boldsymbol{\Gamma}(t^{-1}, v_2) \cdot \dots \cdot 1 \right\rangle_{q,t}.$$

Moreover, for any Young diagram λ

$$\boldsymbol{\theta}_\mu^{(q,t)}(\lambda) = \begin{cases} \left\langle p_\mu, \boldsymbol{\nabla} h^\perp_{|\lambda|-|\mu|}\left[\frac{X}{1-t}\right] \cdot t^{-n(\lambda)} J_\lambda^{(q,t)} \right\rangle_{q,t} & \text{if } |\mu| \leq |\lambda| \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

In particular, when $|\mu| = |\lambda|$ the characters $\boldsymbol{\theta}_\mu^{(q,t)}(\lambda)$ are given by the power-sum expansion of $J_\lambda^{(q,t)}$:

$$(-1)^{|\lambda|} q^{n(\lambda')} t^{-2n(\lambda)} J_\lambda^{(q,t)} = \sum_{\mu \vdash |\lambda|} \frac{\boldsymbol{\theta}_\mu^{(q,t)}(\lambda)}{z_\mu(q,t)} p_\mu. \tag{3.5}$$

We give here a characterization of $\boldsymbol{\theta}_\mu^{(q,t)}$, which has been observed by Féray in the case of Jack polynomials, and proved very useful in practice in this case (see [2, Theorem 2.5]).

Theorem 3.5. *Let μ be a partition. Then $\boldsymbol{\theta}_\mu^{(q,t)}$ is the unique shifted symmetric function degree $|\mu|$ whose top homogeneous part is $p_\mu(v_1, t^{-1}v_2, t^{-2}v_3, \dots)$ and such that $\boldsymbol{\theta}_\mu^{(q,t)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.*

4 Macdonald generalization of some Jack conjectures

4.1 A normalization related to Jack polynomials

Jack polynomials can be obtained from the integral form of Macdonald polynomials as follows (see [17, Chapter VI, eq (10.23)])

$$\lim_{t \to 1} \frac{J_\lambda^{(q=1+\alpha(t-1),t)}}{(1-t)^{|\lambda|}} = J_\lambda^{(\alpha)}. \tag{4.1}$$

In the following, we introduce the parameters (α, γ) related to (q, t) by

$$\begin{cases} q = 1 + \gamma\alpha \\ t = 1 + \gamma \end{cases} \longleftrightarrow \begin{cases} \alpha = \frac{1-q}{1-t} \\ \gamma = t - 1 \end{cases} .$$

We consider the following normalization of Macdonald polynomials

$$\mathfrak{J}_\lambda^{(\alpha,\gamma)} := \frac{J_\lambda^{(q,t)}}{(1-t)^{|\lambda|}}.$$

Hence, this normalization is directly related to Jack polynomials by $\mathfrak{J}_\lambda^{(\alpha,\gamma=0)} = J_\lambda^{(\alpha)}$.

Remark 2. Unlike the integral form $J_\lambda^{(q,t)}$, the coefficients of $\mathfrak{J}_\lambda^{(\alpha,\gamma)}$ in the monomial basis are positive in γ and u. This can be seen using the combinatorial interpretation of Macdonald polynomials given in [12, Proposition 8.1].

We observed that the parameterization (α, γ) allows to generalize several conjectures about Jack polynomials to Macdonald polynomials which we now formulate.

4.2 Weak Lassalle's conjecture

We consider the following normalization of Macdonald characters defined in Section 3.3

$$\theta_\mu^{(\alpha,\gamma)}(s_1, s_2, \dots) := \frac{1}{z_\mu(q,t)\gamma^{|\mu|}} \boldsymbol{\theta}_\mu^{(q,t)}(1 + \alpha\gamma s_1, 1 + \alpha\gamma s_2, \dots).$$

For any partition λ, we denote

$$\theta_\mu^{(\alpha,\gamma)}(\lambda) := \theta_\mu^{(\alpha,\gamma)}\left(\frac{q^{\lambda_1}-1}{\alpha\gamma}, \frac{q^{\lambda_2}-1}{\alpha\gamma}, \dots\right) = \frac{\boldsymbol{\theta}_\mu^{(q,t)}(\lambda)}{z_\mu(q,t)\gamma^{|\mu|}}. \tag{4.2}$$

Jack characters, introduced by Lassalle [16], are obtained from the characters $\theta_\mu^{(\alpha,\gamma)}$ by specializing $\gamma = 0$. We formulate the following conjecture[3], tested for $k \leq 3$ and $|\mu| \leq 7$.

Conjecture 1. *Fix $k \geq 1$ and a partition μ. Then, $(-1)^{|\mu|} t^{(k-1)|\mu|} z_\mu(q,t) \theta_\mu^{(\alpha,\gamma)}(s_1, s_2, \dots, s_k)$ is a polynomial in $\gamma, b := \alpha - 1, -\alpha s_1, -\alpha s_2, \dots, -\alpha s_k$ with non-negative integer coefficients.*

Computer tests also suggest that the action of the operator $\boldsymbol{\Gamma}$ on the power-sum basis satisfies a positivity property that would imply the positivity part in Conjecture 1.

[3] This is a generalization of a weak version of Lassalle's conjecture on Jack characters, in which we keep one alphabet $(s_1, s_2, \dots)$ instead of two alphabets associated to the multirectangular coordinates of λ.

4.3 Structure coefficients of $\theta_\mu^{(\alpha,\gamma)}$

It follows from the definition of the characters $\boldsymbol{\theta}_\mu^{(q,t)}$ that they form a basis of the space of shifted symmetric functions Λ^*. Moreover, $\theta_\mu^{(\alpha,\gamma)}(\lambda)$ is obtained from $\boldsymbol{\theta}_\mu^{(q,t)}$ by a normalization by a scalar and a change of variables (see Equation (4.2)) and by consequence their structure coefficients are well defined:

$$\theta_\mu^{(\alpha,\gamma)}\theta_\nu^{(\alpha,\gamma)} = \sum_\pi g_{\mu,\nu}^\pi(\alpha,\gamma)\theta_\pi^{(\alpha,\gamma)}. \tag{4.3}$$

The coefficients $g_{\mu,\nu}^\pi(\alpha,\gamma)$ are a two parameter generalization of structure coefficients of Jack characters $\theta_\mu^{(\alpha)}$ introduced by Dołęga and Féray in [6] (see also [19]).

Let f be the function defined on triplets of non-negative integers by

$$f(n_1,n_2,k) := (N-n)(N+n-k) + n(n-1) - (k-N)(k-N-1),$$

where $N := \max(n_1,n_2)$ and $n = \min(n_1,n_2)$. We make the following conjecture which extends [19, Conjecture 2.2].

Conjecture 2. *Let π,μ,ν be three partitions. Then, the coefficients $(1+\gamma)^{f(|\mu|,|\nu|,|\pi|)}z_\mu z_\nu g_{\mu,\nu}^\pi$ are polynomials in $b := \alpha-1$ and γ with non-negative integer coefficients.*

4.4 Generalized Goulden and Jackson's conjectures

We define the coefficients $c_{\mu,\nu}^\pi$ and $h_{\mu,\nu}^\pi$ for partitions π,μ and ν of the same size by

$$\sum_{\lambda\in\mathbb{Y}} u^{|\lambda|}t^{-2n(\lambda)}q^{n'(\lambda)}\frac{\mathfrak{J}_\lambda^{(\alpha,\gamma)}[X]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Y]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Z]}{j_\lambda^{(q,t)}\gamma^{-2|\lambda|}} = \sum_{n\geq 0}\sum_{\pi,\mu,\nu\vdash n}\frac{u^n c_{\mu,\nu}^\pi(\alpha,\gamma)}{z_\pi(q,t)}p_\pi[X]p_\mu[Y]p_\nu[Z],$$

$$\log\left(\sum_{\lambda\in\mathbb{Y}} u^{|\lambda|}t^{-2n(\lambda)}q^{n'(\lambda)}\frac{\mathfrak{J}_\lambda^{(\alpha,\gamma)}[X]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Y]\mathfrak{J}_\lambda^{(\alpha,\gamma)}[Z]}{j_\lambda^{(q,t)}\gamma^{-2|\lambda|}}\right) = \sum_{n\geq 0}\sum_{\pi,\mu,\nu\vdash n}\frac{u^n h_{\mu,\nu}^\pi(\alpha,u)}{\alpha[n]_q}p_\pi[X]p_\mu[Y]p_\nu[Z],$$

where $[n]_q := 1+q+\cdots+q^{n-1}$. By taking $\gamma=0$, the coefficients $c_{\mu,\nu}^\pi(\alpha,\gamma)$ and $h_{\mu,\nu}^\pi(\alpha,\gamma)$ give the coefficients of the celebrated Matchings-Jack and b-conjectures formulated by Goulden and Jackson in [11]. The coefficients $c_{\mu,\nu}^\pi$ are actually a special case of $g_{\mu,\nu}^\pi$.

Proposition 4.1. *Let π,μ and ν be three partitions of the same size. Then*

$$c_{\mu,\nu}^\pi(\alpha,\gamma) = g_{\mu,\nu}^\pi(\alpha,\gamma).$$

We consider here a Macdonald analog of Goulden and Jackson conjectures.

Conjecture 3 (Macdonald generalization of the Matchings-Jack conjecture)**.** *For any positive integer n and partitions π, μ, ν of n, the quantity $(1+\gamma)^{n(n-1)} z_\mu z_\nu c^\pi_{\mu,\nu}(\alpha, \gamma)$ is a polynomial in b and γ with non-negative integer coefficients.*

Conjecture 4 (Macdonald generalization of the b-conjecture)**.** *For any positive integer n and partitions π, μ, ν of n, the quantity $(1+\gamma)^{n(n-1)} z_\pi z_\mu z_\nu h^\pi_{\mu,\nu}(\alpha, \gamma)$ is a polynomial in b and γ with non-negative integer coefficients.*

Conjecture 3 has been tested for $n \le 8$ and Conjecture 4 for $n \le 9$. Notice that by Proposition 4.1, Conjecture 3 is a special case of Conjecture 2.

Remark 3. Given the integrality result of [1], it is easy to see that Conjecture 3 implies the Matchings-Jack conjecture [11, Conjecture 4.2]. Similarly, Conjecture 4 implies the positivity in the b-conjecture [11, Conjecture 6.2].

4.5 A generalization of Stanley's conjecture

We conclude with a generalization of Stanley's conjecture about the structures coefficients of Jack polynomials. While not directly related to Macdonald characters, this conjecture is also obtained from the new parameterization in α and γ.

Conjecture 5 (Macdonald version of Stanley's conjecture)**.** *Given λ, μ, ν partitions, the quantity $\langle \mathfrak{J}^{(\alpha,\gamma)}_\lambda \mathfrak{J}^{(\alpha,\gamma)}_\mu, \mathfrak{J}^{(\alpha,\gamma)}_\nu \rangle_{q,t}$ is a polynomial in the parameters α and γ with integer non-negative coefficients.*

This conjecture has been tested for $|\nu| \le 9$. Stanley's conjecture [20, Conjecture 8.5] corresponds to the case $\gamma = 0$ of Conjecture 5.

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Antichains in the representation theory of finite Lattices

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Abstract. The interface between the combinatorics of a partially ordered set (poset) and the representation theory of its incidence algebra has been studied for a long time. Antichains naturally arise as encoding certain representations of combinatorial nature. In this paper, we study antichains with extra properties motivated by the search for good bases for the Coxeter matrix of a poset and the hope of categorifying its properties. We then turn to a concrete example where our methods apply nicely and solve a conjecture on the poset of cominuscule roots.

Résumé. Les interactions entre la combinatoire d'un ensemble ordonné et la théorie des représentations de son algèbre d'incidence forment un sujet bien étudié. Les antichaines apparaissent comme décrivant certaines représentations de nature combinatoire. Dans cet article nous étudions des antichaines satisfaisant des hypothèses de rigidité, motivé par la recherche de bonnes bases pour la matrice de Coxeter d'un poset et l'espoir de catégorifier ses propriétés. On traite ensuite un exemple concret où nos méthodes s'appliquent élégamment et nous permettent de résoudre une conjecture sur des ensembles ordonnés de racines cominuscules.

Keywords: antichain, Calabi–Yau category, Coxeter matrix, distributive lattice

1 Introduction

Fractionally Calabi–Yau posets are fascinating objects in part due to a hypothetical relation to product formulas due to Chapoton [3]. In combinatorics, many families of sets $(E_n)_{n\in\mathbb{N}}$ can be counted by product formulas $|E_n| = \Pi_{i=1}^{n}\frac{D-d_i}{d_i}$ where the sum of the numerator and denominator is constant and equal to D. Such families include the Catalan numbers, the number of alternating sign matrices, the West family and the Tamari intervals family. Chapoton's conjectural explanation is that there should exist a partial order on E_n whose derived category is equivalent to a triangulated Calabi–Yau category constructed from the data of D and the d_i coefficients. That category should be geometric in nature, a type of Fukaya category. This explanation provides with predictions about the Calabi–Yau dimension of the incidence algebra of the poset as well as its Coxeter polynomial that can be tested, *e.g.* with a computer.

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Observe that the binomial $\binom{m+n}{m}$ can be written as

$$\frac{m+n}{1}\frac{m+n-1}{2}\cdot\dots\cdot\frac{m+1}{n} \tag{1.1}$$

where $D = m+n+1$. This is probably the most natural example of product formula as discussed above. The poset of order ideals of a product of total orders of length m and n has cardinality $\binom{n+m}{m}$. We write it $J_{m,n}$. Using our results on *boolean antichains* we are able to confirm Chapoton's prediction about the Calabi–Yau dimension of these posets.

Theorem 1. The bounded derived category of $J_{m,n}$ is $\frac{mn}{m+n+1}$-Calabi–Yau.

Moreover we provide a link to a type of Fukaya category.

Theorem 2. The bounded derived category of the algebra of the poset $J_{m,n}$ is equivalent to the partially wrapped Fukaya category of the $m-1^{th}$ symmetric power of the disc with $n+1$ marked points on the boundary, $\mathcal{W}(\mathrm{Sym}^{m-1}\mathbb{D}, \lambda_{n+1}^{(m-1)})$.

We prove this equivalence through an intermediate category, the derived category of the higher Auslander algebra A_{m+1}^{n-1} [5] which appears at the intersection of several hot topics in contemporary representation theory [6], [9]. This algebra is known to be equivalent to the Fukaya category which appears in Theorem 2 [4]. As a corollary to Theorem 1 we give a positive answer to the Chapoton-Yıldırım conjecture [11] on cominuscule root posets of type A and B.

Corollary 1. *The bounded derived category of the order ideals of cominuscule posets of type A, B are fractionally Calabi–Yau.*

2 Representations of partially ordered sets

Let $\Bbbk$ be a field and X a finite poset. Define its incidence algebra $\mathcal{A} = \mathcal{A}_{\Bbbk}(X)$ over $\Bbbk$ to be the $\Bbbk$ vector space with basis pairs (x, y) such that $x \leq y$ with multiplication defined by

$$(x,y)(z,t) = \begin{cases} (x,t) & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in X$ we write $e_x = (x, x)$ the usual primitive idempotent. Then we have $1_{\mathcal{A}} = \sum_{x\in X} e_x$. Throughout this work we consider left modules over $\mathcal{A}$. For every element $x \in X$ the associated simple module is $S_x \cong \Bbbk$ with action $(y,t)\cdot 1_{\Bbbk} = 0$ unless $y = t = x$ in which case $e_x \cdot 1_{\Bbbk} = 1_{\Bbbk}$. Its projective cover $P_x = \mathcal{A}\cdot e_x$ has basis $\{(y,x)|y \leq x\}$. Its injective hull is the injective indecomposable $I_x = (e_x \cdot \mathcal{A})^*$ and has basis $\{(x,y)^*|x \leq y\}$.

Recall that morphisms between the projective indecomposable modules are characterised by

$$\mathrm{Hom}_{\mathcal{A}}(P_x, P_y) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}e_x, \mathcal{A}e_y) \cong \begin{cases} e_x \mathcal{A} e_y \cong \Bbbk & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the canonical inclusion as $\iota_x^y : P_x \hookrightarrow P_y$ whenever $x \leq y$: this inclusion is nothing more than the right multiplication by (x, y). More generally for any left $\mathcal{A}$-module M, we have $\mathrm{Hom}_{\mathcal{A}}(P_x, M) \cong e_x M$. This isomorphism makes the following diagram commute

$$\begin{tikzcd} f \arrow[d, mapsto] & \in & \mathrm{Hom}_{\mathcal{A}}(P_x, M) \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(P_y, M) \arrow[l, "\circ \iota_x^y"'] \arrow[d] & \ni & g \arrow[d, mapsto] \\ f(e_x) & \in & e_x M & e_y M \arrow[l, "(x,y)\cdot"] & \ni & g(e_y) \end{tikzcd} \tag{2.1}$$

The total hom complex $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(C, M)$ where C is a chain complex $C = ((C_n)_n, (\partial_n))$ of $\mathcal{A}$ modules and M is an $\mathcal{A}$-module, is the complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(C_n, M) \xrightarrow{\partial^*_{n+1}} \mathrm{Hom}_{\mathcal{A}}(C_{n+1}, M) \to \ldots$$

Assuming that $C_n = \bigoplus_{x \in S_n} P_x$ with $S_n \subseteq X$ and taking its cohomology gives shifted morphisms in the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ [12, Lemma 3.7.10]:

$$\mathrm{H}^i(\mathrm{Hom}^{\bullet}_{\mathcal{A}}(C, M)) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}(C, M[i]) \tag{2.2}$$

Moreover, using equation (2.1) we have an isomorphism of cochain complexes

$$\begin{tikzcd} \ldots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(\bigoplus_{x \in S_n} P_x, M) \arrow[r, "\partial^*_{n+1}"] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(\bigoplus_{x \in S_{n+1}} P_x, M) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \bigoplus_{x \in S_n} e_x M \arrow[r] & \bigoplus_{x \in S_{n+1}} e_x M \arrow[r] & \ldots \end{tikzcd} \tag{2.3}$$

The boundary maps of the bottom complex are linear combinations of left multiplication by elements (x, y) of the algebra with coefficients inherited from the top complex.

3 Doing homological algebra with antichains

3.1 Antichains

Let $(L, \wedge, \vee)$ be a finite lattice. We write $\hat{1}$ its maximum and $\hat{0}$ its minimum. Let C be an *antichain* in L *i.e.* a subset C of L that consists of pairwise incomparable elements of L. We say an antichain C is below an element α of L if for all $c \in C$, we have $c \leq \alpha$, and when needed we record this information in the notation C_α. Following [7, Proposition 2.1] we associate to an antichain $C = \{c_1, \dots, c_r\}$ the submodule $N_C = \sum_{i=1}^r A \cdot (c_i, \hat{1})$ of the projective $P_{\hat{1}}$ generated by the antichain. It follows directly from the same proposition that there is a one to one correspondence between antichains and submodules of $P_{\hat{1}}$. The *antichain module* associated to C is defined by $M_C := P_{\hat{1}}/N_C$. We will talk of antichain modules below $\alpha \in L$ by restricting to the sublattice $[\hat{0}, \alpha]$ of L. Then α is the greatest element of this lattice and there is a bijection between submodules of P_α and antichains below α. The corresponding modules will be denoted N_C^α and M_C^α. As our main example consider $a \leq b$ in L. The maxima of the set of elements of L which are strictly less than b but not above a form an antichain C and the antichain module below b associated to C has support the interval $[a, b]$. The corresponding antichain module is usually called an *interval module*. In the rest of the paper we identify intervals with their interval modules.

Lemma 1. Intervals are antichain modules.

With the conventions of the previous paragraph, morphisms between interval modules follow a simple rule

$$\mathrm{Hom}_{\mathcal{A}}([a,b],[c,d]) = \begin{cases} \Bbbk & \text{if } a \leq c \leq b \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

By [7, Theorem 2.2], for every antichain C of cardinal r of a lattice L the associated antichain module M_C has a projective resolution $\mathcal{P}_C$ of the form

$$0 \to P_r \to \cdots \to P_0 \to M_C \text{ where } P_0 = P_{\hat{1}} \text{ and } P_l = \bigoplus_{\substack{S \subseteq C \\ |S|=l}} P_{\wedge S} \text{ for } 1 \leq l \leq r.$$

Similarly, we obtain a projective resolution $\mathcal{P}_C^\alpha$ for the antichain module M_C^α below α. The boundary maps are defined by fixing an arbitrary total ordering of elements in C and, in each degree, setting the following maps between the indecomposable summands of the source and target in each degree:

$$\begin{array}{ccc} P_{\wedge S} & \to & P_{\wedge T} \\ (x, \wedge S) & \mapsto & \begin{cases} (-1)^{|i|_S}(x, \wedge T) & \text{if } T \sqcup \{i\} = S, \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{3.2}$$

for each $S = \{i_1, \dots, i_k\}$ and $\big(\wedge S, \wedge T\big) \in P_{\wedge T}$ where $|i|_S = |\{j \in S | j \leq i\}|$.

3.2 Boolean antichains

Note that in degree i of the projective resolution $\mathcal{P}_C^\alpha$ of M_C^α there are $\binom{r}{i}$ indecomposable components in the direct sum. If one forgets the modules, the complex has the shape of the power set of C, however the indices of the modules in each degree are not necessarily in bijection with the lattice $(\mathcal{P}(C), \subseteq, \cup, \cap)$ (see Figures (1) to (4)).

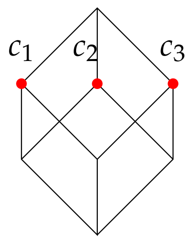


Figure 1: Boolean antichain

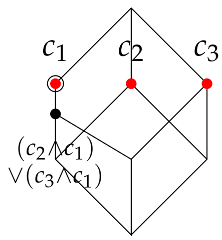


Figure 2: Strong not intersective

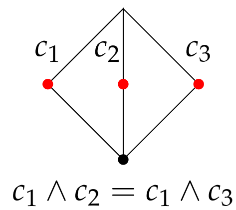


Figure 3: Intersective not strong

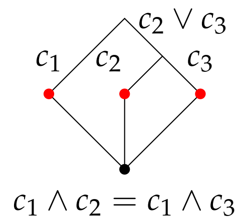


Figure 4: An antichain that is neither

To make this statement precise, let us introduce four definitions regarding an antichain C.

Inclusive antichain. For all subsets S and S' of C, if $\wedge S \leq \wedge S'$ then $S' \subseteq S$.

Intersective antichain. For all subsets S and S' of C, we have $(\wedge S) \vee (\wedge S') = \wedge(S \cap S')$.

Strong antichain. For all S, S' subsets of C of the same cardinal, $\wedge S$ and $\wedge S'$ are incomparable *i.e.* if $\wedge S \leq \wedge S'$ then $S = S'$.

Boolean antichain C is both inclusive and intersective.

If C is below α, we say that C_α satisfies one of these properties if it satisfies it in the lattice $[\hat{0}, \alpha]$. Note that intersectivity depends on the choice of a top element α whereas inclusivity and strength do not. Note also the following lemma.

Lemma 2. An antichain is **inclusive** if and only if it is **strong**.

Proof. The inclusion condition gives the strong antichain condition when the subsets S and S' have the same cardinal. To see the converse, assume that the antichain C is a strong antichain and let S and S' be two subsets of C such that $\wedge S' \leq \wedge S$. Suppose at first that $|S| + n = |S'|$ with $n > 0$. Then there exists $s_1, \ldots, s_n \in S' \setminus S$. Set $S'' = S \sqcup \{s_1, \ldots, s_n\}$. Because the inequalities $\wedge S' \leq \wedge S$ and $\wedge S' \leq \wedge\{s_1, \ldots, s_n\}$ hold, we have

$$\wedge S' \leq (\wedge S) \wedge (\wedge\{s_1, \ldots, s_n\}) = \wedge S''.$$

Because $|S'| = |S''|$, the strong incomparability condition yields $S' = S''$ hence $S \subseteq S'$. Next if $|S| = |S'| + n$, again with $n > 0$, then take $s_1, \ldots, s_n$ in $S \setminus S'$ and set $S'' = S \cup \{s_1, \ldots, s_n\}$. Then we have

$$\wedge S'' = (\wedge S') \wedge (\wedge\{s_1, \ldots, s_n\}) \leq \wedge S.$$

By the strong incomparability condition $S = S''$. Then $S' \subseteq S$ so $\wedge S' \geq \wedge S$ and thus $\wedge S' = \wedge S$. Using the first part of the proof we get $S = S'$. This contradicts the assumption on the integer n. $\square$

Remark 1. If an antichain is strong, then it follows that for each n, the set $\{\wedge S | S \subseteq C \text{ with } |S| = n\}$ is an antichain.

Denote $\langle C\rangle^{\alpha}_{\vee,\wedge}$ the sublattice of $[\hat{0}, \alpha]$ generated by the elements of C and α.

Lemma 3. An antichain is boolean if and only if the map

$$\begin{array}{ccc} (\mathcal{P}(C), \cap, \cup) & \xrightarrow{\phi} & (\langle C\rangle^{\alpha}_{\vee,\wedge}, \wedge, \vee) \\ S & \mapsto & \wedge S \end{array}$$

is a lattice anti-isomorphism.

Proof. Assume that the map ϕ is a lattice anti-isomorphism. Then C_α is intersective because ϕ sends $\cap$ to $\vee$. Now consider $S, S' \subseteq C$ such that $\wedge S \leq \wedge S'$. This is equivalent to the following equality

$$\wedge S = (\wedge S) \wedge (\wedge S').$$

The left hand side is equal to $\wedge(S \cup S')$ and because ϕ is a bijection, $S = S \cup S'$ meaning that $S' \subseteq S$. Thus C has the inclusion property as well. Conversely assume C has both the inclusion and the intersection property below α. The fact that ϕ sends $\cup$ to $\wedge$ is true for any subset of a lattice. The intersection property makes ϕ send $\cap$ to $\vee$. To see that ϕ is injective, note that if $\wedge S = \wedge S'$ then the inclusion property forces $S = S'$. To see that the map is surjective, notice that the image of ϕ, $\mathrm{Im}(\phi) = \{\wedge S | S \subseteq C\}$ is a lattice, using the properties we just exhibited. Moreover, any sublattice of L containing C contains $\mathrm{Im}(\phi)$. It is thus the sublattice of L generated by C, *i.e.*, $\langle C\rangle^{\alpha}_{\vee,\wedge} = \{\wedge S | S \subseteq C\}$ and ϕ is surjective. $\square$

Recall that finite lattices are boolean if and only if they are isomorphic to the powerset of a finite set; this is the real reason for our terminology.

3.3 Morphisms

It is a recurring theme in algebra (and mathematics that use categories) that morphisms are more important than objects. For antichains with the properties we just introduced, we can compute morphisms between their corresponding objects in the derived category more easily as per the following two results.

Proposition 1. *Let C be an antichain of a lattice L and let $I \subseteq L$ be an interval. Suppose the set $E = \{S \subseteq C | \wedge S \in I\}$ is an interval of the lattice $\mathcal{P}(C)$. Then there exists at most one integer p such that $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(M_C, I[p])$ is non zero. When such an integer exists, the hom space is one dimensional.*

Sketch of proof. Because the set E is an interval, we can show that the total hom complex has the shape:

$$0 \leftarrow \Bbbk \leftarrow \cdots \leftarrow \Bbbk^{\binom{|E|}{j}} \leftarrow \cdots \leftarrow \Bbbk \leftarrow 0. \tag{3.3}$$

Where $\Bbbk^{\binom{|E|}{j}}$ is the term in degree j. This is the shape of the simplicial resolution associated to the standard simplex. By reindexing the components of the boundary maps we show that these match the standard simplex resolution as well. In other words this is the Koszul complex $\bigotimes(\Bbbk \xrightarrow{id} \Bbbk)$. By the Künneth's formula [1, Chapter 6.3], it is thus either acyclic or concentrated in one degree when E contains only one element. □

According to the proof, there exists a non trivial morphism if and only if the set E is a singleton *i.e.* there exists a unique $S \subseteq C$ such that $\wedge S \in I$. In this case, the morphism is concentrated in degree $p = |S|$. When the antichain is boolean, we can show that the set E is always an interval which leads to a proof of the following theorem.

Theorem 3. Let C be a *boolean* antichain of a lattice L. Let $I \subseteq L$ be an interval. There exists at most one integer p such that $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(M_C, I[p])$ is non zero. Moreover in this case it is of dimension 1.

Example 1. Consider the lattice in Figure 2 and the strong antichain $C = \{c_1, c_2, c_3\}$ below $\hat{1}$. Its associated module is the simple $S_{\hat{1}}$. Consider moreover the interval $I = [c_1 \wedge c_2, (c_1 \wedge c_2) \vee (c_1 \wedge c_3)]$. The set $E = \{S \subseteq C | \wedge S \in I\}$ is the singleton $\{\{c_1, c_2\}\}$ which is an interval. So Proposition 1 applies and $\dim \mathrm{Hom}_{\mathrm{D^b}}(\mathcal{P}_C, I[2]) = 1$ while for any other shift of the interval it is 0.

Example 2. Consider the lattice in Figure 1 and the antichain $C = \{c_1, c_2\}$ below $\hat{1}$. Its associated antichain module is the interval $[c_3, \hat{1}]$. Consider moreover the interval $I = [\hat{0}, c_1]$. Because the antichain is boolean, Theorem 3 applies. We can check that the set $E = \{S \subseteq C | \wedge S \in I\}$ is the interval $[c_1 \wedge c_2, c_1]$. In that case, $\dim \mathrm{Hom}_{\mathrm{D^b}}(\mathcal{P}_C, I[p]) = 0$ for all integer p as E is not a singleton.

4 Fractionally Calabi–Yau Lattices

For a finite poset with incidence matrix I, the Coxeter matrix is defined as $C = -I \times (I^{-1})^t$. If the poset is a lattice, then it is closely related to its rowmotion bijection [8]. The notion of Calabi–Yau categories was introduced by Kontsevich in the late nineties. A triangulated category $\mathcal{T}$ with a Serre functor $\mathbb{S}$ is said to be fractionally Calabi–Yau if there exists ℓ and d such that $\mathbb{S}^\ell$ is isomorphic as a functor to the suspension functor applied d times. We say that $\mathcal{T}$ is $\frac{d}{\ell}$-Calabi–Yau. When the triangulated category is the derived category of the incidence algebra of a poset, the action of the Serre functor on the Grothendieck group of the category $\mathcal{T}$ coincides with the opposite of the Coxeter matrix. We can safely say that the Serre functor categorifies the opposite of the Coxeter matrix. When the category $\mathcal{T}$ is fractionally Calabi–Yau, its Coxeter matrix has finite order and its characteristic polynomial is a product of cyclotomic polynomials. Both properties can be checked using a computer. In the case of posets with a unique maximal element or a unique minimal element, [10, Theorem 3.1] enables one to prove the fractional Calabi–Yau property by looking at the action of the Serre functor on the projective indecomposable modules only. However, for a given poset the computation itself is still in general very hard, see [2] and [11] for example. Using *strong antichains* as described previously, we are able to provide a relaxation of [10, Theorem 3.1] to help overcome that difficulty.

Theorem 4. Let L be a finite lattice, let m and n be integers and let $(C_\alpha)_{\alpha\in L}$ be a family of antichains in L. For all $\alpha \in L$, consider the following assumptions.

1. The antichain C_α is below α.
2. The module M_C^α is non zero and there is an isomorphism $\mathbb{S}^n M_{C_\alpha}^\alpha \cong M_{C_\alpha}^\alpha[m]$ in $\mathrm{D}^\mathrm{b}(\mathcal{A})$.
3. The antichain C_α is strong.

If there exists a family of antichains $(C_\alpha)_{\alpha\in L}$ satisfying these assumptions then $D^b(\mathcal{A})$ is $\frac{m}{n}$-Calabi–Yau.

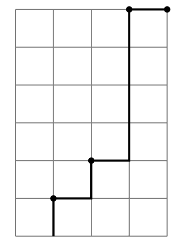
Remark 2. When $C_\alpha = \varnothing$ for all α, we recover [10, Theorem 3.1]. If C_α is the set of all the elements covered by α then $M_C^\alpha = S_\alpha$. Such antichains will often be strong in the examples that we consider. When it is the case the theorem can also be applied to a family of modules which combines some projective indecomposables and some simples. As we will see, the theorem can be applied to less obvious candidates as well.

5 Application: The lattice of order ideals of a grid and its enhancements

In this section we discuss the lattice $J_{m,n}$ in more details and we apply Theorems 3 and 4 on non trivial families of antichain modules in order to prove Theorems 1 and 2 of the introduction.

5.1 Families of antichains

Recall that an order ideal I of a poset P is a subset $I \subseteq P$ which is downward closed, *i.e.* if $x \in I$ and $y \leq x$ then $y \in I$. Order ideals of a poset can be ordered by inclusion and form a distributive lattice when equipped with the union and the intersection of subsets.



We now consider an element I of $J_{m,n}$ *i.e.* an order ideal of the product of total orders $[m] \times [n]$. We can draw the ideal I as a path in an $m \times n$ grid as in the figure on the right. The elements of the order ideal are the points of the grid which lie below the path in the picture. Because I is closed downward, counting the number of points in each column that belong to I, say with increasing first value, gives a monotone sequence which completely determines the ideal. We thus obtain a bijection

$$J_{m,n} \cong \{(a_1, \ldots, a_m) | a_i \in \{0, \ldots, n\} \text{ and } a_1 \leq \ldots, \leq a_n\} \tag{5.1}$$

with non decreasing sequences. If the second set is equipped with term wise comparison this is an isomorphism of posets. We call these non decreasing sequences *partitions*. They can also be written as $(\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r})$ with $\sum_i \mu_i = m$, where μ_i encodes the multiplicity of the value λ_i and $\lambda_i \neq \lambda_j$ if $i \neq j$.

To apply Theorems 3 and 4, we consider several antichains below x for each $x \in J_{m,n}$. These antichains can be encoded as or *enhancements* of x.

Definition 1. An *enhanced partition* is a sequence $(\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ where multiplicities sum to m. We allow $\mu_{r+1} = 0$. If $\mu_{r+1} \neq 0$ we say the partition is strictly enhanced. A partition with $\mu_{r+1} = 0$ is called plain. Call $E_{m,n}$ the set of enhanced partitions. We easily count $\binom{m+n+1}{m}$ enhanced partitions.

For an enhanced partition $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ define the *mutable coefficients* to be $S_\alpha = \{\epsilon, \ldots, r\}$ the indices corresponding to nonzero coefficients. The number ϵ is either 1 or 2. Please remark that this excludes the coefficients beyond the enhancement bar. Similarly, we can define a *left enhanced partition* $(0^{\mu_0} | \lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r})$.

Definition 2. Take an enhanced partition $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$. For any subset J of S_α define a new partition $q_J(\alpha) = ((\lambda_1')^{\mu_1}, \ldots, (\lambda_r')^{\mu_r} | n^{\mu_{r+1}})$ by

$$\lambda_i' = \begin{cases} \lambda_i - 1 & \text{if } i \in J \\ \lambda_i & \text{otherwise.} \end{cases}$$

Consider now the set $C_\alpha = \{q_i(\alpha) | i \in S_\alpha\}$. Because $q_i(\alpha)$ and $q_j(\alpha)$ differ from α at different indices, their associated plain partitions form an antichain. We denote $\mathcal{P}_\alpha$ the resolution associated to it. Here is another family of transformations which leads to antichains.

Definition 3. Let $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ and take $i \in S_\alpha - \{r\}$. We set

$$p_i(\alpha) = \begin{cases} (0^1, \lambda_1^{\mu_1 - 1}, \lambda_2^{\mu_2} \ldots | n^{\mu_{n+1}}) & \text{if } i = 0, \\ (\lambda_1^{\mu_1} \ldots \lambda_i^{\mu_i + 1}, \lambda_{i+1}^{\mu_{i+1} - 1} \ldots | n^{\mu_{n+1}}) & \text{otherwise.} \end{cases} \tag{5.2}$$

For $J = (j_1, \ldots, j_k)$ a sequence of elements $S_\alpha - \{r\}$, set $p_J = p_{j_1} \circ \cdots \circ p_{j_k}$.

For all enhanced partition α consider the set $C^\vee_\alpha = \{p_i(\alpha) | i \in \{0, \dots, r-1\}$ or $i = r$ if $\mu_{r+1>0}\}$. The corresponding plain partitions form an antichain below α. It is easy to prove that C_α and $C^\vee_\alpha$ are boolean antichains below α. We recover these two results of [11]. The second one is an easy categorification of [11, Proposition 4.2].

Proposition 2 ([11, Proposition 2.13])**.** *Let α be a right enhanced partition. Then $\mathcal{P}_\alpha$ is a projective resolution of the interval $[f(\alpha), \alpha]$ where the function f is defined by*

$$f : (\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r} | n^{\mu_{r+1}}) \mapsto (0^{\mu_1 - 1} | \lambda_1^{\mu_2}, \dots, \lambda_r^{\mu_{r+1}}). \tag{5.3}$$

Proposition 3. *Let α be a right enhanced partition. Then $\mathbb{S}^{m+n+1}(\mathcal{P}_\alpha) \cong \mathcal{P}_\alpha[mn]$.*

Combining this with Theorem 4 we obtain a proof of Theorem 1. What remains of this section is dedicated to the description of the full subcategory of the derived category $D^b(J_{m,n})$ whose objects are the $\mathcal{P}_\alpha$ with $\alpha \in E_{m,n}$, and their shifts. We write it $\mathcal{Y}_{m,n}$.

5.2 Elementary morphisms

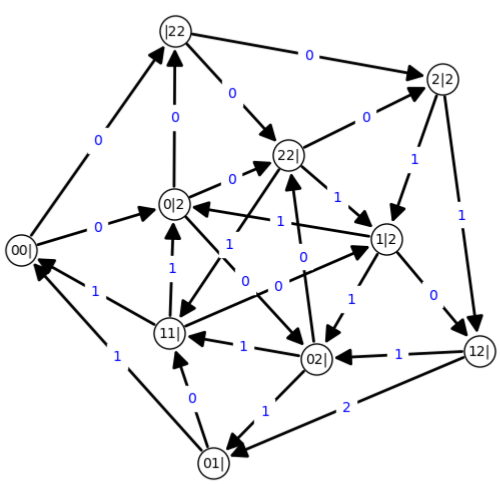


Figure 5: Graph of the category $\mathcal{Y}_{2,2}$

We first describe the morphisms $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta$ using the combinatorics of the partitions introduced in the previous subsection combined with Theorem 3.

Proposition 4. *Let $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta[i]$ be a non zero morphism in $\mathcal{Y}_{m,n}$. Then there exists a unique subset J of S_α such that ϕ factors through $\mathcal{P}_{q_J(\alpha)}[i]$ and $|J| = i$ completing the following diagram.*

$$\begin{tikzcd}
\mathcal{P}_\alpha \arrow[rr, "\phi"] \arrow[dr, "u"'] & & \mathcal{P}_\beta[|J|] \\
& \mathcal{P}_{q_J(\alpha)}[|J|] \arrow[ur, "\mu"'] &
\end{tikzcd}$$

If we order $J = \{j_1, \dots j_k\}$ such that $j_t < j_{t+1}$, then the extension $u : \mathcal{P}_\alpha \to \mathcal{P}_{q_J(\alpha)}$ decomposes as $\mathcal{P}_\alpha \to \mathcal{P}_{q_{\{j_1\}}(\alpha)}[1] \to \dots \to \mathcal{P}_{q_J(\alpha)}[|J|]$ up to signs. Similarly there exists a sequence $d_0, \dots, d_s$ such that for all $0 \leq j < s$, $0 \leq d_j < m_{j+1}$, for $j = s$, $-m_s < d_s \leq m_{s+1}$, $\alpha = p_s^{d_s} \circ \dots \circ p_0^{d_0}(\beta)$ and the degree 0 morphism $\mu : \mathcal{P}_{q_J(\alpha)} \to \mathcal{P}_\beta$ factors through each of the objects associated to the intermediate partitions.

Theorem 2 follows from this proposition. More precisely, we construct a tilting complex with the set of plain partitions and show that its endomorphism algebra is the higher Auslander algebra A^{m-1}_{n+1}.

5.3 Through the lens of configurations

Using a clever bijection of Yıldırım we can give a more satisfying description of the category $\mathcal{Y}_{m,n}$. Let $\mathcal{Z} = \{-m, \ldots, -1, 0, 1, \ldots, n\}$ be a set of representatives of $\mathbb{Z}/(m+n)\mathbb{Z}$. A *configuration* $C = \{c_1 < \cdots < c_m\}$ is a strictly increasing sequence of m elements in $\mathcal{Z}$. We write $C_{m,n}$ the set of configurations of length m on $\mathcal{Z}$. It is easy to see that $|C_{m,n}| = \binom{m+n+1}{m}$. Given a partition α we can construct a configuration containing α's coefficients in its *nonnegative side* and encoding the *multiplicities* of α in its *negative side*. Write x_i to record the index of the last occurrence of the i^{th} coefficient. It will be called the *ending index*. The negative side is thought of as the indices of the elements of the sequence α but with a minus sign. Remove the opposite of the ending index of each coefficient. Call the resulting configuration the *right configuration* associated to α, write it R_α. We denote ϕ the map sending α to R_α.

Example 3. Take $n = 7$, $m = 5$ and consider the partitions $a = (0, 2, 3, 7|7)$. For this partition, $r = 4$ and coefficients end at indices 1, 2, 3 and 4. The associated right configuration is $\{-5 < 0 < 2 < 3 < 7\}$, containing the values $0, 2, 3, 7$ and omitting the opposite of the ending coefficients $-1, -2, -3$ and -4. It is represented in an abacus as follows:

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
•					•		•	•				•

Note that an enhanced partition is plain if and only if the column $-m$ of its abacus is empty.

Proposition 5 ([11, Proposition 3.3])**.** *The map ϕ is a bijection between $E_{m,n}$ and $C_{m,n}$.*

Indeed, partitions are entirely determined by their coefficients and multiplicities which can be recovered from the positive elements of a configurations and its negative gaps.

Definition 4. Consider two sequences $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of length k in $[\![0, N]\!]$. Define a sequence on $\mathbb{Z}$ by setting $y_l = y_r + q \cdot N$ where $l = g \times k + r$ is the euclidian division. We say that they *interpolate circularly* if for all integers h, f and l such that $h \equiv l \equiv f - 1 \mod [k]$ we have $x_h \leq y_l < x_f$.

Example 4. The configurations $\{-5 < 0 < 2 < 3 < 7\}$ and $\{-5 < 0 < 2 < 3 < 6\}$ interpolate circularly.

Definition 5. Let n and m be integers. Define the category $\mathcal{I}_{m,n}$ as follows :

- set $\mathrm{Ob}(\mathcal{I}_{m,n}) = \{$increasing sequences of length m in $[\![0, m+n]\!]\}$;
- given two increasing sequences a and b in $\mathrm{Ob}(\mathcal{I}_{m,n})$, set

$$\mathcal{I}_{m,n}(a, b) = \begin{cases} \Bbbk & \text{if } a \text{ and } b \text{ interpolate circularly,} \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 1. *The categories $\mathcal{Y}_{m,n}$ and $\mathcal{I}_{m,n}$ are equivalent.*

Theorem 2 also follows from this conjecture, however its proof only involves part of $\mathcal{Y}_{m,n}$ and A_{n+1}^{m-1} is Morita equivalent to the corresponding subcategory of $\mathcal{I}_{m,n}$.

Update This form of Conjecture 1 is not exactly right. A correct statement is proven in the thesis of this author. It requires a subtle choice of numbering for the elements of the configurations.

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On the scaling of random Tamari intervals and Schnyder woods of random triangulations (with an asymptotic D-finite trick)

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Abstract. We consider a Tamari interval of size n (i.e., a pair of Tamari-comparable Dyck paths) chosen uniformly at random. We show that the typical height of points on the paths scales as $n^{3/4}$, with an explicit limit law. By the Bernardi-Bonichon bijection, this also applies to Schnyder trees of random plane triangulations. The exact solution of the model is based on polynomial equations with one and two catalytic variables. To deduce convergence in law, we use a simple analytic method based on D-finiteness, which is essentially automatic.

1 Introduction and main results

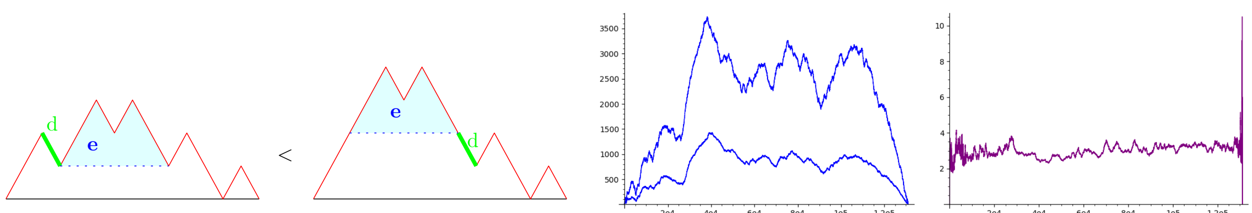

Figure 1: Left: The covering relation defining the Tamari lattice. Right: a uniform random Tamari interval (P_n, Q_n) of size $n = 65536$ (blue) generated with a Python code generously provided by Wenjie Fang, and the plot of Q_n/P_n (purple).

For $n \geq 0$, a *Dyck path of size n* is a lattice path made of n up-steps and n-down steps, starting (and ending) at height 0, and whose height stays always nonnegative. The set $\mathcal{D}_n$ of Dyck paths of size n is endowed with the *Tamari partial order*, whose covering relation is described as follows. Let $\mathfrak{p}$ be a Dyck path, and let d be a down-step in $\mathfrak{p}$, followed by an up-step. Let $\mathfrak{e}$ be the shortest excursion following d in $\mathfrak{p}$ (an excursion is a path staying higher than its starting point except for its last point). Then the Dyck path

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$\mathfrak{q}$ obtained from $\mathfrak{p}$ by exchanging d and $\mathfrak{e}$ is declared *larger* than $\mathfrak{p}$ for the Tamari order. The reflexive transitive closure of this relation defines the Tamari lattice. See Figure 1.

The Tamari lattice plays an important role in many facets of algebraic combinatorics and discrete mathematics, in relation with polytopes (the associahedron), flip graphs, hyperbolic geometry, or mixing times of Markov chains.

This paper is motivated by another famous connection, between Tamari intervals and planar maps. An *interval* of size n in the Tamari lattice is a pair $(\mathfrak{p}, \mathfrak{q}) \in (\mathcal{D}_n)^2$ such that $\mathfrak{p} \leq \mathfrak{q}$ (for the Tamari partial order). We let $\mathcal{I}_n$ be the set of Tamari intervals of size n. In a famous paper, Chapoton proved that the number of elements of $\mathcal{I}_n$ was given by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$, which is also the number of rooted planar triangulations of size n [13]. An elegant, and deep, direct bijective proof has been given by Bernardi and Bonichon in [1]. Since Chapoton's discovery, the analogy between Tamari intervals and planar maps has been much developped, from the existence of refined product formulas [2] strongly resembling the ones appearing in enumerative geometry, to numerous coincidences between the enumeration of special families of intervals and planar maps (e.g., [8]). These phenomena are still very partially understood and are the subject of active research.

Although large random planar maps have been intensely studied in the last decades (see e.g. [6, 10]), it seems that the behaviour of random Tamari intervals has not been studied, and we are not sure to know why. It is however natural to ask this question:

What does a large uniformly random Tamari interval look like?

In this paper we give a first answer to this question. If $P \in \mathcal{D}_n$ and $i \in [0..2n]$, we write $P(i)$ for the height of the point of P lying at abscissa i. We show:

Theorem 1.1 (Main result)**.** *Let (P, Q) be a Tamari interval of size n, chosen uniformly at random in $\mathcal{I}_n$. Let $I \in [0..2n]$ be an integer chosen uniformly at random. Then we have the convergence in law*

$$\frac{Q_n(I)}{n^{3/4}} \longrightarrow Z = (XY)^{1/4} \tag{1.1}$$

when n goes to infinity, where $X \sim \beta(\frac{1}{3}, \frac{1}{6})$ and $Y \sim \Gamma(\frac{2}{3}, \frac{1}{2})$ are independent random variables. In fact, we have the convergence of all moments: for integer $k \geq 0$,

$$\mathbf{E}\left[\left(\frac{Q_n(I)}{n^{3/4}}\right)^k\right] \longrightarrow \frac{\sqrt{3} \cdot 2^{-\frac{k}{4}-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4}k + \frac{1}{3})\Gamma(\frac{1}{4}k + \frac{2}{3})}{\Gamma(\frac{1}{4}k + \frac{1}{2})}. \tag{1.2}$$

For the lower path we have similarly, with again the convergence of all moments,

$$\frac{P_n(I)}{n^{3/4}} \longrightarrow \frac{Z}{3}. \tag{1.3}$$

We recall that the random variables $\beta(a, b)$ and $\Gamma(\ell, \theta)$ have respective densities $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ on $(0, 1)$ and $\frac{x^{k-1}e^{-x/\theta}}{\Gamma(k)\theta^k}$ on $\mathbb{R}_+$. Their respective k-th moments are

$\frac{\Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}$, and $\frac{\theta^k\Gamma(l+k)}{\Gamma(\ell)}$, so it is direct to check that (1.2) with k substituted by $4k$ is indeed equal to the k-th moment of XY. Interestingly, the random variable Z^4 already appears in a (seemingly unrelated) physics context, see [11] or OEIS:A064352.

In view of Theorem 1.1 and simulations (Figure 1) it is natural to suspect that $P_n(I)$ is close to $\frac{1}{3}Q_n(I)$ with high probability. Unfortunately our methods based on functional equations make it hard to track the joint law of $(P_n(I), Q_n(I))$. However, they can handle the joint law of $(\tilde{P}_n(J), \tilde{Q}_n(J))$ where J is uniform on $[1, n]$ and where for a path P we write $\tilde{P}(j)$ for the height of the j-th up-step of P. In the long version of this paper we will prove with similar methods[1] that indeed, in probability,

$$\tilde{P}_n(J) = \frac{1}{3}\tilde{Q}_n(J) + o(\tilde{Q}_n(J)). \tag{1.4}$$

To prevent confusion, we mention that the *individual* convergence of $\tilde{P}_n(J)/n^{3/4}$ and $\tilde{Q}_n(J)/n^{3/4}$ to Z and $Z/3$ follow from Theorem 1.1, using a coupling between I and J.

As we said, Bernardi and Bonichon [1] provided an explicit bijection between intervals in $\mathcal{I}_n$ and rooted plane triangulations of size n. Such a triangulation can always be equipped with a *canonical Schnyder wood*, which is a partition of its internal edges into three trees (say red, blue, green) with certain conditions. See Figure 2. It is then not too hard to deduce the following from (1.3):

Corollary 1.2. *Let T_n be a rooted plane triangulation of size n, chosen uniformly at random, and let $(T_n^{(1)}, T_n^{(2)}, T_n^{(3)})$ be its canonical Schnyder wood, that is to say, the one associated to its minimal orientation in the sense of [1]. Let V be a uniform random internal vertex of T_n and let $H_n^{(i)}$ be the height of the vertex V in the tree $T_n^{(i)}$. Then, for any $i \in [3]$ we have*

$$\frac{H_n^{(i)}}{n^{3/4}} \longrightarrow \frac{1}{3}Z.$$

The proof of each half of Theorem 1.1 (namely (1.1) and (1.3)) has two parts: the first one consists in solving "exactly" the model, by obtaining an algebraic equation for the generating function $f(t,s)$ of intervals having a marked abscissa, with control on the size n, and on the lower or upper height. In each case this requires to solve an equation with catalytic variables. The second part is to deduce the asymptotic of moments from there, which is a problem of analytic combinatorics in 2 variables, for which we need to find good tools. We describe below a simple method that will do the trick.

[1] One can write a functional equation for the generating function of intervals (P, Q) of size n with a marked $j \in [n]$ with control on the parameter $\tilde{Q}(j) - 3\tilde{P}(j)$, very similarly to Equation (3.1). The resolution of this equation is very similar to what is done in Section 3.2, with the difference that the equation with catalytic variable to solve in the second step is now quadratic – and can be solved with the quadratic method. At the time of writing, we have not performed the full asymptotics of moments, but from the solution it is immediate to show that $\mathbf{E}[(\tilde{Q}_n(J) - 3\tilde{P}_n(J))^2] = O(n)$, which is enough to conclude (1.4) by the Chebyshev inequality. For the impatient reader, we have already included all details in [5].

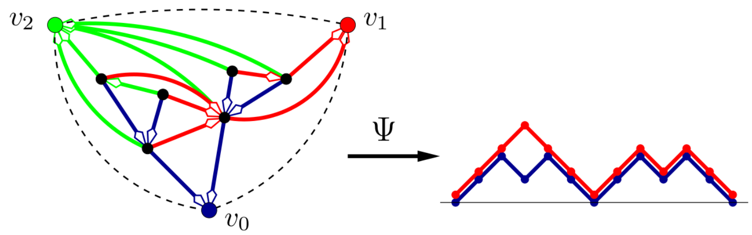

Figure 2: A rooted planar triangulation equipped with its minimal Schnyder-wood, and its image (a Tamari interval) by the Bernardi-Bonichon bijection. The lower path is nothing but the contour function of the blue tree. Figure taken from [1].

1.1 A method to prove moment convergence from algebraic equations

Assume that we have a combinatorial class $\mathcal{A}$ equipped with a size function $|\cdot|$ and an integer statistic μ, and consider the random variable $X_n = \mu(A_n)$ where A_n is an object of size n in $\mathcal{A}$ chosen uniformly at random. Consider the generating function

$$f(t,s) = \sum_{n\geq 0}\sum_{p\in\mathbb{Z}} a_{n,p} t^n s^p = \sum_{n\geq 0}\sum_{p\in\mathbb{Z}} a_n \mathbf{P}[X_n = p] t^n s^p, \tag{1.5}$$

with $a_{n,p} = |\{\alpha \in \mathcal{A}, |\alpha| = n, \mu(\alpha) = p\}|$, and $a_n = |\{\alpha \in \mathcal{A}, |\alpha| = n\}|$, and assume that the generating function $f(t,s)$ is algebraic. For $k \geq 0$ we consider the generating function of factorial moments

$$f^{(k)} \equiv f^{(k)}(t) := \left(\left(\frac{d}{ds}\right)^k f(t,s)\right)\Bigg|_{s=1} = \sum_{n\geq 0} a_n \mathbf{E}[(X_n)_{(k)}] t^n,$$

with $(X_n)_{(k)} := X_n(X_n-1)\dots(X_n-k+1)$. To study the convergence of the random variable X_n, a standard way called the method of moments consists in studying the asymptotics for fixed k of its k-th moment $\mathbf{E}[(X_n)^k]$ (or factorial moment $\mathbf{E}[(X_n)_{(k)}]$). In our setting, this is equivalent to studying the asymptotics of the coefficient $[t^n]f^{(k)}(t)$, for fixed k. Now, since all the functions $f^{(k)}(t)$ are algebraic, they are amenable to singularity analysis in the sense of [9]. Therefore, to study the asymptotics of their coefficients it is enough to determine the nature of the dominant singularity(ies) of $f^{(k)}$ for each $k \geq 0$.

Here comes the main trick: since the function $f(t,s)$ is algebraic, it is also D-finite, i.e. the coefficients of its expansion in any variable satisfy a linear equation with polynomial coefficients (see e.g. [7]). We will apply this to the coefficients of the expansion[2] in the variable r such that $s = r+1$. These coefficients are, up to a factorial factor, the functions

[2]The function $f(t,1+r)$ is algebraic, therefore it is D-finite in the variable r. No notion of convergence is required to say this. Of course, one has to be careful about which branch of this function one considers when performing actual calculations.

$f^{(k)}$. It follows that one can compute the $f^{(k)}$ by induction, with a recurrence of the form

$$f^{(k)}(t) = \sum_{d=1}^{L} h_d(t,k) f^{(k-d)}(t) \quad , \quad k \geq L, \tag{1.6}$$

for some $L > 0$, where for each $d \in [L]$, $h_d(t,k)$ is a rational function in k whose coefficients are algebraic functions of t (we could assume that the h_k are rational in t, but for applications this weaker asumption is convenient: it will enable us to work under some algebraic change of variables). This leads us to:

Method 1.3 (D-finite trick for moment pumping)**.** Given a bivariate algebraic function f, obtain a linear equation for its derivatives $f^{(k)}$ of the form (1.6) using standard computer algebra tools. Then, use it to determine the asymptotic of $f^{(k)}$ near $t = \rho$ by induction on k. Deduce the asymptotics of $[t^n]f^{(k)}$ using the transfer theorem.

The idea of this method is quite general, but of course one has to be careful to carry the analytical details in the induction. We give below a simple framework of application whose proof is essentially immediate. For a function g and $\alpha, c \in \mathbb{R}$, $\rho > 0$ we write $g(t) \hat{\sim} c(1 - t/\rho)^\alpha$ if $g(t)$, as an analytic function, has no singularity on the closed circle of radius ρ except maybe at $t = \rho$, and if $g(t)$ has a Puiseux expansion of the form $g(t) = P(t) + c(1 - t/\rho)^\alpha + o((1 - t/\rho)^\alpha)$ near $t = \rho$, where P is a polynomial.

Theorem 1.4 (D-finite trick for moment pumping, an instance)**.** *Let $f(t,s)$ be a generating function of the form* (1.5)*, and assume that $f(t,s)$ is an algebraic function. Then f is D-finite in the variable $s-1$, and it satisfies an equation of the form* (1.6)*, with $L \geq 1$ and where for $d \in [L]$, $h_d(t,k)$ is a rational function of k whose coefficients are algebraic functions of t. Assume that:*
(i) *There is $\beta > 0$ such that for $d \in [L]$, $h_d(t,k)(1 - t/\rho)^{\beta d} \to a_d(k)$ for $t \to \rho$, with possibly $a_d(k) = 0$. Moreover, $h_d(k)$ has no singularity other than ρ on the circle of radius ρ.*
(ii) *"Initial conditions": there is $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and numbers c_ℓ, with $c_0 > 0$, such that $f^{(\ell)} \hat{\sim} c_\ell (1 - t/\rho)^{\alpha - \beta\ell}$ for all $0 \leq \ell \leq \ell_0$, with $\ell_0 := L + \max(\lfloor \alpha/\beta \rfloor, -1)$. Moreover, $c_\ell = 0$ if $\alpha - \beta\ell \in \mathbb{N}$.*

Then one has $f^{(k)} \hat{\sim} c_k (1 - t/\rho)^{\alpha - \beta k}$ for any $k \geq 0$, where c_k is given by the recurrence

$$c_k = \sum_{d=1}^{L} a_d(k) c_{k-d} \quad , \quad k > \ell_0, \tag{1.7}$$

where the values of c_k for $k \leq \ell_0$ are given by the initial conditions. Moreover, for each $k \geq 0$,

$$\frac{\mathbf{E}[(X_n)^k]}{n^{\beta k}} \to \frac{c_k \Gamma(-\alpha)}{c_0 \Gamma(\beta k - \alpha)}. \tag{1.8}$$

The proof is ommitted in this extended abstract. It consists in showing that $f^{(k)} \hat{\sim} c_k (1 - t/\rho)^{\alpha - \beta k}$ for any $k \geq 0$, which in fact follows from a direction induction, and applying the standard theorems for algebraic functions [9]. The reason why the recurrence starts only at ℓ_0 is because we want $\alpha - \beta\ell$ to be negative for all ℓ such that c_ℓ appears in (1.7), so that the symbol $\hat{\sim}$ is in fact a true equivalent, in order for the induction to work.

Remark 1.5. We do not try to provide minimal hypotheses in this theorem: the only requirement for the method to work is that dominant singularities of $f^{(k)}$ are "not too hard" to track by induction from (1.6). Also, it is conceivable to use this technique in more than bivariate examples, provided the corresponding multivariate generating function is algebraic.

We insist that Theorem 1.4 if applicable, is essentially automatic. Indeed, computer algebra softwares (e.g. the Maple package GFUN [12]) are able to provide a recurrence of the form (1.6), and to check the initial conditions, automatically from an algebraic equation for f. Apart from this, the major interest of this method is that it allows a wide variety of limit laws (including non Gaussian) as our main applications show.

To conclude this discussion, let us recall that using D-finiteness to compute coefficients of algebraic functions in the univariate case is a well-known trick going back at least to Comtet [7]. We are only recycling this idea in the context of bivariate asymptotics.

Example. As a simple application, we invite our reader to rediscover the well known limit law for the height H_n of a uniform random point on a uniform random Dyck path of size n. Using standard path decompositions one easily obtains a quadratic equation for the corresponding generating function $f(t,s)$, and from there it is immediate, with Maple, to check that Theorem 1.4 applies (with $-\alpha = \beta = \frac{1}{2}$). The recurrence (1.7) becomes $c_k = \frac{1}{4}k(k-1)c_{k-2}$, and one thus directly shows that $H_n/\sqrt{n}$ converges to a Rayleigh law, of k-th moment $\Gamma(\frac{k}{2}+1)$. All calculations and checks are available in [5]!

Plan of the paper. In Section 2 we solve the exact counting problem which underlies the first half of Theorem 1.1 (Equation (1.1) by studying the classical equation with one catalytic variable for Tamari intervals, which we enrich by an extra variable marking the height. See Theorem 2.3. In Section 3 we solve the exact counting problem which underlies the second half (Equation (1.3)) see Theorem 3.2. Our proof is more technical, as we only manage to write an equation with *two* catalytic variables for this problem (enriched, once again, by an extra variable for the height). In Section 4, we sketch the proof of Theorem 1.1, which given these results directly follows from Theorem 1.4 and computer algebra calculations.

All the calculations supporting our results (including results whose proofs are not fully presented in this abstract) are available in the accompanying Maple worksheet [5].

2 Upper path: exact solution

2.1 The classical equation and its solution, after [4, 3]

Following [3], we call *contact* of a path a vertex of that path lying on the x-axis. We now present a recursive decomposition of Tamari intervals based on contacts of the lower path, following [4, 3].

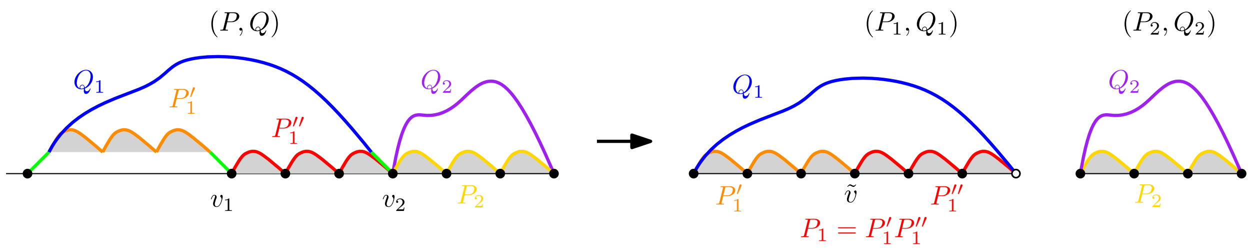

Figure 3: The classical decomposition of Tamari intervals. To the left, an interval of size $n+1$, where v_1, v_2 are the first contacts of the lower and upper path, respectively. The decomposition gives rises, to the right, to two Tamari intervals of total size n, the first of which has a marked contact, called here $\tilde{v}$. This construction is bijective.

Let $(P,Q) \in \mathcal{I}_{n+1}$ be a Tamari interval, for $n \geq 0$, and let v_1 and v_2 be respectively the leftmost contact of P and Q different from the origin. By deleting the first up step of P and Q, and the downstep of P and Q preceding respectively v_1 and v_2, one obtains two paths that can be naturally seen as the concatenation of two pairs of paths (P_1, Q_1) and (P_2, Q_2) as on Figure 3. It is proved in [4, 3] that (P_1, Q_1) and (P_2, Q_2) are two Tamari intervals. Moreover, this operation is a *bijection* (!) between intervals of size $n+1$ and pairs of intervals of total size n, such that the first interval of the pair has a distinguished contact on its lower path – inherited from the vertex v_1. To translate this recursive construction into an equation for enumeration, one needs to introduce a two-parameter generating function:

$$F(x) \equiv F(t,x) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} x^{\mathrm{contact}(P)}$$

where contact(P) is the number of contacts of the lower path P.

Note that, if a path $\tilde{P}_1$ has k contacts, there are k possible ways to mark a contact in $\tilde{P}_1$, thus decomposing it as $\tilde{P}_1 = \tilde{P}_1'\tilde{P}_1''$. When going through these k choices, the number of *non-final* contacts of the path $\tilde{P}_0 := u\tilde{P}_1'd\tilde{P}_1''$ goes through the values $1, 2, \ldots, k$, see Figure 4(up). At the level of generating functions, the corresponding operator is

$$\Delta_x : x^k \longmapsto x^k + \cdots + x^1 = x\frac{x^k - 1}{x-1} \;\; , \;\; \Delta_x : f(x) \longmapsto x\frac{f(x)-1}{x-1}. \tag{2.1}$$

This observation being made, the recursive decomposition immediately translates into the following functional equation [4, 3]

$$F(x) = x + xtF(x)\frac{F(x)-F(1)}{x-1}. \tag{2.2}$$

Indeed, the first term accounts for the empty path (of size $n=0$), and in the second term the factor $F(x)$ is the contribution of the interval (P_2, Q_2), and the factor $x\frac{F(x)-F(1)}{x-1} = \Delta_x F(x)$ is the contribution of the interval (P_1, Q_1).

The solution can be written especially nicely with a rational parametrization.

Proposition 2.1 ([3, Thm. 10])**.** *The functions $F(x)$ and $F(1)$ are given by*

$$F(x) = \frac{1+u}{(1+zu)(1-z)^3}(1-2z-z^2u) \quad , \quad F(1) = \frac{1-2z}{(1-z)^3}, \tag{2.3}$$

with

$$t = z(1-z)^3 \ \ , \ \ x = \frac{1+u}{(1+zu)^2}. \tag{2.4}$$

2.2 The enriched equation and its solution

We now study intervals in which the upper path carries a marked point. We let

$$H(x,s) \equiv H(t,x,s) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} x^{\mathrm{contact}(P)} \sum_{i=0}^{2n} s^{Q(i)}.$$

It is clear that the height of the marked point can be tracked in the decomposition above. This leads to the functional equation:

Proposition 2.2. *The generating function $H(x,s)$ is solution of the equation:*

$$H(x,s) = F(x) + sxt\frac{H(x,s)-H(1,s)}{x-1}F(x) + xt\frac{F(x)-F(1)}{x-1}H(x,s). \tag{2.5}$$

Proof. This follows directly from the combinatorial decomposition of Figure 3, applied to intervals with a mark abscissa. The first term accounts for the case where the marked abscissa (and height) is zero. The second term accounts for the case where the marked abscissa appears before vertex v_2. Through the decomposition the corresponding vertex of the upper path becomes a marked vertex of the path Q_1, and its height is shifted by 1, hence a contribution of $s\Delta_x H(s,x)$ for the interval (P_1,Q_1), while the interval (P_2,Q_2) has contribution $F(x)$ as before. The third term accounts for the case where the marked abscissa appears at v_2 of after, in which case there is no shift in the height and the second interval has contribution $H(s,x)$, while the first has contribution $\Delta_x F(x)$ as before. □

Equation (2.5) is easily solved via the kernel method (see e.g. [9, p. 508]). First, let us work under the change of variables $(t,x) \leftrightarrow (u,z)$ of (2.4), so we can consider $F(1)$ and $F(x)$ known. We can then write (2.5) in "kernel" form $K(x,s)H(x,s) = R.H.S.$ where the kernel $K(x,s)$ is an explicit rational function of z and u. One easily checks that there is a unique series $U = U(z)$ cancelling the kernel, given by

$$s = U(1-z)^3/[z(1+U)^2(1-Uz^2-2z)]. \tag{2.6}$$

Solving the R.H.S. for $H(1)$ we obtain (see [5] for full calculations and checks)

Theorem 2.3. *The series $H(1) \equiv H(t,s,1)$ of Tamari intervals with a marked abscissa, where t marks the size, and s the upper height at this abscissa, has the following rational parametrisation:*

$$H(1) = (1-2z-Uz^2)^2(1+U)/(1-z)^6 \tag{2.7}$$

with the change of variables $(t,s) \leftrightarrow (z,U)$ given by $t = z(1-z)^3$ and by (2.6).

3 Lower path: exact solution

We will now apply the same decomposition as in the previous sections, but keep track of the height of points on the lower path P. In order to do this, we will have to treat differently the contacts of P which appear before or after the marked abscissa, which will force us to work with *two* catalytic variables. We write $\mathrm{contact}_{<i}(P), \mathrm{contact}_{\geq i}(P)$ for the number of contacts of P strictly before, or weakly after, abscissa i, respectively. We introduce the generating function

$$G(x,y) \equiv G(t,x,y,w) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} \sum_{i=0}^{2n} w^{P(i)} x^{\mathrm{contact}_{<i}(P)} y^{\mathrm{contact}_{\geq i}(P)}.$$

3.1 The enriched equation

Proposition 3.1. *The generating function $G(x,y) \equiv G(t,x,y,w)$ of Tamari intervals with a marked abscissa where w marks the lower height is solution of the equation:*

$$\begin{aligned} G(x,y) = F(y) &+ txw\frac{G(1,y)-G(1,1)}{y-1}F(y) + tx\frac{F(y)-yF(1)}{y-1}F(y) \\ &+ t\frac{x^2}{y}\frac{G(x,y)-\frac{y}{x}F(x)-G(1,y)+yF(1)}{x-1}F(y) + tx\frac{F(x)-F(1)}{x-1}G(x,y). \end{aligned} \tag{3.1}$$

Idea of the proof. Given an interval (P,Q) of size $n+1$ with a marked abscissa, we apply again the decomposition of Figure 3. We let i_1, i_2 be the abscissa of the first nonzero contact of P and Q respectively, and i the marked abscissa. We will distinguish fives cases depending on the fact that i belongs to $\{0\}, [1, i_1-1], \{i_1\}, [i_1+1, i_2-1], [i_2, 2n]$. These correspond (from left to right) to the five terms in (3.1).

In this extended abstract, we will only address the second case, which illustrates in a simple way why we need two catalytic variables. If $i \in [1, i_1-1]$, in the decomposition of Figure 3, the corresponding vertex of P becomes a marked vertex of P_1 with a shift of 1 in the height, hence a contribution of w. Moreover, configurations in this case are obtained by applying the construction of Figure 4(top) but restricting it to contacts appearing after the marked abscissa, see Figure 4(center). Therefore, an interval $(\tilde{P}_1, \tilde{Q}_1)$ having a marked abscissa, with respectively k and ℓ contacts strictly before, and weakly

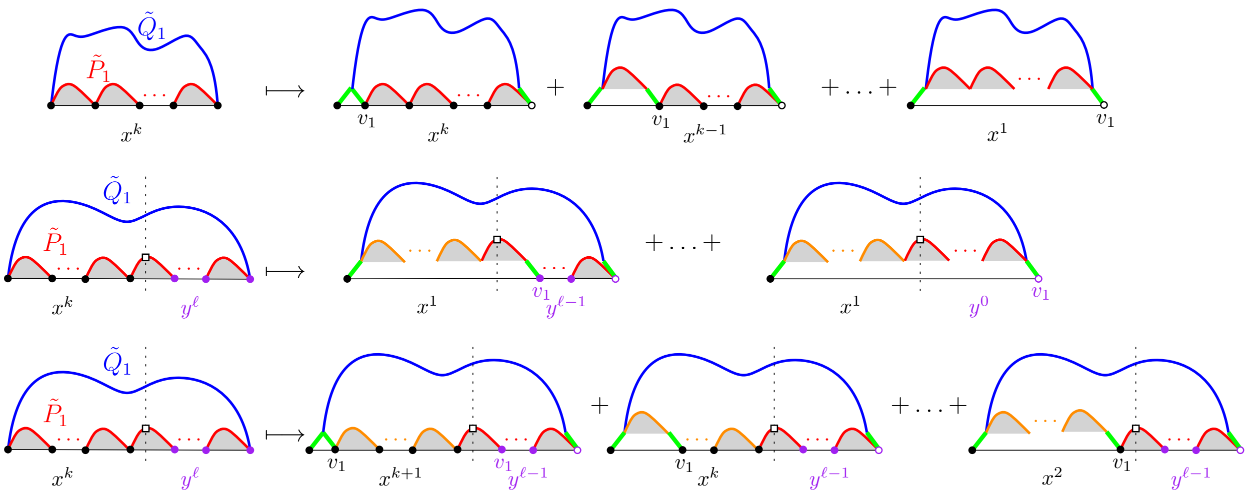

Figure 4: Top: How the divided difference operator appears. On the left, the power of x marks all contacts, while on the right it only marks contacts which are not the last one. Center and Bottom: Refinement to distinguish the case where the marked abscissa is before (upper figure) or after (lower figure) the contact v_1.

after this abscissa (thus having a contribution of $x^k y^\ell$ in $G(x,y)$) gives rise to ℓ intervals contributing to this case, with a contribution of $x^1y^{\ell-1} + x^1y^{\ell-2} + \cdots + x^1y^0 = x\frac{y^\ell-1}{y-1}$. In total, the contribution for the first interval is thus $x\frac{G(1,y)-G(1,1)}{y-1}$. The contribution of the second interval (P_2, Q_2) is just $F(y)$, since all corresponding contacts appear after the marked abscissa. In total, this gives the second term in (3.1).

We omit other cases, but we point out that the fourth case requires a similar discussion where now the catalytic variable x plays the main role, see Figure 4(bottom). Some care is needed since the last vertex of the path $\tilde{Q}_1$ cannot be marked in this case, hence the slightly more complicated numerator in this term. □

3.2 Solution

Although equations with two catalytic variables are notoriously difficult, Equation (3.1) is of a very particular kind, as it involves $G(x,y)$, $G(1,y)$, and $G(1,1)$, but not $G(x,1)$. This will enable us to treat (3.1) as two nested equations, each having only one catalytic variable. In what follows we sketch the resolution, see [5] for more details.
First step: eliminating the variable x (or u). We will work under the change of variables (2.4), with a new variable v which is to y what u is to x, namely

$$t = z(1-z)^3 \ , \ x = \frac{1+u}{(1+zu)^2} \ , \ y = \frac{1+v}{(1+zv)^2}. \tag{3.2}$$

We write respectively $\tilde{G}(u,v), \tilde{G}_1(v), \tilde{G}_{11}$ for the quantities $G(x,y), G(1,y), G(1,1)$ expressed in the variables z, u, v after the substitutions (3.2).

Making the substitutions (3.2) and using the known expressions of $F(x)$, $F(y)$, Equation (3.1) takes the form

$$\tilde{K}_2(z,u,v)\tilde{G}(u,v) = \tilde{L}_2(z,u,v,\tilde{G}_1(v),\tilde{G}_{11})$$

for some rational functions $\tilde{K}_2, \tilde{L}_2$ that can be written explicitly [5]. One easily checks that the kernel $\tilde{K}_2$ has a unique root $U_0 \equiv U_0(z,v)$ which is a power series in z. Substituting $u = U_0$ in the last equation, we cancel the left-hand side, hence we also cancel the right-hand side. We are thus left with the following polynomial equation:

$$\tilde{L}_2(z,U_0(z,v),v,\tilde{G}_1(v),\tilde{G}_{11}) = 0. \tag{3.3}$$

The numerator of this equation has 91 terms, but it has only degree one in $\tilde{G}_1(v)$. At this stage, we have eliminated the unknown $\tilde{G}(u,v)$ and the variable u.

Second step: eliminating the variable y (or v). The last equation, (3.3), is nothing but an equation with one catalytic variable, which is now the variable v (or x)! Since it is linear in $\tilde{G}_1(v)$, we can just use the kernel method again: one first checks that there is a unique series $V_0(z)$ cancelling the kernel, thus giving two equations: the linear and constant coefficient in $\tilde{G}_1(v)$ in (3.3), which both vanish when $v = V_0$. Eliminating V_0, we obtain a polynomial equation for the function $\tilde{G}_{11}$ which is not even so big. We obtain:

Theorem 3.2. *The function $G(1,1)$ after the change of variables $z \leftrightarrow t$ given by* (2.4) *satisfies the polynomial equation $C(G(1,1),z,w) = 0$ with*

$$\begin{aligned} C(h,z,w) &= wz(-1+z)^9h^3 + (-1+z)^6(2w^2z^2 - w^2z + 2z^2 + w - z)h^2 + 4wz^2 - 4wz + w \\ &\quad -(-1+z)^3(w^2z^3 - 3w^2z^2 - 2wz^3 + w^2z - 2wz^2 + z^3 + 5wz - 3z^2 - 2w + z)h. \end{aligned} \tag{3.4}$$

4 Asymptotics of moments

The two parts of Theorem 1.1 are direct applications of Theorem 1.4, up to computer algebra calculations done in [5]. In both cases we have $\rho = \frac{27}{256}$, corresponding to $z = \frac{1}{4}$.

In the case of the upper path, we start from (2.7) in Theorem 2.3, which tells us that the function $f(t,s) = H(1,s)$ is algebraic. With GFUN [12], we directly obtain [5] a recurrence formula for the derivatives at $s = 1$ of the form (1.6) with $L = 6$, where the $h_d(t,k)$ for $d = 1..6$ are Laurent polynomials in $\delta = \sqrt{1-4z}$ and rational functions of k. It is automatic to check that the hypotheses of Theorem 1.4 holds with $\beta = \frac{3}{4}$ and $\alpha = \frac{1}{2}$. One can explicitly check the values of the corresponding constants $a_d(k)$, which are nothing but the coefficients of δ^{-3d} in h_d, up to a scaling factor. They are given [5] by

$$a_1(k), \ldots, a_6(k) = 0, \frac{\sqrt{6}}{96}(3k-4)(3k-8), 0, 0, 0, 0, \tag{4.1}$$

so the main recurrence formula (1.7) becomes

$$c_k = \frac{\sqrt{6}(3k-4)(3k-8)}{96} c_{k-2} \ , \ \ k > 6.$$

The initial conditions require to estimate the main singularity of $f^{(1)}, \ldots, f^{(6)}$, which is easily done automatically, and it is then a direct check that the solution is given by

$$c_k = 16/(27\pi)\, \Gamma\left(\frac{k}{2}+\frac{1}{3}\right) \Gamma\left(\frac{k}{2}-\frac{1}{3}\right) \sqrt{2} \cdot 4^{-k} 6^{3/4\,k}.$$

for all $k \geq 0$. Applying Theorem 1.4, we directly obtain (1.2), and (1.1) follows for example from the Carleman criterion. See [5] for full calculations.

The proof of the second half (i.e. (1.3)) follows similarly from Theorem 3.2, this time with $\alpha = \frac{1}{2}, \beta = \frac{3}{4}$, and $L = 9$. See [5] again.

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Extended Schur Functions and Bases Related by Involutions

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Abstract. The extended Schur and shin functions are Schur-like bases of $QSym$ and $NSym$. We define a creation operator and a Jacobi-Trudi rule for certain shin functions and show that a similar Jacobi-Trudi rule does not exist for every shin function. We also define the skew extended Schur functions and relate them to the multiplicative structure of the shin basis. Then, we introduce two new pairs of dual bases that result from applying the ρ and ω involutions to the extended Schur and shin functions. These bases are defined combinatorially via variations on shin-tableaux much like the row-strict extended Schur functions.

Keywords: Schur-like, QSym, NSym, extended Schur Function, Shin function

1 Introduction

There has been considerable interest over the last decade in studying Schur-like bases of $NSym$ and $QSym$. A basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$ is generally considered ***Schur-like basis*** if $\chi(\mathcal{S}_\lambda) = s_\lambda$ for any partition λ where the ***forgetful map*** $\chi : NSym \to Sym$ gives the commutative image of an element in $NSym$. A Schur-like basis $\{\mathcal{S}^*_\alpha\}_\alpha$ of $QSym$ is informally defined as a basis dual to a Schur-like basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$. These bases are usually defined combinatorially in terms of tableaux that resemble or generalize the semistandard Young tableaux. The canonical Schur-like bases of $NSym$ and $QSym$ are respectively the immaculate basis [3], the Young noncommutative Schur basis [8], and the shin basis [6], as well as the dual immaculate basis, the Young quasisymmetric Schur basis, and the extended Schur functions.

The shin and extended Schur functions, which are dual bases, are unique among the Schur-like bases for having arguably the most natural relationship with the Schur functions. In $NSym$, the commutative image of a shin function indexed by a partition is a Schur function, while the commutative image of any other shin function is 0. In $QSym$, the extended Schur functions indexed by partitions are equal to Schur functions [6]. The goal of this extended abstract is to answer questions about basis expansions, skew functions, and multiplicative properties of these two bases and introduce new, related

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bases. In the full paper, we present additional results including a second type of skew function and connections to the antipode map on $NSym$ [7].

1.1 The Shin and Extended Schur Functions

The dual shin functions were introduced by Campbell, Feldman, Light, Shuldiner, and Xu in [6] as the duals to the shin functions and defined independently by Assaf and Searles in [2] as the extended Schur functions, which are the stable limits of polynomials related to Kohnert diagrams. We use the name “extended Schur functions” but otherwise retain the notation and terminology of the dual shin functions.

Definition 1.1. Let α and β be a composition and weak composition of n, respectively. A ***shin-tableau*** of shape α and type β is a labeling of the boxes of the diagram of α by positive integers such that the number of boxes labeled by i is β_i, the sequence of entries in each row is weakly increasing from left to right, and the sequence of entries in each column is strictly increasing from top to bottom.

Note that shin-tableaux are a direct generalization of semistandard Young tableaux to composition shapes.

Example 1.2. The shin-tableaux of shape $(3,4)$ and type $(1,2,1,1,2)$ are

$$\begin{array}{|c|c|c|c} \hline 1 & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & 5 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c} \hline 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & 5 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c} \hline 1 & 2 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 5 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

A shin-tableau of n boxes is *standard* if each number 1 through n appears exactly once. The *descent set* is defined as $Des_{ש}(U) = \{i : i+1 \text{ is strictly below } i \text{ in } U\}$ for a standard shin-tableau U. Each entry i in $Des_{ש}(U)$ is called a *descent* of U. The *descent composition* of U is defined $co_{ש}(U) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d)$ for $Des_{ש}(U) = \{i_1, \dots, i_d\}$.

The *shin reading word* of a shin-tableau T, denoted $rw_{ש}(T)$, is the word obtained by reading the rows of T from left to right starting with the bottom row and moving up. To *standardize* a shin-tableau T of size n, we will replace entries in T with the numbers 1 through n to obtain a standard shin-tableau. First, replace the 1’s in T with $1, 2, \dots$ in the order they appear in $rw_{ש}(T)$. Then replace the 2’s with the next consecutive numbers, again in the order they appear in $rw_{ש}(T)$, then the 3’s, etc.

For a composition α, the ***extended Schur function*** is defined as $ש^*_\alpha = \sum_T x^T$ where the sum runs over shin-tableaux T of shape α. Their positive expansions into the monomial and fundamental bases in terms of shin-tableaux are known [2, 6]. For a composition α,

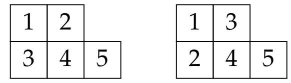
$$ש^*_\alpha = \sum_\beta \mathcal{K}_{\alpha,\beta} M_\beta \qquad \text{and} \qquad ש^*_\alpha = \sum_\beta \mathcal{L}_{\alpha,\beta} F_\beta, \tag{1.1}$$

ש is the hebrew character *shin*.

where $\mathcal{K}_{\alpha,\beta}$ denotes the number of shin-tableaux of shape α and type β, and $\mathcal{L}_{\alpha,\beta}$ denotes the number of standard shin-tableaux of shape α with descent composition β.

Example 1.3. The F-expansion of the extended Schur function $\mathfrak{w}^*_{(2,3)}$:

$$\mathfrak{w}^*_{(2,3)} = F_{(2,3)} + F_{(1,2,2)}$$



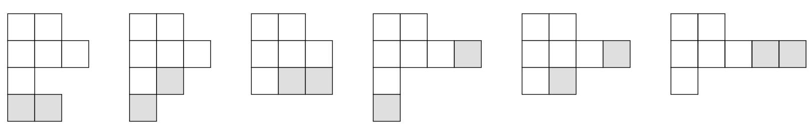
The shin basis of *NSym* was introduced by Campbell, Feldman, Light, Shuldiner, and Xu in [6]. Let α and β be compositions. Then β is said to differ from α by a *shin-horizontal strip* of size r, denoted $\alpha \subset^{\mathfrak{w}}_r \beta$, provided for all i, we have $\beta_i \geq \alpha_i$, $|\beta| = |\alpha| + r$, and for any $i \in \mathbb{N}$ if $\beta_i > \alpha_i$ then for all $j > i$, we have $\beta_j \leq \alpha_i$. The shin functions are defined recursively based on a right Pieri rule using shin-horizontal strips.

Definition 1.4. The ***shin basis*** $\{\mathfrak{w}_\alpha\}_\alpha$ of *NSym* is defined as the unique set of functions $\mathfrak{w}_\alpha$ such that $\mathfrak{w}_\alpha H_r = \sum_{\alpha \subset^{\mathfrak{w}}_r \beta} \mathfrak{w}_\beta$, where the sum runs over all compositions β which differ from α by a shin-horizontal strip of size r.

Intuitively, the compositions β in the summation are given by taking diagrams of α and adding r blocks on the right such that if you add boxes to some row i then no row below i is longer than the original row i. This is referred to as the *overhang rule*.

Example 1.5. The following expression can be visualized with the tableaux below.

$$\mathfrak{w}_{(2,3,1)} H_{(2)} = \mathfrak{w}_{(2,3,1,2)} + \mathfrak{w}_{(2,3,2,1)} + \mathfrak{w}_{(2,4,1,1)} + \mathfrak{w}_{(2,4,2)} + \mathfrak{w}_{(2,5,1)}$$



Repeated application of this right Pieri rule yields the expansion of a complete homogeneous noncommutative symmetric function in terms of the shin functions. This expansion verifies that the extended Schur functions and the shin functions are dual bases. This allows us to expand the ribbon functions into the shin basis dually to the expansion of the extended Schur functions expanded into the fundamental basis [2, 6].

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathcal{K}_{\alpha,\beta} \mathfrak{w}_\alpha \qquad \text{and} \qquad R_\beta = \sum_{\beta \leq_\ell \alpha} \mathcal{L}_{\alpha,\beta} \mathfrak{w}_\alpha. \tag{1.2}$$

The extended Schur functions have the special property that $\mathfrak{w}^*_\lambda = s_\lambda$ for a partition λ. Since the forgetful map χ is dual to the inclusion map from *Sym* to *QSym*, $\chi(\mathfrak{w}_\lambda) = s_\lambda$ when λ is a partition and $\chi(\mathfrak{w}_\alpha) = 0$ otherwise. Another interesting feature of the shin functions is their relationship with the other two canonical Schur-like bases, the immaculate functions and the Young noncommutative Schur functions. Given a partition λ, the immaculate function $\mathfrak{S}_\lambda$ equals the Young noncommutative Schur function $\hat{\mathbf{s}}_\lambda$, but the shin function $\mathfrak{w}_\lambda$ differs from the two.

2 A Creation Operator for Certain Shin Functions

The Schur functions and the immaculate functions can both be defined using ***creation operators***. In fact, the immaculate basis was originally defined in terms of noncommutative Bernstein operators [3]. It is using these operators that one can prove various properties of the immaculate basis including the Jacobi-Trudi rule [3], a left Pieri rule [5], a combinatorial interpretation of the inverse Kostka matrix [1], and a partial Littlewood Richardson rule [4]. Here we give similar creation operators for certain shin functions which then allow us to define a ***Jacobi-Trudi rule***. This rule is especially useful because there is currently no other combinatorial way to expand shin functions into the complete homogeneous basis.

Definition 2.1. For a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $k \geq 1$ and a positive integer m, define the action of the linear operator $\beth_m$ on the complete homogeneous basis by

$$\beth_m(1) = H_m \qquad \text{and} \qquad \beth_m(H_\alpha) = H_{m,\alpha_1,\alpha_2,\dots} - H_{\alpha_1,m,\alpha_2,\dots}.$$

Theorem 2.2. *If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $k \geq 1$ and $0 < m < \alpha_1$, then $\beth_m(\mathfrak{w}_\alpha) = \mathfrak{w}_{m,\alpha}$.*

This theorem follows from showing inductively that the functions given by $\beth_m(\mathfrak{w}_\alpha)$ satisfy the right Pieri rule defining the shin functions, and then showing $\beth_m(\mathfrak{w}_\alpha)$ equals $\mathfrak{w}_{(m,\alpha)}$ by recursive calculation. These operators allow us to construct shin functions indexed by strictly increasing compositions from the ground up.

Corollary 2.3. *Let $\beta = (\beta_1, \dots, \beta_k)$ where $\beta_i < \beta_{i+1}$. Then, $\beth_{\beta_1} \cdots \beth_{\beta_k}(1) = \mathfrak{w}_\beta$.*

Example 2.4. These creation operators can be used to build up $\mathfrak{w}_{(1,3,4)}$ as follows:

$$\beth_1\beth_3\beth_4(1) = \beth_1\beth_3(H_4) = \beth_1(H_{(3,4)} - H_{(4,3)}) = H_{(1,3,4)} - H_{(1,4,3)} - H_{(3,1,4)} + H_{(4,1,3)}.$$

Using these operators, we define the following Jacobi-Trudi rule to express these same shin functions as matrix determinants. Let $S_k^{\geq}(-1)$ be the set of permutations $\sigma \in S_k$ such that $\sigma(i) \geq i - 1$ for all $i \in [k]$.

Theorem 2.5. *Let $\beta = (\beta_1, \dots, \beta_k)$ be a composition such that $\beta_i < \beta_{i+1}$ for all i. Then,*

$$\mathfrak{w}_\beta = \sum_{\sigma \in S_k^{\geq}(-1)} (-1)^\sigma H_{\beta_{\sigma(1)}} \cdots H_{\beta_{\sigma(k)}}.$$

Equivalently, $\mathfrak{w}_\beta$ can be expressed as the matrix determinant

$\beth$ is the hebrew character ***beth***.

$$\mathfrak{w}_\beta = det \begin{vmatrix} H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & 0 & \cdots & 0 & H_{\beta_{k-1}} & H_{\beta_k} \end{vmatrix}$$

using the noncommutative determinant obtained by expanding along the first row.

We can show by counterexample that there is not a matrix rule of this form for every shin function, not even those indexed by partitions [7]. It remains open to find a combinatorial or algebraic way of understanding the expansion of the shin basis into the complete homogeneous basis for the general case.

3 Skew Extended Schur Functions

To define skew extended Schur functions, we first use an algebraic approach, and then connect it to tableaux combinatorics. For $F \in QSym$, the operator $F^\perp$ acts on elements $H \in NSym$ based on the relation $\langle H, FG\rangle = \langle F^\perp H, G\rangle$. For dual bases $\{A_\alpha\}_\alpha$ of $QSym$ and $\{B_\alpha\}_\alpha$ of $NSym$ this expands as $F^\perp(H) = \sum_\alpha \langle H, FA_\alpha\rangle B_\alpha$.

Definition 3.1. For compositions α and β with $\beta \subseteq \alpha$, the ***skew extended Schur functions*** are defined as $\mathfrak{w}^*_{\alpha/\beta} = \mathfrak{w}^\perp_\beta(\mathfrak{w}^*_\alpha)$.

By the equation for $F^\perp$ above, we expand $\mathfrak{w}^*_{\alpha/\beta}$ into various bases as follows.

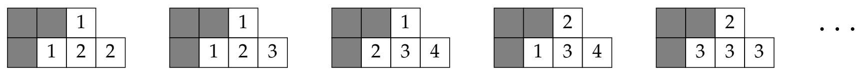
Proposition 3.2. *For compositions $\beta \subseteq \alpha$, $\mathfrak{w}^*_{\alpha/\beta} = \sum_\gamma \langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha\rangle M_\gamma = \sum_\gamma \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle \mathfrak{w}^*_\gamma$. Furthermore, let $C^\alpha_{\beta,\gamma} := \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle$. Then, $\mathfrak{w}_\beta \mathfrak{w}_\gamma = \sum_\alpha C^\alpha_{\beta,\gamma} \mathfrak{w}_\alpha$.*

Using the properties of the forgetful map and the shin basis, we have the following statement about the coefficients that appear in the skew extended Schur functions.

Proposition 3.3. *Let α, β, γ be compositions that are not partitions and let λ, μ, ν be partitions. Then, $C^\nu_{\lambda,\beta} = C^\nu_{\alpha,\mu} = C^\nu_{\alpha,\beta} = 0$ and $C^\nu_{\lambda,\mu} = c^\nu_{\lambda,\mu}$, where $c^\nu_{\lambda,\mu}$ are the usual Littlewood-Richardson coefficients.*

The skew extended Schur functions can also be expressed in terms of skew shin-tableaux.

Proposition 3.4. *For compositions α and β such that $\beta \subseteq \alpha$, $\langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha\rangle$ is equal to the number of skew shin-tableau of shape α/β and type γ. Moreover, $\mathfrak{w}^*_{\alpha/\beta} = \sum_T x^T$, where the sum runs over skew shin-tableau T of shape α/β.*



$$\text{ש}^*_{(3,4)/(2,1)} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_3x_4 + x_2x_3^3 + \dots$$

Skew shin-tableaux of shape λ/μ where λ and μ are partitions are simply skew semistandard Young tableaux. By Proposition 3.4, these skew extended Schur functions are equal to the usual skew Schur functions, $\text{ש}^*_{\lambda/\mu} = s_{\lambda/\mu}$.

4 Involutions on *QSym* and *Nsym*

In *QSym*, we consider three involutions defined on the fundamental basis and their dual maps in *NSym* [8]. They each are defined as extensions of involutions on compositions. The *complement* of a composition α is defined $\alpha^c = comp(set(\alpha)^c)$, where $set((\alpha_1,\dots,\alpha_k)) = \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\dots+\alpha_{k-1}\}$ if α is a composition of n, and $comp(\{s_1,\dots,s_j\}) = (s_1, s_2-s_1, \dots, s_j-s_{j-1}, n-s_j)$, for $\{s_1,\dots,s_j\} \subseteq [n-1]$. The *reverse* of $(\alpha_1,\dots,\alpha_k)$ is $\alpha^r = (\alpha_k,\dots,\alpha_1)$. The *transpose* of α is defined by $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

Definition 4.1. The involutions ψ, ρ and ω on *QSym* and *NSym* are defined as

$$\psi(F_\alpha) = F_{\alpha^c} \qquad \rho(F_\alpha) = F_{\alpha^r} \qquad \omega(F_\alpha) = F_{\alpha^t}$$
$$\psi(R_\alpha) = R_{\alpha^c} \qquad \rho(R_\alpha) = R_{\alpha^r} \qquad \omega(R_\alpha) = R_{\alpha^t},$$

extended linearly. All three involutions on *QSym* and ψ on *NSym* are automorphisms, while ρ and ω on *NSym* are anti-automorphisms.

Note that we use the same notation for the corresponding involutions on *QSym* and *NSym*. These automorphisms commute and $\omega = \rho \circ \psi = \psi \circ \rho$. When ω and ψ are restricted to *Sym*, they are both equivalent to the classical involution $\omega : Sym \to Sym$ which acts on the Schur functions by $\omega(s_\lambda) = \lambda'$ where λ' is the conjugate of λ. The conjugate of a partition λ is found by flipping the diagram of λ over the diagonal.

Applying ψ to the extended Schur and shin functions recovers the row strict extended Schur and row strict extended shin functions ($\mathfrak{R}\text{ש}$) of Niese, Sundaram, van Willigenburg, Vega, and Wang in [9]. We define two new pairs of dual bases ($\rotatebox{180}{ש}$ and $\mathfrak{R}\rotatebox{180}{ש}$) in *QSym* and *NSym* by applying ρ and ω to the extended Schur and shin functions as

$$\psi(\text{ש}^*_\alpha) = \mathfrak{R}\text{ש}^*_\alpha \qquad \rho(\text{ש}^*_\alpha) = \rotatebox{180}{ש}^*_{\alpha^r} \qquad \omega(\text{ש}^*_\alpha) = \mathfrak{R}\rotatebox{180}{ש}^*_{\alpha^r}$$
$$\psi(\text{ש}_\alpha) = \mathfrak{R}\text{ש}_\alpha \qquad \rho(\text{ש}_\alpha) = \rotatebox{180}{ש}_{\alpha^r} \qquad \omega(\text{ש}_\alpha) = \mathfrak{R}\rotatebox{180}{ש}_{\alpha^r}.$$

We give combinatorial interpretations of these two new pairs of bases in terms of variations on shin-tableaux. While specific definitions are to follow, we first describe intuitively how ψ, ρ, and ω act on the tableaux defining each basis. Recall that shin-tableaux have weakly increasing columns and strictly increasing rows. The ψ map switches whether the strictly changing condition is on rows or columns (while the other has a weakly changing condition). The ρ map switches the row condition from increasing to decreasing, or vice versa. The ω map does both. Through this combinatorial interpretation, each of the four pairs of dual bases is related to any other by one of the three involutions ψ, ρ, or ω as shown in the figure below.

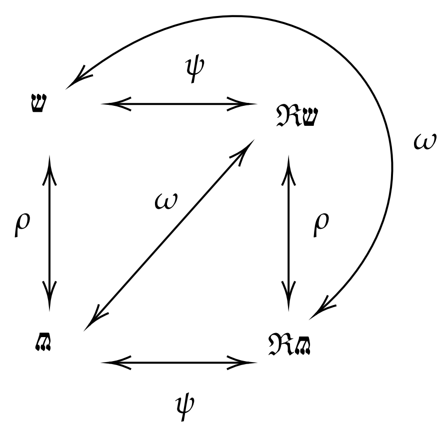


Figure 1: Mappings between shin variants in *NSym*.

Essentially, ψ, ρ, and ω in *QSym* and *NSym* collectively serve as the analogue to the classical ω in *Sym*. In *Sym*, the Schur basis is its own image under ω but in *QSym* and *NSym* our Schur-like basis is instead part of a system of four related bases that are in a sense closed under the three involutions ψ, ρ, and ω. While the combinatorics of these bases are similar, they may have very different applications. For example, the quasisymmetric Schur basis and the Young quasisymmetric Schur basis are related by the involution ρ but the former is much more compatible with Macdonald polynomials while the latter is more useful when working with Schur functions [8].

The table below serves to summarize the tableaux defined over the course of this section. It lists each type of tableaux, the position of $i + 1$ relative to i that makes i a descent, the order the boxes appear in the reading word (Left, Right, Top, Bottom), the condition on entries of each row, and the condition on entries in each column.

	Descent	**Reading Word**	**Rows**	**Columns**
Shin	strictly below	L to R, B to T	weakly increasing	strictly incr.
Row-strict	weakly above	L to R, T to B	strictly increasing	weakly incr.
Reverse	strictly below	R to L, B to T	weakly decreasing	strictly incr.
Row-strict rev.	weakly above	R to L, T to B	strictly decreasing	weakly incr.

We now briefly review row-strict extended Schur and row-strict shin functions but reserve more details for the full paper [7]. Let α be a composition and let β be a weak composition, allowing for zero entries. A ***row-strict shin-tableaux*** (RSST) of shape α and type β is a filling of the composition diagram of α with positive integers such that each row strictly increases from left to right, each column weakly increases from top to bottom, and each integer i appears β_i times. A *standard* row-strict shin-tableaux (SRSST) with n boxes is one containing the entries 1 through n each exactly once. For a composition α, define the ***row strict extended Schur function*** as $\mathfrak{Rw}^*_\alpha = \sum_T x^T$, where the sum runs over all row-strict shin-tableaux T of shape α. The ***row strict shin functions*** are defined as the duals in $NSym$ to the row strict extended Schur functions in $QSym$.

For a standard row-strict shin-tableau U, the *descent set* is defined to be $Des_{\mathfrak{Rw}}(U) = \{i : i+1 \text{ is weakly above } i \text{ in } U\}$. Each entry i in $Des_{\mathfrak{Rw}}(U)$ is called a *descent* of U. The *descent composition* of U is defined to be $co_{\mathfrak{Rw}}(U) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{Rw}}(U) = \{i_1, \dots, i_d\}$. Equivalently, the descent composition is found by counting the number of entries in U (in the order they are numbered) between each descent. Note that the set of standard row-strict shin-tableaux is exactly the same as the set of standard shin-tableaux. Using the framework of standard row-strict shin-tableaux, it is shown in [9] that for a composition α, the row-strict extended Schur function expands into the fundamental basis as $\mathfrak{Rw}^*_\alpha = \sum_U F_{co_{\mathfrak{Rw}}(U)}$, where the sum runs over all standard row-strict shin-tableaux.

Example 4.2. The F-expansion of the row-strict extended Schur function $\mathfrak{Rw}^*_{(2,3)}$:

$$\mathfrak{Rw}^*_{(2,3)} = F_{(1,2,1,1)} + F_{(2,2,1)}$$

1	2	
3	4	5

1	3	
2	4	5

We can now relate the extended Schur and row-strict extended Schur functions by using ψ on their F-expansions. This relationship follows from the fact that the set of standard tableaux is the same but the definitions of descent sets are in a sense complementary and the map ψ is using complements. For all compositions α, $\psi(\mathfrak{w}^*_\alpha) = \mathfrak{Rw}^*_\alpha$, and $\{\mathfrak{Rw}^*_\alpha\}_\alpha$ is a basis of $QSym$. Additionally, $\psi(\mathfrak{w}_\alpha) = \mathfrak{Rw}_\alpha$ and $\{\mathfrak{Rw}_\alpha\}_\alpha$ is a basis of $NSym$.

4.1 Reverse extended Schur and shin functions.

Let α be a composition and β a weak composition. A *reverse shin-tableau* of shape α and type β is a composition diagram α filled with positive integers that weakly decrease along the rows and strictly increase along the columns (from top to bottom) where each positive integer i appears β_i times.

Definition 4.3. For a composition α, the ***reverse extended Schur function*** is defined as $\reflectbox{$\mathfrak{w}$}^*_\alpha = \sum_T x^T$, where the sum runs over all reverse shin-tableaux T of shape α.

A *standard* reverse shin-tableau of shape α is one containing the entries 1 through n each exactly once. For a standard reverse shin-tableau S, the *descent set* is defined as $Des_{\reflectbox{\mathfrak{w}}}(S) = \{i : i+1 \text{ is strictly below } i \text{ in } S\}$. Each entry i in $Des_{\reflectbox{\mathfrak{w}}}(S)$ is called a *descent* of S. The *descent composition* of S is defined $co_{\reflectbox{\mathfrak{w}}}(S) = comp(Des_{\reflectbox{\mathfrak{w}}}(S))$. Define $flip(S)$ to be the tableau U obtained by flipping S horizontally (in other words, reversing the order of the rows of S) and then replacing each entry i with $n-i$. It is easy to see that the map $flip$ is an involution between the set of standard shin-tableaux and the set of standard reverse shin-tableaux.

$$flip\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2} \end{array} \right) = \begin{array}{|c|c|c|}\cline{1-2} 4 & 1 & \\ \hline 5 & 3 & 2 \\ \hline \end{array}$$

The ***reverse shin-reading word*** of a reverse shin-tableau T, denoted $rw_{\reflectbox{\mathfrak{w}}}(T)$, is the word obtained by reading the rows of T from right to left starting with the top row and moving down. To *standardize* a reverse shin-tableau T, replace the 1's with $1, 2, \ldots$ in the order they appear in $rw_{\reflectbox{\mathfrak{w}}}(T)$, then the 2's starting with the next consecutive number, etc.

Proposition 4.4. *For a composition α, $\reflectbox{\mathfrak{w}}^*_\alpha = \sum_S F_{co_{\reflectbox{\mathfrak{w}}}(S)}$, where the sum runs over standard reverse shin-tableaux U of shape α.*

Example 4.5. The F-expansion of the reverse extended Schur function $\reflectbox{\mathfrak{w}}^*_{(3,2)}$:

$$\reflectbox{\mathfrak{w}}^*_{(3,2)} = F_{(3,2)} + F_{(2,2,1)} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2} \end{array}$$

The descent composition of a standard shin-tableau U is the reverse of the descent composition of the standard reverse shin-tableau given by $flip(U)$. Using this fact, we show that the reverse extended Schur functions are the image of the extended Schur functions under ρ.

Theorem 4.6. *For a composition α, $\rho(\mathfrak{w}^*_\alpha) = \reflectbox{\mathfrak{w}}^*_{\alpha^r}$, and $\{\reflectbox{\mathfrak{w}}_\alpha\}_\alpha$ is a basis of QSym.*

Now, we define the reverse shin basis by applying ρ to the shin basis.

Definition 4.7. For a composition α, the ***reverse shin function*** is defined as $\reflectbox{\mathfrak{w}}_\alpha = \rho(\mathfrak{w}_{\alpha^r})$.

By the invariance of ρ under duality, we have that the reverse shin functions are the dual basis to the reverse extended Schur functions, that is $\langle \reflectbox{\mathfrak{w}}_\alpha, \reflectbox{\mathfrak{w}}^*_\beta \rangle = \delta_{\alpha,\beta}$.

Let $\mathcal{K}^{\reflectbox{\mathfrak{w}}}_{\alpha,\beta}$ be the number of reverse shin-tableaux of shape α and type β, and let $\mathcal{L}^{\reflectbox{\mathfrak{w}}}_{\alpha,\beta}$ be the number of reverse shin-tableaux with shape α and descent composition β. Then,

$$\reflectbox{\mathfrak{w}}^*_\alpha = \sum_\beta \mathcal{K}^{\reflectbox{\mathfrak{w}}}_{\alpha,\beta} M_\beta = \sum_\beta \mathcal{L}^{\reflectbox{\mathfrak{w}}}_{\alpha,\beta} F_\beta \qquad \text{and} \qquad H_\beta = \sum_\alpha \mathcal{K}^{\reflectbox{\mathfrak{w}}}_{\alpha,\beta} \reflectbox{\mathfrak{w}}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\reflectbox{\mathfrak{w}}}_{\alpha,\beta} \reflectbox{\mathfrak{w}}_\alpha.$$

By applying ρ, we can translate many of the results on the shin functions to the reverse shin functions.

Theorem 4.8. *For compositions α, β and a positive integer m,*

1. $H_m \shin_\alpha = \sum_{\alpha^r \subset_m^{\shin} \beta^r} \shin_\beta.$

2. $H_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \shin_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^r} \shin_\alpha.$

3. $\shin^*_{\lambda^r} = s_\lambda$. *Also,* $\chi(\shin_{\lambda^r}) = s_\lambda$ *and* $\chi(\shin_\alpha) = 0$ *when* α^r *is not a partition.*

4. *For a composition γ such that $\gamma_i > \gamma_{i+1}$ for all $1 \leq i \leq \ell(\gamma)$,*

$$\shin_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma H_{\gamma_{\sigma(1)}} \cdots H_{\gamma_{\sigma(\ell(\gamma))}},$$

where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \geq i-1$ for all $i \in [\ell(\gamma)]$.

4.2 Row-strict reverse extended Schur and shin functions.

Let α be a composition and β be a weak composition. A *row-strict reverse shin-tableau* (BST) of shape α and type β is a filling of the diagram of α with positive integers such that the entries in each row are strictly decreasing from left to right and the entries in each column are weakly increasing from top to bottom where each integer i appears β_i times. These are essentially a row-strict version of the reverse shin-tableaux.

Definition 4.9. For a composition α, the ***row-strict reverse extended Schur function*** is defined as $\mathfrak{R}\shin^*_\alpha = \sum_T x^T$, where the sum runs over all row-strict reverse shin-tableaux T of shape α.

A row-strict reverse shin-tableau of shape α is *standard* if it includes the entries 1 through n each exactly once. For a standard row-strict reverse shin-tableau S, the *descent set* is defined to be $Des_{\mathfrak{R}\shin}(S) = \{i : i+1 \text{ is weakly above } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{R}\shin}(S)$ is called a *descent* of S. Then, we define the *descent composition* of S to be $co_{\mathfrak{R}\shin}(S) = comp(Des_{\mathfrak{R}\shin}(S))$. Equivalently, the descent composition is found by counting the number of entries in S (in the order they are numbered) between each descent. Note that the set of standard row-strict reverse shin-tableaux is exactly the same as the set of standard reverse shin-tableaux.

The *row-strict reverse shin reading word* of a row-strict reverse shin-tableau T, denoted $rw_{\mathfrak{R}\shin}(T)$ is the word obtained by reading the rows of T from right to left starting with the bottom row and moving up. We can *standardize* a standard row-strict reverse shin-tableau as follows. To *standardize* a row-strict reverse shin-tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{R}\shin}(T)$, then the 2's starting with the next consecutive integer, then 3's, etc.

Proposition 4.10. *For a composition α, $\mathfrak{R}\boldsymbol{\varpi}^*_\alpha = \sum_S F_{co_{\mathfrak{R}\boldsymbol{\varpi}}(S)}$, where the sum runs over standard row-strict reverse shin-tableaux S.*

Example 4.11. The F-expansion of the reverse extended Schur function $\mathfrak{R}\boldsymbol{\varpi}^*_{(3,2)}$:

$$\mathfrak{R}\boldsymbol{\varpi}^*_{(3,2)} = F_{(1,1,2,1)} + F_{(1,2,2)} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array}$$

The descent compositions of row-strict reverse shin tableaux of shape α^r are complementary to the descent compositions of reverse tableaux of shape α^r, thus $\psi(\boldsymbol{\varpi}_{\alpha^r}) = \mathfrak{R}\boldsymbol{\varpi}^*_{\alpha^r}$. Given that $\boldsymbol{\varpi}^*_{\alpha^r} = \rho(\mathfrak{w}^*_\alpha)$ and $\psi \circ \rho = \omega$, we have the following result.

Theorem 4.12. *For a composition α, $\omega(\mathfrak{w}^*_\alpha) = \mathfrak{R}\boldsymbol{\varpi}^*_{\alpha^r}$ and $\{\mathfrak{R}\boldsymbol{\varpi}^*_\alpha\}_\alpha$ is a basis of QSym.*

The row-strict reverse extended Schur basis is not equivalent to the extended Schur basis, the row-strict extended Schur basis, or the reverse extended Schur basis. Again, it is simple to check that there exist row-strict reverse extended Schur functions that are not elements in the extended Schur, row-strict extended Schur, or reverse extended Schur bases. Like with ψ and ρ, it follows from the dual definitions of ω in *NSym* and *QSym* that ω is invariant under duality. Thus, the row-strict reverse extended Schur functions are dual to the row-strict reverse shin functions when defined as follows.

Definition 4.13. For a composition α, define the *row-strict reverse shin function* $\mathfrak{R}\boldsymbol{\varpi}_\alpha = \omega(\mathfrak{w}_{\alpha^r})$.

Let $\mathcal{K}^{\mathfrak{R}\boldsymbol{\varpi}}_{\alpha,\beta}$ be the number of row-strict reverse shin-tableaux of shape α and type β, and let $\mathcal{L}^{\mathfrak{R}\boldsymbol{\varpi}}_{\alpha,\beta}$ be the number of standard row-strict reverse shin-tableaux with shape α and descent composition β. The expansions of the row-strict reverse extended Schur functions into the monomial and fundamental bases follow those of the extended Schur functions. That is,

$$\mathfrak{R}\boldsymbol{\varpi}^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{R}\boldsymbol{\varpi}}_{\alpha,\beta} M_\beta = \sum_\beta \mathcal{L}^{\mathfrak{R}\boldsymbol{\varpi}}_{\alpha,\beta} F_\beta, \qquad \text{and} \qquad H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{R}\boldsymbol{\varpi}}_{\alpha,\beta} \mathfrak{R}\boldsymbol{\varpi}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{R}\boldsymbol{\varpi}}_{\alpha,\beta} \mathfrak{R}\boldsymbol{\varpi}_\alpha.$$

Now, we apply ω to the various results on the shin and extended Schur bases and find analogous results on the row-strict reverse shin and row-strict reverse extended Schur bases.

Theorem 4.14. *For compositions α, β and a positive integer m,*

1. $E_m \mathfrak{R}\boldsymbol{\varpi}_\alpha = \sum_{\alpha^r \subset^{\mathfrak{w}}_m \beta^r} \mathfrak{R}\boldsymbol{\varpi}_\beta.$

2. $E_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \mathfrak{R}\boldsymbol{\varpi}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^t} \mathfrak{R}\boldsymbol{\varpi}_\alpha$

3. $\mathfrak{R}\mathfrak{a}^*_{\lambda^r} = s_{\lambda'}$. *Also,* $\chi(\mathfrak{R}\mathfrak{a}_{\lambda^r}) = s_{\lambda'}$ *and* $\chi(\mathfrak{R}\mathfrak{a}_{\alpha}) = 0$ *when* α^r *is not a partition.*

4. *For a composition* γ *such that* $\gamma_i > \gamma_{i+1}$,

$$\mathfrak{R}\mathfrak{a}_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma E_{\gamma_{\sigma(1)}} E_{\gamma_{\sigma(2)}} \cdots E_{\gamma_{\sigma(\ell(\gamma))}},$$

where the sum runs over $\sigma \in S_{\ell(\gamma)}$ *such that* $\sigma(i) \geq i-1$ *for all* $i \in [\ell(\gamma)]$.

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Supersolvable posets and fiber-type arrangements

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Abstract. We develop a theory of modularity and supersolvability for chain-finite geometric posets, extending that of Stanley for finite lattices and building a new connection between combinatorics and topology. From a combinatorial point of view, our theory features results about factorizations of the characteristic polynomials, dovetails with established notions on geometric semilattices, and behaves well under quotients by translative group actions. We also establish a topological counterpart in the context of toric and abelian arrangements, akin to Terao's fibration theorem connecting bundles of hyperplane arrangements to supersolvability of their intersection lattice. From this, we obtain a combinatorially determined class of $K(\pi, 1)$ toric arrangements. Moreover, we characterize combinatorially when our toric arrangement bundles are pulled back from Fadell–Neuwirth's bundles of configuration spaces, and establish an analogue of Falk–Randell's formula relating the Poincaré polynomial to the lower central series of the fundamental group.

Keywords: supersolvable lattice, hyperplane arrangement, configuration space

1 Introduction

The theory of **supersolvable lattices** is a cornerstone of enumerative, algebraic and topological combinatorics. Its foundations were laid in work by Stanley [20, 19], motivated by the study of subgroups in supersolvable groups and building on the classical notion of **modular elements** in lattices.

Supersolvable lattices arise in a variety of contexts, e.g., the study of factorizations of characteristic polynomials [16] and of shellable posets [6], as well as in convex geometry [1] and representation theory [8]. More generally, modularity is a key concept in lattice theory, see [5, II.§7]. Several of these connections interact in the context of matroid theory, where modular flats exhibit a rich structure [7].

A seminal result by Terao [21] shows the equivalence between supersolvability of the lattice of flats of a matroid and an inductive fibration property of the complement manifold of any **arrangement of hyperplanes** that realizes the given matroid over $\mathbb{C}$. This opens up a powerful bridge between combinatorics and topology.

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More precisely, Terao's result states that the intersection lattice of a complex hyperplane arrangement $\mathcal{A}$ is supersolvable if and only if the arrangement $\mathcal{A}$ is "fiber type", i.e., there is a tower of arrangements $\emptyset = \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \ldots \subsetneq \mathcal{A}_d = \mathcal{A}$ such that the natural projection of complements $M(\mathcal{A}_i) \to M(\mathcal{A}_{i-1})$ is a linear fibration. Falk and Randell's study of fiber-type arrangements [14] unveiled a wealth of combinatorial-topological structure echoeing classical features of configuration spaces. This includes for instance a combinatorial formula for the lower central series of the fundamental group of the arrangement's complement [14, Theorem 4.1] and the proof that the fibrations arising in fiber-type arrangements are pullbacks from the classical Fadell-Neuwirth bundle for the configuration space of points in the plane [9].

A major point of interest of fiber-type arrangements is related to the long-standing $K(\pi, 1)$-problem for hyperplane arrangements, asking for a combinatorial characterization of asphericity of the arrangement's complement. Indeed, fiber-type hyperplane arrangements have aspherical complements, and they are characterized by a combinatorial condition: supersolvability of the intersection lattice.

Here[1] we devise a general theory of modular elements and supersolvability for posets beyond lattices – so-called "geometric posets" (see Definition 1). On the combinatorial side we derive some fundamental results about factorization of characteristic polynomials (Theorem 2) and quotients by poset automorphisms (Theorem 1).

When the geometric poset is a semilattice, our definition of supersolvability agrees with that given by Falk and Terao [15] in studying intersection posets of affine hyperplane arrangements. Moreover, we prove that a geometric semilattice is supersolvable if and only if its canonical extension to a geometric lattice is supersolvable [2, Theorem 4.2.4]. This leads to a first topological consequence of our work: an affine analogue of Terao's fibration theorem [2, Theorem 4.3.3].

Just as for classical lattice supersolvability, our theory has a strong topological counterpart in terms of **toric arrangements** (see Definition 6). Indeed, the notion of "geometric poset" Definition 1 seems to provide the right level of generality for studying intersection data of an arrangement of subtori in a complex torus.

The study of toric arrangements is a recent field of research that has given rise to combinatorial structures such as arithmetic Tutte polynomials, arithmetic matroids, matroid schemes and group actions on semimatroids [18, 10, 4], to name a few. In particular, the poset of intersections of a toric arrangement has been studied from different points of view [17, 12]. Our notion of supersolvability applies to intersection posets of toric arrangements, and has several implications for the topology of the arrangement complement, see §3.

[1] This is an extended abstract of [2].

2 Supersolvable posets

We recall basic ideas about posets and supersolvable geometric lattices. We then define M- and TM-ideals (Definition 3) and the corresponding notions of supersolvability (Definition 4).

2.1 Generalities about posets

Let $\mathcal{P}$ be a partially ordered set (or "poset") with partial order relation $\leq$. For $x, y \in \mathcal{P}$ write $x < y$ when $x \leq y$ and $x \neq y$, and $x \lessdot y$ when $x < y$ and $x \leq z < y$ implies $x = z$. Given any $x \in \mathcal{P}$ let $\mathcal{P}_{<x} := \{y \in \mathcal{P} \colon y < x\}$, partially ordered by the restriction of $<$. The posets $\mathcal{P}_{\leq x}$, $\mathcal{P}_{>x}$ and $\mathcal{P}_{\geq x}$ are defined analogously. The *interval* between two elements $x, y \in \mathcal{P}$ is the set $[x, y] := \mathcal{P}_{\geq x} \cap \mathcal{P}_{\leq y}$. We refer to [20] for standard poset terminology and notation. Departing slightly from standard notation, for any two elements $x, y \in \mathcal{P}$, we define $x \vee y$ to be the *set* of minimal upper bounds and $x \wedge y$ to be the set of maximal lower bounds. That is:

$$x \vee y := \min\{z \in \mathcal{P} \colon z \geq x \text{ and } z \geq y\}, \quad x \wedge y := \max\{z \in \mathcal{P} \colon z \leq x \text{ and } z \leq y\}.$$

More generally, denote by $\bigvee T$ and $\bigwedge T$ the sets of minimal upper bounds and maximal lower bounds of a set $T \subseteq \mathcal{P}$.

A *complement* of an element x in a chain-finite poset $\mathcal{P}$ is any $z \in \mathcal{P}$ such that $x \vee z \subseteq \max \mathcal{P}$ and $x \wedge z \subseteq \min \mathcal{P}$. Given a subset $X \subseteq \mathcal{P}$ we say that $z \in \mathcal{P}$ is a complement to X if z is a complement of every $x \in X$.[2]

2.2 Locally geometric posets

Recall that a chain-finite lattice $\mathcal{L}$ is called **geometric** if and only if, for all $x, y \in \mathcal{L}$:

$$x \lessdot y \text{ if and only if there is an atom } a \in A(\mathcal{L}) \text{ with } a \not\leq x,\ y = x \vee a.$$

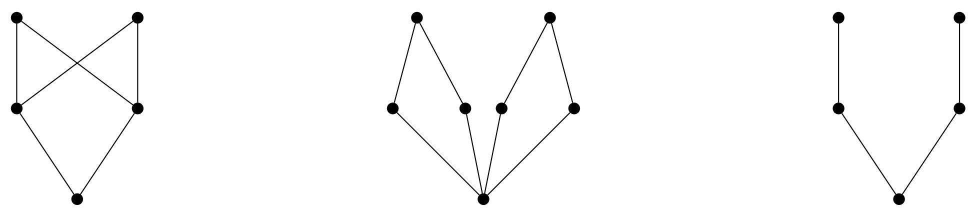
Definition 1 (Locally geometric and geometric posets)**.** A graded, bounded below poset $\mathcal{P}$ is **locally geometric** if, for every $x \in \mathcal{P}$, the subposet $\mathcal{P}_{\leq x}$ is a geometric lattice. A locally geometric poset $\mathcal{P}$ is **geometric** if for all $x, y \in \mathcal{P}$:

(‡‡) if $\mathrm{rk}(x) < \mathrm{rk}(y)$ and $I \subseteq A(\mathcal{P})$ is such that $\bigvee I \ni y$ and $|I| = \mathrm{rk}(y)$, then there is $a \in I$ such that $a \not\leq x$ and $a \vee x \neq \emptyset$.

Remark 1. We do not require $\mathcal{P}$ itself to even be a (semi)lattice. If $\mathcal{P}$ is a lattice, then it is locally geometric if and only if it is geometric. A geometric (semi)lattice in the sense of [22] is precisely a (semi)lattice satisfying condition (‡‡). Further note that if $\mathcal{P}$ is locally geometric, then so are $\mathcal{P}_{\leq x}$ and $\mathcal{P}_{\geq x}$ for any $x \in \mathcal{P}$.

[2] Notice that this definition generalizes the usual one for lattices.

Example 1. A classical example of a geometric lattice is a Boolean lattice B_n, the set of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion. A simplicial poset, in which every closed interval is isomorphic to a Boolean lattice, is then a locally geometric poset. One such example is depicted in Figure 1a: this is a geometric poset that is not a lattice nor a semilattice.



(a) A geometric poset.

(b) A locally geometric but non-geometric poset

(c) A ranked, but not locally geometric, poset.

Figure 1

Example 2. The poset of Figure 1b is locally geometric but not geometric.

2.3 Supersolvable geometric lattices

There are several equivalent definitions for a modular element in a geometric lattice (see eg. [7, Theorem 3.3]). The one we state below is the most useful for our setting and is due to Stanley [19]. We also extend Stanley's definition of supersolvable lattices [20, Corollary 2.3] to the context of *chain-finite* lattices.

Let $\mathcal{L}$ be a chain-finite lattice. Then $\mathcal{L}$ has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Let $x \in \mathcal{L}$. The *complements* of x in $\mathcal{L}$ are the elements $y \in \mathcal{L}$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$.

Definition 2. An element x in a geometric lattice $\mathcal{L}$ is **modular** if the complements of x form an antichain. A geometric lattice $\mathcal{L}$ is **supersolvable** if there is a chain $\hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$ where each y_i is a modular element with $\mathrm{rk}(y_i) = i$.

2.4 Ideals in locally geometric posets

Let $\mathcal{P}$ be a locally geometric poset. An *order ideal* in $\mathcal{P}$ is a downward-closed subset. An order ideal is *pure* if all maximal elements have the same rank. An order ideal $\mathcal{Q}$ is *join-closed* if $T \subseteq \mathcal{Q}$ implies $\bigvee T \subseteq \mathcal{Q}$.

Definition 3 (M-ideals and TM-ideals)**.** An **M-ideal** of a locally geometric poset $\mathcal{P}$ is a pure, join-closed order ideal $\mathcal{Q} \subseteq \mathcal{P}$ such that:

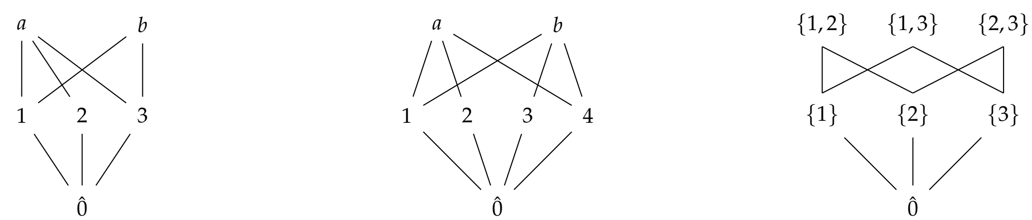
(1) if $y \in \mathcal{Q}$ and $a \in A(\mathcal{P})$ such that $a \vee y = \emptyset$ then $a \in \mathcal{Q}$, and

(2) for every $x \in \max(\mathcal{P})$, there is some $y \in \max(\mathcal{Q})$ such that y is a modular element in the geometric lattice $\mathcal{P}_{\leq x}$.

An M-ideal $\mathcal{Q}$ in a locally geometric poset $\mathcal{P}$ is a **TM-ideal** if $|a \vee y| = 1$ for all $y \in \mathcal{Q}$ and all $a \in A(\mathcal{P}) \setminus A(\mathcal{Q})$.

Remark 2. Our definition of an M-ideal extends Definition 2: An order ideal $\mathcal{Q}$ in a geometric lattice $\mathcal{L}$ is an M-ideal if and only if $\mathcal{Q} = \mathcal{L}_{\leq y}$ for some modular element y.

Example 3. Consider the poset $\mathcal{P}$ in Figure 2b. The subposet $\{\hat{0}, 2, 3\}$ is a TM-ideal; the subposet $\{\hat{0}, 4\}$ is an M-ideal that is not a TM-ideal. Note that in every locally geometric poset $\mathcal{P}$, both $\mathcal{P}$ and $\{\hat{0}\}$ are M-ideals.



(a) A supersolvable, but not strictly supersolvable, poset.

(b) A strictly supersolvable poset.

(c) A "locally" supersolvable, but not supersolvable, poset.

Figure 2

2.5 Supersolvability in geometric posets

We are now prepared to present our definition of a supersolvable locally geometric poset, which extends the definition of a supersolvable geometric lattice (cf. Definition 2).

Definition 4. A locally geometric poset $\mathcal{P}$ is **supersolvable** if there is a chain

$$\hat{0} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_n = \mathcal{P}$$

where each $\mathcal{Q}_i$ is an M-ideal of $\mathcal{Q}_{i+1}$ with $\operatorname{rk}(\mathcal{Q}_i) = i$. If moreover every $\mathcal{Q}_i$ is a TM-ideal of $\mathcal{Q}_{i+1}$ we call $\mathcal{P}$ **strictly supersolvable**.

Example 4. Any rank-one locally geometric poset is strictly supersolvable. The poset $\mathcal{P}$ from Example 3 is strictly supersolvable via the chain $\hat{0} \subset \{\hat{0}, 2, 3\} \subset \mathcal{P}$.

Example 5. The poset $\mathcal{P}$ from Figure 2a is not strictly supersolvable: its only proper M-ideals are $\mathcal{P}_{\leq 1}$ and $\mathcal{P}_{\leq 3}$, and the fact that $|1 \vee 3| = 2$ means neither is a TM-ideal.

If a locally geometric poset is supersolvable, then every closed interval $\mathcal{P}_{\leq x}$ is a supersolvable geometric lattice. However, this "local" supersolvability is not enough for $\mathcal{P}$ itself to be supersolvable, as demonstrated in the following example.

Example 6. Consider the poset $\mathcal{P}$ depicted in Figure 2c. Every closed interval in $\mathcal{P}$ is supersolvable (since every Boolean lattice is), however it is not itself supersolvable. Indeed, the only proper order ideals which are pure and join-closed are principal, that is, $\mathcal{P}_{\leq x}$ for some rank-one element x. However, such an order ideal cannot satisfy Definition 3.(2) since no single element is covered by all maximal elements.

Remark 3. A geometric lattice $\mathcal{L}$ satisfies Definition 4 if and only if it satisfies the supersolvability criterion of Definition 2. In a geometric semilattice $\mathcal{L}$, M-ideals and TM-ideals are equivalent, thus $\mathcal{L}$ is supersolvable if and only if it is strictly supersolvable.

For geometric posets, M-ideals can be characterized using partitions of atoms [2, Theorem 4.1.2.], providing the following characterization of supersolvability, reminiscent of [15, Remark 2.6] for geometric semilattices.

Proposition 1. *([2, Corollary 4.1.3]) Let $\mathcal{P}$ be a geometric poset. Then $\mathcal{P}$ is supersolvable if and only if there is a chain $\{\hat{0}\} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \ldots \subset \mathcal{Q}_n = \mathcal{P}$ of pure, join-closed order ideals of $\mathcal{P}$ with $\mathrm{rk}(\mathcal{Q}_i) = i$ and so that for every $i = 1, \ldots, n$ and any two distinct $a_1, a_2 \in A(\mathcal{Q}_i) \setminus A(\mathcal{Q}_{i-1})$ and every $x \in a_1 \vee a_2$ there is $a_3 \in A(\mathcal{Q}_{i-1})$ with $x > a_3$.*

Example 7. Dowling posets [3] form a class of locally geometric posets that generalize partition lattices and Dowling lattices, which are known to be supersolvable geometric lattices [11, 20]. We can show [2, Proposition 2.6.1.] that, for any positive integer n, finite group G, and finite G-set S, the Dowling poset $D_n(G, S)$ is strictly supersolvable.

2.6 Group actions

Let G be a group. An *action* of G on a poset $\mathcal{P}$ is any group homomorphism $G \to \mathrm{Aut}(\mathcal{P})$ from G to the group of automorphisms of $\mathcal{P}$. Given a group element $g \in G$ it is customary to denote the associated automorphism by $g : \mathcal{P} \to \mathcal{P}$. For $x \in \mathcal{P}$ we will write gx for $g(x)$. Following [10], we focus on the following special type of action.

Definition 5. Let $\mathcal{P}$ be a poset with an action of a group G. We call the action **translative** if $x \vee gx \neq \emptyset$ implies $x = gx$ for every $x \in \mathcal{P}$ and every $g \in G$.

Write $Gx = \{gx \colon g \in G\}$ for the orbit of an $x \in \mathcal{P}$ under G. On the set of orbits $\mathcal{P}/G := \{Gx \colon x \in \mathcal{P}\}$ we consider the relation given by $Gx \leq Gy$ if there is $g \in G$ with $x \leq gy$. If the action is translative, this is a partial order relation on $\mathcal{P}/G$.

Theorem 1. *Let $\mathcal{P}$ be a locally geometric poset with a translative action of a group G and let $\mathcal{Q}$ be a G-invariant subposet of $\mathcal{P}$. If $\mathcal{Q}$ is an M-ideal in $\mathcal{P}$, then $\mathcal{Q}/G$ is an M-ideal in $\mathcal{P}/G$. Moreover, if $\mathcal{P}$ satisfies (‡‡), the converse also holds.*

2.7 Characteristic polynomial

The *characteristic polynomial* of any bounded-below poset $\mathcal{P}$ with a rank function rk is

$$\chi_{\mathcal{P}}(t) := \sum_{x \in \mathcal{P}} \mu_{\mathcal{P}}(x) t^{\mathrm{rk}(\mathcal{P}) - \mathrm{rk}(x)},$$

where $\mu_{\mathcal{P}}$ is the Möbius function of $\mathcal{P}$. A feature of supersolvable geometric lattices is that their characteristic polynomial decomposes into linear factors over $\mathbb{Z}$. We show that this is true also for *strictly* supersolvable posets.

Theorem 2. *Let $\mathcal{Q}$ be a TM-ideal of a locally geometric poset $\mathcal{P}$ with $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{P}) - 1$, and let $a = |A(\mathcal{P}) \setminus A(\mathcal{Q})|$. Then*

$$\chi_{\mathcal{P}}(t) = \chi_{\mathcal{Q}}(t) \cdot (t - a).$$

In particular, if $\mathcal{P}$ is strictly supersolvable via the chain of TM-ideals $\hat{0} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_n = \mathcal{P}$, and $a_i = |A(\mathcal{Q}_i) \setminus A(\mathcal{Q}_{i-1})|$ for each i, then

$$\chi_{\mathcal{P}}(t) = \prod_{i=1}^{n} (t - a_i).$$

Remark 4. The assumption that $\mathcal{Q}$ is a TM-ideal in Theorem 2 is necessary, as demonstrated in the following examples. Accordingly, a poset being supersolvable is not enough for its characteristic polynomial to factor completely over $\mathbb{Z}$.

Example 8. Consider the poset $\mathcal{P}$ depicted in Figure 2 (see also Example 12). Its characteristic polynomial is

$$\chi_{\mathcal{P}}(t) = t^2 - 4t + 4 = (t-2)(t-2).$$

This agrees with the fact that the TM-ideal $\mathcal{Q} = \{\hat{0}, 2, 3\}$ in Figure 2b has $\chi_{\mathcal{Q}}(t) = t - 2$ and $|A(\mathcal{P}) \setminus A(\mathcal{Q})| = 2$.

Example 9. Consider again the poset $\mathcal{P}$ in Figure 2a. It is supersolvable, with $\{\hat{0}, 1\}$ and $\{\hat{0}, 3\}$ both M-ideals. However, it is not strictly supersolvable and its characteristic polynomial $\chi_{\mathcal{P}}(t) = t^2 - 3t + 3$ does not factor over the integers.

3 Toric Arrangement Bundles

3.1 Toric Arrangements

Fix a finitely generated free abelian group $\Gamma \cong \mathbb{Z}^d$ and let $T = \mathrm{Hom}(\Gamma, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^d$ be the complex torus.

Definition 6. A **toric arrangement** is a collection $\{H_\alpha \colon \alpha \in \mathcal{A}\}$ for some finite set $\mathcal{A} \subseteq \Gamma$, where

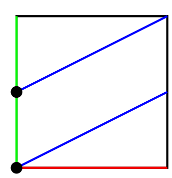
$$H_\alpha := \{t \in T \colon t(\alpha) = 0\}.$$

We will often refer to an arrangement $\{H_\alpha \colon \alpha \in \mathcal{A}\}$ simply by $\mathcal{A}$ when there is no confusion. The **complement** of $\mathcal{A}$ is denoted by

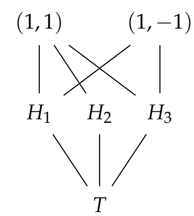
$$M(\mathcal{A}) := T \setminus \bigcup_{\alpha \in \mathcal{A}} H_\alpha.$$

We only consider toric arrangements that are **essential**, i.e., where $\mathcal{A}$ generates a full subgroup of Γ.

Example 10. Let $\Gamma = \mathbb{Z}^2$. The arrangement $\mathcal{A} = \{\alpha_1 = (1,0), \alpha_2 = (0,1), \alpha_3 = (1,2)\}$ yields three subtori in $\mathbb{C}^\times \times \mathbb{C}^\times$, cut out by equations $x = 1$, $y = 1$, and $xy^2 = 1$. The real part, in $S^1 \times S^1$, is depicted in Figure 3a.



(a) The arrangement $\mathcal{A}$ from Example 10 is depicted in $S^1 \times S^1$, with H_1 in green, H_2 in red, and H_3 in blue.



(b) The poset of layers $\mathcal{P}(\mathcal{A})$.

Figure 3

3.2 Poset of layers

The intersection data of a toric arrangement may be encoded in a geometric poset.

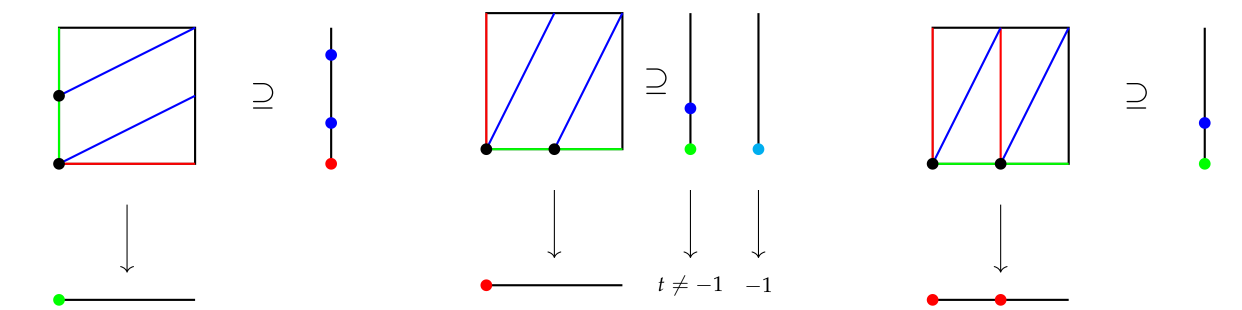
Definition 7. The **poset of layers** of an arrangement $\mathcal{A}$ is the set $\mathcal{P}(\mathcal{A})$ whose elements are the nonempty connected components of intersections $\bigcap_{\alpha \in S} H_\alpha$ where $S \subseteq \mathcal{A}$, partially ordered by reverse inclusion.

By convention, T is the unique minimum element of $\mathcal{P}(\mathcal{A})$. The atoms of $\mathcal{P}(\mathcal{A})$ are precisely the connected components of the H_α, where $\alpha \in \mathcal{A}$.

Remark 5. The poset of layers for a toric arrangement is indeed a geometric poset. The lift of all H_α, $\alpha \in \mathcal{A}$, to the universal cover of T is an arrangement $\mathcal{A}^\upharpoonright$ of affine subspaces in $\mathbb{C}^d$. Its poset of layers $\mathcal{P}(\mathcal{A}^\upharpoonright)$ is a geometric semilattice and the action on $\mathcal{A}^\upharpoonright$ of the group of deck transformations induces a translative action of $\mathbb{Z}^d$ on $\mathcal{P}(\mathcal{A}^\upharpoonright)$. Then $\mathcal{P}(\mathcal{A})$ is isomorphic to the quotient $\mathcal{P}(\mathcal{A}^\upharpoonright)/\mathbb{Z}^d$ (see [10, Lemma 9.8]), and thus it is geometric (via Theorem 1).

Example 11. Let $\mathcal{A}$ be the toric arrangement from Example 10. The Hasse diagram for its poset of layers $\mathcal{P}(\mathcal{A})$ is depicted in Figure 3b. Notice that this poset was seen in Figure 2a and is supersolvable, with M-ideal given by $\mathcal{P}_{\leq H_1}$ or $\mathcal{P}_{\leq H_3}$ (see Example 5).

Example 12. Let $\Gamma = \mathbb{Z}^2$ and $\mathcal{A} = \{\alpha_1 = (1,0), \alpha_2 = (0,2), \alpha_3 = (1,2)\}$. Figure 4c depicts the corresponding H_1, H_2, and H_3 in $S^1 \times S^1$ and Figure 2b depicts the Hasse diagram for its poset of layers. As seen in Example 3 this poset is strictly supersolvable; the maximal elements of its proper TM-ideal are the two connected components of H_2.



(a) A fibration whose fibers are homeomorphic to S^1 with three punctures.

(b) This is not a fibration, as indicated by the two non-homeomorphic fibers.

(c) A fibration whose fibers are homeomorphic to S^1 with two punctures.

Figure 4: Each figure represents a restriction of the projection $S^1 \times S^1 \to S^1$.

3.3 Characterization of fibrations

A subgroup Y of T will be called **admissible** if there is a rank-one direct summand $\Gamma' \subseteq \Gamma$ such that Y is the image of the injection $\varepsilon^* : \mathrm{Hom}(\Gamma', \mathbb{C}^\times) \to \mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ induced by the projection $\varepsilon : \Gamma \to \Gamma'$. When Y is admissible, the corresponding projection

$$p : T \to T/Y \cong \mathrm{Hom}(\Gamma/\Gamma', \mathbb{C}^\times)$$

is a section of the map induced by the quotient $q : \Gamma \to \Gamma/\Gamma'$. This allows us to define toric arrangements

$$\mathcal{A}_Y := \{\alpha \in \mathcal{A} \colon H_\alpha \supseteq Y\} \qquad\qquad \mathcal{A}/Y := q(\mathcal{A}_Y) \subseteq \Gamma/\Gamma',$$

in T and in T/Y, respectively. Then the projection $p : T \to T/Y$ restricts to a map on arrangement complements $\bar{p} : M(\mathcal{A}) \to M(\mathcal{A}/Y)$ and induces an isomorphism of posets $\mathcal{P}(\mathcal{A}_Y) \cong \mathcal{P}(\mathcal{A}/Y)$.

Definition 8. An arrangement $\mathcal{A}$ is **fiber-type** if there is a chain of subgroups $Y_1 \subseteq \ldots Y_{d-1} \subseteq Y_d = (\mathbb{C}^\times)^d$ such that for every projection $p_i : Y_i \to Y_i/Y_{i-1}$ the induced map $\bar{p}_i$ is a fibration on arrangement complements whose fiber is homeomorphic to $\mathbb{C}^\times$ minus a finite set of points.

Remark 6. The poset of layers $\mathcal{P}(\mathcal{A}_Y)$ may be viewed as a subposet of $\mathcal{P}(\mathcal{A})$. Its atoms are the atoms of $\mathcal{A}$ that either contain Y or are disjoint from it. For any $\alpha \notin \mathcal{A}_Y$, every connected component of H_α will intersect Y nontrivially. Moreover, if $Y \in \mathcal{P}(\mathcal{A})$, then the maximal elements of $\mathcal{P}(\mathcal{A}_Y)$ are cosets of Y.

We prove in [2, Theorem 3.3.1.] that $\mathcal{P}(\mathcal{A}_Y)$ is an M-ideal of $\mathcal{P}(\mathcal{A})$ if and only if there is an integer ℓ such that the fibers of the projection $M(\mathcal{A}) \to M(\mathcal{A}/Y)$ are all homeomorphic to $\mathbb{C}^\times$ with ℓ points removed. The number of punctures can be counted by examining how the hypersurfaces not in $\mathcal{P}(\mathcal{A}_Y)$ intersect Y or its translates. When $\mathcal{P}(\mathcal{A}_Y)$ is an M-ideal, the map is moreover locally trivial. Iterating this then yields:

Theorem 3 (Fibration Theorem [2, Theorem A])**.** *An essential toric arrangement is fiber-type if and only if its poset of layers is supersolvable.*

Example 13. Consider the arrangement from Example 10 (see also Figure 2a). Letting $Y = H_1$, the projection $M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is depicted in Figure 4a. As the picture suggests, this map is a fibration with fiber homeomorphic to T with three points removed. On the other hand, letting $Y = H_2$ the projection $M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is not a fibration. This is evident in Figure 4b, which depicts two non-homeomorphic fibers.

This agrees with our observation in Example 3 that in the poset of layers $\mathcal{P} = \mathcal{P}(\mathcal{A})$, the order ideal $\mathcal{P}_{\leq H_1}$ is an M-ideal while $\mathcal{P}_{\leq H_2}$ is not.

From Theorem 3, Falk and Randell's arguments in [14] can be adapted to prove the following results.

Theorem 4 (Asphericity, [2, Corollary B])**.** *If the poset of layers of a toric arrangement is supersolvable, then the arrangement complement is a $K(\pi, 1)$ space. If the poset is strictly supersolvable, then the fundamental group is an iterated semidirect product of free groups.*

Theorem 5 (Lower Central Series Formula, [2, Theorem D])**.** *Let $\mathcal{A}$ be a strictly supersolvable toric arrangement with complement $M(\mathcal{A})$, let $\mathcal{A}_0 = \emptyset \subsetneq \mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_n$ be the associated tower of arrangements and set $a_i := |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|$ for $i = 1, \ldots, n-1$. For $j \geq 1$, let φ_j be the rank of the jth successive quotient in the lower central series of the fundamental group $\pi_1(M(\mathcal{A}))$. Then*

$$\prod_{j=1}^{\infty}(1 - t^j)^{\varphi_j} = \prod_{i=1}^{n}(1 - (a_i + 1)t). \tag{3.1}$$

The right-hand side of (3.1) encodes the Betti numbers, and is a specialization of the characteristic polynomial for the associated poset of layers (see Theorem 2).

3.4 Pullback of Fadell–Neuwirth bundles

Suppose that $\bar{p}\colon M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is a toric arrangement bundle arising from a TM-ideal $\mathcal{Q} = \mathcal{P}(\mathcal{A}_Y)$, and fix an order $H_1, H_2, \ldots, H_\ell$ of the atoms in $\mathcal{P}(\mathcal{A})$ that are not in $\mathcal{Q}$. The definition of a TM-ideal implies that for any $x \in M(\mathcal{A}/Y)$, and any $1 \leq i \leq \ell$, there is a unique point in $H_i \cap p^{-1}(x)$. Through the identification $p^{-1}(x) \cong \mathbb{C}^\times \subseteq \mathbb{C}$, this defines a continuous map $g_i\colon M(\mathcal{A}) \to \mathbb{C}$. Via Proposition 1, the points $g_1(x), \ldots, g_\ell(x)$ must be distinct and nonzero, thus determining a point in the ordered configuration space $\mathrm{Conf}_{\ell+1}(\mathbb{C}) = \{(z_0, \ldots, z_\ell) \in \mathbb{C}^{\ell+1}\colon z_i \neq z_j \text{ when } i \neq j\}$.

In fact, the bundle $\bar{p}$ is pulled back from Fadell–Neuwirth's bundle of configuration spaces (as in [13]) through this map g. Consequently, properties of Fadell–Neuwirth's bundles (eg. existence of a section, trivial homological monodromy) may thus be pulled back to obtain properties of toric arrangement bundles.

Theorem 6 ([2, Theorem 5.3.1]). *The map $g : M(\mathcal{A}) \to \mathrm{Conf}_{\ell+1}(\mathbb{C})$ given by $g(x) = (0, g_1(x), \ldots, g_\ell(x))$ is continuous and yields the following pullback diagram.*

$$\begin{tikzcd} M(\mathcal{A}) \arrow[r, "h"] \arrow[d, "\bar{p}"] & \mathrm{Conf}_{\ell+2}(\mathbb{C}) \arrow[d, "\pi"] \\ M(\mathcal{A}/Y) \arrow[r, "g"] & \mathrm{Conf}_{\ell+1}(\mathbb{C}) \end{tikzcd}$$

Example 14. Let $\mathcal{A}$ be the arrangement of Example 12. The fiber bundle depicted in Figure 4c is pulled back through the map $g : M(\mathcal{A}/Y) \to \mathrm{Conf}_3(\mathbb{C})$, $g(x) = (0, 1, x^2)$.

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Levi-spherical varieties and Demazure characters

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Abstract. We prove a short, root-system uniform, combinatorial classification of Levi-spherical Schubert varieties for any generalized flag variety G/B of finite Lie type. We apply this to the study of multiplicity-free decompositions of Demazure modules and their characters.

Keywords: Demazure characters, multiplicity-free, Schubert varieties, Levi subgroups, spherical varieties, toric varieties

1 Introduction

1.1 History and motivation

In his essay [17] on representation theory and invariant theory, R. Howe discusses the significance of multiplicity-free actions as an organizing principle for the subject. Classical invariant theory focuses on actions of a reductive group G on symmetric algebras, which is to say, coordinate rings of vector spaces. Now one also considers G-actions on varieties X and their coordinate rings $\mathbb{C}[X]$. Such an action is multiplicity-free if $\mathbb{C}[X]$ decomposes, as a G-module, into irreducible G-modules each with multiplicity one. An important example is when X is the *base affine space* of a complex, semisimple algebraic group G [3]; in this case the coordinate ring is a multiplicity-free direct sum of the irreducible representations of G. Lustzig's theory of dual canonical bases [24] provides a basis for it. In the early 2000s, understanding this basis was a motivation for S. Fomin and A. Zelevinsky's theory of Cluster algebras [11].

The notion of multiplicity-free actions is closely connected to that of *spherical varieties*. Let G be a connected, complex, reductive algebraic group; we say that a variety X is a G-variety if X is equipped with an action of G that is a morphism of varieties. A spherical variety is a normal G-variety where a Borel subgroup of G has an open, and therefore dense, orbit. A normal, affine G-variety X is spherical if and only if $\mathbb{C}[X]$ decomposes into irreducible G-modules each with multiplicity one [31]. If X is instead a normal,

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projective G-variety then one can still recover one direction of this implication. That is, if the induced G-action on the homogeneous coordinate ring of X is multiplicity-free, then X is G-spherical [15, Proposition 4.0.1].

Spherical varieties possess numerous nice properties. For instance, projective spherical varieties are Mori Dream Spaces. Moreover, Luna-Vust theory describes all the birational models of a spherical variety in terms of colored fans; these fans generalize the notion of fans used to classify toric varieties (which are themselves spherical varieties). N. Perrin's excellent survey covers additional background on spherical varieties [27].

It is an open problem to classify all spherical actions on products of flag varieties. This is solved in the case of Levi subgroups; we point to the work of P. Littelmann [23], P. Magyar–J. Weyman–A. Zelevinsky [25, 26], J. Stembridge [29, 30], R. Avdeev–A. Petukhov [1, 2]. Connecting back to the representation-theoretic perspective of [17], in [29, 30], J. Stembridge relates this classification problem to the multiplicity-freeness of restrictions of *Weyl modules* [12, Lecture 6]. Indeed, the homogeneous coordinate ring of a single flag variety is a multiplicity-free sum of spaces of global sections on the variety with respect to line bundles associated to each dominant integral weight. By the Borel-Weil-Bott theorem, these spaces are isomorphic to the irreducible representations of G. This is the central object of interest in *Standard Monomial Theory* [22] and is closely related to the coordinate ring of base affine space mentioned above. As remarked above a product of flag varieties is G-spherical if its homogeneous coordinate ring is multiplicity-free as an G-module.

This paper solves a related problem. We classify all *Levi-spherical* Schubert varieties in a single flag variety; that is, Schubert varieties that are spherical for the action of a Levi subgroup. Here, the relevant ring is the homogeneous coordinate ring of a Schubert variety and the attendant representation theory is that of *Demazure modules* [10], which are Borel subgroup representations. Critically for this paper, they are also Levi subgroup representations. Multiplicity-freeness in this setting refers to the decomposition of these modules into irreducible Levi subgroup representations. This study was initiated in [16] and the authors solved the problem for the GL_n case in [14]. In [13] we conjectured an answer for all finite rank Lie types; this paper proves that conjecture.[1]

1.2 Background

Throughout, let G be a complex, connected, reductive algebraic group and let $B \leq G$ be a choice of Borel subgroup along with a maximal torus T contained in B. The *Weyl group* is $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. The orbits of the homogeneous space G/B under the action of B by left translations are the *Schubert cells*

[1]During the completion of this article, we learned that M. Can-P. Saha [4] independently proved the conjecture.

$X^\circ_w, w \in W$. Their Zariski closures

$$X_w := \overline{X^\circ_w}$$

are the *Schubert varieties*. It is relevant below that these varieties are normal [9, 28].

A *parabolic subgroup* of G is a closed subgroup $B \subset P \subsetneq G$ such that G/P is a projective variety. Each such P admits a *Levi decomposition*

$$P = L \ltimes R_u(P)$$

where L is a reductive subgroup called a *Levi subgroup* of P and $R_u(P)$ is the unipotent radical. One parabolic subgroup is $P_w := \mathrm{stab}_G(X_w)$. Any of the parabolic subgroups $B \subseteq Q \subseteq P_w$ act on X_w.

Let L_Q be a Levi subgroup of Q. A variety X is *H-spherical* for the action of a complex reductive algebraic group H if it is normal and contains an open, and therefore dense, orbit of a Borel subgroup of H. Our reference for spherical varieties is [27]; toric varieties are examples of spherical varieties.

Definition 1.1 ([16, Definition 1.8])**.** Let $B \subseteq Q \subseteq P_w$ be a parabolic subgroup of G. $X_w \subseteq G/B$ is *L_Q-spherical* if X_w has a dense, open orbit of a Borel subgroup of L_Q under left-translations.

1.3 The main result

We give a root-system uniform combinatorial criterion to decide if X_w is L_Q-spherical. Let $\Phi := \Phi(\mathfrak{g}, T)$ be the root system of weights for the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G. It has a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\Delta \subset \Phi^+$ be the base of simple roots. The parabolic subgroups $Q = P_I \supset B$ are in bijection with subsets I of Δ; let $L_I := L_Q$. The set of *left descents* of w is

$$\mathcal{D}_L(w) = \{\beta \in \Delta : \ell(s_\beta w) < \ell(w)\},$$

where $\ell(w) = \dim X_w$ is the *Coxeter length* of w. Under the bijection, $P_w = P_{\mathcal{D}_L(w)}$, and $B \subset Q \subseteq P_w = P_{\mathcal{D}_L(w)}$ satisfy $Q = P_I$ for some $I \subseteq \mathcal{D}_L(w)$.

For $I \subset \Delta$, a *parabolic subgroup* $W_I \subseteq W$ is the subgroup generated by $S_I := \{s_\beta : \beta \in I\}$. A *standard Coxeter element* $c \in W_I$ is any product of the elements of S_I listed in some order. Let $w_0(I)$ be the longest element of W_I. The following definition was given in type A in [14, Definition 1.1] and in general type in [13, Section 4]:

Definition 1.2. Let $w \in W$ and $I \subseteq \mathcal{D}_L(w)$ be fixed. Then w is I-spherical if $w_0(I)w$ is a standard Coxeter element for W_J where $J \subseteq \Delta$.

We first note that if $I \subseteq \mathcal{D}_L(w)$, then the left inversion set $\mathcal{I}(w)$, defined in Section 3, contains all the positive roots in the root subsystem generated by I, and thus $w = w_0(I)d$ is a length-additive expression for some $d \in W$, by Proposition 3.1.3 in [5].

Theorem 1.3. *Fix $w \in W$ and $I \subseteq \mathcal{D}_L(w)$. X_w is L_I-spherical if and only if w is I-spherical.*

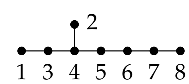
Theorem 1.3 resolves the main conjecture of the authors' earlier work [13, Conjecture 4.1] and completes the main goal set forth in [16]. In [14], Theorem 1.3 was established in the case $G = GL_n$ using essentially algebraic combinatorial methods concerning *Demazure characters* (or in their type A embodiment, the *key polynomials*). In contrast, the geometric arguments of this paper are quite different, significantly shorter, but require more background of the reader in algebraic groups. Theorem 1.3 is a generalization of work of P. Karuppuchamy [21] that characterizes Schubert varieties that are toric, which in our setup means spherical for the action of $L_\emptyset = T$. Using work of R. S. Avdeev–A. V. Petukhov [1], Theorem 1.3 may also be seen as a generalization of some results of P. Magyar–J. Weyman–A. Zelevinsky [25] and J. Stembridge [29, 30] on spherical actions on G/B; see [16, Theorem 2.4]. Previously, there was not even a finite algorithm to decide L_I-sphericality of X_w in general.

1.4 Organization

Examples of the main result are given in Section 2. Sections 3 and 4 prove Theorem 1.3. Section 5 applies our main result to the study of Demazure modules [10].

2 Examples of Theorem 1.3

We begin with a few examples illustrating Theorem 1.3.

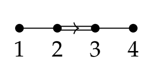
Example 2.1 (E_8 cf. [16, Example 1.3]). The E_8 Dynkin diagram is . One associates the simple roots β_i ($1 \le i \le 8$) with this labeling and writes $s_i := s_{\beta_i}$. Suppose

$$w = s_2s_3s_4s_2s_3s_4s_5s_4s_2s_3s_1s_4s_5s_6s_7s_6s_8s_7s_6 \in W.$$

Then $\mathcal{D}_L(w) = \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_8\}$. Let $I = \mathcal{D}_L(w)$. Here

$$w_0(I) = s_3s_2s_4s_3s_2s_4s_5s_4s_3s_2s_4s_5 \cdot s_7s_8s_7 \text{ and } w_0(I)w = s_1s_6s_7s_8.$$

Since $w = w_0(I)c$ where $c = s_1s_6s_7s_8$ is a standard Coxeter element, Theorem 1.3 asserts that X_w is L_I-spherical in the complete flag variety for E_8.

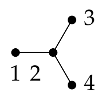
Example 2.2 (F_4 cf. [16, Example 1.5]). The F_4 diagram is . First suppose

$$w = s_4s_3s_4s_2s_3s_4s_2s_3s_2s_1s_2s_3s_4 \; (I = \mathcal{D}_L(w) = \{\beta_2, \beta_3, \beta_4\}).$$

Then $w_0(I) = s_2s_3s_2s_3s_4s_3s_2s_3s_4$ and $w_0(I)w = s_1s_2s_3s_4$ is standard Coxeter. Hence X_w is L_I-spherical. On the other hand if

$$w' = s_2s_1s_4s_3s_2s_1s_3s_2s_4s_3s_2s_1 \ (I = \mathcal{D}_L(w') = \{\beta_2, \beta_4\}),$$

then $w_0(I) = s_2s_4$ and $w_0(I)w = s_1s_3s_2s_1s_3s_2s_4s_3s_2s_1$ is not standard Coxeter and X_w is not L_I-spherical.

Example 2.3 (D_4)**.** The D_4 diagram is 

. Let

$$w = s_3s_2s_3s_4s_2s_1s_2 \ (I = \mathcal{D}_L(w) = \{\beta_2, \beta_3\}).$$

Thus $w_0(I) = s_2s_3s_2$ and $w_0(I)w = s_4s_2s_1s_2$ is not standard Coxeter. Hence X_w is not L_I-spherical. The interested reader can check w is I-spherical in the (different) sense of [16, Definition 1.2]. Therefore, this w provides a counterexample to [16, Conjecture 1.9] in type D_4. This counterexample was also (implicitly) verified in [13] using a different method, namely Demazure character computations, the topic of Section 5.

3 An equivariant isomorphism

The primary goal of this section is to construct a torus equivariant isomorphism from a specified affine subspace of the open cell of a Schubert variety to the open cell of a distinguished Schubert subvariety. In what follows, we assume standard facts from the theory of algebraic groups. References we draw upon are [18, 6, 22].

Let $w \in W$. Let n_w be a coset representative of w in $N_G(T)$. By definition of $N_G(T)$ being the normalizer of T in G, $t \mapsto n_w t n_w^{-1}$ defines an automorphism $\gamma_w : T \to T$.

Lemma 3.1. *The automorphism γ_w does not depend on our choice of coset representative n_w.*

In light of Lemma 3.1, henceforth for $w \in W$ we will also let w denote a coset representative of w in $N_G(T)$. Let X be a T-variety with action denoted by $\cdot$. For each $w \in W$ we define an action $\cdot_w$ on X by $t \cdot_w x = \gamma_w(t) \cdot x$ for all $x \in X$ and $t \in T$.

Lemma 3.2. *For all $w \in W$, the T-variety X has an open, dense T-orbit for the action $\cdot$ if and only if it has an open, dense T-orbit for the action $\cdot_w$. Indeed, the set of T-orbits in X for these two actions is identical.*

For the remainder, we fix $\cdot$ to be the restriction to T of the action of G on G/B by left multiplication. The *left inversion set* of $w \in W$ is

$$\mathcal{I}(w) := \Phi^+ \cap w(\Phi^-) = \{\alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^-\}.$$

Recall two standard facts regarding left inversion sets [19, Chapter 1]. For $w \in W$,

$$|\mathcal{I}(w)| = \ell(w) = \dim_{\mathbb{C}} X_w, \tag{3.1}$$

and

$$\mathcal{I}(w_0(I)) = \Phi^+(I), \tag{3.2}$$

where $\Phi(I) = \Phi(\mathfrak{l}_I, T)$ is the root system for the adjoint action of T on $\mathfrak{l}_I = \mathrm{Lie}(L_I)$.

We say that an algebraic group H is *directly spanned* by its closed subgroups $H_1, \ldots, H_n$, in the given order, if the product morphism

$$H_1 \times \cdots \times H_n \to H$$

is bijective. For $w \in W$, define $U_w := U \cap wU^- w^{-1}$, where U consists of the unipotent elements of B and similarly, U^- is the unipotent part of $B^- := w_0 B w_0$. This is a subgroup of U that is closed and normalized by T. Hence, by [6, §14.4], U_w is directly spanned, in any order, by the *root subgroups* U_α, $\alpha \in \Phi^+$, contained in U_w. Since by [20, Part II, 1.4(5)],

$$wU_\alpha w^{-1} = U_{w(\alpha)}, \tag{3.3}$$

these are the U_α such that $\alpha \in \Phi^+ \cap w(\Phi^-) = \mathcal{I}(w)$. Thus

$$U_w = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha, \tag{3.4}$$

where the products U_α may be taken in any order.

Lemma 3.3. *For a coset $wB \in G/B$, we have*

$$X_w^\circ := BwB = U_w wB = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha \; wB. \tag{3.5}$$

Moreover, X_w° is isomorphic to the affine space $\mathbb{A}^{\ell(w)}$ (as varieties).

We say that $w = uv \in W$ is *length additive* if $\ell(uv) = \ell(u) + \ell(v)$. Under this hypothesis, by [7, Ch. VI, §1, Cor. 2 of Prop. 17] one has

$$\mathcal{I}(uv) = \mathcal{I}(u) \sqcup u(\mathcal{I}(v)).$$

Therefore, in particular, if we assume $w_0(I)d \in W$ is *length additive*, then

$$\mathcal{I}(w_0(I)d) = \mathcal{I}(w_0(I)) \sqcup w_0(I)(\mathcal{I}(d)). \tag{3.6}$$

Define

$$V_d := w_0(I) U_d w_0(I)^{-1} = w_0(I) U_d w_0(I).$$

Lemma 3.4. *V_d is a closed subgroup of $U_{w_0(I)d}$ that is normalized by T.*

Lemma 3.5. *$U_{w_0(I)d}$ is directly spanned by $U_{w_0(I)}$ and V_d:*

$$U_{w_0(I)d} = U_{w_0(I)} V_d = V_d U_{w_0(I)}. \tag{3.7}$$

Define

$$\tilde{O} := V_d w_0(I) d B \subseteq G/B.$$

Lemma 3.6. *$\tilde{O}$ is T-stable for the action $\bullet$.*

The following is the main point of this section:

Proposition 3.7. *If $w_0(I)d \in W$ is length additive then*

$$X^\circ_{w_0(I)d} = U_{w_0(I)d}\, w_0(I)dB.$$

Hence $\tilde{O} \subset X^\circ_{w_0(I)d}$. Moreover, $\tilde{O}$ with the T-action $\bullet$ is T-equivariantly isomorphic to X°_d with the T-action $\bullet_{w_0(I)}$.

4 Proof of the main result

We need a lemma examining the L_I-action on $\tilde{O}$. This lemma is then used in conjunction with Proposition 3.7 to prove our main result.

Let $B_{L_I} = L_I \cap B$ and let U_{L_I} be the unipotent radical of B_{L_I}. Then B_{L_I} is a Borel subgroup in L_I [6, §14.17] with $U_{L_I} = B_{L_I} \cap U$ and $B_{L_I} = T \ltimes U_{L_I}$. Since L_I is the subgroup of G generated by T and $\{U_\alpha \mid \alpha \in \Phi(I)\}$ [22, §3.2.2], it is straightforward to show that

$$U_{L_I} = \prod_{\alpha \in \Phi^+(I)} U_\alpha,$$

where the product is taken in any order [6, §14.4].

Lemma 4.1. *Let $w = w_0(I)d \in W$ be length additive. Let $x \in X^\circ_{w_0(I)d} \setminus \tilde{O}$ and $y, z \in \tilde{O}$.*

(i) $uy \notin \tilde{O}$ for all $u \in U_{L_I}$ with $u \neq e$.

(ii) $tx \notin \tilde{O}$ for all $t \in T$.

(iii) There exists $b \in B_{L_I}$ such that $by = z$ if and only if there exists $t \in T$ such that $ty = z$.

We now have the necessary ingredients to complete the proof of Theorem 1.3.

5 Application to Demazure modules

As an application of these results we give a sufficient condition for a Demazure module to be a multiplicity-free L_I-module; equivalently, a sufficient condition for a Demazure character to be multiplicity-free with respect to the basis of irreducible L_I-characters.

Let $\mathfrak{X}(T)$ denote the lattice of weights of T; our fixed Borel subgroup B determines a subset of dominant integral weights $\mathfrak{X}(T)^+ \subset \mathfrak{X}(T)$. The finite-dimensional irreducible G-representations are indexed by $\lambda \in \mathfrak{X}(T)^+$. Denoting the associated representation by V_λ, there is a class of B-submodules of V_λ, first introduced by Demazure [10], that are indexed by $w \in W$. If v_λ is a nonzero highest weight vector, then the *Demazure module* V_λ^w is the minimal B-submodule of V_λ containing wv_λ.

There is a geometric construction of these Demazure modules. For $\lambda \in \mathfrak{X}(T)^+$, let $\mathfrak{L}_\lambda$ be the associated line bundle on G/B. For $w \in W$, we write $\mathfrak{L}_\lambda|_{X_w}$ for the restriction of $\mathfrak{L}_\lambda$ to the Schubert subvariety $X_w \subseteq G/B$. Then the Demazure module V_λ^w is isomorphic to the dual of the space of global sections of $\mathfrak{L}_\lambda|_{X_w}$, that is

$$V_\lambda^w \cong H^0(X_w, \mathfrak{L}_\lambda|_{X_w})^*.$$

This geometric perspective highlights the fact that V_λ^w is not just a B-module, but is in fact also a L_I-module via the action induced on $H^0(X_w, \mathfrak{L}_\lambda|_{X_w})$ by the left multiplication action of L_I on X_w.

As L_I is a reductive group over characteristic zero, any L_I-module decomposes into a direct sum of irreducible L_I-modules. Let $\mathfrak{X}_{L_I}(T)^+$ be the set of dominant integral weights with respect to the choice of maximal torus and Borel subgroup $T \subseteq B_I \subseteq L_I$. For $\mu \in \mathfrak{X}_{L_I}(T)^+$, let $V_{L_I,\mu}$ be the associated irreducible L_I-module. If M is a L_I-module and

$$M = \bigoplus_{\mu \in \mathfrak{X}_{L_I}(T)^+} V_{L_I,\mu}^{\oplus m_{L_I,\mu}}$$

is the decomposition into irreducible L_I-modules, then we say that M is a *multiplicity-free L_I-module* if $m_{L_I,\mu} \in \{0,1\}$. Similarly, if $\mathrm{char}(M)$ is the formal T-character of M and

$$\mathrm{char}(M) = \sum_{\mu \in \mathfrak{X}_{L_I}(T)^+} m_{L_I,\mu} \mathrm{char}(V_{L_I,\mu}),$$

then we say that $\mathrm{char}(M)$ is *I-multiplicity-free* if $m_{L_I,\mu} \in \{0,1\}$.

The following argument was given for type A in [16, Theorem 4.13(II)]. We prove the general type argument (which is essentially the same) for sake of completeness:

Theorem 5.1. *Let $w \in W$ with $I \subseteq D_L(w)$. Then X_w is L_I-spherical if and only if for all $\lambda \in \mathfrak{X}(T)^+$, the Demazure module V_λ^w is multiplicity-free L_I-module.*

Corollary 5.2. *Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure module V_λ^w is a multiplicity-free L_I-module.*

Corollary 5.3. *Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure character* $\mathrm{char}(V_\lambda^w)$ *is I-multiplicity-free.*

These two corollaries appear non-trivial from a combinatorial perspective, even for a *specific choice* of dominant weight λ with fixed $w \in W$. The Demazure character can be recursively computed using Demazure operators. There is also a combinatorial rule for the character in terms of crystal bases (in instantiations such as the *Littelmann path model* or the *alcove walk model*); see, e.g., the textbook [8]. However, an argument based on these methods eludes in general type, although we have an argument in type A [14].

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Non-symmetric Cauchy kernels, Demazure measures, and LPP

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Abstract. We use non-symmetric Cauchy kernel identities to get the laws of last passage percolation (LPP) models in terms of Demazure characters. The construction is based on the restrictions of the RSK correspondence to augmented stair (Young) shape matrices and rephrased in a unified way compatible with crystal bases.

Keywords: Non-symmetric Cauchy identity, Demazure character, crystal, percolation.

1 Introduction

We introduce the Demazure measure on nonnegative vectors corresponding to the directed last passage percolation (LPP) model on matrices of Young shape, that is, nonnegative integer matrices whose positive entries fit a Young shape. A nonnegative integer vector is always in the Weyl orbit of some partition and therefore all nonnegative vectors in a same Weyl orbit share the size of a largest entry which is the length of a longest row of the unique partition in its orbit. When the Young shape is a rectangle, we recover the Okounkov's Schur measure [4, Chapter 4], [17] on the unique partition of each Weyl orbit, corresponding to the LPP model on nonnegative integer matrices. Our main contribution is the use of Demazure characters, in general, non symmetric polynomials, to study LPP problems: this has only been carried out for models with more symmetries using symmetric polynomials, in particular, Schur polynomials or Weyl characters or geometric analogues as incarnations of Whittaker functions ([6, 7, 16, 19] and references therein). Crystal theory allows the compatibility of Robinson–Schensted–Knuth (RSK) correspondence with non-symmetric Cauchy identities by Lascoux [13] and thus, in particular, the Cauchy identity (1.1). This interpretation was discovered by Choi–Kwon [8]

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for the non-symmetric case on stair cases (3.1). We complete the picture with the truncated and augmented stair shape. This extended abstract is organized in four sections. In §2 we gather relevant definitions on crystals, in §3 present our contributions, and in §4 provide an example for our main result. We refer the reader to the full version [3], accepted for publication, for details and proofs, containing the results hereby presented.

Given two sets of indeterminates $x = \{x_1, \ldots, x_m\}$ and $y = \{y_1, \ldots, y_n\}$ the Cauchy identity asserts that

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_i y_j} = \sum_{\lambda\in\mathcal{P}_{\min(m,n)}} s_\lambda(x)s_\lambda(y) \tag{1.1}$$

where $\mathcal{P}_{\min(m,n)}$ is the set of partitions with at most $\min(m,n)$ parts and, for each such partition λ, $s_\lambda(x)$ and $s_\lambda(y)$ are the Schur polynomials in the indeterminates x and y, respectively. This identity has several interpretations, applications and generalizations (see [9] and references therein). In particular, one can understand the left hand side as the character of polynomial functions on the space $\mathcal{M}_{m\times n}$ of matrices with m rows, n columns and entries in $\mathbb{Z}_{\geq 0}$ and decompose this space into a direct sum of $\mathfrak{gl}_m \times \mathfrak{gl}_n$ bimodules. The products of Schur functions $s_\lambda(x)$ and $s_\lambda(y)$ on the righthand side show this approach as the characters of the tensor product of irreducibles finite dimensional representations of highest weight λ for the linear Lie algebras $\mathfrak{gl}_m(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$. In fact $\mathcal{M}_{m,n}$ is a realization of the the bicrystal of the symmetric space $S(\mathbb{C}^m\otimes\mathbb{C}^n)$ as a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-module (see [8] and references therein). The identity (1.1) can also be proved using the RSK correspondence [10, 18]. This is a one-to-one map ψ between the set $\mathcal{M}_{m,n}$ and the set $\bigsqcup_{\lambda\in\mathcal{P}_{\min(m,n)}} SSYT(\lambda, m)\times SSYT(\lambda, n)$ of pairs (P, Q) of semistandard tableaux of the same shape λ, and entries in $[m] := \{1, \ldots, m\}$ and $[n] := \{1, \ldots, n\}$, respectively. (The convention that we use agrees with that of Kashiwara [12] to which we refer for another description of the RSK procedure and the connection with biwords. See §4 and [10] for variations on RSK.) Regarding $SSYT(\lambda, k)$ as the tableau realization for the $\mathfrak{gl}_k$-crystal $B(\lambda, k)$ of highest weight λ, then

$$\begin{aligned}\psi : \mathcal{M}_{m,n} \to & \bigsqcup_{\lambda\in\mathcal{P}_{\min(m,n)}} B(\lambda, m)\times B(\lambda, n)\\ A \mapsto & \ \psi(A) = (P(A), Q(A))\end{aligned} \tag{1.2}$$

is a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystal isomorphism where the bicrystal structure on $\mathcal{M}_{m,n}$ is afforded from $B(\lambda, m)\times B(\lambda, n)$ by ψ^{-1}, that is, by reverse column Schensted insertion. The RSK correspondence has interesting properties. For each matrix A in $\mathcal{M}_{m,n}$, the greatest integer $p(A)$ obtained by summing up the entries in all the possible paths π starting at position $(1, n)$ and ending at position $(m, 1)$ with steps $\longleftarrow$ or $\downarrow$

$$p(A) := \max_{\pi \text{ path in } A} \sum_{(i,j)\in\pi} a_{ij} \tag{1.3}$$

coincides with the common largest row length of the tableaux $P(A)$ and $Q(A)$ in (1.2). (We consider the paths which are compatible with the version of RSK that is used here. See §4.) It is then natural to study percolation models based on the RSK correspondence where random matrices whose entries follow independent geometric laws are considered [4]. This type of model, in the case of identical and independent geometric distribution, has been deeply studied by Johansson in [11], who proved that the fluctuations of the previous last passage percolation, once correctly normalized, are controlled by the Tracy-Widom distribution (defined from the study of the largest eigenvalues of random Hermitian matrices). The Schur measure, introduced by Okounkov based on the Cauchy kernel identity, is an extension of the probability measure on the partitions corresponding to the directed last passage percolation model with the independent and identical geometric distribution of Johansson in [11], [4, Chapter 4]. Let u_i, v_j be real numbers in $[0, 1)$, for $1 \leq i \leq m$, $1 \leq j \leq n$. Considering an array $\mathcal{W} = \{W_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ of independent random variables, with values in $\mathbb{Z}_{\geq 0}$, called weights, geometrically distributed as

$$\mathbb{P}(W_{ij} = k) = (1 - u_i v_j)(u_i v_j)^k, \text{ for any } k \in \mathbb{Z}_{\geq 0}, \tag{1.4}$$

with parameter $u_i v_j$, $\mathcal{W}$ is a random matrix with values in $\mathcal{M}_{m,n}$. We then get

$$\mathbb{P}(\mathcal{W} = A) = \left(\prod_{1 \leq i \leq m, 1 \leq j \leq n} (1 - u_i v_j) \right) (uv)^A$$

where $(uv)^A = \prod_{1 \leq i \leq m, 1 \leq j \leq n} (u_i v_j)^{a_{i,j}}$. The Last Passage Percolation (LPP) time G of $\mathcal{W}$ is defined to be the random variable $G := p \circ \mathcal{W}$. Applying the RSK correspondence, its properties and the Cauchy identity (1.1), one obtains the law of the random variable G, for any $k \in \mathbb{Z}_{\geq 0}$, in terms of Schur polynomials,

$$\mathbb{P}(G = k) = \prod_{1 \leq i,j \leq n} (1 - u_i v_j) \sum_{\lambda \in \mathcal{P}_{\min(m,n)} | \lambda_1 = k} s_\lambda(u_1, \ldots, u_m) s_\lambda(v_1, \ldots, v_n)$$

where the sum is over partitions λ with largest part k. Johansson [11] has established this result in the special case of identical geometric distribution, $u_i = v_j = \sqrt{q}$, $1 \leq i, j \leq n$, for a fixed $q \in]0, 1[$, a special case of the Schur measure on partitions (see [4, Chapter 10]). The RSK correspondence admits various generalizations and geometric versions which can also be used to get interesting last passage percolation models involving symmetric polynomials, in particular, characters of representations of Lie algebras other than $\mathfrak{gl}_n$ (symmetric with respect to the Weyl group) and geometric analogues [6, 7, 16, 19].

2 Crystal and Demazure modules

The finite dimensional irreducible polynomial representations of $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ are parameterized by the partitions λ in $\mathcal{P}_n$. To each partition $\lambda \in \mathcal{P}_n$ corresponds a finite dimensional representation $V(\lambda)$ (or $\mathfrak{gl}_n$-module), and a crystal graph $B(\lambda)$ which can be regarded as the combinatorial skeleton of the simple module $V(\lambda)$. The vertices of $B(\lambda)$ label a distinguished basis of $V(\lambda)$. On the other hand, $B(\lambda)$ has various combinatorial realizations (*i.e.*, vertex labelings) in terms of semistandard tableaux, Littelmann's paths [14] or semiskylines [15]. The (abstract) crystal $B(\lambda)$ is a graph whose set of vertices is endowed with a weight function $wt : B(\lambda) \to \mathbb{Z}^n$ and with the structure of a coloured and oriented graph given by the action of the crystal operators $\tilde{f}_i$ and $\tilde{e}_i$ with $i \in I = [n-1]$. One has an oriented arrow $b \xrightarrow{i} b'$ between two vertices b and b' in $B(\lambda)$ if and only if $b' = \tilde{f}_i(b) \Leftrightarrow b = \tilde{e}_i(b')$ in which case $wt(b') = wt(b) - \alpha_i$, with α_i a simple root of $\mathfrak{gl}_n$. The crystal $B(\lambda)$ is generated by the actions of the lowering (resp. raising) operators $\tilde{f}_i$ (respect. $\tilde{e}_i$) on the unique highest (resp. lowest) weight vertex b_λ (resp. $b_{\sigma_0\lambda}$) where one has $wt(b_\lambda) = \lambda$, and σ_0 is the longest element of the Weyl group W here the symmetric group $\mathfrak{S}_n =< s_1, \ldots, s_{n-1} >$ (unless mentioned differently).

For $\lambda \in \mathcal{P}_n$, W_λ is the stabilizer of λ under the action of W, and W^λ collects the unique minimal length representative of each coset in W/W_λ. Let $\lambda \in \mathcal{P}_n$ and $\sigma \in W$. Up to a scalar in $\mathbb{C}$, there exists a unique vector $v_{\sigma\lambda}$ in $V(\lambda)$ of weight $\sigma\lambda$. Recall the triangular decomposition $\mathfrak{gl}_n = \mathfrak{gl}_n^+ \oplus \mathfrak{h} \oplus \mathfrak{gl}_n^-$ of $\mathfrak{gl}_n$ into its upper, diagonal and lower parts. The Demazure module associated to $v_{\sigma\lambda}$ is the $U(\mathfrak{gl}_n^+)$-module defined by $V_\sigma(\lambda) := U(\mathfrak{gl}_n^+) \cdot v_{\sigma\lambda}$. Demazure introduced the character $\kappa_{\sigma,\lambda}$ of $V_\sigma(\lambda)$ and showed that it can be computed by applying to x^λ a sequence of divided difference operators $D_{i_1} \cdots D_{i_\ell}$ given by any reduced decomposition of $\sigma = s_{i_1} \cdots s_{i_\ell} \in W$ where ℓ is the length of σ. For $i \in I$, D_i is a certain linear operator on $\mathbb{Z}[x_1, \ldots, x_n]$ (see [3] and references therein) satisfying the relations

$$D_i^2 = D_i \text{ for any } i = 1, \ldots, n-1, \quad D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \text{ for any } i = 1, \ldots, n-2,$$
$$D_i D_j = D_j D_i \text{ for any } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1.$$

Thus, by Mastumoto's Lemma, the operator $D_\sigma = D_{i_1} \cdots D_{i_\ell}$ only depends on σ and not on the chosen reduced decomposition, and $\kappa_{\sigma,\lambda} = D_\sigma(x^\lambda) \in \mathbb{Z}[x_1, \ldots, x_n]$ is the (Demazure) character of $V_\sigma(\lambda)$. In particular, we have $\kappa_{id,\lambda} = x^\lambda$ and $\kappa_{\sigma_{0,\lambda}} = s_\lambda$.

Kashiwara [12] and Littelmann [14] defined a relevant notion of crystals for the Demazure modules. Recall $O(\lambda) = \{\sigma \cdot b_\lambda = b_{\sigma\lambda} \mid \sigma \in W/W_\lambda\}$ the orbit of the highest weight vertex b_λ of $B(\lambda)$. Its elements, uniquely determined by their weight, are called the keys of $B(\lambda)$. (In this sense we may identify $O(\lambda)$ with $W\lambda$.) Given $\sigma, \sigma' \in W/W_\lambda$, we write $\sigma \leq \sigma'$ for the Bruhat order on the cosets in W/W_λ to mean that their unique minimal (maximal) coset representatives satisfy the same relation in the strong Bruhat order restricted to W^λ. We also write $b_{\sigma\lambda} \leq b_{\sigma'\lambda}$ when $\sigma \leq \sigma'$ in W/W_λ. From the dilatation of crystals [12] each vertex b of $B(\lambda)$ carries a pair of keys $K^+(b) \geq K^-(b)$, right, respectively, left key of b, in $O(\lambda)$. For any $\sigma \in W$, consider the Demazure atom

$$\overline{B}_\sigma(\lambda) = \{b \in B(\lambda) \mid K^+(b) = b_{\sigma\lambda}\}, \text{ where } \overline{B}_{id}(\lambda) = \{b_\lambda\}. \tag{2.1}$$

For any $\sigma \in W$, the opposite Demazure module, is defined to be $V^\sigma(\lambda) := U_q(\mathfrak{gl}_n^-) \cdot v_{\sigma\lambda}$, for which we define the opposite Demazure atom

$$\overline{B}^\sigma(\lambda) = \{b \in B(\lambda) \mid K^-(b) = b_{\sigma\lambda}\}, \text{ where } \overline{B}^{\sigma_0}(\lambda) = \{b_{\sigma_0\lambda}\}. \tag{2.2}$$

By definition we have $\overline{B}_\sigma(\lambda) = \overline{B}_{\sigma'}(\lambda)$ and $\overline{B}^\sigma(\lambda) = \overline{B}^{\sigma'}(\lambda)$ whenever σ and σ' belong to the same left coset of W/W_λ. We then get $B(\lambda) = \bigsqcup_{\sigma\in W^\lambda} \overline{B}_\sigma(\lambda) = \bigsqcup_{\sigma\in W^\lambda} \overline{B}^\sigma(\lambda)$. The Demazure crystal $B_\sigma(\lambda)$ and its opposite Demazure crystal $B^\sigma(\lambda)$ are then defined by

$$B_\sigma(\lambda) = \bigsqcup_{\sigma'\in W^\lambda,\, \sigma' W_\lambda \le \sigma W_\lambda} \overline{B}_{\sigma'}(\lambda) = \{b \in B(\lambda) \mid K^+(b) \le b_{\sigma\lambda}\},\ B_{id}(\lambda) = \{b_\lambda\} \tag{2.3}$$

$$B^\sigma(\lambda) = \bigsqcup_{\sigma'\in W^\lambda,\, \sigma W_\lambda \le \sigma' W_\lambda} \overline{B}^{\sigma'}(\lambda) = \{b \in B(\lambda) \mid K^-(b) \ge b_{\sigma\lambda}\},\ B^{\sigma_0}(\lambda) = \{b_{\sigma_0\lambda}\}. \tag{2.4}$$

In particular, we have $B_{\sigma_0}(\lambda) = B(\lambda) = B^{id}(\lambda)$. We then note that for a given $\lambda \in \mathcal{P}_n$,

$$\bigsqcup_{\sigma\in W^\lambda} \overline{B}^\sigma(\lambda) \times B_\sigma(\lambda) = \{(b, b') \in B(\lambda) \times B(\lambda) : K^-(b) \ge K^+(b')\} \simeq B(2\lambda). \tag{2.5}$$

We refer to [8], for the translation of (2.5) to the crystal of Lakshmibai-Seshadri paths. The Demazure crystals respectively atoms and their opposite, are connected via the Lusztig-Schützenberger involution ι on the crystal $B(\lambda)$, a realization of the action of the longest element of W on finite irreducible representations. The map ι is a set involution on $B(\lambda)$ reversing the arrows, flipping the labels i and $n-i$, and reversing the weight. We then have $K^-(b) = \sigma_0.K^+(\iota(b))$ and we get

$$B^\sigma(\lambda) = \iota(B_{\sigma_0\sigma}(\lambda)), \text{ or equivalently } B^{\sigma_0\sigma}(\lambda) = \iota B_\sigma(\lambda), \quad \overline{B}^\sigma(\lambda) = \iota(\overline{B}_{\sigma_0\sigma}(\lambda)). \tag{2.6}$$

Demazure (resp. opposite) crystals can also be generated by the actions of the lowering (resp. raising) operators given by the reduced words in W^λ (resp. $\sigma_0 W^\lambda \sigma_0$) on the highest (resp. lowest) vertex of $B(\lambda)$. The Demazure character $\kappa_{\sigma,\lambda}(x)$ of the Demazure module $V^\sigma(\lambda)$ satisfies $\kappa_{\sigma,\lambda}(x) = \sum_{b\in \mathrm{B}_\sigma(\lambda)} x^{wt(b)}$, and the opposite Demazure character $\kappa^\sigma_\lambda(x)$ for the opposite Demazure module $V^\sigma(\lambda)$ satisfies $\kappa^\sigma_\lambda(x) = \sum_{b\in B^\sigma(\lambda)} x^{wt(b)}$. Using the involution ι and (2.6), we have $\kappa^\sigma_\lambda(x_1, \ldots, x_n) = \kappa_{\sigma_0\sigma\lambda}(x_n, \ldots, x_1)$ and $\overline{\kappa}^\sigma_\lambda(x_1, \ldots, x_n) = \overline{\kappa}_{\sigma_0\sigma\lambda}(x_n, \ldots, x_1) = \sum_{b\in\overline{B}^\sigma(\lambda)} x^{wt(b)}$. Alternatively we may also label the Demazure crystals and the Demazure characters of $B(\lambda)$ directly by the elements in the orbit of λ, $W\lambda$. Given $\mu \in W\lambda$ where $\mu = \sigma\lambda$ and $\sigma \in W^\lambda$, we write B_μ, $B^\mu = \iota B_{\sigma_0\mu}$ instead of $B_\sigma(\lambda)$, $B^\sigma(\lambda)$ respectively, and $\kappa_\mu, \kappa^\mu = \kappa_{\sigma_0\mu}, \overline{\kappa}_\mu, \overline{\kappa}^\mu = \overline{\kappa}_{\sigma_0\mu}$ instead of $\kappa_{\sigma,\lambda}, \kappa^\sigma_\lambda$ and $\overline{\kappa}_{\sigma,\lambda}$, $\overline{\kappa}^\sigma_\lambda$ respectively. The operators D_i act on Demazure characters κ_μ and Demazure atoms $\overline{\kappa}_\mu$ as follows

$$D_i(\kappa_\mu) = \begin{cases} \kappa_{s_i\mu} & \text{if } \mu_i > \mu_{i+1} \\ \kappa_\mu & \text{if } \mu_i \le \mu_{i+1}, \end{cases} \qquad D_i(\overline{\kappa}_\mu) = \begin{cases} \overline{\kappa}_{s_i\mu} + \overline{\kappa}_\mu & \text{if } \mu_i > \mu_{i+1} \\ \overline{\kappa}_\mu & \text{if } \mu_i = \mu_{i+1} \\ 0, & \text{else.} \end{cases} \tag{2.7}$$

For $i \in [n-1]$, we define below Δ_i and $\dot{\Delta}_i$ as operators on Demazure respectively Demazure atom crystals to mimic the action of the operator D_i on Demazure respectively on Demazure atom charaters (2.7), and we then always have $char(\Delta_i(B_\mu)) = D_i(\kappa_\mu)$, and $char(\dot{\Delta}_i(\overline{B}_\mu)) = D_i(\overline{\kappa}_\mu)$,

$$\Delta_i(B_\mu) = \begin{cases} B_{s_i\mu} \text{ if } \mu_i > \mu_{i+1} \\ B_\mu \text{ otherwise,} \end{cases} \quad \dot{\Delta}_i(\overline{B}_\mu) = \begin{cases} \Delta_i(\overline{B}_\mu) = \overline{B}_\mu \bigsqcup \overline{B}_{s_i\mu} \text{ if } \mu_i > \mu_{i+1} \\ \Delta_i(\overline{B}_\mu) = \overline{B}_\mu \text{ if } \mu_i = \mu_{i+1} \\ \emptyset \text{ if } \mu_i < \mu_{i+1}. \end{cases} \tag{2.8}$$

3 Non-symmetric Cauchy kernels, RSK on Young shapes and LPP

We now consider last passage percolation models based on the non-symmetric Cauchy kernel (3.1) as studied by Lascoux in [13] and its extensions to augmented stair shapes. Demazure crystals with their opposite Demazure atoms, and certain parabolic subcrystals will describe the image of RSK, as a bicrystal isomorphism, restricted to stair shape, truncated stair shape and to augmented stair shape matrices. We detach the truncated case from the general Young shape due to its more explicit as well interesting structure.

3.1 LPP, staircase and Demazure measure

The ordinary Cauchy identity (1.1) is then replaced by its non-symmetric analogue

$$\prod_{1 \le j \le i \le n} \frac{1}{1 - x_i y_j} = \sum_{\mu \in \mathbb{Z}^n_{\ge 0}} \overline{\kappa}^\mu(x) \kappa_\mu(y) \tag{3.1}$$

where $\overline{\kappa}^\mu(x)$ and $\kappa_\mu(y)$ are this time (opposite) Demazure atoms and Demazure characters in the indeterminates x and y (with $m = n$). These polynomials are not symmetric in x and y. They correspond to characters of representations for subalgebras of the enveloping algebra $U(\mathfrak{gl}_n)$. It was proved in [13] that the identity (3.1) can be obtained by restricting the RSK correspondence ψ to the set of lower triangular matrices. (The convention of our paper differs from that in [13] which considers matrices with nonzero entries in positions (i, j) with $1 \le i + j \le n + 1$ rather than lower-triangular matrices.) Since then, other proofs have been proposed using combinatorial objects which explicitly carry the pairs of right and left keys [10]. More precisely, [1, Theorem 3, Corollary 2] uses the combinatorics of Mason's semiskyline augmented fillings [15], and [8] uses the combinatorics of crystal bases, in particular, the combinatorial model of Lakshmibai-Seshadri paths [14]. Recently Assaf-Schilling provided an explicit tableau crystal for MasonâĂŹs semiskyline augmented fillings [15], replacing the former objects by equivalent ones, termed semistandard key tableaux (see [3] and references therein). Here we stand in the tableau model for $\mathfrak{gl}_n$ crystals where one has the effective Lascoux's jeu de taquin procedure [10] to compute the right key $K^+(T)$ and left key $K^-(T)$ of a tableau T.

Let D be any subset of $[n] \times [m]$ and write $\mathcal{M}_{m,n}^D$ for the subset of $\mathcal{M}_{m,n}$ containing the matrices A such that $a_{i,j} \neq 0$ only if $(i,j) \in D$. For D in general, the set $\mathcal{M}_{m,n}^D$ is not stable for the $\mathfrak{gl}_m \times \mathfrak{gl}_n$-crystals operators. Nevertheless, when D corresponds to the Young diagram of a fixed partition Λ, see (3.7), $D = D_\Lambda$ is stable under the action of the crystal raising operators. When $m = n$ and $\varrho = (n, n-1, \dots, 1)$, we get in matrix coordinates $D_\varrho = \{(i,j) \mid 1 \leq j \leq i \leq n\}$. Then the bijection ψ (1.2) restricts to a bijection from the set $\mathcal{M}_{m,n}^{D_\varrho}$ of $n \times n$ lower triangular matrices to the set of pairs (P, Q) of semistandard Young tableaux of the same shape on the alphabet $[n]$ such that $K^-(P) \geq K^+(Q)$ (entrywise comparison). (See also [1, Corollary 2] for the Knuth version of RSK.) This means that the image of this restriction, for a fixed $\lambda \in \mathcal{P}_n$, is $\bigsqcup_{\sigma \in W^\lambda} \overline{B}^\sigma(\lambda) \times B_\sigma(\lambda)$ (2.5). Thus the restriction of RSK correspondence ψ to D_ϱ gives

$$\psi : \mathcal{M}_{n,n}^{D_\varrho} \to \bigsqcup_{\lambda \in \mathcal{P}_n} \bigsqcup_{\sigma \in W^\lambda} \overline{B}^\sigma(\lambda) \times B_\sigma(\lambda) \tag{3.2}$$

$$A \mapsto \psi(A) = (P(A), Q(A)) : K^+(Q(A)) \leq K^-(P(A)), \tag{3.3}$$

where $B_\sigma(\lambda)$ is a Demazure crystal (2.3) and $\overline{B}^\sigma(\lambda)$ its opposite Demazure atom (2.2). This time, we only consider independent random variables $W_{i,j}$ when $1 \leq j \leq i \leq n$ with geometric distributions as in (1.4). This defines a lower triangular random square matrix $\mathcal{L}$ with nonnegative integer entries. In this model we consider paths from position $(1, n)$ to position $(n, 1)$ where only the entries in the lower part of A contribute to the length of the paths. We define the random variable $L = p \circ \mathcal{L}$ and determine its law. Since (3.2) gives a bijective correspondence obtained as the restriction of the RSK map ψ (1.2) to lower triangular matrices, the value of L still corresponds to the length of the largest part of the partitions on the right hand side of (3.3).

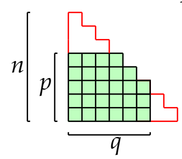
Theorem 1. *For any $k \in \mathbb{Z}_{\geq 0}$, we have the law*

$$\mathbb{P}(L = k) = \prod_{1 \leq j \leq i \leq n} (1 - u_i v_j) \sum_{\mu \in \mathbb{Z}_{\geq 0}^n | \max(\mu) = k} \overline{\kappa}^\mu(u_1, \dots, u_n) \kappa_\mu(v_1, \dots, v_n). \tag{3.4}$$

This law was also obtained by Baik-Rains [5, 6, Section 4] when $u_i = v_i$. In this case, (2.5), and (3.1) with $x_i = y_i$, together give a refinement of a Littlewood identity: $\prod_{1 \leq j \leq i \leq n} (1 - x_i x_j)^{-1} = \sum_{\mu \in \mathbb{Z}_{\geq 0}^n} \overline{\kappa}^\mu(x) \kappa_\mu(x) = \sum_{\lambda \in \mathcal{P}_n} s_{2\lambda}(x)$. In [6] it is called a law in the point-to-line last passage percolation in zero temperature limit. However this formula is not produced in [6] by the geometric RSK but rather one in terms of a symplectic Cauchy like identity.

3.2 Main results: LPP on Young shapes and Demazure measure

Lascoux [13] also established generalizations of the formula (3.1) where positions with nonzero entries are allowed in the matrices outside their lower triangular part. These augmented staircase formulas below $(*)$ were then obtained just by computations on polynomials and thus not related to the RSK correspondence. This connection was partially done in [2] where certain truncated staircases formulas are proved to be compatible with the RSK correspondence using the combinatorics of semiskyline augmented fillings [15]. More precisely, this applies to the case where nonzero entries are authorized only in positions (i,j) with $n-p\le i\le j\le q$, for p and q two nonnegative integers such that $n\ge q\ge p\ge 1$. We consider the Young diagram $D_{p,q}=\{(i,j)\mid n-p+1\le i\le n, 1\le j\le q\}\cap D_{\varrho}$ defined by using the matrix coordinates (i,j). It is the intersection of D_{ϱ} with a quarter of plane defined by the lines $i=p$ and $j=q$ (in Cartesian coordinates). When $n-p+1\le q$, we get the Young diagram $D_{p,q}=D_{\Lambda(p,q)}$ with $\Lambda(p,q)=(q^{n-q+1},q-1,\dots,n-p+1)$, and $D_{n,n}=D_{\Lambda(n,n)}=D_{\varrho}$. Below one illustrates the truncated Young shape $D_{\Lambda(p,q)}$, in green, fitting the p by q rectangle so that the staircase D_{ϱ} of size n, in red, is the smallest one containing $D_{\Lambda(p,q)}$. If $p\le q$, $D_{(p,p-1,\dots,1)}$ is the biggest staircase inside $D_{\Lambda(p,q)}$.



We write $B_p(\lambda)$ for the subcrystal of the $\mathfrak{gl}_n$-crystal $B(\lambda,0^{n-p})$ with $\lambda\in\mathcal{P}_p$, obtained by keeping only the vertices connected to its highest weight vertex by i-arrows with $i\in[p-1]$. Given $u\in\mathfrak{S}_p$, $B_{p,u}(\lambda), B_p^u(\lambda), \overline{B}_{p,u}(\lambda)$ and $\overline{B}_p^u(\lambda)$ denote the Demazure, its opposite, respectively, atom and its opposite crystals associated to u in the $\mathfrak{gl}_p$-crystal $B_p(\lambda)$. See Example 4 and (4.1), (4.2). The restriction of the map ψ from $\mathcal{M}_{n,n}^{D_\varrho}$ (3.3) to $\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}$ gives

$$\psi(\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}})=\bigsqcup_{\lambda\in\mathcal{P}_n}\bigsqcup_{\sigma\in W^\lambda}\overline{B}^{\sigma}(\lambda)\cap B^p(\lambda)\times B_\sigma(\lambda)\cap B_q(\lambda)=\bigsqcup_{\mu\in\mathbb{Z}^n_{\ge0}}\overline{B}^{\mu}\cap B^p(\lambda)\times B_\mu\cap B_q(\lambda).$$

By the Borel-Weil theorem, Demazure crystals and Schubert varieties are in natural correspondence. Let $\sigma\in\mathfrak{S}_n$ and $\sigma_0^{[q]}$ be the longest element of $\mathfrak{S}_q$. From the Billey–Fan–Losonczy parabolic map (see [3, Algorithm 3.1, Proposition 3.4] and references therein) the set $\{v\in\mathfrak{S}_q\mid v\le\sigma\}$ has a unique maximal element σ^{I_q} for the Bruhat order $\le$ in W. For $\sigma\in W^\lambda$, the intersections $S_{\sigma_0^{[q]}}\cap S_\sigma=S_{\sigma^{I_q}}$ and $B_\sigma(\lambda)\cap B_q(\lambda)=B_\sigma(\lambda)\cap B_{\sigma_0^{[q]}}(\lambda)=\mathrm{B}_{q,\sigma^{I_q}}(\lambda)$ translate into each other, where S_σ is a Schubert variety of the flag variety G/B with B a Borel subgroup of the reductive group G with Weyl group W [10, Chapter III]. However $\overline{B}^{\sigma}(\lambda)\cap B^p(\lambda)=\emptyset$ unless $\sigma\in\sigma_0\mathfrak{S}_p^\lambda$, $\lambda\in\mathcal{P}_p$ and then

$\overline{B}^{\sigma}(\lambda) \cap B^p(\lambda) = \iota \overline{B}_{p,\sigma_0\sigma}(\lambda)$ [3]. In this case $B_\sigma(\lambda) \cap B_q(\lambda) = B_{q,\sigma^{I_q}}(\lambda)$. The restriction of the RSK correspondence ψ to $\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}$ then gives a one-to-one correspondence

$$\psi : \mathcal{M}_{n,n}^{D_{\Lambda(p,q)}} \to \bigsqcup_{\mu \in \mathbb{Z}^p_{\geq 0}} \iota(\overline{B}_{p,\mu}) \times B_{q,\tilde{\mu}}, \text{ and} \tag{3.5}$$

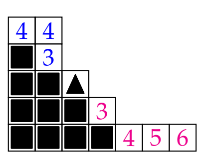
$$\prod_{(i,j)\in D_{\Lambda(p,q)}} \frac{1}{1 - x_i y_j} = \sum_{(\mu_1,\ldots,\mu_p)\in \mathbb{Z}^p_{\geq 0}} \overline{\kappa}_{(\mu_p,\ldots,\mu_1)}(x_n, \ldots, x_{n-p+1}) \kappa_{\tilde{\mu}}(y_1, \ldots, y_q), \tag{3.6}$$

where for each $\mu \in \mathbb{Z}^p_{\geq 0}$, the vector $\tilde{\mu} = (\sigma_0\tau)^{I_q}(\lambda, 0^{q-p}, 0^{n-q})$ with $\tau \in \mathfrak{S}^\lambda_p$ is such that $\mu = \tau\lambda$. It can also be explicitly computed by a simple algorithm in [1, 3, Theorem 3.20] (see also examples in [3, Section 3.1]). One can then similarly use (3.5) to study the percolation model on random matrices $\mathcal{T}_{p,q}$ with nonnegative random integer coefficients having zero entries in each position (i,j) such that $i \leq n-p$ and $j > q$. Each random variable $W_{i,j}$ with $i \geq n-p+1$ and $j \leq q$ follows a geometric distribution of parameter $u_i v_j$. Using the same arguments as before, we obtain the law of the random variable $T_{p,q} = p \circ \mathcal{T}_{p,q}$.

Theorem 2. *For any nonnegative integer k, we have for $v = (v_1, \ldots, v_q)$*

$$\mathbb{P}(T_{p,q} = k) = \prod_{(i,j)\in D_{\Lambda(p,q)}} (1 - u_i v_j) \sum_{(\mu_1,\ldots,\mu_p)\in \mathbb{Z}^p_{\geq 0} | \max(\mu) = k} \overline{\kappa}_{(\mu_p,\ldots,\mu_1)}(u_n, \ldots, u_{n-p+1}) \kappa_{\tilde{\mu}}(v).$$

In [13] Lascoux gave other non-symmetric Cauchy type identities for any partition $\Lambda \in \mathcal{P}_n$. One considers the largest staircase $\rho_\Lambda = (m, m-1, \ldots, 1)$ contained in the Young diagram of Λ. Then one chooses a box at position (i_0, j_0), in Cartesian coordinates, in the augmented staircase $(m+1, m, \ldots, 1)$ which is not in Λ. The diagonal $L_{i,j} : j - i = j_0 - i_0$, in Cartesian coordinates, cuts Λ in a northwest part and a southeast part corresponding to the boxes above and below $L_{i,j}$, respectively. Now fill the boxes (i,j), in the $n \times n$ matrix convention, of the NW part of Λ by $n - i$ (*i.e.*, by the $n \times n$ matrix reverse row index (equivalently counting rows from bottom to top) minus one), and the boxes (i,j) of the SE part by $j - 1$ (*i.e.*, by the index of the column minus one). Let $\sigma(\Lambda, NW) = s_{i_1} \cdots s_{i_a}$ be the element of W where the word $i_1 \cdots i_a$ is obtained from right to left column reading of the NW part of Λ, each column being read from top to bottom. Similarly, let $\sigma(\Lambda, SE) = s_{j_1} \cdots s_{j_b}$ be the element of W where the word $j_1 \cdots j_b$ is obtained from top to bottom row reading of the SE part of Λ, each row being read from right to left. For instance, let $n = 8$ and $\Lambda = (7,4,2,2,2)$. Take $(i_0, j_0) = (3,3)$ (the box with ▲). Hence $m = 4$, $\rho_\Lambda = (4,3,2,1)$, and $\sigma(\Lambda, NW) = s_4 s_3 s_4$, $\sigma(\Lambda, SE) = s_3 s_6 s_5 s_4$,



(3.7)

The following identity is established in [13] and reproved for near stair shapes in [2],

$$(*)\quad \prod_{(i,j)\in\Lambda} \frac{1}{1-x_iy_j} = \sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m} D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\ldots,\mu_1)}(x_n,\ldots,x_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\ldots,\mu_m)}(y),$$

where $y=(y_1,\ldots,y_m)$, and $D_{\sigma(\Lambda,NW)} = D_{i_1}\cdots D_{i_a}$, $D_{\sigma(\Lambda,SE)} = D_{j_1}\cdots D_{j_b}$ are compositions of Demazure operators (2.7).

Theorem 3. *The restriction of the RSK correspondence ψ to $\mathcal{M}_{n,n}^{D_\Lambda}$ gives the one-to-one correspondence*

$$\psi : \mathcal{M}_{n,n}^{D_\Lambda} \to \bigsqcup_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m_{\geq 0}} \iota\left(\dot{\Delta}_{\sigma(\Lambda,NW)}(\overline{B}_{(\mu_m,\ldots,\mu_1)})\right) \times \Delta_{\sigma(\Lambda,SE)}\left(B_{(\mu_1,\ldots,\mu_m)}\right) \tag{3.8}$$

where $\Delta_{\sigma(\Lambda,SE)} = \Delta_{j_1}\cdots\Delta_{j_b}$ and $\dot{\Delta}_{\sigma(\Lambda,NW)} = \dot{\Delta}_{i_1}\cdots\dot{\Delta}_{i_a}$ (2.8). *(As usual $\emptyset \times U = \emptyset$.)*

Now, for a fixed partition Λ in $\mathcal{P}_n$, we consider random matrices $\mathcal{A}_\Lambda$ with nonnegative random integer coefficients having zero entries in each position (i,j) such that $(i,j)\notin\Lambda$. Here again each random variable $W_{i,j}$ for $(i,j)\in\Lambda$ follows a geometric distribution of parameter u_iv_j. Define the random variable $A_\Lambda = p\circ\mathcal{A}_\Lambda$. Then, by $(*)$ and (3.8), we get the law of A_Λ.

Theorem 4. *For any nonnegative integer k,*

$$\mathbb{P}(A_\Lambda = k) = \prod_{(i,j)\in D_\Lambda}(1-u_iv_j).$$
$$\cdot \sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m|\max(\mu)=k} D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\ldots,\mu_1)}(u_n,\ldots,u_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\ldots,\mu_m)}(v_1,\ldots,v_m).$$

4 An example for RSK on augmented stair shapes

Let us resume to the setting of (3.7) with $n=8$, $\Lambda=(7,4,2,2,2)$, and $\sigma(\Lambda,NW) = s_4s_3s_4$, $\sigma(\Lambda,SE) = s_3s_6s_5s_4$. Let ψ be the RSK restricted to $\mathcal{M}_{8,8}^{D_\Lambda}$. Then (3.8) gives for $m=4$

$$\psi : \mathcal{M}_{8,8}^{D_{(7,4,2,2,2)}} \to \bigsqcup_{(\mu_1,\ldots,\mu_4)\in\mathbb{Z}^4_{\geq 0}} \iota\left(\dot{\Delta}_4\dot{\Delta}_3\dot{\Delta}_4(\overline{B}_{(\mu_4,\ldots,\mu_1)})\right) \times \Delta_3\Delta_6\Delta_5\Delta_4\left(B_{(\mu_1,\ldots,\mu_4)}\right)$$
$$A \mapsto \psi(A) = (P,Q).$$

Let $A=\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \mathbf{0}&\mathbf{1}&0&0&0&0&0&0\\ \mathbf{1}&\mathbf{1}&0&0&0&0&0&0\\ \mathbf{0}&\mathbf{0}&0&0&0&0&0&0\\ \mathbf{2}&\mathbf{0}&\mathbf{1}&\mathbf{1}&0&0&0&0\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{2}&0 \end{pmatrix}\in\mathcal{M}_{8,8}^{D_\Lambda}$ encoded as a tensor product of row tableaux $577\otimes 45\otimes 7\otimes 7\otimes 8\otimes\emptyset\otimes 88\otimes\emptyset$ on the alphabet $[8]$ where $a_{i,j}$ is the number of letters i in the tensor j-th component. One then applies the column insertion procedure from left to right. This means that we begin by reading the first column (of A) 775 and compute the column insertions $5 \to 7 \to 7$ to get 577 then read the second column 54 and compute the column insertion $4 \to 5 \to 577$ to get 45577, then $7 \to 45577$ to get $\begin{smallmatrix}7 \\ 4&5&5&7&7\end{smallmatrix}$, and eventually get the tableau P below. The "recording tableau" Q is obtained by filling with letters j the new boxes appearing during the insertion of column j of A,

$$P = \begin{array}{|c|c|c|c|c|} \hline 8&8 \\ \cline{1-3} 7&7&8 \\ \hline 4&5&5&7&7 \\ \hline \end{array} \qquad K^-(P) = \begin{array}{|c|c|c|c|c|} \hline 8&8 \\ \cline{1-3} 7&7&8 \\ \hline 4&4&4&4&4 \\ \hline \end{array} = K(0^3,5,0^2,2,3) \tag{4.1}$$

$$Q = \begin{array}{|c|c|c|c|c|} \hline 5&7 \\ \cline{1-3} 3&4&7 \\ \hline 1&1&1&2&2 \\ \hline \end{array} \qquad K^+(Q) = \begin{array}{|c|c|c|c|c|} \hline 7&7 \\ \cline{1-3} 4&4&7 \\ \hline 2&2&2&2&2 \\ \hline \end{array} = K(0,5,0,2,0^2,3,0). \tag{4.2}$$

We show that there exists $\mu = (\mu_1,\mu_2,\mu_3,\mu_4) \in \mathbb{Z}^4_{\geq 0}$ such that $\psi(A) = (P,Q) \in \iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(\sigma_0\mu,0^4)}) \times \Delta_3\Delta_6\Delta_5\Delta_4 B_{(\mu,0^4)}$, where $\sigma_0 \in \mathfrak{S}_4$ and ι is the Schützenberger (evacuation) involution. From (2.8) one has $\iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(\mu_4,\dots,\mu_1,0^4)}) =$

$$= \begin{cases} \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,0,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_2,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0^2,\mu_2,0^3)}, \text{ if } \mu_2 > \mu_1 = 0 \quad (**) \\ \iota\overline{B}_{(\mu_4,\mu_3,0,0,0^4)}, \text{ if } \mu_1 = \mu_2 = 0 \\ \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,\mu_2,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,0,\mu_2,0^3)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_2,\mu_2,0^3)}, \text{ if } \mu_1 = \mu_2 > 0 \\ \emptyset, \text{ if } \mu_1 > \mu_2 \geq 0 \\ \iota\overline{B}_{(\mu_4,\dots,\mu_1,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_1,\mu_2,0^4)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)} \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,0,\mu_1,\mu_2,0^3)} \\ \bigsqcup \iota\overline{B}_{(\mu_4,\mu_3,\mu_1,0,\mu_2,0^3)}, \text{ if } \mu_2 > \mu_1 > 0. \end{cases}$$

Then, by (2.2), $K^-(P) = K(0^3,5,0^2,2,3) \Leftrightarrow P \in \overline{B}^{(0^3,5,0^2,2,3)} = \iota\overline{B}_{(3,2,0^2,5,0^3)}$, and we are in case $(**)$, where $\mu_2 = 5 > \mu_1 = 0$, $\mu_3 = 2$, $\mu_4 = 3$. Hence, $\mu = (0,5,2,3)$ and $\iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(3,2,5,0,0^4)}) = \iota\overline{B}_{(3,2,5,0,0^4)} \bigsqcup \iota\overline{B}_{(3,2,0,5,0^4)} \bigsqcup \iota\overline{B}_{(3,2,0^2,5,0^3)}$. Therefore, by the LHS of (2.8), $\Delta_3\Delta_6\Delta_5\Delta_4 B_{(\mu,0^4)} = B_{(0,5,0,2,0,0,3,0)}$. Indeed $K_+(Q) \leq K(0,5,0,2,0^2,3,0)$ and from (2.4), $Q \in B_{(0,5,0,2,0^2,3,0)}$. Hence, $(P,Q) \in \overline{B}^{(0^3,5,0^2,2,3)} \times B_{(0,5,0,2,0^2,3,0)}$ and $\psi(A) \in \iota(\dot\Delta_4\dot\Delta_3\dot\Delta_4\overline{B}_{(3,2,5,0,0^4)}) \times \Delta_3\Delta_6\Delta_5\Delta_4 B_{(0,5,2,3,0^4)}$.

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Tamari intervals and blossoming trees

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Abstract. We introduce a simple bijection between Tamari intervals and the blossoming trees (Poulalhon and Schaeffer, 2006) encoding planar triangulations, using a new meandering representation of such trees. Its specializations to the families of synchronized, Kreweras, new/modern, and infinitely modern intervals give a combinatorial proof of the counting formula for each family. Compared to (Bernardi and Bonichon, 2009), our bijection behaves well with the duality of Tamari intervals, enabling also the counting of self-dual intervals.

Résumé. Nous donnons une nouvelle bijection simple entre les intervalles de Tamari et les arbres bourgeonnants (Poulalhon et Schaeffer, 2006) qui encodent les triangulations planaires, en passant par une nouvelle représentation méandrique de ces arbres. Les spécialisations aux familles des intervalles synchrones, Kreweras, nouveaux/modernes, et infiniment modernes donnent des preuves combinatoires des formules de comptage pour ces familles. Par rapport à (Bernardi et Bonichon, 2009), notre bijection se comporte bien vis-à-vis de la dualité des intervalles de Tamari, nous permettant de compter les intervalles auto-duaux.

Keywords: Tamari intervals, blossoming trees, enumeration, duality

1 Introduction

The Tamari lattice Tam_n is a well-known poset on Catalan objects of size n, that plays an important role in several domains, such as representation theory [1, 5], polyhedral combinatorics and Hopf algebras [3, 13]. Partially motivated by such links, the enumeration of intervals in the Tamari lattice was first considered by Chapoton [6] who discovered the beautiful formula

$$I_n = \frac{2}{n(n+1)}\binom{4n+1}{n-1} \tag{1.1}$$

for the number of intervals in Tam_n. The subject has attracted much attention since then, with strikingly simple counting formulas found for several other families [4, 5, 10].

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Regarding combinatorial proofs, Bernardi and Bonichon [2] gave a bijection from Tamari intervals to planar (simple) triangulations via Schnyder woods. Then, a bijection by Poulalhon and Schaeffer [15] encodes the same triangulations by a class of blossoming trees, which yields (1.1). The bijection in [2] can be specialized to some subfamilies of Tamari intervals, such as Kreweras intervals [2] and synchronized Tamari intervals [11]. Another strategy, for instance in [10], is to construct bijections between Tamari intervals and planar maps inspired by their recursive decompositions.

In this extended abstract, we present a more direct bijection between Tamari intervals and the blossoming trees from [15]. Readers are referred to [9] for the full version. Our construction, presented in Sections 2 and 3, starts from a suitable planar representation of an interval as a pair of binary trees. With simple local operations, we get a "meandering representation" of the interval, closely related to interval-posets of Châtel and Pons [7]. Such a representation can be seen as a folded version of a blossoming tree. When unfolded, the blossoming tree is characterized by local conditions as in [15]. We also find it convenient to give a certain bicoloring to half-edges in blossoming trees, which breaks symmetries.

Due to its simplicity, our bijection is also well-suited for specializations to known subfamilies of Tamari intervals, by characterizing the blossoming trees in each case (Theorem 4.5). In addition to synchronized intervals, whose specialization is much simpler than that in [11], and Kreweras intervals, already given in [2], our bijection also specializes to new/modern intervals [6, 16] and infinitely modern intervals [16]. Compared to [2], our bijection has also the advantage that it transfers the duality involution on Tamari intervals in a simple way, which amounts to a color-switch in blossoming trees (Proposition 4.6). Self-dual intervals thus correspond to blossoming trees with a half-turn symmetry, which are easy to count for each family we consider (see Table 1), leading to counting formulas that are new to our knowledge, except for Kreweras intervals, for which it is known.

The following statement summarizes our main results.

Theorem 1.1. *There is a bijection Φ between intervals in* Tam_n *and bicolored blossoming trees of size n that sends self-dual intervals to blossoming trees with a half-turn symmetry. Its specialization to synchronized, Kreweras, modern/new, and infinitely modern intervals yields combinatorial proofs of counting formulas for intervals and self-dual intervals in each case, see Table 1.*

Finally, besides color switch, another natural involution on blossoming trees is to apply a reflection. This yields a new involution on Tamari intervals with interesting properties, see Remark 4.7.

2 Tamari intervals and their meandering representation

Let $\mathcal{T}_n$ be the set of rooted binary trees with n nodes. Recall that the Tamari lattice Tam_n is the poset $(\mathcal{T}_n, \leq)$ whose covering relations are given by right rotations, i.e., changing a subtree of the form $((T_1, T_2), T_3))$ into $(T_1, (T_2, T_3))$. An *interval* in Tam_n is a pair (T, T') such that $T \leq T'$. Let $\mathcal{X}_n = \mathcal{T}_n \times \mathcal{T}_n$, and $\mathcal{I}_n \subseteq \mathcal{X}_n$ the set of intervals in Tam_n. In the following, we denote by $[n]$ the set $\{1, \ldots, n\} \subset \mathbb{N}$.

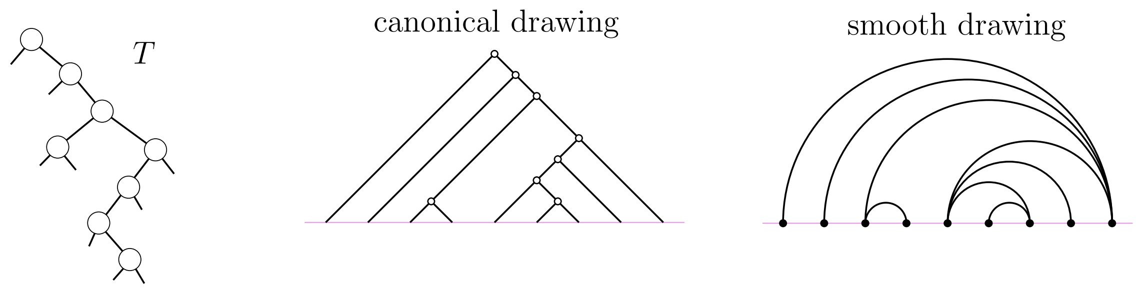

Figure 1: A binary tree T with its canonical drawing and smooth drawing.

We first review some representations and encodings of binary trees. For $T \in \mathcal{T}_n$, the *canonical drawing* of T is the crossing-free drawing of T with its $n+1$ leaves placed from left to right at the points of abscissas $0, \ldots, n$ on the x-axis, its nodes in the upper half-plane, and its left (resp. right) branches being segments of slope 1 (resp. -1). The *smooth drawing* of T is obtained by removing all lines, then for each node u, adding a semi-circle in the upper half-plane linking the leftmost and the rightmost leaves of the subtree induced by u, see Figure 1. For $t \in [n]$, let A_t be the unique arc covering the unit-segment $[t-1, t]$ and visible from it, see the left-part of Figure 2.

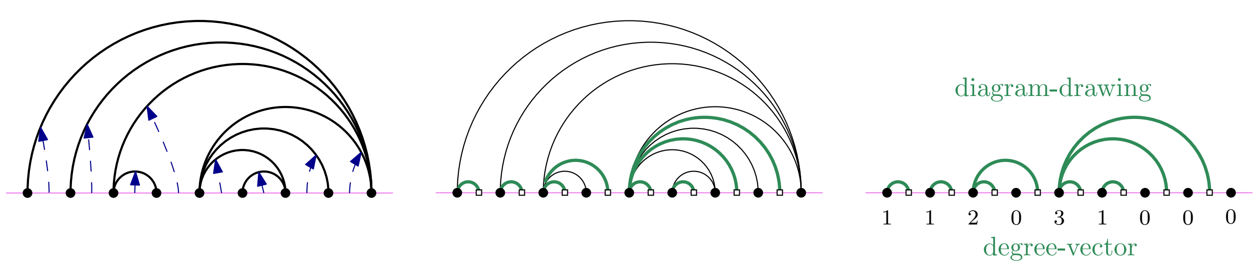

Figure 2: Construction of the diagram-drawing from the smooth drawing, with its degree-vector.

For $T \in \mathcal{T}_n$, the *diagram-drawing* $\widehat{T}$ of T is obtained from the smooth drawing of T as follows. For each $t \in [n]$, we add a white point at $t - \frac{1}{2}$, and we replace A_t by an arc

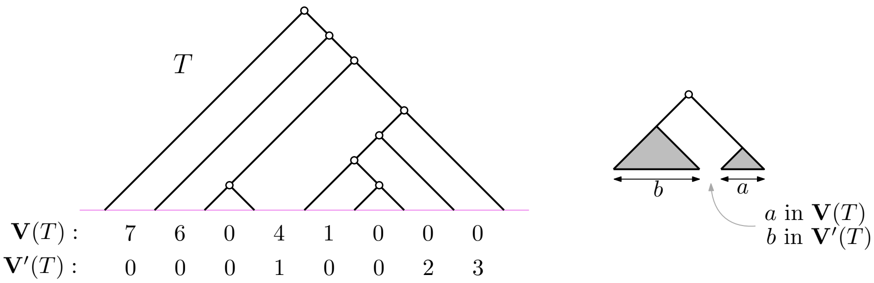

Figure 3: A binary tree T, its bracket-vector $\mathbf{V}(T)$ and dual bracket-vector $\mathbf{V}'(T)$.

from this white point to the black point at the left end of A_t, see Figure 2. To recover the smooth drawing from $\widehat{T}$, for each white point w of $\widehat{T}$, its *right-attachment point* b is the black point at $x = n$ if there is no arc above w, and is the black point to the left of w' if w is covered by an arc $b' \to w'$. Then, to obtain the smooth drawing of T, each arc $b \to w$ in $\widehat{T}$ is replaced by an arc connecting b to the right-attachment point of w.

For $T \in \mathcal{T}_n$, the *degree-vector* of T is the vector $\mathrm{Deg}_{\nearrow}(T) = (d_0, \ldots, d_n)$ such that d_i is the number of arcs incident to the black point b at $x = i$ in the diagram-drawing $\widehat{T}$ for $0 \leq i \leq n$. We see that d_i is also the right-degree of b in the smooth drawing of T, and is the length of the left branch of T ending at the leaf at abscissa i in the canonical drawing. The diagram-drawing of T is easily recovered from its degree-vector.

Finally, we recall the bracket-vector and dual bracket-vector encoding of a binary tree $T \in \mathcal{T}_n$. We label the nodes of T by left-to-right infix order, with v_i the node of label $i \in [n]$. Let a_i (resp. b_i) be the size of the right (resp. left) subtree of v_i. The *bracket-vector* of T is $\mathbf{V}(T) = (a_1, \ldots, a_n)$, and the *dual bracket-vector* of T is $\mathbf{V}'(T) = (b_1, \ldots, b_n)$, see Figure 3 for an illustration. These vectors can also be specified by inequality constraints, which we do not reproduce here, see [12]. The bracket-vector encoding is convenient to characterize Tamari intervals. For $(T, T') \in \mathcal{X}_n$, it is known [12] that $(T, T') \in \mathcal{I}_n$ if and only if $\mathbf{V}(T) \leq \mathbf{V}(T')$ componentwise, or equivalently, $\mathbf{V}'(T) \geq \mathbf{V}'(T')$ componentwise.

Remark 2.1. The dual bracket-vector is closely related to the diagram drawing. For $T \in \mathcal{T}_n$ and $t \in [n]$, the unique arc at the white point $t - \frac{1}{2}$ is connected to the black vertex at $x = t - 1 - b_t$.

The *mirror* of a binary tree T, denoted by $\mathrm{mir}(T)$, is the mirror image of T exchanging left and right. The *mirror canonical drawing* (resp. *mirror smooth drawing*) of T is the canonical drawing (resp. smooth drawing) of $\mathrm{mir}(T)$ rotated by a half-turn, which preserves the left-to-right order of leaves of T.

For $X = (T, T') \in \mathcal{X}_n$, the *canonical drawing* (resp. *smooth drawing*) of X is the superimposition of the canonical (resp. smooth) drawing of T' with the mirror canonical (resp. smooth) drawing of T, see Figure 4. In this case, the *upper diagram-drawing* of X is the diagram-drawing of T', while the *lower diagram-drawing* of X is the diagram-drawing

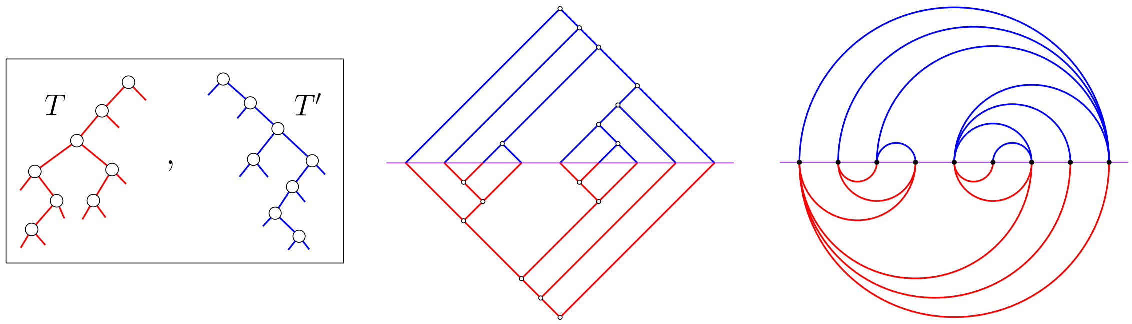

Figure 4: A pair (T, T') of binary trees of the same size, its canonical drawing, and its smooth drawing.

of $\text{mir}(T)$ rotated by a half-turn. The *diagram-drawing* of X is the superimposition of the upper and lower diagram-drawings of X. As a convention, in each of the 3 representations of X, the arcs are blue (resp. red) in the upper (resp. lower) part. Let ϕ be the mapping that sends $X \in \mathcal{X}_n$ to its diagram-drawing, see Figure 5.

Definition 2.2. A *meandering diagram* of size n is a non-crossing arc-diagram M with $2n+1$ points, at $0, \frac{1}{2}, 1, \ldots, n-\frac{1}{2}, n$ on the x-axis, colored black for integral points and white for half-integral ones, with all upper (resp. lower) arcs having a black (resp. white) left end and a white (resp. black) right end, such that each white point is incident to exactly one upper (resp. lower) arc. The *underlying graph* of M is the graph with black points as vertices, where each white point yields an edge connecting the black endpoints of its incident upper and lower arcs. A *meandering tree* is a meandering diagram whose underlying graph is a tree. Let $\mathcal{MD}_n$ (resp. $\mathcal{MT}_n$) be the set of meandering diagrams (resp. meandering trees) of size n.

Proposition 2.3. *For $n \geq 1$, the mapping ϕ is a bijection between $\mathcal{X}_n$ and $\mathcal{MD}_n$. It specializes to a bijection between $\mathcal{I}_n$ and $\mathcal{MT}_n$.*

Sketch of proof. For the first statement, the inverse ψ of ϕ is obtained by the equivalence between the representations of binary trees discussed above. For $M \in \mathcal{MD}_n$, we consider the upper part of M as an upper diagram-drawing, from which we compute the corresponding smooth drawing, and turn it into the canonical drawing of a binary tree T'. We do the same for the half-turn of the lower diagram-drawing, yielding a binary tree, with T its mirror. Then we take $\psi(M) = (T, T')$.

For the second statement, we use the fact that $X \in \mathcal{X}_n$ is in $\mathcal{I}_n$ if and only if its smooth drawing has no pair of arcs as on the left side of Figure 6, which follows from the bracket-vector characterization of Tamari intervals, and is closely related to the Tamari diagrams

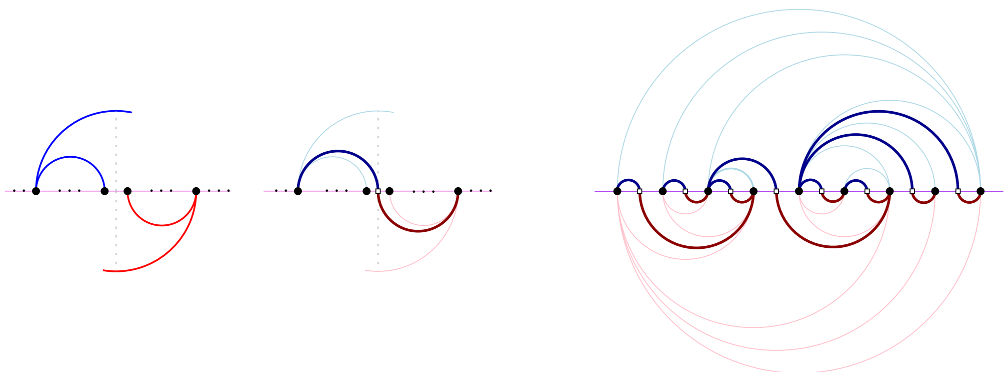

Figure 5: Left: The action of ϕ on each segment of consecutive points on the x-axis of the smooth drawing of $X \in \mathcal{X}_n$ (the shorter blue and red arcs in the left drawing may be reduced to a point). Right: the diagram-drawing $M = \phi(X)$ for the pair X in Figure 4, which is a meandering tree, meaning $X \in \mathcal{I}_n$ by Proposition 2.3.

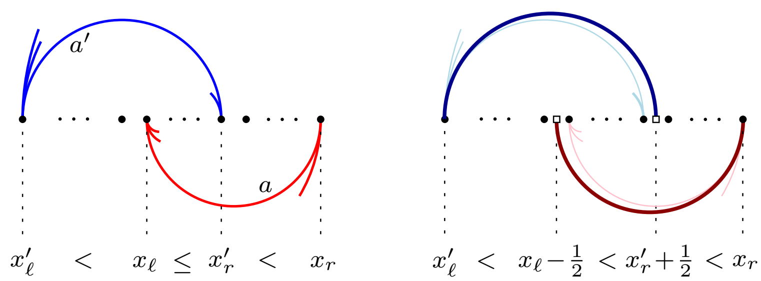

Figure 6: The forbidden pattern for $X \in \mathcal{X}_n$ to be in $\mathcal{I}_n$ (left) corresponds via ϕ to the forbidden pattern for $M \in \mathcal{MD}_n$ to be in $\mathcal{MT}_n$ (right).

in [8]. We then show that $M \in \mathcal{MD}_n$ is in $\mathcal{MT}_n$ if and only if it has no pair of arcs as on the right side of Figure 6, and that these patterns are in correspondence via ϕ. □

Remark 2.4. Each diagram $M \in \mathcal{MD}_n$ yields a relation R on integers in $[n]$ where $i \, R \, j$ if the edge $\{a_j, b_j\}$ of the underlying graph in Definition 2.2 associated to the white point $j - \frac{1}{2}$ satisfies $[i-1, i] \subseteq [a_j, b_j]$. It can be shown that R defines a poset if and only if $M \in \mathcal{MT}_n$. In this case, by construction, $([n], R)$ is an interval-poset defined in [7]. Let $I = \psi(M)$, by Proposition 2.3, we have $I \in \mathcal{I}_n$. We checked that $([n], R)$ is the interval-poset of I under the bijection in [7].

Recalling Remark 2.1, the mapping ϕ can be formulated simply in terms of the bracket-vector of T and dual bracket-vector of T'.

Proposition 2.5. *Let $(T, T') \in \mathcal{X}_n$, $\mathbf{V}(T) = (a_1, \ldots, a_n)$, and $\mathbf{V}'(T') = (b_1, \ldots, b_n)$. Then $\phi(X)$ is given by its lower arcs $(t - \frac{1}{2}, t + a_t)$ and upper arcs $(t - \frac{1}{2}, t - b_t - 1)$ for all $t \in [n]$.*

3 Blossoming trees and their meandering representation

We consider the following trees, which are in bijection with simple triangulations [15].

Definition 3.1. A *blossoming tree* B is an unrooted plane tree such that each *node*, that is, vertex of degree at least 2, has exactly two neighbors that are *leaves*, which are vertices of degree 1. We only consider such trees with at least two nodes. Edges incident to leaves are called *buds*, drawn as an outgoing arrow, and all other edges are called *plain edges*. The *size* of B is its number of plain edges, which is also its number of nodes minus 1.

A blossoming tree is *bicolored* if each plain edge has one half-edge colored red and the other blue, such that the half-edges at each node are separated by the two incident buds into a group of blue and a group of red, one of the groups being possibly empty. See Figure 8(a) for an example. We note that a blossoming tree yields at most two bicolored blossoming trees, since the bicoloring is uniquely determined once the color of a half-edge is fixed. It yields just one if and only if it possesses the half-turn symmetry. We denote by $\mathcal{B}_n$ the set of bicolored blossoming trees of size n.

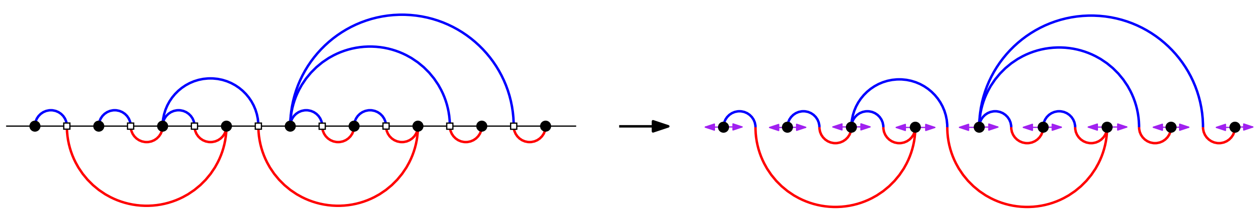

Figure 7: A meandering tree and the corresponding bicolored blossoming tree.

For $M \in \mathcal{MT}_n$, we construct $B \in \mathcal{B}_n$ by adding a "left" and a "right" bud at each black point along the x-axis, while keeping the colors of arcs, which are turned into half-edges of plain edges in B, see Figure 7. Let γ be the mapping sending M to B.

Conversely, given a bicolored blossoming tree B, its *closure*, denoted by $\overline{B}$, is constructed as follows, see Figure 8. For each plain edge e, we insert an *edge-vertex* v_e in its middle, and we attach to v_e two unmatched half-edges called *legs*, one on each side of e. The counterclockwise-contour of B yields a cyclic word of parentheses, whose opening (resp. closing) ones are given by buds (resp. legs). We then match buds and legs in a planar way, see Figure 8(b). Since B has $2n+2$ buds and $2n$ legs, two buds are left unmatched. It is easily checked that the two unmatched buds of the closure $\overline{B}$ are at distinct vertices, which are called the *extremal vertices* of $\overline{B}$.

Lemma 3.2. *For $B \in \mathcal{B}_n$, let $\overline{B}$ be the closure of B, and π the subgraph of $\overline{B}$ induced by all closure-edges, that is, those obtained by matching a bud with a leg. Then*

- *π is a Hamiltonian path of $\overline{B}$ whose ends are the two extremal vertices;*

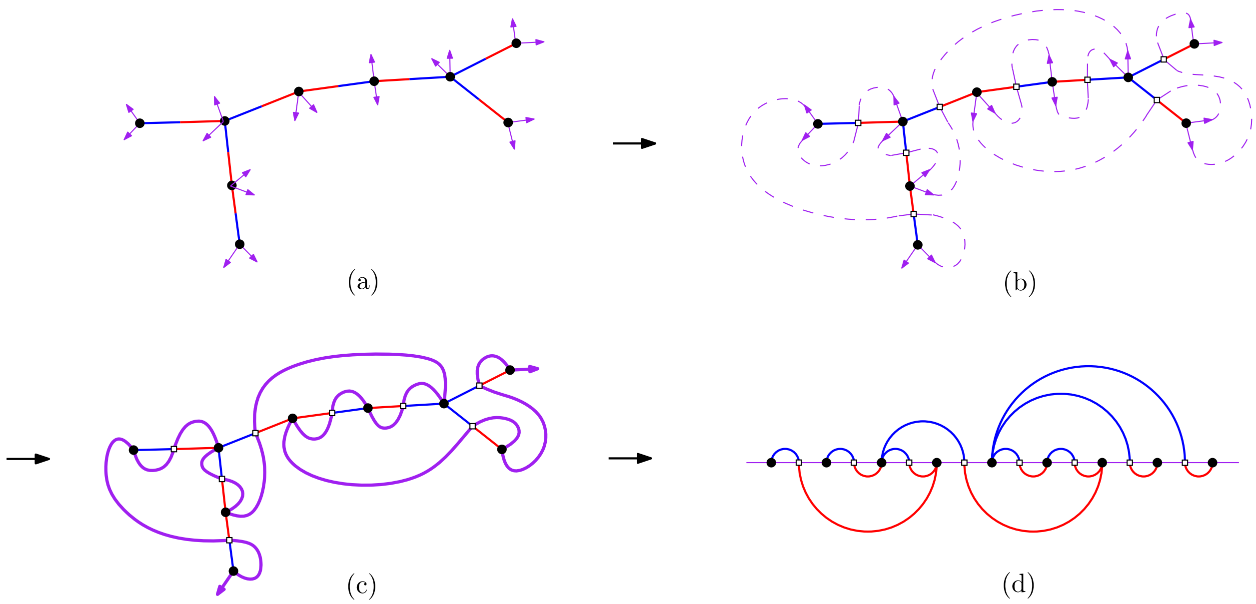

Figure 8: (a) A bicolored blossoming tree B; (b) the matching of buds with legs; (c) the closure $\overline{B}$ of B, where the meandric path is shown in bold; (d) the meandering tree $M = \delta(B)$ obtained by stretching the meandric path.

- *π splits half-edges of B by color;*
- *For any edge $e = \{u, v\}$ of $\overline{B}$ corresponding to a half-edge of plain edge of B, with v the edge-vertex end, let π_e be the unique subpath of π from u to v, and $\sigma_e = \pi_e \cup \{e\}$, which is a cycle. Then, the interior of σ_e is on the right of e traversed from u to v.*

The Hamiltonian path π of $\overline{B}$ in Lemma 3.2 is called the *meandric path* of $\overline{B}$. From the first statement of Lemma 3.2, for $B \in \mathcal{B}_n$, we may stretch the meandric path of $\overline{B}$ into the horizontal segment $\{0 \leq x \leq n, y = 0\}$ with $2n + 1$ equally-spaced vertices, along with arcs as semi-circles. By the second statement of Lemma 3.2, this can be done in a unique way with the blue (resp. red) half-edges of B turned into the arcs above (resp. below) the segment. Let M be the arc-diagram thus obtained, then the third statement of Lemma 3.2 ensures that $M \in \mathcal{MT}_n$. We define δ as the mapping that sends B to M.

Proposition 3.3. *For $n \geq 1$, the mapping γ is a bijection from $\mathcal{MT}_n$ to $\mathcal{B}_n$, with δ its inverse.*

4 The main bijection: properties and enumeration results

Combining Propositions 2.3 and 3.3, we obtain the following.

Theorem 4.1. *The mapping $\Phi := \gamma \circ \phi$ is a bijection from $\mathcal{I}_n$ to $\mathcal{B}_n$. Its inverse is $\Psi := \psi \circ \delta$.*

It is possible to track several parameters via the bijection. For $X = (T, T') \in \mathcal{X}_n$ given as it canonical drawing, and for $0 \leq i \leq n$, the *bi-degree* of X at i is the pair (d, d') such that the right branch of T (resp. left branch of T') ending at i on the horizontal axis has d (resp. d') nodes. In other words, $\mathrm{Deg}_{\nearrow}(T')_i = d'$ and $\mathrm{Deg}_{\nearrow}(\mathrm{mir}(T))_{n-i} = d$. The *canopy-type* of X at i is $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$, where $s = \mathbb{1}_{d'>0}$ and $r = \mathbb{1}_{d=0}$ are given by indicator functions. We note that, if $X \in \mathcal{I}_n$, then the canopy-type $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ can not occur. For $B \in \mathcal{B}_n$, the *bi-degree* of a node $v \in B$ is the pair (d, d') such that d (resp. d') is the number of red (resp. blue) half-edges at v, and the *canopy-type* of v is $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ where $s = \mathbb{1}_{d'>0}$ and $r = \mathbb{1}_{d=0}$.

Proposition 4.2. *For $X \in \mathcal{I}_n$ and $B = \Phi(X)$, each index $0 \leq i \leq n$ corresponds to a node $v \in B$ of same bi-degree, and thus same canopy-type.*

Remark 4.3. It seems harder to read the lengths of left branches of T and right branches of T'. A clear way to read these lengths could yield bijective proofs of the counting formulas for m-Tamari intervals [5] (via [14, Prop.72]) and for labeled Tamari intervals [4].

The number of entries of each canopy-type is then easy to track in bicolored blossoming trees using a root-decomposition, yielding the following counting formulas.

Corollary 4.4. *We denote by $J_{i,j}(n)$ the number of Tamari intervals of size n with $i+1$ canopy-entries $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $j+1$ canopy-entries $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, and thus $n-1-i-j$ canopy-entries $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Let $A \equiv A(t; x, y)$ and $B \equiv B(t; x, y)$ be the trivariate series specified by*

$$A = \frac{t}{(1-B)^2}\left(y + \frac{A}{1-A}\right), \quad B = \frac{t}{(1-A)^2}\left(x + \frac{B}{1-B}\right). \tag{4.1}$$

Then we have

$$J_{i,j}(n) = \frac{1}{n}[t^{n+1}x^{i+1}y^{j+1}]AB. \tag{4.2}$$

In particular, using Lagrange inversion, the coefficients $S_{i,j} := J_{i,j}(i+j+1)$ and $J_k(n) := \sum_{i+j=k} J_{i,j}(n)$ are given by

$$S_{i,j} = \frac{1}{(i+1)(j+1)}\binom{2i+j+1}{j}\binom{2j+i+1}{i}, \quad J_k(n) = \frac{2}{n(n+1)}\binom{3n}{k-2}\binom{n+1}{k}, \tag{4.3}$$

where $S_{i,j}$ counts synchronized Tamari intervals, i.e., those with no canopy-entry of type $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ (cf. [10, Section 2]).

The expression of $S_{i,j}$ in Equation (4.3) can be obtained using bijections in [10] or in [11] to planar non-separable maps counted by vertices and faces. On the other hand, the coefficients $J_k(n)$ count Tamari intervals by size and number of synchronized entries (type $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$), and it has been recently computed in [3] by solving functional equations, and via the Bernardi-Bonichon bijection building upon [11]. The derivation with our bijection is however more direct.

Besides synchronized intervals, the bijection Φ can also be specialized to other known families of Tamari intervals, as listed in Table 1, namely:

- modern intervals [16], i.e., intervals $I = (T, T')$ whose "rise" $((T, \epsilon), (\epsilon, T'))$ is also a Tamari interval. In this case, the rise is a "new" Tamari interval defined in [6],
- infinitely modern intervals [16], i.e., intervals such that all iterated rises are also Tamari intervals,
- Tamari intervals corresponding to Kreweras intervals (*cf.* [2] and references therein) under the standard bijection from binary trees to non-crossing partitions: the parts are given by right branches of the binary tree with nodes labeled by infix order.

Theorem 4.5. *For each of the following families of Tamari intervals, the associated bicolored blossoming trees given by Φ are characterized by the following conditions:*

- **Synchronized***: for each node, its two incident buds are consecutive in cyclic order.*
- **Modern***: for every plain edge, at least one end is followed by a bud in clockwise order.*
- **Infinitely modern***: for every path of plain edges, at least one end is followed by a bud in clockwise-order.*
- **Kreweras***: for every path of plain edges, at least one end is followed by a bud in counter-clockwise order.*

These conditions amount to forbidding in bicolored blossoming trees the patterns illustrated in Figure 9. In each case, a decomposition of the corresponding trees yields a combinatorial proof of the known counting formula, given by the first column in Table 1.

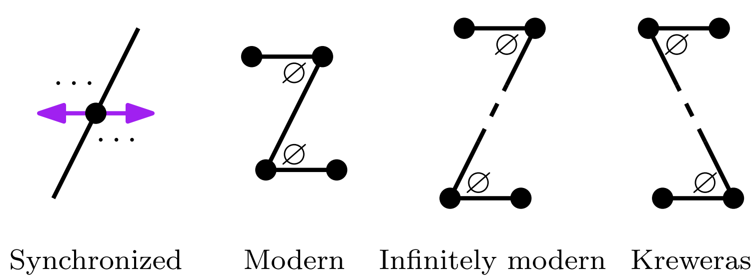

Figure 9: Forbidden patterns of blossoming trees for subfamilies of Tamari intervals.

We define $\mathrm{mir}(X)$ for $X = (T, T') \in \mathcal{X}_n$ as $\mathrm{mir}(X) = (\mathrm{mir}(T'), \mathrm{mir}(T))$. We call mir the *duality* on $\mathcal{X}_n$, it is an involution on $\mathcal{X}_n$ and on $\mathcal{I}_n$. Its name comes from the fact that mir on binary trees is the duality map for Tam_n.

Types	All, size n	Self-dual, size $2m$	Self-dual, size $2m+1$
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	$\frac{1}{3m+1}\binom{4m}{m}$	$\frac{1}{m+1}\binom{4m+2}{m}$
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	0	$\frac{1}{m+1}\binom{3m+1}{m}$
Modern / New (for size-1)	$\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$	$\frac{2^{m-1}}{m+1}\binom{2m}{m}$	$\frac{2^{m}}{m+1}\binom{2m}{m}$
Modern and synchronized	$\frac{1}{n+1}\binom{2n}{n}$	0	$\frac{1}{m+1}\binom{2m}{m}$
Inf. modern / Kreweras	$\frac{1}{2n+1}\binom{3n}{n}$	$\frac{1}{2m+1}\binom{3m}{m}$	$\frac{1}{m+1}\binom{3m+1}{m}$

Table 1: Counting formulas for Tamari intervals and self-dual ones.

Proposition 4.6. *For $I \in \mathcal{I}_n$, we obtain $\Phi(\mathrm{mir}(I))$ by switching colors of half-edges in $\Phi(I)$. Hence, self-dual intervals are mapped by Φ to blossoming trees with half-turn symmetry.*

For each of the families in Table 1, one can easily count the corresponding blossoming trees that are half-turn symmetric. This yields the formulas shown in the second and the third column in Table 1, which are new to our knowledge, except for Kreweras.

Remark 4.7. We define the *reflection* of a blossoming tree to be its mirror image. It is clear that reflection commutes with color switch on blossoming trees, and it is transferred by Ψ to an involution on Tamari intervals. Combined with Theorem 4.5, with forbidden patterns illustrated in Figure 9, we see that synchronized intervals are stable by this new involution, while infinitely modern intervals are matched with Kreweras intervals.

It was previously known that both infinitely modern intervals and Kreweras intervals are equinumerous to ternary trees [16], but they seem to have very different structures. Our involution somehow relates these two families. We plan to further explore properties of this new involution.

Remark 4.8. Regarding the counting formulas for self-dual intervals in Table 1, one observes that, in the cases of general and synchronized intervals, they are given by a simple q-analogue of the formula for all intervals taken at $q = -1$. It would be nice to have a natural explanation of this fact. This may come from a combinatorial analysis of blossoming trees.

Acknowledgements

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Eulerian Polynomials for Digraphs

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Abstract. Given an n-vertex digraph D and a labeling $\sigma : V(D) \to [n]$, we say that an arc $u \to v$ of D is a descent of σ if $\sigma(u) > \sigma(v)$. Foata and Zeilberger introduced a generating function $A_D(t)$ for labelings of D weighted by descents, which simultaneously generalizes both Eulerian polynomials and Mahonian polynomials. Motivated by work of Kalai, we look at problems related to -1 evaluations of $A_D(t)$. In particular, we give a combinatorial interpretation of $|A_D(-1)|$ in terms of "generalized alternating permutations" whenever the underlying graph of D is bipartite.

Keywords: Eulerian polynomial, alternating permutations, combinatorial reciprocity

1 Introduction

Descents and inversions are two of the oldest and most well-studied *permutation statistics* dating back to work of MacMahon [15, 14]. A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ on the set $[n] := \{1, 2, \ldots, n\}$ is an index $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$, and an inversion is a pair of integers (i, j) with $1 \le i < j \le n$ such that $\sigma(i) > \sigma(j)$. The number of descents and inversions of σ are denoted by $\operatorname{des}(\sigma)$ and $\operatorname{inv}(\sigma)$, respectively.

The generating functions

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)} \quad M_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{inv}(\sigma)}$$

are called the *Eulerian* and *Mahonian* polynomials respectively. Both of these polynomials are important objects of study in many branches of combinatorics and have been generalized in many different ways. In this paper, we consider a polynomial due to Foata and Zeilberger [9] which generalizes both the Eulerian and Mahonian polynomials via directed graphs.

A *permutation* of an n-vertex digraph $D = (V, E)$ is a bijection $\sigma : V \to [n]$. We will use the notation $\mathfrak{S}_D$, $\mathfrak{S}_V$, or $\mathfrak{S}_n$ to denote the set of permutations of D. For a given

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directed graph $D = (V, E)$ and a permutation σ of D, a *D-descent* (or just *descent* when D is understood) is an arc $u \to v$ such that $\sigma(u) > \sigma(v)$. The total number of D-descents of a permutation σ is denoted by $\mathrm{des}_D(\sigma)$; see Figure 1 for an example.

Figure 1: Two labelings $\pi : V(D) \to [5]$ where descent arcs are marked by red dashed lines.

These statistics generalize both of des and inv as Figure 2 shows.

(a) $\mathrm{des}(23154) = 2$ (b) $\mathrm{inv}(23154) = 3$

Figure 2: The Eulerian polynomial $A_D(t)$ generalizes both descents and inversions.

With all this in mind, we can now define the central object of study for this paper: the *Eulerian polynomial* of a digraph $D = (V, E)$ is the generating function

$$A_D(t) = \sum_{\sigma \in \mathfrak{S}_D} t^{\mathrm{des}_D(\sigma)}. \tag{1.1}$$

In particular, we have $A_{\vec{P_n}}(t) = A_n(t)$ and $A_{\vec{K_n}}(t) = M_n(t)$.

This polynomial can be seen in other work: as a *weighted-inversion* generating function as in [11, 5]; as an *Eulerian polynomial for a (particular) family $\mathfrak{B}_n$ of digraphs* [1]; as a specialization of the *chromatic quasisymmetric function for digraph* [6] and *B-polynomial* [2]. There are also a myriad of other objects generalizing Eulerian polynomials which are related by varying degrees to ours.

The primary objective of this extended abstract is to study evaluations of $A_D(t)$ at -1. See [4] for the full paper. This is a problem in the area of *combinatorial reciprocity*, which studies combinatorial polynomials evaluated at negative integers. For example, the classical Eulerian and Mahonian polynomials both have good combinatorial interpretations for their evaluation at -1: the former being the number of *alternating permutations* [8] and the latter being the number of *correct proofs of the Riemann hypothesis*[1]. Many more results on combinatorial reciprocity can be found in the book by Beck and Sanyal [3].

[1]As of the time of writing.

Kalai [12, Section 8.1] makes a critical observation about $A_D(-1)$.

Proposition 1.1. *If D, D' are orientations of the same graph G, then $|A_D(-1)| = |A_{D'}(-1)|$.*

With Proposition 1.1 in mind, for any graph G we can define

$$\nu(G) := |A_D(-1)|,$$

where D is any orientation of G. The problem of studying $\nu(G)$ was first introduced by Kalai [12] due to its relation with the Condorcet paradox in social choice theory, and a few basic properties of $\nu(G)$ were established by Even-Zohar [7]. Outside of this, nothing seems to be known about $\nu(G)$ despite Kalai raising the problem over 20 years ago.

In this extended abstract, we prove three types of results related to $\nu(G)$: we give combinatorial interpretations for $\nu(G)$ for a large class of graphs G, we determine the maximum and minimum values achieved by $\nu(G)$ amongst n vertex trees, and we consider the refined problem of determining the multiplicity of -1 as a root of $A_D(t)$.

2 Combinatorial Interpretations for $\nu(G)$

A classical result of Foata and Schützenberger [8] (see also [16, Exercise 135]) states that for odd n the Eulerian polynomial $A_n(t)$ evaluated at $t = -1$ is equal (up to sign) to the number of *alternating permutations* of length n, i.e. the number of permutations σ and $\sigma(1) < \sigma(2) > \sigma(3) < \cdots > \sigma(n)$. Because $A_n(t) = A_{\overrightarrow{P}_n}(t)$ for $\overrightarrow{P}_n$ the directed path, this result implies $\nu(P_n)$ is equal to the number of alternating permutations of size n.

Given this observation, it is natural to expect $\nu(G)$ to count "alternating permutations for graphs" for some generalized notion of alternating permutations. There are many such generalizations one could consider, for example, one could force every maximal path of G to be an alternating permutation. However, it turns out that the definition we will want to consider is the following (non-obvious) generalization.

Definition 2.1. Given an n-vertex graph G, we say that an ordering $\pi = (\pi_1, \ldots, \pi_n)$ of the vertex set $V(G)$ is an *even sequence* if each of the subgraphs $G[\pi_1, \ldots, \pi_i]$ induced by the first i vertices of π have an even number of edges for all $1 \le i \le n$. We let $\eta(G)$ denote the number of even sequences of G.

Lemma 2.2. *For any graph G,*

(a) $\nu(G) \le \sum_{v \in V(G)} \nu(G - v)$.

(b) If G has an odd number of edges, $\eta(G) = 0$. Otherwise, $\eta(G) = \sum_{v \in V(G)} \eta(G - v)$.

(c) $\nu(G) \le \eta(G)$.

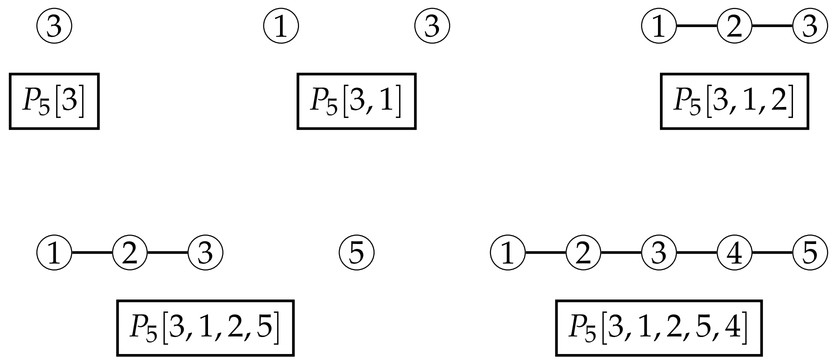


Figure 3: A depiction of the induced subgraphs $P_5[\pi_1,\ldots,\pi_i]$ for the ordering $\pi = (3,1,2,5,4)$ of the path graph P_5. Note that π is an even sequence since each of these induced subgraphs have an even number of edges. We also observe that $\pi^{-1} = (2,3,1,5,4)$ is an alternating permutation.

One can verify that even sequences for the path graph P_n with vertex set $[n]$ are exactly inverses of alternating permutations of size n, so $\nu(P_n) = \eta(P_n)$ in this case. Our main result shows that this equality holds for a substantially larger class of graphs.

To state this result, we remind the reader that a graph is *complete multipartite* if one can partition its vertices into sets $V_1,\ldots,V_r$ such that u and v are adjacent if and only if $u \in V_i, v \in V_j$ for some $i \neq j$. We say that a graph is a *blowup of a cycle* if one can partition its vertices into sets $V_1,\ldots,V_r$ such that u and v are adjacent if and only if $u \in V_i$ and $v \in V_{i+1}$ for some i (with the indices written mod r).

Theorem 2.3. *If G is a graph which is either bipartite, complete multipartite, or a blowup of a cycle, then $\nu(G) = \eta(G)$.*

The proofs for each of these cases follows the same basic strategy: We first show that for some "natural" orientation D of G, we can easily predict the sign of $A_D(-1)$. From this we deduce $\nu(G) = \sum \nu(G-v)$, and hence that $\nu(G) = \eta(G)$ since the statistics ν, η satisfy the same recurrence relation. Accordingly, we will only discuss the proof for bipartite graphs in this extended abstract, leaving the other classes of graphs for the full paper.

Lemma 2.4. *Let D be a digraph such that one can partition its vertex set into $U \cup V$ such that every arc $u \to v$ of D has $u \in U$ and $v \in V$. Then*

$$A_D(-1) \geq 0,$$

and if D has an even number of arcs, then

$$A_D(-1) = \sum_{v \in V(D)} A_{D-v}(-1).$$

Corollary 2.5. *If G is a bipartite graph with an odd number of edges, then $\nu(G) = 0$, and otherwise $\nu(G) = \sum_v \nu(G - v)$.*

Proof of Theorem 2.3. We aim to show that $\nu(G) = \eta(G)$ whenever G is bipartite, complete multipartite, or a blowup of a cycle. We first consider the case that G is bipartite. We prove this result by induction on $|V(G)|$, the base case $\nu(K_1) = \eta(K_1) = 1$ being trivial. By Corollary 2.5 and Lemma 2.2, if G has an odd number of edges then $\nu(G) = \eta(G) = 0$, and otherwise

$$\nu(G) = \sum_{v \in V(G)} \nu(G - v) = \sum_{v \in V(G)} \eta(G - v) = \eta(G),$$

where the middle equality used the inductive hypothesis (and that $G - v$ is bipartite whenever G is). □

It is tempting to try to generalize this approach by finding "natural" orientations of other graphs in order to show $\nu(G) = \sum \nu(G - v)$; see for example Conjecture 5.2. However, the following theorem shows that the inductive proof of Theorem 2.3 can not be extended beyond the class of graphs which are bipartite, complete multipartite, or blowups of cycles.

Theorem 2.6. *If G is a connected graph such that $\nu(G') = \eta(G')$ for all induced subgraphs $G' \subseteq G$, then G is either bipartite, complete multipartite, or a blowup of a cycle.*

Our proof of Theorem 2.6 relies on a structural graph theory result which may be of independent interest. The *odd pan graph* C^*_{2k+1} is defined to be the graph obtained by taking the odd cycle C_{2k+1} and then adding a new vertex u adjacent to exactly one vertex of C_{2k+1}. We say that a graph G is *odd pan-free* if it contains no induced subgraph which is isomorphic to C^*_{2k+1} for any $k \geq 1$.

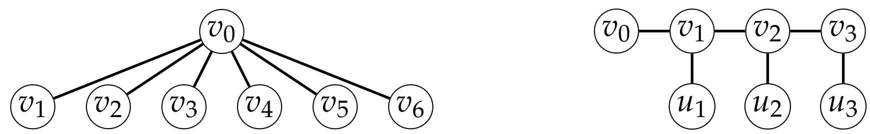
Proposition 2.7. *If G is a connected graph, then G is odd pan-free if and only if it is either bipartite, complete multipartite, or a blowup of a cycle.*

While we do not have a full understanding of $\nu(G)$ for arbitrary graphs, we are able to prove several other results regarding $\nu(G)$, such as the general bound $\nu(G) \leq \eta(G)$ in the full paper.

3 Upper and lower bounds of $\nu(G)$ and $\eta(G)$

We next turn to the extremal problem of studying the largest and smallest possible values of $\nu(G)$ and $\eta(G)$. For arbitrary n-vertex graphs this is an uninteresting problem, since $\nu(\overline{K_n}) = \eta(\overline{K_n}) = n!$ and $\nu(K_n) = \eta(K_n) = 0$ for $n \geq 2$ are easily seen to achieve the maximum and minimum possible values. However, this problem becomes non-trivial when one looks at smaller classes of graphs. To this end, we consider these extremal problems for trees.

To state our result, we recall that a tree is a *star* $K_{1,n}$ if there is a single-non leaf vertex; see Figure 4a. We say that a tree is a *hairbrush* if it consists of a path $v_0v_1 \cdots v_n$ such that each vertex v_i with $i \geq 1$ is adjacent to a leaf u_i; see Figure 4b.



(a) The star $K_{1,6}$ (b) The hairbrush H_3

Theorem 3.1. *If T is a tree on $2n+1$ vertices, then*

$$n!2^n \leq \nu(T) = \eta(T) \leq (2n)!$$

Moreover, equality holds in the lower bound if and only if T is a hairbrush, and equality holds in the upper bound if and only if T is a star.

To aid with our proofs, given a tree T, we define

$$\widetilde{X}(T) = \{x \in V(T) : \text{each component of } T - x \text{ has an even number of edges}\},$$

and we will denote this simply by $\widetilde{X}$ whenever T is understood. Our motivation for this definition is the following.

Lemma 3.2. *If T is a tree with an even number of edges, then*

$$\nu(T) = \sum_{x \in \widetilde{X}} \nu(T - x).$$

With this lemma in mind, the idea for the proofs of the upper and lower bounds is as follows: we first apply Lemma 3.2 and then use induction to bound each of the terms $\nu(T-x)$ in the sum. Finally, we bound our total sum in terms of $|\widetilde{X}|$ and show that equality can only occur when $|\widetilde{X}| = 1$. In particular, we can show that

$$n!2^n \leq \nu(T-x) \leq \frac{1}{2|\widetilde{X}|-1}(2n)! \tag{3.1}$$

for all trees with an even number of edges and $x \in \tilde{X}$ and so the result follows.

Remark 3.3. Our proofs yield slightly stronger bounds on $\nu(T)$ whenever $\widetilde{X}$ is large. For example, (3.1) gives the lower bound $\nu(T) \geq |\widetilde{X}|n!2^n$. Bounds of this form are known as *stability results* in extremal graph theory, which roughly are results saying that bounds for a graph T can be substantially improved if T is "far" from a unique extremal construction. Here, T being "far" from H_n and $K_{1,2n}$ is measured by having $|\widetilde{X}|$ large.

4 Multiplicity of Roots

Lastly, we consider the problem of determining the multiplicity of -1 as a root of $A_D(t)$, and we denote this quantity by $\mathrm{mult}(A_D(t), -1)$.

One of the first questions one might ask in this setting is how large $\mathrm{mult}(A_D(t), -1)$ can be amongst all n-vertex digraphs? Trivially, $\mathrm{mult}(A_D(t), -1) \leq e(D)$ (since the degree of $A_D(t)$ is at most $e(D)$), which implies $\mathrm{mult}(A_D(t), -1) \leq \binom{n}{2}$ if D has n vertices. We prove a substantially stronger upper bound which turns out to be sharp.

Theorem 4.1. *If D is an n-vertex digraph, then*

$$\mathrm{mult}(A_D(t), -1) \leq n - s_2(n),$$

where $s_2(n)$ denotes the number of 1's in the binary expansion of n. Moreover, for all n, there exist n-vertex digraphs D with $\mathrm{mult}(A_D(t), -1) = n - s_2(n)$.

The upper bound can be achieved with the following construction. Given digraphs D_1, D_2, and a root vertex $v \in D_2$, the *rooted product digraph*, denoted $D_1 \circ_v D_2$, is obtained by gluing a copy of D_2 at v to each vertex of D_1, see Figure 5 for an example.

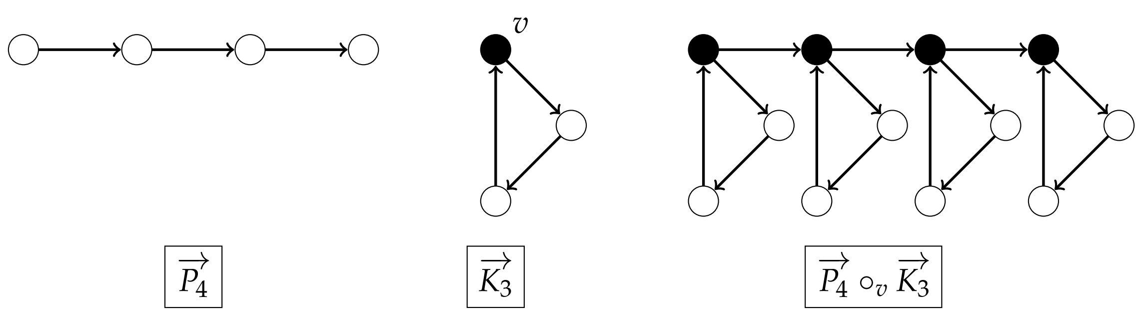


Figure 5: The rooted product digraph $\overrightarrow{P_4} \circ_v \overrightarrow{K_3}$ with the vertex v highlighted in black.

This product was first defined by Godsil and McKay [10], and it turns out that this operation plays very nicely with the Eulerian polynomial.

Proposition 4.2. *Let D_1 and D_2 be two digraphs on m and n vertices respectively. If $v \in D_2$, then*

$$A_{D_1 \circ_v D_2}(t) = \frac{1}{m!}\binom{mn}{n, \ldots, n} A_{D_1}(t) A_{D_2}(t)^m.$$

In particular, the polynomial is the same for any choice of root $v \in D_2$.

Remark 4.3. The last line of the statement implies that there are non-isomorphic digraphs with the same Eulerian polynomial.

With this, we first consider the case when $n = 2^m$ for some $m \geq 1$. Let P_2 be the graph on vertices v_1, v_2 with a single arc $v_1 \to v_2$. Define a sequence of digraphs $\{L_m\}_{m\in\mathbb{N}}$ by

$$L_1 = P_2 \quad \text{and} \quad L_{m+1} = L_m \circ_{v_1} P_2.$$

We observe that L_m has 2^m vertices and $2^m - 1$ arcs. Then from Proposition 4.2, we have

$$A_{L_m}(t) = (2^m)! \left(\frac{1+t}{2}\right)^{2^m-1}.$$

Since $s_2(2^m) = 1$, this gives the desired construction when n is a power of two. For arbitrary n, we let $a_1, \ldots, a_\ell$ be the indices of nonzero powers of 2 in the binary expansion of n and then define D to be the disjoint union of the digraphs $L_{a_1}, \ldots, L_{a_\ell}$. Then $A_D(t)$ gives the desired upper bound.

We also obtain a general lower bound on $\mathrm{mult}(A_D(t), -1)$.

Proposition 4.4. *Let D be an orientation of an n-vertex graph G. If every matching in the complement of G has size at most m, then* $\mathrm{mult}(A_D(t), -1) \geq \lfloor \frac{n}{2} \rfloor - m$.

Roughly speaking, Proposition 4.4 says that if G is "dense" (i.e. if the complement of G contains small only matchings), then $\mathrm{mult}(A_D(t), -1)$ will be large. While Proposition 4.4 is not tight in general, it turns out to be tight if D is an orientation of the complete graph as we show now.

Let $OP(\alpha)$ denote the set of all ordered set partitions of type α, and let $SP(\lambda)$ denote the set of all unordered set partitions with type λ. For two sets S, T of vertices in a digraph D, let $e_D(S, T)$ be the number of edges which start in S and end in T. For a digraph D and an ordered set partition $P = (P_1, \ldots, P_k)$ of the vertices of D of length k and $i \in [k]$, define the *i-th forward sequence number* of P to be

$$FS_{D,P}(i) = \sum_{j=i+1}^{k} e_D(P_i, P_j)$$

and the *i-th reverse sequence number* of P to be

$$RS_{D,P}(i) = \sum_{j=i+1}^{k} e_D(P_j, P_i)$$

where we set $FS_{D,P}(k) = 0$ and $RS_{D,P}(1) = 0$.

With this notation in hand, we can factor the Eulerian polynomial.

Lemma 4.5. *If D is a tournament on the vertex set $[n]$ and α is the integer composition (2^k) of n if n is even and $(1, 2^k)$ if n is odd, then*

$$A_D(t) = (1+t)^k \frac{1}{2^k} \sum_{P\in OP(\alpha)} \prod_{i=1}^{k} t^{FS_{D,P}(i)} + t^{RS_{D,P}(i)}.$$

A parity argument shows that the sum in the lemma does not have -1 as a root. Therefore, we obtain the following.

Theorem 4.6. *If D is a tournament on n vertices, then* $\operatorname{mult}(A_D(t), -1) = \lfloor \frac{n}{2} \rfloor$.

More generally, we suspect that Proposition 4.4 is tight for orientations of complete multipartite graphs; see Conjecture 5.4 for more.

Given Theorem 4.6 and the fact that $|A_D(-1)| = |A_{D'}(-1)|$ whenever D, D' are orientations of the same graph, it is perhaps natural to guess that $\operatorname{mult}(A_D(t), -1)$ depends only on the underlying graph of D. This turns out to be false; see Figure 6 for a counterexample.

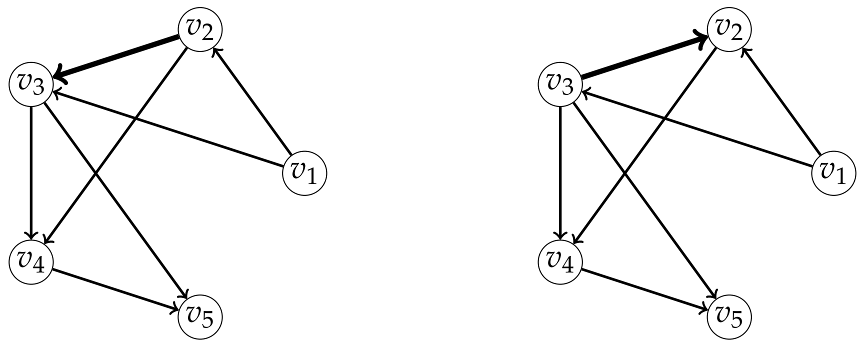


Figure 6: Two orientations of the same graph with different -1 multiplicities. The digraph on the left has $A_{D_1}(t) = (1+t)^3(1+t+11t^2+t^3+t^4)$ while the one on the right has $A_{D_2}(t) = (1+t)(1+5t+16t^2+16t^3+16t^4+5t^5+t^6)$.

5 Concluding Remarks and Open Problems

In this extended abstract, we studied a notion of Eulerian polynomials $A_D(t)$ for digraphs D and proved a number of results related to evaluations at $t = -1$. We conclude by listing a number of remaining open problems themed around interpreting $\nu(G)$ and multiplicities of -1 as a root of $A_D(t)$.

Interpretations for $\nu(G)$. Recall that for any graph G we define $\nu(G) = |A_D(-1)|$ where D is any orientation of G. While Theorem 2.3 provides a combinatorial interpretation for $\nu(G)$ when G is bipartite, complete bipartite, or a blowup of a cycle, we are still far from understanding this quantity for general graphs, which we leave as the main open problem for this paper.

Question 5.1. *Can one give a combinatorial interpretation for $\nu(G)$ for arbitrary graphs G?*

In view of Theorem 2.3 and the bound $\nu(G) \le \eta(G)$ from Lemma 2.2(a), we suspect that in general $\nu(G)$ should count even sequences of G with some special properties, but what these properties should be remains a mystery.

To answer Question 5.1, it might be useful to establish which graphs G satisfy $\nu(G) = \sum_v \nu(G-v)$, as recurrences of this form were a key step in proving Theorem 2.3. In particular, computational evidence suggests that the following could hold, where here we recall that a graph is *Eulerian* if all of its degrees are even.

Conjecture 5.2. *If G is an Eulerian graph, then $\nu(G) = \sum_v \nu(G-v)$.*

We note that an Eulerian graph has a "natural" orientation via orienting each edge according to an Eulerian tour. Given that e.g. our proof of Corollary 2.5 relied on "natural" orientations of bipartite graphs, it is plausible that this natural orientation for Eulerian graphs could be used to prove Conjecture 5.2.

Our proof of Theorem 2.3 is non-combinatorial, and it would be interesting to have a more direct combinatorial proof of this fact, say for bipartite graphs.

Problem 5.3. *For any bipartite graph $G = ([n], E)$ and orientation D of G, construct an explicit involution $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ such that*

(a) The set of fixed points $\mathcal{F}_\varphi$ of φ is the set of (inverses of) even sequences of G, and

(b) $(-1)^{\mathrm{des}_D(\sigma)} = -(-1)^{\mathrm{des}_D(\varphi(\sigma))}$ for all $\sigma \notin \mathcal{F}_\varphi$.

Such an involution is known to exist when $G = P_n$ (i.e. when inverses of even sequences are exactly alternating permutations), but this involution is somewhat complex; see [16, Exercise 135] for more.

Multiplicity of Roots. In Theorem 4.6 we showed every n vertex tournament D has -1 as a root of $A_D(t)$ with multiplicity exactly $\lfloor \frac{n}{2} \rfloor$. A natural generalization of this result would be the following.

Conjecture 5.4. *If D is the orientation of a complete multipartite graph which has r parts of odd size, then* $\mathrm{mult}(A_D(t), -1) = \lfloor \frac{r}{2} \rfloor$.

Observe that the bound $\mathrm{mult}(A_D(t), -1) \ge \lfloor \frac{r}{2} \rfloor$ follows from Proposition 4.4, so the difficulty lies in proving the upper bound.

Another direction is to look at the more general quantity $\mathrm{mult}(A_D(t), \alpha)$, which is defined to be the multiplicity of α as a root of $A_D(t)$. For example, it is not difficult to see that $\mathrm{mult}(A_D(t), 0)$ is equal to the minimum number of arcs that one must remove in D to obtain an acyclic digraph. Such a set of arcs is known as a *minimum feedback arc set*, and determining the size of such a set is well known to be an NP-Complete problem [13].

This connection to feedback arc sets, together with the results of this paper, establishes a number of results for $\mathrm{mult}(A_D(t), \alpha)$ when $\alpha \in \{0, -1\}$, and it is natural to ask what can be said about other integral values of α. An immediate obstacle to this is the following.

Question 5.5. *Does there exist a digraph D such that $A_D(t)$ has an integral root which is not equal to either 0 or -1?*

We have verified that no such digraph exists on at most 5 vertices. We also note that there exist digraphs with real roots of magnitude larger than 2, so the obstruction to finding these integral roots is not that their magnitudes are too large.

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Framed polytopes and higher categories

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Abstract. Pasting diagrams form an important special class of higher categories. In 1991, Kapranov and Voevodsky announced that any d-polytope in $\mathbb{R}^d$, when equipped with a generic frame of $\mathbb{R}^d$, naturally defines a d-dimensional pasting diagram. Our main result is a counterexample to this claim.

After translating this category-theoretic statement into a purely convex-geometric one, we were led to the study of globular structures and higher cellular strings on polytopes. Specifically, the absence of cellular loops is a necessary condition for the claim. We strongly disprove it by constructing polytopes for which every frame leads to a cellular loop.

An important infinite family of framed polytopes without cellular loops is defined by the canonically framed cyclic simplices. These happen to be exceptional since we show that, as the dimension of a canonically framed random simplex grows, the probability that it has a cellular loop tends to 1.

We conclude this work relating globular structures on simplices to oriented flag matroids, and use this connection to prove a universality theorem showing how complicated the moduli space of frames can be.

Keywords: Framed polytopes, n-categories, pasting diagrams, cellular strings, globular structures, random polytopes, oriented flag matroids, Mnëv's universality theorem

1 Introduction

Higher categories offer a powerful framework for systematizing complex hierarchies. Polytopes were first introduced into higher category theory to organize coherence relations. Kapranov and Voevodsky significantly expanded the connection between convex

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geometry and higher category theory announcing several intriguing results in [7], including the following insightful idea. Consider a convex d-polytope $P \subseteq \mathbb{R}^d$ and a generic ordered basis B of $\mathbb{R}^d$, which we refer to as a *frame*. Using the frame we define, for each face F, two distinct subsets of its k-faces: its *k-source* $\mathrm{s}_k(F)$ and *k-target* $\mathrm{t}_k(F)$. Kapranov and Voevodsky conjectured [7, Thm. 2.3] that the data consisting of all sources and targets, referred to as the *globular structure* of (P, B), defines a d-dimensional *pasting diagram*, a special and important type of d-dimensional categories. Using ideas of Steiner [10], we show in the full version of this article that this claim holds if and only if the framed polytope has no cellular loops, a notion we now define. A *cellular k-string* in a framed polytope is a sequence $F_1, \dots, F_\ell$ of faces such that two consecutive faces F_i and F_{i+1} share a k-face G with $\mathrm{t}_k(F_i) \cap \mathrm{s}_k(F_{i+1}) = G$. We say it is a *cellular loop* if and $F_i = F_j$ for some $i \neq j$.

The first contribution we discuss in this paper are counterexamples to [7, Thm. 2.3]. More precisely, in Section 3 we provide examples showing the following.

Theorem 1.1. *Starting in dimension 4 there exist framed polytopes with cellular loops.*

We also considered whether the following weaker version of their claim could be true: For any polytope there is a frame making it into a pasting diagram. However, this weaker version also fails since we provide in Section 4 a construction establishing the following.

Theorem 1.2. *Starting in dimension 4 there exist polytopes for which all frames lead to cellular loops.*

An important infinite family of framed polytopes, which was studied by Kapranov–Voevodsky, is given by the *canonically framed cyclic simplices* $(C(d), \{e_1, \dots, e_d\})$, where $\{e_1, \dots, e_d\}$ is the canonical frame of $\mathbb{R}^d$ and $C(d)$ is the convex closure of $d + 1$ distinct points in the *moment curve* $t \mapsto (t, t^2, \dots, t^d)$. In an insightful observation [7, Thm. 2.5], they announced that $(C(d), \{e_k\})$ has no cellular loops and recover Street's free d-category on the d-simplex, a fundamental object in higher category theory [11]. We were able to verify this claim after replacing the canonical frame by $\{e_1, -e_2, e_3, -e_4, \dots\}$. These framed polytopes are rare and special in the following probabilistic sense.

A *Gaussian d-simplex* is the convex hull of $d + 1$ independent random points in $\mathbb{R}^d$, each chosen according to a d-dimensional standard normal distribution. In Section 6 we prove the following.

Theorem 1.3. *The probability that a canonically framed Gaussian d-simplex has a cellular loop tends to* 1 *as d tends to* ∞.

We next turn our attention to the moduli of frames of a simplex Δ_d under the equivalence relation induced by globular structures. Our aim is to quantify the complexity of the *realization space* of a globular structure on Δ_d, that is, the set of all frames of

Δ_d inducing it. Using a celebrated result of N. E. Mnëv [8], in Section 8 we show the following.

Theorem 1.4. *For every open primary basic semi-algebraic set S defined over $\mathbb{Z}$ there is a globular structure on some simplex Δ_d whose realization space is stably equivalent to S.*

A key step in the proof of this result is the following theorem–presented in Section 7–which we consider noteworthy in its own right.

Theorem 1.5. *Globular structures of framed simplices are in bijection with uniform acyclic realizable full flag chirotopes.*

For reasons of scope and extension, we do not discuss our formalization of the Kapranov–Voevodsky idea, nor the applications within higher category theory of this connection with convex geometry, simply mentioning that the resulting d-categories are gaunt, an important type of higher categories that fully-faithfully embed into any model of (∞, d)-categories. Our focus here will remain primarily with polytopes. For further details, including proofs and discussions of the aforementioned topics, we invite the interested reader to consult the full version of this article, which will become available soon. We believe that, beyond our initial motivation, the results presented herein hold intrinsic value from a combinatorial-geometric standpoint. Indeed, some important research topics in combinatorial polytope theory, such as the Baues problem, were originally motivated by questions in algebraic topology and category theory, with our work extending these connections to higher category theory.

2 Definitions and preliminaries

A *polytope* P is a subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$ obtained as the convex hull of a finite set of points. A *face* F of P is a subset of P maximizing some linear functional. Its *dimension* is that of its affine span. A d-dimensional polytope is called a *d-polytope* and a k-dimensional face is called a *k-face*. We denote the set of k-faces of P by $\mathcal{L}_k(P)$ and the set of all faces by $\mathcal{L}(P)$. As usual, if P is a d-polytope, its $(d-1)$-faces are called *facets*, and the outer-pointing *normal* vector of a facet F is denoted n_F^P.

A *frame* B is an ordered basis $(v_1, \ldots, v_d)$ of $\mathbb{R}^d$. The *canonical frame* $(e_1, \ldots, e_d)$ consists of the standard basis vectors. The *system of projections* of a frame is the collection $\{\pi_k\}_{k \in \mathbb{N}}$ with

$$\pi_k \colon \mathbb{R}^d \to V_k \overset{\text{def}}{=} \operatorname{Span}(v_1, \ldots, v_k) \quad ; \quad \pi_k(v_i) = \begin{cases} v_i & \text{if } i \leq k, \\ 0 & \text{if } i > k. \end{cases}$$

A frame is said to be *P-admissible* if for any k-face F of P the restriction $\pi_k \colon \operatorname{Lin} F \to V_k$ is a linear isomorphism. We remark that the property of being P-admissible is stable under small perturbations.

A *framed polytope* is a pair (P, B) consisting of a polytope P and a P-admissible frame B. We will typically omit the frame from the notation. We remark that B is $\pi_k(P)$-admissible, so $\pi_k(P)$ is canonically framed for any $k \in \mathbb{N}$.

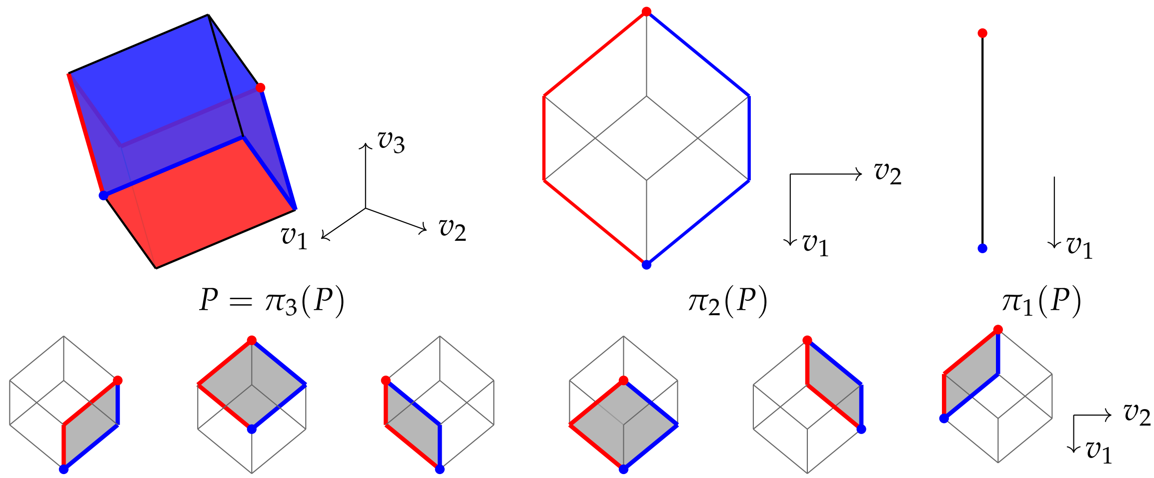

Figure 1: A globular structure on a 3-cube P given by a frame (v_1, v_2, v_3) of $\mathbb{R}^3$. The first row depicts P and its projections $\pi_2(P)$ and $\pi_1(P)$. The faces in $s_0(P)$, $s_1(P)$ and $s_2(P)$, and their projections, are in red, while the faces in $t_0(P)$, $t_1(P)$ and $t_2(P)$, and their projections, are in blue. The second row shows the 0- and 1-sources and targets of the 2-faces, projected onto the $\langle v_1, v_2 \rangle$ plane. The 0-sources and targets of the 1-faces are computed similarly.

Let (P, B) be a framed polytope. Its *k-boundary* $\partial^{(k)} P$ is the subset of k-faces of P consisting of the faces F such that $\pi_{k+1}(F)$ is in the boundary of the polytope $\pi_{k+1}(P)$. The *k-source* $\mathrm{s}_k(P)$ (resp. *k-target* $\mathrm{t}_k(P)$) of a framed polytope $(P, \{v_k\})$ is the subset of $\partial^{(k)} P$ containing one such F if

$$\left\langle \mathrm{n}_{\pi_{k+1}(F)}^{\pi_{k+1}(P)}, v_{k+1} \right\rangle < 0 \text{ (resp. } > 0).$$

See an example in Figure 1. Similar definitions apply to all faces of P using the induced frame.

The data of all sources and targets of faces of P is called the *globular structure* on P induced by B. Two P-admissible frames are said to be *P-equivalent* if they induce the same globular structure on P.

Lemma 2.1. *If a frame $\{v'_1, v'_2, \ldots\}$ is obtained from a P-admissible frame $\{v_1, v_2, \ldots\}$ via a positive lower triangular transformation, meaning that there exist $\lambda_{pq} \in \mathbb{R}$ for $p > q$ and $\lambda_i \in \mathbb{R}_+$ such that $v'_q = \lambda_q v_q + \sum_{p>q} \lambda_{pq} v_p$, then these frames are P-equivalent.*

Corollary 2.2. *Every P-admissible frame is P-equivalent to an orthonormal frame.*

3 Cellular loops

Let (P, B) be a framed polytope. A *cellular k-string* in P is a sequence $F_1, \dots, F_m$ of faces of P satisfying $\mathrm{t}_k(F_i) \cap \mathrm{s}_k(F_{i+1}) \neq \emptyset$ for every $i \in \{1, \dots, m-1\}$. We remark that this intersection is precisely a single k-face. Figure 2 depicts two examples of cellular strings. Note that cellular 0-strings starting at $\mathrm{s}_0(P)$ and ending at $\mathrm{t}_0(P)$ are precisely the cellular strings defined in [1].

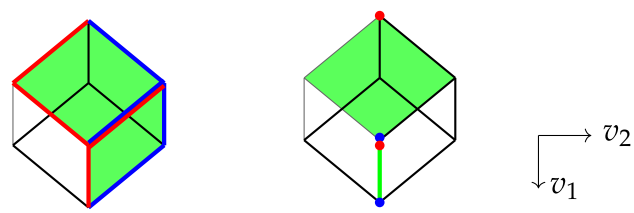

Figure 2: A cellular 1-string and a cellular 0-string on the example of Figure 1

A *cellular k-loop* is a cellular k-string $F_1, \dots, F_m$ with $F_i = F_j$ for some $i \neq j$.

3.1 A cellular 1-loop in the 5-simplex

We describe a 5-simplex P_5 for which the canonical frame is admissible and induces a 1-loop. Consider the 6 points $p_1, \dots, p_6$ in $\mathbb{R}^5$ whose coordinates are the columns of matrix

$$\begin{pmatrix} -3 & -2 & -1 & 1 & 2 & 3 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since these are affinely independent, their convex hull P_5 is a 5-simplex, and one can easily see that the canonical frame of $\mathbb{R}^5$ is P_5-admissible. This framed polytope contains the following cellular 1-loop of 2-faces:

$$[p_1p_2p_3],\ [p_2p_3p_6],\ [p_2p_6p_4],\ [p_4p_5p_6],\ [p_1p_4p_5],\ [p_1p_3p_5],\ [p_1p_2p_3]. \tag{3.1}$$

Consulting Figure 3, it is straightforward to check that for each of the triangles t_i, the edge $t_i \cap t_{i+1}$ lies in the 1-target $\mathrm{t}_1(t_i)$ of t_i and in the 1-source $\mathrm{s}_1(t_{i+1})$ of t_{i+1}.

Remark 3.1. All 2-faces involved in (3.1) are also faces of the 4-dimensional polytope $P_4 = \pi_4(P_5)$, which is a *cyclic 4-polytope* with 5 vertices. Therefore, the 4-polytope P_4 together with the canonical frame also has a cellular 1-loop. This example is minimal in dimension since we can prove that all framed n-polytopes for $n < 4$ have no cellular loops. (We prove that there cannot be 0-loops nor $(n-2)$-loops.)

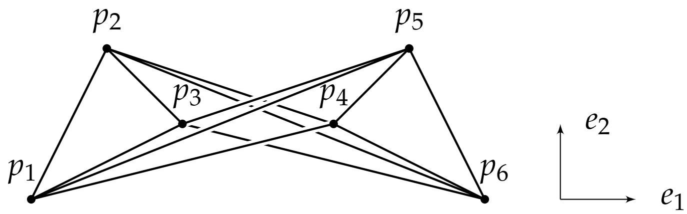

Figure 3: A cellular 1-loop in P_5 formed by 2-faces. It represents the image of the vertices of P_5 and some of its edges under the projection $\pi_2 \colon \mathbb{R}^5 \to \mathbb{R}^2$.

3.2 A cellular 2-loop in the 6-simplex

We now present a cellular 2-loop on a framed 6-simplex. It is a relative of the so-called *mother of all examples* [5, Sec. 7.1]. In contrast with our previous example, the projections of the simplices involved in the loop do not overlap and all the vertices are preserved under the projection.

Consider the 7 points $q_0, q_1, \ldots, q_6$ in $\mathbb{R}^6$ whose coordinates are given by the columns of the matrix Q and the frame B of $\mathbb{R}^6$ given by the columns $v_1, \ldots, v_6$ of the matrix V below

$$Q = \begin{pmatrix} 0 & 10 & 0 & 0 & 7 & 2 & 3 \\ 0 & 0 & 10 & 0 & 3 & 7 & 2 \\ 0 & 0 & 0 & 10 & 2 & 3 & 7 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The columns of Q are affinely independent, and therefore form the vertex set of a simplex Q_6. The frame B is Q_6-admissible, as it can be easily checked by computer, and the resulting framed 6-simplex (Q_6, B) has the following 2-loop of 4-faces:

$$[q_0q_1q_4q_5],\ [q_0q_1q_3q_4],\ [q_0q_3q_4q_6],\ [q_0q_2q_3q_6],\ [q_0q_2q_5q_6],\ [q_0q_1q_2q_5],\ [q_0q_1q_4q_5]. \tag{3.2}$$

Although checking that this is indeed a cellular loop can be done using a computer, it is instructive to understand the geometry of this example. Please refer to Figure 4 as we proceed to present it.

Since $\mathrm{Span}(v_1, v_2, v_3) = \mathrm{Span}(e_1, e_2, e_3)$ and $\mathrm{Span}(v_4, v_5, v_6) = \mathrm{Span}(e_4, e_5, e_6)$, the projection π_3 is given by forgetting the last three coordinates. The 2-loop (3.2) is apparent on $\pi_3(Q_6)$, depicted on the left of our figure. The vector v_3 that determines the 2-sources and 2-targets goes in the direction from q_0 to the center of the equilateral triangles there.

As the points $q_1, \ldots, q_6$ are very close to being coplanar, it is somehow easier to understand the loop in the 2-dimensional picture on the right. Here, q_0 has to be thought

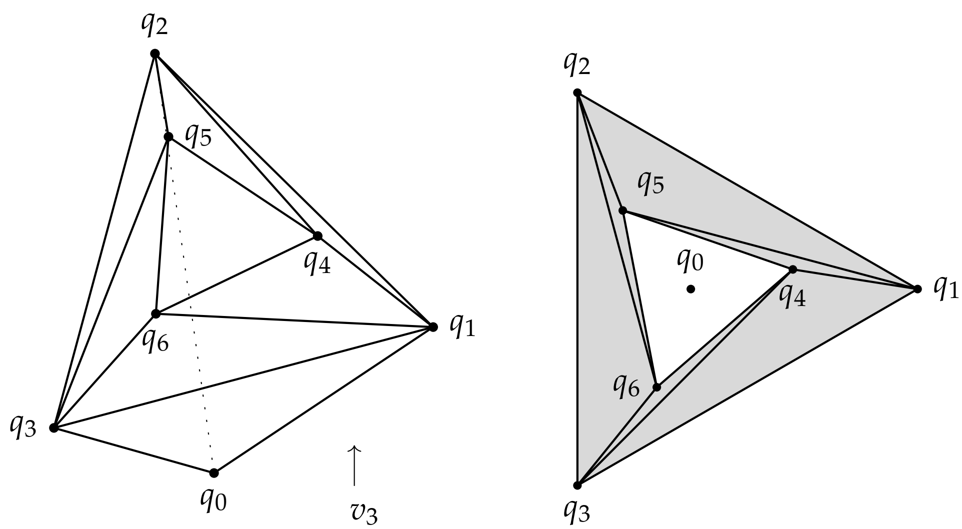

Figure 4: A cellular 2-loop on Q_6. The convex hull of $\pi_3(Q_6)$ is depicted on the left. On the right we see $\pi_2(Q_6)$ and the edges more relevant in the loop (note that they are not the same as in the convex hull).

as being behind the plane spanned by the other points, and v_3 is perpendicular to this plane.

The loop consists of the six tetrahedra arising as the cone over q_0 of each of the shaded triangles in the picture in the right. Topologically, they form a "pinched" solid torus where the interior circle has been collapsed to a point. For each tetrahedron, the facets pointing "downwards" towards q_0 are in the source, and those pointing "upwards" away from q_0 are in the target. For example, for the tetrahedron $[q_0, q_1, q_4, q_5]$, the source is the triangle $[q_0, q_1, q_5]$, and the target is formed by the triangles $[q_1, q_4, q_5]$, $[q_0, q_1, q_4]$, and $[q_0, q_4, q_5]$. Similarly, for the tetrahedron $[q_0, q_1, q_3, q_4]$, the source are the triangles $[q_0, q_1, q_4]$ and $[q_0, q_1, q_3]$, and the target are the triangles $[q_0, q_3, q_4]$ and $[q_1, q_3, q_4]$. The other tetrahedra behave analogously.

To check the loop, notice that the triangle $[q_0, q_1, q_4]$ is in the target of $[q_0, q_1, q_4, q_5]$ and in the source of $[q_0, q_1, q_3, q_4]$. The triangle $[q_0, q_3, q_4]$ is in the target of $[q_0, q_1, q_3, q_4]$ and in the source of $[q_0, q_3, q_4, q_6]$. And so on.

Remark 3.2. It is not hard to prove that if a face or a vertex figure of a polytope P has a frame inducing a loop, then so does P. Combining our counterexamples with these observations we see that every simple or simplicial polytope of dimension ≥ 6 admits a frame inducing a loop.

4 Loop inevitability

The goal of this section is to construct polytopes for which **every** admissible frame induces a cellular loop. The construction is too technical and involved to fit in here, but we will give some indications on the key steps of the proof.

Our main idea is to transform our polytopes via an operation called *flattening* that enlarges the space of loop-inducing frames. And then combine several reflected copies of a flattened polytope, via an operation called *squashing*, to cover the full space of admissible frames.

Let P be a polytope and B a frame inducing a loop on P. Then there is an open neighborhood of B in the space of frames that contains P-equivalent frames to B. The flattening operation edits P so that B still induces a loop, but makes the set of equivalent frames become arbitrarily large.

Lemma 4.1. *Let P be a framed polytope in $\mathbb{R}^d$ with orthonormal frame $B = \{v_1, \ldots, v_d\}$. For any $\varepsilon \overset{\text{def}}{=} (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{R}^d$, let $\Phi_\varepsilon \colon v_i \mapsto \varepsilon_i v_i$ be the map that scales the v_i coordinate by ε_i.*

For every $0 < \delta < 1$ there is a positive $\varepsilon \in \mathbb{R}^d_{>0}$ such that if $B' = \{v'_1, \ldots, v'_d\}$ is an orthonormal frame of $\mathbb{R}^d$ with $\langle v_i\,,\, \tilde{v}'_i \rangle > \delta$ for all $1 \le i \le d$, where $\tilde{v}'_i$ is the projection of v'_i to $V_i = \mathrm{Span}(v_1, \ldots, v_i)$ along $\mathrm{Span}(v'_{i+1}, \ldots, v'_d)$ rescaled so that it is a unit vector[1], then the frames B and B' are $\Phi_\varepsilon(P)$-equivalent.

Figure 5 represents a regular hexagon P, for which the canonical basis (v_1, v_2) and the basis $(v_1, v'_2 := (1,1))$ induce distinct globular structures. However, for $\varepsilon := (1, \frac{1}{4}) \in \mathbb{R}^2$ both bases are $\Phi_\varepsilon(P)$-equivalent.

Figure 5: A regular hexagon P for which the bases (v_1, v_2) and (v_1, v'_2) are not equivalent, and a flattened version $\Phi_\varepsilon(P)$ for which they are. The set of vectors w for which (v_1, w) is P-equivalent to (v_1, v_2) is depicted with a blue cone, and the set of those that are $\Phi_\varepsilon(P)$-equivalent is depicted by a larger red cone.

The observation now is that, if we take ε conveniently, then every frame B will induce a loop in some reflection of $\Phi_\varepsilon(P)$ by the coordinate hyperplanes. The idea is to take a copy of each possible reflection of $\Phi_\varepsilon(P)$ to construct the desired polytope $\tilde{P}$. This does not work directly, because we need the faces of the reflections of $\Phi_\varepsilon(P)$ involved

[1] The fact that this projection is well defined follows inductively from the condition $\langle v_i\,,\, \tilde{v}'_i \rangle > \delta$.

in the loops to be also faces of the convex hull of all these reflected copies. Thus, one has to be careful on where and how to place the reflected copies. In our proof we do so by introducing a new operation on polytopes that we call *squashing*, which is closely related to *connected sums* (see, for example, [9]). And then squashing on top of faces of a barycentric subdivision of a simplex.

Lemma 4.2. *Let $P \subset \mathbb{R}^d$ be a framed d-polytope with a cellular k-loop for some $k \leq d-2$. If all the faces in this loop are faces of faces in $\mathrm{t}_{d-1}(P)$, then there is a d-polytope $\tilde{P}$ such that every $\tilde{P}$-admissible frame induces a k-loop on $\tilde{P}$.*

We conclude by noting that the framed polytope P_4 defined in Remark 3.1 together with the loop (3.1) satisfy the condition of this lemma, from which we conclude the following.

Theorem 4.3. *There is a 4-polytope for which every admissible frame induces a cellular loop.*

5 Canonically framed cyclic simplices

We now turn to an infinite family of framed polytopes with no loops. Consider the *moment curve* $\mathbb{R} \to \mathbb{R}^d$ given by $v_t = (t, t^2, \ldots, t^d)$. A *cyclic simplex* $C(d)$ is the convex hull of $d+1$ distinct points in the moment curve.

A polytope $P \subset \mathbb{R}^d$ is said to be *canonically framed* if it is considered with the canonical frame $\{e_1, \ldots, e_d\}$ which is assumed P-admissible.

It was announced by Kapranov and Voevodsky [7, Thm. 2.5] that the canonically framed cyclic simplices recover Street's pasting diagram structure on standard simplices [11]. We were able to verify this claim after replacing the canonical frame with $\{e_1, -e_2, e_3, -e_4, \ldots\}$.

We can extend the absence of cellular loops to all canonically framed cyclic polytopes. A *cyclic polytope* $C(n,d)$ is the convex hull of n distinct points in the image of the the moment curve in $\mathbb{R}^d$, and we have the following.

Theorem 5.1. *All canonically framed cyclic polytopes have no cellular loops.*

6 Canonically framed Gaussian simplices

We now measure how special the absence of cellular loops is on cyclic simplices compared to random embeddings. A *Gaussian d-simplex* is the convex hull of $d+1$ independent random points in $\mathbb{R}^d$, each chosen according to a d-dimensional standard normal distribution.

Theorem 6.1. *For every $k \geq 1$, the probability that the canonically framed Gaussian d-simplex has a k-loop tends to 1 as d tends to ∞.*

Our proof uses very few hypothesis on the distribution, which could be further relaxed. Mainly that it is supported on $\mathbb{R}^d$ and that the vertices are independently sampled. Therefore, for most usual distributions of random simplices the same kind of result should hold.

We obtain similar results if instead of fixing the frame and choosing the simplex, we fix the simplex and chose the frame. In view of Corollary 2.2, a reasonable approach is to consider a random orthonormal frame chosen with respect to the Haar measure. Let the *standard d-simplex* be the convex hull of the canonical basis of $\mathbb{R}^{d+1}$.

Theorem 6.2. *For every $k \geq 1$, the probability that a uniform random orthonormal frame induces a k-loop on the standard d-simplex tends to 1 as d tends to ∞.*

7 Framed simplices and oriented matroids

A *chirotope* is a non-zero alternating map $\chi\colon \{1,\dots,n\}^d \to \{+,-,0\}$ satisfying the *chirotope axioms* [2, Def. 3.5.3]. We will consider those that are *realizable*, meaning that they are associated to a vector configuration, and hence omit the general combinatorial definition. We refer to [2] for a comprehensive reference on the topic. The *chirotope* associated to a vector configuration $V = (v_1,\dots,v_n) \in \mathbb{R}^{d\times n}$ is the map

$$\begin{aligned}\chi^V\colon \{1,\dots,n\}^d &\to \{+,-,0\}\\ (i_1,\dots,i_d) &\mapsto \operatorname{sign}(\det(v_{i_1},\dots,v_{i_d})).\end{aligned}$$

A realizable chirotope is called *acyclic* if all the vectors of the configuration lie in a common half-space; and *uniform* if $\chi(i_1,\dots,i_d) \neq 0$ whenever $i_1,\dots,i_d$ are pairwise distinct.

A realizable chirotope depends on a frame for the ground vector space, as an orientation reversing change of basis results in a global sign change for the chirotope. An *oriented matroid* can be defined as an equivalence class $\pm\chi = \{\chi,-\chi\}$ of chirotopes up to global reorientation [2, Prop. 3.5.2 and Thm. 3.5.5], where $-\chi$ denotes the chirotope obtained from χ by reversing all the signs. Despite this subtle difference, the two terms *chirotope* and *oriented matroid* are often used interchangeably in the literature.

When we restrict to framed simplices (Δ_d, B), the relation between globular structures and chirotopes is quite satisfying as the next statement shows.

Lemma 7.1. *Let (Δ_d, B) be a framed simplex with vertex set $P = \{p_0,\dots,p_d\}$. The globular structure on Δ_d induced by B determines and is determined by the chirotopes of the point configurations $\pi_k(P) = (\pi_k(p_0),\dots,\pi_k(p_d)) \in \mathbb{R}^{k\times(d+1)}$ for all $0 \leq k \leq d$.*

The core of this correspondence lies in the fact that the orientation of a facet F of a simplex S in a codimension 1 projection can be deduced from the orientation of S and knowing whether F belongs to the source or the target of S. We can therefore compute the chirotope of $\pi_k(\Delta_d)$ from the globular structure and the chirotope of $\pi_{k+1}(\Delta_d)$; and conversely, the k-sources and k-targets can be found by comparing the chirotopes of $\pi_k(\Delta_d)$ and $\pi_{k+1}(\Delta_d)$.

Flag matroids were introduced in [4], and also admit an oriented version. A *flag chirotope*[2] is defined as a sequence $(\chi_1, \ldots, \chi_s)$ of chirotopes related by strong maps (also called quotients), see [6, Example above Thm. D] and [3, Def. 4.1], and also [2, Def. 3.5.3, Thms 3.5.5 and 3.6.2, and Def. 7.7.2] for more details on the definition and the relation with ordinary oriented matroids.

A *realizable full flag chirotope* is a sequence of chirotopes $(\chi_0, \ldots, \chi_d)$, where χ_k is the chirotope of the vector configuration $\{\pi_k(e_1), \ldots, \pi_k(e_d)\}$, $\{e_1, \ldots, e_d\}$ is the canonical frame of $\mathbb{R}^d$, and $\pi_k : \mathbb{R}^d \to V_k$ is the associated system of projections of another frame B of $\mathbb{R}^d$ (see [4, Sec 1.7.5]). We will say that a flag chirotope $(\chi_0, \ldots, \chi_d)$ is *uniform* (resp. *acyclic*) if χ_k is uniform (resp. acyclic) for $0 \leq k \leq d$.

Theorem 7.2. *Globular structures of framed simplices are in bijection with uniform acyclic realizable full flag chirotopes.*

8 Universality

We now study the moduli space of frames under the equivalence relation defined by globular structures. The *realization space* of a globular structure on a polytope P induced by a frame B is the set of P-admissible frames that are P-equivalent to B. Our main result in this section is that Δ_d-equivalence classes of Δ_d-admissible frames are universal in the sense of [8]. To explain this statement we introduce the following notions. A *primary basic semi-algebraic* set is a subset of $\mathbb{R}^d$ defined by integer polynomial equations and strict inequalities. Two semi-algebraic sets S, S' are called *stably equivalent* if they lie in the same equivalence class generated by stable projections and rational equivalence. Here, a projection $\pi \colon S \to S'$ is called *stable* if its fibers are relative interiors of non-empty polyhedra of the same dimension defined by polynomial functions on S' (see [9, Section 2.5] for details, and [12] for the constant dimension constraint).

Theorem 8.1. *For every open primary basic semi-algebraic set S defined over $\mathbb{Z}$ there is a globular structure on some Δ_d whose realization space is stably equivalent to S.*

[2] In the literature, they are usually called *oriented flag matroids*. However, we think that the name *flag chirotopes* is more precise, in view of the (subtle) difference between the classical definitions.

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Poset polytopes and pipe dreams: toric degenerations and beyond

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Abstract. We demonstrate how pipe dreams can be applied to the theory of poset polytopes to produce toric degenerations of flag varieties. Specifically, we present such constructions for marked chain-order polytopes of Dynkin types A and C. These toric degenerations also give rise to further algebraic and geometric objects such as PBW-monomial bases and Newton–Okounkov bodies. We discuss a construction of the former in the type A case and of the latter in type C.

1 Introduction

Recent decades have seen a wide range of new methods for constructing toric degenerations of flag varieties. These methods commonly proceed by attaching a degeneration to every combinatorial or algebraic object of a certain form. Examples of such objects include adapted decompositions in the Weyl group, certain valuations on the function field and certain birational sequences (see [6] for details concerning these results and a partial history of the subject). These correspondences are of great interest for a number of reasons, however, not many explicit constructions are known for the attached objects. This leads to a shortage of concrete recipes that would work in a general situation.

Until recently, the only explicit constructions known to work in the generality of all type A flag varieties were the Gelfand–Tsetlin (GT) degeneration due to [14] and the Feigin–Fourier–Littelmann-Vinberg (FFLV) degeneration due to [10] (as well as slight variations of these two). An important step was made by Fujita in [12] where it is proved that each marked chain-order polytope (MCOP) of the GT poset provides a toric degeneration of a type A flag variety. Each such MCOP $\mathcal{Q}_O(\lambda)$ is given by a subset O of the GT poset P and an integral dominant $\mathfrak{sl}_n$-weight λ. The GT and FFLV polytopes appear as special cases. General MCOPs were defined by Fang and Fourier in [5] and present a far-reaching generalization of the poset polytopes considered by Stanley in [23].

Now, it must be noted that the main objects of study in [12] are Newton–Okounkov bodies, toric degenerations are obtained somewhat indirectly via a general result of [1]

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relating the two notions. This project was initiated with the goal of finding a more direct approach in terms of explicit initial degenerations similar to the classical construction in [14]. Recall that F, the variety of complete flags in $\mathbb{C}^n$, is realized by the Plücker ideal I in the polynomial ring in Plücker variables $X_{i_1,\dots,i_k}$. Meanwhile, the toric variety of $\mathcal{Q}_O(\lambda)$ is realized by a toric ideal I_O in the polynomial ring in variables X_J labeled by order ideals J in P. To obtain a toric initial degeneration of F we may find an isomorphism between the two polynomial rings which would map I_O to an initial ideal of I. The key challenge is then to define this isomorphism, the solution is provided by pipe dreams: a combinatorial rule for associating a permutation w_M with every subset $M \subset P$.

Theorem 0 (cf. Theorem 1)**.** *Fix $O \subset P$. For every order ideal J one can choose $M_J \subset P$ and $k_J \in \mathbb{N}$ so that the map $\psi : X_J \mapsto X_{w_{M_J}(1),\dots,w_{M_J}(k_J)}$ is an isomorphism and $\psi(I_O)$ is an initial ideal of I. Consequently, the toric variety of $\mathcal{Q}_O(\lambda)$ is a flat degeneration of F.*

One reason for the popularity of toric degenerations is that they are accompanied by a collection of other interesting objects: standard monomial theories, Newton–Okounkov bodies, PBW-monomial bases, etc. All of these can also be obtained from our construction, in particular, we explain how PBW-monomial bases are obtained in type A. Consider the irreducible $\mathfrak{sl}_n$-representation V_λ with highest-weight vector v_λ, let $f_{i,j}$ denote the negative root vectors. A basis in V_λ is formed by the vectors $\prod f_{i,j}^{x_{i,j}} v_\lambda$ with x ranging over the lattice points in $\xi(\mathcal{Q}_O(\lambda))$ for a unimodular transformation ξ (see Theorem 2).

We then discuss an extension of our approach to type C. Every integral dominant $\mathfrak{sp}_{2n}$-weight λ and every subset O of the type C GT poset also define an MCOP $\mathcal{Q}_O(\lambda)$. Using a notion of type C pipe dreams (not to be confused with other known symplectic pipe-dream analogs) we state a type C counterpart of Theorem 0, see Theorem 4. A notable new feature of this case is the intermediate degeneration $\widetilde{F}$ of the symplectic flag variety which happens to be a type A Schubert variety (Theorem 3, compare also [3]). All of our toric degenerations are obtained as further degenerations of $\widetilde{F}$.

We also use type C to showcase another aspect of the theory: Newton–Okounkov bodies. Namely, we show how every $\mathcal{Q}_O(\lambda)$ can be realized as a Newton–Okounkov body of the symplectic flag variety (Theorem 5). To us this result is of particular interest because the paper [12] explains in detail why its methods do not extend to type C.

Proofs and further context for the results in type A can be found in [20] (which discusses an extension to semi-infinite Grassmannians); [19] covers types B and C.

2 Type A

2.1 Poset polytopes

Choose an integer $n \geq 2$ and consider the set of pairs $P = \{(i,j)\}_{1\leq i\leq j\leq n}$. We define a partial order $\prec$ on P by setting $(i,j) \preceq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. The poset $(P, \prec)$ is sometimes referred to as the *Gelfand–Tsetlin* (or *GT*) *poset*. We denote $A = \{(i,i)\}_{i\in[1,n]} \subset P$. Let $\mathcal{J}$ be the set of order ideals (lower sets) in $(P, \prec)$. For $k \in [0,n]$ let $\mathcal{J}_k \subset \mathcal{J}$ consist of J such that $|J \cap A| = k$, i.e. J contains $(1,1), \dots, (k,k)$ but not $(k+1, k+1)$.

We now associate a family of polytopes with this poset. Each polytope is determined by a subset of P and a vector in $\mathbb{Z}^{n-1}_{\geq 0}$. For $k \in [1, n-1]$ we let ω_k denote the kth basis vector in $\mathbb{Z}^{n-1}_{\geq 0}$.

Definition 1. *Consider a subset $O \subset P$ such that $A \subset O$. For $J \in \mathcal{J}$ consider the set*

$$M_O(J) = (J \cap O) \cup \max_{\prec}(J)$$

($\max_\prec$ denotes the subset of $\prec$-maximal elements). Let $x_O(J) \in \mathbb{R}^P$ denote the indicator vector $\mathbf{1}_{M_O(J)}$. The marked chain-order polytope (MCOP) $\mathcal{Q}_O(\omega_k)$ is the convex hull of $\{x_O(J)\}_{J\in\mathcal{J}_k}$. For $\lambda = (a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}_{\geq 0}$ the MCOP $\mathcal{Q}_O(\lambda)$ is the Minkowski sum

$$a_1 \mathcal{Q}_O(\omega_1) + \dots + a_{n-1} \mathcal{Q}_O(\omega_{n-1}) \subset \mathbb{R}^P.$$

MCOPs were introduced in [5, 7] in the generality of arbitrary finite posets. The original definition describes the polytope in terms of linear inequalities. The equivalence of the above approach is proved in [11, Subsection 3.5].

The first thing to note is that for $\lambda = (a_1, \dots, a_n)$ and any $x \in \mathcal{Q}_O(\lambda)$ one has $x_{i,i} = a_i + \dots + a_{n-1}$. When $O = P$ one has $M_O(J) = J$. It follows that $\mathcal{Q}_P(\lambda)$ consists of points x with $x_{i,j} \geq x_{i',j'}$ whenever $(i,j) \preceq (i',j')$. Now, identify $\mathbb{Z}^{n-1}$ with the lattice of integral $\mathfrak{sl}_n$-weights by letting ω_k be the kth fundamental weight. Then $\mathcal{Q}_P(\lambda)$ is the GT polytope of [13] corresponding to the integral dominant weight λ (i.e. $\lambda \in \mathbb{Z}^{n-1}_{\geq 0}$). If $O = A$, then $M_O(J)$ is the union of $J \cap A$ and the antichain $\max_\prec J$. One can check that $\mathcal{Q}_A(\lambda)$ consists of points x with all $x_{i,j} \geq 0$ and $\sum_{(i,j)\in K} x_{i,j} \leq a_l + \dots + a_r$ for any chain $K \subset P \backslash A$ starting in $(l, l+1)$ and ending in $(r, r+1)$. This is the FFLV polytope ([9]) of λ. Other MCOPs can be said to interpolate between these two cases.

Note that $|\mathcal{J}_k| = \binom{n}{k}$ and, since $\mathcal{Q}_O(\omega_k)$ is a 0/1-polytope, it has $\binom{n}{k}$ lattice points. More generally, a key property of MCOPs is that the number of lattice points in $\mathcal{Q}_O(\lambda)$ does not depend on O and, moreover, the polytopes with a given λ are pairwise Ehrhart-equivalent ([7, Corollary 2.5]). Now, it is well known that the number of lattice points in the GT or FFLV polytope of λ is $\dim V_\lambda$ where V_λ denotes the irreducible $\mathfrak{sl}_n(\mathbb{C})$-representation with highest weight λ. This immediately provides the following.

Proposition 1. *For any O and λ we have $|\mathcal{Q}_O(\lambda) \cap \mathbb{Z}^P| = \dim V_\lambda$.*

The polytope $\mathcal{Q}_O(\omega_k)$ is normal which means that the associated toric variety is embedded into $\mathbb{P}(\mathbb{C}^{\mathcal{J}_k})$. It is cut out by the kernel of the homomorphism from $\mathbb{C}[X_J]_{J\in\mathcal{J}_k}$ to $\mathbb{C}[P] = \mathbb{C}[z_{i,j}]_{(i,j)\in P}$ mapping X_J to $z^{x_O(J)} = \prod_{(i,j)\in P} z_{i,j}^{x_O(J)_{i,j}}$. For general $\lambda = (a_1, \dots, a_n)$

the definition of $\mathcal{Q}_O(\lambda)$ implies that its normal fan and its toric variety (up to isomorphism) depend only on the set of i for which $a_i > 0$. Hence, for regular λ (all $a_i > 0$) the toric variety coincides with that of $\mathcal{Q}_O(\omega_1) + \cdots + \mathcal{Q}_O(\omega_{n-1})$. The toric variety of a Minkowski sum has a standard multiprojective embedding. Consider the product

$$\mathbb{P}_{\mathcal{J}} = \mathbb{P}(\mathbb{C}^{\mathcal{J}_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{\mathcal{J}_{n-1}})$$

and its multihomogeneous coordinate ring $\mathbb{C}[\mathcal{J}] = \mathbb{C}[X_J]_{J \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{n-1}}$. Let I_O denote the kernel of the homomorphism $X_J \mapsto z^{x_O(J)}$ from $\mathbb{C}[\mathcal{J}]$ to $\mathbb{C}[P]$.

Proposition 2. *For regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is isomorphic to the zero set of I_O in $\mathbb{P}_{\mathcal{J}}$.*

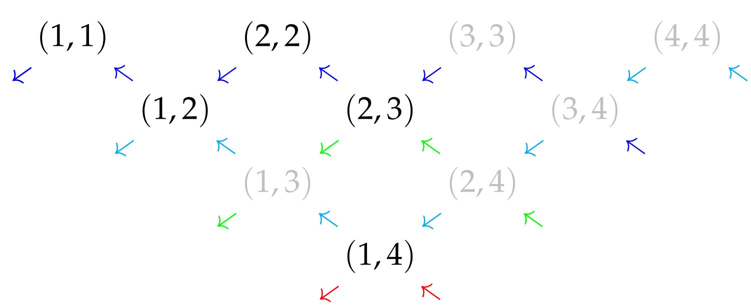
2.2 Pipe dreams

Consider the permutation group $\mathcal{S}_n$ and for $(i, j) \in P$ let $s_{i,j}$ denote the transposition $(i, j) \in \mathcal{S}_n$. In particular, $s_{i,i}$ is always the identity.

Definition 2. *For any subset $M \subset P$ let $w_M \in \mathcal{S}_n$ denote the product of all $s_{i,j}$ with $(i, j) \in M$ ordered first by i increasing from left to right and then by j increasing from left to right.*

Note that w_M is determined by $M \backslash A$ but it is convenient for us to consider subsets of P rather than $P \backslash A$. The term *pipe dream* is due to [17] and refers to a certain diagrammatic interpretation of this correspondence between subsets of P and permutations. The poset P can be visualized as a triangle as shown in (2.1) for $n = 4$. In these terms the pipe dream corresponding to M consists of n polygonal curves or *pipes* described as follows. The ith pipe enters the element (i, n) from the bottom-right, continues in this direction until it reaches an element of $M \cup A$, after which it turns left and continues going to the bottom-left until it reaches an element of M, after which it turns right and again continues to the top-right until it reaches an element of $M \cup A$, etc. The last element passed by the pipe will then be $(1, w_M(i))$.

The pipe dream of the set $M = \{(1, 1), (2, 2), (1, 2), (2, 3), (1, 4)\}$ is shown below, here each pipe is shown in its own colour. Indeed, $s_{1,1}s_{1,2}s_{1,4}s_{2,2}s_{2,3} = (4, 3, 1, 2)$.



(2.1)

We will use a "twisted" version of the correspondence depending on $O \subset P$. For $M \subset P$ we denote $w_M^O = w_O^{-1} w_M$. Diagrammatically, $w_M^O(i) = j$ if the ith pipe of the pipe dream of M ends in the same element as the jth pipe of the pipe dream of O.

2.3 Toric degenerations

For a polynomial ring $\mathbb{C}[x_a]_{a\in A}$ a *monomial order* $<$ on $\mathbb{C}[x_a]_{a\in A}$ is a partial order on the set of monomials that is multiplicative ($M_1 < M_2$ if and only if $M_1 x_a < M_2 x_a$) and weak (incomparability is an equivalence relation). For such an order and a polynomial $p \in \mathbb{C}[x_a]_{a\in A}$ its *initial part* $\mathrm{in}_< p$ is equal to the sum of those monomials occurring in p which are maximal with respect to $<$, taken with the same coefficients as in p. For a subspace $U \subset \mathbb{C}[x_a]_{a\in A}$ its *initial subspace* $\mathrm{in}_< U$ is the linear span of all $\mathrm{in}_< p$ with $p \in U$. The initial subspace of an ideal is an ideal (the *initial ideal*), the initial subspace of a subalgebra is a subalgebra (the *initial subalgebra*). One key property of initial ideals and initial subalgebras is that they define flat degenerations, we explain this phenomenon in the context of flag varieties.

For $n \geq 2$ let F be the variety of complete flags in $\mathbb{C}^n$. The Plücker embedding realizes F as a subvariety in
$$\mathbb{P} = \mathbb{P}(\wedge^1\mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1}\mathbb{C}^n).$$
The multihomogeneous coordinate ring of $\mathbb{P}$ is $S = \mathbb{C}[X_{i_1,\dots,i_k}]_{k\in[1,n-1],1\leq i_1<\dots<i_k\leq n}$ and F is cut out in $\mathbb{P}$ by the *Plücker ideal* $I \subset S$ which can be defined as follows. Consider the $n \times n$ matrix Z with $Z_{i,j} = z_{i,j}$ if $i \leq j$ and $Z_{i,j} = 0$ otherwise. Denote by $D_{i_1,\dots,i_k}$ the minor of Z spanned by rows $1,\dots,k$ and columns $i_1,\dots,i_k$. Then I is the kernel of the homomorphism $\varphi : X_{i_1,\dots,i_k} \mapsto D_{i_1,\dots,i_k}$ from S to $\mathbb{C}[P]$. One can also equip S with a $\mathbb{Z}^{n-1}$-grading grad with $\operatorname{grad} X_{i_1,\dots,i_k} = \omega_k$ and characterize F as $\operatorname{MultiProj} S/I$ with respect to the induced $\mathbb{Z}^{n-1}$-grading. The following fact is essentially classical, for the context of partial monomial orders see [16] (where an algebraic wording is given).

Proposition 3. *For a monomial order $<$ on S the scheme $\operatorname{MultiProj} S/\operatorname{in}_< I$ (i.e. the zero set of $\operatorname{in}_< I$ in $\mathbb{P}$ if the scheme is reduced) is a flat degeneration of F: there exists a flat family $\mathcal{F} \to \mathbb{A}^1$ with fiber over 0 isomorphic to $\operatorname{MultiProj} S/\operatorname{in}_< I$ and all other fibers isomorphic to F.*

Now fix $O \subset P$ containing A. For $J \in \mathcal{J}$ denote $w^O_{M_O(J)} = w^J$. The key ingredient of our first main result is a homomorphism $\psi : \mathbb{C}[\mathcal{J}] \to S$. To define ψ for $J \in \mathcal{J}_k$ we set
$$\psi(X_J) = X_{w^J(1),\dots,w^J(k)}$$
where we use the convention $X_{i_1,\dots,i_k} = (-1)^\sigma X_{i_{\sigma(1)},\dots,i_{\sigma(k)}}$ for $\sigma \in \mathcal{S}_k$. The map ψ encodes a correspondence $J \mapsto (w^J(1),\dots,w^J(k))$ between order ideals and tuples. When $O = P$ the tuples obtained in this way are precisely the increasing tuples. When $O = A$ one obtains the *PBW tuples* defined in [8]. In general, every subset of $[1,n]$ is represented by exactly one of the obtained tuples, this means that ψ is an isomorphism.

Theorem 1. *The map ψ is an isomorphism and there exists a monomial order $<$ on S such that $\psi(I_O) = \mathrm{in}_< I$. In particular, for regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is a flat degeneration of F.*

Sketch of proof. It can be checked that $w^J(i) \geq i$ for $J \in \mathcal{J}_k$ and $i \in [1,k]$. Furthermore, there exists a unimodular transformation $\xi \in SL(\mathbb{Z}^P)$ such that for any $J \in \mathcal{J}_k$ one has

$$\xi(x_O(J)) = \mathbf{1}_{\{(1,w^J(1)),\ldots,(k,w^J(k))\}}.$$

This means that I_O is the kernel of the homomorphism $X_J \mapsto z_{1,w^J(1)} \cdots z_{k,w^J(k)}$.

Using pipe dreams one can define a lexicographic monomial order $\ll$ on $\mathbb{C}[P]$ so that

$$\mathrm{in}_\ll D_{w^J(1),\ldots,w^J(k)} = z_{1,w^J(1)} \cdots z_{k,w^J(k)}$$

for $J \in \mathcal{J}_k$. Since ξ is bijective, the right-hand sides are distinct monomials for distinct J, hence the sets $\{w^J(1),\ldots,w^J(k)\}$ are also pairwise distinct. This provides the isomorphism claim. Note that $\psi(I_O)$ is the kernel of $X_{i_1,\ldots,i_k} \mapsto \mathrm{in}_\ll D_{i_1,\ldots,i_k}$. Moreover, the $\mathrm{in}_\ll D_{i_1,\ldots,i_k}$ generate the initial subalgebra $\mathrm{in}_\ll \varphi(S)$ (i.e. the determinants form a *sagbi basis*). By general properties of initial degenerations, $\ll$ can now be pulled back to a monomial order $<$ on S with the desired property. □

In fact, we could define ψ using the "untwisted" permutation $w_{M(J)}$ and Theorem 1 would still hold since I is invariant under $\mathcal{S}_n$. However, we consider w^J the natural choice because of the property $w^J(i) \geq i$, $i \in [1,k]$ which is also crucial in the next subsection.

2.4 PBW-monomial bases

The map ξ considered in the proof sketch of Theorem 1 maps $\mathcal{Q}_O(\omega_k)$ to $\Pi_O(\omega_k)$: the convex hull of all $\mathbf{1}_{\{(1,w^J(1)),\ldots,(k,w^J(k))\}}$ with $J \in \mathcal{J}_k$. For $\lambda = (a_1,\ldots,a_n)$ let $\Pi_O(\lambda)$ denote the image $\xi(\mathcal{Q}_O(\lambda))$. It equals the Minkowski sum $a_1\Pi_O(\omega_1) + \cdots + a_{n-1}\Pi_O(\omega_{n-1})$.

Next, let us recall some standard Lie-theoretic notation. We have identified $\mathbb{Z}^{n-1}$ with the lattice of integral $\mathfrak{sl}_n$-weights, let $\alpha_1,\ldots,\alpha_{n-1} \in \mathbb{Z}^{n-1}$ denote the simple roots, i.e. $\alpha_i = 2\omega_i - \omega_{i-1} - \omega_{i+1}$ where $\omega_0 = \omega_n = 0$. The positive roots are then $\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$ with $1 \leq i < j \leq n$. Let $f_{i,j} \in \mathfrak{sl}_n(\mathbb{C})$ denote the negative root vector of weight $-\alpha_{i,j}$. For $x \in \mathbb{Z}^P_{\geq 0}$ we write f^x to denote the PBW monomial $\prod_{(i,j)\in P\setminus A} f_{i,j}^{x_{i,j}}$ in $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ ordered first by i increasing from left to right and then by j increasing from left to right. Finally, let v_λ denote a chosen highest-weight vector in V_λ. Another of our main results is as follows.

Theorem 2. *The vectors $f^x v_\lambda$ with $x \in \Pi_O(\lambda) \cap \mathbb{Z}^P$ form a basis in V_λ.*

When $O = A$ the transformation ξ is almost the identity: one has $\xi(x)_{i,j} = x_{i,j}$ for all $i < j$ so that $f^{\xi(x)} = f^x$. Since $O_A(\lambda)$ is the FFLV polytope, one sees that the obtained basis is the FFLV basis of [9]. For $O = P$ the corresponding basis is also known, see, for instance, [22, 21]. Since in this case the tuples $(w^J(1), \dots, w^J(k))$ are increasing, the definition of $\Pi_P(\lambda)$ is particularly simple. The observation that such a polytope $\Pi_P(\lambda)$ is unimodularly equivalent to the GT polytope $\mathcal{Q}_O(\lambda)$ is due to [18].

3 Type C

3.1 Type C poset polytopes

For $n \geq 2$ consider the totally ordered set $(N, \lessdot) = \{1 \lessdot \cdots \lessdot n \lessdot -n \lessdot \cdots \lessdot -1\}$.

Definition 3. *The type C GT poset $(P, \prec)$ consist of pairs of integers (i, j) such that $i \in [1, n]$ and $j \in [i, n] \cup [-n, -i]$. The order relation is given by $(i_1, j_1) \preceq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \lessdot j_2$.*

$(P, <)$ has length $2n$, below is its Hasse diagram for $n = 2$.

$$\begin{tikzcd}[column sep=small, row sep=small]
(1,1) \arrow[dr] & & (2,2) \arrow[dr] & \\
& (1,2) \arrow[ur] \arrow[dr] & & (2,-2) \\
& & (1,-2) \arrow[ur] \arrow[dr] & \\
& & & (1,-1)
\end{tikzcd} \tag{3.1}$$

We use notation similar to type A. Let $A \subset P$ be the set of all (i, i). Let $\mathcal{J}$ denote the set of order ideals in $(P, \prec)$. For $k \in [1, n]$ let $\mathcal{J}_k$ consist of J such that $|J \cap A| = k$. We also consider the lattice $\mathbb{Z}^n$ with ω_k denoting the kth basis vector. The definition of MCOPs is almost identical.

Definition 4. *Consider a subset $O \subset P$ such that $A \subset O$. For $J \in \mathcal{J}$ consider the set*

$$M_O(J) = (J \cap O) \cup \max_{\prec}(J).$$

Let $x_O(J) \in \mathbb{R}^P$ denote the indicator vector $\mathbf{1}_{M_O(J)}$. The MCOP $\mathcal{Q}_O(\omega_k)$ is the convex hull of $\{x_O(J)\}_{J \in \mathcal{J}_k}$. For $\lambda = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ the MCOP $\mathcal{Q}_O(\lambda)$ is the Minkowski sum

$$a_1 \mathcal{Q}_O(\omega_1) + \cdots + a_n \mathcal{Q}_O(\omega_n) \subset \mathbb{R}^P.$$

We identify $\mathbb{Z}^n$ with the lattice of integral $\mathfrak{sp}_{2n}$-weights with ω_k being the kth fundamental weight. Then for an integral dominant weight $\lambda \in \mathbb{Z}^n_{\geq 0}$ one sees that $\mathcal{Q}_P(\lambda)$ is the type C Gelfand-Tsetlin polytope defined in [2] while $\mathcal{Q}_A(\lambda)$ is the type C FFLV polytope defined in [10]. Both of these polytopes are known to parametrize bases in V_λ, the irreducible $\mathfrak{sp}_{2n}(\mathbb{C})$-representation with highest weight λ. This again provides

Proposition 4. *For any O and λ we have $|\mathcal{Q}_O(\lambda) \cap \mathbb{Z}^P| = \dim V_\lambda$.*

We also have multiprojective embeddings for toric varieties. Consider the product

$$\mathbb{P}_{\mathcal{J}} = \mathbb{P}(\mathbb{C}^{\mathcal{J}_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{\mathcal{J}_n})$$

and its multihomogeneous coordinate ring $\mathbb{C}[\mathcal{J}] = \mathbb{C}[X_J]_{J \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_n}$. Let I_O denote the kernel of the homomorphism $\varphi_O : X_J \mapsto z^{x_O(J)}$ from $\mathbb{C}[\mathcal{J}]$ to $\mathbb{C}[P] = \mathbb{C}[z_{i,j}]_{(i,j) \in P}$.

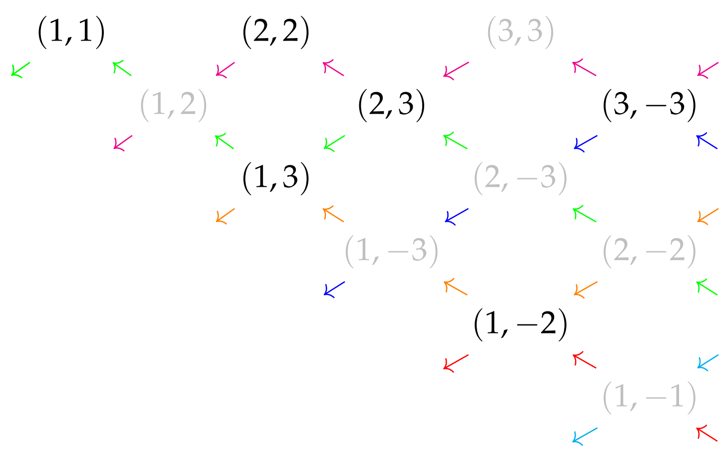
Proposition 5. *For regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is isomorphic to the zero set of I_O in $\mathbb{P}_{\mathcal{J}}$.*

3.2 Type C pipe dreams

Let S_N denote the group of all permutations of the set N. For $(i,j) \in P$ let $s_{i,j} \in S_N$ denote the transposition which exchanges i and j and fixes all other elements ($s_{i,i} = \mathrm{id}$).

Definition 5. *For $M \subset P$ let $w_M \in S_N$ be the product of all $s_{i,j}$ with $(i,j) \in M$ ordered first by i increasing from left to right and then by j increasing with respect to $\prec$ from left to right.*

In this case the pipe dream consists of $2n$ pipes enumerated by N. In terms of the visualization in (3.1) the ith pipe with $i \in [1,n]$ enters the element $(i,-i)$ from the **bottom**-right and turns at elements of $M \cup A$ while the ith pipe with $i \in [-n,-1]$ enters the element $(i,-i)$ from the **top**-right and then also turns at elements of $M \cup A$.



The pipe dream for the set $M = \{(1,1),(1,3),(1,-2),(2,2),(2,3),(3,-3)\}$ is shown above with each pipe in its own colour. One obtains

$$w_M(1,2,3,-3,-2,-1) = (-2,1,-3,2,3,-1)$$

which agrees with $w_M = s_{1,1}s_{1,3}s_{1,-2}s_{2,2}s_{2,3}s_{3,-3}$.

In fact, pipe dreams for type C_n can be viewed as a special case of pipe dreams for type A_{2n-1}, i.e. for $\mathfrak{sl}_{2n}$. Indeed, one may identify the type C_n GT poset with the "left half" of the type A_{2n-1} GT poset. Then the $2n$ pipes in the type C pipe dream of M will just be end parts of the $2n$ pipes in the type A pipe dream of the same set M.

We again introduce a "twisted" version of the correspondence determined by the choice of $O \subset P$: set $w_M^O = w_O^{-1} w_M$.

3.3 The intermediate Schubert degeneration

We now construct a degeneration $\widetilde{F}$ of the symplectic flag variety which will be used as an intermediate step: toric degenerations will be obtained as further degenerations of $\widetilde{F}$.

Definition 6. *A tuple $(i_1, \dots, i_k)$ of elements of N is admissible if for every $l \in [1, n]$ the number of elements with $|i_j| \le l$ does not exceed l. Let Θ denote the set of all admissible tuples of the form $(i_1 \lessdot \dots \lessdot i_k)$ and $\Theta_k \subset \Theta$ denote the subset of k-tuples.*

Consider the space $\mathbb{C}^N \simeq \mathbb{C}^{2n}$ with basis $\{e_i\}_{i \in N}$. One has a standard embedding $V_{\omega_k} \subset \wedge^k \mathbb{C}^N$. Let $\{X_{i_1,\dots,i_k}\}_{i_1 \lessdot \dots \lessdot i_k}$ be the basis in $(\wedge^k \mathbb{C}^N)^*$ dual to $\{e_{i_1} \wedge \dots \wedge e_{i_k}\}_{i_1 \lessdot \dots \lessdot i_k}$. It is known ([4]) that the set $\{X_{i_1,\dots,i_k}\}_{(i_1,\dots,i_k) \in \Theta_k}$ projects to a basis in $V_{\omega_k}^*$.This allows us to identify $V_{\omega_k}^*$ with $\mathbb{C}^{\Theta_k}$. The multihomogeneous coordinate ring of

$$\mathbb{P} = \mathbb{P}(V_{\omega_1}) \times \dots \times \mathbb{P}(V_{\omega_n})$$

is then identified with $\mathbb{C}[\Theta] = \mathbb{C}[X_{i_1,\dots,i_k}]_{(i_1,\dots,i_k) \in \Theta}$. The Plücker embedding of the complete symplectic flag variety $F \hookrightarrow \mathbb{P}$ is defined by the symplectic Plücker ideal $I \subset \mathbb{C}[\Theta]$.

Next, consider the variety F_{A} of type A partial flags in $\mathbb{C}^N$ of signature $(1, \dots, n)$. It is embedded into

$$\mathbb{P}_{\mathrm{A}} = \mathbb{P}(\wedge^1 \mathbb{C}^N) \times \dots \times \mathbb{P}(\wedge^n \mathbb{C}^N)$$

where it is cut out by the Plücker ideal I_{A} in the multihomogeneous coordinate ring $S = \mathbb{C}[X_{i_1,\dots,i_k}]_{k \in [1,n], \{i_1 \lessdot \dots \lessdot i_k\} \subset N}$. Consider the surjection $\pi : S \to \mathbb{C}[\Theta]$ mapping all $X_{i_1,\dots,i_k} \notin \mathbb{C}[\Theta]$ to 0 and fixing $\mathbb{C}[\Theta]$. Set $\widetilde{I} = \pi(I_{\mathrm{A}})$.

Theorem 3. *There exists a monomial order $\widetilde{<}$ on $\mathbb{C}[\Theta]$ such that $\mathrm{in}_{\widetilde{<}} I = \widetilde{I}$.*

This means that $\widetilde{F}$, the zero set of $\widetilde{I}$ in $\mathbb{P}$, is a flat degeneration of F. Also, importantly to us, every initial ideal of $\widetilde{I}$ is an initial ideal of I. The advantage of working with $\widetilde{I}$ instead of degenerating I directly is that the former is more simply expressed as a homomorphism kernel, allowing us to use the technique of sagbi degenerations. Consider Z, the $n \times 2n$ matrix with rows indexed by $[1, n]$ and columns indexed by N such that $Z_{i,j} = z_{i,j}$ if $(i, j) \in P$ and $Z_{i,j} = 0$ otherwise. Denote by $D_{i_1,\dots,i_k}$ the minor of Z spanned by rows $1, \dots, k$ and columns $i_1, \dots, i_k$.

Proposition 6. *$\widetilde{I}$ is the kernel of the homomorphism $X_{i_1,\dots,i_k} \mapsto D_{i_1,\dots,i_k}$ from $\mathbb{C}[\Theta]$ to $\mathbb{C}[P]$.*

A noteworthy property of $\widetilde{F}$ is that it is a type A Schubert variety in F_{A}. Indeed, $\pi^{-1}(\widetilde{I})$ cuts out $\widetilde{F}$ in $\mathbb{P}_{\mathrm{A}}$ and $\pi^{-1}(\widetilde{I})$ is generated by I_{A} and all $X_{i_1,\dots,i_k} \notin \mathbb{C}[\Theta]$. Consider the alternative order $-1 \lessdot' 1 \lessdot' \cdots \lessdot' -n \lessdot n$ on N. One sees that $X_{i_1,\dots,i_k} \notin \mathbb{C}[\Theta]$ if and only if $(i_1,\dots,i_k)$ has a reordering $(j_1 \lessdot' \cdots \lessdot' j_k)$ such that $j_l \lessdot' -l$ for some l. Now one sees that $\pi^{-1}(\widetilde{I})$ is indeed the defining ideal of a Schubert variety in F_{A}. Namely, the Schubert variety corresponding to the Borel subgroup in $SL(\mathbb{C}^N)$ given by the ordering $\lessdot'$ and the torus-fixed point $y \in F_{\mathrm{A}}$ with all multihomogeneous coordinates zero except for $y_{-1,\dots,-k}$ with $k \in [1,n]$.

3.4 Toric degenerations

Fix $O \subset P$ containing A. We can now realize the toric variety of $\mathcal{Q}_O(\lambda)$ as a degeneration of F by identifying I_O with an initial ideal of $\widetilde{I}$. This is again done via an isomorphism between $\mathbb{C}[\mathcal{J}]$ and $\mathbb{C}[\Theta]$. For $J \in \mathcal{J}$ denote $w^O_{M_O(J)} = w^J$.

Lemma 1. *For every $J \in \mathcal{J}_k$ and $i \in [1,k]$ one has $|w^J(i)| \geq i$. In particular, $(w^J(1),\dots,w^J(k))$ is admissible.*

The lemma lets us define a homomorphism $\psi : \mathbb{C}[\mathcal{J}] \to \mathbb{C}[\Theta]$, for $J \in \mathcal{J}_k$ we set

$$\psi(X_J) = X_{w^J(1),\dots,w^J(k)}.$$

Theorem 4. *The map ψ is an isomorphism and for a certain (explicitly defined) monomial order $<$ on S one has $\psi(I_O) = \mathrm{in}_< \widetilde{I}$. In particular, for regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is a flat degeneration of $\widetilde{F}$ and, subsequently, of F.*

3.5 Newton–Okounkov bodies

Following [15] we associate a Newton–Okounkov body of F with a line bundle $\mathcal{L}$, a global section τ of $\mathcal{L}$ and a valuation ν on the function field $\mathbb{C}(F)$. We choose an integral dominant $\lambda = (a_1,\dots,a_n)$ and let $\mathcal{L}$ be the $Sp_{2n}(\mathbb{C})$-equivariant line bundle on F associated with the weight λ. In terms of the multiprojective embedding $F \subset \mathbb{P}$ this is the restriction of $\mathcal{O}(a_1,\dots,a_n)$ to F. Consider the $\mathbb{Z}^N$-grading on $\mathbb{C}[\Theta]$ given by $\operatorname{grad} X_{i_1,\dots,i_k} = \omega_k$ and the induced grading on the Plücker algebra $R = C[\Theta]/I$. Then $H^0(F,\mathcal{L})$ is identified with the homogeneous component of grading λ in R. No we choose $\tau \in H^0(F,\mathcal{L})$ as the image of $\prod_k X^{a_k}_{1,\dots,k}$ in R.

To define the valuation ν we first define a valuation on R. Theorems 3 and 4 provide a monomial order $<_0$ on $\mathbb{C}[\Theta]$ such that $\mathrm{in}_{<_0} I = \psi(I_O)$. The proofs of the theorems show that $<_0$ arises from a total monomial order $\ll$ on $\mathbb{C}[P]$ in the following sense. Consider the homomorphism $\rho : \mathbb{C}[\Theta] \to \mathbb{C}[P]$ mapping the variable $\psi(X_J)$ to $z^{x_O(J)}$, note that

$\rho = \varphi_O\psi^{-1}$ and $\ker\rho = \psi(I_O)$. Then for two monomials one has $M_1 <_0 M_2$ if and only if $\rho(M_1) \ll \rho(M_2)$. The monomial order $\ll$ corresponds to a semigroup order on $\mathbb{Z}^P_{\geq 0}$ which we also denote by $\ll$. We have a $(\mathbb{Z}^P_{\geq 0}, \ll)$-filtration on R with component R_x, $x \in \mathbb{Z}^P_{\geq 0}$ spanned by the images of monomials $M \in \mathbb{C}[\Theta]$ such that $\rho(M) \ll z^x$. By general properties of initial degenerations we then have $\operatorname{gr} R \simeq \mathbb{C}[\Theta] / \operatorname{in}_{<_0} I$ (which is the toric ring of $\mathcal{Q}_O(\lambda)$). For nonzero $p \in R$ we now define $\nu(p)$ as the $\ll$-minimal x such that $p \in R_x$: by definition, such a map is a valuation if and only if $\operatorname{gr} R$ is an integral domain. One sees that ν maps the (image in R of) $\psi(X_J)$ to $x_O(J)$ and, consequently,

$$\nu(H^0(F,\mathcal{L})\backslash\{0\}) = \mathcal{Q}_O(\lambda) \cap \mathbb{Z}^P.$$

Since $\mathbb{C}(F)$ consists of fractions p/q where grad-homogeneous $p, q \in R$ satisfy $\operatorname{grad} p = \operatorname{grad} q$, we can now extend the valuation to $\mathbb{C}(F)$ by $\nu(p/q) = \nu(p) - \nu(q)$.

Definition 7. *The Newton–Okounkov body of F defined by $\mathcal{L}$, τ and ν is the convex hull closure*

$$\Delta = \overline{\operatorname{conv}\left\{ \frac{\nu(\sigma/\tau^m)}{m} \,\middle|\, m \in \mathbb{Z}_{>0}, \sigma \in H^0(F, \mathcal{L}^{\otimes m})\backslash\{0\} \right\}} \subset \mathbb{R}^P.$$

For every $k \in [1,n]$ we have a unique $J \in \mathcal{J}_k$ such that $w^J(\{1,\ldots,k\}) = \{1,\ldots,k\}$, denote $x_k = x_O(J)$. For $\lambda = (a_1,\ldots,a_n)$ denote $x_\lambda = a_1x_1 + \cdots + a_nx_n$. We have $\nu(\tau) = x_\lambda$ and it is now straightforward to deduce

Theorem 5. $\Delta = \mathcal{Q}_O(\lambda) - x_\lambda$.

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On the f-vectors of poset associahedra

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Abstract. For any finite connected poset P, Galashin introduced a simple convex $(|P|-2)$-dimensional polytope $\mathscr{A}(P)$ called the poset associahedron. First, we show that the f-vector of $\mathscr{A}(P)$ only depends on the comparability graph of P. Additionally, for a family of posets called broom posets, whose poset associahedra interpolate between permutohedra and associahedra, we give a simple combinatorial interpretation of the h-vector. The interpretation relates to the theory of stack-sorting and allows us to prove the real-rootedness of some of their h-polynomials.

Keywords: poset associahedra, stack-sorting, real-rootedness

1 Introduction

For a finite connected poset P, Galashin introduced the *poset associahedron* $\mathscr{A}(P)$ (see [4]). The faces of $\mathscr{A}(P)$ correspond to *tubings* of P, and the vertices of $\mathscr{A}(P)$ correspond to *maximal tubings* of P; see Section 2.2 for the definitions. $\mathscr{A}(P)$ can also be described as a compactification of the configuration space of order-preserving maps $P \to \mathbb{R}$.

The *comparability graph* of a poset P is a graph $\mathscr{C}(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of P is said to be *comparability invariant* if it only depends on $\mathscr{C}(P)$. Properties of finite posets known to be comparability invariant include the order polynomial and number of linear extensions [10], the fixed point property [3], and the Dushnik–Miller dimension [11]. Our first main result is the following.

Theorem 3.6. *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

In our study of the f-vectors of poset associahedra, we also consider a rich class of examples whose poset associahedra interpolate between associahedra and permutohedra. A *broom poset* is a poset of the form $A_{n,k} := C_{n+1} \oplus A_k$ where C_n is a chain of n elements,

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A_k is an antichain of k elements, and $\oplus$ denotes ordinal sum. In particular, $A_{0,k}$ is a *claw poset* where $\mathscr{A}(A_{0,k})$ is a permutohedron, and $A_{n,0}$ is a chain where $\mathscr{A}(A_{n,0})$ is an associahedron. Our second main result is to give a combinatorial interpretation of the h-vector of $A_{n,k}$, giving a common interpretation for both permutohedra and associahedra. Our interpretation involves the theory of stack-sorting.

West's stack-sorting map is a function $s : \mathfrak{S}_n \to \mathfrak{S}_n$ which attempts to sort the permutations w in $\mathfrak{S}_n$ in linear time, not always sorting them completely (see Definition 4.1). It is well-known that for the associahedron, h_i counts the number of permutations in $s^{-1}(1 \ldots n)$ with exactly i descents. We give a generalization of this result for all broom poset associahedra. Define

$$\mathfrak{S}_{n,k} := \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}.$$

We prove the following:

Theorem 4.2. *Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents.*

An immediate corollary of Theorem 4.2 is γ-nonnegativity of $\mathscr{A}(A_{n,k})$. In particular, we recall the following result of Bränden.

Theorem 4.4 ([2]). *For $A \subseteq \mathfrak{S}_n$, we have*

$$\sum_{\sigma \in s^{-1}(A)} t^{\mathrm{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(A) \ : \ \mathrm{peak}(\sigma) = m\}|}{2^{n-1-2m}} t^m (1+t)^{n-1-2m},$$

where $\mathrm{peak}(\sigma)$ *is the number of index i such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

Thus, we have the following corollary.

Corollary 4.5. *The γ-vector of $\mathscr{A}(A_{n,k})$ is nonnegative.*

In addition, in the process of proving Theorem 4.2, we find the size of $s^{-1}(\mathfrak{S}_{n,k})$ in terms of $k!$ and the Catalan convolution $C_n^{(k)}$, which will be introduced in Section 4.2.

Corollary 4.3. *For all $n, k \geq 0$, we have*

$$|s^{-1}(\mathfrak{S}_{n,k})| = k! \cdot C_n^{(k)}.$$

Finally, in Section 4.4, we prove the following strengthening of Corollary 4.5:

Theorem 4.10. *Let $H_n(t)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$. Then, $H_n(t)$ is real-rooted.*

This paper is an extended abstract to [5] and [6].

2 Background

2.1 Face numbers

For a d-dimensional polytope P, the sequence $(f_0(P), \ldots, f_d(P))$ is called the *f-vector* of P, where $f_i(P)$ is the number of i-dimensional faces of P and

$$f_P(t) = \sum_{i=0}^{d} f_i(P) t^i$$

is called the *f-polynomial* of P. When P is *simple*, recall that the h-polynomial and γ-polynomial are defined by

$$\begin{aligned} f_P(t) &= h_P(t+1), \\ h_P(t) &= (1+t)^d \gamma\left(\frac{t}{(1+t)^2}\right). \end{aligned}$$

2.2 Poset associahedra

We recall the following definitions.

Definition 2.1. Let $(P, \preceq)$ be a finite poset and let $\sigma, \tau \subseteq P$.

- τ is *connected* if it is connected as an induced subgraph of the Hasse diagram of P.
- τ is *convex* if whenever $x, z \in \tau$ and $y \in P$ such that $x \preceq y \preceq z$, then $y \in \tau$.
- τ is a *tube* of P if it is connected, convex, and $|\tau| > 1$. We say τ is a *proper tube* if additionally $|\tau| < |P|$.
- τ and σ are *nested* if $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$ and they are *disjoint* if $\tau \cap \sigma = \emptyset$.
- We say $\sigma \prec \tau$ if $\sigma \cap \tau = \emptyset$ and there exists $x \in \sigma$ and $y \in \tau$ such that $x \preceq y$.
- A *tubing* T of P is a set of proper tubes such that any pair of tubes in T is either nested or disjoint and there is no subset $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$.
- A tubing T is *maximal* if it is maximal under inclusion, i.e. T is not a proper subset of any other tubing.

Definition 2.2 ([4, Theorem 1.2])**.** For a finite, connected poset P, there exists a simple, convex polytope $\mathscr{A}(P)$ of dimension $|P| - 2$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. The faces of $\mathscr{A}(P)$ correspond to tubings of P, and the vertices of $\mathscr{A}(P)$ correspond to maximal tubings of P. This polytope is called the *poset associahedron* of P.

3 Comparability invariance

The *comparability graph* of a poset P is the graph $\mathscr{C}(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of a poset is said to be *comparability invariant* if it only depends on $\mathscr{C}(P)$. In [3], Dreesen, Poguntke, and Winkler give a powerful characterization of comparability invariance which we recall in this section.

Definition 3.1. Let P and S be posets and let $a \in P$. The *substitution* of a for S is the poset $P(a \to S)$ on the set $(P - \{a\}) \sqcup S$ formed by replacing a with S.

More formally, $x \preceq_{P(a \to S)} y$ if and only if one of the following holds:

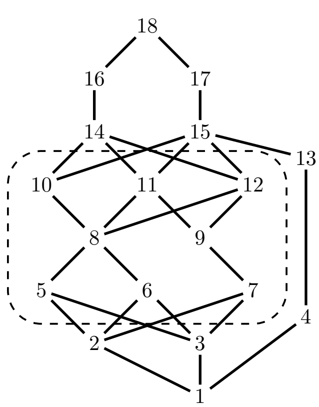
- $x, y \in P - \{a\}$ and $x \preceq_P y$;
- $x, y \in S$ and $x \preceq_S y$;
- $x \in S, y \in P - \{a\}$ and $a \preceq_P y$;
- $y \in S, x \in P - \{a\}$ and $y \preceq_P a$.

Definition 3.2. Let P be a poset and let $S \subseteq P$. S is called *autonomous* if there exists a poset Q and $a \in Q$ such that $P = Q(a \to S)$.

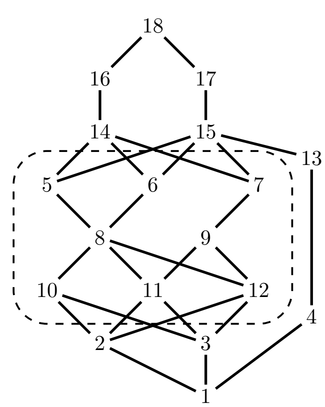
Equivalently, S is autonomous if for all $x, y \in S$ and $z \in P - S$, we have

$$(x \preceq z \Leftrightarrow y \preceq z) \text{ and } (z \preceq x \Leftrightarrow z \preceq y).$$

Definition 3.3. For a poset S, the *dual poset* S^{op} is defined on the same ground set where $x \preceq_S y$ if and only if $y \preceq_{S^{\text{op}}} x$. A *flip* of S in $P = Q(a \to S)$ is the replacement of P by $Q(a \to S^{\text{op}})$.



(a) An autonomous subset S of a poset P.



(b) A flip of S.

Figure 1

See Figure 1a for an example of an autonomous subset and Figure 1b for an example of a flip.

Lemma 3.4 ([3, Theorem 1]). *If P and P' are finite posets such that $\mathscr{C}(P) = \mathscr{C}(P')$ then P and P' are connected by a sequence of flips of autonomous subsets.*

Our main technical lemma is the following.

Lemma 3.5. *Let P be a poset and let $S \subseteq P$ be autonomous, and let P' be the poset obtained by flipping S in P. Then $\mathscr{A}(P)$ and $\mathscr{A}(P')$ have the same f-vector.*

Lemma 3.5 immediately gives our first theorem.

Theorem 3.6. *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

Theorem 3.6 may lead one to ask if $C(P) \simeq C(P')$, then are $\mathscr{A}(P)$ and $\mathscr{A}(P')$ necessarily combinatorially equivalent? We answer this in the negative with the following example:

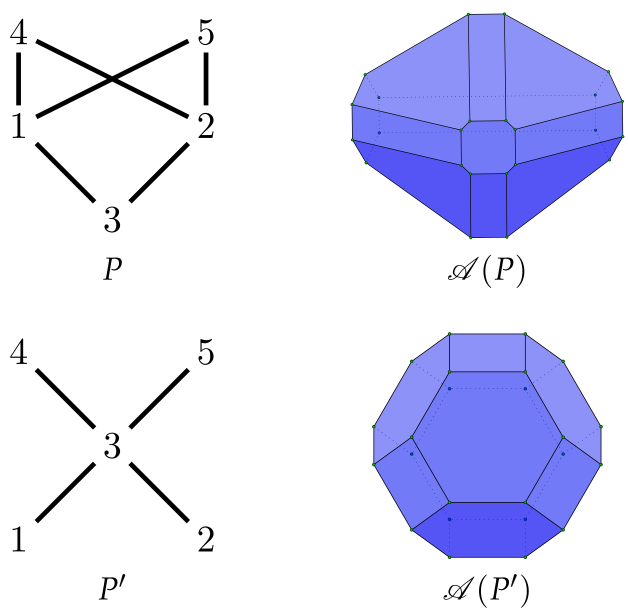


Figure 2: $\mathscr{A}(P)$ has an octagonal face, but $\mathscr{A}(P')$ does not.

3.1 Proof sketch of Lemma 3.5

Let $P = Q(a \to S)$ and $P' = Q(a \to S^{\text{op}})$. By an abuse of notation, we let $\mathscr{A}(P)$ also refer to the set of tubings of P. Our goal is to build a bijection $\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$ such that for any tubing $T \in \mathscr{A}(P)$, we have $|T| = |\Phi_{P,S}(T)|$. Let $T \in \mathscr{A}(P)$. We will describe how to construct $T' := \Phi_{P,S}(T)$.

Definition 3.7. A tube $\tau \in T$ is *good* if $\tau \subseteq P - S$, $\tau \subseteq S$, or $S \subseteq \tau$ and is *bad* otherwise. We denote the set of good tubes by T_{good} and the set of bad tubes by T_{bad}.

The key idea of defining $\Phi_{P,S}$ is to decompose T_{bad} into a triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$ where $\mathcal{L}$ and $\mathcal{U}$ are nested sequences of sets, some of which may be marked, contained in $P - S$ and M is an ordered set partition of S. We build the decomposition in such a way so that we can uniquely recover T_{bad} from $(\mathcal{L}, \mathcal{M}, \mathcal{U})$. Then, we construct T' by keeping T_{good} and replacing T_{bad} by T'_{bad}, which is obtained from $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ where $\overline{\mathcal{M}}$ is the reverse of $\mathcal{M}$. We decompose T_{bad} as follows.

Definition 3.8. A tube $\tau \in T_{\text{bad}}$ is called *lower* (resp. *upper*) if there exist $x \in \tau - S$ and $y \in \tau \cap S$ such that $x \preceq y$ (resp. $y \preceq x$). We denote the set of lower tubes by T_L and the set of upper tubes by T_U.

Lemma 3.9 (Structure Lemma)**.** *T_{bad} is the disjoint union of T_L and T_U. Furthermore, T_L and T_U each form a nested sequence.*

Definition 3.10 (Tubing decomposition)**.** Let $T_L = \{\tau_1, \tau_2, \ldots\}$ where $\tau_i \subset \tau_{i+1}$ for all i. For convenience, we define $\tau_0 = \emptyset$. We define a nested sequence $\mathcal{L} = (L_1, L_2, \ldots)$ and a sequence of disjoint sets $\mathcal{M}_L = (M_L^1, M_L^2, \ldots)$ as follows.

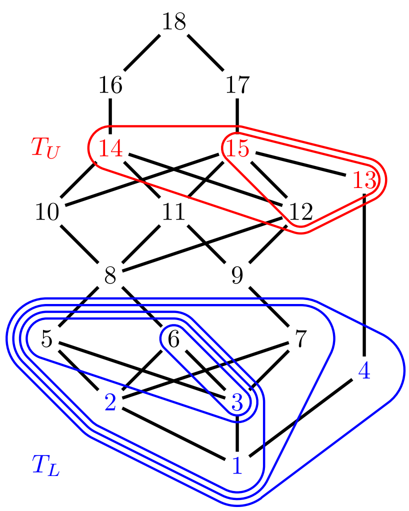
- For each $i \geq 1$, let $L_i = \tau_i - S$, and mark L_i with a star if $(\tau_i - \tau_{i-1}) \cap S \neq \emptyset$.
- If L_i is the j-th starred set, let $M_L^j = (\tau_i - \tau_{i-1}) \cap S$.

We define the sequences $\mathcal{U}$ and $\mathcal{M}_U$ analogously. We make the following definitions.

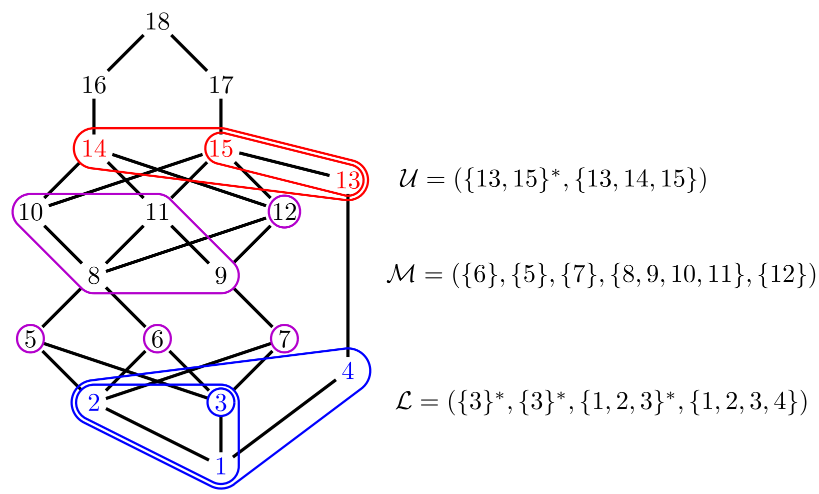
- Let $\hat{M} := S - \bigcup_{\tau \in T_{\text{bad}}} \tau$.
- For sequences $\mathbf{a}$ and $\mathbf{b}$, let the sequence $\mathbf{a} \cdot \mathbf{b}$ be $\mathbf{b}$ appended to $\mathbf{a}$, and let $\overline{\mathbf{a}}$ be the reverse of $\mathbf{a}$.
- We define
$$\mathcal{M} := \begin{cases} \mathcal{M}_L \cdot \overline{\mathcal{M}}_U & \text{if } \hat{M} = \emptyset \\ \mathcal{M}_L \cdot (\hat{M}) \cdot \overline{\mathcal{M}}_U & \text{if } \hat{M} \neq \emptyset \end{cases}$$
where $(\hat{M})$ is the sequence containing $\hat{M}$.
- The *decomposition* of T_{bad} is the triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$.

Figure 3 gives an example of a decomposition.

Lemma 3.11 (Reconstruction algorithm)**.** *T_{bad} can be reconstructed from its decomposition.*



(a) T_L is blue and T_U is red.



(b) $\mathcal{L}$ is blue, $\mathcal{M}$ is purple, and $\mathcal{U}$ is red.

Figure 3: The decomposition of T_{bad}.

Proof. Let $\mathcal{M} = (M_1, \ldots, M_n)$. To reconstruct T_L, we set $\tau_1 = L_1 \cup M_1$ and take

$$\tau_i = \begin{cases} \tau_{i-1} \cup L_i & \text{if } L_i \text{ is not starred} \\ \tau_{i-1} \cup L_i \cup M_j & \text{if } L_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

For T_U, we set $\tau_1 = U_1 \cup M_n$ and

$$\tau_i = \begin{cases} \tau_{i-1} \cup U_i & \text{if } U_i \text{ is not starred} \\ \tau_{i-1} \cup U_i \cup M_{n-j+1} & \text{if } U_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

$\square$

Lemma 3.12. *Applying the reconstruction algorithm to $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ yields a proper tubing T'_{bad} of P' with exactly $|T_{bad}|$ tubes.*

We define $T' := T'_{\text{bad}} \sqcup T_{\text{good}}$ and take $\Phi_{P,S}(T) := T'$.

Lemma 3.13. *T' is a proper tubing of P'. Furthermore, $\Phi_{P',S}(T') = T$ and $|\Phi_{P,S}(T)| = |T|$.*

4 Broom posets

Recall that the *ordinal sum* of two posets $(P, <_P)$ and $(Q, <_Q)$ is the poset $(R, <_R)$ whose elements are those in $P \cup Q$, and $a \leq_R b$ if and only if

- $a, b \in P$ and $a \leq_P b$ or

- $a, b \in Q$ and $a \leq_Q b$ or
- $a \in P$ and $b \in Q$.

We denote the ordinal sum of P and Q as $P \oplus Q$. Let C_n be the chain poset of size n and A_k be the antichain of size k. In this section, we study the *broom posets* $A_{n,k} = C_{n+1} \oplus A_k$. In particular, $A_{n,0}$ is the chain poset C_{n+1}, and $A_{0,k}$ is the claw poset $C_1 \oplus A_k$. Recall that $\mathscr{A}(A_{n,0})$ is the associahedron and $\mathscr{A}(A_{0,k})$ is the permutohedron. We show that the h-vectors of broom posets have a simple combinatorial interpretation in terms of descents of stack-sorting preimages.

4.1 Stack-sorting

In [12], West defined a deterministic version of Knuth's stack-sorting algorithm, which we call the *stack-sorting map* and denote by s. The stack-sorting map is defined as follows.

Definition 4.1 (Stack-sorting)**.** Given a permutation $\pi \in \mathfrak{S}_n$, $s(\pi)$ is obtained through the following procedure. Iterate through the entries of π. In each iteration,

- if the stack is empty or the next entry is smaller than the entry at the top of the stack, push the next entry to the top of the stack;
- otherwise, pop the entry at the top of the stack to the end of the output permutation.

Figure 4 illustrates the stack-sorting process on $\pi = 3142$.

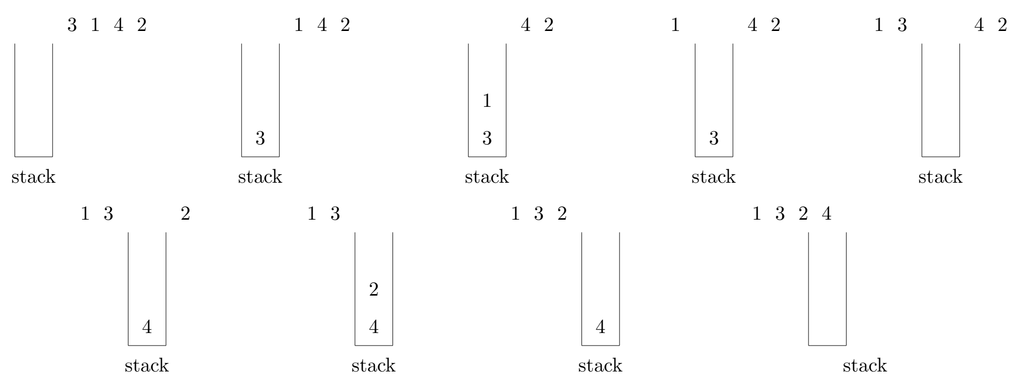


Figure 4: Example of $s(3142)$

4.2 Catalan convolution

Recall that the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ have generating function $\mathcal{C}(t) = \frac{1-\sqrt{1-4t}}{2t}$. The *k-th Catalan convolution* is the sequence with generating function $\mathcal{C}(t)^k$. For convenience, we denote $[t^n]\mathcal{C}(t)^k$ by $C_n^{(k)}$.

The explicit formula for $C_n^{(k)}$ is

$$C_n^{(k)} = \frac{k+1}{n+k+1}\binom{2n+k}{n}.$$

4.3 h-vector

Recall that we defined $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}$. In this section, our main theorem is:

Theorem 4.2. *Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents.*

As a corollary, we obtain the following result.

Corollary 4.3. *For all $n, k \geq 0$, we have*

$$|s^{-1}(\mathfrak{S}_{n,k})| = k! \cdot C_n^{(k)}.$$

Recall also the following result by Brändén.

Theorem 4.4 ([2])**.** *For $A \subseteq \mathfrak{S}_n$, we have*

$$\sum_{\sigma \in s^{-1}(A)} t^{\mathrm{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(A) \ : \ \mathrm{peak}(\sigma) = m\}|}{2^{n-1-2m}} t^m (1+t)^{n-1-2m},$$

where $\mathrm{peak}(\sigma)$ *is the number of index i such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

This gives the following corollary.

Corollary 4.5. *The γ-vector of $\mathscr{A}(A_{n,k})$ is nonnegative.*

Remark 4.6. Corollary 4.5 also follows from the fact that $\mathscr{A}(A_{n,k})$ is isomorphic to the graph associahedra of lollipop graphs, which are chordal. It was shown in [7] that graph associahedra of chordal graphs are γ-nonnegative.

4.4 Real-rootedness

In this section, we will sketch the proof of real-rootedness of the h-polynomial of $\mathscr{A}(A_{n,2})$. We say a polynomial $a_0 + a_1 t + \ldots + a_n t^n$ is real-rooted if all of its zeros are real. We say a sequence $(a_0, a_1, \ldots, a_n)$ is real-rooted if the polynomial $a_0 + a_1 t + \ldots + a_n t^n$ is real-rooted.

Let f and g be real-rooted polynomials with positive leading coefficients and real roots $\{f_i\}$ and $\{g_i\}$, respectively. We say that f *interlaces* g if

$$g_1 \leq f_1 \leq g_2 \leq f_2 \leq \ldots \leq f_{d-1} \leq g_d$$

where $d = \deg g = \deg f + 1$. We say that f *alternates left of* g if

$$f_1 \leq g_1 \leq f_2 \leq g_2 \leq \ldots \leq f_d \leq g_d$$

where $d = \deg g = \deg f$. Finally, we say f *interleaves* g, denoted $f \ll g$, if f either interlaces or alternates left of g.

Recall that the Narayana polynomial $N_n(t)$ is defined by

$$N_n(t) = \sum_{i=0}^{n-1} a_i t^i$$

where a_i counts the number of permutations in $s^{-1}(1 \ldots n)$ with exactly i descents. In other words, $N_n(t)$ is the h-polynomial of $\mathscr{A}(A_{n,0})$ and $\mathscr{A}(A_{n-1,1})$. We have the following result.

Theorem 4.7 ([1]). *For all n, $N_n(t)$ is real-rooted. Furthermore, $N_{n-1}(t) \ll N_n(t)$.*

To prove real-rootedness of the h-polynomial of $\mathscr{A}(A_{n,2})$, we will need the following "happy coincidence".

Proposition 4.8. *The number of permutations in $s^{-1}(2134 \ldots n)$ with exactly i descents is the same as the number of permutations w in $s^{-1}(1 \ldots n)$ with exactly i descents such that $w_1, w_n < n$.*

Proposition 4.8 leads to the following important recurrence.

Proposition 4.9. *Let $H_n(t)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$, and recall that $N_{n+2}(t)$ and $N_{n+1}(t)$ are the h-polynomials of $\mathscr{A}(A_{n+2,0})$ and $\mathscr{A}(A_{n+1,0})$, respectively. We have*

$$H_n(t) = 2N_{n+2}(t) - (1+t)N_{n+1}(t).$$

This recurrence and Theorem 4.7 allows us to prove the following theorem.

Theorem 4.10. *Let $H_n(t)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$. Then, $H_n(t)$ is real-rooted.*

5 Open Questions

Question 5.1. Can we define $f_{\mathscr{A}(P)}(z)$ purely in terms of $C(P)$? It would also be interesting to answer this question even for f_0.

Question 5.2. It remains open to find an interpretation of the h-vector of $\mathscr{A}(P)$ in terms of the combinatorics of P. Can $h(z)$ be defined purely in terms of $C(P)$?

Question 5.3. The map $\Phi_{P,S}$ can be analogously defined for *affine poset cyclohedra* [4], where an autonomous subset S has at most one representative from each residue class. Again, it preserves the f-vector of the affine poset cyclohedron. Does Lemma 3.4 (and hence Theorem 3.6) hold for affine posets?

We have the following conjectured generalization of Proposition 4.9.

Conjecture 5.4. *Let P be a poset with an autonomous subposet S that is a chain of size 2, i.e. $S = C_2$. Let P_1 be the poset obtained from P by replacing S by a singleton. Let P_2 be the poset obtained from P by replacing S by an antichain of size 2, i.e. A_2. Let $h_P(t)$, $h_{P_1}(t)$, $h_{P_2}(t)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\mathscr{A}(P_2)$, respectively. Then,*

$$2h_P(t) = h_{P_2}(t) + (1+t)h_{P_1}(t).$$

As a result, let $\gamma_P(t)$, $\gamma_{P_1}(t)$, $\gamma_{P_2}(t)$ be the γ-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\mathscr{A}(P_2)$, respectively. Then,

$$2\gamma_P(t) = \gamma_{P_2}(t) + \gamma_{P_1}(t).$$

Conjecture 5.4 is useful in proving real-rootedness of the h-polynomials, as shown in Theorem 4.10. Furthermore, the resulting recurrence of the γ-polynomial would also be useful in proving γ-positivity. More generally, we have the following recurrence when S is an antichain of size n.

Conjecture 5.5. *Let P be a poset with an autonomous subposet S that is a chain of size n, i.e. $S = C_n$. For $1 \le i \le n$, let P_i be the poset obtained from P by replacing S by an antichain of size i, i.e. A_i. Let $h_P(t)$, $h_{P_1}(t)$, $\ldots$, $h_{P_n}(t)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\ldots$, $\mathscr{A}(P_n)$, respectively. Then,*

$$h_P(t) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} B_1(t)^{c_1(w)} \ldots B_n(t)^{c_n(w)} h_{P_{\ell(\lambda(w))}}(t) \tag{5.1}$$

where

$$B_k(t) = \sum_{i=0}^{k-1} \binom{k-1}{i}^2 t^i$$

are type B Narayana polynomials, $c_i(w)$ is the number of cycles of size i in w, and $\ell(\lambda(w))$ is the length of the cycle type $\lambda(w)$ of w.

The type B Narayana polynomials above also show up as the rank-generating function of the type B analogue NC_n^B of the lattice of non-crossing partitions (see [8]) and the h-polynomials of type B associahedra (see [9]).

Equation 5.1 bears resemblance to the Frobenius characteristic map. Thus, it is a natural question to ask if there is a representation theory story behind this equation. This is an interesting question for future research.

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Rational Catalan Numbers for Complex Reflection Groups

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Abstract. Assuming standard conjectures, we show that the canonical symmetrizing trace evaluated at powers of a Coxeter element produces rational Catalan numbers for irreducible spetsial complex reflection groups. This extends a technique used by Galashin, Lam, Trinh, and Williams to uniformly prove the enumeration of their noncrossing Catalan objects for finite Coxeter groups.

Keywords: reflection groups, Hecke algebra, parking

1 Introduction

1.1 Catalan combinatorics

This is an extended abstract of [15]. The prototypical noncrossing Coxeter-Catalan objects are the *noncrossing partitions*. In type A with the usual Coxeter element $c = (1, 2, \dots, n)$, these correspond to partitions $\{B_1, \dots, B_k\}$ of the set $\{1, 2, \dots, n\}$ such that there do not exist $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$. These type A noncrossing partitions are counted by $\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$.

In [8], the authors define rational noncrossing Coxeter-Catalan objects called *maximal* $\mathbf{c}^p$*-Deograms*, counted by the *rational Coxeter-Catalan numbers*

$$\mathrm{Cat}_p(W) := \prod_{i=1}^{n} \frac{p + (pe_i \mod h)}{d_i},$$

where $h = d_n$ is the Coxeter number of W, p is coprime to h, $d_1 \leq \cdots \leq d_n$ are the degrees of a set of algebraically independent homogeneous polynomials which generate the algebra of invariants $\mathrm{Sym}(V^*)^W$, and $e_i = d_i - 1$. These objects are defined for finite Coxeter groups. As part of the type-uniform proof of this enumeration, the authors of [8] use Hecke algebra traces to compute the point count of braid Richardson varieties over a finite field, producing q-deformed rational Catalan numbers:

$$\mathrm{Cat}_p(W; q) := \prod_{i=1}^{n} \frac{[p + (pe_i \mod h)]_q}{[d_i]_q}.$$

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Many of the objects used in their proof can also be defined for the well-generated complex reflection groups, so it is natural to try to compute these traces in the complex case. It turns out that the *well-generated* condition is too weak for certain representation-theoretic techniques to work—the necessary condition is for the group to be *spetsial*. These spetsial complex reflection groups (see Definition 3.2) are well-generated complex reflection groups that behave as if they were the Weyl group for some connected reductive algebraic group.

Analogs of unipotent characters, Harish-Chandra theory, and Lusztig's Fourier transform can be defined combinatorially for these groups, which allows techniques from the representation theory of finite groups of Lie type to be extended to spetsial complex reflection groups.

As the main result of our paper, Theorem 8.1, we show that for irreducible spetsial complex reflection groups the trace of a power of a Coxeter element still produces a rational Catalan number even though there are not braid Richardson varieties in this context. Precisely, we prove the following result:

Theorem 1.1. *Let W be an irreducible spetsial complex reflection group with Coxeter number h, and let c be a ζ_h-regular element of W. Let $\mathbf{c} \in B(W)$ be a lift of c such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\tau_q(T_{\mathbf{c}}^{-p}) = q^{-np}(1-q)^n \operatorname{Cat}_p(W;q).$$

1.2 Parking combinatorics

The *noncrossing parking functions* in type A are sets of tuples $\{(B_1, L_1), \ldots, (B_k, L_k)\}$, where $B_i, L_i \subseteq \{1, \ldots, n\}$ and

- $\{B_1, \ldots, B_k\}$ is a noncrossing partition of $\{1, \ldots, n\}$,
- $\{L_1, \ldots, L_k\}$ is a set partition of $\{1, \ldots, n\}$, and
- $|B_i| = |L_i|$ for $i = 1, \ldots, k$.

The number of these noncrossing parking functions is $(n+1)^{n-1}$.

In [8], the authors uniformly defined rational noncrossing parking objects for finite Coxeter groups. These parking objects correspond to certain walks in the Hasse digram of the weak Bruhat order and are counted by p^n. As a corollary of our main result, we prove the following (see Corollary 8.2):

Corollary 1.2. *For W an irreducible spetsial complex reflection group, let $\mathcal{B}$ be a basis of the spetsial Hecke algebra $\mathcal{H}_q(W)$ (adapted to the Wedderburn decomposition), and let $\mathbf{c}$ be a lift of a ζ_h-regular element such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}^p} b) = (q-1)^n [p]_q^n.$$

In the real case, this is a key algebraic step in the proof of the enumeration of rational noncrossing parking functions [8]. Finding a combinatorial interpretation of the left-hand-side of this equation, e.g. rational noncrossing parking functions for spetsial complex reflection groups, is an open problem.

2 Complex reflection groups

Let V be an n-dimensional complex vector space. A linear transformation $g \in \mathrm{GL}(V)$ is a *reflection* if the order of g is finite and the subspace $\mathrm{Fix}(g) := \{v \in V : gv = v\}$ has codimension 1. In this case, $\mathrm{Fix}(g)$ will be called the *reflection hyperplane* of g. A *(finite) complex reflection group* is a finite subgroup of $\mathrm{GL}(V)$ that is generated by reflections. It is said to be *well-generated* if it can be generated by n reflections. For a complex reflection group W, we will use $\mathcal{R}$ to denote the set of reflections in W, and $\mathcal{A}$ will denote the corresponding set of reflecting hyperplanes (for Coxeter groups, $|\mathcal{R}| = |\mathcal{A}|$).

A complex reflection group is *irreducible* if V is an irreducible W-module. By the classification of Shephard and Todd, an irreducible complex reflection group is either in the infinite family $G(m, p, n)$, for p a divisor of m, or is one of 34 exceptional groups labeled 4 to 37.

For an orbit of hyperplanes $\mathcal{C} \in \mathcal{A}/W$, we will let $e_{\mathcal{C}}$ denote the order of the pointwise stabilizer $W_H = \{w \in W : wh = h,\ \forall h \in H\}$ for any $H \in \mathcal{C}$ (the order does not depend on the choice of H). For any $H \in \mathcal{C}$, the group W_H is cyclic with order $e_{\mathcal{C}}$, and there is a reflection $s_H \in W$ with reflecting hyperplane H and determinant $\zeta_{e_{\mathcal{C}}} = \exp(2\pi i / e_{\mathcal{C}})$. Such reflections are called *distinguished reflections*.

The *field of definition* k_W of a complex reflection group W is the field generated by the traces of the elements of W on the reflection representation. The field of definition is a subfield of $\mathbb{R}$ when W is a finite Coxeter group and equals $\mathbb{Q}$ when W is a Weyl group.

The *degrees* of W are defined to be the degrees $d_1 \leq \cdots \leq d_n$ of a collection of algebraically independent homogeneous polynomials that generate the *algebra of invariants* $\mathrm{Sym}(V^*)^W$. The *Poincaré polynomial* P_W is defined by

$$P_W := \prod_{j=1}^{n} [d_j]_q,$$

where $[n]_q$ denotes the q-analog $[n]_q := (q^n - 1)/(q - 1)$ for $n \in \mathbb{Z}$.

Let S_+^W be the ideal of $S := \mathrm{Sym}(V^*)$ generated by the positive degree elements of S^W. Then the action of W on the *coinvariant algebra* S/S_+^W is the regular representation, so S/S_+^W contains exactly r copies of any irreducible representation M of W of dimension r. The *exponents* of M are defined to be the degrees $e_1(M) \leq \cdots \leq e_r(M)$ of the homogeneous components of S/S_+^W containing a copy of M. If χ is the irreducible character corresponding to M, we will also denote $e_i(\chi) := e_i(M)$ and $e_i := e_i(V)$.

Definition 2.1. The *fake degree* $\mathrm{Feg}_\chi(q)$ of an irreducible character χ of W is the graded multiplicity of the irreducible representation with character χ in S/S_+^W:

$$\mathrm{Feg}_\chi(q) = \sum_{i=1}^{r} q^{e_i(\chi)}.$$

3 Hecke algebras

Let $V^{\mathrm{reg}} := V \setminus \bigcup_{H\in\mathcal{A}} H$ denote the hyperplane complement. The *pure braid group* for a complex reflection group W is $P(W) := \pi_1(V^{\mathrm{reg}})$. Its *braid group* is $B(W) := \pi_1(V^{\mathrm{reg}}/W)$. The quotient $\rho : V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ induces a surjection $\varphi : B(W) \to W$, giving a short exact sequence

$$1 \to P(W) \xrightarrow{\rho_*} B(W) \xrightarrow{\varphi} W \to 1,$$

where W can be interpreted as the group of deck transformations of the covering.

The braid group $B(W)$ has a set of generators $\{\mathbf{s}_{H,\gamma}\}$ called *generators of the monodromy* or *braid reflections* [4], such that $\varphi(\mathbf{s}_{H,\gamma}) = s_H$ is a distinguished reflection. Moreover, the pure braid group $P(W)$ is generated by the $\{\mathbf{s}_{H,\gamma}^{e_\mathcal{C}}\}$ (where $H \in \mathcal{C}$), and so $W \cong B(W)/\langle \mathbf{s}_{H,\gamma}^{e_\mathcal{C}} \rangle$.

The *full twist* $\boldsymbol{\pi} \in P(W)$ is given by $t \mapsto v\exp(2\pi i t)$, where the basepoint $v \in V^{\mathrm{reg}}$ is suppressed. The image $\rho_*(\boldsymbol{\pi}) \in B(W)$ is a central element of $B(W)$ and will also be called the full twist and be denoted by $\boldsymbol{\pi}$.

Define $\mathbf{u} := (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W),(0\le j\le e_\mathcal{C}-1)}$, and let $\mathbb{Z}[\mathbf{u}^{\pm 1}]$ be the ring of Laurent polynomials. Let J be the ideal of the group algebra $\mathbb{Z}[\mathbf{u}^\pm]B(W)$ generated by the elements

$$(\mathbf{s}_H - u_{\mathcal{C},0})(\mathbf{s}_H - u_{\mathcal{C},1})\cdots(\mathbf{s}_H - u_{\mathcal{C},e_\mathcal{C}-1}),$$

where $\mathcal{C} \in \mathcal{A}/W$, $H \in \mathcal{C}$, and $\mathbf{s}_H$ is a braid reflection (since generators $\mathbf{s}_{H,\gamma}$ and $\mathbf{s}_{H,\gamma'}$ of the monodromy around H are conjugate in $B(W)$ [4], it suffices to use only one such braid reflection for each H in the above relations to generate J). The *generic Hecke algebra* $\mathcal{H}(W)$ is the quotient $\mathbb{Z}[\mathbf{u}^\pm]B(W)/J$. For $\mathbf{g} \in B(W)$, we'll denote by $T_\mathbf{g}$ the corresponding element in the Hecke algebra.

The *spetsial Hecke algebra* $\mathcal{H}_q(W)$ is the admissible cyclotomic Hecke algebra induced by the map

$$\theta_q : u_{\mathcal{C},j} \mapsto \begin{cases} q & \text{if } j = 0 \\ \zeta_{e_\mathcal{C}}^j & \text{if } j > 0. \end{cases}$$

This is a generalization of the 1-parameter Iwahori-Hecke algebra of Coxeter groups.

The spetsial Hecke algebra $\mathcal{H}_q(W)$ has splitting field $k_W(y)$, where $y^{|\mu(k_W)|} = q$ for $\mu(k_W)$ the group of roots of unity in k_W. By Tits' deformation theorem, there is a bijection between the irreducible characters of $\mathcal{H}_q(W)$ and those of W. We will denote by χ_q the character of $\mathcal{H}_q(W)$ coresponding to $\chi \in \mathrm{Irr}(W)$.

We will make the following assumption, called the BMM symmetrizing trace conjecture:

Assumption 3.1 ([2]). *There exists a $\mathbb{Z}[\mathbf{u}^{\pm}]$-linear map $\tau : \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u}^{\pm}]$ such that:*

1. *τ is a symmetrizing trace; that is, τ is the bilinear form $\mathcal{H}(W) \times \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u}^{\pm}]$ given by $(h, h') \mapsto \tau(hh')$ is symmetric and non-degenerate.*
2. *Through the specialization $u_{\mathcal{C},j} \mapsto \zeta^j_{e_\mathcal{C}}$, the form τ becomes the canonical symmetrizing trace on the group algebra: $w \mapsto \delta_{1w}$.*
3. *For all $\mathbf{b} \in B(W)$,*
$$\tau(T_{\mathbf{b}^{-1}})^\vee = \frac{\tau(T_{\mathbf{b}\boldsymbol{\pi}})}{\tau(T_{\boldsymbol{\pi}})},$$
where $\alpha \mapsto \alpha^\vee$ is the automorphism on $\mathbb{Z}[\mathbf{u}^{\pm}]$ consisting of simultaneous inversion of the indeterminates.

If such a symmetrizing trace exists, it is unique. We will call τ the canonical symmetrizing trace on $\mathcal{H}(W)$, and denote by τ_q the specialization to the spetsial Hecke algebra. The BMM symmetrizing trace conjecture has been proven for the infinite family $G(m, p, n)$ and the finite Coxeter groups, but remains open for some of the exceptional groups.

There exist weights $S_\chi(q) \in \mathbb{Z}_{k_W}[y^{\pm}]$ for $\chi \in \mathrm{Irr}(W)$, called *Schur elements*, such that
$$\tau_q = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{S_\chi(q)} \chi_q,$$
where $\mathbb{Z}_{k_W}$ is the ring of integers of k_W [2]. The Schur elements have been computed even in the cases for which Assumption 3.1 is still open [12].

Using these Schur elements, we can now define the class of spetisal complex reflection groups, which form a subset of the well-generated complex reflection groups.

Definition 3.2. An complex reflection group is called *spetsial* if all of its irreducible components W satisfy any of the following equivalent conditions (equivalence of the conditions is shown in [12]):

1. $S_{\mathbf{1}}(q) = P_W$, where $\mathbf{1}$ denotes the trivial representation of W
2. $P_W / S_\chi(q) \in k_W(q)$ for all $\chi \in \mathrm{Irr}(W)$
3. W is one of the following groups:
$$G(m, 1, n), \quad G(m, m, n), \quad G_i \text{ where } i \in \{4, 6, 8, 14, 23, \ldots, 30, 32, \ldots, 37\}.$$

If W is an irreducible spetsial complex reflection group, define the *generic degree* of an irreducible character χ by $\mathrm{Deg}_\chi(q) := P_W / S_\chi(q) \in k_W(q)$.

4 Coxeter elements

The *Coxeter number* of a complex reflection group W is $h := (|\mathcal{R}| + |\mathcal{A}|)/n$. If W is well-generated, then $h = d_n$. More generally, define the *generalized Coxeter number* h_χ associated to a character χ to be the normalized trace of the central element $\sum_{r\in\mathcal{R}}(1-r)$. That is,

$$h_\chi = |\mathcal{R}| - \frac{1}{\chi(1)} \sum_{r\in\mathcal{R}} \chi(r).$$

These generalized Coxeter numbers are integers, and $h_\phi = h$ when ϕ is the character of the reflection representation.

A vector $v \in V$ is *regular* if it is not contained in any reflection hyperplane. An element $c \in W$ is *regular* if it has a regular eigenvector. Moreover, c is *ζ-regular* if this eigenvector may be chosen to have eigenvalue ζ. In this case, the multiplicative order d of ζ is a *regular number* for W.

If W is well-generated, then there exists a ζ-regular element for every h-th root of unity ζ. For ζ a primitive h-th root of unity, the ζ-regular elements of W are called *Coxeter elements*.

Remark 4.1. Some authors define the Coxeter elements to be only the ζ_h-regular elements. In several cases, we will find it useful to restrict our attention to this subset of the Coxeter elements. It follows easily from the definition that each Coxeter element is a power of some ζ_h-regular element.

Proposition 4.2. *Suppose W is an irreducible spetsial complex reflection group with Coxeter number h, and let c be a ζ_h-regular element of W. Then there exists a lift $\mathbf{c} \in B(W)$ of c such that $\mathbf{c}^h = \boldsymbol{\pi}$, and*

$$\tau_q(T_{\mathbf{c}}^{-p}) = \frac{1}{P_W} \sum_{\chi\in\mathrm{Irr}(W)} q^{(h_\chi - nh)p/h} \operatorname{Feg}_\chi(e^{2\pi i p/h}) \operatorname{Deg}_\chi(q).$$

Proof. It is shown in [1] that the lift $\mathbf{c}$ exists. The trace formula then follows from [5, 18]. □

5 Exterior powers of Galois twists

Let W be a well-generated irreducible complex reflection group with Coxeter number h and reflection representation V of dimension n. The *Galois twist* V^{σ_p} of V is the irreducible representation of W obtained by applying σ_p to the matrices representing the elements $w \in W$ as linear operators on V, where $\sigma_p \in \mathrm{Gal}(\mathbb{Q}(\zeta_h)/\mathbb{Q})$ is defined by $\sigma_p : \zeta_h \mapsto \zeta_h^p$ for some p coprime to h.

Lemma 5.1. *The generalized Coxeter number of $\Lambda^k V^{\sigma_p}$ is kh.*

Proof. This is a straightforward computation. □

Lemma 5.2. *The fake degree of* $\Lambda^k V^{\sigma_p}$ *is*

$$\sum_{i_1<\cdots<i_k} q^{e_{i_1}(V^{\sigma_p})+\cdots+e_{i_k}(V^{\sigma_p})},$$

and the sets $\{e_1(V^{\sigma_p}),\ldots,e_n(V^{\sigma_p})\}$ *and* $\{pe_1 \mod h,\ldots,pe_n \mod h\}$ *coincide.*

Proof. The first part is shown in [17]. For the second part, the result can be checked by computer [16] for the exceptional groups. For the groups $G(m,1,n)$ and $G(m,m,n)$, we use Malle's [11] formulas for the fake degrees of irreducible characters in terms of corresponding *m-symbols*. □

Theorem 5.3. *Suppose W is an irreducible well-generated complex reflection group with Coxeter number h. Then for p relatively prime to h and* χ *an irreducible character of W,*

$$[S_{\mathbf{1}}/S_\chi](e^{2\pi i p/h}) = \begin{cases} (-1)^k & \text{if } \chi = \chi_{k,p}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_{k,p}$ *is the character corresponding to* $\Lambda^k V^{\sigma_p}$.

Proof. For the symmetric groups (in fact, all real groups), the proof is sketched in [8]. For the exceptional irreducible well-generated groups, this is checked by computer [16]. The proof for the groups $G(m,1,n)$ and $G(m,m,n)$ follows the proof of the "untwisted case" in [14], and we will now outline it.

The group $G(m,1,n)$ is generated by the n elements $\{t,s_1,s_2,\ldots,s_{n-1}\}$, where t is given by the matrix $\mathrm{Diag}(\zeta_m,1,\ldots,1)$ and s_i is given by the permutation matrix corresponding to the transposition $(i,i+1)$. The irreducible representations of $G(m,1,n)$ can be parametrized by m-partitions of n, that is, tuples $\boldsymbol{\lambda} = (\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(m-1)})$ of partitions such that $|\lambda^{(0)}|+\cdots+|\lambda^{(m-1)}| = n$. The correspondence is then

$$\boldsymbol{\lambda} \leftrightarrow \chi_{\boldsymbol{\lambda}} = \mathrm{Ind}_{G(m,1,|\lambda^{(0)}|)\times\cdots\times G(m,1,|\lambda^{(m-1)}|)}^{G(m,1,n)} \left((\chi_0\otimes\gamma_0)\boxtimes\cdots\boxtimes(\chi_{m-1}\otimes\gamma_{m-1})\right),$$

where

- γ_k is the linear character of $G(m,1,|\lambda^{(k)}|)$ defined by $t\mapsto\zeta_m^k$ and $s_i\mapsto 1$ for $i=1,\ldots,|\lambda^{(k)}|-1$.
- χ_k is the character of the symmetric group $S_{|\lambda^{(k)}|}$ corresponding to $\lambda^{(k)}$ considered as a character of $G(m,1,|\lambda^{(k)}|)$ via the surjection $G(m,1,|\lambda^{(k)}|)\to S_{|\lambda^{(k)}|}$.

There are explicit combinatorial models for these representations [13] which can be used to show that the m-partition of n corresponding to $\Lambda^k V^{\sigma_p}$ is $(n-k, \varnothing, \ldots, \varnothing, 1^k, \varnothing, \ldots, \varnothing)$, where the 1^k is in the pth slot (mod m; indexing starts with 0).

In [11], Malle gives a combinatorial construction of unipotent characters and generic degrees for the groups $G(m,1,n)$ and $G(m,m,n)$ which enjoy many of the same properties as the corresponding objects for Weyl groups. Using a simplified formula for the Schur elements [7], we show that evaluating Deg_χ at the root of unity for χ an exterior power of a Galois twist produces $(-1)^k$. We then use a case-by-case argument to show that the evaluation of Deg_χ is zero otherwise.

Using Clifford theory, one can describe a relationship between the irreducible characters of $G(m,1,n)$ and those of $G(m,m,n)$: For an m-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(m-1)})$ of n, denote by $\omega(\boldsymbol{\lambda})$ the cyclic permutation $(\lambda^{(m-1)}, \lambda^{(0)}, \ldots, \lambda^{(m-2)})$. Let $\langle\omega\rangle$ denote the cyclic group of order m, and let $s(\boldsymbol{\lambda})$ be the size of the subgroup of $\langle\omega\rangle$ fixing $\boldsymbol{\lambda}$. Then there is a correspondence:

$$\{\chi_{\boldsymbol{\lambda}}, \chi_{\omega(\boldsymbol{\lambda})}, \ldots, \chi_{\omega^{m/s(\boldsymbol{\lambda})-1}(\boldsymbol{\lambda})}\} \in \mathrm{Irr}(G(m,1,n)) \leftrightarrow \left\{\psi, \psi^t, \ldots, \psi^{t^{s(\boldsymbol{\lambda})-1}}\right\} \in \mathrm{Irr}(G(m,m,n))$$
$$(\chi_{\omega^j(\boldsymbol{\lambda})})_{G(m,m,n)} = \psi + \psi^t + \cdots + \psi^{t^{s(\boldsymbol{\lambda})-1}}$$
$$\chi_{\boldsymbol{\lambda}} + \chi_{\omega(\boldsymbol{\lambda})} + \cdots + \chi_{\omega^{m/s(\boldsymbol{\lambda})-1}(\boldsymbol{\lambda})} = \mathrm{Ind}_{G(m,m,n)}^{G(m,1,n)} \psi^{t^j}.$$

Again, Malle has combinatorial constructions for the unipotent characters and generic degrees [11, 10] which we use to evaluate Deg_χ at the root of unity for χ an exterior power of a Galois twist. We again use a case-by-case argument to show that the evaluation of Deg_χ is zero otherwise. □

6 Families of unipotent characters

Let W be an irreducible spetsial complex reflection group. Define the *Rouquier ring* $\mathcal{R}_W(y)$ to be the $\mathbb{Z}_{k_W}$-subalgebra of $k_W(y)$ given by

$$\mathcal{R}_W(y) := \mathbb{Z}_{k_W}[y, y^{-1}, (y^n - 1)^{-1}_{n\geq 1}].$$

To each $\chi \in \mathrm{Irr}(W)$ we can associate a central primitive idempotent e_χ in $k_W(y)\mathcal{H}_q(W)$ given by

$$e_\chi := \frac{1}{S_\chi(q)} \sum_{b\in\mathcal{B}} \chi_q(b) b^\vee,$$

where $\mathcal{B}$ is a basis of $\mathcal{H}_q(W)$ adapted to the Wedderburn decomposition, and the $b^\vee$ form the dual basis with respect to τ_q [6].

There exists a unique partition RB(W) of $\mathrm{Irr}(W)$ such that

- for each $B \in \mathrm{RB}(W)$, the element $e_B := \sum_{\chi \in B} e_\chi$ is a central primitive idempotent in $\mathcal{R}_W(y)\mathcal{H}_q(W)$,
- $1 = \sum_{B \in \mathrm{RB}(W)} e_B$ and for every central idempotent e of $\mathcal{R}_W(y)\mathcal{H}_q(W)$ there exists a subset $\mathrm{RB}(W,e)$ of $\mathrm{RB}(W)$ such that $e = \sum_{B \in \mathrm{RB}(W,e)} e_B$.

We then say that two characters $\chi, \phi \in \mathrm{Irr}(W)$ belong to the same *Rouquier block* of $\mathcal{H}_q(W)$ if they belong to the same element of $\mathrm{RB}(W)$. We then have

Proposition 6.1. *If χ and ψ belong to the same Rouquier block of the spetsial Hecke algebra, then $h_\chi = h_\phi$.*

Proof. The proof that the statistics a_χ and A_χ are constant on Rouquier blocks is described in [6]. It is shown in [2] that $h_\chi = a_\chi + A_\chi$, so the result follows. □

For W be an irreducible Weyl group and q a power of a prime p, let $\mathbf{G}$ be a simple connected reductive group over $\overline{\mathbb{F}}_p$ with connected center which has Weyl group W, and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius map with respect to some $\mathbb{F}_q$-rational structure which acts trivially on W. We denote by $\mathbf{G}^F$ the corresponding finite group of Lie type and fix a maximally split torus $\mathbf{T}_0$.

A character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is called a *unipotent character* if $\langle R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}}), \rho \rangle \neq 0$ for some F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$. Here $1_{\mathbf{T}}$ is the trivial character of $\mathbf{T}^F$ and $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}})$ is the induced Deligne-Lusztig character of $\mathbf{G}^F$. We denote by $\mathrm{Uch}(\mathbf{G}^F)$ the set of all unipotent characters of $\mathbf{G}^F$.

There are two important subsets of the unipotent characters of $\mathbf{G}^F$:

1. For each $\chi \in \mathrm{Irr}(W)$, there is a *unipotent uniform almost character* R_χ which satisfies $R_\chi(1) = \mathrm{Feg}_\chi(q)$
2. For each $\chi \in \mathrm{Irr}(W)$, there is a *unipotent principal series character* ρ_χ which satisfies $\rho_\chi(1) = \mathrm{Deg}_\chi(q)$.

Define a graph on the set of vertices $\mathrm{Uch}(\mathbf{G}^F)$ as follows: two unipotent characters $\rho_1, \rho_2 \in \mathrm{Uch}(\mathbf{G}^F)$ are joined if and only if there is an irreducible character $\chi \in \mathrm{Irr}(W)$ such that $\langle R_\chi, \rho_i \rangle \neq 0$ for $i = 1, 2$. The sets of vertices corresponding to the connected components of the graph are called the *families* in $\mathrm{Uch}(\mathbf{G}^F)$.

These families recover the Rouquier blocks of $\mathrm{Irr}(W)$ via the inclusion $\chi \mapsto \rho_\chi$. For the spetsial groups $G(m,1,n)$ and $G(m,m,n)$, Malle has defined *families* of his unipotent characters $\mathrm{Uch}(W)$ which recover the Rouquier blocks via an inclusion $\mathrm{Irr}(W) \hookrightarrow \mathrm{Uch}(W)$.

For W an exceptional irreducible spetsial complex reflection group, there is a set $\mathrm{Uch}(\mathbb{G})$ defined in [3] for the corresponding split spets. There is also a principal series $\mathrm{Uch}(\mathbb{G},1)$ with bijection $\mathrm{Irr}(W) \to \mathrm{Uch}(\mathbb{G},1)$. Moreover, there is a partition of $\mathrm{Uch}(\mathbb{G})$ into *families* which recovers the Rouquier blocks of $\mathcal{H}_q(W)$ when restricted to the principal series.

7 Lusztig's Fourier transform

Lemma 7.1. *Suppose that there exists a pairing* $\{-,-\}_W : \mathrm{Irr}(W) \times \mathrm{Irr}(W) \to \mathbb{C}$ *satisfying*

(T1) For all $\chi \in \mathrm{Irr}(W)$*, we have*

$$\mathrm{Deg}_\chi(q) = \sum_{\phi \in \mathrm{Irr}(W)} \{\chi, \phi\}_W \, \mathrm{Feg}_\phi(q).$$

(T2) For all $\chi, \phi \in \mathrm{Irr}(W)$*, we have* $\{\chi, \phi\}_W = \{\phi, \chi\}_W$*.*

(T3) For all $\chi, \phi \in \mathrm{Irr}(W)$ *with* $\{\chi, \phi\}_W \neq 0$*, we have* $h_\chi = h_\phi$*.*

Then

$$\sum_{\chi \in \mathrm{Irr}(W)} q_1^{f(\chi)} \, \mathrm{Feg}_\chi(q_2) \, \mathrm{Deg}_\chi(q_3) = \sum_{\chi \in \mathrm{Irr}(W)} q_1^{f(\chi)} \, \mathrm{Feg}_\chi(q_3) \, \mathrm{Deg}_\chi(q_2),$$

if f *satisfies* $(h_\chi = h_\phi \implies f(\chi) = f(\phi))$*.*

Proof. This follows from a double-summation argument. □

These conditions can be interpreted as saying that the pairing transforms fake degrees to generic degrees, is symmetric, and is block diagonal on families. The following conjecture has been proven for W a finite Coxeter group and $W = G(m, 1, n)$.

Conjecture 7.2. *For all irreducible spetsial complex reflection groups* W*, there exists a pairing* $\{-,-\}_W$ *satisfying (T1), (T2), and (T3), which we will call the* truncated Lusztig Fourier transform*.*

For W a Weyl group, the transform is given by the inner product of characters $\langle \rho_\chi, R_\phi \rangle$. For non-Weyl Coxeter groups, the pairing is described in [9]. Malle constructs a Fourier transform in [11] for $G(m, 1, n)$, and it is shown in [10] that it satisfies (T1), (T2), and (T3).

In [10], Lasy conjectures the existence of a Fourier transform for $G(m, m, n)$ and describes its relation to a "pre-Fourier" transform which is a slight modification of the construction in [11]. This conjectured transform will satisfy (T1), (T2), and (T3).

The Fourier transforms for the exceptional groups (and for the families $G(m, 1, n)$ and $G(m, m, n)$) are contained in the data for GAP3, but their properties have not yet appeared in publication. See [3].

8 Rational Catalan numbers

Theorem 8.1. *Let W be an irreducible spetsial complex reflection group with Coxeter number h, and let c be a ζ_h-regular element of W. Let $\mathbf{c} \in B(W)$ be a lift of c such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\tau_q(T_{\mathbf{c}}^{-p}) = q^{-np}(1-q)^n \operatorname{Cat}_p(W;q).$$

Proof. Assuming Conjecture 7.2,

$$\begin{aligned}
\tau_q(T_{\mathbf{c}}^{-p}) &= \frac{1}{P_W} \sum_{\chi \in \operatorname{Irr}(W)} q^{(h_\chi - nh)p/h} \operatorname{Feg}_\chi(e^{2\pi i p/h}) \operatorname{Deg}_\chi(q) \\
&= \frac{1}{P_W} \sum_{\chi \in \operatorname{Irr}(W)} q^{(h_\chi - nh)p/h} \operatorname{Feg}_\chi(q) \operatorname{Deg}_\chi(e^{2\pi i p/h}) \\
&= \frac{1}{P_W} \sum_{k=0}^{n} (-1)^k q^{(k-n)p} \sum_{i_1 < \cdots < i_k} q^{e_{i_1}(V^{\sigma_p}) + \cdots + e_{i_k}(V^{\sigma_p})} \\
&= \frac{1}{P_W} q^{-np} \prod_{i=1}^{n} \left(1 - q^{p + e_i(V^{\sigma_p})}\right) = q^{-np}(1-q)^n \prod_{i=1}^{n} \frac{[p + e_i(V^{\sigma_p})]_q}{[d_i]_q} \\
&= q^{-np}(1-q)^n \operatorname{Cat}_p(W;q).
\end{aligned}$$

□

These "trace techniques" also allow us to extend a result from [8] related to the enumeration of rational parking functions.

Corollary 8.2. *For W an irreducible spetsial complex reflection group, let $\mathcal{B}$ be a basis of the spetsial Hecke algebra $\mathcal{H}_q(W)$ (adapted to the Wedderburn decomposition), and let $\mathbf{c}$ be a lift of a ζ_h-regular element such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}^p} b) = (q-1)^n [p]_q^n.$$

This corollary motivates future work: Are there noncrossing parking objects for spetsial complex reflection groups analogous to those in [8] whose enumeration is related to the sum $\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}^p} b)$?

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Brewing Fubini-Bruhat Orders

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Abstract. The Bruhat order on permutations arises out of the study of Schubert varieties in Grassmannians and flag varieties, which have been important for over 100 years [3, 5, 8, 13, 14]. The purpose of this paper is to study variations on this theme related to subvarieties of the spanning line configurations $X_{n,k}$ as defined by Pawlowski and Rhoades [16]. These subvarieties are indexed by Fubini words, or equivalently by ordered set partitions. Three natural partial orders arise in this context; we refer to them as the decaf, medium roast, and espresso orders. The decaf order is a generalization of the weak order on permutations defined by covering relations using simple transpositions. The medium roast order is a generalization of the (strong) Bruhat order defined by the closure relationship on the subvarieties. The espresso order is the transitive closure of a relation based on intersecting subvarieties. Many properties of Schubert varieties and Bruhat order extend to one or more of the three Fubini-Bruhat orders. We examine some of the many possibilities in this work.

Keywords: Fubini words, ordered set partitions, Schubert varieties, permutations

1 Introduction

For positive integers $k \leq n$, a **Fubini word** $w = w_1 \cdots w_n$ represents a surjective map $w : [n] \to [k]$. We denote a Fubini word by its **one-line notation**, an ordered list $w = w_1 w_2 \cdots w_n$, where $w_i = w(i)$. We denote by $\mathcal{W}_{n,k}$ the Fubini words of length n on the alphabet $[k]$. For $k = n$, a Fubini word $w \in \mathcal{W}_{n,n}$ is exactly a permutation in S_n, and the one-line notation for w is the same whether w is viewed as a Fubini word or a permutation. The bijection between Fubini words and ordered set partitions maps $w \in \mathcal{W}_{n,k}$ to $\mathrm{B}(w) = \mathrm{B}_1 \,|\, \mathrm{B}_2 \,|\, \ldots \,|\, \mathrm{B}_k$ where $\mathrm{B}_i = \{j \in [n] \,|\, w_j = i\}$. Hence the number of Fubini words in $\mathcal{W}_{n,k}$ is $k!S(n,k)$ where $S(n,k)$ is the Stirling number of the second kind [15, A000670, A019538].

Let $\mathcal{F}_{k\times n}(\mathbb{C})$ be the set of full rank $k \times n$ matrices with no zero columns. Such matrices have a Bruhat decomposition into orbits

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$$\mathcal{F}_{k\times n}(\mathbb{C}) = \bigsqcup_{w\in\mathcal{W}_{n,k}} B_-^{(k)} M_w B_+(w) \tag{1.1}$$

where M_w is the analog of a permutation matrix with a 1 in position (w_j, j) and 0's elsewhere, B_- and B_+ are the set of invertible lower and upper triangular matrices respectively and the superscript indicates their size, and $B_+(w)$ is the subgroup of the $n \times n$ invertible upper triangular matrices A such that $M_w A \in \mathcal{F}_{k\times n}(\mathbb{C})$. Every matrix in the double orbit $B_-^{(k)} M_w B_+(w)$ can be written in many ways as a triple product, thus it can be useful to chose canonical representatives. Let $U = U_-^{(k)}$ be the set of lower unitriangular matrices in $GL_k(\mathbb{C})$, and let $T = T^{(n)}$ be the set of diagonal matrices in $GL_n(\mathbb{C})$. Pawlowski and Rhoades [16] defined the **pattern matrices** P_w indexed by words $w \in \mathcal{W}_{n,k}$ to be a specific set of orbit representatives such that each $M \in B_-^{(k)} M_w B_+(w)$ can be written uniquely as a product $M = XYZ$ with $X \in U, Y \in P_w$, and $Z \in T$ [16, Lem. 3.1 and Prop. 3.2]. See Section 2 for more details. Thus, we have an **efficient Bruhat decomposition**

$$\mathcal{F}_{k\times n}(\mathbb{C}) = \bigsqcup_{w\in\mathcal{W}_{n,k}} U P_w T. \tag{1.2}$$

Under right multiplication, every T-orbit of $\mathcal{F}_{k\times n}(\mathbb{C})$ determines an ordered list of n 1-dimensional subspaces whose vector space sum is $\mathbb{C}^k$ via its ordered list of columns. The set of such "lines" in $\mathbb{C}^k$ is the $(k-1)$-dimensional complex projective space $\mathbb{P}^{k-1}$.

Definition 1.1. *[16, Def. 1.3] A* ***spanning line configuration*** $l_\bullet = (l_1, \ldots, l_n)$ *is an ordered n-tuple in the product of projective spaces $(\mathbb{P}^{k-1})^n$ whose vector space sum is $\mathbb{C}^k$. Let*

$$X_{n,k} = \mathcal{F}_{k\times n}(\mathbb{C})/T = \{l_\bullet = (l_1, \ldots, l_n) \in (\mathbb{P}^{k-1})^n \mid l_1 + \cdots + l_n = \mathbb{C}^k\} \tag{1.3}$$

be the ***space of spanning line configurations*** *for $1 \le k \le n$.*

In 2017, Pawlowski and Rhodes initiated the study of the space of spanning line configurations [16]. They observed and proved the following remarkable properties. The projection of $X_{n,n} = GL_n/T$ to the flag variety $GL_n/B_+^{(n)}$ is a homotopy equivalence, so they have isomorphic cohomology rings. More generally, $X_{n,k}$ is an open subvariety of $(\mathbb{P}^{k-1})^n$, hence it is a smooth complex manifold of dimension $n(k-1)$. The cohomology ring of $X_{n,k}$ may be presented as the ring

$$R_{n,k} = \mathbf{Z}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n\rangle$$

defined by Haglund-Rhoades-Shimozono [12], generalizing the coinvariant algebra and Borel's theorem $H^*(GL_n/B) \cong R_{n,n}$. Here, e_i is the i^{th} elementary symmetric function in $x_1, \ldots, x_n$. Furthermore, there is a natural S_n action on n-tuples of lines inducing an

S_n action on the cohomology ring of $X_{n,k}$, which is isomorphic to $R_{n,k}$ as a graded S_n-module. See also [11] for another geometric interpretation of $R_{n,k}$. The efficient Bruhat decomposition gives rise to a cellular decomposition

$$X_{n,k} = \bigsqcup_{w \in \mathcal{W}_{n,k}} UP_w.$$

Let $C_w = UP_w$ for $w \in \mathcal{W}_{n,k}$. Let $\overline{C}_w$ be the closure of the cell C_w in Zariski topology on on $X_{n,k}$. Then the cohomology classes $[\overline{C}_w]$ can be represented by variations on Schubert polynomials and these polynomials descend to a basis of $R_{n,k}$ over $\mathbb{Z}$ [16, Sec. 1, Prop 3.4]. The Poincaré polynomial for $H^*(X_{n,k}, \mathbb{Z})$ is determined by

$$\sum_{w \in \mathcal{W}_{n,k}} q^{\operatorname{codim}(C_w)} = [k]!_q \cdot \text{rev-Stir}_q(n,k), \tag{1.4}$$

where $\text{rev-Stir}_q(n,k)$ is the polynomial obtained by reversing the coefficients of a well-known q-analog of the Stirling numbers of the second kind [4, 17, 19].

Given the impressive results due to Pawlowski and Rhoades, we call $C_w = UP_w$ the **Pawlowski-Rhoades cell** or **PR cell** indexed by $w \in \mathcal{W}_{n,k}$. Similarly, the **PR variety** is denoted $\overline{C}_w$. The PR cells and PR varieties are natural variations on the theory of Schubert cells/varieties extending to $k \times n$ matrices, hence we believe they merit careful study of their own. We have used known theorems for Schubert varieties as inspiration for conjectures and results on PR varieties.

It follows from [16, Sec. 5] that the PR variety $\overline{C}_w$ is defined by certain bounded rank conditions. The rank conditions give rise to the ideal I_w generated by the minor determinants $\Delta_{I,J} \in \mathbb{C}[x_{11}, \ldots, x_{kn}]$ for $I, J \in \binom{[n]}{h}$ with $h \in [k]$ which vanish on every matrix in $C_w = UP_w$. The zero set of these minors is well defined on the orbits in $\mathcal{F}_{k\times n}(\mathbb{C})/T$ since the right action of the diagonal matrices just rescales each such minor. Therefore, the spanning line configurations in $\overline{C}_w$ can be represented by matrices in $\mathcal{F}_{k\times n}(\mathbb{C})$ that vanish for every minor generating I_w.

Definition 1.2. *[16, Sec. 9] The **medium roast Fubini-Bruhat order** $(\mathcal{W}_{n,k}, \leq)$ is defined on Fubini words by $v \leq w$ if and only if one of the following equivalent statements is true:*

1. $I_v \subset I_w$,
2. $\overline{C}_v \supseteq C_w$,
3. $\{(I,J) \mid \Delta_{I,J}(M) = 0 \ \forall M \in C_v\} \subset \{(I,J) \mid \Delta_{I,J}(M) = 0 \ \forall M \in C_w\}$.

One can observe that medium roast order on Fubini words is equivalent to Bruhat order on permutations when $n = k$. As with Bruhat order, it follows by definition that $v < w$ implies $\operatorname{codim}(C_v) < \operatorname{codim}(C_w)$. However, some of the properties for Bruhat order on $S_n = \mathcal{W}_{n,n}$ do not extend to all $\mathcal{W}_{n,k}$. Specifically, if $v \leq w$ in $\mathcal{W}_{n,k}$, then $\overline{C}_v \cap C_w \neq \emptyset$, but the converse does not necessarily hold. For example, using the third condition above and the definition of pattern matrices in Definition 2.4, one can observe that $\overline{C}_{1323}$ contains the matrix $M_{1123} \in C_{1123}$, but C_{1323} and C_{1123} are cells of the same dimension so 1323 and 1123 are unrelated in medium roast order. Since $\overline{C}_v \cap C_w \neq \emptyset$ is a weaker condition than $C_w \subseteq \overline{C}_v$, this suggests a refinement of the medium roast Fubini-Bruhat order, which we will denote by $\preceq$. Note that our notation for $\preceq$ is $\leq'$ in Pawlowski and Rhoades' notation. They use $\preceq$ for the dual order to $\leq$.

Definition 1.3. *For $v, w \in \mathcal{W}_{n,k}$, we say C_v **touches** C_w if $\overline{C}_v \cap C_w \neq \emptyset$, denoted $v \rightharpoonup w$.*

Pawlowski and Rhoades observe in [16, Sec. 9] that unlike the medium roast order relations, the touching relation on Fubini words is not transitive. However, they showed that the transitive closure of the touching relations is acyclic [16, Prop. 9.2], so the touching relations give rise to a poset on $\mathcal{W}_{n,k}$ first studied but not named in [16].

Definition 1.4. *[16, Sec. 9] The **espresso Fubini-Bruhat order** $(\mathcal{W}_{n,k}, \preceq)$ is defined by taking the transitive closure of the relations of the form $v \rightharpoonup w$ if v touches w.*

Observe that for Fubini words $v, w \in \mathcal{W}_{n,k}$, $v \leq w$ implies $v \preceq w$. Thus, the medium roast order is a subposet of the espresso order on the same set of elements.

Pawlowski and Rhoades asked for a combinatorial description of the espresso and medium roast Fubini-Bruhat orders [16, Prob. 9.5]. We address this problem by giving two more sets of defining equations for PR varieties $\overline{C}_w$ inside $X_{n,k}$, see Theorem 1.5 and Theorem 5.4 below. Each set is typically properly contained in the set of all minors that vanish on the PR cell C_w, and hence "more efficient".

Let Δ_J be the **flag minor** associated to columns in J and rows $1, 2, \ldots, |J|$. Such minors are used historically for the Plücker embedding of the flag variety into projective space [8]. Note that the flag minors are invariant under the left action of the unitriangular matrices. Hence, to determine the vanishing/non-vanishing flag minors of $M \in C_w = UP_w$, it suffices to consider the unique U-orbit representative of M in P_w. We can partition the set of all flag minors on $k \times n$ matrices into the **sometimes, truly, and unvanishing flag minors** for w, by defining the indexing sets

$$S_w = \{J \in \binom{[n]}{[k]} \mid \exists\, A, B \in C_w \text{ s.t. } \Delta_J(A) = 0,\ \Delta_J(B) \neq 0\},$$

$$T_w = \{J \in \binom{[n]}{[k]} \mid \Delta_J(M) = 0\ \ \forall\, M \in C_w\}, \text{ and}$$

$$U_w = \{J \in \binom{[n]}{[k]} \mid \Delta_J(M) \neq 0\ \ \forall\, M \in C_w\}.$$

Theorem 1.5. *For any Fubini word $w \in \mathcal{W}_{n,k}$, the PR variety $\overline{C}_w$ is the set of spanning line configurations in $X_{n,k}$ represented by matrices such that all flag minors indexed by T_w vanish, so*

$$\overline{C}_w = \{A \in X_{n,k} \mid \Delta_J(A) = 0 \,\forall\, J \in T_w\}.$$

Note, the ideal J_w generated by the flag minors $\{\Delta_J \mid J \in T_w\}$ is in general not the same as I_w generated by all vanishing minors for C_w. For example, using the definition and example of P_w in Section 2, one can observe that the minor $\Delta_{\{2\},\{1\}} = x_{21}$ is not in the ideal J_w for $w = 31123$, but it does vanish on all of C_w. Note, both I_w and J_w are radical ideals since determinants don't factor, so they determine different affine varieties in $\mathbb{C}^{nk}$, which agree on $X_{n,k}$.

Corollary 1.6. *For any two Fubini words $v, w \in \mathcal{W}_{n,k}$, we have*

1. *$v \leq w$ in medium roast Fubini-Bruhat order if and only if $T_v \subseteq T_w$, and*
2. *$v \rightharpoonup w$ if and only if $T_v \subseteq (S_w \cup T_w)$.*

Identifying vanishing flag minors of C_w is more efficient than calculating all vanishing minors of C_w, but still cumbersome directly from the definition. In fact, we can characterize the sometimes, truly, and unvanishing flag minors via the Gale partial order on certain multisets $\alpha_J(w)$ defined below. We refer to this as the **Alpha Test**. These tests generalize Ehresmann's Criteria for Bruhat order in S_n using the Gale partial order on multisets denoted $A \trianglelefteq B$. See Section 2 for more details.

Definition 1.7. *For any Fubini word $w \in \mathcal{W}_{n,k}$, let $\alpha_i = \alpha_i(w)$ denote the position of the initial i in w for each $i \in [k]$. Call $\alpha(w) = (\alpha_1, \ldots, \alpha_k)$ the **alpha vector** of w. We will sometimes drop the (w) when it is clear from context. Observe that when $k = n$, the alpha vector coincides with the notion of $w^{-1} \in S_n = \mathcal{W}_{n,n}$. For $J \subset [n]$, define the multiset*

$$\alpha_J(w) = \{\alpha_{w(j)} \mid j \in J\}. \tag{1.5}$$

Theorem 1.8. ***(The Alpha Test)*** *Suppose $w \in \mathcal{W}_{n,k}$ and $J \in \binom{[n]}{[k]}$ with $|J| = h$. Then*

1. *$J \in S_w$ if and only if $\{\alpha_1, \ldots, \alpha_h\} \underset{\neq}{\trianglelefteq} \alpha_J(w)$,*
2. *$J \in T_w$ if and only if $\{\alpha_1, \ldots, \alpha_h\} \not\trianglelefteq \alpha_J(w)$, and*
3. *$J \in U_w$ if and only if $\{\alpha_1, \ldots, \alpha_h\} = \alpha_J(w)$.*

For example, let $w = 21231231 \in \mathcal{W}_{8,3}$ and $J = \{2, 6, 8\}$. Then $\alpha(w) = (\alpha_1, \alpha_2, \alpha_3) = (2, 1, 4)$, and $\alpha_J = \{\alpha_{w(2)}, \alpha_{w(6)}, \alpha_{w(8)}\} = \{2, 1, 2\}$. Since $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 2, 4\} \not\trianglelefteq \{1, 2, 2\} = \alpha_J(w)$ in Gale order, we know $J \in T_w$.

Corollary 1.9. *Let $v, w \in \mathcal{W}_{n,k}$. Then, $v \leq w$ in medium roast Fubini-Bruhat order if and only if for each $J \in \binom{[n]}{[k]}$ with $|J| = h \leq k$ such that*

$$\{\alpha_1(w), \ldots, \alpha_h(w)\} \trianglelefteq \alpha_J(w) \tag{1.6}$$

we also have

$$\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \alpha_J(v). \tag{1.7}$$

A similar test for $v \rightharpoonup w$ holds as well based on testing each J such that $\{\alpha_1(w), \ldots, \alpha_h(w)\} = \alpha_J(w)$. Therefore, if $v \leq w$ or $v \rightharpoonup w$, we have $\{\alpha_1(v), \ldots, \alpha_h(v)\} \trianglelefteq \{\alpha_1(w), \ldots, \alpha_h(w)\}$ for all $1 \leq h \leq k$, generalizing the Ehresmann Criterion.

In Section 2, we briefly review our notation and key concepts from the literature. In Section 3, we indicate some of the lemmas needed to prove Theorem 1.5 and its corollaries. In Section 4, we identify certain families of covering relations and use them to define the decaf Fubini-Bruhat order. We also state an analog of the Lifting Property of Bruhat order. In Section 5, we generalize Fulton's essential set for permutations to Fubini words and show this set gives the unique minimal set of rank conditions defining a PR variety, see Corollary 5.5.

2 Background

For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Generalizing the notation for binomial coefficients, we let $\binom{[n]}{k}$ denote all size k subsets of $[n]$ and $\binom{[n]}{[k]} = \bigcup_{h=1}^{k} \binom{[n]}{h}$. The **Gale order** on $\binom{[n]}{k}$ is given by $\{a_1 < \cdots < a_k\} \trianglelefteq \{b_1 < \cdots < b_k\}$ if and only if $a_i \leq b_i$ for all $i \in [k]$ [9]. Gale order can easily be extended to multisets of positive integers of the same size.

Let S_n denote the symmetric group on $[n]$ thought of as bijections $w : [n] \to [n]$. As usual, write a permutation w in **one-line notation** as $w = w_1 \cdots w_n$. Let t_{ij} be the transposition interchanging i and j, and let s_i denote the simple transposition interchanging i and $i+1$. The permutation $t_{ij}w$ is obtained from the one-line notation for w by interchanging the values i and j, while right multiplication wt_{ij} interchanges the values w_i and w_j. The permutation matrix M_w for $w \in S_n$ is the $n \times n$ matrix with a 1 in position (w_j, j) for all $j \in [n]$ and 0's elsewhere. Permutation multiplication agrees with matrix multiplication: $u = vw$ if and only if $M_u = M_v M_w$. Permutation multiplication extends to Fubini words if the corresponding matrices have the correct size.

Schubert varieties X_w for $w \in S_n$ in the flag variety $GL_n / B_+^{(n)}$ are defined via bounded rank conditions on matrices coming from the associated permutation matrices [8]. The **Bruhat order** on S_n is defined by reverse inclusion on Schubert varieties: $v \leq w \iff X_w \subset X_v$. This poset can be characterized as the transitive closure of the relation $w \leq t_{ij}w$

provided $i < j$ and i appears to the left of j in the online notation for w [3]. The covering relations are given by the set of edges $w \leq t_{ij}w$ such that $t_{ij}w$ has exactly one more inversion than w. Ehresmann characterized Bruhat order on S_n in terms of Gale order, decades prior to Gale or Bruhat's work, by the **Ehresmann Criterion** [5]

$$v \leq w \iff \{v_1, v_2, \ldots, v_i\} \trianglelefteq \{w_1, w_2, \ldots, w_i\} \; \forall i \in [n]. \tag{2.1}$$

Suppose $v \leq w$ in Bruhat order on S_n, $i \in [n-1]$ and $i+1$ precedes i in both v and w. Then, the **Lifting Property of Bruhat order** [3, Prop. 2.2.7] implies that $s_i v \leq s_i w$.

Definition 2.1. *The **Rothe diagram** of a permutation $w \in S_n$ is the subset of $[n] \times [n]$ in matrix coordinates given by $D(w) = \{(w_j, i) \mid i < j \text{ and } w_i > w_j\}$. Define the **essential set** of w, denoted $Ess(w)$, to include all $(i, j) \in D(w)$ such that $(i+1, j), (i, j+1) \notin D(w)$.*

The Rothe diagrams are used extensively in the theory of Schubert varieties. In particular, Fulton showed that the rank conditions coming from the coordinates $(i, j) \in Ess(w)$ determine the unique minimal set of bounded rank equations defining the Schubert variety X_w [7]. Eriksson-Linusson showed that the average size of the essential set is $n^2/36$ for $w \in S_n$ [6].

Much of the notation for permutations defined above has an analog for Fubini words. For $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$, let M_w be the matrix obtained from the $k \times n$ all zeros matrix by setting the (w_j, j) entry to be 1 for all $j \in [n]$. Note that M_w has exactly one 1 in each column and at least one 1 in each row, but it may have many 1's in any row. Recall from Definition 1.7 that $\alpha_i(w) = \alpha_i$ is the position of the first letter i in w for $i \in [k]$.

Definition 2.2. *[16, §3] For a word $w \in \mathcal{W}_{n,k}$, the **initial positions** of w are the set $in(w) = \{\alpha_1, \ldots, \alpha_k\}$. A **redundant position** of w is any position that is not initial. An **initial letter** is a letter appearing in an initial position, and a **redundant letter** is a letter appearing in a redundant position.*

Definition 2.3. *[16, §3] For $w \in \mathcal{W}_{n,k}$, the **initial permutation**, $\pi(w) \in S_k$, is obtained from w by deleting the redundant letters from the one-line notation.*

Definition 2.4. *[16, §3] For $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$, the **pattern matrix** P_w is a $k \times n$ matrix with entries 0, 1, or $\star$. Obtain P_w by starting with M_w and replacing the 0 by a $\star$ in each position (w_i, j) such that $i \in in(w)$, $i < \alpha_{w(j)}$, $1 \leq j \leq n$, and either $j \in in(w)$ and $w_i < w_j$, or $j \notin in(w)$.*

*A matrix is said to **fit the pattern of** w if that matrix can be obtained by replacing the $\star$'s in the pattern matrix of w with complex numbers. We will abuse notation and consider P_w both as a $k \times n$ matrix with entries in $\{0, 1, \star\}$ and as the set of all matrices fitting the pattern of w.*

Definition 2.5. *[16, Eq. (3.6)] The **dimension** of $w \in \mathcal{W}_{n,k}$, denoted $\dim(w)$, is the number $\star$'s in its pattern matrix P_w.*

Example 2.6. The pattern matrices of $v = 31422$ and $w = 31424$ in $\mathcal{W}_{5,4}$ are

$$P_{31422} = \begin{pmatrix} 0 & 1 & \star & \star & \star \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & \star & 0 & \star \\ 0 & 0 & 1 & 0 & \star \end{pmatrix} \text{ and } P_{31424} = \begin{pmatrix} 0 & 1 & \star & \star & \star \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & \star & 0 & \star \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Therefore, $\dim(31422) = 6$ and $\dim(31424) = 5$.

If $w \in \mathcal{W}_{n,k}$, then the dimension of the PR cell C_w is $\dim(w) + \binom{k}{2}$. The unique largest dimensional cell in $X_{n,k}$ is $C_{123\cdots kk^{n-k}}$ and $\dim(12\cdots kk^{n-k}) = \binom{k}{2} + (n-1)(k-1)$. Hence, $X_{n,k} = \overline{C}_{12\cdots kk^{n-k}}$ has dimension $n(k-1) = 2\binom{k}{2} + (n-1)(k-1)$ and $12\cdots kk^{n-k}$ is the unique minimal element in all three Fubini-Bruhat orders. Since Fubini words are in bijection with ordered set partitions, the dimension generating function gives a natural q-analog of the Stirling numbers of the second kind $\sum_{w\in\mathcal{W}_{n,k}} q^{\dim(w)} = [k]!_q \cdot \mathrm{Stir}_q(n,k)$. Reversing the coefficients in this generating function gives (1.4).

3 Outlines of Proofs

We outline the proofs of Theorem 1.5 and Theorem 1.8. These statements form the basis from which the covering relations and other Fubini-Bruhat order properties can be proved.

Lemma 3.1. *Given $A \in \mathcal{F}_{k\times n}(\mathbb{C})$, the projective coordinates $P(A) = (\Delta_J(A) \mid J \in \binom{[n]}{[k]})$ determine both the unique $w \in \mathcal{W}_{n,k}$ such that $A \in UP_wT^{(n)}$ and $A' \in P_w$ such that $A \in UA'$.*

Corollary 3.2. *The set T_w of truly vanishing flag minors on the PR cell C_w determines $w \in \mathcal{W}_{n,k}$, and therefore the rank conditions defining $\overline{C}_w$ as a subset of $X_{n,k}$.*

Corollary 3.2 says there is enough information in the set T_w to recover w. To make the relationship between T_w and $\overline{C}_w$ precise, we observe several relations among minors that hold specifically on PR cells and spanning line configurations.

Lemma 3.3. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, $J \subset [n]$, and $1 \leq h \leq k$. Let $\mathrm{rank}_w^{(h)}(J)$ be the largest value r such that there exist subsets $I \subset [h]$ and $J' \subset J$ such that $r = |I| = |J'|$ and $\Delta_{I,J'}(A) \neq 0$ for some $A \in C_w$. The following conditions are equivalent.*

1. *We have $\mathrm{rank}_w^{(h)}(J) < |J|$.*
2. *For every $I \subseteq [h]$ such that $|I| = |J|$, the (I,J)-minor vanishes on C_w.*
3. *For all subsets $K \in \binom{[n]}{h}$ such that $J \subset K$, we have $K \in T_w$.*

Corollary 3.4. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word, $I \subseteq [k]$ and $J \subseteq [n]$ are sets of the same size, and $h = \max(I)$. If the (I, J)-minor vanishes on C_w, then at least one of the following hold.*

1. *For every $j \in J$, the $(I \setminus \{h\}, J \setminus \{j\})$-minor vanishes on C_w.*
2. *For all subsets K such that $J \subseteq K \in \binom{[n]}{h}$, we have $K \in T_w$.*

Corollary 3.4 follows from Lemma 3.3. Theorem 1.5 follows by induction on the number of rows of a minor of C_w using Corollary 3.4, and by Lemma 3.3.

Lemma 3.5. *Suppose $w \in \mathcal{W}_{n,k}$ is a Fubini word and $J \in \binom{[n]}{[k]}$ with $h = |J|$. Then, $J \in U_w$ if and only if the submatrix $M_w[[h], J]$ is a permutation matrix.*

Lemma 3.6. *Let $w \in \mathcal{W}_{n,k}$, $I \subseteq [k]$ and $J \in \binom{[n]}{[k]}$ such that $|I| = |J|$ and $\Delta_{I,J}(A) = 0$ for all A in the PR cell C_w. Then (H, J) indexes a vanishing minor on C_w for any H such that $|H| = |I|$ and $H \leq_L I$ in lex order. In particular, $\Delta_{[|I|],J}$ is a vanishing flag minor on C_w, so $J \in T_w$.*

Lemmas 3.5 and 3.6, together with the earlier lemmas can be used to prove Corollary 1.6. Corollary 1.6 and Lemma 3.5 imply Theorem 1.8.

4 Covering Relations and the Decaf Order

The following rules describe some families of covering relations for the medium roast and espresso Fubini-Bruhat orders, giving a partial answer to Problem 9.5 in [16]. The Transposition Rule and the Pushback Rule allow us to define the decaf Fubini-Bruhat order, the only ranked Fubini-Bruhat order. We also discuss a generalization of the Lifting Property from Bruhat order.

We start with two observations on covering relations that follow from the definition of medium roast order, pattern matrices, and Corollary 1.6. Let $w = w_1 \cdots w_n \in \mathcal{W}_{n,k}$ with initial permutation $\pi(w) = \pi_1 \cdots \pi_k$.

1. **The Transposition Rule.** For $1 \leq i < j \leq k$, we have $w < t_{ij}w$ in medium roast Fubini-Bruhat order if and only if $\alpha_i(w) < \alpha_j(w)$. In particular, $t_{ij}w$ covers w in medium roast Fubini-Bruhat order if and only if $\pi(t_{ij}w)$ covers $\pi(w)$ in Bruhat order on S_k.
2. **The Pushback Rule.** Suppose $w_j = \pi_i$ is a redundant letter in w for $i \in [k-1]$ and $j \in [n]$. Let v be the Fubini word obtained from w by replacing w_j by π_{i+1}. Then, w covers v in medium roast Fubini-Bruhat order. See Example 2.6 for an example of $v < w$ satisfying the pushback covering relation.

Definition 4.1. *The **decaf Fubini-Bruhat order** on $\mathcal{W}_{n,k}$ is the transitive closure of the covering relations given by the Transposition Rule and the Pushback Rule.*

The decaf order has many nice properties. It is the product of Bruhat order for S_k and the poset determined by pushbacks on the subset $\{w \in \mathcal{W}_{n,k} \mid \pi(w) = id\}$. The decaf order is a ranked poset on $\mathcal{W}_{n,k}$, and its rank generating function is the same as the Poincaré polynomial in (1.4). The medium roast and espresso orders are not ranked posets in general. For $n \geq 5$ and most values of k, there are covering relations in the medium roast Fubini-Bruhat order $(\mathcal{W}_{n,k}, \leq)$ with a dimension difference of 2 or more, causing the medium roast Fubini-Bruhat order to be unranked in general. For example, in $\mathcal{W}_{5,4}$, 44312 covers 41321, but 44312 has dimension 1, and 41321 has dimension 3.

Theorem 4.2. ***The Superpushback Rule.*** *Suppose $w \in \mathcal{W}_{n,k}$, $i \in [k-1]$, and $j \in [n]$ such that $w_j = \pi_i$ is a redundant letter in w. If $i + p \leq k$ and v is obtained from w by replacing w_j by $\pi_{i+p}(w)$, then $v \rightharpoonup w$ and this is a covering relation in both espresso and medium roast orders.*

Theorem 4.3. ***The Lifting Property.*** *Suppose $v, w \in \mathcal{W}_{n,k}$, $i \in [k-1]$, $\alpha_{i+1}(v) < \alpha_i(v)$, and $\alpha_{i+1}(w) < \alpha_i(w)$. If $v \leq w$ in medium roast Fubini-Bruhat order, then $s_i v \leq s_i w$. Furthermore, if $v \rightharpoonup w$, then $s_i v \rightharpoonup s_i w$.*

5 Essential Sets

We extend the notion of a Rothe diagram from Definition 2.1 to Fubini words. This allows us to define the essential set for a Fubini word. We then show the essential set determines a minimal set of rank equations on the corresponding PR variety, generalizing Fulton's essential set for permutations and Schubert varieties [7]. This leads to an essential set characterization of $v \leq w$ in medium roast order.

Definition 5.1. *[16] A Fubini word $w \in \mathcal{W}_{n,k}$ is called **convex** if $h < j$ and $w_h = w_j$ implies that $w_i = w_j$ for every i such that $h < i < j$. Then the **convexification** of w, denoted by $conv(w)$, is the unique convex word such that $\pi(conv(w)) = \pi(w)$ and the content of w and $conv(w)$ are the same as multisets. The **standardization** of w, denoted $std(w) \in S_n$, is obtained by replacing the $n-k$ redundant letters of w with $k+1, k+2, \ldots, n$ from left to right.*

Deduce from Definition 5.1 that two Fubini words $v, w \in \mathcal{W}_{n,k}$ have the same convexification, $\text{conv}(v) = \text{conv}(w)$, if and only if $\pi(v) = \pi(w)$ and they have the same multiset of letters.

Definition 5.2. *Given Fubini word $w \in \mathcal{W}_{n,k}$, define the **diagram** of w to be $D(std(conv(w)))$.*

One can observe that $D(\text{std}(\text{conv}(w))) \subset [k] \times [n]$, as none of the bottom $n-k$ rows will contribute any elements to $D(\text{std}(\text{conv}(w)))$. Thus, the diagram of a Fubini word in $\mathcal{W}_{n,k}$ can be drawn as a $k \times n$ grid of dots. For example, the convexification of $w = 44253136541 \in \mathcal{W}_{11,6}$ is 44425533116, and $\text{std}(44425533116) = [4, 7, 8, 2, 5, 9, 3, 10, 1, 11, 6]$. So the diagram for w is $D([4, 7, 8, 2, 5, 9, 3, 10, 1, 11, 6])$. See Figure 1.

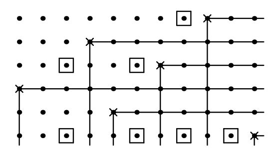

Figure 1: Diagram of 44253136541 with cells in the essential set boxed.

In analogy with the alpha vector, define the **beta vector** $\beta(w) = (\beta_1(w), \ldots, \beta_k(w))$ for $w \in W_{n,k}$ by $\beta_i(w) = \beta_i = \{j \in [n] \mid w_j \in \{\pi_1, \ldots, \pi_i\}\}$ where $\pi(w) = (\pi_1, \ldots, \pi_k) \in S_k$ is the initial permutation. Note that $\beta_1 \subset \cdots \subset \beta_k$. For example, if $w = 12123123 \in \mathcal{W}_{8,3}$, we observe $\beta_1 = \{1,3,6\}$, $\beta_2 = \{1,2,3,4,6,7\}$, and $\beta_3 = [8]$.

Given any Fubini word $w \in \mathcal{W}_{n,k}$, define its **rank function** to be the map $r_w : [k] \times [k] \to \mathbb{Z}_{\geq 0}$ that sends (h,i) to the maximum value of the rank of the submatrix $A[[h], \beta_i]$ over all $A \in C_w$. This function can be determined directly from the Fubini word w as with permutations, but the statement is more complicated so we have omitted it for brevity. From the pattern matrix definition, one can observe that the jumps in the rank functions of matrices in a PR variety are determined by the sets in the beta vector.

Definition 5.3. *Given any Fubini word $w \in \mathcal{W}_{n,k}$, define the **ranked essential set** of w to be*

$$Ess^*(w) = \{(h, \beta_i, r) \mid (h, |\beta_i|) \in Ess(std(conv(w))), r = r_w(h,i)\}.$$

Theorem 5.4. *A matrix $A \in \mathcal{F}_{k\times n}(\mathbb{C})$ is in the PR variety $\overline{C}_w$ if and only if the rank of the top h rows of A in the columns $\beta_i(w)$ is at most r for each $(h, \beta_i(w), r) \in Ess^*(w)$, and no smaller set of rank conditions will suffice.*

Corollary 5.5. *Let $v, w \in \mathcal{W}_{n,k}$. Then $v \leq w$ if and only if for every $(m, \beta_j(v), s) \in Ess^*(v)$, there exists an $(h, \beta_i(w), r) \in Ess^*(w)$ such that $max(0, m-h) + |\beta_j(v) \setminus \beta_i(w)| \leq s - r$.*

Björner-Brenti gave an improvement on the Ehresmann Criterion for Bruhat order on permutations in [2]. Similar improvements on the Alpha Test for medium and espresso orders exist as well. Such improvements also lead to a reduction in the number of equations necessary to define a PR variety. In recent work, Gao-Yong found a minimal number of equations defining a Schubert variety in the flag variety [10]. Thus, it would be interesting to consider the following problem.

Open Problem 5.6. *Identify a minimal set of equations defining a PR variety.*

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This paper grew out of the Ph.D. thesis of the second author [1, 18]. Further details and more background can be found there. Many thanks to Brendan Pawlowski, Brendan Rhoades, Jordan Weaver, and the referees for insightful suggestions on this project.

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From geometry to generating functions: rectangulations and permutations

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Abstract. We enumerate several classes of pattern-avoiding rectangulations. We establish new bijective links with pattern-avoiding permutations, prove that their generating functions are algebraic, and confirm several conjectures by Merino and Mütze. We also analyse a new class of rectangulations, called whirls, using a generating tree.

Keywords: Rectangulations, permutations, pattern avoidance, generating functions

1 Introduction

A *rectangulation* of size n is a tiling of a rectangle by n rectangles such that no four rectangles meet in a point. In the literature, rectangulations are also called *floorplans* or *rectangular dissections*. See Section 2 and [3, 9, 15] for basic definitions and results.

Such structures appear naturally for architectural building plans, integrated circuits (see Figure 1), and were investigated since the 70s with some graph theory, computational geometry, and combinatorial optimization point of views [16, 18]. Then, in the 2000s, rectangulations began to be investigated with more combinatorial approaches [1, 2, 4, 13, 17]: it was shown that some important families of rectangulations are enumerated by famous integer sequences (e.g., Baxter, Schröder, Catalan numbers) and that they have strong links with pattern-avoiding permutations (as studied in the seminal article [11]).

Figure 1: (a) VLSI are rectangulations playing an important role for integrated circuits.
(b) The artwork *Composition décentralisée*, 1924, by Theo van Doesburg (1883–1931).
(c) A book on the geometry of building plans [18]. Its cover is not a rectangulation, since it contains instances of 4 rectangles meeting in a point.
(d) The minimal solution of Tutte's "Squaring the square" is a rectangulation [12].

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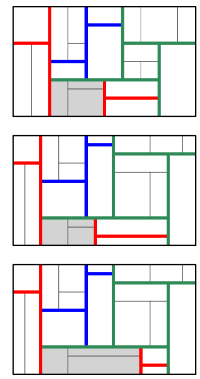
2 Patterns in rectangulations and summary of our results



Two rectangulations are *equivalent* ("strongly equivalent" in [3]) if one can translate (horizontally or vertically) some of their segments (without meeting an endpoint of any other segment) so that they coincide. In the drawings on the right, only the first two rectangulations are equivalent.

In this article we deal with *patterns in rectangulations*. In each drawing, we highlight an occurrence of the pattern ┴┬ (in green), ┤├ (in blue), ┤|├ (in red). A rectangulation contains ┤|├ if there is a (possibly further partitioned) rectangle (here in gray) such that the segment containing its left side has an adjacent horizontal segment on the left, and the segment containing its right side has an adjacent horizontal segment on its right.

We are interested in the enumeration of different natural classes of rectangulations, where the goal is to count the number of non-equivalent rectangulations of size n. E.g., ┤├ ┬┴ -avoiding rectangulations are enumerated by Baxter numbers [1, 11].

Recently, Arturo Merino and Torsten Mütze tackled the question of the exhaustive generation of rectangulations avoiding any subset of {┬┴, ┴┬, ┤├, ┬┴, ┐└, ┴┬, ┤|├, ╪}. In [15], they present an efficient algorithm to generate such rectangulations[1]. This led to a surprising observation: many sequences coincide (at least up to size 12) with integer sequences which already appeared in the literature, for apparently unrelated problems.

In Theorem 1, we solve all the cases related to rectangulations avoiding ┬┴ ┴┬ ┤├ ┬┴ . These are *guillotine diagonal rectangulations*, that correspond to *separable permutations*. When they avoid further patterns among ┐└ ┴┬ ┤|├ ╪ , we obtain the following table[2], and provide generating functions for these cases. (See [6] for the notion of *vincular patterns*.)

Entry in [15, Table 3]	Guillotine diagonal rectangulations avoiding...	Separable permutations avoiding...	G.f.	OEIS
1234	∅	∅	alg.	A006318
12345	┐└	2<u>14</u>3	alg.	A106228
12347	┤|├	21354	alg.	A363809
123456	┐└ ┴┬	2<u>14</u>3, 3<u>41</u>2	alg.	A078482
123457	┐└ ┤|├	2143	alg.	A033321
123458	┐└ ╪	2<u>14</u>3, 45312	alg.	A363810
123478	┤|├ ╪	21354, 45312	rat.	A363811
1234567	┐└ ┴┬ ┤|├	2143, 3<u>41</u>2	alg.	A363812
1234578	┐└ ┤|├ ╪	2143, 45312	rat.	A363813
12345678	┐└ ┴┬ ┤|├ ╪	2143, 3412	rat.	A006012

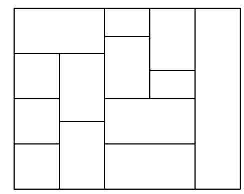
In Section 4, we additionally prove algebraicity of some non-guillotine models, such as *vortex rectangulations* (A026029, case 1345678 in [15]) and *whirls* (A002057).

[1] Let us here advertise the section dedicated to rectangulations in the nice Combinatorial Object Server, created by Frank Ruskey, and now handled by Torsten Mütze, Joe Sawada, and Aaron Williams.

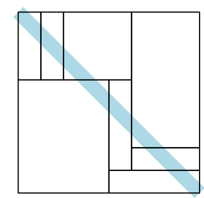
[2] All other cases are equivalent to those presented here via straightforward symmetries.

3 Guillotine diagonal rectangulations

The patterns $P_1 =$, $P_2 =$, $P_3 =$, $P_4 =$ were considered in some earlier work (for example [1, 9]) since they characterize some special kinds of rectangulations.



A rectangulation $\mathcal{R}$ is *guillotine* if it is of size 1, or if it has a *cut* (a segment whose endpoints lie on opposite sides of R) that splits it into two guillotine rectangulations. It is well known [1] that a rectangulation is guillotine if and only if it avoids $P_1 =$ and $P_2 =$ (these two patterns are called *windmills*).



A rectangulation $\mathcal{R}$ is called *diagonal* if it avoids $P_3 =$ and $P_4 =$. This notion is due to the fact that such a rectangulation can be drawn so that the NW–SE diagonal of R intersects all the rectangles. At the same time, diagonal rectangulations are frequently seen as canonical representatives of rectangulations up to the "weak equivalence" [3, 9].

These two classes have — in different ways — stronger structural properties than the general case. Therefore we expected that families that avoid these four patterns and any other subset of patterns of { , , , } will yield noteworthy results. There are essentially ten different such models, all listed in [15, Table 3]. Amongst these 10 cases, 3 of them can be solved by ad-hoc bijections with trees (see [2, 4]), 2 are conjectured by Merino and Mütze to lead to algebraic generating functions, and for the remaining 5 no conjectures were provided. Below we present a unified framework which allows us to solve these 10 cases (confirming en passant the conjectures of Merino and Mütze). The main result of this section is Theorem 1, which, in particular, states that all these cases are in fact algebraic!

Theorem 1. *The generating functions for the ten guillotine cases are algebraic.*

1. *The generating function of rectangulations avoiding is*

$$F(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2}.$$

2. *The generating function of rectangulations avoiding satisfies*

$$tF^3 + 2tF^2 + (2t - 1)F + t = 0.$$

3. *The generating function of rectangulations avoiding) satisfies*
$$t^4(t-2)^2F^4+t(t-2)(4t^3-7t^2+6t-1)F^3+(2t^4-t^3-2t^2+5t-1)F^2-(4t^3-7t^2+6t-1)F+t^2=0.$$

4. *The generating function of rectangulations avoiding is*

$$F(t) = \frac{1 - 3t + t^2 - \sqrt{1 - 6t + 7t^2 - 2t^3 + t^4}}{2t}.$$

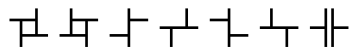
5. *The generating function of rectangulations avoiding*  *is*

$$F(t) = \frac{(1-t)(1-2t) - \sqrt{(1-t)(1-5t)}}{2t(2-t)}.$$

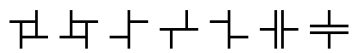
6. *The generating function of rectangulations avoiding*  *satisfies* $t^8(t-2)^2F^4 - t^3(t^2-3t+2)(t^5-7t^4+4t^3-6t^2+5t-1)F^3 - t(t-1)(4t^7-22t^6+37t^5-42t^4+53t^3-35t^2+10t-1)F^2 - (5t^6-16t^5+15t^4-28t^3+23t^2-8t+1)(t-1)^2F - (2t^5-5t^4+4t^3-10t^2+6t-1)(t-1)^2 = 0.$

7. *The generating function of rectangulations avoiding*  *is*

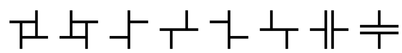
$$F(t) = \frac{t(1-16t+11t^2-434t^3+1045t^4-1590t^5+1508t^6-846t^7+252t^8-30t^9)}{(1-2t)^4(1-3t+t^2)^2(1-4t+2t^2)}.$$

8. *The generating function of rectangulations avoiding*  *is*

$$F(t) = \frac{1-3t-t^2+2t^3-\sqrt{1-6t+7t^2+2t^3+t^4}}{2t^2(2-t)}.$$

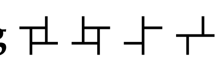
9. *The generating function of rectangulations avoiding*  *is*

$$F(t) = \frac{t(1-t)(1-7t+16t^2-11t^3+2t^4)}{(1-4t+2t^2)(1-3t+t^2)^2}.$$

10. *The generating function of rectangulations avoiding*  *is*

$$F(t) = \frac{t(1-2t)}{1-4t+2t^2}.$$

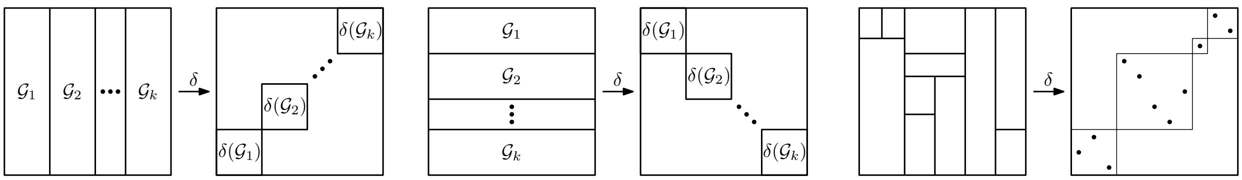
We now present *separable permutations* — a fundamental class which will be used in the proof of this theorem. This notion was coined in [7].

3.1 Separable permutations and rectangulations avoiding 

A permutation π is *separable* if it is either of size 1 (a *singleton*), or if it is (recursively) a direct sum of separable permutations (in this case π is called *ascending separable*) or a skew sum of separable permutations (in this case π is called *descending separable*). We refer to [6] for these notions. Accordingly, separable permutations are precisely the non-empty $(2413, 3142)$-avoiding permutations [7].

The first key step in the proof of Theorem 1 is "translating" (sets of) geometric patterns into (sets of) permutation patterns. In all 10 cases we obtain a bijection between a subclass of guillotine rectangulations and a subclass of separable permutations. We provide details for the first three cases, and just give the key decompositions for the other cases.

Case 1: Guillotine diagonal rectangulations. They are in bijection with separable permutations (see, e.g., [1, 4]). Here is a natural recursive bijection: the rectangulation of size 1 is mapped to the permutation of size 1, and the recursive steps are illustrated in the following drawing. The left and the middle illustrations describe the transformation for horizontal and vertical cuts, and the right illustration is an example of size 11.

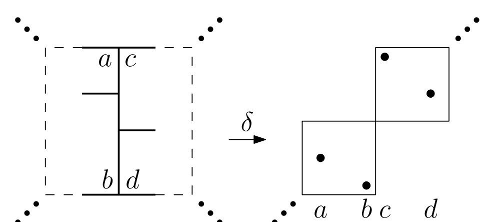


The recursive definition of separable permutations translates directly to a system of equations that binds $A(t)$, $D(t)$, and $F(t) = t + A(t) + D(t)$, the generating functions for ascending, descending, and all separable permutations. Since an ascending (resp. descending) separable permutation can be seen as a sequence of singletons and descending (resp. ascending) separable permutations ("blocks"), we obtain the system $\left\{A = \frac{(t+D)^2}{1-(t+D)},\ \ D = \frac{(t+A)^2}{1-(t+A)}\right\}$. Due to the symmetry $A(t) = D(t)$, we have $A = \frac{(t+A)^2}{1-(t+A)}$. This yields $F(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2}$, the generating function of Schröder numbers (`A006318`).

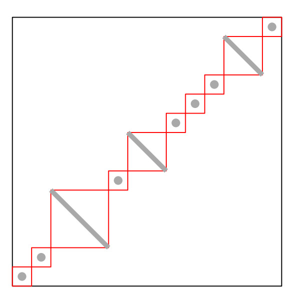
Case 2: -avoiding guillotine diagonal rectangulations.

Lemma 2. *A guillotine diagonal rectangulation $\mathcal{R}$ avoids if and only if $\delta(\mathcal{R})$ avoids $2\underline{14}3$.*

Proof (sketch). This result follows from the bijection δ described above. An occurrence of in $\mathcal{R}$ means that there are four rectangles a, b, c, d as in the drawing, where the segment that separates a and b from c and d is a cut at some step of the recursive decomposition of $\mathcal{R}$. It follows that in $\delta(\mathcal{R})$ we have four indices $a < b < c < d$ such that a and b belong to a descending block, and c and d belong to the next descending block. This yields an occurrence of $2\underline{14}3$ in $\delta(R)$. The converse direction is based on similar considerations. □



Now we enumerate $2\underline{14}3$-avoiding separable permutations. Let π be such a permutation. If π is ascending, then it either consists of at least two singletons, or it has one or several descending blocks, which are separated by at least one singleton (see the drawing). For descending permutations, the decomposition is identical to Case 1, since the skew sum of $2\underline{14}3$-avoiding ascending blocks cannot create a new occurrence of $2\underline{14}3$. This leads to the system $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2}\frac{1}{1-\frac{tD}{1-t}} - 1\right)D,\ \ D = \frac{(t+A)^2}{1-(t+A)}\right\}$. Solving this system (for example by computer algebra) yields Theorem 1(2).



Case 3: -avoiding guillotine diagonal rectangulations.

First we show that a guillotine diagonal rectangulation $\mathcal{R}$ avoids if and only if $\delta(\mathcal{R})$ avoids 21354. As in Lemma 2, the result directly follows from the definition of δ.

Thus, we need to enumerate 21354-avoiding separable permutations. Let π be an ascending separable permutation. If π has just one descending block, then π avoids 21354 if and only if this block avoids 21354. If π has at least three descending blocks, then it contains 21354. If π has precisely two descending blocks, then π is 21354-avoiding if and only if they are adjacent, the first one is 213-avoiding, and the second is 132-avoiding. An ascending 213-avoiding permutation is either the identity permutation of size $\geqslant 2$, or has at least one singleton and precisely one 213-avoiding descending block (the last one). For descending permutations, an occurrence of 21354 implies its occurrence in one of its ascending blocks: hence, the decomposition is again identical to Case 1. Let S_A and S_D be generating functions for ascending and, respectively, descending 213-avoiding (and, equivalently, 132-avoiding) permutations. Then, we have $\left\{S_A = \frac{t^2}{1-t} + \frac{tS_D}{1-t},\ \ S_D = \frac{(t+S_A)^2}{1-(t+S_A)}\right\}$, and for 21354-avoiding separable permutations $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) D + \frac{S_D^2}{(1-t)^2},\ \ D = \frac{(t+A)^2}{1-(t+A)}\right\}$. These systems yield Theorem 1(3).

The treatment of other cases in Theorem 1 is similar. We first translate geometric patterns into permutation patterns, obtaining some subclass of separable permutations. Then its combinatorial specification yields a system of equations that binds $A(t)$, the generating functions for ascending permutations in this class, and $D(t)$ for descending permutations. In some cases we use an auxiliary family (as in Case 3 above). Here we omit the details and only list permutation patterns, systems that bind $A(t)$ and $D(t)$, and, when relevant, auxiliary families and systems for their generating functions S_A and S_D.

Case 4: { , }-avoiding guillotine diagonal rectangulations. Such rectangulations are called *one-sided guillotine rectangulations* [14]. This family corresponds to $(2\underline{14}3, 3\underline{41}2)$-avoiding separable permutations. Due to the symmetry of the model, we have $A(t) = D(t)$, and, therefore, just *one* equation: $A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2}\frac{1}{1-\frac{tA}{1-t}} - 1\right) A$.

Case 5: { , }-avoiding guillotine diagonal rectangulations. They correspond to 2143-avoiding separable permutations, the system is $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) D,\ \ D = \frac{(t+A)^2}{1-(t+A)}\right\}$.

Case 6: { , }-avoiding guillotine diagonal rectangulations. They correspond to $(2\underline{14}3, 45312)$-avoiding separable permutations. The auxiliary class is $(2\underline{14}3, 231)$-avoiding permutations. The system for the auxiliary class is $\left\{S_A = \frac{t^2}{1-t} + \left(\frac{1}{1-\frac{tS_D}{1-t}}\frac{1}{(1-t)^2} - 1\right) S_D,\right.$ $\left. S_D = \frac{t^2}{1-t} + \frac{tS_A}{1-t}\right\}$. The system for $(2\underline{14}3, 45312)$-avoiding separable permutations is $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2}\frac{1}{1-\frac{tD}{1-t}} - 1\right) D,\ \ D = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) A + \frac{S_A^2}{(1-t)^2}\right\}$.

Case 7: { , , }-avoiding guillotine diagonal rectangulations. They correspond to $(21354, 45312)$-avoiding separable permutations. The auxiliary class is $(45312, 213)$-avoiding permutations. The system for the auxiliary class is $\left\{S_A = \frac{t^2}{1-t} + \frac{S_D}{1-t},\ \ S_D = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) S_A + \left(\frac{1}{1-t}\frac{t^2}{1-2t}\right)^2\right\}$. The equation for $(21354, 45312)$-avoiding separable permutations is $A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) A + \frac{S_D^2}{(1-t)^2}$.

Case 8: { , , }-avoiding guillotine diagonal rectangulations. They correspond to $(2143, 3\underline{41}2)$-avoiding separable permutations. This leads to the following system $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) D,\ \ D = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2}\frac{1}{1-\frac{tA}{1-t}} - 1\right) A\right\}$.

Case 9: { , , }-avoiding guillotine diagonal rectangulations. They correspond to $(2143, 45312)$-avoiding separable permutations. The auxiliary class is $(2143, 231)$-avoiding permutations. The system for the auxiliary class is $\left\{S_A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) S_D,\ \ S_D = \frac{t^2}{1-t} + \frac{tS_A}{1-t}\right\}$. The system for $(2143, 45312)$-avoiding separable permutations is $\left\{A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) D,\ \ D = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) A + \frac{S_A^2}{(1-t)^2}\right\}$.

Case 10: { , , , }-avoiding guillotine diagonal rectangulations. They correspond to $(2143, 3412)$-avoiding separable permutations. The equation for this symmetric model is $A = \frac{t^2}{1-t} + \left(\frac{1}{(1-t)^2} - 1\right) A$.

4 Vortex rectangulations and whirls

In this section we consider a class harder to enumerate, as it is not a guillotine case: rectangulations that avoid { } (that is, we forbid all our patterns except $P_2 =$). We denote this class of rectangulations by $\mathcal{V}$, and call them *vortex* rectangulations. Our goal is to prove the conjecture of Merino and Mütze [15].

Theorem 3. *The generating function of $\mathcal{V}$ is $V(t) = tC^2(t)\big(1 + t^2C^4(t)\big)$, where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function of Catalan numbers. The enumerating sequence of $\mathcal{V}$ is* `A026029`.

A vortex either avoids or contains the pattern $P_2 =$. Vortices that avoid constitute Case 10 from Theorem 1. It remains to enumerate vortices with at least one : such rectangulations will be called *whirls*. The *interior* of a windmill is the (possibly further partitioned) rectangular area bounded by its segments. The entire rectangle being partitioned by a given rectangulation will be denoted by R.

Lemma 4. *If a whirl contains several windmills, then they are all nested. In other words: for any two windmills, one of them entirely lies in the interior of the other.*

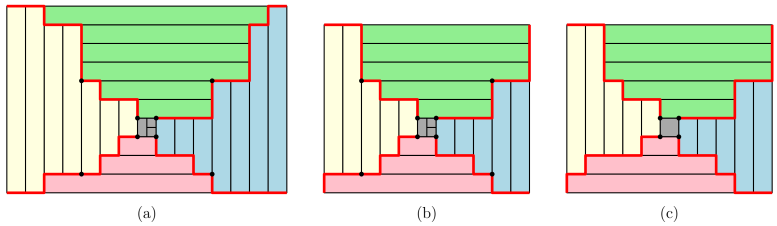


Figure 2: Three whirls: (a) is peelable, (b) is non-peelable, (c) is simple.

Proof (sketch). Let W be a whirl, and consider some specific occurrence of ┴┬. Starting from the right vertical segment of this windmill, we alternately go along the segments downwards to their lower endpoint and rightwards to their right endpoint, until we reach the SE corner of R. Similarly we define four *alternating paths*: see Figure 2 where they are shown by red.

These alternating paths partition R into five regions: R_1, R_2, R_3, R_4, and the interior of the windmill. In our drawings we colour these regions by blue, red, yellow, green, and grey. Then every rectangle in R_1 and in R_3 has its top and bottom sides on the alternating paths, and every rectangle in R_2 and in R_4 has its left and right sides on the alternating paths (see Figure 2). To prove this, for example for R_1, one scans this region from the left to the right: then the assumption that some rectangle in R_1 violates this condition leads to an occurrence of ┤├, ┤├ or ┤├. Moreover, for every rectangle in R_1 its NW corner has the shape $\top$ and its SW corner has the shape $\perp$. It follows that if another windmill — not in the interior of the given one — exists, then its segments belong to four different regions R_1, R_2, R_3, R_4. Hence, the given windmill is entirely included in this another one. $\square$

A whirl with an empty interior can be drawn so that all the rectangles in R_1 and R_3 have width 1, and all the rectangles in R_2 and R_4 have height 1, and such a representation is unique. To see that, we modify the whirl so that its segments belong to consecutive vertical and horizontal grid lines. See Figure 2(a) for an example of a whirl which has two nested windmills (the corners of their interiors are shown by small dots).

A whirl is *peelable* if it has a rectangle that extends from the top to the bottom or from the left to the right side of R. From every peelable whirl it is possible to obtain a unique non-peelable whirl by *peeling* (i.e., successively deleting such rectangles). Figure 2(b) shows a non-peelable whirl which is obtained from 2(a) by peeling.

Finally, a *simple* whirl is a non-peelable whirl with precisely one windmill ┴┬ whose interior is not further partitioned. See Figure 2(c) for an example of a simple whirl.

4.1 Enumeration of simple whirls

In this section we prove the following remarkable result: simple whirls are enumerated by $t^5C^4(t)$. Our proof combines geometric-structural considerations, the generating tree method [5], and solving a functional equation with catalytic variables. It would be interesting to find an independent bijective proof.

4.1.1 Generating tree for simple whirls

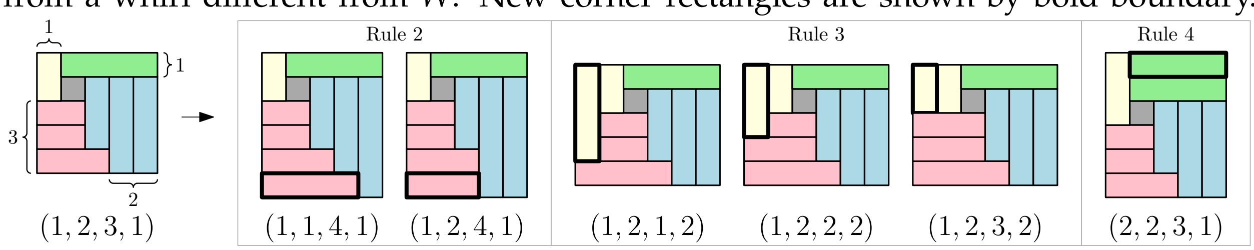
Denote by S_1, S_2, S_3, S_4 the eastern, southern, western, northern sides of R. A *signature* of a simple whirl is the quadruple (s_1, s_2, s_3, s_4), where s_i is the number of rectangles from R_{i-1} that touch S_i. (The addition of indices of R_i's and S_i's is mod 4.) For example, the signature of the simple whirl from Figure 2(c) is $(3, 2, 1, 2)$.

Given a simple whirl W, it has exactly four *corner rectangles* touching one corner of W. Since W avoids and is not peelable, the corner rectangle touching the sides S_i and S_{i+1} has the same colour as the region R_i. Then, as illustrated in the figure at the bottom of this page, from a simple whirl W of size n, we can construct a simple whirl W' of size $n+1$, by adding a new corner rectangle to R_i of length larger or equal to the length of the former corner rectangle (and not touching the side S_{i-1}, to avoid creating a peelable whirl). Some rectangles of R_{i-1} are then extended to reach the modified S_i. In W', s_{i+1} thus increases by 1, and s_i can assume all the values from 1 to the (original) s_i. This generation algorithm has the drawback that some simple whirls are generated several times.

To generate every simple whirl precisely once, we consider only those possibilities in which the added corner rectangle of W' belongs to R_i with the largest possible i (that is, the largest i such that in W' we have $s_{i+1} > 1$). The new generation algorithm thus starts from the initial configuration $(1, 1, 1, 1)$ (only the unique simple whirl of size 5 has this signature), and applies the following rewriting rules

1. $(a,\ b,\ 1,\ 1) \longrightarrow (1,\ b+1,\ 1,\ 1)$,
2. $(1,\ b,\ c,\ 1) \longrightarrow (1,\ [1..b],\ c+1,\ 1)$,
3. $(1,\ b,\ c,\ d) \longrightarrow (1,\ b,\ [1..c],\ d+1)$,
4. $(a,\ b,\ c,\ d) \longrightarrow (a+1,\ b,\ c,\ [1..d])$.

The notation $[1..b]$ means that we generate b signatures where this component takes the values $1, 2, \ldots, b$. The rules are not mutually exclusive: for example, all four rules can be applied on quadruples of the form $(1, b, 1, 1)$. The figure below shows all the descendants of a simple whirl W with signature $(1, 2, 3, 1)$ on which the second, the third, and the fourth rules can be applied. The first rule does not apply since the resulting whirl W' is obtained from a whirl different from W. New corner rectangles are shown by bold boundary.



4.1.2 An intriguing functional equation

Theorem 5 (Algebraicity of simple whirls)**.** *Let $F(t, x_1, x_2, x_3, x_4)$ be the multivariate generating function of simple whirls, where z counts their size, and each x_i counts the number of rectangles of colour i touching their border. This generating function is algebraic and given by*

$$F(t, x_1, x_2, x_3, x_4) = t^5 \frac{1}{2\alpha} \left(\beta - \sqrt{\beta^2 - 4\alpha e_4^2} \right) \tag{4.1}$$

with $\alpha := \prod_{i=1}^4 (1 - x_i + t x_i^2)$ and $\beta := (2e_4 t^2 - t(4e_4 - 3e_3 + 2e_2) + e_4 - e_3 + e_2 - e_1 + 2)e_4$, where $e_m := [t^m] \prod_{i=1}^4 (1 + t x_i)$ is the elementary symmetric polynomial of total degree m.

In particular, the generating function of simple whirls is $F(t) = F(t, 1, 1, 1, 1) = t^5 C(t)^4$, where $C(t)$ is the generating function of Catalan numbers.

Proof. The generating tree from Section 4.1.1 translates to the functional equation

$$\begin{aligned} F(t, x_1, x_2, x_3, x_4) &= t^5 x_1 x_2 x_3 x_4 + t x_1 x_2 x_3 x_4 [x_3 x_4] F(t, 1, x_2, x_3, x_4) \\ &+ t x_1 x_2 x_3 x_4 \frac{[x_1 x_4] F(t, x_1, x_2, x_3, x_4) - [x_1 x_4] F(t, x_1, 1, x_3, x_4)}{x_2 - 1} \\ &+ t x_1 x_3 x_4 \frac{[x_1] F(t, x_1, x_2, x_3, x_4) - [x_1] F(t, x_1, x_2, 1, x_4)}{x_3 - 1} \\ &+ t x_1 x_4 \frac{F(t, x_1, x_2, x_3, x_4) - F(t, x_1, x_2, x_3, 1)}{x_4 - 1}. \end{aligned} \tag{4.2}$$

Unfortunately, there are currently no generic methods to solve this type of catalytic functional equation. Luckily, in our case, we were able to solve this equation. First, recall that the valuation of a series $f(t) = \sum_{n \geqslant 0} f_n t^n$ is the smallest integer n such that $f_n \neq 0$ (and $\mathrm{val}(f(t)) = +\infty$ if $f(t) = 0$). Thus, Equation (4.2) is a contraction in the metric space of formal power series (equipped with the distance $d(f(t), g(t)) = 2^{-\mathrm{val}(f(t) - g(t))}$). Therefore, the Brouwer fixed-point theorem ensures that there is a unique series F satisfying Equation (4.2). Now, it can be checked (by substitution) that the closed form (4.1) satisfies the functional equation (4.2); this proves the theorem.

Let us also explain how we guessed this closed form, as it offers a useful heuristic for dealing with similar equations. The classical guessing technique using Padé approximants is too costly, so, instead, we used linear algebra to identify an algebraic equation of degree 2 (in F) and degree 2 (in x_1) for $F(t, x_1, 11, 31, 71)$. It is not obvious from the functional equation that $F(t, x_1, x_2, x_3, x_4)$ is a symmetric function in the x_i's — yet, this follows from the fact that any rotation of a whirl is still a whirl. Therefore, its minimal polynomial should also have symmetric coefficients in the x_i's. Then, when one obtains a monomial like $532642 x_1 = 2x_1 \times 11^2 \times 31 \times 71$, it makes sense to rewrite it as $2x_1 x_2^2 x_3 x_4$, and all the symmetric versions of this monomial will also appear as coefficients. This leads to the minimal polynomial $\alpha G^2 - \beta G + e_2^4$ (for $G = F/t^5$) , and thus to the closed form (4.1). □

4.2 Enumeration of whirls and vortices

We go back from simple whirls to possibly peelable whirls with empty interior by alternately adding sequences of rectangles on the two horizontal and the two vertical sides. This yields the generating function $P(t) = t^5C^4(t)\left(\frac{2}{1-\left(\frac{1}{(1-t)^2}-1\right)} - 1\right)$. Such whirls can be transformed into a whirl with > 1 windmills by substituting the interior by another whirl (see Figure 2). Thus whirls $\mathcal{W}$ with empty interior of the innermost windmill are enumerated as a sequence of $P(t)/t$. So we obtain the generating function $W(t) = \frac{1}{1-P(t)/t}$.

Finally, to get the family $\mathcal{V}$ of vortices (i.e., the rectangulations that avoid , , , , , ,), we replace the innermost windmill by a rectangulation that avoids all eight patterns (i.e., Case 10 from Theorem 1, counted by $Z(t) = \frac{t(1-2t)}{1-4t+2t^2}$). This leads to

$$V(t) = W(t)Z(t) = (1-2t)\big(1-4t+2t^2+(1-2t)\sqrt{1-4t}\,\big)/(2t^3) = tC^2(t)(1+t^2C^4(t)),$$

which is exactly the generating function of the sequence A026029, as conjectured in [15, Table 3, entry 1345678]. This concludes the proof of Theorem 3.

As for any algebraic generating function, the corresponding sequence satisfies a linear recurrence, $(n+4)v_n - 6(n+2)v_{n-1} + 4(2n-1)v_{n-2} = 0$, from which one can compute v_n in time $O(\sqrt{n}\ln n)$ and singularity analysis gives $v_n \sim 4^{n+2}/\sqrt{\pi}n^{-3/2}$.

5 Conclusion

In this article, we solved several conjectures related to families of pattern-avoiding rectangulations and permutations. We proved that all our generating functions are $\mathbb{N}$-algebraic[3], and we provide an interesting example of $\mathbb{N}$-algebraic structure (the simple whirls, counted by $t^5C^4(t)$) for which no context-free grammar is known.

Merino and Mütze [15, Table 3] mention a few more families of rectangulations for which enumeration is still open. Some are in fact tractable with variants of methods presented here. These results will be included in the full version. It would also be of interest to consider further forbidden patterns, e.g., to determine which patterns lead to algebraic, D-finite, D-algebraic generating functions. Is it the case that they all lead to a Stanley–Wilf-like conjecture: is the number of such rectangulations bounded by A^n, for some constant A? In conclusion, rectangulations, while having a very simple definition, are an inexhaustible source of challenging problems for generating function lovers!

[3]This is the class of generating functions counting words of length n generated by unambiguous context-free grammars. It has many noteworthy structural and asymptotic properties [8, 10].

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Growth Diagrams for Schubert RSK

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Abstract. Motivated by classical combinatorial Schubert calculus on the Grassmannian, Huang–Pylyavskyy introduced a generalized theory of Robinson–Schensted–Knuth (RSK) correspondence for studying Schubert calculus on the complete flag variety, via insertion algorithms. The inputs of the correspondence are certain biwords, the insertion objects are bumpless pipe dreams, and the recording objects are certain chains in Bruhat order. In particular, they defined plactic biwords and showed that classical Knuth relations can be generalized to these. In this extended abstract, we give an analogue of Fomin's growth diagrams for this generalized RSK correspondence on plactic biwords. We show that this growth diagram recovers the bijection between pipe dreams and bumpless pipe dreams of Gao–Huang.

The general philosophy of a *growth diagram* can be thought of as translating a temporal object, i.e., an algorithm, to a spatial object, i.e., a diagrammatic encoding of the algorithm, so as to provide a powerful tool to study the algorithm, as well as an interface between combinatorial algorithms and algebraic or geometric phenomena.[1] The most classical example of a growth diagram is of the classical Robinson-Schensted (RS) correspondence, a bijection between a permutation and a pair of standard Young tableaux. The Robinson-Schensted-Knuth (RSK) correspondence is a generalization of the RS correspondence and is of central importance in symmetric function theory. Each variation of these correspondences has its corresponding growth diagram version. The RS correspondence is originally defined as an insertion algorithm on pairs of standard tableaux. The algorithm iteratively scans the permutation, inserting each time a number to the insertion tableaux, and records the position of the new entry in the recording tableaux. The growth diagram first introduced by Fomin [2, 3], however, is a two dimensional grid that can be roughly thought of as an "enriched" permutation matrix, with the extra information determined by certain local "growth rules." Although far from apparent at a first glance, the growth diagram is a lossless encoding of the insertion algorithm. Furthermore, the growth diagram manifests many non-obvious properties of the insertion algorithm. For example, the property $w \overset{RS}{\longleftrightarrow} (P, Q)$ implies $w^{-1} \overset{RS}{\longleftrightarrow} (Q, P)$ can be easily seen by transposing the growth diagram.

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[1]We learned this philosophy from Allen Knutson.

It is possible to give the RSK correspondence an operator theoretic interpretation through growth diagrams, and as a consequence obtain a noncommutative version of Cauchy's identity [4]. Furthermore, growth diagrams for the RS correspondence has beautiful geometric and representation-theoretic interpretations [11, 14, 15].

Beyond classical RSK, there are many examples in the literature of expressing combinatorial algorithms using growth diagrams, see, e.g., [9, 12, 13, 16].

In [7] and [8], the first author and Pylyavskyy introduced a generalization of the classical RSK correspondence for Schubert polynomials, called bumpless pipe dream (BPD) RSK. As in the classical case, this generalization of RSK is defined via insertion algorithms. The algorithm takes as input a certain biword, iteratively inserts it into a bumpless pipe dream, and records the insertion via chains in mixed k-Bruhat order. An analogue of Knuth moves was discovered for a more restrictive set of biwords, called *plactic biwords*. It is then natural to pursue a growth diagram version of his generalized RSK correspondence on plactic biwords. In this extended abstract, we describe these new growth diagrams for the RSK correspondence for plactic biwords. As an application, our growth diagram manifests the canonical bijection between pipe dreams and bumpless pipedreams of the first author and Gao [5]. We also hope that this opens up a venue for connecting the combinatorics of this generalized RSK to its algebraic or geometric interpretations.

1 Plactic biwords and growth rules

1.1 Bumpless pipe dreams

In this subsection we recall the basic definition of bumpless pipe dreams [10]. A (reduced) **bumpless pipe dream** for a permutation $\pi \in S_n$ is a tiling of an $n \times n$ grid with allowable tiles ⊞, □, ⊟, ◫, ◲, and ◰, such that n "pipes" traveling from the bottom of the grid to the right of the grid form, and no two pipes cross twice. The bottom of the grid is labeled with $1, \cdots, n$, and a permutation read from the pipe labels from the top to bottom on the right edge of the grid is π. We denote the set of bumpless pipe dreams for $\pi \in S_n$ with $\mathrm{BPD}(\pi)$. For example, Figure 1 shows a bumpless pipe dream in BPD(14253). The natural embedding of permutations $S_n \hookrightarrow S_{n+1}$ gives rise to a natural embedding of bumpless pipe dreams in the $n \times n$ grid to those in the $(n+1) \times (n+1)$ grid.

1.2 Generalized Knuth relations on plactic biwords

Definition 1.1 ([8])**.** A biletter is a pair of positive integers $\binom{a}{k}$ where $a \leq k$. A **plactic biword** is a word of biletters $\binom{\mathbf{a}}{\mathbf{k}} = \begin{pmatrix} a_1 & \cdots & a_\ell \\ k_1 & \cdots & k_\ell \end{pmatrix}$, where $k_i \geq k_{i+1}$ for each i.

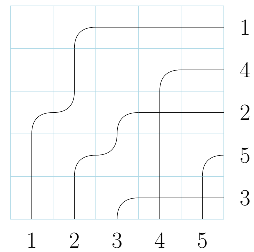


Figure 1: A bumpless pipe dream in BPD(14253)

Definition 1.2 ([8]). We define the **generalized Knuth relations** on plactic biwords as follows:

(1) $\left(\begin{smallmatrix} \cdots & b & a & c & \cdots \\ \cdots & k & k & k & \cdots \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} \cdots & b & c & a & \cdots \\ \cdots & k & k & k & \cdots \end{smallmatrix}\right)$ if $a < b \leq c$

(2) $\left(\begin{smallmatrix} \cdots & a & c & b & \cdots \\ \cdots & k & k & k & \cdots \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} \cdots & c & a & b & \cdots \\ \cdots & k & k & k & \cdots \end{smallmatrix}\right)$ if $a \leq b < c$

(3) $\left(\begin{smallmatrix} \cdots & a & b & \cdots \\ \cdots & k & k & \cdots \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} \cdots & a & b & \cdots \\ \cdots & k+1 & k & \cdots \end{smallmatrix}\right)$ if $a \leq b$

(4) $\left(\begin{smallmatrix} \cdots & b & a & \cdots \\ \cdots & k+1 & k+1 & \cdots \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} \cdots & b & a & \cdots \\ \cdots & k+1 & k & \cdots \end{smallmatrix}\right)$ if $a < b$.

Notice that these relations are only defined on plactic biwords. We do not apply the relation (3) or (4) if the resulting word is no longer plactic.

Given a biword $Q = \left(\begin{smallmatrix} b_1 & b_2 & \dots & b_\ell \\ k_1 & k_2 & \dots & k_\ell \end{smallmatrix}\right)$, [7] defines a map $\mathcal{L}(Q) = (\varphi_L(Q), ch_L(Q))$ where $\varphi_L(Q)$ is the BPD obtained by reading Q from right to left and successively performing left insertion, and $ch_L(Q)$ is the recording chain in mixed k-Bruhat order with edge labels $k_\ell, \cdots, k_1$, as well as a map $\mathcal{R}(Q) = (\varphi_R(Q), ch_R(Q))$ where $\varphi_R(Q)$ is the BPD obtained by reading Q from left to right and successively performing right insertion, and $ch_R(Q)$ is the recording chain in mixed k-Bruhat order with edge labels $k_1, \cdots, k_\ell$. For details of these insertion algorithms see [7, Section 3]. Furthermore, by [8, Proposition 1.2], the insertion BPD is well-defined regardless of the choice of insertion algorithms, so we write $\varphi(D) := \varphi_R(D) = \varphi_L(D)$. For the analysis of the insertion algorithm in this paper we use $\mathcal{R}$, the right insertion algorithm.

Theorem 1.3 ([8]). For any $D \in \mathrm{BPD}(\pi)$, the set of plactic biwords

$$\mathrm{words}(D) := \{Q : \varphi(Q) = D\}$$

is connected by the generalized Knuth relations.

For a biword Q, we define $Q_{>i}$ to be the biword obtained from Q by removing all biletters $\binom{a_j}{k_j}$ with $a_j \leq i$. In particular, $Q_{>0}$ is Q. We have the following lemma.

Lemma 1.4. Suppose Q and Q' are connected by the generalized Knuth relations, then for all i, $Q_{>i}$ and $Q'_{>i}$ are connected by the generalized Knuth relations.

Proof. It suffices to consider the case where Q and Q' are connected by one generalized Knuth relation. Observe that in all relations, if we remove the biletters $\binom{a}{k}$ and $\binom{a}{k+1}$, then the remaining biwords are the same. Thus, we can iteratively remove all biletters $\binom{1}{*}, \binom{2}{*}, \ldots, \binom{i}{*}$, and after each step, either the remaining biwords are connected by the same generalized Knuth relation or they are the same biword. $\square$

As a result of Lemma 1.4, for any $D \in \mathrm{BPD}(\pi)$ and any i, the set of plactic words $\{Q_{>i} \mid Q \in \mathrm{words}(D)\}$ is also connected by the generalized Knuth relations. Therefore, for any $Q \in \mathrm{words}(D)$, $\varphi(Q_{>i})$ is the same BPD.

Remark 1.5. One could similarly define $Q_{<i}$ to be the biword obtained from Q by removing all biletters $\binom{a_j}{k_j}$ with $a_j \geq i$ and ask if $Q \sim Q'$ implies $Q_{<i} \sim Q'_{<i}$ for all i. The answer is unfortunately no. One small example is $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 3 & 3 & 2 \end{pmatrix}$ but $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ are not connected by generalized Knuth relations. The reason is that if Q and Q' are connected by the generalized Knuth relation (3) or (4), then removing $\binom{b}{*}$ yields two different biwords.

1.3 Jeu de taquin on BPDs

Given $D \in \mathrm{BPD}(\pi)$ with $\ell(\pi) > 0$, [5, Definition 3.1] produces another bumpless pipe dream $\nabla D \in \mathrm{BPD}(\pi')$ where $\ell(\pi') = \ell(\pi) - 1$. We call the ∇ operator **jeu de taquin** on BPDs. The justification of this name is that, after applying a direct bijection between (skew) semistandard tableaux and BPDs for Grassmaninan permutations, the jeu de taquin algorithm on tableaux can be realized as a corresponding algorithm on BPDs. See [6] for a detailed description. We will sometimes use the notation $\mathrm{jdt}(b, r)$ instead of ∇ to emphasize that jeu de taquin starts from position (b, r). See Figure 4 for an illustration.

For each BPD D, let b be the smallest row with an empty square $\square$, define $D' = \mathrm{rect}(D)$ be the BPD obtained from D by performing jdt on all empty squares on row b from right to left. Suppose π and μ are the permutations of D' and D, respectively, then by [5], we have $\mu = s_{i_j} \ldots s_{i_1} \pi$, where $i_j > \ldots > i_1$.

Theorem 1.6. Let D be the BPD corresponding to a biword $w = \begin{pmatrix} b_1 & b_2 & \ldots & b_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{pmatrix}$ and $b = \min\{b_1, \ldots, b_\ell\}$, and let D' be the BPD corresponding to w' obtained by removing all biletter $\binom{b}{k_i}$ from w. Then $D' = \mathrm{rect}(D)$.

The following corollary is immediate from Theorem 1.6 by [5].

Corollary 1.7. With the same notation as in Theorem 1.6, let π and μ be the permutations of D' and D, respectively, then

$$\mu = s_{i_j} \ldots s_{i_1} \pi$$

where $i_j > \ldots > i_1$.

1.4 Growth diagrams

1.4.1 Defining growth diagrams

Given a plactic biword $\begin{pmatrix} b_1 & b_2 & \dots & b_\ell \\ k_1 & k_2 & \dots & k_\ell \end{pmatrix}$ and let $a = \max\{b_i \mid 1 \le i \le \ell\}$. We define a growth diagram to be a matrix of permutations $\pi_{i,j}$ with $0 \le i \le a$ and $0 \le j \le \ell$. The **initial condition** is $\pi_{i,0} = \mathrm{id}$ for all i and $\pi_{a,j} = \mathrm{id}$ for all j. The figure below shows a generic **square** of the growth diagram.

$$\begin{tikzcd}
\pi_{i,j-1} \arrow[r, dash] \arrow[d, dash] & \pi_{i,j} \arrow[d, dash] \\
\pi_{i-1,j-1} \arrow[r, dash] & \pi_{i-1,j}
\end{tikzcd}$$

We fill the squares of the growth diagram as follows. For each biletter $\binom{b_i}{k_i}$, we put an $\times_{k_i}$ in the square whose corners are $\pi_{b_i,i-1}, \pi_{b_i,i}, \pi_{b_i-1,i-1}, \pi_{b_i-1,i}$. In addition, in every other square between columns $i-1$ and i, we put a subscript k_i. The following figure shows an example where the biword is $\begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \end{pmatrix}$.

$$\begin{tikzcd}
\pi_{3,0} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & \pi_{3,1} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_3"] & \pi_{3,2} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & \pi_{3,3} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & \pi_{3,4} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "1"] & \pi_{3,5} \arrow[d, dash] \\
\pi_{2,0} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & \pi_{2,1} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & \pi_{2,2} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & \pi_{2,3} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_2"] & \pi_{2,4} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "1"] & \pi_{2,5} \arrow[d, dash] \\
\pi_{1,0} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_3"] & \pi_{1,1} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & \pi_{1,2} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_2"] & \pi_{1,3} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & \pi_{1,4} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_1"] & \pi_{1,5} \arrow[d, dash] \\
\pi_{0,0} \arrow[r, dash] & \pi_{0,1} \arrow[r, dash] & \pi_{0,2} \arrow[r, dash] & \pi_{0,3} \arrow[r, dash] & \pi_{0,4} \arrow[r, dash] & \pi_{0,5}
\end{tikzcd}$$

For each point (i,j) in the growth diagram, let $w(i,j)$ be the biword obtained from reading from left to right the X's to the NW of (i,j). Formally speaking, $w(i,j)$ is obtained from $\begin{pmatrix} b_1 & b_2 & \dots & b_\ell \\ k_1 & k_2 & \dots & k_\ell \end{pmatrix}$ by removing all biletter $\binom{b_s}{k_s}$ with $b_s \le i$ or $s > j$. For example, in the above growth diagram, $w(1,4) = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$. Define $\pi_{i,j}$ to be the permutation of $\varphi(w(i,j))$, the bumpless pipe dream obtained by inserting $w(i,j)$.

Remark 1.8. When $k_1 = \cdots = k_\ell = k$, we recover a version of classical growth diagrams for the RSK correspondence, where the input is a word with letters in positive numbers, the insertion object is a semistandard tableau, and the recording object is a standard tableau. However for classical Knuth relations, deleting either all of the smallest letter in a word, or all of the largest letter in a word, preserves Knuth classes. However in our generalized RSK, we may only delete the biletters with the smallest b_i, as stated in Lemma 1.4.

1.4.2 Local rules

Theorem 1.9. Given a square with subscript k as follows:

$$\begin{tikzcd}
\pi \arrow[r, dash] \arrow[d, dash] & \sigma \arrow[d, dash] \\
\mu \arrow[r, dash] & \rho
\end{tikzcd}$$

Then one can get ρ from π, μ, and σ by the following rules:

1. If there is no $\times$:
   (a) If $\pi = \sigma$ then $\rho = \mu$.
   (b) If $\pi = \mu$ then $\rho = \sigma$.
   (c) If $\pi \neq \sigma, \mu$, then $\mu = s_{i_j} \ldots s_{i_1} \pi$ where $I = \{i_j > \ldots > i_1\}$, and $\sigma = t_{\alpha\beta}\pi$ such that $\pi^{-1}(\alpha) \leq k < \pi^{-1}(\beta)$ for some $\alpha < \beta$. Let $x := \min(I^C \cap [\alpha, \beta))$, and $A := (I^C \cap [\beta, \infty)) \cup \{x\} = \{j_1 < j_2 < \ldots\}$. Then $\rho = t_{j_\ell, j_{\ell+1}}\mu$ where ℓ is the smallest index such that $\mu^{-1}(j_\ell) \leq k < \mu^{-1}(j_{\ell+1})$.
2. If there is an $\times$, then $\pi = \sigma$ and $\mu = s_{i_j} \ldots s_{i_1} \pi$ where $I = \{i_j > \ldots > i_1\}$. Let $I^C = \{j_1 < j_2 < \ldots\}$, then $\rho = t_{j_\ell, j_{\ell+1}}\mu$ where ℓ is the smallest index such that $\mu^{-1}(j_\ell) \leq k < \mu^{-1}(j_{\ell+1})$.

Example 1.10. Let the biword be $\left(\begin{smallmatrix}1&3&1&2&2&1\\3&3&2&2&1&1\end{smallmatrix}\right)$, using the rules in Theorem 1.9, we have the following growth diagram.

$$\begin{tikzcd}
12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & 12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_3"] & 12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & 12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & 12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "1"] & 12345 \arrow[d, dash] \\
12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & 12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & 12435 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & 12435 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_2"] & 12435 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "1"] & 12435 \arrow[d, dash] \\
12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_3"] & 12345 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "3"] & 12435 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_2"] & 12435 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & 13425 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_1"] & 13425 \arrow[d, dash] \\
12345 \arrow[r, dash] & 12435 \arrow[r, dash] & 12534 \arrow[r, dash] & 13524 \arrow[r, dash] & 15324 \arrow[r, dash] & 25314
\end{tikzcd}$$

Notice that in the square

$$\begin{tikzcd}
\pi = 12435 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "2"] & \sigma = 13425 \arrow[d, dash] \\
\mu = 13524 \arrow[r, dash] & \rho = 15324
\end{tikzcd}$$

we use rule (1c) of Theorem 1.9. In particular, we have $\pi \neq \sigma, \mu$ and $\mu = s_4 s_2 \pi$. Thus, $I = \{2, 4\}$. Also, $\sigma = t_{23}\pi$, so $A = \{3, 5, 6, \ldots\}$. Since $\mu^{-1}(3) \leq k = 2 < \mu^{-1}(5)$, we have $\rho = t_{35}\mu = 15324$. On the other hand, in the square

$$\begin{tikzcd}
\pi = 13425 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\times_1"] & \sigma = 13425 \arrow[d, dash] \\
\mu = 15324 \arrow[r, dash] & \rho = 25314
\end{tikzcd}$$

we use rule (2) of Theorem 1.9. We have $\mu = s_4 s_3 \pi$, so $I = \{3, 4\}$. Thus, $I^C = \{1, 2, 5, 6, \ldots\}$. We have $\mu^{-1}(1) \leq k = 1 < \mu^{-1}(2)$, so $\rho = t_{12}\mu = 25314$.

To check that the above growth diagram is correct, we can go through the insertion process. Figure 2 shows the insertion process of this biword. One can check that the permutations we obtain along the way are exactly the permutations on the bottom row of the growth diagram.

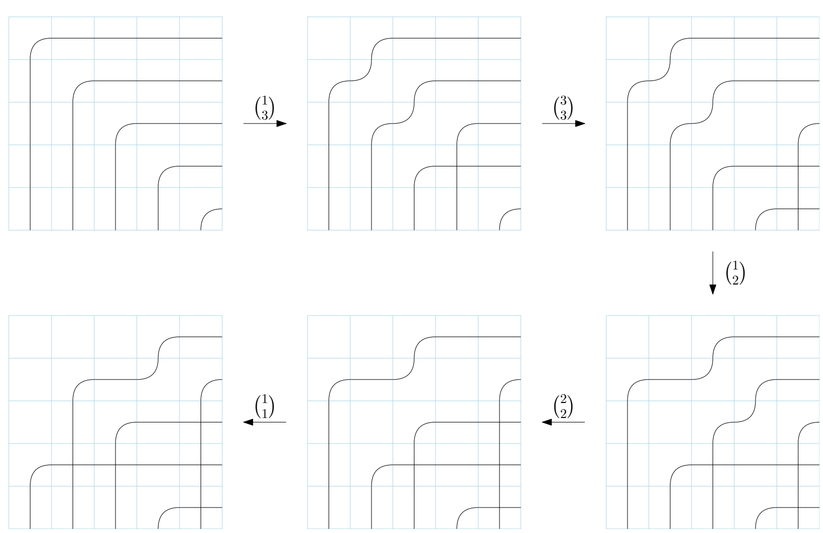


Figure 2: Insertion of $\begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \end{pmatrix}$

On the other hand, removing all biletters $\binom{1}{k}$ in the original biword, we obtain the biword $\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$. The BPD of this biword is shown in Figure 3.

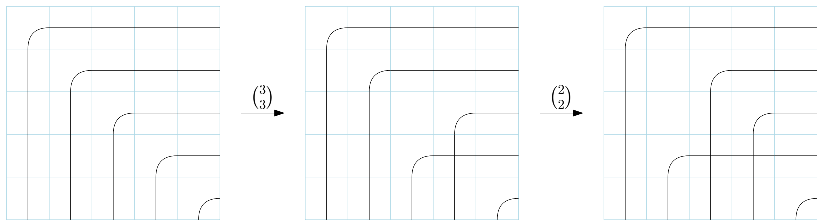


Figure 3: Insertion of $\binom{3\ 2}{3\ 2}$

Let D be the BPD corresponding to the original biword $\binom{1\ 3\ 1\ 2\ 1}{3\ 3\ 2\ 2\ 1}$ (in Figure 2), and D' be the BPD corresponding to the new biword $\binom{3\ 2}{3\ 2}$ (in Figure 3). Theorem 1.6 says that $D' = \mathrm{rect}(D)$. This is indeed the case as shown in Figure 4.

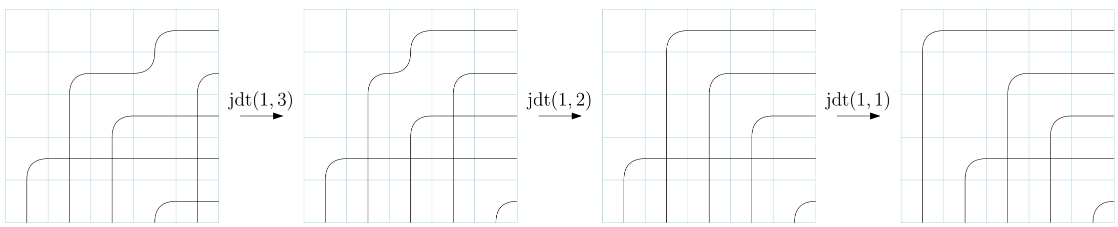


Figure 4

Definition 1.11 ([1])**.** For a permutation π with $\ell(\pi) = \ell$, a pair of integer sequences $\big(\mathbf{a} = (a_1, \ldots, a_\ell), \mathbf{r} = (r_1, \ldots, r_\ell)\big)$ is a **bounded reduced compatible sequence** of π if $s_{a_1} \cdots s_{a_\ell}$ is a reduced word of π, $r_1 \leq \cdots \leq r_\ell$ is weakly increasing, $r_j \leq a_j$ for $j = 1, \ldots, \ell$, and $r_j < r_{j+1}$ if $a_j < a_{j+1}$.

Theorem 1.12. Let $Q := \begin{pmatrix} b_1 & b_2 & \ldots & b_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{pmatrix}$ and let $a = \max\{b_i \mid 1 \leq i \leq \ell\}$, and $(\pi_{i,j})_{0 \leq i \leq a, 0 \leq j \leq \ell}$ be the growth diagram of Q. Then the rightmost vertical chain

$$\mathrm{id} = \pi_{a,\ell} \lessdot \cdots \lessdot \pi_{0,\ell}$$

uniquely recovers a bounded reduced compatible sequence, and this bijects to $\varphi(Q)$ under the bijection in [5].

Explicitly, by Corollary 1.7, for each $1 \leq i \leq a$, we have $s_{i,m_i}, \cdots s_{i,1} \pi_{i,\ell} = \pi_{i-1,\ell}$., where $s_{i,1} > \cdots > s_{i,m_i}$. Then the compatible sequence that corresponds to Q is

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} s_{0,1} & \cdots & s_{0,m_1} & s_{1,1} & \cdots & s_{1,m_1} & \cdots & s_{a-1,1} & \cdots & s_{a-1,m_{a-1}} \\ 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots & a & \cdots & a \end{pmatrix}.$$

Example 1.13. Continuing Example 1.10, the compatible sequence that corresponds to the chain

$$12345 \lessdot 12435 \lessdot 13425 \lessdot 25314$$

is
$$\binom{\mathbf{a}}{\mathbf{r}} = \begin{pmatrix} s_4 & s_3 & s_1 & s_2 & s_3 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}.$$

2 Summary of proofs

Theorem 1.6 follows from the following lemma, which can be proven by a technical analysis of the algorithms.

Lemma 2.1. Let $D \in \mathrm{BPD}(\pi)$ and $D' = \nabla(D)$. Let c be the smallest such that row c contains a blank tile in D. Given $b \geq c$ and k such that the smallest descent in π is at least k. Then
$$\nabla\left(D \leftarrow \binom{b}{k}\right) = D' \leftarrow \binom{b}{k}.$$

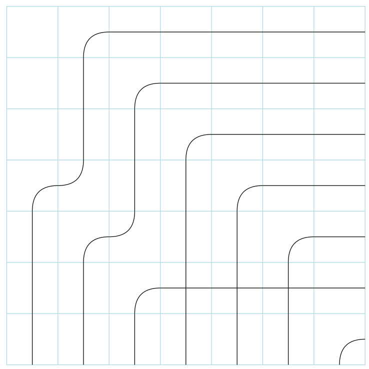
For Theorem 1.9, cases (1a) and (1b) follow directly from the definition of the growth diagram. It remains to prove cases (1c) and (2). The key lemma to prove these two cases is the following. We use a notion of "insertion path" and do a careful analysis of how the insertion algorithms interact with the pipes in D and D'.

Lemma 2.2. Let $D \in \mathrm{BPD}(\pi)$, and $D' = \nabla(D)$. Suppose $\mathrm{pop}(D) = (i, c)$, then by definition $D' \in \mathrm{BPD}(\sigma)$ where $\sigma = s_i\pi$. Given $b \geq c$ and k such that the smallest descent in π is at least k. Suppose the insertion path of $D' \leftarrow \binom{b}{k}$ goes through pipes $p_1 < p_2 < \ldots < p_\ell$. Let $P := \{p_1, p_2, \ldots, p_\ell\}$, then

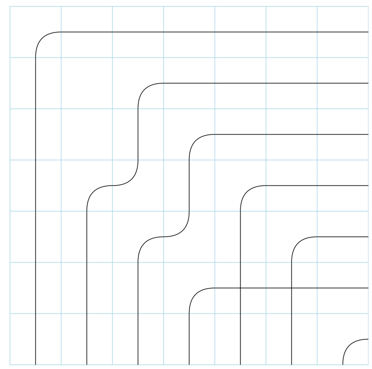
1. if $i = p_j$ and $i + 1 \neq p_{j+1}$ for some $1 \leq j \leq \ell - 1$, then $D \leftarrow \binom{b}{k}$ goes through pipes $p_1, \ldots, p_{j-1}, p_j + 1, p_{j+1}, \ldots, p_\ell$;
2. if $i = p_{\ell-1}$ and $i + 1 = p_\ell$, then $D \leftarrow \binom{b}{k}$ goes through pipes $p_1, \ldots, p_{\ell-2}, p_\ell, p_\ell + 1, p_\ell + 2, \ldots$ until it terminates;
3. if $i = p_\ell$ then $D \leftarrow \binom{b}{k}$ goes through pipes $p_1, \ldots, p_{\ell-1}, p_\ell + 1$;
4. otherwise, $D \leftarrow \binom{b}{k}$ goes through pipes $p_1, \ldots, p_\ell$.

In particular, unless $i = p_{\ell-1}$ or $i = p_\ell$, the last two pipes of $D \leftarrow \binom{b}{k}$ are still $p_{\ell-1}$ and p_ℓ.

Let us give some examples of Lemma 2.2. In Figure 5, we have a BPD D with $\mathrm{pop}(D) = (3, 1)$. In $D' = \nabla(D)$, the insertion path of $D' \leftarrow \binom{2}{5}$ goes through pipes $2, 3, 5, 6, 7$. Since $i = 3$ is one of the pipes, but $i + 1 = 4$ is not, the insertion path of $D \leftarrow \binom{2}{5}$ goes through pipes $2, 4, 5, 6, 7$. This is case (1) of Lemma 2.2. On the other hand, the insertion path of $D' \leftarrow \binom{1}{5}$ goes through pipes $1, 2, 3, 4$. Since i and $i + 1$ are the last



D



D'

Figure 5

two pipes, the insertion path of $D \leftarrow \binom{1}{5}$ goes through pipes $1, 2, 4, 5, 6, 7$. This is case (2) in Lemma 2.2. Finally, the insertion path of $D' \leftarrow \binom{2}{2}$ goes through pipes 2 and 3. Thus, the insertion path of $D \leftarrow \binom{2}{2}$ goes through pipes 2 and 4. This is case (3) in Lemma 2.2.

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A signed e-expansion of the chromatic symmetric function and some new e-positive graphs

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Abstract. We prove a new signed elementary symmetric function expansion of the chromatic symmetric function of any unit interval graph. We then use sign-reversing involutions to prove new combinatorial formulas for many families of graphs, including the K-chains studied by Gebhard and Sagan, formed by joining cliques at single vertices, and for graphs obtained from them by removing any number of edges from any of the cut vertices. We also introduce a version for the quasisymmetric refinement of Shareshian and Wachs.

Keywords: chromatic symmetric function, elementary symmetric function, Stanley–Stembridge conjecture, unit interval graph

The Stanley–Stembridge conjecture [26, 27] is one of the most actively researched open problems in algebraic combinatorics today. It asserts that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then G is *e-positive*, meaning that the chromatic symmetric function $X_G(\boldsymbol{x})$ defined by Stanley [26] is a nonnegative linear combination of elementary symmetric functions. Several authors have proven that certain graphs are e-positive [3, 6, 9, 11, 14, 16, 21, 29, 30, 31], studied other positivity properties of $X_G(\boldsymbol{x})$, [4, 12, 13, 15, 17, 19, 22], defined generalizations of the chromatic symmetric function, [10, 18, 25], and explored implications of the Stanley–Stembridge conjecture to immanants of Jacobi–Trudi matrices [27], cohomology of Hessenberg varieties [1, 5, 7, 20, 24], and characters of Hecke algebras [8].

In this extended abstract, we give a signed elementary function expansion of $X_G(\boldsymbol{x})$ for any unit interval graph G, in terms of objects called *forest triples*. We then show how sign-reversing involutions on forest triples can be used to prove combinatorial formulas for many classes of unit interval graphs, including the K-chains proven to be e-positive by Gebhard and Sagan [18] and *melting K-chains* obtained from them by removing any number of edges from any of the cut vertices. Melting K-chains were not previously known to be e-positive. We also present a generalization of our forest triple formula for the chromatic quasisymmetric function of Shareshian and Wachs [25].

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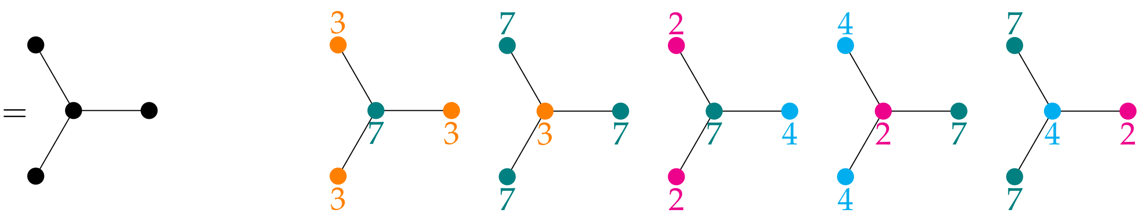
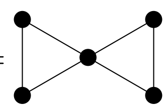
Figure 1: The claw graph G, five proper colourings of G, the corresponding monomials, the chromatic symmetric function $X_G(\boldsymbol{x})$, the bowtie graph H, and the chromatic symmetric function $X_H(\boldsymbol{x})$

$G =$ 

$$X_G(\boldsymbol{x}) = \cdots + x_3^3 x_7 + x_3 x_7^3 + \cdots + x_2^2 x_4 x_7 + x_2 x_4^2 x_7 + x_2 x_4 x_7^2 + \cdots \tag{1.3}$$
$$= e_{211} - 2e_{22} + 5e_{31} + 4e_4 \tag{1.4}$$

$H =$ 

$$X_H(\boldsymbol{x}) = 4e_{32} + 12e_{41} + 20e_5 \tag{1.5}$$

1 Background

Let $G = (V, E)$ be a graph. A *colouring* of G is a function $\kappa : V \to \mathbb{P} = \{1, 2, 3, \ldots\}$ and we say that κ is *proper* if $\kappa(i) \neq \kappa(j)$ whenever $(i, j) \in E$. The *chromatic symmetric function* of G is the formal power series in infinitely many variables $\boldsymbol{x} = (x_1, x_2, x_3, \ldots)$ given by [26, Definition 2.1]

$$X_G(\boldsymbol{x}) = \sum_{\kappa : V \to \mathbb{P} \text{ proper}} \boldsymbol{x}^\kappa, \text{ where } \boldsymbol{x}^\kappa = \prod_{v \in V} x_{\kappa(v)}. \tag{1.1}$$

We are interested in expanding the symmetric function $X_G(\boldsymbol{x})$ in the basis $\{e_\lambda\}$ of *elementary symmetric functions* indexed by integer partitions $\lambda = \lambda_1 \cdots \lambda_\ell$, defined by

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}, \text{ where } e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}. \tag{1.2}$$

We say that G is *e-positive* if the chromatic symmetric function $X_G(\boldsymbol{x})$ is a nonnegative linear combination of elementary symmetric functions.

Example 1. *Figure 1 shows the claw graph G. Some proper colourings of G, the corresponding monomials of $X_G(\boldsymbol{x})$, and the e-expansion of $X_G(\boldsymbol{x})$ are given. Because of the negative coefficient* *on the term $-2e_{22}$, the graph G is not e-positive. By contrast, the bowtie graph H is e-positive. For the complete graph K_n, proper colourings must use n distinct colours and given n distinct colours there are $n!$ proper colourings, so $X_{K_n}(\boldsymbol{x}) = n!e_n$ and K_n is e-positive.*

There has been considerable interest in characterizing e-positive graphs. The most prominent open problem in this direction is the Stanley–Stembridge conjecture [26, Corollary 5.1], equivalently [27, Corollary 5.5], which by a result of Guay-Paquet [19, Theorem 5.1] can be equivalently stated for *unit interval graphs* G, which are graphs whose vertices can be labelled 1 through n so that

$$\text{for } i < j < k, \text{ if } (i,k) \in E(G), \text{ then } (i,j) \in E(G) \text{ and } (j,k) \in E(G). \tag{1.6}$$

Conjecture 1. *(Stanley–Stembridge conjecture) All unit interval graphs are e-positive.*

2 A signed formula

Let $G = ([n], E)$ be a *natural unit interval graph*, meaning it satisfies (1.6). We give a signed combinatorial formula for the elementary symmetric function expansion of $X_G(\boldsymbol{x})$.

Definition 1. *A subtree T of G is* decreasing *if every vertex $v \in V(T)$ has at most one larger neighbour. A subforest F of G is* decreasing *if all of its trees are decreasing.*

Definition 2. *A* tree triple *of G is an object $\mathcal{T} = (T, \alpha, r)$ consisting of the following data.*

- *T is a decreasing subtree of G.*
- *α is an integer composition with size $|\alpha| = |V(T)|$.*
- *r is a positive integer with $1 \leq r \leq \alpha_1$, the first part of α.*

A forest triple *of G is a set of tree triples $\mathcal{F} = \{\mathcal{T}_i = (T_i, \alpha^{(i)}, r_i)\}_{i=1}^m$ with $\sqcup_{i=1}^m V(T_i) = [n]$, so the set of trees is a decreasing spanning forest of G. The* type *of $\mathcal{F}$ is the integer partition*

$$type(\mathcal{F}) = sort(\alpha^{(1)} \cdots \alpha^{(m)}) \tag{2.1}$$

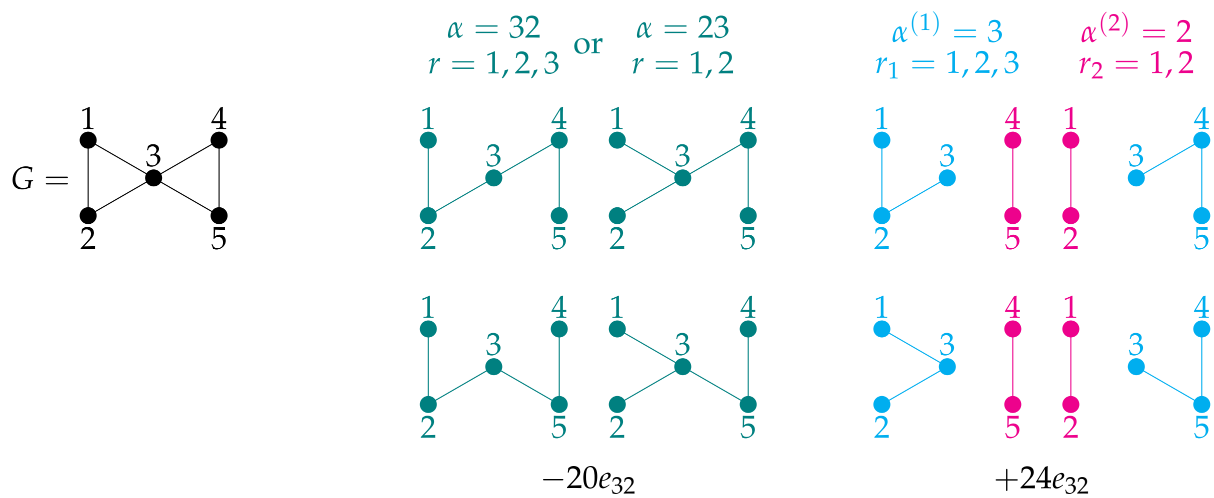
formed by concatenating the compositions and then sorting to form a partition. The sign *of $\mathcal{F}$ is the integer*

$$sign(\mathcal{F}) = (-1)^{\sum_{i=1}^m (\ell(\alpha^{(i)})-1)} = (-1)^{\ell(type(\mathcal{F}))-m}, \tag{2.2}$$

where $\ell(\alpha)$ is the length of a composition α. We denote by $FT(G)$ the set of forest triples of G and by $FT_\mu(G)$ the set of forest triples of G of type μ.

We now state our combinatorial formula. It was proven by first expanding $X_G(\boldsymbol{x})$ in the power sum basis and then applying a change-of-basis to the elementary symmetric function basis. The technique of studying properties of $X_G(\boldsymbol{x})$ by converting between different bases is explored in upcoming work of Sagan and the author [23].

Figure 2: The bowtie graph G and the forest triples of G of type 32



Theorem 1. *[28, Theorem 4.3] Let G be a natural unit interval graph. The chromatic symmetric function $X_G(\boldsymbol{x})$ satisfies*

$$X_G(\boldsymbol{x}) = \sum_{\mathcal{F} \in FT(G)} sign(\mathcal{F}) e_{type(\mathcal{F})} = \sum_{\mu} \left(\sum_{\mathcal{F} \in FT_\mu(G)} sign(\mathcal{F}) \right) e_\mu. \tag{2.3}$$

Example 2. *Figure 2 shows the forest triples of type 32 for the bowtie graph. We can have a single tree triple $\mathcal{T} = (T, \alpha, r)$, in which case α is either 32 or 23 and there are either 3 or 2 choices for r. Alternatively, we can have two tree triples $\mathcal{T}_1 = (T_1, \alpha^{(1)}, r_1)$ and $\mathcal{T}_2 = (T_2, \alpha^{(2)}, r_2)$ with $\alpha^{(1)} = 3 = |V(T_1)|$ and $\alpha^{(2)} = 2 = |V(T_2)|$, and there are 3 choices of r_1 and 2 choices of r_2.*

Example 3. *For the case of $\mu = n$, forest triples $\mathcal{F} \in FT_n(G)$ consist of a single tree triple $\mathcal{T} = (T, \alpha, r)$, where T is a decreasing spanning tree, we must have $\alpha = n$ so $sign(\mathcal{F}) = 1$, and we can have any value of $1 \leq r \leq n$. Because a decreasing spanning tree can be identified by specifying the unique larger neighbour of each vertex $1 \leq i \leq n-1$, we have that the coefficient of e_n in $X_G(\boldsymbol{x})$ is $nd_1 \cdots d_{n-1}$, where d_i is the number of larger neighbours of vertex i in G.*

Now our goal is to find a sign-reversing involution on forest triples of G to combinatorially prove that G is e-positive. The structure of forest triples suggests the following approach. Let us say that a tree triple $\mathcal{T} = (T, \alpha, r)$ is *breakable* if $\ell(\alpha) \geq 2$. In this case, we would like to somehow define a pair of forest triples $\text{break}(\mathcal{T}) = (\mathcal{S}_1, \mathcal{S}_2)$ of the form

$$\mathcal{S}_1 = (S_1, \alpha \setminus \alpha_\ell, r) \text{ and } \mathcal{S}_2 = (S_2, \alpha_\ell, r_2) \tag{2.4}$$

for some decreasing trees S_1 and S_2 with $V(S_1) \sqcup V(S_2) = V(T)$ and some integer $1 \leq r_2 \leq \alpha_\ell$, where α_ℓ is the last part of α and $\alpha \setminus \alpha_\ell$ denotes the composition with α_ℓ removed. Let us say that the pair $(\mathcal{S}_1, \mathcal{S}_2)$ is *joinable* if it is of the form $\mathrm{break}(\mathcal{T})$ for some unique $\mathcal{T}$, which we will denote $\mathrm{join}(\mathcal{S}_1, \mathcal{S}_2)$. Then we would like to somehow define a map φ on $\mathrm{FT}(G)$ by either replacing some breakable tree triple $\mathcal{T}$ by $\mathrm{break}(\mathcal{T})$ or by replacing some joinable pair of tree triples $(\mathcal{S}_1, \mathcal{S}_2)$ by $\mathrm{join}(\mathcal{S}_1, \mathcal{S}_2)$, if one exists.

If we can systematically choose which tree triples to replace so that φ is an involution, then it would reverse sign because it changes the total number of tree triples by one, it would preserve type by construction, and fixed points $\mathcal{F}$ must have no breakable tree triples or joinable pairs of tree triples so in particular $\mathrm{sign}(\mathcal{F}) = (-1)^{\sum_{i=1}^{m}(1-1)} = 1$. Therefore, we would prove that G is e-positive, and we would also get a combinatorial formula for the chromatic symmetric function $X_G(\boldsymbol{x})$ in terms of the fixed points of φ.

We now demonstrate this method in the case of paths. More general results are known [25, Section 5], [26, Proposition 5.3] but this proof technique is new.

Proposition 1. *The chromatic symmetric function of a path P_n is given by*

$$X_{P_n}(\boldsymbol{x}) = \sum_{\alpha \vDash n} \alpha_1(\alpha_2 - 1) \cdots (\alpha_\ell - 1) e_{sort(\alpha)}, \tag{2.5}$$

where the notation $\alpha \vDash n$ means that α is a composition with size n. In particular, P_n is e-positive.

Proof. We label the vertices of P_n so that its edges are of the form $(i, i+1)$, so decreasing subtrees of P_n are paths from some i to some $j > i$, which we will denote $P_{i \to j}$. For a breakable tree triple $\mathcal{T} = (P_{i \to j}, \alpha, r)$ of P_n, we define $\mathrm{break}(\mathcal{T}) = (\mathcal{S}_1, \mathcal{S}_2)$, where

$$\mathcal{S}_1 = (P_{i \to j - \alpha_\ell}, \alpha \setminus \alpha_\ell, r) \text{ and } S_2 = (P_{j - \alpha_\ell + 1 \to j}, \alpha_\ell, 1), \tag{2.6}$$

and we define a pair of tree triples $(\mathcal{S}_1 = (P_{i \to j}, \alpha^{(1)}, r_1), \mathcal{S}_2 = (P_{i' \to j'}, \alpha^{(2)}, r_2))$ to be *joinable* if $\ell(\alpha^{(2)}) = 1$, $i' = j + 1$, and $r_2 = 1$, in which case we define the tree triple

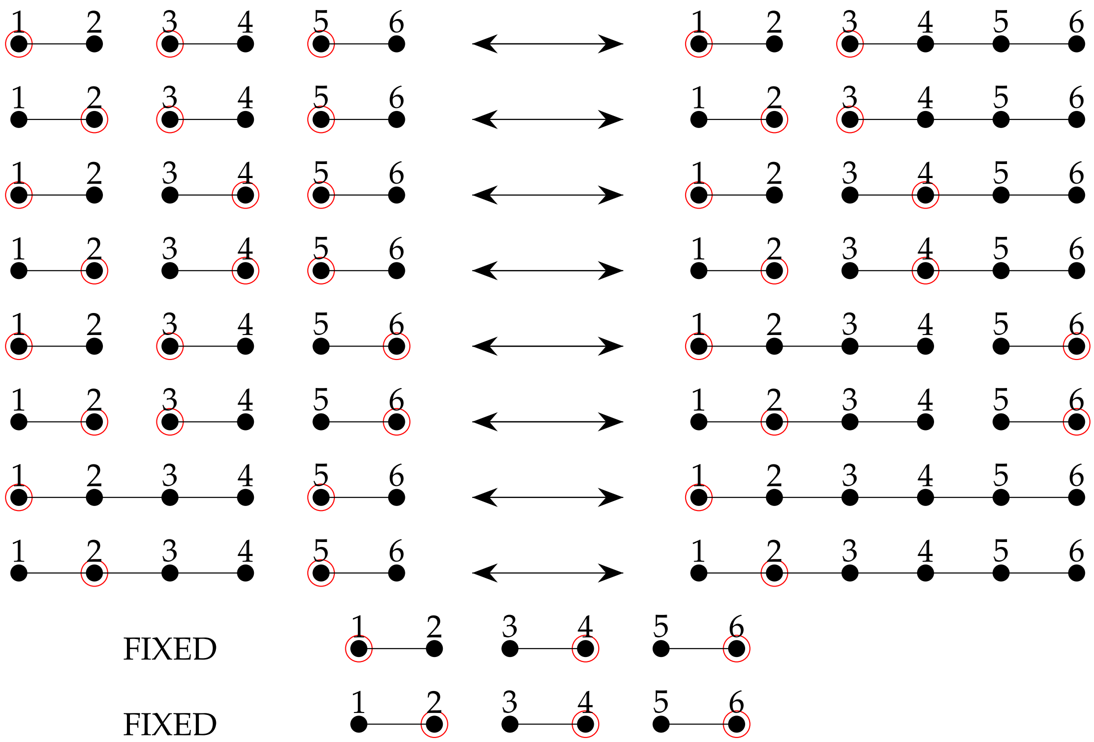
$$\mathrm{join}(\mathcal{S}_1, \mathcal{S}_2) = (P_{i \to j'}, \alpha^{(1)} \cdot \alpha^{(2)}, r_1). \tag{2.7}$$

Note that $\mathcal{T}$ is breakable with $\mathrm{break}(\mathcal{T}) = (\mathcal{S}_1, \mathcal{S}_2)$ if and only if $(\mathcal{S}_1, \mathcal{S}_2)$ is joinable with $\mathrm{join}(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{T}$. Now given a forest triple

$$\mathcal{F} = \{\mathcal{T}_1 = (P_{i_1 \to i_2 - 1}, \alpha^{(1)}, r_1), \mathcal{T}_2 = (P_{i_2 \to i_3 - 1}, \alpha^{(2)}, r_2), \ldots, \mathcal{T}_\ell = (P_{i_\ell \to n}, \alpha^{(\ell)}, r_\ell)\}, \tag{2.8}$$

we let j be maximal so that either $\mathcal{T}_j$ is breakable or the pair $(\mathcal{T}_{j-1}, \mathcal{T}_j)$ is joinable, if such a j exists, and we define $\varphi(\mathcal{F})$ by either breaking $\mathcal{T}_j$, joining $(\mathcal{T}_{j-1}, \mathcal{T}_j)$, or doing nothing if no such j exists. We can check that φ is a sign-reversing involution, the fixed points can be associated with a composition $\alpha \vDash n$ by reading the tree sizes from left to right, they have type $\mathrm{sort}(\alpha)$, and in order for no pair to be joinable, we must have $r_i \geq 2$ for every $i \geq 2$, so there are $\alpha_1(\alpha_2 - 1) \cdots (\alpha_\ell - 1)$ choices of the r_i. □

Figure 3: The forest triples of P_6 of type 222 paired under our sign-reversing involution φ



Example 4. *Figure 3 shows all of the forest triples of P_6 of type 222 paired under our sign-reversing involution φ. The compositions are not written but they are 2, 22, or 222. We have indicated whether each r_i is 1 or 2 by circling the r_i-th smallest vertex of the corresponding tree. There are $2(2-1)(2-1) = 2$ fixed points, so the coefficient of e_{222} in $X_{P_6}(\boldsymbol{x})$ is 2.*

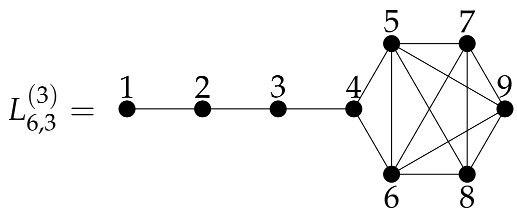
3 Positive formulas

We are able to use sign-reversing involutions on forest triples to prove several combinatorial e-positive expansions of unit interval graphs.

Definition 3. *Let $L^{(t)}_{a,b}$ denote the unit interval graph where a path of length b is joined to a clique of size a, and then t edges incident to the joined vertex are removed from the clique. An example is given in Figure 4. Such graphs are called* melting lollipops.

Huh, Nam, and Yoo showed that melting lollipops are e-positive [21, Theorem 4.9]. A result of Aliniaeifard, Wang, and van Willigenburg [3, Proposition 3.1] implies that

Figure 4: The melting lollipop graph $L_{6,3}^{(3)}$



the sequence $(X_{L_{a,b}^{(t)}}(\boldsymbol{x}))_{t=0}^{a-1}$ forms an arithmetic progression. We were able to use forest triples to get a new explicit formula and see this arithmetic progression directly.

Theorem 2. *[28, Theorem 5.2] The chromatic symmetric function of a melting lollipop $L_{a,b}^{(t)}$ is given by*

$$X_{L_{a,b}^{(t)}}(\boldsymbol{x}) = t(a-2)! \sum_{\substack{\alpha \vDash n \\ \alpha_\ell = a-1}} \alpha_1(\alpha_2 - 1)\cdots(\alpha_{\ell-1}-1)e_{sort(\alpha)} \\ + (a-t-1)(a-2)! \sum_{\substack{\alpha \vDash n \\ \alpha_\ell \geq a}} \alpha_1(\alpha_2-1)\cdots(\alpha_\ell - 1)e_{sort(\alpha)}. \tag{3.1}$$

In particular, $L_{a,b}^{(t)}$ is e-positive.

Definition 4. *For a composition γ with all parts at least 2, let K_γ denote the unit interval graph where cliques of sizes $\gamma_1, \ldots, \gamma_\ell$ are successively joined end to end at single vertices. An example is given in Figure 5. Such graphs are called K-*chains.

Gebhard and Sagan showed that K-chains are e-positive [18, Corollary 7.7] by using a generalization of the chromatic symmetric function in noncommuting variables. We were able to use forest triples to get a new explicit formula as a sum over a certain set A_γ of weak compositions.

Theorem 3. *[28, Theorem 6.13] The chromatic symmetric function of a K-chain K_γ is given by*

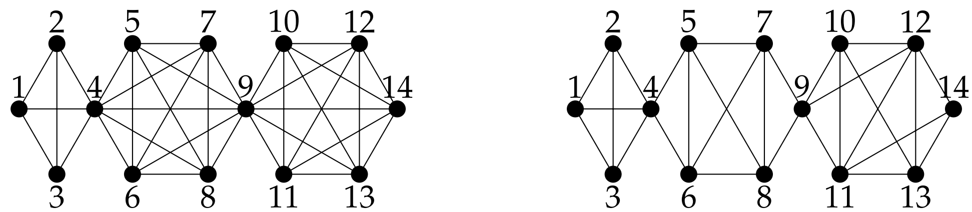
$$X_{K_\gamma}(\boldsymbol{x}) = (\gamma_1 - 2)!\cdots(\gamma_{\ell-1}-2)!(\gamma_\ell - 1)! \sum_{\alpha \in A_\gamma} \left(\alpha_1 \prod_{i=2}^{\ell(\gamma)} |\alpha_i - (\gamma_{i-1}-1)| \right) e_{sort(\alpha)} \tag{3.2}$$

In particular, K_γ is e-positive.

Example 5. *If $\gamma = ab$ has length 2, we get that*

$$X_{K_{ab}}(\boldsymbol{x}) = (a-1)!(b-1)! \sum_{k=\max\{a,b\}}^{n} (2k-n)e_{k,n-k}. \tag{3.3}$$

Figure 5: The K-chain K_{466} and the melting K-chain $K_{466}^{(032,032)}$



Definition 5. *Let γ be a composition with all parts at least 2, and let ϵ and ζ be weak compositions with $\ell(\epsilon) = \ell(\zeta) = \ell(\gamma)$ such that $0 \leq \epsilon_t, \zeta_t \leq \gamma_t - 2$ for all t and $\epsilon_t = 0$ if and only if $\zeta_t = 0$. Let $K_\gamma^{(\epsilon,\zeta)}$ denote the unit interval graph formed by removing edges from the K-chain K_γ so that for all t, the t-th clique has ϵ_t edges absent from the smallest vertex and ζ_t edges absent from the largest vertex. An example is given in Figure 5. Such graphs are called* melting K-chains, *and if every $\epsilon_t, \zeta_t \in \{0, 1\}$ (so $\epsilon = \zeta$), they are called* slightly melting K-chains.

Aliniaeifard, Wang, and van Willigenburg showed that slightly melting K-chains are e-positive [3, Proposition 5.5]. We were able to use forest triples to get a new explicit formula as a sum over a certain set $A_\gamma^{(\epsilon)}$ of weak compositions.

Theorem 4. *[28, Theorem 7.9] The chromatic symmetric function of a slightly melting K-chain $K_\gamma^{(\epsilon,\epsilon)}(\boldsymbol{x})$ is given by*

$$X_{K_\gamma^{(\epsilon,\epsilon)}}(\boldsymbol{x}) = (\gamma_1 - 2)! \cdots (\gamma_\ell - 2)! \sum_{\alpha \in A_\gamma^\epsilon} \left(\alpha_1 \prod_{i=1}^{\ell(\gamma)} |\alpha_{i+1} - (\gamma_i - 1 - \epsilon_i)| \right) e_{sort(\alpha)}. \tag{3.4}$$

In particular, $K_\gamma^{(\epsilon,\epsilon)}$ is e-positive.

We also proved the new result that all melting K-chains are e-positive. We have a combinatorial description of the fixed points but they are much more complicated to describe and enumerate.

Theorem 5. *[28, Theorem 8.3] All melting K-chains $K_\gamma^{(\epsilon,\zeta)}$ are e-positive.*

It would be interesting to see whether sign-reversing involutions on forest triples could be used to show e-positivity of other unit interval graphs. Alternatively, we could take a dual approach where we fix μ and show that the coefficient of e_μ is nonnegative for every unit interval graph. This is done by Hwang [22, Theorem 5.13] if $\mu_2 = 1$, by Abreu and Nigro [1, Corollary 1.10] if $\ell(\mu) = 2$, and in upcoming work by Sagan and the author [23] if $\mu_1 \leq 3$. If we can prove the following inequality, the forest triple formula would give another proof of nonnegativity for all two-part partitions.

Problem 1. *Let $G = ([n], E)$ be a natural unit interval graph and let $1 \le k \le n-1$. Let $s_k(G)$ be the number of decreasing spanning forests (T_1, T_2) of G, where $|V(T_2)| = k$ and $1 \in V(T_1)$. Let $s(G)$ be the number of decreasing spanning trees of G. Prove that $ks_k(G) \ge s(G)$.*

The author checked by computer that this inequality holds for all unit interval graphs G with $n \le 10$ vertices.

4 A quasisymmetric generalization

We also generalize our forest triple formula for the *chromatic quasisymmetric function* defined by Shareshian and Wachs [25, Definition 1.2].

Definition 6. *The* chromatic quasisymmetric function *of a labelled graph $G = ([n], E)$ is the formal power series*

$$X_G(\boldsymbol{x}; q) = \sum_{\substack{\kappa:[n]\to\mathbb{P} \\ \kappa \text{ proper}}} q^{asc(\kappa)} \boldsymbol{x}^{\kappa}, \tag{4.1}$$

where $asc(\kappa) = |\{(i,j) \in E(G) : i < j,\ \kappa(i) < \kappa(j)\}|$.

Alexandersson used the following idea to study e-positivity of LLT polynomials [2].

Definition 7. *Let $\theta \subseteq E(G)$. For a vertex $u \in [n]$, let $hrv_\theta(u)$ be the highest $v \in [n]$ reachable from u by an increasing path in $([n], \theta)$ and let $\{[u_1]_\theta, \ldots, [u_m]_\theta\}$ be the set of equivalence classes of $[n]$ under the relation $u \sim_\theta u'$ if $hrv_\theta(u) = hrv_\theta(u')$. Let $\theta' \subseteq \theta$ be the subset of edges used by the increasing paths from every u to $hrv_\theta(u)$ that go to the largest possible vertex at each step. We let $U(\theta) = \theta \setminus \theta'$ and the elements of $U(\theta)$ are called* unnecessary edges*.*

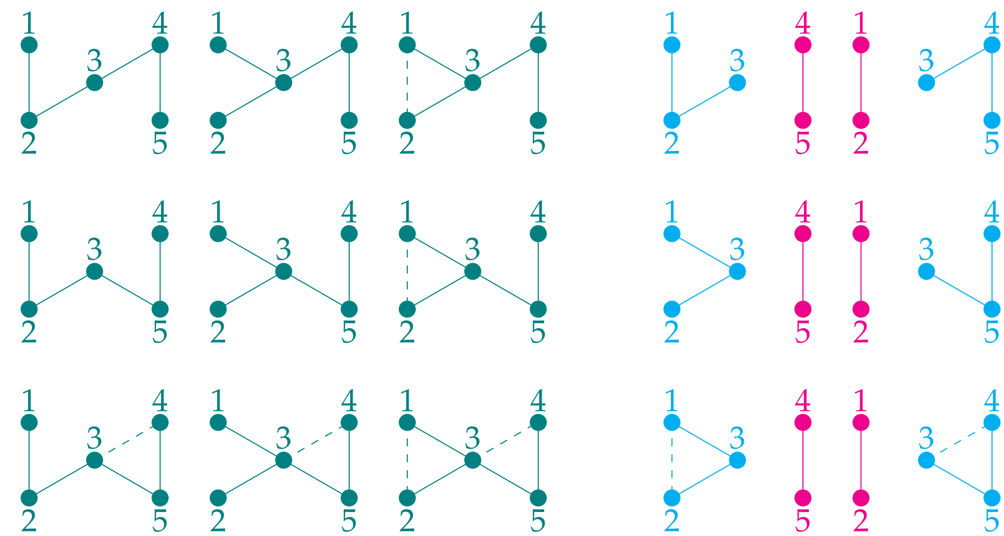
Definition 8. *A* subgraph quadruple *of G is an object $\mathcal{S} = (\theta, f, \boldsymbol{\alpha}, \boldsymbol{r})$ consisting of the following data.*

- *$\theta \subseteq E(G)$ is a subset of the edges of G.*
- *$f : U(\theta) \to \{q, -1\}$ is a function that assigns either a q or a (-1) to each unnecessary edge.*
- *$\boldsymbol{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(m)})$ is a sequence of compositions such that each $|\alpha^{(i)}| = |[u_i]_\theta|$.*
- *$\boldsymbol{r} = (r_1, \ldots, r_m)$ is a sequence of positive integers such that each $1 \le r_i \le \alpha_1^{(i)}$.*

The type *of $\mathcal{S}$ is the partition*

$$type(\mathcal{S}) = sort(\alpha^{(1)} \cdots \alpha^{(m)}), \tag{4.2}$$

Figure 6: The edges of the subgraph quadruples of type 32 for the bowtie graph G



$-(4+4(q-1)+(q-1)^2)(1+q+q^2+1+q)e_{32}$ $\quad$ $(4+2(q-1))(1+q+q^2)(1+q)e_{32}$

the sign *of $\mathcal{S}$ is the integer*

$$sign(\mathcal{S}) = (-1)^{\sum_{i=1}^{m}(\ell(\alpha^{(i)})-1)}(-1)^{|\{e\in U(\theta):f(e)=-1\}|}, \tag{4.3}$$

and the weight *of $\mathcal{S}$ is the integer*

$$weight(\mathcal{S}) = \sum_{i=1}^{m}(r_i-1) + |\{e \in U(\theta) : f(e) = q\}|. \tag{4.4}$$

We denote by $SQ(G)$ the set of subgraph quadruples of G.

Theorem 6. *[28, Theorem 9.5] The chromatic quasisymmetric function $X_G(\boldsymbol{x};q)$ of a natural unit interval graph G satisfies*

$$X_G(\boldsymbol{x};q) = \sum_{\mathcal{S}\in SQ(G)} sign(\mathcal{S})q^{weight(\mathcal{S})}e_{type(\mathcal{S})}. \tag{4.5}$$

Example 6. *Figure 6 shows the edges of the subgraph quadruples of type 32 for the bowtie graph, where the unnecessary edges are shown by dotted lines and would each be assigned a q or a (-1). If we have a single equivalence class, then α is either 32 or 23 and there are either 3 or 2 choices for r. If we have two, then $\alpha^{(1)} = 3$, $\alpha^{(2)} = 2$, there are 3 choices of r_1, and 2 choices of r_2. We have written the contributions to the coefficient of e_{32}, taking into account the choices of f and $\boldsymbol{r}$.*

We can apply the earlier sign-reversing involution to subgraph quadruples to prove the following combinatorial e-expansion of Shareshian and Wachs [25, Section 5].

Proposition 2. *The chromatic quasisymmetric function of the path P_n is given by*

$$X_{P_n}(\boldsymbol{x};q) = \sum_{\alpha \vDash n} q^{\ell(\alpha)-1}[\alpha_1]_q[\alpha_2 - 1]_q \cdots [\alpha_\ell - 1]_q e_{sort(\alpha)}, \tag{4.6}$$

where the vertices of P_n are labelled so that (1.6) *holds and we define $[k]_q = 1 + q + \cdots + q^{k-1}$.*

We could try to adapt our other sign-reversing involutions to subgraph quadruples.

Problem 2. *Use subgraph quadruples to prove combinatorial e-positive expansions for the chromatic quasisymmetric functions $X_{K_\gamma}(\boldsymbol{x};q)$ and $X_{K_\gamma^{(\epsilon,\zeta)}}(\boldsymbol{x};q)$.*

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All Kronecker coefficients are reduced Kronecker coefficients

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Abstract. We settle the question of where exactly do the reduced Kronecker coefficients lie on the spectrum between the Littlewood-Richardson and Kronecker coefficients by showing that every Kronecker coefficient of the symmetric group is equal to a reduced Kronecker coefficient by an explicit construction. This implies the equivalence of a question by Stanley from 2000 and a question by Kirillov from 2004 about combinatorial interpretations of these two families of coefficients. Moreover, as a corollary, we deduce that deciding the positivity of reduced Kronecker coefficients is NP-hard, and computing them is $\#\mathsf{P}$-hard under parsimonious many-one reductions.

Keywords: Kronecker coefficients, reduced Kronecker coefficients, representation theory, symmetric group, general linear group

1 Introduction

The *Kronecker coefficients* $\mathtt{k}(\lambda,\mu,\nu)$ of the symmetric group S_n are some of the most classical, yet largely mysterious, quantities in Algebraic Combinatorics and Representation Theory. The Kronecker coefficient is the multiplicity of the irreducible S_n representation $\mathbb{S}_\nu$ in the tensor product $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$ of two other irreducible S_n representations. Murnaghan defined them in 1938 as an analogue of the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ of the general linear group GL_N, which are the multiplicities of the irreducible Weyl modules V_λ in the tensor products $V_\mu \otimes V_\nu$. Yet, the analogy has not translated far into their properties. The Littlewood-Richardson coefficients have a beautiful positive combinatorial interpretation and their positivity is "easy" to decide, formally it is in P. However, positive combinatorial formulas for the Kronecker coefficients have eluded us so far, see Section 1.2, and their positivity is hard to decide.

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The *reduced Kronecker coefficients* $\overline{\mathtt{k}}(\alpha,\beta,\gamma)$ are defined as the stable limit of the ordinary Kronecker coefficients

$$\overline{\mathtt{k}}(\alpha,\beta,\gamma) := \lim_{n\to\infty} \mathtt{k}(\,(n-|\alpha|,\alpha),\ (n-|\beta|,\beta),\ (n-|\gamma|,\gamma)\,).$$

These coefficients are called *extended Littlewood-Richardson numbers* in [13], since in the special case when $|\alpha| = |\beta| + |\gamma|$ we have $\overline{\mathtt{k}}(\alpha,\beta,\gamma) = c^{\alpha}_{\beta,\gamma}$, the Littlewood-Richardson coefficient. Problem 2.32 in [13] asks for a combinatorial interpretation of $\overline{\mathtt{k}}(\alpha,\beta,\gamma)$. As such they have been considered as an intermediate, an interpolation, between the Littlewood-Richardson and Kronecker coefficients. They have been an object of independent interest, see [16, 17, 4, 27, 13, 3, 2, 5, 15, 24, 10, 21, 18, 19], and considered better behaved than the ordinary Kronecker coefficients.

This is, however, not the case. As we show, every Kronecker coefficient is equal to an explicit reduced Kronecker coefficient of not much larger partitions, and in particular:

Theorem 1. *For all partitions λ, μ, ν of equal sizes, we have*

$$\mathtt{k}(\lambda,\mu,\nu) \ = \ \overline{\mathtt{k}}\big(\,\nu_1^{\ell(\lambda)} + \lambda,\ \nu_1^{\ell(\mu)} + \mu,\ (\nu_1^{\ell(\lambda)+\ell(\mu)},\nu)\,\big).$$

Here $a^b := (\underbrace{a,\ldots,a}_{b \text{ many}})$ and $(\nu_1^b,\nu) := (\underbrace{\nu_1,\ldots,\nu_1}_{b \text{ many}},\nu_1,\nu_2,\nu_3,\ldots)$.

This implies that in a very strong sense, on the spectrum between Littlewood-Richardson and Kronecker coefficients, the reduced Kronecker coefficients are at the same point as the ordinary Kronecker coefficients. In particular, Theorem 1 implies that Problem 2.32 in [13] is equivalent to Problem 10 in [26]: Finding a combinatorial interpretation for the Kronecker coefficient or for the reduced Kronecker coefficient are the same problem. Formally, Conjecture 9.1 and 9.4 in [20] are the same. Our result can be interpreted in a positive or in a negative way. On the one hand, the reduced Kronecker coefficients cannot be easier to understand than the ordinary Kronecker coefficients. On the other hand, understanding the reduced Kronecker coefficients is sufficient to understand all ordinary Kronecker coefficients.

We thus settle the conjecture from [21, §4.4] on the hardness of deciding positivity:

Corollary 1. *Given α,β,γ in unary, deciding if $\overline{\mathtt{k}}(\alpha,\beta,\gamma) > 0$ is* NP*-hard.*

Moreover, by the same immediate argument it is now clear that computing the reduced Kronecker coefficient is strongly #P-hard under parsimonious many-one reductions (the argument in [21] gives only the #P-hardness under Turing reductions).

1.1 Background and definitions

We refer to [12, 25, 23] for basic definitions and properties from algebraic combinatorics and representation theory, and employ the following notation. For a *partition*

$\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, denoted $\lambda \vdash n$, its size is denoted $|\lambda| := \sum_i \lambda_i$ and length $\ell(\lambda) = \max\{i \mid \lambda_i > 0\}$. We write λ' do denote the *transpose* partition, i.e., the partition that arises from reflecting the Young diagram at the main diagonal. Formally, $\lambda'_j := \max\{i \mid \lambda_i \geq j\}$. We add partitions row-wise: $(\lambda+\mu)_i = \lambda_i + \mu_i$. We define $\lambda \diamond \mu := (\lambda' + \mu')'$, adding partitions column-wise as Young diagrams. The Specht modules $\mathbb{S}_\lambda$ for $\lambda \vdash n$ are the irreducible representation of the symmetric group S_n, see [12, 25, 23].

The *Kronecker coefficient* $\mathtt{k}(\lambda, \mu, \nu)$ is the structure constant[1] defined as

$$\mathbb{S}_\nu \otimes \mathbb{S}_\mu = \bigoplus_\lambda \mathbb{S}_\lambda^{\oplus \mathtt{k}(\lambda,\mu,\nu)}$$

via Specht modules, giving that $\mathtt{k}(\lambda, \mu, \nu)$ is a nonnegative integer. Yet the problem of finding a combinatorial interpretation of $\mathtt{k}(\lambda, \mu, \nu)$ is wide open [26, 9, 22]. The Kronecker coefficients were defined by Murnaghan [16] in 1938 as the analogues of the *Littlewood-Richardson coefficients* $c^\lambda_{\mu\nu}$, which are the structure constants in the ring of irreducible GL_N representations, the Weyl modules V_λ, given as $V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{\oplus c^\lambda_{\mu\nu}}$. Some simple properties, see [12, 23] include the transposition invariance $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}(\lambda', \mu', \nu)$ and permutation of the terms. We define $\mathtt{k}'(\lambda, \mu, \nu) := \mathtt{k}(\lambda', \mu', \nu') = \mathtt{k}(\lambda', \mu, \nu) = \mathtt{k}(\lambda, \mu', \nu) = \mathtt{k}(\lambda, \mu, \nu')$. It is known that $\mathtt{k}(\lambda, \mu, \nu) = 0$ if $\ell(\lambda) > \ell(\mu) \cdot \ell(\nu)$ [6], which also follows by combining $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}(\lambda, \mu', \nu')$ with Lemma 3. We define the *stable range* as the set of triples (λ, μ, ν) that satisfy $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}\big(\lambda + (i),\ \mu + (i),\ \nu + (i)\big)$ for all $i \geq 0$. The *reduced Kronecker coefficient* is defined as this limit value:

$$\overline{\mathtt{k}}(\alpha, \beta, \gamma) := \lim_{n\to\infty} \mathtt{k}\big(\,(n - |\alpha|, \alpha),\ (n - |\beta|, \beta),\ (n - |\gamma|, \gamma)\,\big)$$

for arbitrary partitions α, β, γ (in particular, we do *not* require $|\alpha| = |\beta| = |\gamma|$). When $|\alpha| = |\beta| + |\gamma|$, then $\overline{\mathtt{k}}(\alpha, \beta, \gamma) = c^\alpha_{\beta,\gamma}$. For a full list of definitions and properties we refer to the full version of this paper [11].

1.2 Related work

The Littlewood-Richardson (LR) coefficients can be computed by the Littlewood-Richardson rule, stated in 1934 and proven formally about 40 years later. It says that $c^\lambda_{\mu\nu}$ is equal to the number of LR tableaux of shape λ/μ and content ν. The apparent analogy in definitions motivates the community to search for such interpretations for the Kronecker coefficients. Interest in efficient ways to compute $\mathtt{k}(\lambda, \mu, \nu)$ and $\overline{\mathtt{k}}(\alpha, \beta\gamma)$ dates back at least to Murnaghan [16]. Specific interest in nonnegative combinatorial interpretations of $\mathtt{k}(\lambda, \mu, \nu)$ can be found in [Lascoux 1979, Garsia-Remmel 1985] and

[1] In the combinatorics literature these coefficients have usually been denoted by g, e.g. $g(\lambda, \mu, \nu)$, but here we use $\mathtt{k}$ to avoid overlap with the notation used for the representation theory of GL_N.

was formulated clearly again by Stanley as Problem 10 in his list "Open Problems in Algebraic Combinatorics" [26]. See also [22] for a detailed discussion on this topic.

Despite its natural and fundamental nature and the variety of efforts, this question has seen relatively little progress. The state of the art is combinatorial interpretations for specific classes of partitions (ν being a hook, or μ, ν being two-row partitions, etc). It was shown by Murnaghan [17] that the reduced Kronecker coefficients generalize the Littlewood-Richardson coefficients as

$$\overline{\mathtt{k}}(\alpha, \beta, \gamma) = c^{\alpha}_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta| + |\gamma|,$$

which motivates Kirillov's naming of $\overline{\mathtt{k}}$ as "extended Littlewood-Richardson numbers". This relationship and other properties have motivated an independent interest in the reduced Kronecker coefficients as intermediates between Littlewood-Richardson and ordinary Kronecker coefficients. Some special cases of combinatorial interpretations can be derived from the existing ones for the ordinary Kronecker coefficients. In [5] a combinatorial interpretation was given when μ, ν are rectangles and λ is one row. A combinatorial interpretation of $\overline{\mathtt{k}}(\alpha, \beta, \gamma)$ in the subcase where $\ell(\alpha) = 1$ was obtained in [1]. Methods to compute them have been discussed in [16, 17] and have been developed in a series of papers, see [3, 2, 18, 19]. As observed in [2] the reduced Kronecker coefficients are also the structure constants for the ring of so called character polynomials. The reduced Kronecker coefficients are a special case of a more general stability phenomenon that if $\mathtt{k}(i\alpha, i\beta, i\gamma) = 1$ for all i, then $\mathtt{k}(\lambda + N\alpha, \mu + N\beta, \nu + N\gamma)$ stabilizes as $N \to \infty$ as seen in [24, 27].

The Kronecker coefficients can be expressed as a small alternating sum of reduced Kronecker coefficients, and reduced Kronecker coefficients are certain sums of ordinary Kronecker coefficients for smaller partitions, see [3]. These relationships showed that reduced Kronecker coefficients are also #P-hard to compute, see [21]. However, these relations did not imply that deciding positivity of reduced Kronecker coefficients is NP-hard.

It is important to note that deciding if $c^{\lambda}_{\mu\nu} > 0$ is in P, since they count integer points in a polytope that has an integral vertex whenever it is nonempty, a consequence of Knutson-Tao's proof of the saturation property: $c^{N\lambda}_{N\mu,N\nu} > 0 \iff c^{\lambda}_{\mu\nu} > 0$. The Kronecker coefficients do not satisfy the saturation property, because $\mathtt{k}(2^2, 2^2, 2^2) = 1$, but $\mathtt{k}(1^2, 1^2, 1^2) = 0$. Until recently it was believed that the reduced Kronecker coefficients have the saturation property: It was conjectured in [13, 14] that if $\overline{\mathtt{k}}(N\alpha, N\beta, N\gamma) > 0$ for some $N > 0$, then $\overline{\mathtt{k}}(\alpha, \beta, \gamma) > 0$. This was disproved in [21] in 2020 and moved the reduced Kronecker coefficients away from the Littlewood-Richardson coefficients on that spectrum.

2 Setting up the proof of Theorem 1

We discovered Theorem 1 using the natural interpretation of $\mathtt{k}(\lambda, \mu, \nu)$ via the general linear group, see §3, and the relationship with 3-dimensional binary contingency arrays. We set the proof up in this section, reducing to a more general Theorem 2, which has a short proof via GL_N and two short, self-contained proofs using basic symmetric function techniques. The complete proofs are available in [11].

Lemma 1. *Let λ, μ, ν be partitions with $\ell(\lambda) \leq l$, $\ell(\mu) \leq m$. Then*

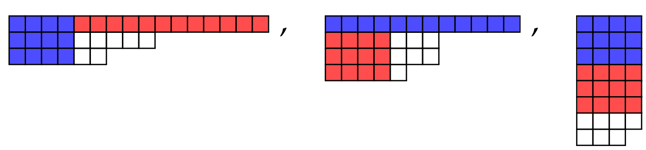
$$\mathtt{k}(\lambda, \mu, \nu) \; = \; \mathtt{k}(\, m^l + \lambda,\; l^m + \mu,\; 1^{lm} + \nu\,).$$

The following Lemma 2 is proved by applying Lemma 1 twice, in different directions.

Lemma 2. *Let λ, μ, ν be partitions of the same size, and let $l \geq \ell(\lambda)$, $m \geq \ell(\mu)$ and $c \geq \nu_1$. Let $d = (m+1)c$, $e = (l+1)c$. Then*

$$\mathtt{k}(\lambda, \mu, \nu) \; = \; \mathtt{k}\big(\,(d) \diamond (c^l + \lambda),\; (e) \diamond (c^m + \mu),\; c^{l+m+1} \diamond \nu\,\big).$$

In lieu of a proof we illustrate this by example with $\lambda = (5, 2)$, $\mu = (3, 3, 1)$ and $\nu = (4, 3)$, with $l = 2$, $m = 3$ and $c = 4$. The red boxes are the addition from the first application of Lemma 1 and the blue boxes are the second application.



Theorem 2. *Let λ, μ, ν be partitions of the same size, such that $\lambda_1 \geq \ell(\mu) \cdot \nu_1$ and $\mu_1 \geq \ell(\lambda) \cdot \nu_1$. Then for every $h \geq 0$ we have*

$$\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}(\, \lambda + h,\; \mu + h,\; \nu + h\,).$$

Our proofs use an observation on 3-dimensional contingency arrays Q with zeros and ones as entries (Lemma 4), applied differently. We identify subsets $Q \subseteq \mathbb{N}^3$ with their characteristic functions $Q : \mathbb{N}^3 \to \{0, 1\}$, and we call Q a binary or $\{0,1\}$-contingency array. This means, we interpret Q as a function to $\{0, 1\}$, and as the point set of its support. The interpretation will always be clear from the context. The 2-dimensional marginals of Q are defined as $Q_{i**} := \sum_{j,k} Q_{i,j,k} = |Q \cap (\{i\} \times \mathbb{N} \times \mathbb{N})|$, $Q_{*i*} := \sum_{j,k} Q_{j,i,k} = |Q \cap (\mathbb{N} \times \{i\} \times \mathbb{N})|$, $Q_{**i} := \sum_{j,k} Q_{j,k,i} = |Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\})|$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, $\beta \in \mathbb{N}^{\mathbb{N}}$, $\gamma \in \mathbb{N}^{\mathbb{N}}$, $|\alpha| = |\beta| = |\gamma| < \infty$, we denote by

$$\mathcal{C}(\alpha, \beta, \gamma) := \{Q \subseteq \mathbb{N}^3 \mid Q_{i**} = \alpha_i,\; Q_{*i*} = \beta_i,\; Q_{**i} = \gamma_i \text{ for every } i\}.$$

There is a close connection to the Kronecker coefficients via the following (see e.g. §4):

Lemma 3. *For partitions α, β, γ of equal size, we have $\mathtt{k}'(\alpha,\beta,\gamma) \le |\mathcal{C}(\alpha,\beta,\gamma)|$.*

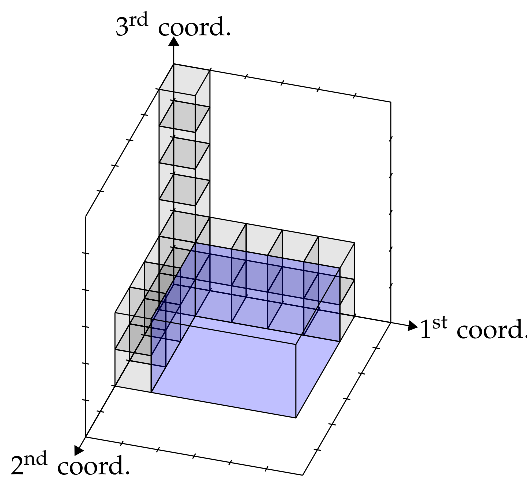
Restrictions on the marginals can result in strong restrictions on the sets Q:

Lemma 4. *Let α, β, γ be compositions with $|\alpha| = |\beta| = |\gamma|$. Let $a \ge \ell(\alpha)$, $b \ge \ell(\beta)$, and let the integers c, h be such that $c + h \ge \ell(\gamma)$ and $\sum_{i>c} \gamma_i \le h$. Furthermore, let $\alpha_1 \ge bc + h$, $\beta_1 \ge ac + h$. Then, for every $Q \in \mathcal{C}(\alpha,\beta,\gamma)$ we have*

$$\{1\} \times [b] \times [c] \subseteq Q, \quad [a] \times \{1\} \times [c] \subseteq Q, \quad \{1\} \times \{1\} \times [c+h] \subseteq Q, \text{ and}$$
$$Q \cap (\mathbb{N} \times \mathbb{N} \times [c+1, c+h]) = \{1\} \times \{1\} \times [c+1, c+h].$$

In particular, if $\mathcal{C}(\alpha,\beta,\gamma)$ is non-empty, then $a = \ell(\alpha)$, $b = \ell(\beta)$, $\gamma_i = 1$ for all $c+1 \le i \le c+h$, and $\alpha_1 = bc + h$, $\beta_1 = ac + h$, $\alpha_2 \le bc$, and $\beta_2 \le ac$.

In other words, if we have 3d point configurations with such marginals, then the walls consist of two rectangles and a long column as depicted in the figure below.



Proof: Assume that there exists a binary contingency array $Q \in \mathcal{C}(\alpha,\beta,\gamma)$. Let $B_\cup := \{1\} \times [b] \times [c+h] \;\cup\; [a] \times \{1\} \times [c+h]$ be the set of points in the planes $x = 1$ and $y = 1$, and let $B_\cap := \{1\} \times \{1\} \times [c+h]$ be the set of points on the line $x = y = 1$. Let $H_i := Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) \cap B_\cup$ be the entries of Q in $B_\cup$ at the section with the plane $z = i$. In particular, $\sum_{i=1}^{c+h} |H_i| = |Q \cap B_\cup|$. We have $\sum_{i=c+1}^{c+h} |H_i| \le \sum_{i=c+1}^{c+h} \gamma_i \le h$, $|H_i| \le a + b - 1$ for all $0 < i \le c$ and $|Q \cap B_\cap| \le c + h$. All these inequalities must be met with equality, because

$$\begin{aligned}
\alpha_1 + \beta_1 &= |Q \cap B_\cap| + |Q \cap B_\cup| = |Q \cap B_\cap| + \textstyle\sum_{i=1}^{c+h} |H_i| \\
&= |Q \cap B_\cap| + \textstyle\sum_{i=1}^{c} |H_i| + \sum_{i=c+1}^{c+h} |H_i| \\
&\le (c+h) + (a+b-1)c + h = (a+b)c + 2h \le \alpha_1 + \beta_1.
\end{aligned}$$

We thus have the following equalities: $|Q \cap B_\cap| = c + h = |B_\cap|$ and $\forall i \in [c]$ we have $|H_i| = a + b - 1 = |(\mathbb{N} \times \mathbb{N} \times \{i\}) \cap B_\cup|$. Thus we have $B_\cap \subseteq Q$, and $\{1\} \times [b] \times [c] \subseteq Q$, and $[a] \times \{1\} \times [c] \subseteq Q$, and $Q \cap (\mathbb{N} \times \mathbb{N} \times [c+1, c+h]) = \{1\} \times \{1\} \times [c+1, c+h]$. This gives the desired marginals and the claim follows. □

Proof of Theorem 1. Let $\ell(\lambda) = l$, $\ell(\mu) = m$ and $\nu_1 = c$ and set $d = mc + c$, $e = lc + c$.

Suppose first that $\lambda_1 \le mc$ and $\mu_1 \le lc$. We apply Lemma 2, and obtain

$$\mathtt{k}(\lambda,\mu,\nu) = \mathtt{k}\big(\underbrace{(d) \diamond (c^l + \lambda)}_{=:\hat\lambda}, \underbrace{(e) \diamond (c^m + \mu)}_{=:\hat\mu}, \underbrace{c^{l+m+1} \diamond \nu}_{=:\hat\nu}\big).$$

The top rows of $\hat{\lambda}, \hat{\mu}, \hat{\nu}$ are d, e, c respectively and thus Theorem 2 gives that for all $h \in \mathbb{N}$ we have $\mathtt{k}(\hat{\lambda},\, \hat{\mu},\, \hat{\nu}) = \mathtt{k}(\hat{\lambda}+h,\, \hat{\mu}+h,\, \hat{\nu}+h) =$

$$= \mathtt{k}(\,(d+h)\diamond(c^l+\lambda),\,(e+h)\diamond(c^m+\mu),\,(c+h)\diamond c^{l+m}\diamond\nu\,) = \overline{\mathtt{k}}(c^l+\lambda,\, c^m+\mu,\, c^{l+m}\diamond\nu),$$

where the last identity follows by letting $h \to \infty$. This proves Theorem 1 in the first case.

Suppose now that $\lambda_1 > mc$, the case $\mu_1 > lc$ is completely analogous. Set $b := m+1$. Then we have $\mathtt{k}(\lambda,\mu,\nu) = \mathtt{k}(\lambda',\mu,\nu') = 0$ since $\ell(\lambda') = \lambda_1 > mc = \ell(\mu)\ell(\nu')$. On the other hand, the reduced Kronecker coefficient is obtained by adding long first rows, $cm+c+h,\, cl+c+h,\, c+h$ respectively, so $\overline{\mathtt{k}}(c^l+\lambda,\, c^m+\mu,\, c^{l+m}\diamond\nu) =$

$$\begin{aligned}
&= \mathtt{k}\big((cm+c+h)\diamond(c^l+\lambda),\,(lc+c+h)\diamond(c^m+\mu),\,(c+h)\diamond c^{l+m}\diamond\nu)\big) \\
&= \mathtt{k}'\big(\,\underbrace{(cm+c+h)\diamond(c^l+\lambda)}_{=:\,\alpha},\,\underbrace{(lc+c+h)\diamond(c^m+\mu)}_{=:\,\beta},\,\underbrace{((l+b)^c+\nu')\diamond(1^h)}_{=:\,\gamma}\,\big)
\end{aligned}$$

for sufficiently large h. Let $\hat{\gamma} = (l+b)^c + \nu'$ be γ without the h many trailing 1s. We observe that $\alpha_2 = \lambda_1 + c$, $\ell(\beta) = b$, and $\ell(\hat{\gamma}) = c$. From $\lambda_1 > mc$ we conclude $\alpha_2 > bc$. Lemma 4 shows that $\mathcal{C}(\alpha,\beta,\gamma) = \emptyset$. Hence $\mathtt{k}'(\alpha,\beta,\gamma) = 0$ by Lemma 3. $\square$

3 Proofs via the general linear group

We refer to [7, §8] for the basic properties of the irreducible representations of the general linear group. The Kronecker coefficients have an interpretation as the structure coefficients arising when decomposing irreducible GL_{ab} representations as $\mathrm{GL}_a \times \mathrm{GL}_b$ representations, which can be seen directly from Schur-Weyl duality:

$$V_\nu(\mathbb{C}^{ab}) \simeq \bigoplus_{\lambda\vdash_a|\nu|,\,\mu\vdash_b|\nu|} \Big(V_\lambda(\mathbb{C}^a)\otimes V_\mu(\mathbb{C}^b)\Big)^{\oplus\mathtt{k}(\lambda,\mu,\nu)}.$$

Another formulation is via the multiplicity of the irreducible $G := \mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$ representation $V_\alpha(\mathbb{C}^a)\otimes V_\beta(\mathbb{C}^b)\otimes V_\gamma(\mathbb{C}^c)$ in the D-th wedge power of $\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c$, see [8]. Formally for partitions $\alpha,\beta,\gamma \vdash D$ we have

$$\mathtt{k}'(\alpha,\beta,\gamma) \;:=\; \mathtt{k}(\alpha,\beta,\gamma') \;=\; \mathrm{mult}_{\alpha,\beta,\gamma}\big(\textstyle\bigwedge^D(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)\big).$$

A vector v for which $\big(\mathrm{diag}(r_1,\dots,r_a),\mathrm{diag}(s_1,\dots,s_b),\mathrm{diag}(t_1,\dots,t_c)\big)v = r_1^{\lambda_1}\cdots r_a^{\lambda_a}\cdot s_1^{\mu_1}\cdots s_b^{\mu_b}\cdot t_1^{\nu_1}\cdots t_c^{\nu_c}v$ is called a *weight vector* of weight (λ,μ,ν).

For $(A,B,C)\in\mathbb{C}^{a\times a}\times\mathbb{C}^{b\times b}\times\mathbb{C}^{c\times c}$, the Lie algebra action on $\bigwedge^D(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)$ is defined as $(A,B,C).v := \lim_{\varepsilon\to 0}\varepsilon^{-1}((\varepsilon(A,B,C)+(\mathrm{id}_a,\mathrm{id}_b,\mathrm{id}_c))v - v)$. A raising operator is the Lie algebra action of $(E_{i-1,i},0,0)$, where $E_{i,j}$ is the matrix with a 1 at position (i,j) and zeros everywhere else. The other raising operators are $(0,E_{i-1,i},0)$ and

$(0,0,E_{i-1,i})$. Let $e_i := (0,\dots,0,1,0,\dots,0)^T$ and let $e_{i,j,k} := e_i \otimes e_j \otimes e_k$. Then, for example, $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise. A highest weight vector (HWV) of weight (α,β,γ) is a nonzero weight vector of weight (α,β,γ) that is mapped to zero by all raising operators. The irreducible $\mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$ representation $V_\alpha \otimes V_\beta \otimes V_\gamma$ contains exactly one HWV (up to scale), and that is of weight (α,β,γ). Hence ([8, Lemma 2.1]),

$$\mathtt{k}'(\alpha,\beta,\gamma) = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma}\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right),$$

where $\mathrm{HWV}_{\alpha,\beta,\gamma}$ denotes the space of HWVs of weight (α,β,γ). Note that each standard basis vector in $\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ is a weight vector, and hence for each weight vector space of weight w we have a basis given by the set of standard basis vectors of weight w. Let $e_{i,j,k} := e_i \otimes e_j \otimes e_k$, and for a list of points $Q \in (\mathbb{N}^3)^D$ we define $\psi_Q := e_{Q_1} \wedge e_{Q_2} \wedge \cdots \wedge e_{Q_D}$. If Q has marginals (α,β,γ), then ψ_Q has weight (α,β,γ). This immediately implies the result of Lemma 3.

Proof of Theorem 2 via contingeny arrays and highest weight vectors. Let $a := \ell(\lambda)$, $b := \ell(\mu)$, $c := \nu_1$. Let $\gamma := \nu'$, so $\ell(\gamma) = c$. We have $\lambda_1 \geq bc$ and $\mu_1 \geq ac$. Observe that $\mathtt{k}(\lambda,\mu,\nu) = \mathtt{k}'(\lambda,\mu,\gamma)$. Let $\widetilde{\lambda} = \lambda + (h)$, $\widetilde{\mu} = \mu + (h)$, $\widetilde{\gamma} = \gamma \diamond (1^h)$. We define an injective linear map φ as follows.

$$\begin{array}{rrcl} \varphi: & \bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) & \to & \bigwedge^{D+h}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^{c+h}) \\ & v & \mapsto & v \wedge e_{1,1,c+1} \wedge e_{1,1,c+2} \wedge \cdots \wedge e_{1,1,c+h} \end{array}$$

Note that φ maps vectors of weight (λ,μ,γ) to vectors of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$. It remains to show that φ maps HWVs to HWVs, and that every HWV of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$ has a preimage under φ.

We first prove that φ sends HWVs to HWVs. By construction of φ, we observe that for $1 \leq i < i' \leq a$, we have

$$(E_{i,i'},0,0)\varphi(u) = \varphi((E_{i,i'},0,0)u) = \varphi(0) = 0.$$

Analogously, $(0,E_{j,j'},0)\varphi(u) = 0$ for $1 \leq j < j' \leq b$, and $(0,0,E_{k,k'})\varphi(u) = 0$ for $1 \leq k < k' \leq c$. The remaining raising operators also vanish by construction of φ, because

$$\begin{aligned} & (0,0,E_{k,k'})(v \wedge e_{1,1,c+1} \wedge \cdots \wedge e_{1,1,c+h}) \\ = \; & v \wedge e_{1,1,c+1} \wedge \cdots \wedge \widehat{e_{1,1,c+k}} \wedge e_{1,1,c+k'} \wedge e_{1,1,c+k'} \wedge \cdots \wedge e_{1,1,c+h} = 0 \end{aligned}$$

because of the repeated factor $e_{1,1,c+k'}$. Here the $\widehat{e_{1,1,c+k}}$ means omission of that factor.

We now show that every weight vector of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$ has a preimage under φ, which finishes the proof. It is sufficient to show this for basis vectors. Let $u = \psi_P$ be a basis weight vector of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$, i.e., $Q \subseteq \mathbb{N}^3$ with marginals $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$. We apply Lemma 4 to see that $\{1\} \times \{1\} \times [c+1,c+h] \subset Q$ and $Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) = \{(1,1,i)\}$ for all $c+1 \leq i \leq c+h$. Therefore, ψ_Q has a preimage under φ, namely ψ_P, where P arises from Q by deleting all points with 3rd coordinate $> c$. $\square$

4 Proofs via symmetric functions

Here we use basic definitions and facts from symmetric function theory, see [25, 23] and will skip the definitions of SSYTs, Schur function etc.. The *multi-LR coefficients* $c^{\lambda}_{\alpha^1\cdots\alpha^k}$ are defined as

$$c^{\lambda}_{\alpha^1\cdots\alpha^k} := \langle s_\lambda, s_{\alpha_1}s_{\alpha^2}\cdots s_{\alpha^k}\rangle = \sum_{\beta^1,\beta^2,\ldots} c^{\lambda}_{\alpha^1\beta^1}c^{\beta^1}_{\alpha^2\beta^2}\cdots c^{\beta^{k-1}}_{\alpha^{k-1}\alpha^k} \tag{4.1}$$

from where it is easy to see that they count SSYTs T of shape λ and type $(\alpha^1 \diamond \alpha^2 \diamond \cdots)$, such that the reading word of each skew subtableau corresponding to the entries with values between $1+\sum_{i=1}^{r}\ell(\alpha^i)$ and $\sum_{i=1}^{r+1}\ell(\alpha^i)$ is a lattice permutation for every $r = 1,\ldots,k-1$. For example, $\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&4&4&6\\\hline 2&2&2&4&5&7\\\cline{1-6} 3&5&5&6&6\\\cline{1-5}\end{array}$ and $\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&4&4&6\\\hline 2&2&2&4&6&6\\\cline{1-6} 3&5&5&5&7\\\cline{1-5}\end{array}$ are two multi-LR tableaux of shape $\lambda = (7,6,5)$ and types $\alpha^1 = (4,3,1)$, $\alpha^2 = (3,3)$, $\alpha^3 = (3,1)$.

The Kronecker coefficient can be studied via the following two [equivalent] identities

$$s_\lambda[x\cdot y] = \textstyle\sum_{\mu,\nu} \mathtt{k}(\lambda,\mu,\nu)s_\mu(x)s_\nu(y), \quad \sum_{\lambda,\mu,\nu}\mathtt{k}(\lambda,\mu,\nu)s_\lambda(x)s_\mu(y)s_{\nu'}(z) = \prod_{i,j,k}(1+x_iy_jz_k).$$

Extracting coefficients in both gives us the following formulas via multi-LRs:

$$\mathtt{k}(\lambda,\mu,\nu) = \sum_{\sigma\in S_\ell}\operatorname{sgn}(\sigma)\sum_{\alpha^i\vdash\lambda_i-i+\sigma_i} c^{\mu}_{\alpha^1\cdots\alpha^k}c^{\nu}_{\alpha^1\cdots\alpha^k}. \tag{4.2}$$

and via 3d point configurations with given marginals:

$$\textstyle\sum_{\lambda,\mu,\nu}\mathtt{k}(\lambda,\mu,\nu)s_\lambda(x)s_\mu(y)s_{\nu'}(z) = \sum_{\alpha,\beta,\gamma}C(\alpha,\beta,\gamma)x^\alpha y^\beta z^\gamma \tag{4.3}$$

Note that this identity immediately gives the upper bound in Lemma 3 by comparing coefficients at $x^\lambda y^\mu z^{\nu'}$ on both sides. Replacing the Schurs by Weyl determinantal formula and extracting monomials gives

$$\mathtt{k}(\lambda,\mu,\nu) = \sum_{\sigma\in S_a,\,\pi\in S_b,\,\rho\in S_c}\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\rho)C(\lambda+\sigma-\mathrm{id},\mu+\pi-\mathrm{id},\nu'+\rho-\mathrm{id}). \tag{4.4}$$

where a permutation σ is interpreted as the vector $(\sigma(1),\ldots,\sigma(a))$ and $\mathrm{id} = (1,2,\ldots)$.

Proof of Theorem 2 via contingency arrays and symmetric functions. From now on we will use formula (4.4) and Lemma 4 to show that the only possible contingency arrays are the ones depicted there. Consider now $\mathtt{k}(\lambda+h,\mu+h,\nu+h)$ as in the problem, and let $\alpha = (\lambda+h), \beta = (\mu+h)$, $\gamma = (\nu+h)'$ so that $\mathtt{k}(\alpha,\beta,\gamma') = \mathtt{k}(\lambda+h,\mu+h,\nu+h)$. Let $\nu_1 = c$, $\ell(\lambda) = a$ and $\ell(\mu) = b$, so we have $\alpha_1 \geq bc+h, \beta_1 \geq ac+h$, $\gamma_i = 1$ for $i = c+1,\ldots,c+h$ and

$$\mathtt{k}(\alpha,\beta,\gamma') = \sum_{\sigma\in S_a,\,\pi\in S_b,\,\rho\in S_{c+h}}\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\rho)C(\alpha+\sigma-\mathrm{id},\beta+\pi-\mathrm{id},\gamma+\rho-\mathrm{id}). \tag{4.5}$$

In formula (4.5) we then consider $\{0,1\}$-contingency arrays Q with marginals

$$Q_{1**} := \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h, \quad Q_{*1*} := \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h,$$
$$Q_{**k} := \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \ldots, c+h.$$

Note that then we have $\sum_{k>c} Q_{**k} = h + \sum_{k=c+1}^{c+h} \rho_k - \sum_{k=c+1}^{c+h} k \leq h$, and the support of the array is in $[1,a] \times [1,b] \times [1,c+h]$, so we can apply Lemma 4 and conclude that $Q_{1,j,k} = 0$ iff $(j,k) \in [2,b] \times [c+1,c+h]$ and $Q_{i,1,k} = 0$ iff $(i,k) \in [2,a] \times [c+1,c+h]$. Thus, we must have $Q_{1**} = bc + h$, $Q_{*1*} = ac + h$ and so $\sigma_1 = \pi_1 = 1$, $\{\rho_{c+1}, \ldots, \rho_{c+h}\} = \{c+1, \ldots, c+h\}$ and for $k \in [c+1, c+h]$ we must have $Q_{i,j,k} = 0$ unless $i = j = 1$. This also forces us to have $Q_{1,1,k} = 1$ for all these k, and so $\rho_k = k$ for $k = c+1, \ldots, c+h$.

This completely determines $Q_{i,j,k}$ for $k > c$, as well as ρ_k for $k > c$, and $\rho = \bar{\rho}, (c+1), \ldots, (c+h)$ for $\bar{\rho} \in S_c$. We can thus write formula (4.5) as

$$\begin{aligned}
&\mathtt{k}(\lambda + h, \mu + h, \nu + h) \\
&\quad = \sum_{\sigma \in S_a, \pi \in S_b, \rho \in S_{c+h}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\pi) \operatorname{sgn}(\rho)\, C(\alpha + \sigma - \mathrm{id}, \beta + \pi - \mathrm{id}, \gamma + \rho - \mathrm{id}) \\
&\quad = \sum_{\sigma \in S_a, \pi \in S_b, \eta \in S_c} \operatorname{sgn}(\sigma) \operatorname{sgn}(\pi) \operatorname{sgn}(\eta)\, C(\bar{\alpha} + \sigma - \mathrm{id}, \bar{\beta} + \pi - \mathrm{id}, \bar{\gamma} + \eta - \mathrm{id}),
\end{aligned}$$

where $\bar{\alpha} = \alpha - (h) = \lambda$, $\bar{\beta} = \beta - (h) = \mu$ and $\bar{\gamma} = (\gamma_1 \ldots, \gamma_c) = \nu'$. As the last part coincides with the expression for $\mathtt{k}(\lambda, \mu, \nu)$ in (4.4), we get the desired identity. □

Proof of Theorem 2 via Littlewood-Richardson coefficients.
Let again $\ell(\lambda) = a$, $\ell(\mu) = b$ and $\nu_1 = c$.

We have that $\mathtt{k}(\lambda + h, \mu + h, \nu + h) = \mathtt{k}(\nu' \diamond (1^h), \lambda' \diamond (1^h), \mu + h)$ and we are going to apply formula (4.2) with that triple of partitions. Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \diamond (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \diamond (1^h)(\nu + h)'$. Here $\ell(\nu' \diamond (1^h)) = c + h$, so

$$\mathtt{k}(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

From the iterated definition of the multi-LR coefficients (4.1) we see that in order for the coefficients to be nonzero, we must have $\alpha^i \subset \hat{\mu}$ and $\alpha^i \subset \hat{\lambda}$. Tthen $\ell(\alpha^i) \leq \ell(\mu) = b$ and $\alpha^i_1 \leq \hat{\lambda}_1 = a$. Note that multi-LR coefficients count certain SSYTs of type $(\alpha^1 \diamond \alpha^2 \diamond \ldots \diamond \alpha^c \diamond \ldots)$ and thus in the shape $\hat{\lambda}$ the first column will have at most $\ell(\alpha^1) + \cdots + \ell(\alpha^c) \leq bc$ many entries from the first c partitions. So there are at least h boxes in the first column which need to be covered by the partitions $\alpha^{c+1}, \ldots, \alpha^{c+h}$. We then have

$$h \leq \ell(\alpha^{c+1}) + \cdots + \ell(\alpha^{c+h}) \leq |\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = \sum_{i=c+1}^{c+h} 1 - i + \sigma_i \leq h,$$

as $\sigma_{c+1} + \cdots + \sigma_{c+h} \leq c + 1 + \cdots c + h$. Thus we need to have equalities, and so

$$|\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = h, \ell(\alpha^i) = |\alpha^i|,$$

so α^i are single column partitions, possibly empty. Further, we have $\alpha^i \leq a$, $\alpha^i \subset \hat{\mu}$. As there is a multi-LR of type $(\alpha^1 \diamond \alpha^2 \cdots)$, the first row of that tableaux can only be occupied by the smallest entries of each type. So we must have

$$ac + h = \hat{\mu}_1 \leq \sum_i \alpha_1^i \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha_1^i.$$

Thus $\alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. Since $\alpha_1^i \leq 1$ by the above consideration, we must have $\alpha^i = (1)$ for all $i > c$. So $\sigma_i = i$ for $i = c+1, \ldots, c+h$. Then

$$c^{\hat{\lambda}}_{\alpha^1\alpha^2\cdots\alpha^{c+h}} = c^{\lambda'}_{\alpha^1\cdots\alpha^c} \quad \text{and} \quad c^{\hat{\mu}}_{\alpha^1\alpha^2\cdots\alpha^{c+h}} = c^{\mu}_{\alpha^1\cdots\alpha^c}.$$

We thus get that

$$\mathtt{k}(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1\alpha^2\cdots} c^{\hat{\mu}}_{\alpha^1\alpha^2\cdots}$$

$$= \sum_{\sigma \in S_c} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \nu'_i - i + \sigma_i} c^{\lambda'}_{\alpha^1\alpha^2\cdots} c^{\mu}_{\alpha^1\alpha^2\cdots} = \mathtt{k}(\nu', \lambda', \mu) = \mathtt{k}(\lambda, \mu, \nu). \quad \square$$

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Quasipartition and planar quasipartition algebras

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Abstract. The quasi-partition algebras were introduced by Daugherty and the first author in 2014, as centralizers of the symmetric group. Here we provide a more general construction using idempotents which allows us to define the half quasi-partition algebra. Our construction allows us to describe the planar analogues of these quasi-partition algebras. In this case the planar subalgebras are centralizer algebras of the quantum group $U_q(\mathfrak{sl}_2)$ and of dimensions equal to Motzkin and Riordan numbers. We use a Bratteli-like diagram to describe how the representation theories of these algebras are related.

Keywords: diagram algebras, representation theory, Motzkin and Riordan numbers

1 Introduction

The partition algebra was defined independently in the work of Martin and his coauthors [10, 11] and Jones [9] in the early 1990s as a natural generalization of centralizer algebras such as the Brauer and Temperley-Lieb algebras. It is of interest for combinatorial representation theory because it provides a dual approach to resolving some of the open combinatorial problems related to the representation theory of the symmetric group. The partition algebras are related to the Kronecker coefficients [3] and to the restriction and plethysm coefficients [12].

For integers k and n with $n \geq 2k$ and $V_n = \mathbb{C}^n$, then $V_n^{\otimes k}$ is an S_n module with the diagonal action of a permutation on the k tensors. The centralizer of this action is isomorphic to the partition algebra $\mathsf{P}_k(n)$. It is possible to understand the tensor products of permutation modules as sequences of restriction and induction [4] of the trivial S_n-module, $\mathbb{S}^{(n)}$, since we have $V_n^{\otimes k} \cong (\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n})^k\mathbb{S}^{(n)}$. Following [8], we denote the half-partition algebras as $\mathsf{P}_{k+\frac{1}{2}}(n)$. These algebras lie in between two partition algebras and are isomorphic to centralizers of the symmetric group S_{n-1} acting on $\mathrm{Res}_{S_{n-1}}^{S_n}V_n^{\otimes k}$. This defines a structure of embeddings and inclusions as

$$P_0(n) \hookrightarrow \mathsf{P}_{\frac{1}{2}}(n) \subseteq \mathsf{P}_1(n) \hookrightarrow \mathsf{P}_{\frac{3}{2}}(n) \subseteq \mathsf{P}_2(n) \hookrightarrow \cdots$$

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that makes it possible to construct the irreducible representations using what is known as the "basic construction" (see Section 4 of [8]).

The quasi-partition algebra was introduced by Daugherty and the first author [5] by considering the centralizer of the symmetric groups when they act instead on $(\mathbb{S}^{(n-1,1)})^{\otimes k}$ where $\mathbb{S}^{(n-1,1)}$ is the simple S_n-module indexed by $(n-1,1)$. It is well known that $V_n \cong \mathbb{S}^{(n-1,1)} \oplus \mathbb{S}^{(n)}$. Let proj denote the projection that maps V_n to $\mathbb{S}^{(n-1,1)}$. We have that

$$(\mathbb{S}^{(n-1,1)})^{\otimes k} \cong (\text{proj} \circ \text{Ind}_{S_{n-1}}^{S_n} \text{Res}_{S_{n-1}}^{S_n})^k \mathbb{S}^{(n)} \,.$$

This decomposition into three operations gives rise to three families of quasi-partition algebras. Notably there is a half quasi-partition algebra that is the centralizer of the symmetric group S_{n-1} when it acts on $\text{Res}_{S_{n-1}}^{S_n}(\mathbb{S}^{(n-1,1)})^{\otimes k}$ and another algebra that is the centralizer when S_n acts diagonally on $\text{Ind}_{S_{n-1}}^{S_n}\text{Res}_{S_{n-1}}^{S_n}(\mathbb{S}^{(n-1,1)})^{\otimes k} \cong (\mathbb{S}^{(n-1,1)})^{\otimes k} \otimes V_n$.

This paper develops the quasi-partition algebras both as centralizer algebras (Theorem 3.9) and as projections of the partition algebra multiplied on the left and right by an idempotent (Equation (3.4)). The main results are the construction of a tower of quasi-partition algebras (Subsection 3.3) and an explicit description of bases of the simple modules of $\mathsf{QP}_k(n)$ (Section 3.1). The tower of algebras is used to relate the dimensions of the irreducibles of these families using the inclusions and projections (Theorem 3.10).

A motivation for introducing these algebras is to gain a better understanding of the representation theory of the symmetric group. An important insight from the reference [3] is that reduced Kronecker coefficients arise as multiplicities in the restriction and induction of simple partition algebra modules. In analogy, it can be shown that the coefficients occurring in the restriction/induction of simple quasi-partition algebras are also the reduced Kronecker coefficients. This is because there is a see-saw pair which relates these coefficients:

$$\begin{tikzcd}
\mathsf{QP}_{k+\ell}(n) \arrow[dr, dash] & & S_n \times S_n \arrow[dl, dash] \\
& \left(\mathbb{S}^{(n-1,1)}\right)^{\otimes k+\ell} & \\
\mathsf{QP}_k(n) \otimes \mathsf{QP}_\ell(n) \arrow[uu, hook] \arrow[ur, dash] & & S_n \arrow[uu, hook] \arrow[ul, dash]
\end{tikzcd} .$$

This relationship implies that the reduced Kronecker coefficients, which are multiplicities of the restriction of an $S_n \times S_n$ module to S_n, are also the multiplicities of a simple $\mathsf{QP}_k(n) \otimes \mathsf{QP}_\ell(n)$ module in the restriction of a simple $\mathsf{QP}_{k+\ell}(n)$ module.

We conclude by describing the planar quasi-partition and half quasi-partition algebras. These algebras are isomorphic to centralizer algebras of the quantum group $U_q(\mathfrak{sl}_2)$ and have dimensions which are given by the Motzkin and Riordan numbers.

2 Preliminaries

The partition algebra was originally defined by Martin in [11]. All the results in this section are due to Martin and his collaborators, see [10] and references therein. For a nice survey on the partition algebra see [8].

For $k \in \mathbb{Z}_{>0}$, $x \in \mathbb{C}$, we let $\mathsf{P}_k(x)$ denote the complex vector space with bases given by all set partitions of $[k] \cup [\overline{k}] := \{1, 2, \ldots, k, \overline{1}, \overline{2}, \ldots, \overline{k}\}$. A part of a set partition is called a *block*. For a given block B, the set $B \cap [\overline{k}]$ denotes the subset of all barred elements of B are referred to as the *bottom* of B and the set $B \cap [k]$ denotes the subset of all unbarred elements of B and is referred to as the *top* of B. Notice that for a given set partition d on $2k$ elements, then $d \cap [k]$ and $d \cap [\overline{k}]$ are set partitions on k elements. We will let $\widehat{\mathsf{P}}_k$ denote the set of all set partitions of $\{1, 2, \ldots, k, \overline{1}, \overline{2}, \ldots, \overline{k}\}$.

Blocks with a single element will be referred to as *singletons*. Blocks containing at least one element from $[k]$ and one element from $[\overline{k}]$ will be called *propagating blocks*; all other blocks will be called *non-propagating blocks*.

For example,

$$d = \{\{1, 2, 4, \overline{2}, \overline{5}\}, \{3\}, \{5, 6, 7, \overline{3}, \overline{4}, \overline{6}, \overline{7}\}, \{8, \overline{8}\}, \{\overline{1}\}\},$$

is a set partition (for $k = 8$) with 5 blocks. The block $B = \{1, 2, 4, \overline{2}, \overline{5}\}$ is propagating. The block $\{3\}$ is a singleton.

A set partition in $\widehat{\mathsf{P}}_k$ can be represented by a *partition diagram* consisting of a frame with k distinguished points on the top and bottom boundaries, which we call vertices. We number the top vertices from left to right by $1, 2, \ldots, k$ and the bottom vertices similarly by $\overline{1}, \overline{2}, \ldots, \overline{k}$. We create a graph with connected components corresponding to the blocks of the set partition such that there is a path of edges between two vertices if they belong to the same block. A partition diagram is an equivalence class of graphs, where the equivalence is given by having the same connected components. In displaying the diagrams, we often omit the numbering on the vertices in the interest of keeping the diagrams less cluttered.

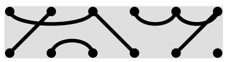
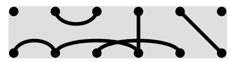
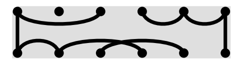
We will use the word *diagram* to refer to any element of $\widehat{\mathsf{P}}_k$ or equivalently its partition diagram. Examples of set partitions represented as diagrams are given in Example 2.1.

We define an internal product, $d_1 \cdot d_2$, of two diagrams d_1 and d_2 using the concatenation of d_1 above d_2, where we identify the bottom vertices of d_1 with the top vertices of d_2. If there are m connected components consisting only of middle vertices, then

$$d_1 \cdot d_2 = x^m d_3$$

where d_3 is the diagram with the middle vertices components removed.

Example 2.1. Consider the set partitions $d_1 = \{\{1, 3, \overline{4}\}, \{2, \overline{1}\}, \{4, 5, 6, \overline{5}\}, \{\overline{2}, \overline{3}\}, \{\overline{6}\}\}$ and $d_2 = \{\{1\}, \{2, 3\}, \{4, \overline{1}, \overline{2}, \overline{4}\}, \{5, \overline{6}\}, \{6\}, \{\overline{3}, \overline{5}\}\}$ in $\mathsf{P}_6(x)$. Which have the diagram representation given below. When we stack d_1 on top of d_2, there are two components containing only middle vertices, hence the coefficient x^2 in the product.

$d_1 =$  and $d_2 =$  then $d_1 d_2 = x^2$ .

Extending this by linearity defines a multiplication on $\mathsf{P}_k(x)$. With this product, $\mathsf{P}_k(x)$ becomes an associative algebra with unit of dimension $B(2k)$, the *Bell number* which enumerates the number of set partitions of a set with $2k$ elements.

A diagram is *planar* if the blocks of the diagram can be drawn so they do not intersect (to be clear, the blocks are not permitted to leave the bounding box). The span of the planar diagrams of size k is a subalgebra of the partition algebra $\mathsf{P}_k(x)$ which we denote $\mathsf{PP}_k(x)$. This subalgebra is of dimension equal to the Catalan number C_{2k}.

Both $\mathsf{P}_{k+1}(x)$ and $\mathsf{PP}_{k+1}(x)$ have a subalgebra spanned by the diagrams with $k+1$ and $\overline{k+1}$ are in the same block. These subalgebras will be denoted $\mathsf{P}_{k+\frac{1}{2}}(x)$ and $\mathsf{PP}_{k+\frac{1}{2}}(x)$ and have dimensions equal to $B(2k+1)$ and C_{2k+1} respectively. The planar partition algebras $\mathsf{PP}_k(x)$ and $\mathsf{PP}_{k+\frac{1}{2}}(x)$ are known to be isomorphic to the Temperley-Lieb algebras $\mathsf{TL}_{2k}(\sqrt{x})$ and $\mathsf{TL}_{2k+1}(\sqrt{x})$ respectively.

Given diagrams $d_1 \in \widehat{\mathsf{P}}_{k_1}$ and $d_2 \in \widehat{\mathsf{P}}_{k_2}$, we denote by $d_1 \otimes d_2$ the diagram in $\widehat{\mathsf{P}}_{k_1+k_2}$ obtained by placing d_2 to the right of d_1. Alternatively, in terms of set partition notation

$$d_1 \otimes d_2 = d_1 \cup \{\{b + k_1 : b \in B\} : B \in d_2\} .$$

This external product is extended linearly to a product of elements from $\mathsf{P}_{k_1}(x)$ and $\mathsf{P}_{k_2}(x)$ with the result being an element in $\mathsf{P}_{k_1+k_2}(x)$.

Let $\mathbf{1} := \{\{1, \overline{1}\}\}$ and $\mathsf{p} := \{\{1\}, \{\overline{1}\}\}$ denote special elements of $\hat{\mathsf{P}}_1$. For a fixed k, we denote the identity element of $\mathsf{P}_k(x)$ by $\mathbf{1}^{\otimes k}$ and the elements $\mathsf{p}_j := \mathbf{1}^{\otimes j-1} \otimes \mathsf{p} \otimes \mathbf{1}^{\otimes k-j} \in \hat{\mathsf{P}}_k$ for $1 \leq j \leq k$. For a complete presentation of the partition algebra see Theorem 1.11 in [8].

Let $V_n = \mathbb{C}^n$, the symmetric group acts on V_n via the permutation matrices

$$\sigma \cdot v_i = v_{\sigma(i)}, \quad \text{for } \sigma \in S_n.$$

Thus, S_n acts diagonally on a basis of simple tensors of $V_n^{\otimes k}$,

$$\sigma \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)} .$$

There is an action of $\mathsf{P}_r(x)$ on an element of $V_n^{\otimes k}$ which we do not explicitly use and so we do not state it here. Using this action, we have that for $n \geq 2k$, $k \in \mathbb{Z}_{\geq 0}$, $\mathsf{P}_k(n) \cong \mathrm{End}_{S_n}(V_n^{\otimes k})$ and for $n \geq 2k+1$ and $k \in \mathbb{Z}_{\geq 0}$, $\mathsf{P}_{k+\frac{1}{2}}(n) \cong \mathrm{End}_{S_{n-1}}(\mathrm{Res}^{S_n}_{S_{n-1}} V_n^{\otimes k})$. For details and proofs see [4, 8, 9].

The planar partition algebras (through the isomorphism with the Temperley-Lieb algebra) have a similar interpretation as centralizer of the quantized universal enveloping algebra when acting on the $2k$-fold tensor of the defining representation $V(1)^{\otimes 2k}$ [6].

3 Quasi-partition algebras

For $k \in \mathbb{Z}_{>0}$, the quasi-partition algebra $\mathsf{QP}_k(n)$ was introduced in [5] as the centralizer algebra $\mathrm{End}_{S_n}((\mathbb{S}^{(n-1,1)})^{\otimes k})$, where $\mathbb{S}^{(n-1,1)}$ is the irreducible representation of the symmetric group, S_n. In this section, we give a more general definition and introduce the half quasi-partition algebras, $\mathsf{QP}_{k+\frac{1}{2}}(x)$.

Let $k \in \mathbb{Z}_{\geq 0}$ and let J be any subset of $[k] = \{1, 2, \ldots, k\}$, we set $\mathsf{p}_\emptyset := \mathbf{1}^{\otimes k}$ and $\mathsf{p}_J := \prod_{j \in J} \mathsf{p}_j$. We define $\pi := \mathbf{1} - \frac{1}{x}\mathsf{p}$ and using tensor notation we define an idempotent $\pi^{\otimes k}$ in $\mathsf{P}_k(x)$ as follows

$$\pi^{\otimes k} := \left(\mathbf{1}^{\otimes k} - \frac{1}{x}\mathsf{p}_1\right)\left(\mathbf{1}^{\otimes k} - \frac{1}{x}\mathsf{p}_2\right)\cdots\left(\mathbf{1}^{\otimes k} - \frac{1}{x}\mathsf{p}_k\right) = \sum_{J \subseteq [k]} \frac{1}{(-x)^{|J|}}\mathsf{p}_J\,. \tag{3.1}$$

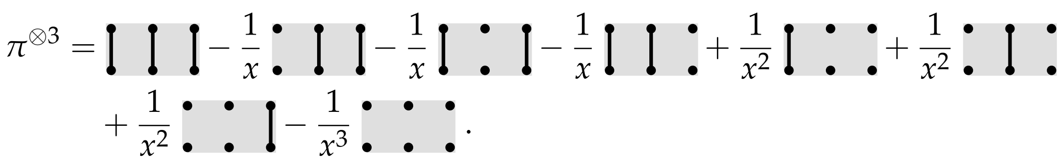
The corresponding idempotent in $\mathsf{P}_{k+\frac{1}{2}}(x) \subseteq \mathsf{P}_{k+1}(x)$ will be denoted by $\pi_{k+1}^{\otimes k} := \pi^{\otimes k} \otimes \mathbf{1}$ to indicate that it is contained in the larger algebra. For $k \in \mathbb{Z}_{\geq 0}$ and any diagram $d \in \widehat{\mathsf{P}}_k$, we define

$$\overline{d} = \pi^{\otimes k} d \pi^{\otimes k}.$$

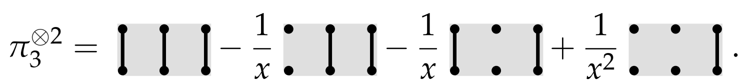
And similarly, for $d \in \widehat{\mathsf{P}}_{k+\frac{1}{2}}$, we define $\overline{d} = \pi_{k+1}^{\otimes k} d \pi_{k+1}^{\otimes k}$. For integers $k \geq 0$, and $d \in \mathsf{P}_{k+1}(x)$, we define

$$\tilde{d} = \pi_{k+1}^{\otimes k} d \pi_{k+1}^{\otimes k}\,.$$

Example 3.1. The idempotent in $\mathsf{P}_3(x)$ is $\pi^{\otimes 3} = \mathbf{1}^{\otimes 3} - \frac{1}{x}\mathsf{p}_1 - \frac{1}{x}\mathsf{p}_2 - \frac{1}{x}\mathsf{p}_3 + \frac{1}{x^2}\mathsf{p}_1\mathsf{p}_2 + \frac{1}{x^2}\mathsf{p}_1\mathsf{p}_3 + \frac{1}{x^2}\mathsf{p}_2\mathsf{p}_3 - \frac{1}{x^3}\mathsf{p}_1\mathsf{p}_2\mathsf{p}_3$. This element expressed using diagrams is



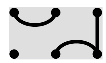
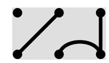
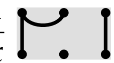
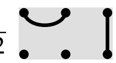
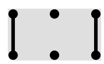
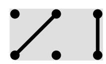
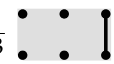
The idempotent in $\mathsf{P}_{2+\frac{1}{2}}(x)$ is $\pi_3^{\otimes 2} = \mathbf{1}^{\otimes 3} - \frac{1}{x}\mathsf{p}_1 - \frac{1}{x}\mathsf{p}_2 + \frac{1}{x^2}\mathsf{p}_1\mathsf{p}_2$ and this expression in terms of diagrams is



Lemma 3.2. *For $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ if $d \in \widehat{\mathsf{P}}_r$ is a diagram with one or more singletons, then $\overline{d} = 0$.*

Let d be a diagram without singletons, we note that $\overline{d}$ is equal to a sum of elements d plus other terms with at least one singleton.

Example 3.3. For $d = \{\{1, 2, \overline{1}\}, \{3, \overline{2}, \overline{3}\}\}$ a diagram in $\mathsf{P}_{2+\frac{1}{2}}(x)$, we compute directly:

$\overline{d} =$  $- \frac{1}{x}$  $- \frac{1}{x}$  $- \frac{1}{x}$  $- \frac{1}{x}$  $+ \frac{1}{x^2}$ 
$+ \frac{1}{x^2}$  $+ \frac{1}{x^2}$  $+ \frac{2}{x^2}$  $- \frac{2}{x^3}$  .

For $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we set $\widehat{\mathsf{D}}_r = \{d : d \in \widehat{\mathsf{P}}_r \text{ without singletons}\}$, and define

$$\mathsf{QP}_r(x) = \mathbb{C}(x)\text{-Span}\{\overline{d} \,|\, d \in \widehat{\mathsf{D}}_r\}. \tag{3.2}$$

If r is an integer, we call $\mathsf{QP}_r(x)$ the *quasi-partition algebra* and if r is half an integer, the *half quasi-partition algebra*.

Now consider the subalgebra of $\mathsf{P}_{k+1}(x)$,

$$\widetilde{\mathsf{QP}}_{k+1}(x) = \mathbb{C}(x)\text{-Span}\{\tilde{d} \,|\, d \in \mathsf{P}_{k+1}(x)\} .$$

We note that the basis of $\widetilde{\mathsf{QP}}_{k+1}(x)$ is:

$$\{\tilde{d} : d \in \widehat{\mathsf{P}}_{k+1} \text{ has no singletons in } [k] \cup [\overline{k}]\} . \tag{3.3}$$

The index set are the diagrams which have no singletons in the first k positions but that may have singletons in the last position.

Hence, the first step is the natural inclusion of $\mathsf{QP}_{k+\frac{1}{2}}(x)$ in $\widetilde{\mathsf{QP}}_{k+1}(x)$. It should be made clear that $\widetilde{\mathsf{QP}}_{k+1}(x)$ is larger than both $\mathsf{QP}_{k+\frac{1}{2}}(x)$ and $\mathsf{QP}_{k+1}(x)$ since, for instance, $\pi_{k+1}^{\otimes k}\mathsf{p}_{k+1} \in \widetilde{\mathsf{QP}}_{k+1}(x)$ but it is not an element of either $\mathsf{QP}_{k+\frac{1}{2}}(x)$ or $\mathsf{QP}_{k+1}(x)$.

Thus far we have introduced algebras in our tower so that for each $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \mathsf{QP}_k(x) &= \pi^{\otimes k}\mathsf{P}_k(x)\pi^{\otimes k} , \\ \mathsf{QP}_{k+\frac{1}{2}}(x) &= \pi_{k+1}^{\otimes k}\mathsf{P}_{k+\frac{1}{2}}(x)\pi_{k+1}^{\otimes k} , \\ \widetilde{\mathsf{QP}}_{k+1}(x) &= \pi_{k+1}^{\otimes k}\mathsf{P}_{k+1}(x)\pi_{k+1}^{\otimes k} . \end{aligned} \tag{3.4}$$

The second step is to explain how they are related. There is a projection from $\widetilde{\mathsf{QP}}_{k+1}(x)$ to $\mathsf{QP}_{k+1}(x)$ which, for each $d \in \widehat{\mathsf{P}}_{k+1}$, $\tilde{d} \in \widetilde{\mathsf{QP}}_{k+1}(x)$ is sent to $\overline{d} = (\mathbf{1}^{\otimes k+1} - \frac{1}{x}\mathsf{p}_{k+1})\tilde{d}(\mathbf{1}^{\otimes k+1} - \frac{1}{x}\mathsf{p}_{k+1}) \in \mathsf{QP}_{k+1}(x)$.

Therefore we have the following chain of inclusions and projections:

$$\mathsf{QP}_0(x) \hookrightarrow \mathsf{QP}_{\frac{1}{2}}(x) \subseteq \widetilde{\mathsf{QP}}_1(x) \twoheadrightarrow \mathsf{QP}_1(x) \hookrightarrow \mathsf{QP}_{1+\frac{1}{2}}(x) \subseteq \widetilde{\mathsf{QP}}_2(x) \twoheadrightarrow \mathsf{QP}_2(x) \hookrightarrow \cdots . \tag{3.5}$$

The dimensions of these algebras are determined by counting the elements in Equations (3.2) and (3.3). The dimension of $\mathsf{QP}_k(x)$, $\dim(\mathsf{QP}_k(x))$, is equal to the number of set partitions of $[k] \cup [\overline{k}]$ without blocks of size one and is equal to (see [5])

$$\dim(\mathsf{QP}_k(x)) = \sum_{j=1}^{2k} (-1)^{j-1} B(2k-j) + 1$$

and are every other term in [15] sequence A000296, while $\dim(\mathsf{QP}_{k+\frac{1}{2}}(x)) = B(2k)$ and is every other term in [15] sequence A000110. The sequence of dimensions of $\widetilde{\mathsf{QP}}_{k+1}(x)$ is given by [15] sequence A207978. Using the standard counting technique of inclusion-exclusion we deduce that

$$\dim(\widetilde{\mathsf{QP}}_{k+1}(x)) = \sum_{s=0}^{2k}(-1)^s\binom{2k}{s}B(2k+2-s)\ .$$

Example 3.4. The sequence of dimensions of the algebras for $0 \le k \le 6$ is given in the table below.

k	0	1	2	3	4	5	6
$\dim(\mathsf{QP}_k(x))$	1	1	4	41	715	17722	580317
$\dim(\mathsf{QP}_{k+\frac{1}{2}}(x))$	1	2	15	203	4140	115975	4213597
$\dim(\widetilde{\mathsf{QP}}_{k+1}(x))$	2	7	67	1080	25287	794545	31858034

3.1 Representations of quasi-partition algebras

For this section, let $x = n \in \mathbb{Z}_{\ge 0}$ with $n \ge 2k$.

Recall that a block B is called *propagating* if it contains at least one element from each $[k]$ and $[\overline{k}]$. Define $V(k, m)$ to be the vector space spanned by the diagrams corresponding to set partitions of $[k] \cup [\overline{k}]$ with $\overline{m+1}, \ldots, \overline{k}$ in singleton blocks, and all other $\overline{j}$ are in propagating blocks where $\overline{j}$ is the only barred element in its block. We call these *(k,m)-diagrams*. For a diagram $d \in \hat{\mathsf{P}}_k$, let $p(d)$ denote the number of propagating blocks. A (k,m)-diagram is called *(k,m)-standard* if its propagating blocks $B_{\overline{1}}, \ldots, B_{\overline{m}}$ satisfy $\max(B_{\overline{j-1}} \cap [k]) < \max(B_{\overline{j}} \cap [k])$ for all $1 \le j \le m$.

For $0 \le m \le k$ and $\nu \vdash m$, a basis of the simple $\mathsf{P}_k(x)$ module $\Delta_k(\nu)$ is defined by

$$\mathcal{B}_k(\nu) = \{d \otimes T \mid d \text{ is a } (k,m)\text{-standard and } T \text{ is a standard tableau of shape } \nu\}. \quad (3.6)$$

A diagram $d \in \mathsf{P}_k(x)$ acts on a basis element $d' \otimes T$ of $\Delta_k(\nu)$ by left multiplication,

$$d \cdot d' \otimes T = \begin{cases} dd' \otimes T & \text{if } p(dd') = m \\ 0 & \text{otherwise} \end{cases}, \quad (3.7)$$

in the case that $p(dd') = m$, we may factor $dd' = x^a d_1 \tau$ where d_1 is a (k,m)-standard diagram and $\tau \in S_m$. Hence, $dd' \otimes T = x^a d_1 \otimes \tau \cdot T$, where τ acts on T by permuting the entries of the tableau and $\tau \cdot T$ might not be standard, but can be written as a linear combination of standard tableaux using the Garnir straightening algorithm for Specht modules (see for instance [14]).

Following [7] the elements of $\mathcal{B}_k(\nu)$ can be combined into a single object that is represented by a set valued tableau.

Definition 3.5. For $k \in \mathbb{Z}_{\geq 0}$, $n \geq 2k$ and $0 \leq i \leq k$, let λ be a partition of n, a *$[k]$-set valued tableau T of shape λ* satisfies the following conditions:

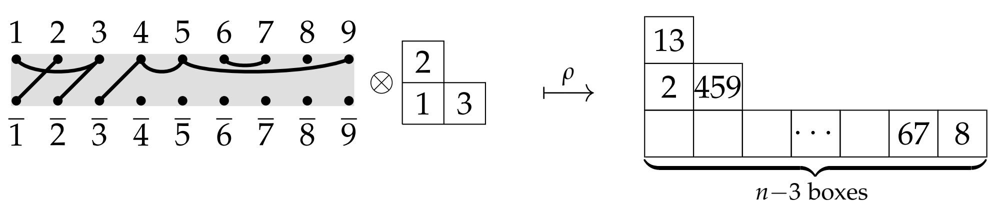
1. The sets filling the boxes of the Young diagram of λ form a set partition α of $[k]$, the sets in α are called blocks.

2. Every box in rows $\lambda_2, \ldots, \lambda_\ell$ is filled with a block in α.

3. Boxes at the end of the first row of λ could contain blocks of α and, because of the condition that $n \geq 2k$, there are at least k empty boxes preceding the boxes containing sets.

Let $\mathcal{T}_k(\lambda)$ denote the set of all $[k]$-set valued tableaux of shape λ.

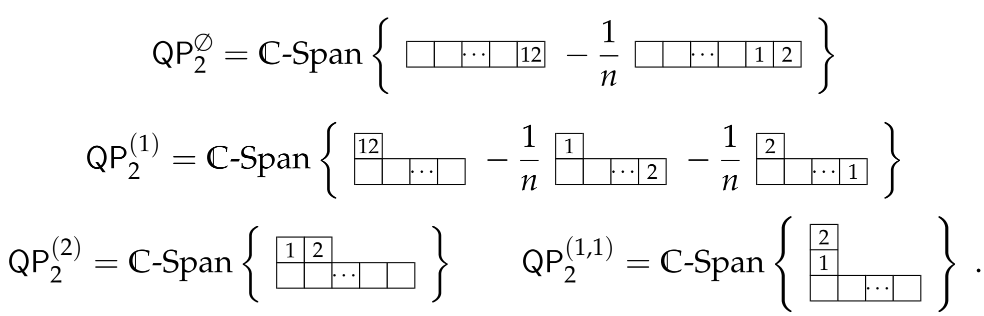
Example 3.6. Correspondence between a basis element $d \otimes T \in \Delta_9((2,1))$ and a $[9]$-set valued tableau of shape $(n-3,2,1)$.



We define $\mathcal{Q}_k(\nu)$ to be the set of nonzero $\pi^{\otimes k} d \otimes T$ (that is, d has no singletons in the top row), for $d \otimes T \in \mathcal{B}_k(\nu)$. Define QP_k^ν to be the $\mathbb{C}(x)$-Span of the elements in $\mathcal{Q}_k(\nu)$ for every $\nu \vdash m$ and $0 \leq m \leq k$.

Theorem 3.7. *Let $k \in \mathbb{Z}_{\geq 0}$, the set $\{\mathsf{QP}_k^\nu \mid \nu \vdash m \text{ where } 0 \leq m \leq k\}$ forms a complete set of mutually non-isomorphic simple modules for $\mathsf{QP}_k(x)$.*

Example 3.8. For $k = 2$ there are four simple modules of $\mathsf{QP}_2(n)$ and all are of dimension one and we display them using the correspondence with set valued tableaux:



In [13] we give a similar description of the simple modules of the half partition algebras. From that construction, it is possible to give a similar description of the simple modules of $\mathsf{QP}_{k+\frac{1}{2}}(x)$. However we will see in the next section that $\mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathsf{P}_k(n-1)$.

3.2 Quasi-partition algebras as centralizers

Let $V_n = \mathbb{C}\text{-Span}\{v_1, v_2, \ldots, v_n\}$, then it is well known that $\mathbb{S}_{S_n}^{(n-1,1)} \cong \mathbb{C}\text{-Span}\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}$ and $\mathbb{S}_{S_n}^{(n)} \cong \mathbb{C}\text{-Span}\{v_1 + v_2 + \cdots + v_n\}$ and that $V_n \cong \mathbb{S}_{S_n}^{(n-1,1)} \oplus \mathbb{S}_{S_n}^{(n)}$ as an S_n-module.

We refer the reader to [1, 2, 4] for the action of the elements $\mathsf{P}_k(n)$ when it acts on $V_n^{\otimes k}$. This action realizes the partition algebra as a centralizer algebra $\mathsf{P}_k(n) \cong \mathrm{End}_{S_n}(V_n^{\otimes k})$. In this section we state the corresponding realizations of the quasi-partition algebras as centralizer algebras.

Theorem 3.9. *For $n, k \in \mathbb{Z}_{>0}$, if $n \geq 2k$, then*

$$\mathsf{QP}_k(n) \cong \mathrm{End}_{S_n}\left(\left(\mathbb{S}_{S_n}^{(n-1,1)}\right)^{\otimes k}\right), \qquad \mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathrm{End}_{S_{n-1}}\left(\mathrm{Res}_{S_{n-1}}^{S_n}\left(\mathbb{S}_{S_n}^{(n-1,1)}\right)^{\otimes k}\right),$$

$$\textit{and} \qquad \widetilde{\mathsf{QP}}_{k+1}(n) \cong \mathrm{End}_{S_n}\left(\left(\mathbb{S}_{S_n}^{(n-1,1)}\right)^{\otimes k} \otimes V_n\right).$$

Since we have that

$$\mathrm{Res}_{S_{n-1}}^{S_n}\mathbb{S}_{S_n}^{(n-1,1)} \cong \mathbb{S}_{S_{n-1}}^{(n-2,1)} \oplus \mathbb{S}_{S_{n-1}}^{(n-1)} \cong V_{n-1}$$

it follows that $\mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathsf{P}_k(n-1)$.

3.3 Dimensions of irreducible modules and a Bratteli diagram

The interpretation of the quasi-partition algebras as centralizer algebras allows us to relate the dimensions of the irreducibles in the following recursive formulae.

Theorem 3.10. *Let $n \geq 2k+1$, then for $\mu \vdash n-1$ such that $|\overline{\mu}| < k$, then*

$$\dim\left(\mathsf{QP}^{\mu}_{k+\frac{1}{2}}(n)\right) = \sum_{\lambda \leftarrow \mu} \dim(\mathsf{QP}^{\lambda}_k(n)), \quad \dim(\widetilde{\mathsf{QP}}^{\lambda}_k(n)) = \sum_{\mu \to \lambda} \dim\left(\mathsf{QP}^{\mu}_{k+\frac{1}{2}}(n)\right), \tag{3.8}$$

$$\dim(\mathsf{QP}^{\lambda}_k(n)) = \dim(\widetilde{\mathsf{QP}}^{\lambda}_k(n)) - \dim(\mathsf{QP}^{\lambda}_{k-1}(n)). \tag{3.9}$$

Each row of the diagram on the left in Figure 1 displays partitions $\overline{\lambda}$ where λ is in the index set of the irreducible representations of the chain algebras from Theorem 3.10. The irreducible representations of $\mathsf{QP}_k(n)$ are displayed in red, $\mathsf{QP}_{k+\frac{1}{2}}(n)$ are displayed in blue, $\widetilde{\mathsf{QP}}_{k+1}(n)$ are displayed in green.

Let $\lambda \to \mu$ represent the relation that λ is obtained from μ by removing a cell. The relations between the irreducibles in the rows of the diagram are summarized as follows:

- (Equation (3.8)) Between the $\mathsf{QP}_k(n)$ and $\mathsf{QP}_{k+\frac{1}{2}}(n)$ rows there is an edge from $\overline{\lambda}$ to $\overline{\mu}$ if $\overline{\mu} = \overline{\lambda}$ or $\overline{\mu} \to \overline{\lambda}$ (alternatively, if $\mu \to \lambda$).
- (Equation (3.8)) Between the $\mathsf{QP}_{k+\frac{1}{2}}(n)$ and $\widetilde{\mathsf{QP}}_{k+1}(n)$ rows there is an edge from $\overline{\mu}$ to $\overline{\lambda}$ if $\overline{\lambda} = \overline{\mu}$ or $\overline{\lambda} \leftarrow \overline{\mu}$ (alternatively, these two conditions may be stated as 'if $\lambda \leftarrow \mu$').
- (Equation (3.9)) Between the $\widetilde{\mathsf{QP}}_{k+1}(n)$ and $\mathsf{QP}_k(n)$ rows there is an edge from $\overline{\lambda}$ to $\overline{\lambda}$ but the dimension of the irreducible $\overline{\lambda}$ is equal to the dimension of the irreducible $\overline{\lambda}$ minus the dimension of the irreducible $\overline{\lambda}$ at $k-1$.

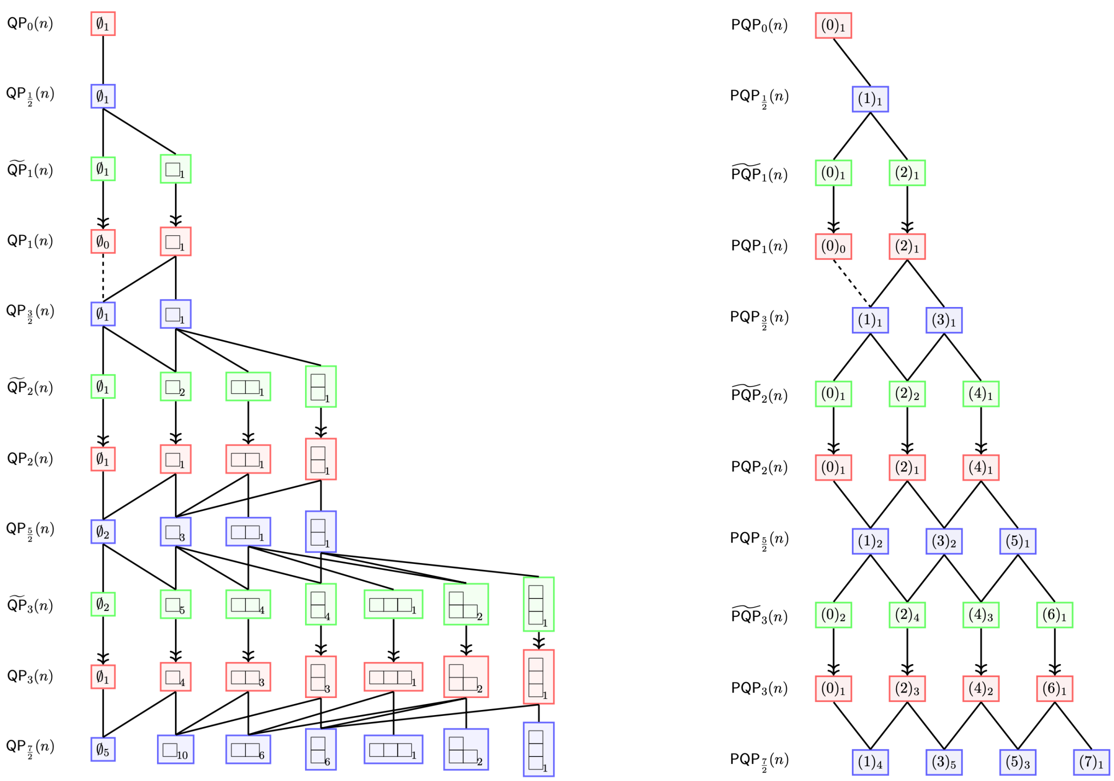


Figure 1: On the left, a Bratteli like-diagram showing the relations of the tower of algebras in Equation (3.5); and on the right the corresponding diagram for the planar counterpart. The subscripts within the colored boxes indicate the dimensions of the irreducibles.

The diagram is similar to a Bratteli diagram except that, because of the projection operation from $\widetilde{\mathsf{QP}}_{k+1}(n)$ to $\mathsf{QP}_{k+1}(n)$, the dimension is no longer the number of paths in the diagram and is instead something slightly more complex.

4 Planar quasi-partition algebras

We now proceed to develop a similar construction to the quasi-partition algebra by considering a subalgebra of the planar partition algebra. Due to space considerations and that we have provided details on the quasi-partition algebras already, our presentation of these algebras here will be briefer, but analogues of the results for the quasi-partition algebras in this setting can be shown using similar methods.

We define three subalgebras of $\mathsf{PP}_r(x)$ for $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} \mathsf{PQP}_k(x) &= \pi^{\otimes k}\mathsf{PP}_k(x)\pi^{\otimes k}\,, \\ \mathsf{PQP}_{k+\frac{1}{2}}(x) &= \pi^{\otimes k}_{k+1}\mathsf{PP}_{k+\frac{1}{2}}(x)\pi^{\otimes k}_{k+1}\,, \\ \widetilde{\mathsf{PQP}}_{k+1}(x) &= \pi^{\otimes k}_{k+1}\mathsf{PP}_{k+1}(x)\pi^{\otimes k}_{k+1}\,. \end{aligned} \tag{4.1}$$

Example 4.1. The sequence of dimensions of $\mathsf{PQP}_k(x)$, $\mathsf{PQP}_{k+\frac{1}{2}}(x)$ and $\widetilde{\mathsf{PQP}}_{k+1}(x)$ for $0 \leq k \leq 13$ is

k	0	1	2	3	4	5	6
$\dim(\mathsf{PQP}_k(x))$	1	1	3	15	91	603	4213
$\dim(\mathsf{PQP}_{k+\frac{1}{2}}(x))$	1	2	9	51	323	2188	15511
$\dim(\widetilde{\mathsf{PQP}}_{k+1}(x))$	2	6	30	178	1158	7986	57346

The first row of this table is given by [15] sequence A099251 and is every other term of the Riordan numbers (A005043). The second row of this table is given by [15] sequence A026945 which is every other term of the Motzkin numbers (A001006). The third row of this table is every other term in the [15] sequence A005554 which are a sum of two successive Motzkin numbers.

The planar partition algebra $\mathsf{PP}_k(x)$ is isomorphic to the Temperley-Lieb algebra $TL_{2k}(\sqrt{x})$. For $q \in \mathbb{C}$, let $U_q(\mathfrak{sl}_2)$ denote the quantum group of the Lie algebra $\mathfrak{sl}_2$ and recall that its simple modules are classically denoted by $V(i)$, where i is a nonnegative integer. For example, $V(0)$ is the trivial representation, $V(1) \cong \mathbb{C}^2$ and $V(2)$ is the adjoint representation. It is well known that $TL_k \cong \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V(1)^{\otimes k})$ see [6] for more details. Using well known tensor rules, we have that $\mathbb{V} := V(1)^{\otimes 2} \cong V(0) \oplus V(2)$.

Theorem 4.2. *Let r be a nonzero integer and $0 \neq q \in \mathbb{C}$ is not a root of unity, and set $\mathbb{V} = V(0) \oplus V(2)$, then we have the following*

$$PQP_r((q+q^{-1})^2) \cong End_{U_q(\mathfrak{sl}_2)}(V(2)^{\otimes r}),$$

$$PQP_{r+\frac{1}{2}}((q+q^{-1})^2) \cong End_{U_q(\mathfrak{sl}_2)}(V(2)^{\otimes r} \otimes V(1)),$$

and

$$\widetilde{PQP}_{r+1}((q+q^{-1})^2) \cong End_{U_q(\mathfrak{sl}_2)}(V(2)^{\otimes r} \otimes \mathbb{V}).$$

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Invariant theory for the face algebra of the braid arrangement

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Abstract. The faces of the braid arrangement form a monoid. The associated monoid algebra – the face algebra– is well-studied, especially in relation to card shuffling and other Markov chains. In this abstract, we explore the action of the symmetric group on the face algebra from the perspective of invariant theory. Bidigare proved the invariant subalgebra of the face algebra is (anti)isomorphic to Solomon's descent algebra. We answer the more general question: what is the structure of the face algebra as a simultaneous representation of the symmetric group and Solomon's descent algebra?

Special cases of our main theorem recover the Cartan invariants of Solomon's descent algebra discovered by Garsia–Reutenauer and work of Uyemura-Reyes on certain shuffling representations. Our proof techniques involve the homology of intervals in the lattice of set partitions.

Keywords: descent algebra, higher Lie characters, plethysm, finite dimensional algebras, poset topology, reflection arrangements

1 Background

1.1 The braid arrangement and its face algebra

Write $\mathbf{x} := (x_1, x_2, \cdots, x_n)$ to denote an element of the vector space $\mathbb{R}^n$. The *braid arrangement $\mathcal{B}_n$* is the hyperplane arrangement in $\mathbb{R}^n$ consisting of the hyperplanes $\{\mathbf{x} : x_i = x_j\}$ for all $1 \leq i < j \leq n$. Each hyperplane $\{\mathbf{x} : x_i = x_j\}$ partitions $\mathbb{R}^n$ into three subsets: the halfspace $H_{ij}^+ = \{\mathbf{x} : x_i > x_j\}$, the halfspace $H_{ij}^- = \{\mathbf{x} : x_i < x_j\}$, and the hyperplane itself H_{ij}^0. The *faces* of $\mathcal{B}_n$ are the nonempty intersections of the form

$$\bigcap_{1\leq i<j\leq n} H_{ij}^{\mathrm{sgn}_{ij}}$$

for some set of choices $\mathrm{sgn}_{ij} \in \{+, -, 0\}$.

The faces of $\mathcal{B}_n$ naturally correspond to strings of inequalities relating all coordinates. For example, one face F of $\mathcal{B}_7$ corresponds to the string $x_4 < x_1 = x_5 < x_7 < x_2 = x_3 =$

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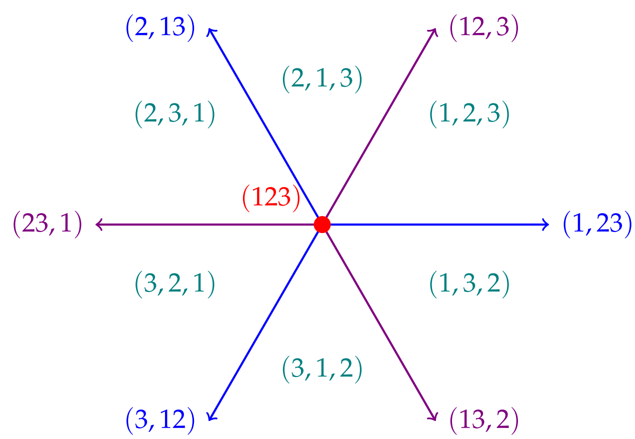
x_6. Combinatorially, these strings (and their corresponding faces) are *ordered set partitions* of the set $[n] := \{1, 2, \cdots, n\}$. For example, F is labelled by the ordered set partition $(\{4\}, \{1,5\}, \{7\}, \{2,3,6\})$, which we write as $(4, 15, 7, 236)$. The symmetric group S_n acts on the faces of B_n by $\pi(P_1, P_2, \cdots, P_k) := (\pi(P_1), \pi(P_2), \cdots, \pi(P_k))$.

Example 1. *The braid arrangement $\mathcal{B}_3$ (intersected with the plane $x_1 + x_2 + x_3 = 0$) is shown below. The colors point out the four S_3-orbits of faces.*



The faces of $\mathcal{B}_n$ have an associative multiplicative structure. This product was first considered by Tits in [22]. In terms of ordered set partitions,

$$(P_1, P_2, \cdots, P_k) \cdot (Q_1, Q_2, \cdots, Q_\ell) := (P_1 \cap Q_1, P_1 \cap Q_2, \cdots, P_1 \cap Q_\ell, P_2 \cap Q_1, \cdots P_k \cap Q_\ell)^\wedge,$$

where $\wedge$ indicates the removal of empty sets. For example, in $\mathcal{B}_7$,

$$(4, 15, 7, 236) \cdot (245, 367, 1) = (4, 5, 1, 7, 2, 36).$$

The ordered set partition with a single block $(12 \cdots n)$ is an identity element, so the faces form a monoid, which we denote by $\mathcal{F}_n$. We are primarily interested in the *face algebra* $\mathbb{C}\mathcal{F}_n$ which is the free $\mathbb{C}-$module with basis $\mathcal{F}_n$ and multiplication

$$\left(\sum_{F \in \mathcal{F}_n} c_F F\right) \cdot \left(\sum_{G \in \mathcal{F}_n} d_G G\right) := \sum_{F, G \in \mathcal{F}_n} c_F d_G F \cdot G.$$

It is straightforward to check the symmetric group action on $\mathcal{F}_n$ is by monoid homomorphisms. Hence, S_n acts on $\mathbb{C}\mathcal{F}_n$ by algebra homomorphisms. The structure of $\mathbb{C}\mathcal{F}_n$ as an S_n-representation is also simple to check. Throughout this abstract, let ch denote the *Frobenius characteristic map* from characters of symmetric groups to the ring of symmetric functions. We write $\alpha \vDash n$ if $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is an *integer composition* of n (a sequence of positive integers summing to n). Using $h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_k}$ for h_i the complete homogeneous symmetric function of degree i, we have $\mathrm{ch}(\mathbb{C}\mathcal{F}_n) = \sum_{\alpha \vDash n} h_\alpha$.

In [5], Bidigare–Hanlon–Rockmore discovered that the face algebra has rich connections to card shuffling and other Markov chains. These connections were studied further by many others, including Uyemura-Reyes in [23] and Reiner–Saliola–Welker in [13]. In addition, the face algebra has been studied as an interesting algebra in its own right; for instance, see work of Bidigare in [6], Saliola in [14, 15], Aguiar–Mahajan in [1], and Schocker in [16].

1.2 Solomon's descent algebra and Bidigare's theorem

Each permutation $\pi \in S_n$ has an associated (right) *descent set* $\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\} \subseteq [n-1]$. For each subset $J \subseteq [n-1]$, define an element $\mathbf{x}_J$ in the group algebra $\mathbb{C}S_n$ by

$$\mathbf{x_J} := \sum_{\pi:\mathrm{Des}(\pi)\subseteq J} \pi.$$

In [17], Solomon proved that the $\mathbb{C}$- span of the elements $\{x_J : J \subseteq [n-1]\}$ is closed under multiplication, so it is a subalgebra of $\mathbb{C}S_n$. This subalgebra is known as *Solomon's descent algebra*, which we will denote by Σ_n. The descent algebra is intimately linked to the face algebra. For a subset $J = \{a_1 < a_2 < \cdots < a_k\} \subseteq [n-1]$, write $\alpha(J)$ to be the integer composition $(a_1, a_2 - a_1, a_3 - a_2, \cdots, a_k - a_{k-1}, n - a_k)$. Bidigare proved the following connection in [6, Theorem 3.8.1].

Theorem 2. *(Bidigare) The S_n-invariant subalgebra of the face algebra is antiisomorphic to Solomon's descent algebra via the map*

$$\Phi : x_J \mapsto \sum_{\substack{\text{Faces } F\\ \text{with block}\\ \text{sizes } \alpha(J)}} F.$$

Example 3. *Using one-line notation, the element $x_{\{1\}} = 1 + 21 + 312 \in \Sigma_3$ is mapped under Bidigare's antiisomorphism to the sum of the three rays colored blue in Example 1.*

2 Our question

By Maschke's theorem, the face algebra $\mathbb{C}\mathcal{F}_n$ decomposes into a direct sum of irreducible S_n-representations. Although this decomposition is not unique, the sums of irreducibles of the same isomorphism type, called the *isotypic subspaces*, are. The irreducible representations of S_n are indexed by partitions ν of n, written $\nu \vdash n$. Hence, there is an S_n-representation decomposition

$$\mathbb{C}\mathcal{F}_n = \bigoplus_{\nu \vdash n} (\mathbb{C}\mathcal{F}_n)^{\nu},$$

where $(\mathbb{C}\mathcal{F}_n)^\nu$ is the S_n-isotypic subspace associated to the irreducible labelled by ν.

The trivial isotypic subspace $(\mathbb{C}\mathcal{F}_n)^{(n)}$ is precisely the invariant subalgebra $(\mathbb{C}\mathcal{F}_n)^{S_n}$. So, it is a natural extension of Bidigare's theorem to consider what the other S_n-isotypic subspaces look like. In fact, in [6, §3.5.3], Bidigare studied the *sign isotypic subspace* $(\mathbb{C}\mathcal{F}_n)^{1^n}$, which he proved is a one-dimensional nilpotent subalgebra of $\mathbb{C}\mathcal{F}_n$.

Moreover, the isotypic subspaces are not only S_n-representations; each carries an additional, rich structure as a left module over $(\mathbb{C}\mathcal{F}_n)^{S_n}$. Hence, by Theorem 2, each isotypic subspace is actually a (right) module over the descent algebra Σ_n by the action

$$f \cdot x := \Phi(x) f \text{ for } f \in (\mathbb{C}\mathcal{F}_n)^\nu, x \in \Sigma_n.$$

This brings us to our main question.

Question 4. *What is the structure of each S_n-isotypic subspace $(\mathbb{C}\mathcal{F}_n)^\nu$ as a Σ_n-module?*

We will answer Question 4 with Theorem 7. Specifically, we will reduce Question 4 to understanding specific symmetric group representations, which we analyze up to longstanding open problems. To explain this problem conversion and our answer, we must first say a bit about the representation theory of the descent algebra.

2.1 Representation theory of Solomon's descent algebra

The (right) representation theory of the descent algebra has been studied in great depth by Garsia and Reutenauer in [9]. Although Σ_n is not semisimple, its representation theory is still quite nice. The simple Σ_n-modules are all one-dimensional and are indexed by integer partitions of n. Let M_λ denote the Σ_n-simple associated to the partition λ. From the theory of finite dimensional algebras, we have that as Σ_n-modules,

$$\Sigma_n \cong \bigoplus_{\lambda \vdash n} P_\lambda,$$

where P_λ is the projective indecomposable Σ_n-module with top M_λ.

Any complete family of primitive orthogonal idempotents *(cfpoi)* for the descent algebra Σ_n is necessarily indexed by integer partitions of n too. For notational convenience, we write $\{E_\lambda : \lambda \vdash n\}$ to denote the images of such idempotents under the Bidigare antiisomorphism Φ (so in $(\mathbb{C}\mathcal{F}_n)^{S_n}$ rather than Σ_n). We choose the indexing appropriately so that $P_\lambda \cong \Phi^{-1}(E_\lambda)\Sigma_n \cong (\mathbb{C}\mathcal{F}_n)^{S_n} E_\lambda$ as right Σ_n-modules.

In a similar fashion, any cfpoi for the face algebra $\mathbb{C}\mathcal{F}_n$ is indexed by (unordered) set partitions of $[n]$. We write Π_n to denote the set partition lattice ordered under refinement and say a set partition $X \in \lambda$ if it has block sizes λ. In [14], Saliola constructed cfpois $\{E_X : X \in \Pi_n\}$ for $\mathbb{C}\mathcal{F}_n$ for which $\pi(E_X) = E_{\pi(X)}$ for $\pi \in S_n$. He proved the orbit sums of such families, $\{\sum_{X \in \lambda} E_X : \lambda \vdash n\}$, form cfpois for $(\mathbb{C}\mathcal{F}_n)^{S_n}$.

Any two cfpois for the invariant subalgebra $(\mathbb{C}\mathcal{F}_n)^{S_n}$ are conjugate by an invertible element of $(\mathbb{C}\mathcal{F}_n)^{S_n}$ (see [1, Lemma D.26]). By conjugating Saliola's idempotents[1], any cfpoi for $(\mathbb{C}\mathcal{F}_n)^{S_n}$ can be written as the S_n-orbit sums of some cfpoi for $\mathbb{C}\mathcal{F}_n$ permuted by S_n. For these reasons, our choice of a cfpoi for $(\mathbb{C}\mathcal{F}_n)^{S_n}$ turns out to not matter. For the remainder of this abstract, let $\{E_\lambda : \lambda \vdash n\}$ be a cfpoi for $(\mathbb{C}\mathcal{F}_n)^{S_n}$ and let $\{E_X : X \in \Pi_n\}$ be a cfpoi for $\mathbb{C}\mathcal{F}_n$ which is permuted by S_n and has orbit sums $\{E_\lambda\}$.

2.2 Problem conversion

As a first step towards answering Question 4, we decompose each S_n-isotypic subspace $(\mathbb{C}\mathcal{F}_n)^\nu$ into a direct sum of smaller Σ_n-modules. Write f^ν to denote the number of standard Young tableaux of shape ν, write $\alpha \sim \mu$ if a composition α rearranges to a partition μ, and write $K_{\nu,\mu}$ to denote the *Kostka number* which counts the number of semistandard Young tableaux of shape ν and content μ.

Proposition 1. *As (right) Σ_n-modules,*

$$(\mathbb{C}\mathcal{F}_n)^\nu = \bigoplus_{\mu \vdash n} (\mathbb{C}\mathcal{F}_n E_\mu)^\nu, \text{ and}$$

$$\dim_{\mathbb{C}} (\mathbb{C}\mathcal{F}_n E_\mu)^\nu = f^\nu \cdot \#\{\alpha \vDash n \mid \alpha \sim \mu\} \cdot K_{\nu,\mu}.$$

Proposition 1 reduces Question 4 to understanding each Σ_n-module $\left(\mathbb{C}\mathcal{F}_n E_\mu\right)^\nu$ for any two partitions μ, ν of n. Since Σ_n is not semisimple, we are unable in general to decompose each Σ_n-module $\left(\mathbb{C}\mathcal{F}_n E_\mu\right)^\nu$ into a direct sum of simples. However, by the Jordan-Hölder theorem, we can take the alternative approach of understanding the Σ_n-*composition factors* of each Σ_n-module $\left(\mathbb{C}\mathcal{F}_n E_\mu\right)^\nu$. The number of times a Σ_n-simple M_λ appears as a composition factor of a Σ_n-module V is the *composition multiplicity* of M_λ in V, written $[V : M_\lambda]$. Thus, we have converted Question 4 to the following question.

Question 5. *For partitions μ, ν, λ of n, what is the composition multiplicity $[(\mathbb{C}\mathcal{F}_n E_\mu)^\nu : M_\lambda]$?*

The proposition below follows from the theory of finite dimensional algebras.

Proposition 2. *The composition multiplicity of the Σ_n- simple M_λ in $\left(\mathbb{C}\mathcal{F}_n E_\mu\right)^\nu$ is*

$$\left[(\mathbb{C}\mathcal{F}_n E_\mu)^\nu : M_\lambda\right] = f^\nu \cdot \left\langle s_\nu, \mathrm{ch}\left(E_\lambda \mathbb{C}\mathcal{F}_n E_\mu\right)\right\rangle,$$

where s_ν is the Schur function associated to the partition ν and $\langle \cdot, \cdot \rangle$ is the Hall inner product.

Hence, our final conversion of Question 4 is the question below.

Question 6. *What is the S_n-representation theoretic structure of $E_\lambda \mathbb{C}\mathcal{F}_n E_\mu$?*

[1] Aguiar and Mahajan further study and characterize such idempotents very thoroughly in [1, §16.8].

3 Our answer

Thrall studied a collection of S_n-representations in [21] which are (also) indexed by partitions of n and often called the *higher Lie representations*. We write L_λ to denote the Frobenius image of the higher Lie representation associated to λ. These representations have many interpretations and are closely tied to the free Lie algebra. For our purposes, it is most revealing to define L_n as the Frobenius image of the S_n-representation carried by the top homology of (the proper part of) the set partition lattice Π_n tensored with the sign representation[2]. More generally, for a partition $\lambda = 1^{m_1}2^{m_2}\cdots k^{m_k}$, let

$$L_\lambda := \prod_{i=1}^{k} h_{m_i}[L_i],$$

where the brackets denote *plethysm*. Positively expanding L_λ into Schur functions is a longstanding open problem, known as *Thrall's problem*.

A *Lyndon word* is a nonempty finite word on $\{1,2,\cdots\}$ that is lexicographically *strictly* smaller than all of its cyclic rearrangements. For an integer composition, partition, or word $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ on $\{1,2,\cdots\}$, we write $|\alpha|$ to denote the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_k$. For any infinite variable set $\mathbf{y} = \{y_1, y_2, \cdots\}$, let $\mathbf{y}_\alpha$ denote the product $\mathbf{y}_\alpha := y_{\alpha_1} y_{\alpha_2} \cdots y_{\alpha_k}$. The scaling of α by an integer m is $\alpha \cdot m := (\alpha_1 \cdot m, \alpha_2 \cdot m, \cdots, \alpha_k \cdot m)$ and raising α to an integer, α^m, means repeated concatenation of α.

We now have the necessary definitions to state our main theorem.

Theorem 7. *There is an equality of generating functions*

$$\sum_{n\geq 0} \sum_{\substack{\lambda \vdash n \\ \mu \vdash n}} \mathbf{y}_\lambda \mathbf{z}_\mu \cdot \mathrm{ch}(E_\lambda \mathbb{C}\mathcal{F}_n E_\mu) = \prod_{\substack{\text{Lyndon} \\ w}} \sum_{\substack{\text{partition} \\ \rho}} \mathbf{y}_{\rho\cdot|w|} \mathbf{z}_{w^{|\rho|}} L_\rho[h_w]. \tag{3.1}$$

Let F be the generating function on the right side of Equation (3.1). Theorem 7 explains the structure of $\mathbb{C}\mathcal{F}_n$ as a module over S_n and Σ_n simultaneously, answering Question 4. Indeed, Proposition 2 and Theorem 7 combine to give

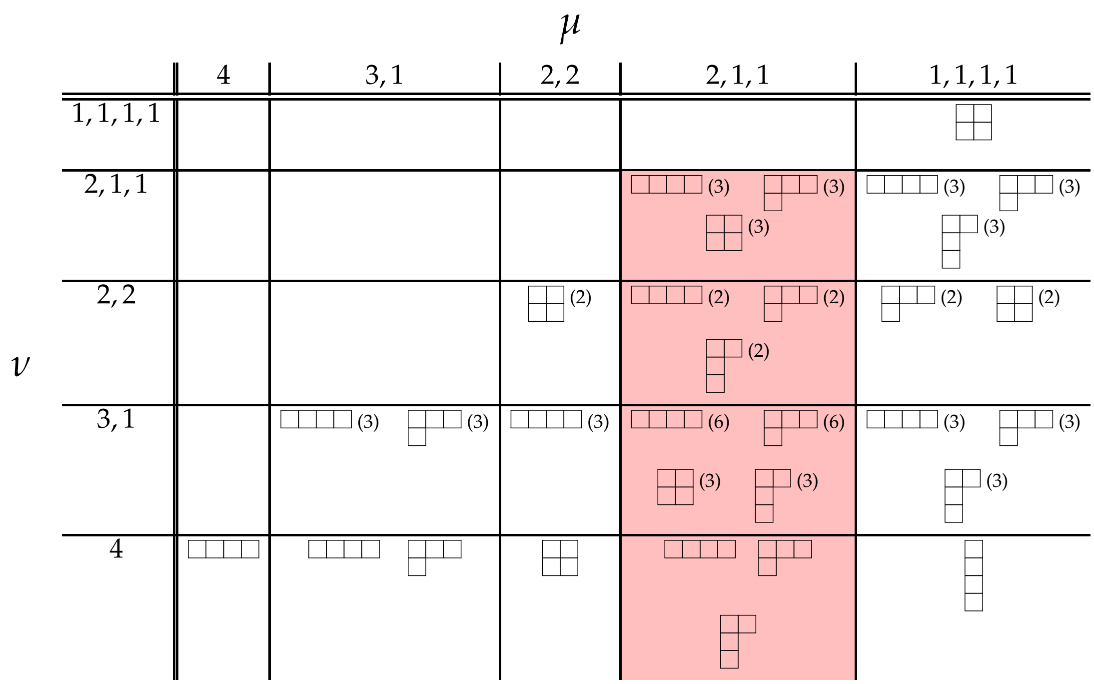
$$\left[\left(\mathbb{C}\mathcal{F}_n E_\mu\right)^\nu : M_\lambda \right] = f^\nu \cdot \left\langle s_\nu, \text{coefficient of } \mathbf{y}_\lambda \mathbf{z}_\mu \text{ in } F \right\rangle. \tag{3.2}$$

Since Thrall's problem and understanding plethysm coefficients are longstanding open problems, this is as far as we are able to simplify our answer for now.

3.1 Example

As an example of Theorem 7, we analyze the case $n = 4$ in the table below. The box in row ν and column μ is filled with $[(\mathbb{C}\mathcal{F}_4 E_\mu)^\nu : M_\lambda]$ copies of λ, where the numbers in parentheses indicate multiplicities.

[2]This is equivalent to the standard definition by work of Stanley [18], Hanlon [11], and Klyachko [12].



Each term in the expansion of F is formed by choosing one (potentially empty) partition ρ for each Lyndon word factor w and multiplying the corresponding terms $\mathbf{y}_{\rho\cdot|w|}\mathbf{z}_{w^{|\rho|}}L_\rho[h_w]$. To obtain terms with $\mathbf{z}-$weight $\mathbf{z}_{211} = z_2z_1^2$, the only Lyndon words w for which one can choose a nonempty partition ρ are $w = 1, w = 2, w = 12$, and $w = 112$. With these relevant factors first, the generating function F is:

$$\underbrace{\left(\sum_\rho \mathbf{y}_\rho \mathbf{z}_1^{|\rho|} L_\rho[h_1]\right)}_{w=1}\underbrace{\left(\sum_\rho \mathbf{y}_{\rho\cdot 2} \mathbf{z}_2^{|\rho|} L_\rho[h_2]\right)}_{w=2}\underbrace{\left(\sum_\rho \mathbf{y}_{\rho\cdot 3} \mathbf{z}_{12}^{|\rho|} L_\rho[h_{12}]\right)}_{w=12}\underbrace{\left(\sum_\rho \mathbf{y}_{\rho\cdot 4} \mathbf{z}_{112}^{|\rho|} L_\rho[h_{112}]\right)}_{w=112}\cdots.$$

Labelling by the w for which a nonempty ρ was chosen, the coefficient of $\mathbf{z}_{211}$ is

$$\begin{aligned}
&\underbrace{\mathbf{y}_2L_2[h_1]}_{\substack{w=1\\ \rho=2}}\cdot\underbrace{\mathbf{y}_2L_1[h_2]}_{\substack{w=2\\ \rho=1}}+\underbrace{\mathbf{y}_{11}L_{11}[h_1]}_{\substack{w=1\\ \rho=11}}\cdot\underbrace{\mathbf{y}_2L_1[h_2]}_{\substack{w=2\\ \rho=1}}+\underbrace{\mathbf{y}_1L_1[h_1]}_{\substack{w=1\\ \rho=1}}\cdot\underbrace{\mathbf{y}_3L_1[h_{12}]}_{\substack{w=12\\ \rho=1}}+\underbrace{\mathbf{y}_4L_1[h_{112}]}_{\substack{w=112\\ \rho=1}}\\
&= \mathbf{y}_{22}\left(L_2[h_1]L_1[h_2]\right)+\mathbf{y}_{211}\left(L_{11}[h_1]L_1[h_2]\right)+\mathbf{y}_{31}\left(L_1[h_1]L_1[h_{12}]\right)+\mathbf{y}_4\left(L_1[h_{112}]\right)\\
&= \mathbf{y}_{22}\left(s_{211}+s_{31}\right)+\mathbf{y}_{211}\left(s_4+s_{22}+s_{31}\right)+\mathbf{y}_{31}\left(s_4+2s_{31}+s_{22}+s_{211}\right)\\
&\quad+\mathbf{y}_4\left(s_4+2s_{31}+s_{22}+s_{211}\right),
\end{aligned}$$

where the final equality can be computed with SageMath. This process reveals how to fill each box of the pink column. For instance, the composition multiplicity of M_4 in $(\mathbb{C}\mathcal{F}_4E_{211})^{31}$ is $6 = 3\cdot 2$ because $f^{(3,1)} = 3$ and the coefficient of $\mathbf{y}_4s_{31}$ in the above equation is 2, as indicated by the coloring.

3.2 Recovering results of Garsia–Reutenauer and Uyemura-Reyes

As further examples, we explain how Theorem 7 specializes to recover results of Garsia–Reutenauer in [9] and Uyemura-Reyes in [23].

3.2.1 The bottom row ($\nu = (n)$): Garsia–Reutenauer's Cartan invariants of Σ_n

In [9, Theorem 5.4], Garsia and Reutenauer discovered the Cartan invariants[3] of the descent algebra. To state their result, let $\mathrm{type}(\alpha)$ for a composition α be the partition obtained by reordering $|w_1|, |w_2|, \cdots, |w_k|$ where $w_1 w_2 \cdots w_k$ is the unique factorization of α into weakly decreasing (lexicographically) Lyndon words (see [9, Proposition 5.3]).

Theorem 8 (Garsia–Reutenauer). *The composition multiplicity*

$$[P_\mu : M_\lambda] = \#\{\alpha \sim \mu : \mathrm{type}(\alpha) = \lambda\}.$$

Example 9. *The compositions rearranging to* $(2,1,1)$ *are* $(2,1,1)$, $(1,2,1)$, *and* $(1,1,2)$. *As the table below illustrates, the composition factors of* P_{211} *are one copy each of* M_{211}, M_{31}, *and* M_4. *Compare this to the box in row* (4) *and column* $(2,1,1)$ *of the table in Section 3.1.*

α	Lyndon Factorization	$\mathrm{type}(\alpha)$
$(2,1,1)$	$(2)(1)(1)$	$(2,1,1)$
$(1,2,1)$	$(1,2)(1)$	$(3,1)$
$(1,1,2)$	$(1,1,2)$	(4)

As descent algebra modules, $P_\mu \cong \left(\mathbb{C}\mathcal{F}_n E_\mu\right)^{(n)}$. So, by Equation (3.2), the following proposition recovers Garsia–Reutenauer's discovery.

Proposition 3. *For* λ, μ *partitions of* n,

$$\left\langle s_n, [\mathbf{y}_\lambda \mathbf{z}_\mu] \prod_{\substack{\text{Lyndon}\\ w}} \sum_{\substack{\text{partition}\\ \rho}} \mathbf{y}_{\rho\cdot|w|} \mathbf{z}_{w^{|\rho|}} L_\rho[h_w] \right\rangle = \#\{\alpha \sim \mu : \mathrm{type}(\alpha) = \lambda\}.$$

Proof Sketch. From properties of plethysm and higher Lie representations, one can show

$$\left\langle s_n, \prod_w L_{\nu^w}[h_w] \right\rangle = 0$$

unless each partition ν^w is of the form 1^{m_w} for some integer m_w with $\sum_w |w| m_w = n$, in which case it is one. Hence, the left side of the proposition statement simplifies to

$$[\mathbf{y}_\lambda \mathbf{z}_\mu] \prod_{\substack{\text{Lyndon}\\ w}} \sum_{m \geq 0} \mathbf{y}_{|w|^m} \mathbf{z}_{w^m}.$$

[3]They actually proved a stronger result by finding bases for the spaces $\Phi^{-1}(E_\mu)\Sigma_n\Phi^{-1}(E_\lambda)$.

A straightforward combinatorial argument using Lyndon factorization recovers that

$$\prod_{\substack{\text{Lyndon}\\ w}} \sum_{m\geq 0} \mathbf{y}_{|w|^m}\mathbf{z}_{w^m} = \sum_{\substack{\text{partitions}\\ \lambda,\mu}} \#\{\alpha \sim \mu : \text{type}(\alpha) = \lambda\}\mathbf{y}_\lambda \mathbf{z}_\mu. \qquad \square$$

3.2.2 The rightmost column ($\mu = 1^n$): Uyemura-Reyes's shuffling representations

In his PhD thesis (see [23, Theorem 4.1]), Uyemura-Reyes studied certain shuffling eigenspaces indexed by partitions $\lambda \vdash n$, which turn out to be the spaces $E_\lambda \mathbb{C}\mathcal{F}_n E_{1^n}$. He proved the λ-eigenspace has Frobenius characteristic L_λ. Theorem 7 recovers this result, since it is simple to check that the coefficient of $\mathbf{z}_{1^n}$ in Equation (3.1) is

$$\sum_{\lambda\vdash n} \mathbf{y}_\lambda L_\lambda[h_1] = \sum_{\lambda\vdash n} \mathbf{y}_\lambda L_\lambda.$$

3.3 More explicit answer for the sign isotypic subspace

Recall from Section 2 that the sign isotypic subspace $(\mathbb{C}\mathcal{F}_n)^{1^n}$ is always a one-dimensional subspace. Hence, it must be a simple Σ_n-module.

Proposition 4. *As Σ_n-modules, the sign isotypic subspace $(\mathbb{C}\mathcal{F}_n)^{1^n}$ of the face algebra is isomorphic to M_λ where $\lambda = \left(2^{\frac{n}{2}}\right)$ if n is even and $\lambda = \left(2^{\frac{n-1}{2}}, 1\right)$ if n is odd.*

This follows from a result of Gessel–Reutenauer [10, Theorem 2.1] which (as a special case) shows $\langle L_\lambda, s_{1^n}\rangle$ counts permutations in S_n with cycle type λ and descent set $[n-1]$. Hence, the scalar product is zero except for when λ is the cycle type of the longest word.

3.4 Outline of the proof of Theorem 7

Although the complete proof of Theorem 7 is quite long, a nice range of combinatorics is involved. So, we briefly outline the important ideas for the curious reader. For the details, see the full version of this abstract in [8].

3.4.1 Reduction to homology of intervals in the set partition lattice Π_n

The proposition below relies on special properties holding for any cfpoi of $\mathbb{C}\mathcal{F}_n$. Let $\text{Stab}_{S_n}(X)$ denote the S_n-stabilizer subgroup of the set partition X.

Proposition 5. *If μ does not refine λ, then $E_\lambda \mathbb{C}\mathcal{F}_n E_\mu = 0$. Otherwise, as S_n-representations,*

$$E_\lambda \mathbb{C}\mathcal{F}_n E_\mu \cong \bigoplus_{[X\leq Y]} E_Y \mathbb{C}\mathcal{F}_n E_X \Big\uparrow_{\text{Stab}_{S_n}(X)\cap \text{Stab}_{S_n}(Y)}^{S_n}$$

where the direct sum is over S_n-orbits of pairs $X \leq Y$ in Π_n with $X \in \mu, Y \in \lambda$.

A twisting character appears when studying the spaces $E_Y \mathbb{C}\mathcal{F}_n E_X$. The set partition lattice is (S_n-equivariantly) isomorphic to the lattice of intersections of the hyperplanes of $\mathcal{B}_n$. Let $\det(Y)$ be the $\mathrm{Stab}_{S_n}(Y)-$character sending g to $+1$ if it preserves orientation on the intersection associated to Y and -1 otherwise.

Saliola proves the non-equivariant version of the following proposition in [15, §10.2]. The twists making it equivariant appear implicitly in his work in [14, Theorem 6.2]. Aguiar and Mahajan also explain it in [1, Proposition 14.44].

Proposition 6. *Assume $X, Y \in \Pi_n$ with X refining Y. As representations of* $\mathrm{Stab}_{S_n}(X) \cap \mathrm{Stab}_{S_n}(Y)$,

$$E_Y \mathbb{C}\mathcal{F}_n E_X \cong \tilde{H}^{\mathrm{top}}\left(X, Y\right) \otimes \det(Y) \otimes \det(X),$$

with $\tilde{H}(X, Y)$ the poset cohomology of the open interval in Π_n, using the convention that $\tilde{H}^{\mathrm{top}}\left(X, X\right)$ is the trivial representation.

By properties of dual representations and induction, we are able to consider the *homology* of intervals $\tilde{H}_{\mathrm{top}}\left(X, Y\right)$ instead when combining Proposition 5 and Proposition 6.

3.4.2 Base case: the spaces $E_n \mathbb{C}\mathcal{F}_n E_\mu$

A key step in proving Theorem 7 is to understand the case $\lambda = n$. If $\{X_\mu\}$ are set partitions with block sizes μ and $\hat{1}$ denotes the maximal element of Π_n, then by Section 3.4.1,

$$\sum_{\mu \neq \emptyset} \mathbf{z}_\mu \cdot \mathrm{ch}(E_{|\mu|} \mathbb{C}\mathcal{F}_{|\mu|} E_\mu) = \sum_{\mu \neq \emptyset} \mathbf{z}_\mu \cdot \mathrm{ch}\left(\tilde{H}_{\mathrm{top}}\left(X_\mu, \hat{1}\right) \otimes \det\left(X_\mu\right) \Big\uparrow_{\mathrm{Stab}_{S_{|\mu|}}\left(X_\mu\right)}^{S_{|\mu|}}\right). \tag{3.3}$$

Sundaram studied the homology of the partition lattice in great depth. In [20, proof of Thm 1.4], she studies the $\mathrm{Stab}_{S_{|\mu|}}\left(X_\mu\right)-$representations $\tilde{H}_{\mathrm{top}}\left(X_\mu, \hat{1}\right)$. Adjusting her work with the $\det(X_\mu)$ twists reframes Equation (3.3) as

$$\sum_{r \geq 1} L_r[z_1 h_1 + z_2 h_2 + \cdots]. \tag{3.4}$$

In [10, Equation 2.1], Gessel–Reutenauer interpret the symmetric functions L_r with necklaces. Using their interpretation, we construct a necklace bijection to rewrite (3.4) as

$$\sum_{\substack{\text{Lyndon} \\ w}} \sum_{m \geq 1} \mathbf{z}_{w^m} L_m[h_w].$$

3.4.3 General case: the spaces $E_\lambda \mathbb{C}\mathcal{F}_n E_\mu$

The general case comes down to understanding the action of the subgroups $\mathrm{Stab}_{S_n}(X) \cap \mathrm{Stab}_{S_n}(Y)$ on $\tilde{\mathcal{H}}_{\mathrm{top}}\left(X, Y\right) \otimes \det(X) \otimes \det(Y)$. By identifying the intersections of these stabilizer subgroups, we recast this action as the action of (wreath) products of smaller subgroups on products of smaller partition lattices (with appropriate twists). Sundaram's work in [19, Prop 2.1, 2.3] is helpful in reducing these representations to our base case. Then, a generating function argument comes into play.

4 A note on other Coxeter types

Much of this work holds in all Coxeter types. The face algebra, the descent algebra, and Bidigare's theorem were each originally defined or proved in all types. The representation theory of the descent algebra has been studied in other types thoroughly in [2, 3, 4]. Saliola's work in [14] is also for general type, so analogues of Proposition 5 and Proposition 6 hold. In fact, an analogue of Proposition 4 holds for all types. In [7, Thm 1.1], Blessenohl–Hohlweg–Schocker generalize Gessel–Reutenauer's result to general type. Their work helps us prove that as a descent algebra module, the sign isotypic subspace of the face algebra is the simple indexed by the cycle type[4] of the longest word. Unfortunately, we do not have analogues of Theorem 7 since our proof relies heavily on the structure of the partition lattice and symmetric functions.

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The Newton polytope of the Kronecker product

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Abstract. We study the Kronecker product of two Schur functions $s_\lambda * s_\mu$, whose Schur expansion is given by the Kronecker coefficients $g(\lambda, \mu, \nu)$ of the symmetric group. We prove special cases of a conjecture of Monical–Tokcan–Yong that its monomial expansion has a saturated Newton polytope. Our proofs employ the Horn inequalities for positivity of Littlewood–Richardson coefficients and imply necessary conditions for the positivity of Kronecker coefficients.

Keywords: Kronecker coefficients, saturated Newton polytope, Symmetric group representations

1 Introduction

The Kronecker coefficients $g(\lambda, \mu, \nu)$ of the symmetric group present an 85 year old mystery in Algebraic Combinatorics and Representation Theory. They are defined as the multiplicities of an irreducible S_n-module $\mathbb{S}_\nu$ in the tensor product of two other irreducibles: $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$. Originally introduced by Murnaghan in 1938 [10, 11], the question for their computation has been reiterated many times since the 1980s. Stanley's 10th open problem in Algebraic Combinatorics [18] is to find a manifestly positive combinatorial interpretation for the Kronecker coefficients. Yet, over the years, very little progress has been made and only for special cases, see [13] for an overview. Their importance has been reinforced by their role in Geometric Complexity Theory, a program aimed at establishing computational lower bounds and ultimately separating complexity classes like P vs NP, see [14] and references therein. While no positive combinatorial formula exists, we also lack understanding for when such coefficients would be positive. The possibility of answering these questions in a "nice" way is explored using computational complexity theory, see [12, 14].

In a different direction, [8] initiated the study of the Newton polytopes of important polynomials in Algebraic Combinatorics. It has since been established that some of the main polynomials of interest have the *saturated Newton polytope* (SNP) property.

Definition 1.1. *A multivariate polynomial with nonnegative coefficients $f(x_1, \ldots, x_k) = \sum_\alpha c_\alpha x^\alpha$ has a saturated Newton polytope (SNP) if the set of points $M_k(f) := \{(\alpha_1, \cdots, \alpha_k) : c_\alpha > 0\}$ coincides with its convex hull in $\mathbb{Z}^k$.*

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Given a symmetric function f, let $f(x_1,\ldots,x_k)$ denote the specialization of f to the variables $x_1,\ldots,x_k$ that sets $x_m = 0$ for all $m \geq k+1$.

Definition 1.2. *A symmetric function f has a saturated Newton polytope (SNP) if $f(x_1,\ldots,x_k)$ has a SNP for all $k \geq 1$.*

1.1 SNP for the Kronecker product

The Kronecker coefficients of S_n, denoted by $g(\lambda,\mu,\nu)$, give the multiplicities of one Specht module in the tensor product of the other two, namely

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu\vdash n}\mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}.$$

The Kronecker product $*$ of symmetric functions is defined on the Schur basis as

$$s_\lambda * s_\mu := \sum_\nu g(\lambda,\mu,\nu)s_\nu,$$

and extended by linearity. It is equivalent to the inner product of S_n characters under the characteristic map.

Conjecture 1.3 ([8]). *The Kronecker product $s_\lambda * s_\mu$ has a saturated Newton polytope.*

We prove this conjecture for partitions of lengths 2 and 3 and various truncations.

Theorem 1.4. *Let $\lambda,\mu \vdash n$ with $\ell(\lambda) \leq 2, \ell(\mu) \leq 3$, and $\mu_1 \geq \lambda_1$ then $s_\lambda * s_\mu(x_1,\ldots,x_k)$ has a saturated Newton polytope for every $k \in \mathbb{N}$.*

This theorem follows from the Kronecker product containing a term s_ν, where ν dominates all other partitions in the product. As a result, the degree vectors of the monomials are the integer points $(a_1,\ldots,a_k)$ that, when sorted, satisfy $\mathrm{sort}(a_1,\ldots,a_k) \preceq \nu$ in the dominance order, ensuring the polytope is saturated. However, it is not always the case that there is a unique maximal term with respect to the dominance order. The first instance where no such dominant partition exists is covered in the following theorem.

Theorem 1.5. *Let $\lambda,\mu \vdash n$ with $\ell(\lambda) \leq 3$ and $\ell(\mu) \leq 2$. Then $s_\lambda * s_\mu(x_1,x_2,x_3)$ has a saturated Newton polytope.*

The difficulty with this problem in the general case is the lack of any criterion for the positivity of the Kronecker coefficients. We express the Kronecker product in the monomial basis as sums of products of multi-Littlewood–Richardson coefficients. Using the Horn inequalities, which determine when a Littlewood–Richardson coefficient is nonzero, we construct a polytope $\mathcal{P}(\lambda,\mu;\mathbf{a})$ parametrized by λ,μ and $\mathbf{a} = (a_1,\ldots,a_k)$ for the monomial of interest $x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}$. A monomial appears in $s_\lambda * s_\mu$ if and only if $\mathcal{P}(\lambda,\mu;\mathbf{a})$ has an integer point, and we can infer the following.

Proposition 1.6. *Let $\mu, \lambda \vdash n$. The Kronecker product $s_\lambda * s_\mu(x_1, \ldots, x_k)$ has a saturated Newton polytope if and only if for every $\mathbf{a} \in \mathbb{Z}^k$ the polytope $\mathcal{P}(\lambda, \mu; \mathbf{a})$ is either empty or has an integer point.*

It is not hard to see that $\mathcal{P}(\lambda, \lambda; \mathbf{a})$ is always nonempty and has an integer point. However, it is far from clear how to characterize when $\mathcal{P}(\lambda, \mu; \mathbf{a}) \neq \emptyset$ once $\mu \neq \lambda$ and the number of variables k grows, and further to determine if there is an integer point. It is also not apparent whether these polytopes have an integer vertex as the relevant inequalities result in many non-integral vertices.

The limiting version of Conjecture 1.3 holds in general.

Theorem 1.7. *Let λ, μ be partitions of the same size and $k \in \mathbb{N}$. Then the set of points*

$$\bigcup_{p=1}^{\infty} \frac{1}{p} M_k(s_{p\lambda} * s_{p\mu})$$

is a convex subset of $\mathbb{Q}^k$.

This is not surprising since the set of triples $\frac{1}{|\lambda|}(\lambda, \mu, \nu)$ for which there is a p, such that $g(p\lambda, p\mu, p\nu) > 0$, forms a polytope known as the Moment polytope, see [19, 1].

1.2 Positivity implications

Suppose that $g(\lambda, \mu, \alpha) > 0$ and $g(\lambda, \mu, \beta) > 0$ for some partitions α, β. Then the monomials with powers α and β appear in $s_\lambda * s_\mu$. Suppose that $\gamma = t\alpha + (1-t) * \beta \in \mathbb{Z}^k$ for some $t = \frac{p}{q} \in \mathbb{Q}$ with $p < q$. The SNP property would imply that γ appears as a monomial, and thus there is a partition $\theta \succ \gamma$, such that $g(\lambda, \mu, \theta) > 0$. By the semigroup property we have that $g(p\lambda, p\mu, p\alpha) > 0$, $g((q-p)\lambda, (q-p)\mu, (q-p)\beta) > 0$ and thus $g(q\lambda, q\mu, q\gamma) > 0$. However, the Kronecker coefficients do not, in general, possess the saturation property, so we cannot expect $g(\lambda, \mu, \gamma) > 0$ and in fact this is not always true[1]. We can generalize the above reasoning into the following.

Proposition 1.8. *Suppose that $s_\lambda * s_\mu$ has a saturated Newton polytope. Then for every collection of partitions $\alpha^1, \alpha^2, \ldots$, s.t. $g(\lambda, \mu, \alpha^i) > 0$ and $\sum_i t_i \alpha^i$ has integer parts for some $t_i \in [0,1]$ with $t_1 + t_2 + \cdots = 1$, there exists a partition $\theta \succeq \sum_i t_i \alpha^i$ in the dominance order, such that $g(\lambda, \mu, \theta) > 0$.*

Our methods and the Horn inequalities also give some necessary conditions for a Kronecker coefficient to be positive. We cannot expect easy necessary and sufficient

[1] Let $\lambda = (8,8)$ and $\mu = (5,3,1,1,1,1,1,1,1,1)$. Let $\alpha = (7,3,2,2,2)$, $\beta = (5,5,2,2,2)$ and $\nu = (6,4,2,2,2)$. We have that $g(\lambda, \mu, \alpha) = g(\lambda, \mu, \beta) = 1$, but $g(\lambda, \mu, \frac{\alpha+\beta}{2}) = 0$, and $s_\lambda * s_\mu$ does not have a unique dominant term.

criteria for positivity since this decision problem is NP-hard by [3]. The general statement is Theorem 6.1 stated in Section 6 in terms of the so-called LR-consistent triples. We illustrate the criteria with a simplified version below in the case of one two-row partition.

Proposition 1.9. *Suppose that $g(\lambda,\mu,\nu) > 0$ and $\ell(\mu) = 2$, $k = \ell(\lambda)$. Then there exist nonnegative integers $y_i \in [0, \lfloor \lambda_i/2 \rfloor]$ for $i \in [k]$, such that*

$$\sum_{i\in A\cup C} \lambda_i + \sum_{i\in B} y_i - \sum_{i\in C} y_i \le \min\{\sum_{j\in J} \mu_j, \sum_{j\in J} \nu_j\} \tag{1.1}$$

for all triples of mutually disjoint sets $A \sqcup B \sqcup C \subset [k]$ and $J = \{1,\dots,r,r+2,\dots,r+b+1\}$ or $J = \{1,\dots,r+b-1,r+2b\}$, where $r = 2|A|+|C|$ and $b = |B|$.

The details of the above results, along with full proofs, computations, and additional discussions will appear in the full version of this abstract, available in [15].

2 Definitions and tools

2.1 Basic notions from algebraic combinatorics

We use standard notation from [7] and [17, §7] throughout the paper.

The irreducible representations of the *symmetric group* S_n are the Specht modules $\mathbb{S}_\lambda$ and are indexed by partitions $\lambda \vdash n$. The irreducible polynomial representations of $GL_N(\mathbb{C})$ are the *Weyl modules* V_λ and are indexed by all partitions with $\ell(\lambda) \le N$. Their characters are the Schur functions $s_\lambda(x_1,\dots,x_N)$, where $x_1,\dots,x_N$ are the eigenvalues of $g \in GL_N(\mathbb{C})$.

We will use the standard bases for the ring of symmetric functions Λ: the monomial symmetric functions

$$m_\alpha(x_1,x_2,\dots,x_k) = \sum_\sigma x_{\sigma_1}^{\alpha_1} x_{\sigma_2}^{\alpha_2}\cdots,$$

where the sum goes over all permutations σ giving different monomials.

The Schur functions $s_\lambda(x_1,\dots)$ can be defined as the generating functions for SSYTs of shape λ, i.e., $s_\lambda = \sum_\alpha K_{\lambda\alpha} m_\alpha$. We will also use the homogeneous symmetric functions h_λ defined as $h_k := s_{(k)} = \sum_{i_1\le\cdots\le i_k} x_{i_1}\cdots x_{i_k}$ and $h_\lambda := h_{\lambda_1}h_{\lambda_2}\cdots$.

The Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ are defined as structure constants in Λ for the Schur basis, and also as the multiplicities in the GL-module decomposition $V_\mu \otimes V_\nu = \oplus_\lambda V_\lambda^{c^\lambda_{\mu\nu}}$. We have

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

They can be evaluated by the Littlewood–Richardson rule as a positive sum of skew SSYT of shape λ/μ and type (weight) ν whose reverse reading word is a ballot sequence.

Their positivity can be decided in polynomial time as $c^{\lambda}_{\mu\nu} > 0$ if and only if its corresponding polytope is nonempty (see [5, 9]). The multi-LR coefficients can be defined recursively as

$$c^{\lambda}_{\nu^1,\nu^2,\ldots,\nu^k} := \langle s_\lambda, s_{\nu^1}s_{\nu^2}\cdots s_{\nu^k}\rangle = \sum_{\tau^1,\tau^2,\ldots,\tau^k} c^{\lambda}_{\nu^1\tau^1}c^{\tau^1}_{\nu^2\tau^2}\cdots c^{\tau^{k-1}}_{\nu^k\tau^k}.$$

2.2 The Kronecker product

The Kronecker product, denoted by $*$, of symmetric functions can be defined on the basis of the Schur functions and extended by linearity: $s_\lambda * s_\mu = \sum_\nu g(\lambda,\mu,\nu)s_\nu$.

It is also $\mathrm{ch}(\chi^\lambda\chi^\mu)$, where χ are the S_n characters and ch is the Frobenius characteristic map. The Kronecker coefficients can be equivalently defined as the coefficients in the expansion

$$s_\lambda[x\cdot y] = \sum_{\mu,\nu} g(\lambda,\mu,\nu)s_\mu(x)s_\nu(y), \tag{2.1}$$

where $[x\cdot y] := (x_1y_1, x_1y_2, \ldots, x_2y_1, \ldots)$ denote all pairwise products of the two sets of variables.

Via Schur–Weyl duality the Kronecker coefficients can be interpreted as the dimensions of GL highest weight spaces, which then makes the following semigroup property, see [2], apparent:

If $\alpha^1,\beta^1,\gamma^1 \vdash n$ and $\alpha^2,\beta^2,\gamma^2 \vdash m$ satisfy $g(\alpha^i,\beta^i,\gamma^i) > 0$ for $i = 1,2$, then $g(\alpha^1 + \alpha^2, \beta^1+\beta^2, \gamma^1+\gamma^2) \geq \max\{g(\alpha^1,\beta^1,\gamma^1), g(\alpha^2,\beta^2,\gamma^2)\}$.

Here we will be concerned with the monomial expansion. Since the homogeneous and monomial bases are orthogonal to each other, i.e. $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$ we have that

$$s_\lambda * s_\mu = \sum_\nu g(\lambda,\mu,\nu)s_\nu = \sum_{\nu,\alpha} g(\lambda,\mu,\nu)K_{\nu\alpha}m_\alpha = \sum_{\alpha\vdash n}\langle s_\lambda * s_\mu, h_\alpha\rangle m_\alpha. \tag{2.2}$$

In Section 4 we will see further ways of finding the monomial expansion.

2.3 Newton polytopes

Let $f(x_1,\ldots,x_k) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial with nonnegative coefficients, where $x^\alpha := x_1^{\alpha_1}\cdots x_k^{\alpha_k}$ and $\alpha \in \mathbb{Z}^k_{\geq 0}$ is the degree vector. We denote by $M_k(f) := \{\alpha \in \mathbb{Z}^k_{\geq 0} : c_\alpha > 0\}$ the set of vectors, for which the corresponding monomial appears in $f(x_1,\ldots,x_k)$. For brevity we will say "monomial α appears in f". We denote by $N_k(f) := Conv(M_k(f))$ the convex hull of $M_k(f)$, this is the Newton polytope of $f(x_1,\ldots,x_k)$. Thus, a polynomial f has a saturated Newton polytope if and only if $M_k(f) = N_k(f)$. In particular, a polynomial f has an SNP if and only if the following condition holds:

For every $k+1$-tuple of compositions $(\alpha^1,\ldots,\alpha^{k+1})$, such that $c_{\alpha^i} > 0$, and

(snp)

weights $t_i \in [0,1]$, such that $t_1 + \cdots + t_{k+1} = 1$ and $\gamma := \sum_{i=1}^{k+1} t_i \alpha^i \in \mathbb{Z}^k$, we have $c_\gamma > 0$.

Note that it is enough to check the convex combination of $k+1$ points in k-dimensional space by Caratheodory's theorem.

As noted in [8] many of the important symmetric polynomials have SNP. Since Kostka coefficients $K_{\lambda\mu}$ are positive if and only if $\lambda \succ \mu$ in the dominance order, we get an immediate characterization of $M_k(s_\lambda)$ and the following important statement.

Proposition 2.1 ([8])**.** *The Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ has a saturated Newton polytope and $M_k(f) = conv\{(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_k})$ for all $\sigma \in S_k\}$.*

3 Two and three-row partitions

In this section, we deduce the SNP property for certain cases from existing formulas. In the cases treated here we will see that there will be a unique partition ν, s.t. $g(\lambda, \mu, \nu) > 0$ and if $g(\lambda, \mu, \alpha) > 0$ then $\nu \succ \alpha$ and so $s_\lambda * s_\mu$ will contain all monomials $\alpha \prec \nu$, as observed in [8].

First, let $\ell(\lambda), \ell(\mu) = 2$ and the number of variables be arbitrary. In [16], Rosas computed the Kronecker product of two two-row partitions. In particular, [16, Corollary 5] gives a formula for Kronecker coefficients indexed by 3 two-row partitions. We could then show that $N(s_\lambda * s_\mu; k) = N(s_\nu; k)$ for a certain partition ν.

Lemma 3.1. *Let $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, and $\nu = (\nu_1, \nu_2)$ be two-row partitions of n. Without loss of generality, suppose that $\lambda_2 \geq \mu_2$. Then $\langle s_\lambda * s_\mu, h_\nu \rangle > 0$ if and only if $\nu_2 \geq \lambda_2 - \mu_2$.*

By equation (2.2) this means that m_ν appears with a nonzero coefficient in that Kronecker product.

We now move to a more general case and invoke the full Theorem from [16]. Specifically, [16, Theorem 5] gives a formula for Kronecker products of 2 two-row partitions, allowing us to show that there is a unique maximal term in dominance order in the Kronecker product $s_\lambda * s_\mu$ in the following case.

Proposition 3.2 (Theorem 1.4)**.** *Let λ and μ be partitions of n, where $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2, \mu_3)$, such that $\mu_1 \geq \lambda_1$. Then the Kronecker product $s_\lambda * s_\mu(x_1, \ldots, x_k)$ has a saturated Newton polytope for every k.*

Remark 3.3. We cannot expect to have unique maximal terms in general. For instance, $s_{(6,6)} * s_{(8,2,1,1)} = s_{(4,4,2,1,1)} + s_{(4,4,3,1)} + s_{(5,3,1,1,1,1)} + s_{(5,3,2,1,1)} + s_{(5,3,2,2)} + s_{(5,3,3,1)} + 2s_{(5,4,1,1,1)} + 3s_{(5,4,2,1)} + s_{(5,4,3)} + s_{(5,5,1,1)} + 2s_{(5,5,2)} + s_{(6,2,2,1,1)} + 2s_{(6,3,1,1,1)} + 3s_{(6,3,2,1)} + s_{(6,3,3)} + 4s_{(6,4,1,1)} + 2s_{(6,4,2)} + 2s_{(6,5,1)} + s_{(7,2,1,1,1)} + s_{(7,2,2,1)} + 2s_{(7,3,1,1)} + 2s_{(7,3,2)} + 2s_{(7,4,1)} + s_{(7,5)} + s_{(8,2,1,1)} + s_{(8,3,1)}$. In this product, $(7,5)$ and $(8,3,1)$ are incomparable maximal.

4 Multi-LR coefficients and Horn inequalities

4.1 Monomial expansion via multi-LR coefficients

As we observed, the Kronecker product does not necessarily have a unique dominating term s_ν. Moreover, there are no positive formulas for many other cases we could use. We thus move directly towards the monomial expansion. The coefficient at $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} = (a_1, a_2, \ldots)$ in $s_\lambda * s_\mu$ can be expressed as

$$\langle s_\lambda(y) * s_\mu(z), h_{\mathbf{a}}[yz] \rangle = \langle s_\lambda(y) * s_\mu(z), \prod_i \sum_{\alpha^i \vdash a_i} s_{\alpha^i}(y) s_{\alpha^i}(z) \rangle = \sum_{\alpha^i \vdash a_i, i=1,\ldots} c^{\lambda}_{\alpha^1\alpha^2\ldots} c^{\mu}_{\alpha^1\alpha^2\ldots} \tag{4.1}$$

We now define the following set of points given by the concatenation of the vectors $\alpha^1, \alpha^2, \ldots, \alpha^k$:

$$P(\mu; \mathbf{a}) := \{(\alpha^1, \alpha^2, \cdots, \alpha^k) \in \mathbb{Z}_{\geq 0}^{\ell(\mu)k} : c^{\mu}_{\alpha^1\alpha^2\ldots} > 0 \text{ and } |a^i| = a_i \text{ for all } i = 1, \ldots, k\}. \tag{4.2}$$

Observe that $P(\mu; \mathbf{a}) \neq \emptyset$ for all $\mu, \mathbf{a}$ of the same size. This can be seen either by a greedy algorithm to construct $\alpha^1, \ldots$ a nonzero multi-LR coefficient, or by observing that $s_\mu * s_\mu = s_{(n)} + \cdots$ and contains every monomial of degree n, so for every $\mathbf{a}$ there are some $\alpha^i \vdash a_i$ with $c^{\mu}_{\alpha^1\ldots} > 0$. The monomials appearing in $s_\lambda * s_\mu$ correspond to $\mathbf{a}$, for which there exist $\alpha^1, \cdots$ with $c^{\lambda}_{\alpha^1\ldots} > 0$ and $c^{\mu}_{\alpha^1\ldots} > 0$. Thus

Proposition 4.1. *The set of monomial degrees $\mathbf{a} = (a_1, \ldots, a_k)$ appearing in $s_\lambda * s_\mu$ is given as*

$$M_k(s_\lambda * s_\mu) = \{\mathbf{a} \in \mathbb{Z}^k_{\geq 0} : P(\lambda; \mathbf{a}) \cap P(\mu; \mathbf{a}) \neq \emptyset\}.$$

We turn towards understanding the above set of points, and in particular, whether they would be the set of lattice points of a convex polytope.

4.2 Horn inequalities for multi-LR's

We first reduce our multi-LR positivity problem from (4.1) and (4.2) to the case of regular LR coefficients. Let again $c^{\mu}_{\alpha^1,\alpha^2,\ldots} = \langle s_{\alpha^1} s_{\alpha^2} \cdots, s_\mu \rangle$ be the multi-LR coefficients.

Theorem 4.2 ([6])**.** *Let λ, μ, ν be partitions such that $|\lambda| = |\mu| + |\nu|$. Then $c^{\lambda}_{\mu,\nu} = \langle s_\lambda, s_{\mu \diamond \nu} \rangle$, where $\mu \diamond \nu$ denotes the skew shape $(\nu_1^{\ell(\mu)} + \mu, \nu)/\nu$.*

We can thus generalize Theorem 4.2 as follows.

Lemma 4.3. *Let $\lambda \vdash n$. For a k-tuple of partitions $\alpha^1, \cdots, \alpha^k$ with $\ell(\alpha^i) \leq \ell$, such that $|\alpha^1| + \cdots + |\alpha^k| = n$ we have that $c^{\lambda}_{\alpha^1\cdots\alpha^k} = \langle s_\lambda, s_{\alpha^1 \diamond \alpha^2 \diamond \cdots \diamond \alpha^k} \rangle = c^{\omega(\alpha)}_{\lambda, \delta_k(n,\ell)}$, where $\alpha^1 \diamond \alpha^2 \diamond \alpha^3 \cdots = \alpha^1 \diamond (\alpha^2 \cdots)$ recursively, $\omega(\alpha) := ((n(k-1))^\ell + \alpha^1, (n(k-2))^\ell + \alpha^2, \cdots, \alpha^k)$, and $\delta_k(n, \ell) := \left((n(k-1))^\ell, (n(k-2))^\ell, \cdots, n^\ell\right)$.*

We next turn to LR positivity as described by the Horn inequalities. For a subset $I = \{i_1 < i_2 < \cdots < i_s\} \subset [r]$, let $\rho(I)$ denote the partition $\rho(I) := (i_s - s, \ldots, i_2 - 2, i_1 - 1)$. We say a triple of subsets $I, J, K \subset [r]$ is LR-consistent if they have the same cardinality s and $c^{\rho(I)}_{\rho(J),\rho(K)} = 1$.

Theorem 4.4 ([20, 4, 5])**.** *Let $\lambda, \mu, \nu \in \mathbb{N}^r$ with weakly decreasing component. Then $c^{\lambda}_{\mu,\nu} > 0$ if and only if $|\lambda| = |\mu| + |\nu|$ and $\sum_{i\in I} \lambda_i \le \sum_{j\in J} \mu_j + \sum_{k\in K} \nu_k$ for all LR-consistent triples $I, J, K \subset [r]$.*

For a set $I \subset \{1, \ldots, \ell k\}$ construct the set $D(I) := \{(i, j) \in [k] \times [\ell]$, such that $\ell(i - 1) + j \in I\}$, that is the set of pairs $(\lceil \frac{x}{\ell} \rceil, x\%\ell)$, where $x \in I$ and $x\%\ell$ is its remainder by division by ℓ, adjusted to be in the range from 1 to ℓ. Applying Theorem 4.4 with $\lambda = \omega(\alpha)$, μ and $\nu = \delta_k(n, \ell)$ from Lemma 4.3, and observing that if $m = \ell(i - 1) + j$ then $\omega(\alpha)_m = n(k - i) + \alpha^i_j$ and $(\delta_k(n, \ell))_m = n(k - i)$ we get the following.

Corollary 4.5. *Let $\ell(\mu) = \ell$ and $\mathbf{a} = (a_1, \ldots, a_k)$. Then $P(\mu; \mathbf{a})$ is the set of points $(\alpha^1, \ldots, \alpha^k) \in \mathbb{Z}^{\ell k}_{\ge 0}$ satisfying the following linear conditions.*

$$\sum_j \alpha^i_j = a_i, \qquad \text{for } i \in [k]; \tag{4.3}$$

$$\alpha^i_j \ge \alpha^i_{j+1}, \qquad \text{for } j \in [\ell - 1],\ i \in [k]; \tag{4.4}$$

$$\sum_{(i,j)\in D(I)} \left(n(k - i) + \alpha^i_j\right) \le \sum_{j\in J} \mu_j + \sum_{(d,r)\in D(K)} n(k - d), \tag{4.5}$$

where the last inequalities hold for all LR-consistent triples $I, J, K \in [\ell k]$.

4.3 The case for $k = 3$

As we know the values of LR coefficients for the triples of partitions $\rho(I), \rho(J), \rho(K)$ when $|I| \le 6$, we can write all the linear inequalities defining the set of (λ, μ, ν) with $\ell(\lambda), \ell(\mu), \ell(\nu) \le 6$ and see that they are the integer points in a convex polytope. In general this polytope is quite complicated and it is not known whether it has any integral nonzero vertices. We will approach the first cases beyond Section 3.

We will restrict ourselves to the Kronecker product of a two-row and a three-row partition and monomials $x_1^{a_1} x_2^{a_2} x_3^{a_3}$. Let $\ell(\lambda) = 2$ and $\ell(\mu) = 3$. Our goal is to describe $P(\lambda; a_1, a_2, a_3) \cap P(\mu; a_1, a_2, a_3)$. Applying Corollary 4.5 to λ, (a_1, a_2, a_3) and μ, (a_1, a_2, a_3), we have

$$c^{\mu}_{\alpha^1,\alpha^2,\alpha^3} c^{\lambda}_{\alpha^1,\alpha^2,\alpha^3} > 0 \text{ if and only if} \tag{4.6}$$
$$\max\{\alpha^1_1, \alpha^2_1, \alpha^3_1, \alpha^1_2 + \alpha^2_2, \alpha^1_2 + \alpha^3_2, \alpha^2_2 + \alpha^3_2\} \le \mu_1$$

$$\max\{\alpha_2^1,\alpha_2^2,\alpha_2^3\} \le \mu_2$$
$$\alpha_2^1+\alpha_2^2+\alpha_2^3 \le \lambda_2$$
$$\max\{\alpha_1^1+\alpha_2^2+\alpha_2^3,\alpha_2^1+\alpha_1^2+\alpha_2^3,\alpha_2^1+\alpha_2^2+\alpha_1^3\} \le \min\{\mu_1+\mu_3,\lambda_1\}$$
$$\max\{\alpha_1^1+\alpha_1^2+\alpha_2^3,\alpha_2^1+\alpha_1^2+\alpha_1^3,\alpha_1^1+\alpha_2^2+\alpha_1^3\} \le \mu_1+\mu_2$$
$$\max\{\alpha_1^1+\alpha_2^1+\alpha_2^2+\alpha_2^3,\alpha_2^1+\alpha_1^2+\alpha_2^2+\alpha_2^3,\alpha_2^1+\alpha_2^2+\alpha_1^3+\alpha_2^3\} \le \mu_1+\mu_2.$$

4.4 The set $P(\lambda;\mathbf{a})\cap P(\mu;\mathbf{a})$

The linear inequalities (4.6) describe a polytope in $\mathbb{R}^6$ for the variables $(\alpha_1^1,\alpha_2^1,\ldots)$. By Section 4 a monomial $\mathbf{x}^{\mathbf{a}}$ occurs in $s_\lambda * s_\mu$ if and only if the set $P(\lambda;\mathbf{a})\cap P(\mu;\mathbf{a})$ has a nonzero integer point. This set corresponds to the set of lattice points of the section of the polytope in (4.6) with $\alpha_1^i+\alpha_2^i=a_i$ for $i=1,2,3$, as well as $\alpha_1^i\ge\alpha_2^i$, which comes from α^is being partitions. Let $x:=\alpha_1^1, y:=\alpha_1^2, z:=\alpha_1^3$. Define $\mathcal{P}(\lambda,\mu,\mathbf{a})$ to be that polytope, substituting the new constraints in (4.6), it is defined by the following inequalities

$$\mathcal{P}(\lambda,\mu,\mathbf{a}) := \left\{ \begin{aligned} (x,y,z)\in\mathbb{R}^3 \text{ s.t. } a_1-\min(\mu_2,\lambda_2,\tfrac{a_1}{2}) &\le x \le \min(a_1,\mu_1) && (1)\\ a_2-\min(\mu_2,\lambda_2,\tfrac{a_2}{2}) &\le y \le \min(a_2,\mu_1) && (2)\\ a_3-\min(\mu_2,\lambda_2,\tfrac{a_3}{2}) &\le z \le \min(a_3,\mu_1) && (3)\\ \max(\mu_3,a_1+a_2-\mu_1) &\le x+y && (4)\\ \max(\mu_3,a_1+a_3-\mu_1) &\le x+z && (5)\\ \max(\mu_3,a_2+a_3-\mu_1) &\le y+z && (6)\\ \lambda_1 &\le x+y+z && (7)\\ \max(\mu_2,\lambda_2)-a_1 &\le -x+y+z \le \mu_1+\mu_2-a_1 && (8)\\ \max(\mu_2,\lambda_2)-a_2 &\le x-y+z \le \mu_1+\mu_2-a_2 && (9)\\ \max(\mu_2,\lambda_2)-a_3 &\le x+y-z \le \mu_1+\mu_2-a_3 && (10) \end{aligned} \right\}$$

We can summarize these descriptions and derivations in the following.

Proposition 4.6. *The monomial $\mathbf{x}^{\mathbf{a}}$ occurs in $s_\lambda * s_\mu$ if and only if $P(\lambda;\mathbf{a})\cap P(\mu;\mathbf{a})\neq\emptyset$. When $\ell(\lambda)=2,\ell(\mu)=3$ and $\mu_1<\lambda_1$ this is equivalent to $\mathcal{P}(\lambda,\mu,\mathbf{a})\cap\mathbb{Z}^3\neq\emptyset$.*

5 Integer points in $\mathcal{P}(\lambda,\mu,\mathbf{a})$

We are now ready to prove the counterpart of Proposition 3.2 by analyzing the polytope $\mathcal{P}(\lambda,\mu,\mathbf{a})$. By considering $\mathcal{P}(\lambda,\mu,\mathbf{c})$ as a fiber of a linear projection from a polyhedral cone, we have the following proposition.

Proposition 5.1. *Suppose that $\mathcal{P}(\lambda, \mu, \mathbf{a}^i) \neq \emptyset$ for some vectors $\mathbf{a}^i$, $i = 1, \ldots, 4$ and $\mathbf{c} = \sum_i t_i \mathbf{a}^i$ for some $t_i \in [0, 1]$ with $t_1 + t_2 + t_3 + t_4 = 1$. Then $\mathcal{P}(\lambda, \mu, \mathbf{c}) \neq \emptyset$.*

Proof sketch. The inequalities defining $\mathcal{P}(\lambda, \mu, \mathbf{a})$ can be written in the form $A[x, y, z]^T \leq \mathbf{v}$ for a 3×3 matrix A with entries $\{0, 1, -1\}$ and vector $\mathbf{v} = B_1[\lambda_1, \lambda_2]^T + B_2[\mu_1, \mu_2, \mu_3]^T + B_3[a_1, a_2, a_3]^T$. Assuming $\mathcal{P}(\lambda, \mu, \mathbf{a}^i) \neq \emptyset$ for all i, we can show that $p := \sum_i t_i p_i$ where $p_i \in \mathcal{P}(\lambda, \mu, \mathbf{a}^i)$ satisfies the inequalities for $\mathcal{P}(\lambda, \mu, \mathbf{c})$ and this polytope is hence nonempty. □

We will now show this polytope is nonempty if and only if it has an integer point.

Theorem 5.2. *If $\mathcal{P}(\lambda, \mu, \mathbf{a}) \neq \emptyset$ then it has an integer point, i.e. $\mathcal{P}(\lambda, \mu, \mathbf{a}) \cap \mathbb{Z}^3 \neq \emptyset$.*

Proof sketch. We first show that if a polytope $\mathcal{P} = \mathcal{P}(\lambda, \mu, \mathbf{a})$ is nonempty, it contains a half-integer point by discussing cases for different types of matrices defining the polytope and proving that, in each case, there exists a half-integer point near a vertex of $\mathcal{P}$. We then extend this result by showing that if $\mathcal{P}$ contains a half-integer point, it must contain an integer point. Our proof considers perturbations of a given half-integer point, showing that small adjustments lead to integer points within $\mathcal{P}$. Exploiting the integer bounds of the inequalities is key to bridge the gap between half-integer and integer points. □

Proof of Theorem 1.5. Let $x_1^{a_1^i} x_2^{a_2^i} x_3^{a_3^i}$ be monomials appearing in $s_\lambda * s_\mu(x_1, x_2, x_3)$ with non zero coefficients. By Proposition 4.6 we have that $\mathcal{P}(\lambda, \mu; \mathbf{a}^i) \cap \mathbb{Z}^3 \neq \emptyset$. Suppose that (c_1, c_2, c_3) is in the convex hull of $\{\mathbf{a}^i\}_i$, so $\mathbf{c} = \sum_i t_i \mathbf{a}^i$ for some $t_i \in [0, 1]$ with $t_1 + t_2 + \cdots = 1$. By Proposition 5.1 we have that $\mathcal{P}(\lambda, \mu, \mathbf{c}) \neq \emptyset$. Then if $c_i \in \mathbb{Z}$ by Theorem 5.2 we have $\mathcal{P}(\lambda, \mu; \mathbf{c}) \cap \mathbb{Z}^3 \neq \emptyset$ and thus $\mathbf{x}^{\mathbf{c}}$ appears as a monomial in $s_\lambda * s_\mu$. By the characterization (snp), the polynomial $s_\lambda * s_\mu(x_1, x_2, x_3)$ has a saturated Newton polytope. □

6 Positivity of Kronecker coefficients

First, we will discuss the limiting case of the SNP property.

Proof sketch of Theorem 1.7. By Caratheodory's theorem, it suffices to show that if every point is a convex combination of $k + 1$ points from our set and is contained in the set, then the set is convex. Consider points $\alpha^1, \alpha^2, \cdots, \alpha^{k+1} \in \bigcup_{p=1}^{\infty} \frac{1}{p} M_k(p\lambda, p\mu)$ where $M_k(p\lambda, p\mu) := M_k(s_{p\lambda} * s_{p\mu})$. For each α^i, choose p_i such that $\alpha^i \in \frac{1}{p_i} M_k(p_i\lambda, p_i\mu)$. Let $p = lcm(p_1, \ldots, p_k)$. Employing the semigroup property, establish that $\alpha^i \in \frac{1}{p} M_k(p\lambda, p\mu)$

for all i. Suppose θ is a rational convex combination of $\alpha^1, \alpha^2, \ldots, \alpha^{k+1}$. Apply the semigroup property to show that θ is in $\frac{1}{qp} M_k(qp\lambda, qp\mu)$ for some carefully chosen $q \in \mathbb{Z}$, implying convexity of the set by Caratheodory's theorem. □

We next consider positivity criteria for Kronecker coefficients. Suppose that $g(\lambda, \mu, \nu) > 0$, then s_ν appears in $s_\lambda * s_\mu$, and so its leading monomial m_ν also appears, so $\mathcal{P}(\lambda, \mu, \nu) \cap \mathbb{Z}^r \neq \emptyset$, where $r = \min\{\ell(\lambda), \ell(\mu)\}\ell(\nu)$. Then from Section 4 we must have that $P(\lambda; \nu) \cap P(\mu; \nu)$ has an integer point. We can then apply Corollary 4.5 and its inequalities to infer that the polytope $\mathcal{P}(\lambda, \mu, \nu)$ has an integer point.

We define an mLR-consistent triple (I, J, K) of subsets of $[1, \ldots, \ell k]$ as an LR-consistent triple satisfying the condition that $|I \cap [\ell(j-1)+1, \ldots, \ell j]| = |K \cap [\ell(j-1), \ldots, \ell j]|$ for every $j = 1, \ldots, k$.

Theorem 6.1. *Suppose that $g(\lambda, \mu, \nu) > 0$ and let $\ell = \min\{\ell(\mu), \ell(\nu)\}$ Then there exist nonnegative integers $\{\alpha^i_j\}_{i\in[k], j\in[\ell]}$ satisfying*

$$\sum_j \alpha^i_j = \lambda_i, \qquad \text{for } i \in [k]; \tag{6.1}$$

$$\alpha^i_j \geq \alpha^i_{j+1}, \qquad \text{for } j \in [\ell-1],\ i \in [k]; \tag{6.2}$$

$$\sum_{(i,j)\in D(I)} \alpha^i_j \leq \min\{\sum_{j\in J} \mu_j, \sum_{j \in J} \nu_j\}, \qquad \text{for every mLR-consistent } (I, J, K). \tag{6.3}$$

Proof sketch of Theorem 6.1. For I, J, K to be an LR-consistent triple, we must have $\rho(K) \subset \rho(I)$, which implies that if $I = \{i_1 < i_2 < \cdots < i_s\}$ and $K = \{k_1 < \cdots < k_s\}$ then $k_j \leq i_j$ for all j. Thus in (4.5) we have $\sum_{(d,r)\in D(K)} n(k-d) \geq \sum_{(i,j)\in D(I)} n(k-i)$, with a difference of at least n if the two sums are not equal. If they are not equal then the inequalities are trivially satisfied. Thus we assume that we have equality. Thus $I = \cup I_p$ and $K = \cup K_p$, where $I_j, K_j \subset [\ell(j-1)+1, \ldots, \ell j]$ and $|I_j| = |K_j|$ and for all such sets, and a set J with $|J| = |I|$ and $c^{\rho(I)}_{\rho(J)\rho(K)} = 1$, which is the definition of mLR-consistent. □

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Whirling and rowmotion dynamics on the chain of V's poset

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Abstract. Given a finite poset P, we study the *whirling* action on vertex-labelings of P with the elements $\{0, 1, 2, \ldots, k\}$. When such labelings are (weakly) order-reversing, we call them *k-bounded P-partitions*. We give a general equivariant bijection between k-bounded P-partitions and order ideals of the poset $P \times [k]$ which conveys whirling to the well-studied *rowmotion* operator. As an application, we derive periodicity and homomesy results for rowmotion acting on the *chain of V's* poset $\mathsf{V} \times [k]$. We are able to generalize some of these results to the more complicated dynamics of rowmotion on $\mathsf{C}_n \times [k]$, where C_n is the **claw poset** with n unrelated elements each covering $\widehat{0}$.

Keywords: posets, chain of V's, dynamical algebraic combinatorics, homomesy, P-partitions, rowmotion, whirling.

1 Introduction

We connect the well-studied operation of *rowmotion* on the order ideals of a finite poset with the less familiar *whirling* action on P-partitions with bounded labels. One of our main results is an equivariant bijection that carries one to the other for any finite poset P. We then leverage this to study the rowmotion action on the "chain of V's" poset $\mathsf{V}_k := \mathsf{V} \times [k]$ (a 3-element V-shaped poset cross a finite chain, see Figure 2), which has surprisingly good dynamical properties. We also generalize this to the case where we replace V with a ***n*-claw**, a poset with a single minimal element covered by exactly n incomparable elements. In both cases we obtain both periodicity results and homomesy.

Let P be a finite poset, and $\mathcal{J}(P)$ be the set of order ideals of P. (For basic poset definitions, we refer the reader to Stanley [9, Ch. 3].) *Combinatorial rowmotion* is an invertible map $\rho : \mathcal{J}(P) \to \mathcal{J}(P)$ which takes each ideal $I \in \mathcal{J}(P)$ to the order ideal generated by the minimal elements of the complement of I in P. The periodicity of this map on products of chains was first studied by Brouwer and Schrijver [2], and Cameron and Fon-der-Flaass [3]. Later Striker and Williams [10] considered it as one element of the "toggle group" of a poset and related it to a kind of "promotion" operator on order

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ideals. Around the same time, Armstrong, Stump, and Thomas [1] studied rowmotion on *root posets*, relating it to "Kreweras complementation" on noncrossing partitions, and used this to prove a conjecture of Panyushev about the equality of the average cardinality of antichains for each rowmotion orbit.

Propp and Roby [7] noticed that this conjecture was merely one instance of a much broader phenomenon which they dubbed *homomesy*. Given a finite set S, a "statistic" $f : S \to \mathbb{C}$, and an invertible map φ on S, we call f *homomesic* if the average value of f is the same for every φ-orbit $\mathcal{R}$, i.e., $\dfrac{1}{\#\mathcal{R}} \sum_{x \in \mathcal{R}} f(x) = c$, where c is a constant not dependent on the choice of orbit $\mathcal{R}$. The confluence of all this work was the beginning of *dynamical algebraic combinatorics* as a distinct area within algebraic combinatorics (with antecedents going back to the Robinson–Schensted–Knuth correspondence and related operations on Young tableaux such as promotion, evacuation, and cyclage). In the past decade, the subfield has grown in a number of directions, and the study of rowmotion has been of continuing interest. For more background information, see the survey articles of Hopkins [4], Roby [8], and Striker [11].

Cameron and Fon-der-Flaass [3] were the first to describe rowmotion as a product of involutions called *toggles*, as detailed in Section 1.1. A natural generalization of toggling at a poset element x is "*whirling* at x," which cycles the label at x among j possible values. (Toggles are the case when $j = 2$.) Joseph, Propp, and Roby defined these and the operation of *whirling* on sets of functions between finite sets, obtaining various homomesy results for various classes of functions (injective, surjective, etc.) [6]. This is described in Section 2.

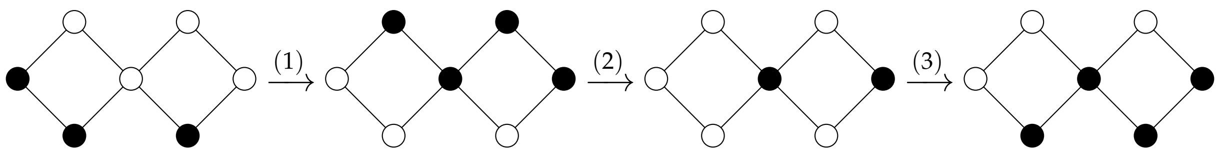
A bijective function $f : P \to [p]$ (with $\#P = p$) such that $f(x) < f(y)$ whenever $x <_P y$ is called a *linear extension*. We denote by $\mathcal{L}(P)$ the set of *all linear extensions* of P; its cardinality, $e(P)$, is an important numerical invariant of a poset. Its refinement, the *order polynomial* $\Omega_P(k)$, counts the number of k-bounded P-partitions. For some special posets P, mainly ones connected with Lie theory (root and minuscule posets) and those of partition or shifted shapes, product formulae for $\Omega_P(k)$ are known. Hopkins surveys these posets, the formulae, and gives the heuristic: *Posets with order-polynomial product formulae are the same as the posets with good dynamical behavior.* The one poset in his list whose rowmotion dynamics were relatively unexplored is $\mathsf{V} \times [k]$, a gap this paper fills. In separate work Hopkins and Rubey study the dynamics of Schützenberger promotion on linear extensions of $\mathsf{V} \times [k]$, which also exhibit unusually good behavior [5].

This paper is organized as follows. In Section 1 after the introduction, we review the toggling definition of rowmotion. Section 2 describes whirling, and includes the equivariant bijection which allows us to study rowmotion on V_k as whirling on k-bounded P-partitions. Section 3 contains our main periodicity and homomesy results for rowmotion on V_k, which use decompositions of the "orbit board" of the corresponding whirling action into "whorms". Finally, Section 4 contains the periodicity and homomesy results which generalize to rowmotion on the "chain of claws" graph, $\mathsf{C}_n \times [k]$. A version of this paper with full proofs will appear soon on the arXiv.

1.1 Rowmotion as a product of toggles

Definition 1.1. We define the (order-ideal) rowmotion map, $\rho : \mathcal{J}(P) \to \mathcal{J}(P)$ as follows: For any $I \in \mathcal{J}(P)$, $\rho(I)$ is the order ideal generated by the minimal elements of the complement of I, as in the example below.

Example 1.2. Here is one iteration of ρ on an order ideal with the action broken down into its three steps: (1) complement, (2) take minimal elements, (3) saturate down.

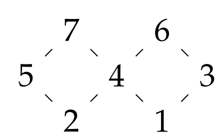


Rowmotion has an alternate definition as a composition of toggling involutions, which has proven useful for understanding and generalizing many of its properties. Cameron and Fon-der-Flaass [3] showed that for any finite poset P, rowmotion can be realized as "toggling once at each element of P along any linear extension (from top to bottom)". Other toggling orders also lead to interesting maps, such as Striker–Williams "promotion" (of order ideals) of a poset, which is toggling from left-to-right along "files" of a poset [10].

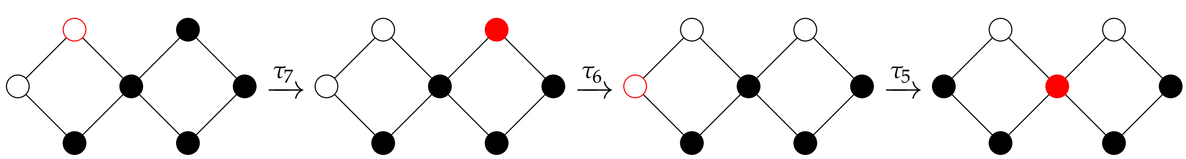
Definition 1.3. For each fixed $x \in P$ define the *(order-ideal) toggle* $\tau_x : \mathcal{J}(P) \to \mathcal{J}(P)$ by

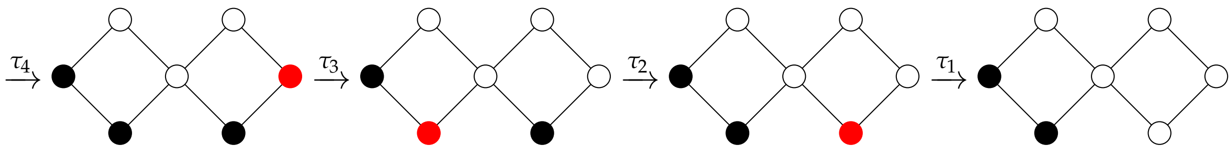
$$\tau_x(I) = \begin{cases} I \setminus \{x\} & \text{if } x \in I \text{ and } I \setminus \{x\} \in \mathcal{J}(P) \\ I \cup \{x\} & \text{if } x \notin I \text{ and } I \cup \{x\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

It is an easy exercise to show that order-ideal toggles [3, §2] are involutions, and that toggles at incomparable elements commute (a special case of Prop 2.7).

Example 1.4. We will toggle each node down the following fixed linear extension: at each step we consider whether or not to toggle the red node in or out.  .

For this linear extension we toggle the elements from top-to-bottom, then left-to-right.





Proposition 1.5 ([3, Lemma 1]). *Let $x_1, x_2, \ldots, x_p$ be any linear extension (i.e., any order-preserving listing of the elements) of a finite poset P with p elements. Then the composite map $\tau_{x_1}\tau_{x_2}\cdots\tau_{x_p}$ coincides with the rowmotion operation ρ.*

2 Whirling

2.1 Whirling function between finite sets

Let $\mathcal{F} \subseteq [k]^{[n]}$ be a family of functions $f : [n] \to [k]$. For the rest of section 2.1, we use $\{1, \ldots, k\} = [k]$ to represent the congruence classes of $\mathbb{Z}/k\mathbb{Z}$, as opposed to the usual $\{0, 1, \ldots, k-1\}$. For fixed values of k and n, we represent such functions in *one-line* notation, e.g., $f = 21344$ represents the function $f \in [4]^{[5]}$ with $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, $f(4) = 4$, and $f(5) = 4$.

Definition 2.1 ([6, Definition 2.3]). For $f \in \mathcal{F}$ we define the *whirl* $w_i : \mathcal{F} \to \mathcal{F}$ at index i as follows: repeatedly add 1 (modulo k) to the value of $f(i)$ until we get a function in $\mathcal{F}$.

Example 2.2. Let $\mathcal{F} = \{f \in [4]^{[5]} : f(1) \neq f(2)\}$. If we apply w_2 to $f = 21344$, adding 1 in the second position gives 22344, but this is not in $\mathcal{F}$. Adding 1 again in this position gives the result: $w_2(f) = 23344$.

4	1	5
6	2	1
3	4	2
5	6	3
1	2	4
3	5	6
4	1	2
5	3	4
6	5	1
2	6	3

Figure 1

We will now highlight some specific results from the paper where whirling was first introduced. Let $\mathrm{Inj}_m(n,k)$ be the set of *m-injective functions*, that is, functions $f : [n] \to [k]$ such that $\#f^{-1}(t) \leq m$ for all $t \in [k]$. Similarly, let $\mathrm{Sur}_m(n,k)$ be the set of *m-surjective functions*, that is, $f : [n] \to [k]$ such that $\#f^{-1}(t) \geq m$ for all $t \in [k]$. Note that injective functions are 1-injections and surjective functions are 1-surjections. We also define the statistic $\eta_j(f) = \#f^{-1}(\{j\})$.

Theorem 2.3. *[6, Theorem 2.11] Fix $\mathcal{F}$ to be either $\mathrm{Inj}_m(n,k)$ or $\mathrm{Sur}_1(n,k)$ for given $n, k, m \in \mathbb{P}$. Then under the action of $\mathbf{w} = w_n \circ w_{n-1} \circ \cdots \circ w_1$ on $\mathcal{F}$, η_j is $\frac{n}{k}$-mesic for any $j \in [k]$*

This result is conjectured to hold for $\mathrm{Sur}_m(n,k)$, but is still open for $m > 1$. Proof details can be found in Sections 2.2–2.4 of [6].

Example 2.4. Here is the orbit of $\mathbf{w}$ on $\mathrm{Inj}_1(3,6)$ containing $f = 415$.

$$415 \xrightarrow{\mathbf{w}} 621 \xrightarrow{\mathbf{w}} 342 \xrightarrow{\mathbf{w}} 563 \xrightarrow{\mathbf{w}} 124 \xrightarrow{\mathbf{w}} 356 \xrightarrow{\mathbf{w}} 412 \xrightarrow{\mathbf{w}} 534 \xrightarrow{\mathbf{w}} 651 \xrightarrow{\mathbf{w}} 263 \xrightarrow{\mathbf{w}}\!\downarrow$$

Figure 1 shows the corresponding *orbit board* (a matrix whose rows are the successive orbit elements) partitioned into chunks. Notice that each value $1, 2, \ldots, 6$ appear exactly 5 times in this orbit of size 10, in accordance with the $1/2$-mesy of Theorem 2.3.

2.2 k-bounded P-partitions

Now we extend the definition of whirling to k-bounded P-partitions. Throughout the rest of the paper, P will denote a finite poset. Define $[0, k] := \{0, 1, 2, \ldots, k\}$.

A *P-partition* is a map σ from P to $\mathbb{N}$ such that if $x <_P y$, then $\sigma(x) \geq \sigma(y)$ [9, Ch. 3].

Definition 2.5. A *k-bounded P-partition* is a function $f : P \to [0, k]$ such that if $x \leq_P y$, then $f(x) \geq f(y)$. Let $\mathcal{F}_k(P)$ be the set of all such functions.

Throughout the rest of the paper we use $\{0, 1, \ldots, k\}$ to represent the congruence classes of $\mathbb{Z}/(k+1)\mathbb{Z}$, as usual.

Definition 2.6. For $f \in \mathcal{F}_k(P)$ and $x \in P$, define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at x*, as follows: repeatedly add 1 (mod $k+1$) to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

The case $k = 1$ of the above definition recovers toggling of order ideals (Def. 1.3).

Proposition 2.7. *If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.*

Definition 2.8. Let $(x_1, x_2, \ldots x_p)$ be a linear extension of P. Define $w : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$ by $w := w_{x_1} w_{x_2} \ldots w_{x_p}$. The above proposition shows that this is well-defined, since one can get from any linear extension to any other by a sequence of interchanges of incomparable elements.

Example 2.9. Let P be the V poset with labels $\begin{array}{ccc}\ell&&r\\&\diagdown\;\diagup&\\&c&\end{array}$, $k = 2$, and $w = w_c w_r w_\ell$.

$$\begin{array}{ccc}0&&2\\&\diagdown\;\diagup&\\&2&\end{array} \xrightarrow{w_\ell} \begin{array}{ccc}1&&2\\&\diagdown\;\diagup&\\&2&\end{array} \xrightarrow{w_r} \begin{array}{ccc}1&&0\\&\diagdown\;\diagup&\\&2&\end{array} \xrightarrow{w_c} \begin{array}{ccc}1&&0\\&\diagdown\;\diagup&\\&1&\end{array}$$

There is a natural bijection between order ideals of a poset P and 1-bounded P-partitions in $\mathcal{F}_1(P)$. Specifically, a 1-bounded P-partition in $\mathcal{F}_1(P)$ is simply the indicator function of an order ideal $I \in J(P)$. We extend this to an equivariant bijection $\mathcal{F}_k(P) \to \mathcal{J}(\mathcal{P} \times [k])$ which sends w to ρ, meaning the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_k(P) \arrow[r, "w"] \arrow[d, leftrightarrow] & \mathcal{F}_k(P) \arrow[d, leftrightarrow] \\
\mathcal{J}(P \times [k]) \arrow[r, "\rho"] & \mathcal{J}(P \times [k])
\end{tikzcd}$$

We will call the chains $\{(x,1),(x,2),\ldots,(x,k)\} \subseteq P \times [k]$, for $x \in P$, the *fibers* of $P \times [k]$, and construct an equivariant bijection that first sends w_x to order-ideal toggling down the fiber $\{(x,1),(x,2),\ldots,(x,k)\}$.

Lemma 2.10. *There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w_x to the toggle product $\tau_{(x,1)}\tau_{(x,2)}\cdots\tau_{(x,k)}$.*

Theorem 2.11. *Fix any linear extension $(x_1, x_2, \ldots, x_p) \in \mathcal{L}(P)$. There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends whirling, $w = w_{x_1}w_{x_2}\cdots w_{x_p}$, to rowmotion on $\mathcal{J}(P \times [k])$.*

The following definitions will allow us to partition orbit boards of whirling into subsets called *whorms*.

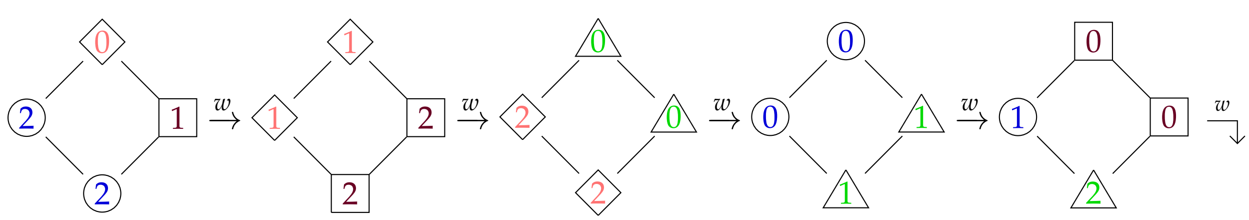
Definition 2.12. For any $x \in P$ and $f \in \mathcal{F}_k(P)$, define (x, f) to be a *whirl element*. The whirl element (y, g) is *whirl successive* of (x, f) if either:

1. $y = x$ and $g(y) = w(f)(x) = f(x) + 1$, or
2. x covers y, $f = g$, and $f(x) = g(y)$.

We consider whirl-successive elements to be whirl elements which are one step away from each other, either by moving one covering relation down the poset or by whirling the function at the element, and ending one label greater. *While we must consider the entire P-partitions f and g to check whether two whirl elements are whorm connected, we think of whirl elements as being simply $(x, f(x))$, the location and its label, and indicate them in this way in the examples that follow.*

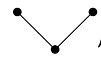
Definition 2.13. Two whirl elements (x, f) and (y, g) are *whorm-connected* if there exists a sequence of whirl-successive elements $\{(x, f) = (x_0, f_0), (x_1, f_1), \ldots, (x_p, f_p) = (y, g)\}$. A *whorm* is a maximal set of whorm-connected whirl elements, that is, if (x, f) is in a whorm and (x, f) is whorm-connected to (y, g), then (y, g) is in the whorm.

Example 2.14. An orbit of whirling P-partitions (for $P = [2] \times [2]$) with its four whorms indicated by the same color and (redundantly) node-shape.



3 Periodicity and homomesy for rowmotion on $\mathsf{V} \times [k]$

In this section we consider the dynamics of rowmotion acting on the order ideals of the chain of V's poset V_k, establishing its periodicity and finding interesting examples of homomesy.

Definition 3.1. Let V be the 3-element poset with Hasse diagram , and define $\mathsf{V}_k = \mathsf{V} \times [k]$, where $[k]$ is the chain poset. We call V_k the *chain of V's poset*.

Example 3.2. Figure 2 shows the Hasse diagram of V_k with our vertex-labeling convention.

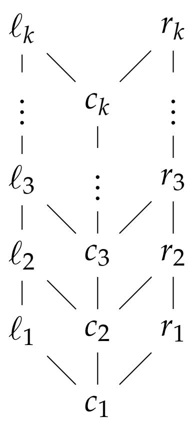


Figure 2

Our main goals for this section are the following theorems. We will leverage the equivariant bijection and the notion of whorms from the last section.

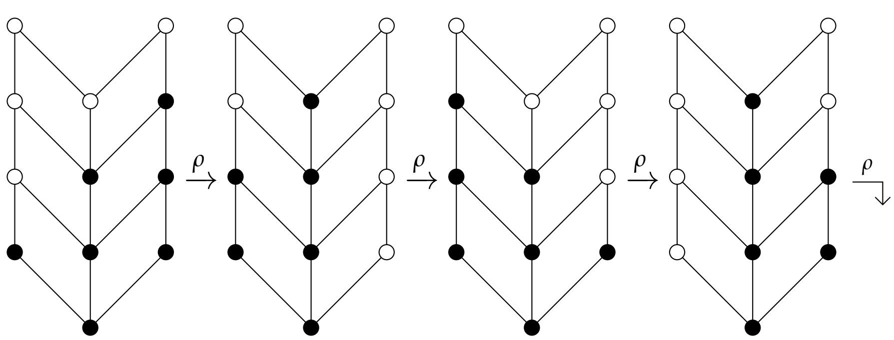
Theorem 3.3. *The order of rowmotion on $\mathcal{J}(\mathsf{V}_k)$ is $2(k+2)$.*

Theorem 3.4. *Let χ_s be the indicator function for $s \in \mathsf{V}_k$. We have the following homomesies for the action of ρ on $\mathcal{J}(\mathsf{V}_k)$*

1. *The statistic $\chi_{\ell_i} - \chi_{r_i}$ is 0-mesic for all $i \in [k]$.*

2. *The statistic $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ is $\frac{2(k-1)}{k+2}$-mesic.*

Example 3.5. This ρ-orbit on $\mathcal{J}(\mathsf{V}_4)$ has size 4, which divides $2(4+2) = 12$. The homomesies are also easily checked, e.g., across the orbit the total number of elements at rank 1 in the side fibers is 6, minus the two at the top of the center fiber, for an average of $\frac{6-2}{4} = 1 = \frac{2(4-1)}{4+2}$, agreeing with Theorem 3.4(2).



To prove these theorems we utilize our equivariant bijection (Theorem 2.11) from $\mathcal{J}(\mathsf{V}_k)$ to $\mathcal{F}_k(\mathsf{V})$, then represent the latter by triples $f = (\ell, c, r)$ with $\ell \leq c$ and $r \leq c$. This bijection ϕ sends an order ideal I to a triple (ℓ, c, r), counting the number of elements of the order ideal in the left, center, and right fibers respectively.

Example 3.6. Here is the orbit of $\mathcal{F}_4(\mathsf{V})$ corresponding to Example 3.5.

$$(1,3,3) \xrightarrow{w} (2,4,0) \xrightarrow{w} (3,3,1) \xrightarrow{w} (0,4,2) \xrightarrow{w}\!\!\lrcorner$$

Proposition 3.7. *The number of order ideals of* V_k *is given by* $|\mathcal{J}(\mathsf{V}_k)| = \frac{k(k+1)(2k+1)}{6}$.

3.1 Center-seeking whorms

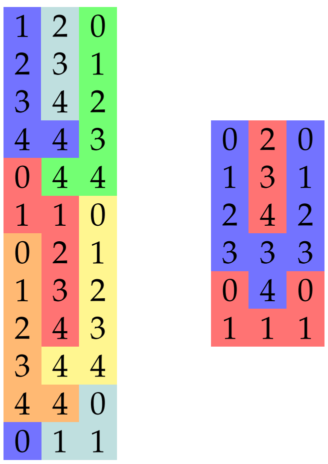


Figure 3

To show that the order of ρ on $\mathcal{J}(\mathsf{V}_k)$ is $2(k+2)$ we end up proving something stronger, namely that $\rho^{k+2}(I)$ is the reflection of I across the the center chain. Our method is to investigate the whorms that arise from repeatedly whirling a k-bounded P-partition.

Recall from Definition 2.13 that, given a whirling orbit board, $\mathcal{O} = \{f, w(f), w^2(f), \dots\}$ of w on $\mathcal{F}_k(\mathsf{V})$, a whorm ς is a maximal set of whorm-connected elements. Figure 3 shows two orbit boards of $\mathcal{F}_4(\mathsf{V})$, one with six whorms and one with two whorms. Notice that each whorm in the second orbit has two "starting" positions.

Each whorm in an orbit board of $\mathsf{V} \times [k]$ starts on the left, or the right, or both left and right; we call the former *one-tailed* and the later *two-tailed*. Since these whorms move down the orbit board at every step, except for one move to the center, we consider them as a sequence of function values in the orbit board which start at 0 and end at k, where one value is repeated when moving into the center. We call these *center-seeking whorms*. (Since an orbit board is actually a cylinder, we have a "can of worms" to deal with.) In the left orbit of Figure 4 we isolate one example of a left whorm: $\varsigma = \{(\ell,(0,3,3)), (\ell,(1,4,0)), (\ell,(2,2,1)), (c,(2,2,1)), (c,(0,3,2)), (c,(1,4,3))\}$, visualized within an orbit board of $\mathcal{F}_4(\mathsf{V})$. It is easy to see that an orbit board is tiled either entirely by one-tailed whorms or entirely by two-tailed whorms. (See the discussion at the start of Section 4.)

We first observe that all whorms have $k+2$ elements, since each contains the $k+1$ elements $0, ..., k$, exactly one of which is doubled.

Define $\mathsf{b}(\varsigma) := 1 + \min\{f(c) : (c,f) \in \varsigma\}$, the number of elements in the outer columns and $\mathsf{e}(\varsigma) := k+2-\mathsf{b}(\varsigma)$, the number of elements in the center column. For the red whorm in the orbit on the left of Figure 4, $\mathsf{b}(\varsigma) = 3$ and $\mathsf{e}(\varsigma) = 3$.

Example 3.8. The right orbit board in Figure 4 is the previous example with all the whorms colored. The number of elements in the left column of the yellow, red, and orange whorms are $5, 3, 4$ respectively, and the orbit board is of length 12.

It follows that the order of whirling divides the sum of $\mathsf{b}(\varsigma)$ over all whorms $\varsigma \in S$. In the setting of $\mathcal{F}_k(\mathsf{V})$, as long as we know $\mathsf{b}(\varsigma)$ and whether $f(\ell) = 0$, $f(r) = 0$, or both, then we can recover the entire whorm.

Definition 3.9. We place a circular order on the whorms. Let ς_1 and ς_2 be whorms in an orbit board of $\mathcal{F}_k(\mathsf{V})$. If there exists $(c, f) \in \varsigma_1$ with $f(c) = k$ such that $(c, w(f)) \in \varsigma_2$, then we say ς_2 is *in front of* ς_1. We call a sequence of whorms *consecutive* if each is in front of the next. In a one-tailed orbit board, consecutive whorms alternate starting from the left and right.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

Figure 4

Example 3.10. In Figure 4 the blue (horizontal lines) whorm is in front of the red (crosshatch) whorm, which is in front of the green (northwest lines) whorm.

Lemma 3.11. *Assume an orbit board $\mathcal{O}$ of w on $\mathcal{F}_k(\mathsf{V})$ has all one-tailed whorms. Let ς_1, ς_2, and ς_3 be three consecutive whorms, that is, ς_3 is in front of ς_2 which is in front of ς_1 in $\mathcal{O}$. Then,* $\mathsf{b}(\varsigma_1) + \mathsf{b}(\varsigma_2) + \mathsf{b}(\varsigma_3) = 2(k+2)$. *Otherwise, if there are two-tailed whorms, then* $\mathsf{b}(\varsigma_1) + \mathsf{b}(\varsigma_2) = k + 2$.

In fact, the entire orbit board can be reconstructed simply from knowing the values of $\mathsf{b}(\varsigma_1)$ and $\mathsf{b}(\varsigma_2)$ for two consecutive whorms in the one-tailed case, and from a single $\mathsf{b}(\varsigma_1)$ in the two-tailed case.

Example 3.12. In Figure 4 we have $k = 4$, $\mathsf{b}(\text{green}) = 4$, $\mathsf{b}(\text{red}) = 3$, and $\mathsf{b}(\text{blue}) = 5$, which sum to $12 = 2(4+2)$.

Lemma 3.13. *Given an orbit board with one-tailed whorms, let $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ be consecutive, then* $\mathsf{b}(\varsigma_4) = \mathsf{b}(\varsigma_1)$. *Furthermore, if the orbit board contains two-tailed whorms, then* $\mathsf{b}(\varsigma_1) = \mathsf{b}(\varsigma_3)$.

Notice that for orbits with one-tailed whorms, we are *not* claiming the board starts to repeat; since whorms alternate sides, ς_4 will start on the opposite side from ς_1. If we keep applying the previous Lemma to even more consecutive whorms, we see $\mathsf{b}(\varsigma_5) = \mathsf{b}(\varsigma_2)$ and $\mathsf{b}(\varsigma_6) = \mathsf{b}(\varsigma_3)$. Finally we get $\mathsf{b}(\varsigma_7) = \mathsf{b}(\varsigma_1)$ and the pattern repeats. Therefore there are at most six unique whorms in a one-tailed orbit board.

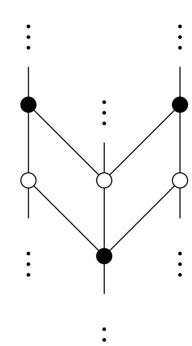


Figure 5

Lemma 3.14. *Given an orbit board with one-tailed whorms, there are at most six distinct whorms.*

Theorem 3.15. *Let $(x, y, z) \in \mathcal{F}_k(\mathsf{V})$, then $w^{k+2}(x, y, z) = (z, y, x)$.*

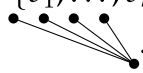
Corollary 3.16. *The order of w on $\mathcal{F}_k(\mathsf{V})$ divides $2(k+2)$.*

Lemma 3.17. *Under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V}_k)$, the difference of successive flux-capacitor indicator functions, $F_i - F_{i+1}$ is $\frac{3}{k+2}$-mesic for $i \in [2, k-1]$.*

This lemma can be generalized to the following theorem.

Theorem 3.18. *For $k > 1$. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$. Under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V}_k)$, the difference of arbitrary flux- capacitors is $F_i - F_j$ is $\frac{3(j-i)}{k+2}$-mesic.*

4 Periodicity and homomesy for rowmotion on $\mathsf{C}_n \times [k]$

We define the *claw poset* $\mathsf{C}_n = \{b_1, \ldots, b_n, \widehat{0}\}$ where each b_i covers $\widehat{0}$. For example, the Hasse diagram of C_4 would be .

Using the established equivariant bijection between $\mathcal{J}(\mathsf{C}_n \times [k])$ and k-bounded P-partitions $\mathcal{F}_k(\mathsf{C}_n)$ that sends rowmotion to whirling, we can prove similar homomesies and periodicity to that of $\mathsf{C}_2 = \mathsf{V}$. Now instead of *triples* of numbers, we will consider orbit boards of $(n+1)$-*tuples* on $[0, k]$, $\Big(f(b_1), f(b_2), \ldots, f(b_n), f(\widehat{0})\Big)$, satisfying $f(b_i) \leq f(\widehat{0})$ for each $i \in [n]$.

1	0	1	2	3
2	1	2	3	3
3	2	3	0	3
0	3	0	1	3
1	0	1	2	2
2	1	2	0	3
3	2	3	1	3
0	3	0	2	3
1	0	1	3	3
2	1	2	0	2
0	2	0	1	3
1	3	1	2	3
2	0	2	3	3
3	1	3	0	3
0	2	0	1	2

Figure 6

In Figure 6, note that if two entries are the same among the first n in a given row, then those positions (columns) remain the same throughout the *entire* orbit board. This is because the entries $b_1, \ldots, b_n$ represent the result of whirling at *incomporable* elements of the poset C_n. Furthermore, these two entries must belong to the same whorm, because each will be whorm-connected via $\widehat{0}$ exactly when their value matches the value of the last entry. These observations will allow us to generalize our peridocity and homomesy results from V to C_n.

Definition 4.1. For $A \subseteq [0, k]$, define the family of order-reversing maps $\mathcal{F}_k^A(\mathsf{C}_n) = \{f : f \in \mathcal{F}_k(\mathsf{C}_n) \text{ and } f(b_j) \in A \text{ for all } j \in [n]\}$. For any fixed A we denote $\overline{w}$ to be whirling on the non-$\widehat{0}$ elements of order-reversing maps $f \in \mathcal{F}_k^A(\mathsf{C}_n)$. Which is equivalent to incrementing each non-$\widehat{0}$ value, but only allowing values within A.

Given $f \in \mathcal{F}_k(\mathsf{C}_n)$, set $A(f) = \{a : f(b_j) = a \text{ for some } j \in [n]\}$, the set of values that the P-partition f attains on the non-$\widehat{0}$ elements of C_n. Set $\alpha = \#A$ and $\alpha(f) = \#A(f)$. For any $f, g \in \mathcal{F}_k(\mathsf{C}_n)$, if $g = w^j(f)$ for some $j \in \mathbb{N}$, then $\alpha(f) = \alpha(g)$. So we may sometimes write just α when an orbit is fixed. For this section, we impose $A = A(f)$ when computing $\overline{w} : \mathcal{F}_k(\mathsf{C}_n) \to \mathcal{F}_k(\mathsf{C}_n)$ of an order-reversing map f.

Example 4.2. Consider $f = (1, 3, 3, 0, 4, 1, 6) \in \mathcal{F}_9(\mathsf{C}_6)$. We see $A(f) = \{0, 1, 3, 4\}$ so

$$\overline{w}(1, 3, 3, 0, 4, 1, 6) = (3, 4, 4, 1, 0, 3, 6).$$

The last entry remains unchanged and the earlier entries are increasing cyclically within the set $A(f) = \{0, 1, 3, 4\}$. In the special case where V $(= \mathsf{C}_2)$ are set within any orbit A will have at most two elements, hence $\overline{w}$ will just toggle between those two values at the left and the right. This means that $\overline{w}$ is the same as reflecting values across the center of V, which we already saw was the effect of w^{k+2}. Our next result generalizes this to the case C_n.

Lemma 4.3. *Let $f \in \mathcal{F}_k(\mathsf{C}_n)$ and $\alpha = \alpha(f)$. If $\varsigma_1, \ldots, \varsigma_{\alpha+1}$ are $\alpha + 1$ consecutive whorms, then*

$$\mathsf{b}(\varsigma_1) + \cdots + \mathsf{b}(\varsigma_{\alpha+1}) = \alpha(k+2).$$

Proposition 4.4. *Let w be whirling k-bounded P-partitions on $\mathcal{F}_k(\mathsf{C}_n)$. For any $f \in \mathcal{F}_k(\mathsf{C}_n)$ and $A = A(f)$, we have $w^{k+2}(f) = \overline{w}(f)$.*

The proof of this theorem can be approached with whorms. Define $\mathsf{b}(\varsigma) = 1 + \min\{f(\hat{0}) : (\hat{0}, f) \in \varsigma\}$. If there exists $(\hat{0}, f) \in \varsigma_1$ with $f(\hat{0}) = k$ such that $(\hat{0}, w(f)) \in \varsigma_2$, then we say ς_2 is *in front of* ς_1. In Figure 6, the pink snake is in front of the red snake.

Corollary 4.5. *Let $f \in \mathcal{F}_k(\mathsf{C}_n)$ and $\alpha = \alpha(f)$. If $\varsigma_1, \ldots, \varsigma_{\alpha+2}$ are consecutive whorms, then* $\mathsf{b}(\varsigma_1) = \mathsf{b}(\varsigma_{\alpha+2})$.

If $f \in \mathcal{F}^k(\mathsf{C}_n)$ satisfies $f(\widehat{0}) \notin A(f)$, then f will contain entries from $\alpha + 1$ distinct whorms. From Proposition 4.4, we will have at most $\alpha(\alpha+1)$ whirls in an orbit board (each action of $\widehat{w}$ resulting in $\alpha + 1$ whorms potentially distinct from those previous, as in Figure 6). On the other hands, consider the orbit board of $\mathcal{F}_3(\mathsf{C}_4)$ in Figure 7 with $\alpha = 4$. Here $w(f) = w^5(f)$ so the orbit is only 4 rows long with 4 distinct whorms. In general, we can extend this to a super orbit board with $\alpha(\alpha+1)$ whorms.

0	1	2	3	3
1	2	3	0	3
2	3	0	1	3
3	0	1	2	3

Figure 7

Theorem 4.6. *Let $m = \min(k, n)$. The order of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$ divides $m!(k+2)$.*

Only the analogue of the first homomesy in Theorem 3.4 holds.

Theorem 4.7. *Let $\chi_{(i,j)}$ be the indicator function for $(i, j) \in \mathsf{C}_n \times [k]$. Then for the action of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$, the statistic $\chi_{(i,a)} - \chi_{(j,a)}$ is 0-mesic for all $i, j \in [n]$ and $a \in [k]$.*

Remark 4.8. The average of the statistic $\left(\sum_{i=1}^{n} \chi_{(i,1)}\right) - \chi_{(\hat{0},k)}$ (analogous to Theorem 3.4(2)) turns out to be **dependent** on $\alpha(f)$ (for any $f \in \mathcal{O}$) and can be computed as

$$\frac{n(\alpha)(k+2) - (n+\alpha)(\alpha+1)}{(\alpha)(k+2)}.$$

Consider the super-orbit with $n\alpha(k+2)$ entries among the non-minimal elements. We know $\chi_{(i,1)}(I) = 0$ if and only if for corresponding f, $f(i) = 0$. But this is counted by the number of whorm beginnings, that is $n(\alpha+1)$. Furthermore, $\chi_{(\widehat{0},k)}(I)=1$ if and only if for corresponding f, $f(\widehat{0})=k$, which is counted by the number of whorm endings, that is $\alpha(\alpha+1)$. Therefore the average is obtained.

The "flux-capacitor" homomesy of Theorem 3.18 also generalizes to the claw-graph setting, and has a similar proof.

Theorem 4.9. *Let* $B_i=\chi_{(i-1,\widehat{0})}+\sum_{j=1}^{n}\chi_{(i,j)}$. *Then for the action of rowmotion on* $\mathcal{J}(\mathsf{C}_n\times[k])$, B_i-B_j *is* $\frac{(j-i)(n+1)}{k+2}$*-mesic for all* $i,j\in[n]$.

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A Whitney polynomial for hypermaps

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Abstract. We introduce a Whitney polynomial for hypermaps and use it to generalize the results connecting the circuit partition polynomial to the Martin polynomial and the results on several graph invariants.

Résumé. Nous introduisons un polynôme de Whitney pour les hypercartes et nous l'utilisons pour généraliser les résultats liant le polynôme des partitions de circuit aux polynômes de Martin et les résultats de plusieurs invariants de graphes.

Keywords: set partitions, noncrossing partitions, genus of a hypermap, Tutte polynomial, Whitney polynomial, medial map, circuit partition, characteristic polynomial, chromatic polynomial, flow polynomial

Introduction

The Tutte polynomial is a key invariant of graph theory, which has been generalized to matroids, polymatroids, signed graphs and hypergraphs in many ways. A partial list of recent topological graph and hypergraph generalizations includes [2, 3, 4, 19, 21].

The work we present [8] generalizes a variant of the Tutte polynomial, the Whitney rank generating function to *hypermaps* which encode hypergraphs topologically embedded in a surface. This polynomial may be recursively computed using a generalized deletion-contraction formula, and many of the famous special substitutions (for instance, counting spanning subsets of edges, or trees contained in the a graph) may be easily generalized to this setting. Our approach seems to be most amenable to generalize results on the Eulerian circuit partition polynomials, but we also have a promising generalization of the characteristic polynomial. This last generalization (involving the Möbius function of the noncrossing partition lattice) also indicates that, for hypermaps many invariants cannot be obtained by a simple negative substitution into some generalized Tutte polynomial. For similar reasons, a generalized Whitney polynomial seems to work better than a generalized Tutte polynomial, and the difference between the two should not be thought of as a mere linear shift.

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Our work is organized as follows. After the Preliminaries, the key definition of our Whitney polynomial is contained in Section 2. We may generalize the well-known deletion-contraction recurrence formulas from graphs to this setting. This section also contains several important specializations and the proof of the fact that taking the dual of a planar hypermap amounts to swapping the two variables in its Whitney polynomial.

We introduce the directed medial map of a hypermap in Section 3. Every directed Eulerian graph arises as the directed medial map of a collection of hypermaps. In this section and in Section 5, we present analogues and generalizations of several results of Arratia, Bollobás, Ellis-Monaghan, Martin and Sorkin [1, 5, 12, 13, 15, 14, 25, 26] on the circuit partition polynomials of Eulerian digraphs and the medial graph of a plane graph.

The visually most appealing part of our work is in Section 4. Here we extend the circuit partition approach to a refined count which keeps track of circuits bounding external ("wet") and internal ("dry") faces and define a process that allows the computation of the Whitney polynomial of a planar hypermap using paper and a scissor.

Finally, in Section 6 we introduce a characteristic polynomial for hypermaps which generalizes the characteristic polynomial of a map, as well as of a graded poset. We show that for hypermaps whose hyperedges have length at most three, this variant of the characteristic polynomial is still a chromatic polynomial, counting the admissible colorings of the vertices.

1 Preliminaries

A *hypermap* is a pair of permutations (σ, α) acting on the same finite set of labels, generating a transitive permutation group. It encodes a hypergraph, topologically embedded in a surface. Fig. 1 represents the planar hypermap (σ, α) for $\sigma = (1,4)(2,5)(3)$ and $\alpha = (1,2,3)(4,5)$. The cycles of σ are the *vertices*, the cycles of α are the *hyperedges* and the cycles of $\alpha^{-1}\sigma = (1,5)(2,4,3)$ are the *faces*. A hypermap is a *map* if the length of each cycle in α is at most 2. In terms of the function $z(\pi)$, counting the cycles of the

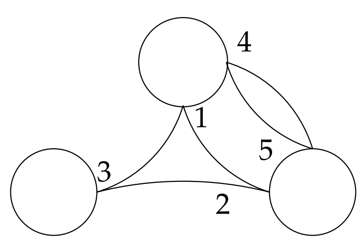

Figure 1: The hypermap (σ, α)

permutation π, the *genus* $g(\sigma, \alpha)$ of a hypermap may be computed using the equation

$n+2-2g(\sigma,\alpha) = z(\sigma)+z(\alpha)+z(\alpha^{-1}\sigma)$ due to Jacques [17]. The present work was motivated by the study of the *spanning hypertrees of a hypermap*, initiated in [6, 9, 10, 24] and continued in [7]. A hypermap (σ,α) is *unicellular* if it has only one face, and it is a *hypertree* if it also has genus zero. A permutation β is a *refinement* of a permutation α if β is obtained by replacing each cycle α_i of α by a permutation β_i acting on the same set of points in such a way that $g(\alpha_i,\beta_i)=0$. We will use the notation $\beta\leq\alpha$ to denote that β is a refinement of α. A hypermap (σ,β) *spans* the hypermap (σ,α) if β is a refinement of α. A *hyperdeletion* is the operation of replacing a hypermap (σ,α) with the hypermap $(\sigma,\alpha\delta)$ where $\delta=(i,j)$ is a transposition *disconnecting* α, that is, i and j must belong to the same cycle. This time we will work with *collections of hypermaps* (defined in Section 2) hence we may perform hyperdeletions even if the permutation group generated by the pair $(\sigma,\alpha\delta)$ has more orbits than the one generated by the pair (σ,α). For maps the deletion operation corresponds to deleting an edge (i,j). A *hypercontraction* is the operation of replacing a hypermap (σ,α) with the hypermap $(\gamma\sigma,\gamma\alpha)$ where $\gamma=(i,j)$ is a transposition disconnecting α. All hypercontractions considered in this work will be *topological*: i and j have to belong to different cycles of σ, for maps this operation corresponds to contracting an edge that is not a loop.

2 A Whitney polynomial of a collection of hypermaps

Definition 1. *A* collection of hypermaps (σ,α) *is an ordered pair of permutations acting on the same set of points. We call the orbits of the permutation group generated by σ and α the* connected components of (σ,α) *and denote their number by $\kappa(\sigma,\alpha)$.*

Definition 2. *The* Whitney polynomial $R(\sigma,\alpha;u,v)$ *of a collection of hypermaps (σ,α) on a set of n points is defined by the formula*

$$R(\sigma,\alpha;u,v)=\sum_{\beta\leq\alpha} u^{\kappa(\sigma,\beta)-\kappa(\sigma,\alpha)}\cdot v^{\kappa(\sigma,\beta)+n-z(\beta)-z(\sigma)}$$

Here the summation is over all permutations β refining α.

For maps we recover the usual definition of the Whitney polynomial of the underlying graph. This invariant is multiplicative for a pair of collections of hypermaps on disjoint sets of points. The function $R(\sigma,\alpha;u,v)$ may be computed recursively using the following generalization of the the well-known deletion-contraction recurrence for the Whitney polynomial $R(G;u,v)$ of a graph G.

Theorem 3. *Let $H=(\sigma,\alpha)$ be a collection of hypermaps on the set $\{1,2,\ldots,n\}$ and assume that $(1,2,\ldots,m)$ is a cycle of α of length at least 2. Then the Whitney polynomial $R(H;u,v)$ is given by the sum*

$$R(H;u,v)=\sum_{k=1}^{m} R(\phi_k(H);u,v)\cdot w_k,$$

where each $\phi_k(H)$ is a collection of hypermaps and each w_k is a monomial from the set $\{1, u, v, uv\}$, according to the following rules:

$$\phi_k(H) = \begin{cases} ((1,k)\sigma, (1,k)\alpha(1,k-1)) & \text{if } z((1,k)\sigma \leq z(\sigma), \\ (\sigma, (1,k)\alpha(1,k-1)) & \text{otherwise.} \end{cases} \tag{2.1}$$

$$w_k = \begin{cases} u^{\kappa(\phi_k(H))-\kappa(H)} & \text{if } z((1,k)\sigma \leq z(\sigma), \\ u^{\kappa(\phi_k(H))-\kappa(H)}v & \text{otherwise.} \end{cases} \tag{2.2}$$

In rule (2.1) we count modulo m, that is, we replace $k-1$ with m if $k=1$, and we read $(1,1)$ as a shorthand for the identity permutation.

In analogy to the case of maps, Theorem 3 may be modified in such a way that for a hypermap (σ, α) the recurrence only involves hypermaps. This recurrence uses hyperdeletions and hypercontractions.

Example 4. For the hypermap shown in Fig 1, repeated use of Theorem 3 gives

$$R((1,4)(2,5)(3), (1,2,3)(4,5); u, v) = u^2 + uv + 4u + v + 3.$$

Certain substitutions into the Tutte polynomial yield famous graph theoretic invariants. Some of these results carry over easily to the Whitney polynomial of a collection of hypermaps. We define a *hyperforest* as a collection of genus zero unicellular hypermaps and we call a collection of hypermaps (σ, β) associated to some refinement β of α (σ, α) *a spanning collection of hypermaps* if the subgroup generated by σ and β has the same orbits as the permutation group generated by σ and α. Then

1. $R(\sigma, \alpha; 0, 0)$ is the number of spanning hyperforests of (σ, α).
2. $R(\sigma, \alpha; 0, 1)$ is the number of spanning collections of hypermaps of (σ, α).

The Tutte polynomial $T(G; x, y)$ of a graph G (or of a map) is given by $T(G; x, y) = R(G; x-1, y-1)$. Extending this definition to collections of hypermaps the obvious way does not seem to be a good idea because of the following example.

Example 5. Consider the hypermap (σ, α) given by $\sigma = (1)(2)\cdots(n)$ and $\alpha = (1, 2, \ldots, n)$. For this, $R(\sigma, \alpha; u, v)$ and $R(\alpha^{-1}\sigma, \alpha^{-1}; u, v)$ are the *Narayana polynomials* of u and and v, respectively, associated to the noncrossing partitions of $\{1, 2, \ldots, n\}$. For $n = 2$ these are $1 + u$ and $1 + v$ respectively, but they become much more complicated for larger values of n. For $n = 3$ we get $R(\sigma, \alpha; u, v) = u^2 + 3u + 1$, substituting $u = x - 1$ yields $x^2 + x - 1$, a polynomial with a negative coefficient.

On the other hand, we have the following generalized duality result.

Theorem 6. *A collection of hypermaps (σ, α) of genus zero and its dual collection $(\alpha^{-1}\sigma, \alpha^{-1})$ satisfy $R(\sigma, \alpha; u, v) = R(\alpha^{-1}\sigma, \alpha^{-1}; v, u)$.*

3 Medial maps

Definition 7. *Let (σ, α) be a collection of hypermaps on the set of points $\{1, 2, \ldots, n\}$. We define its* medial map $M(\sigma, \alpha)$ *as the following map (σ', α') on $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$:*

1. *the cycles of σ' are all cycles of the form $(i_1^-, i_1^+, i_2^-, i_2^+, \ldots, i_k^-, i_k^+)$ where $(i_1, i_2, \ldots, i_k)$ is a cycle of α;*
2. *the cycles of α' are all cycles of the form $(i^+, \sigma(i)^-)$.*

We obtain a collection of maps (σ', α') satisfying $\sigma'(i^-) = i^+$ for all i, such that the endpoints of each edge have opposite signs. We call each such collection of maps *Eulerian* and define its *underlying Eulerian digraph* by directing each edge (i^+, j^-) from its positive endpoint toward its negative endpoint. The process of creating the medial map

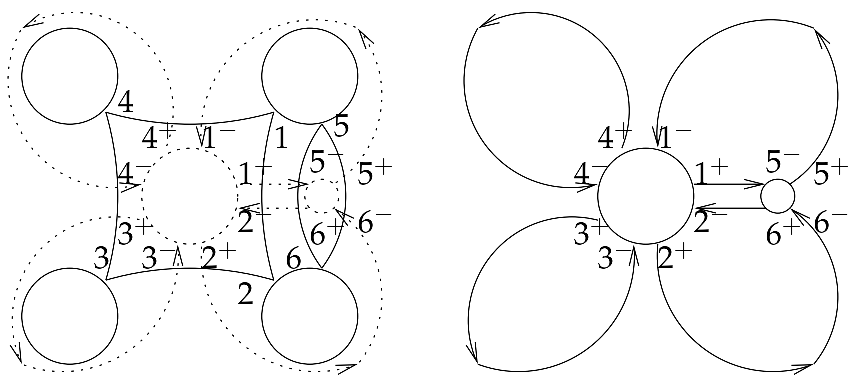

Figure 2: A planar hypermap and its medial map

of $((1,5)(2,6)(3)(4), (1,2,3,4)(5,6))$ is shown in Figure 2.

Proposition 8. *Every directed Eulerian graph arises as the directed medial graph of a collection of hypermaps.*

A hypermap of any genus has a medial map of the same genus. A planar map (σ, α) encodes a plane graph G. In this case $M(\sigma, \alpha)$ is essentially the *directed medial graph* $\overrightarrow{G_m}$ of G, as defined by Martin [25, 26]. Next we generalize the circuit partition polynomials appearing in the works of Ellis-Monaghan [12, 13] and Arratia, Bollobás and Sorkin [1] (see also the Introduction of [5]).

Definition 9. *Let (σ, α) be a collection of Eulerian maps. A* noncrossing Eulerian state *is a partitioning of the edges of the underlying directed medial graph into closed paths in such a way that these paths do not cross at any of the vertices.*

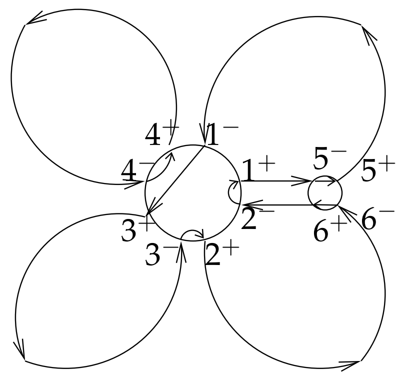

Figure 3: A noncrossing Eulerian state

Figure 3 represents a noncrossing Eulerian state of the Eulerian map shown in the right hand side of Figure 2. We partition the set of edges into closed paths by matching each negative point on a vertex to a positive point on the same vertex. The arrows inside the vertices are the ones pointing from the negative points towards the positive points. A noncrossing Eulerian state is uniquely defined by a *coherent matching* that refines the vertex permutation of the Eulerian map, and matches positive points to negative points.

Definition 10. *We define the* noncrossing circuit partition polynomial *of an Eulerian map* (σ, α) *as*

$$j((\sigma,\alpha);x) = \sum_{k\geq 0} f_k(\sigma,\alpha)x^k.$$

Here $f_k(\sigma, \alpha)$ *is the number of noncrossing Eulerian states with k cycles.*

The following result generalizes Ellis-Monaghan's generalization of Martin's formula [15, Eq. (15)] from planar maps to hypermaps.

Theorem 11. *Let* (σ, α) *be a genus zero collection of hypermaps and* $M(\sigma, \alpha)$ *the collection of its medial maps. Then* $j(M(\sigma, \alpha); x) = x^{\kappa(\sigma,\alpha)}R(\sigma, \alpha; x, x)$ *holds.*

4 A visual computation of $R(\sigma, \alpha; u, v)$ in the planar case

Consider the hypermap (σ, α) given by $\sigma = (1,5,12)(4,11,10)(3,9,8)(2,7,6)$ and $\alpha = (1,2,3,4)(5,6)(7,8)(9,10)(11,12)$, shown in Figure 4. For each point i we add the points i^- and i^+ of the medial map. The vertices of $M(\sigma, \alpha)$ are identified with the hyperedges of (σ, α), and we "shrink" the edges $(i^+, \sigma(i)^-)$ of $M(\sigma, \alpha)$ to follow the outline of the original vertices and the hyperedges. Thus we fatten the outline of the diagram of (σ, α), except for the counterclockwise arcs from i^- to i^+ along the vertices (unless $\sigma(i) = i$) and for he arcs from i^+ to $\alpha(i)^-$ along the hyperedges (unless $\alpha(i) = i$). Next we select

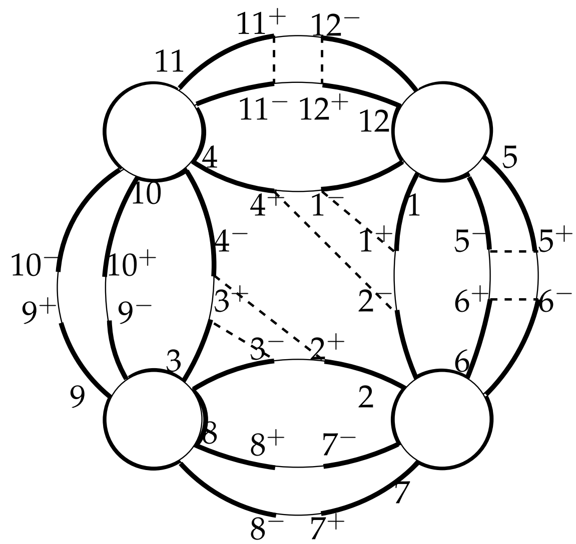

Figure 4: A planar hypermap with the edges of its medial map shrunk to its outline

a coherent matching on the signed points. The *nontrivial choices* are the ones when i^+ is not matched to $\alpha(i)^-$, the remaining choices are trivial. We can think of the diagram of (σ, α) as a paper cutout, with the vertices and the hyperedges being solid and the faces missing. Each nontrivial pair of matched points corresponds then to a cut into the object using a scissor, subject to the following rules:

(R1) Each cut is a simple curve connecting a point i^+ with a point j^-, inside a hyperedge.
(R2) Each point i^+ and j^- may be used at most once.
(R3) The remaining points not used in the cuts must come in pairs $(i^+, \alpha(i)^-)$.
(R4) A new cut cannot cut into the cut-line of a previous cut.

At the end of the process the curves of the outline correspond to the faces $\beta^{-1}\sigma$ of (σ, β) for some $\beta \leq \alpha$, and they are also the circuits of the corresponding circuit partition. Let us think of the unbounded face of the hypermap (σ, α) as "the ocean" with a "wet coastline". After a few nontrivial cuts, we may have several connected components, each has one coastline. Figure 5 illustrates this situation after performing the nontrivial cuts indicated in Figure 4. The shaded regions indicate the faces of σ, α whose border has been merged with a (thickened) wet coastline.

Theorem 12. *Given a planar hypermap (σ, α), we may visually compute its Whitney polynomial by making its model in paper, and performing the above cutting procedure in all possible ways and associating to each outcome u raised to the power of the wet coastlines and v raised to the power of the dry faces. The sum of all weights is $u \cdot R(\sigma, \alpha; u, v)$.*

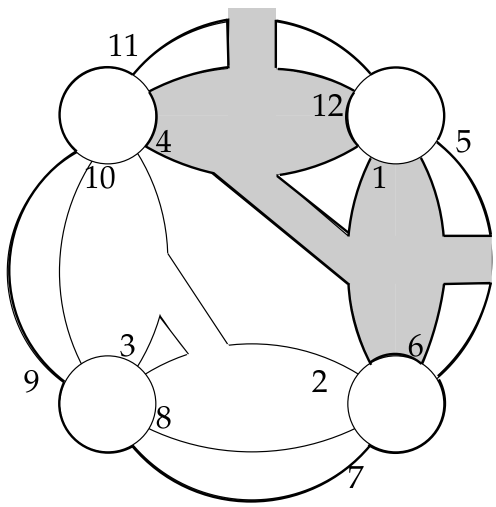

Figure 5: Wet coastlines after a few cuts

5 Counting noncrossing Eulerian colorings

In this section we extend the formula counting the Eulerian colorings of the medial graph of a plane graph [14, Evaluation 6.9] to medial maps of planar hypermaps. The following definition may be found in [14, Definition 4.3].

Definition 13. *An* Eulerian m-coloring *of an Eulerian directed graph* $\overrightarrow{G}$ *is an edge coloring of* $\overrightarrow{G}$ *with m colors so that for each color the (possibly empty) set of all edges of the given color forms an Eulerian subdigraph.*

Consider now a planar hypermap (σ, α) and its directed Eulerian medial map $M(\sigma, \alpha)$. Given an Eulerian m-coloring of the edges, let us color the endpoints of $(i^+, \sigma(i)^-)$ with the color of the edge. We call this coloring of the points the *coloring of the points induced by the Eulerian m-coloring*. In order to relate the count of the Eulerian m-colorings to our Whitney polynomial, we must restrict our attention to *noncrossing Eulerian m-colorings*, defined as follows.

Definition 14. *Let* (σ, α) *be a planar hypermap and let* $M(\sigma, \alpha)$ *be its directed medial map. We call an Eulerian m-coloring* noncrossing *if there is a noncrossing Eulerian state such that all edges of the same connected circuit have the same color.*

Remark 15. If (σ, α) is a map then the above noncrossing condition is automatically satisfied by each Eulerian m-coloring of $M(\sigma, \alpha)$ as all Eulerian states of $M(\sigma, \alpha)$ are noncrossing. The more general case of partitioning the edge set of an Eulerian digraph into Eulerian subdigraphs was addressed in [1, 5, 12].

Using the induced coloring of the points we can verify vertex by vertex whether an Eulerian m-coloring is non-crossing: it is necessary and sufficient to be able to find a coherent matching at each vertex such that only points of the same color are matched. This observation motivates the following definition.

Definition 16. *Let (σ, α) be a planar hypermap on the set of points $\{1, 2 \ldots, n\}$ and $M(\sigma, \alpha) = (\sigma', \alpha')$ its directed medial map on the set of points $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$. We call an m-coloring of the points $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$ a* legal coloring *if it satisfies the following conditions:*

1. *The endpoints of each edge $(i^+, \sigma(i)^-) \in \alpha'$ of $M(\sigma, \alpha)$ have the same color.*
2. *There is a coherent matching of the set $\{1^-, 1^+, 2^-, 2^+, \ldots, n^-, n^+\}$ such that each point is matched to a point of the same color.*

Definition 16 is motivated by the following observation.

Proposition 17. *Given a planar hypermap (σ, α) and its directed medial map $M(\sigma, \alpha)$ a coloring of the set of points of $M(\sigma, \alpha)$ is legal if and only if it is induced by a noncrossing Eulerian m-coloring of the edges of $M(\sigma, \alpha)$.*

Note that for a given m-coloring of the signed points, induced by a coloring of the edges in α', condition (2) may be independently verified at each vertex of σ'. This observation motivates the following definition.

Definition 18. *Let $(i_1^-, i_1^+, i_2^-, i_2^+, \ldots, i_k^-, i_k^+)$ be a cyclic signed permutation and let is fix an m-coloring of its points. We say that the* valence *of this colored cycle is number of coherent matchings of its points that match each point the a point of the same color.*

Now we are able to state the generalization of [14, Evaluation 6.9].

Theorem 19. *Let (σ, α) be a planar hypermap. Then, for a fixed positive integer m, we have*

$$m^{\kappa(\sigma,\alpha)} R(\sigma, \alpha; m, m) = \sum_{\lambda} \prod_{v \in \sigma'} \nu(v, \lambda).$$

Here the summation runs over all Eulerian m-colorings λ of the directed medial map $M(\sigma, \alpha) = (\sigma', \alpha')$, and for each vertex $v \in \sigma'$ the symbol $\nu(v, \lambda)$ represents the valence of v colored by the restriction of the point coloring induced by λ to the points of v.

Example 20. For maps (σ, α), there are essentially two types of vertices in the directed medial map $M(\sigma, \alpha)$: monochromatic vertices and vertices colored with two colors. We can recover the formula [14, Evaluation 6.9]:

$$m^{\kappa(\sigma,\alpha)} R(\sigma, \alpha; m, m) = \sum_{\lambda} 2^{m(\lambda)}$$

where $m(\lambda)$ is the number of monochromatic vertices. Evaluating this formula at $m = 2$ was used by Las Vergnas [23] to describe the exact power of 2 that divides $R(\sigma, \alpha; 2, 2)$ for a map (σ, α), or equivalently the evaluation of its Tutte polynomial at $(3, 3)$.

6 The characteristic polynomial of a hypermap

Proposition 21. *Let α be a permutation of $\{1, 2, \ldots, n\}$ with k cycles of lengths $c_1, \ldots, c_k$. Then the partially ordered set of all refinements of α, ordered by the refinement operation, is a direct product $[\mathrm{id}, \alpha] = \prod_{i=1}^{k} \mathrm{NC}(c_i)$. Here* id *is the identity permutation $(1)(2)\cdots(n)$ and $\mathrm{NC}(c_i)$ is the lattice of noncrossing partitions on c_i elements.*

Using the results in [27], the Möbius function $\mu(\beta, \alpha)$ of any interval $[\beta, \alpha]$ may be expressed in terms of Catalan numbers.

Definition 22. *Given a collection of hypermaps (σ, α) on the set of points $\{1, 2, \ldots, n\}$, we define its* characteristic polynomial *$\chi(\sigma, \alpha; t)$ by*

$$\chi(\sigma, \alpha; t) = \sum_{\beta \le \alpha} \mu(\mathrm{id}, \beta) \cdot t^{\kappa(\sigma,\beta) - \kappa(\sigma,\alpha)}.$$

When (σ, α) is a collection of maps on the set $\{1, 2, \ldots, n\}$, we get that $\chi(\sigma, \alpha; t)$ is the characteristic polynomial of its underlying graph. For a fixed collection of hypermaps (σ, α), let us define the function $X([\alpha_1, \alpha_2]; t)$ on the intervals of the partially ordered set of the refinements of α by

$$X([\alpha_1, \alpha_2]; t) = \sum_{\beta \in [\alpha_1, \alpha_2]} \mu(\alpha_1, \beta) \cdot t^{\kappa(\sigma,\beta)} \tag{6.1}$$

A Möbius inversion formula computation yields $\sum_{\beta \le \alpha} X([\beta, \alpha]; t) = t^{z(\sigma)}$. For maps, this computation implies that the chromatic polynomial $\kappa(\sigma, \alpha)\chi(\sigma, \alpha; t) = X([\mathrm{id}, \alpha]; t)$ is the number of ways to color the vertices using t colors such that adjacent vertices have the same color. This reasoning cannot be extended to arbitrary hypermaps, but it is possible to generalize it to hypermaps with hyperedges containing at most 3 points.

Theorem 23. *Let (σ, α) be a collection of hypermaps such that each cycle of α has length at most 3. Then for any positive integer n, the number $n^{\kappa(\sigma,\alpha)} \cdot \chi(\sigma, \alpha, n)$ is the number of ways to n-color the vertices of (σ, α) in such a way that no two vertices of the same color are incident to the same cycle of α.*

A completely analogous dual reasoning may be developed for the *flow polynomial* $C(\sigma, \alpha, ; t)$ of a collection of hypermaps (σ, α) on the set of points $\{1, 2, \ldots, n\}$, defined by

$$C(\sigma, \alpha; t) = \sum_{\beta \le \alpha} \mu(\beta, \alpha) t^{n + \kappa(\sigma,\beta) - z(\beta) - z(\sigma)}.$$

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The doubly asymmetric simple exclusion process, the colored Boolean process, and the restricted random growth model

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Abstract. The multispecies asymmetric simple exclusion process (mASEP) is a Markov chain in which particles of different species hop along a one-dimensional lattice. This paper studies the doubly asymmetric simple exclusion process DASEP(n, p, q) in which q particles with species $1, \ldots, p$ hop along a circular lattice with n sites, but also the particles are allowed to spontaneously change from one species to another. In this paper, we introduce two related Markov chains called the colored Boolean process and the restricted random growth model, and we show that the DASEP lumps to the colored Boolean process, and the colored Boolean process lumps to the restricted random growth model. This allows us to generalize a theorem of David Ash on the relations between sums of steady state probabilities. We also give explicit formulas for the stationary distribution of DASEP$(n, 2, 2)$.

Keywords: Asymmetric simple exclusion process, Markov chain

1 Introduction

The asymmetric simple exclusion process (ASEP) is a model from statistical mechanics introduced by Macdonald-Gibbs-Pipkin [12] and Spitzer [17], which describes a Markov chain for particles hopping left or right along a one-dimensional lattice such that each site contains at most one particle. It can be used to model traffic flow or translation in protein synthesis. There are many variations of the ASEP: the lattice can have open, half open, closed, or periodic boundaries, and there can be reservoirs (see Liggett [10, 11]). Particles can exhibit different species, and this variation is called the multispecies ASEP (mASEP). The asymmetry can be partial, so that particles are allowed to hop both left and right, but one side is t times more probable, and this is called the partially asymmetric exclusion process (PASEP). The ASEP is closely related to a growth model defined by Kardar-Parizi-Zhang [8], and various methods have been invented to study the ASEP, such as the matrix ansatz introduced by Derrida et al. in [5]. The combinatorics of the ASEP was studied by many people, see [2, 3, 4, 7, 13].

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Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $\lambda_1 \geq \cdots \lambda_n \geq 0$, $|\lambda|$ be the sum of all parts of λ, and $m_i = m_i(\lambda) := \#\{j : \lambda_j = i\}$ be the number of parts of λ that equal i. We also denote λ by $1^{m_1}2^{m_2}\cdots$. Let $\ell(\lambda) = \sum_i m_i(\lambda)$ denote the length of λ. We write $S_n(\lambda)$ as the set of all weak compositions obtained from permuting the parts of λ. The mASEP can be thought of a Markov chain on $S_n(\lambda)$ [4, Definition 1.2], or a coupling of multiple ASEP [13]. The stationary distribution of the mASEP is related to Macdonald polynomials [4] and multiline queues [7].

Let n be the number of sites on the lattice, p be number of types of species, and q be the number of particles. David Ash [1] defined the *doubly asymmetric simple exclusion process* DASEP(n, p, q). The DASEP is a variant of the mASEP but also allows particles to spontaneously change species. This might be applied to biology models involving evolutions, or traffic flow problem that also tracks the gears of the cars. If $p = 1$, DASEP$(n, 1, q)$ is the usual 1-species PASEP on a ring.

Definition 1. [1] Let n, p, q be positive integers with $n > q$, and let $u, t \in [0, 1)$ be constants. The *doubly asymmetric simple exclusion process* DASEP(n, p, q) is a Markov chain on the set of words (or weak compositions) of length n in $0, \ldots, p$ with $n - q$ zeros:

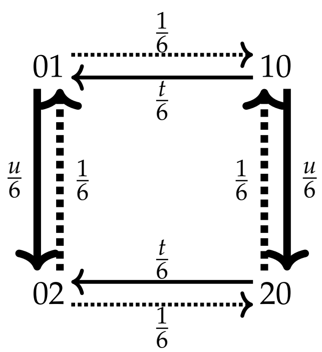
$$\Gamma_n^{p,q} = \bigcup_{\substack{\lambda_1 \leq p, \\ \ell(\lambda)=q}} S_n(\lambda) = \bigcup_{m_1+\cdots+m_p=q} S_n(1^{m_1}\cdots p^{m_p}).$$

The transition probability $P(\mu, \nu)$ on two states μ and ν is as follows:

- If $\mu = AijB$ and $\nu = AjiB$ (where A and B are words in $0, \ldots, p$) with $i \neq j$, then $P(\mu, \nu) = \frac{t}{3n}$ if $i > j$ and $P(\mu, \nu) = \frac{1}{3n}$ if $j > i$.
- If $\mu = iAj$ and $\nu = jAi$ with $i \neq j$, then $P(\mu, \nu) = \frac{t}{3n}$ if $j > i$ and $P(\mu, \nu) = \frac{1}{3n}$ if $i > j$.
- If $\mu = AiB$ and $\nu = A(i+1)B$ with $i \leq p - 1$, then $P(\mu, \nu) = \frac{u}{3n}$.
- If $\mu = A(i+1)B$ and $\nu = AiB$ with $i \geq 1$, then $P(\mu, \nu) = \frac{1}{3n}$.
- Otherwise $P(\mu, \nu) = 0$ for $\mu \neq \nu$ and $P(\mu, \mu) = 1 - \sum_{\nu \neq \mu} P(\mu, \nu)$.

Remark 1. There is an inherent cyclic symmetry in the definition, so that a state has the same dynamic under any cyclic permutation.

This Markov chain is irreducible and aperiodic, so it has a unique stationary distribution π given by rational functions in u, t, which satisfies the global balance equations $\pi(\mu) \sum_{\nu \neq \mu} P(\mu, \nu) = \sum_{\nu \neq \mu} \pi(\nu) P(\nu, \mu)$ for any state μ. For convenience, we clear the denominators and obtain the "unnormalized steady state probabilities" π_{DASEP} which are proportional to the stationary distribution by a factor of the *partition function* $Z_n^{p,q} =$



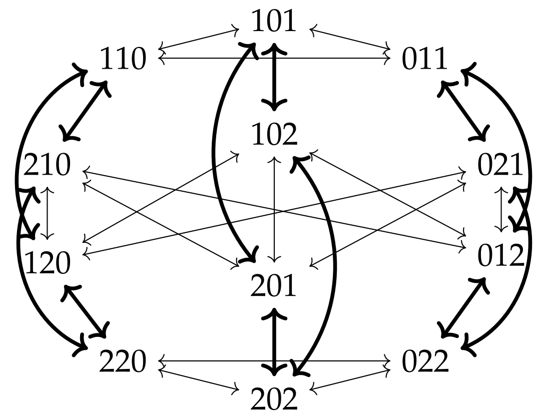


Figure 1: The state diagram of DASEP$(2,2,1)$ and DASEP$(3,2,2)$. Bold edges denote changes in species, while regular edges denote exchanges of particles of different species or between particles and holes.

$\sum_{\mu\in\Gamma_n^{p,q}} \pi_{\text{DASEP}}(\mu)$. We require the unnormalized steady state probabilities to be coprime so they are uniquely defined.

Our first main result concerns the ratio between the sums of certain sets of $\pi_{\text{DASEP}}(\mu)$. For each partition λ with length q and each binary word $w = (w_1, \dots, w_n)$ with q ones and $n-q$ zeros, define

$$S_n^w(\lambda) := \{\mu \in S_n(\lambda) | \mu_i \neq 0 \text{ if and only if } w_i \neq 0\}$$

as the equivalence class of weak compositions μ obtained from permuting λ whose support is equal to w. Then we have $|S_n(1^{m_1}\cdots p^{m_p})| = \binom{n}{n-q,m_1,\dots,m_p}$ and $|S_n^w(1^{m_1}\cdots p^{m_p})| = \binom{q}{m_1,m_2,\dots,m_p}$.

Theorem 1. *Consider* DASEP(n,p,q) *for any positive integers n,p,q with $n > q$.*

(1) For any two binary words $w, w' \in \binom{[n]}{q}$, we have $\pi_{\text{DASEP}}(w) = \pi_{\text{DASEP}}(w')$.

(2) For any binary word $w \in \binom{[n]}{q}$ and partition $\lambda = 1^{m_1}2^{m_2}\cdots p^{m_p}$ with $m_1 + \cdots + m_p = q$, we have

$$\sum_{\mu\in S_n^w(\lambda)} \pi_{\text{DASEP}}(\mu) = u^{|\lambda|-q}\binom{q}{m_1,m_2,\dots,m_p}\pi_{\text{DASEP}}(w).$$

In other words, the average of steady state probabilities over orbits of S_q-action on the particles are all equal up to a power of u. This is a polynomial generalization of a combinatorial phenomenon called *homomesy* defined by Propp and Roby, see[14].

μ	$\pi_{\mathrm{DASEP}}(\mu)$
011	$u+3t+4$
012	$u(u+4t+3)$
021	$u(u+2t+5)$
022	$u^2(u+3t+4)$

μ	$\pi_{\mathrm{DASEP}}(\mu)$
0011	$u+2t+3$
0101	$u+2t+3$
0022	$u^2(u+2t+3)$
0202	$u^2(u+2t+3)$
0012	$u(u+3t+2)$
0102	$u(u+2t+3)$
0021	$u(u+t+4)$

Table 1: The unnormalized steady state probabilities of DASEP$(3,2,2)$ and DASEP$(4,2,2)$. We present all states up to cyclic symmetry.

Remark 2. In the special case of DASEP$(3,p,2)$, Theorem 1 was proved by David Ash [1, Theorem 5.2].

Example 1. For the partition $\lambda=(2,1,0)$ with $|\lambda|=2+1=3$, we have $S_3^{011}((2,1,0))=\{012,021\}$ and $|S_3^{011}((2,1,0))|=\binom{2}{1,1}=2$; also $S_3((2,1,0))=\{012,021,102,201,120,210\}$ and $|S_3((2,1,0))|=\binom{3}{1,1,1}=6$.

The following are direct corollaries of Theorem 1.

Corollary 1. *For the* DASEP(n,p,q) *defined by positive integers $n,p,q,n>q$, and λ,μ two partitions with $\lambda_1\le p,\mu_1\le p,\ell(\lambda)=\ell(\mu)=q$, we have*

$$\frac{\sum_{\nu\in S_n(\lambda)}\pi_{\mathrm{DASEP}}(\nu)}{\sum_{\nu\in S_n(\mu)}\pi_{\mathrm{DASEP}}(\nu)}=\frac{|S_n(\lambda)|}{|S_n(\mu)|}u^{|\lambda|-|\mu|}.$$

Let $t=1$, then our model is symmetric, dubbed the “doubly symmetric simple exclusion process (DSSEP)". It is a generalization of the model considered by Salez in [15], which is an exclusion process on a graph (a circle in our case) with a reservoir of particles at each vertex. Recall that $[p+1]_u=1+u+\cdots+u^p$ denotes the u analog of the integer $p+1$. For DSSEP, it follows from Theorem 1 that

Corollary 2. *The partition function of* DSSEP(n,p,q) *is*

$$\binom{n}{q}(1+u+\cdots+u^p)^q=\binom{n}{q}([p+1]_u)^q.$$

Example 2. For DASEP$(3,2,2)$, by Theorem 1, we have $\pi_{\mathrm{DASEP}}(012)+\pi_{\mathrm{DASEP}}(021)=2u\pi_{\mathrm{DASEP}}(011)$ and $\pi_{\mathrm{DASEP}}(022)=u^2\pi_{\mathrm{DASEP}}(011)$ which can be seen from Table 1.

Similarly, for DASEP$(4,2,2)$, Theorem 1 asserts that $\pi_{\mathrm{DASEP}}(0011)=\pi_{\mathrm{DASEP}}(0101)$, $\pi_{\mathrm{DASEP}}(0012)+\pi_{\mathrm{DASEP}}(0021)=2u\pi_{\mathrm{DASEP}}(0011)$ and $\pi_{\mathrm{DASEP}}(0102)+\pi_{\mathrm{DASEP}}(0201)=$

$2u\pi_{\text{DASEP}}(0011)$. Since 0201 is a cyclic permutation of 0102, their steady state probabilities are equal by Remark 2, and Table 1 shows that it is equal to $u\pi_{\text{DASEP}}(0101)$.

To prove Theorem 1, we introduce a new Markov chain that we call *colored Boolean process* (see Definition 2), and we show that DASEP *lumps* is a colored Boolean process. This gives a relationship between the stationary distribution of the colored Boolean process and the DASEP; see Theorem 2.

In Theorem 6, we give explicit formulas for the stationary distributions of the infinite family DASEP$(n, 2, 2)$, $n \geq 3$ which depend on whether n is odd or even. Both are described by polynomial sequences given by a second-order homogeneous recurrence relation (see Theorem 6). The polynomials sequences are generating functions of matchings of certain graphs (see Figure 4 and Figure 5). When specialized to $u = t = 1$, the polynomial sequences specialize to trinomial transform of Lucas number A082762 and binomial transform of the denominators of continued fraction convergents to $\sqrt{5}$ A084326 [16].

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2 The DASEP lumps to the colored Boolean process

In this section, we define the *colored Boolean process*, and we show that the DASEP *lumps* to the *colored Boolean process*. We compute the ratios between steady states probabilities in the colored Boolean process, leading us to understand the ratios between sums of steady state probabilities of the DASEP.

Definition 2. The *colored Boolean process* is a Markov chain dependent on three positive integers n, p, q with $n > q$ on the set of pairs of binary words and partition in a $q \times p$ rectangle

$$\Omega_n^{p,q} = \{(w, \lambda) | w \in \binom{[n]}{q}, \lambda_1 \leq p, \ell(\lambda) = q\}$$

with the following transition probabilities:

- $Q((w, \lambda), (w, \lambda')) = \frac{m_i(\lambda)u}{3n}$ if λ' is obtained from λ by changing a part equal to $i < p$ to a part equal to $i + 1$, denoted by $\lambda \nearrow_i \lambda'$.

- $Q((w, \lambda), (w, \lambda')) = \frac{m_i(\lambda)}{3n}$ if λ' is obtained from λ by changing a part equal to $i > 1$ to a part equal to $i - 1$, denoted by $\lambda \searrow_i \lambda'$.

- $Q((w,\lambda),(w',\lambda)) = \frac{1}{3n}$ if w' is obtained from w by $01 \to 10$ in a unique position, allowing wrap-around at the end.
- $Q((w,\lambda),(w',\lambda)) = \frac{t}{3n}$ if w' is obtained from w by $10 \to 01$ at a unique position, allowing wrap-around at the end.
- If none of the above applies but $w \neq w'$ or $\lambda \neq \lambda'$, then $Q((w,\lambda),(w',\lambda')) = 0$. Otherwise $Q((w,\lambda),(w,\lambda)) = 1 - \sum_{(w',\lambda')\neq(w,\lambda)} Q((w,\lambda),(w',\lambda'))$.

We denote the stationary distribution of $\Omega_n^{p,q}$ by π_{CBP}. We think of parts of different sizes as particles of different colors, or species; hence the name.

The relation between the colored Boolean process and the DASEP is captured by the following notion.

Definition 3. [9, Section 6.3] Let $\{X_t\}$ be a Markov chain on state space Ω_X with transition matrix P, and let $f : \Omega_X \to \Omega_Y$ be a surjective map. Suppose there is an $|\Omega_Y| \times |\Omega_Y|$ matrix Q such that for all $y_0, y_1 \in \Omega_Y$, if $f(x_0) = y_0$, then

$$\sum_{x: f(x)=y_1} P(x_0, x) = Q(y_0, y_1).$$

Then $\{f(X_t)\}$ is a Markov chain on Ω_Y with transition matrix Q. We say that $\{f(X_t)\}$ is a *lumping* of $\{X_t\}$.

We may use the stationary distribution of $\{X_t\}$ to compute that of its lumping.

Proposition 1. *[9, Section 6.3] Suppose p is a stationary distribution for $\{X_t\}$, and let π be the measure on Ω_Y defined by $\pi(y) = \sum_{x:f(x)=y} p(x)$. Then π is a stationary distribution for $\{f(X_t)\}$.*

Theorem 2. *The projection map on state spaces $f : \Gamma_n^{p,q} \to \Omega_n^{p,q}$ sending each μ to (w,λ) if $\mu \in S_n^w(\lambda)$ is a lumping of* DASEP(n,p,q) *onto the colored Boolean process $\Omega_n^{p,q}$.*

It follows from Proposition 1 that the unnormalized steady state probabilities of the colored Boolean process are proportional to the sums of the unnormalized steady state probabilities of the DASEP as follow:

$$\pi_{\mathrm{CBP}}(w,\lambda) \propto \sum_{\mu \in S_n^w(\lambda)} \pi_{\mathrm{DASEP}}(\mu).$$

Proof. Fix (w_0,λ_0) and (w_1,λ_1), we want to show that for any $\mu_0 \in S_n^{w_0}(\lambda_0)$, the quantity $\sum_{\mu:\mu\in S_n^{w_1}(\lambda_1)} P(\mu_0,\mu)$ is independent of the choice of μ_0 and equal to $Q((w_0,\lambda_0),(w_1,\lambda_1))$. We may assume $(w_0,\lambda_0) \neq (w_1,\lambda_1)$. Note that this quantity is nonzero only in the following cases:

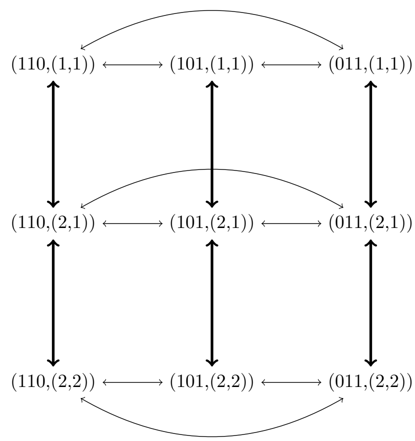

Figure 2: Transition diagram of $\Omega_3^{2,2}$, as a lumping of DASEP$(3, 2, 2)$ as shown on the right hand side in Figure 1. The bold edges denote the changes of species, while the regular edges denote the exchanges between particles of different species or between particles and holes.

- If $w_0 = w_1$ and there exists a unique $i < p$ such that $\lambda_0 \nearrow_i \lambda_1$, we increase the species of a particle from i to $i+1$, and there are m_i ways to do it. For each $\mu \in S_n^{w_1}(\lambda_1)$, we have $P(\mu_0, \mu) = \frac{u}{3n}$, so their sum is equal to $\frac{m_i u}{3n}$.
- If $w_0 = w_1$ and there exists a unique $i > 1$ such that $\lambda_0 \searrow_i \lambda_1$, we decrease the species of a particle from i to $i-1$, and there are m_i ways to do it, so the quantity is equal to $\frac{m_i}{3n}$.
- If $\lambda_0 = \lambda_1$ and w_1 is obtained from w_0 by $01 \to 10$ or $10 \to 01$ at a unique position (allow wraparound). This quantity is equal to $\frac{1}{3n}$ or $\frac{t}{3n}$ respectively.

□

Theorem 3. *Consider the colored Boolean process $\Omega_n^{p,q}$.*

(1) The steady state probabilities of all binary words with the trivial partition are equal, i.e.,

$$\pi_{\mathrm{CBP}}(w, 0^{n-q}1^q) = \pi_{\mathrm{CBP}}(w', 0^{n-q}1^q), \quad \text{for all } w, w' \in \binom{[n]}{q}.$$

(2) The steady state probability of an arbitrary state (w, λ) can be expressed in terms of the steady state probability of the corresponding state $(w, 0^{n-q}1^q)$ with the trivial partition $0^{n-q}1^q$ as follows:

$$\pi_{\mathrm{CBP}}(w, \lambda) = u^{|\lambda|-q} \binom{q}{m_1, \ldots, m_p} \pi_{\mathrm{CBP}}(w, 0^{n-q}1^q). \tag{2.1}$$

Proof. Since the colored Boolean process is irreducible, it suffices to verify the global balance equations. For simplicity of notation, denote $\pi_{\mathrm{CBP}}(w,\lambda)$ by $p_{w,\lambda}$. Let b_w be the number of blocks of consecutive 1's in w (allowing wrap-around).

We first check it for the states given by a binary word and the trivial partition $\lambda_0 = 0^{n-q}1^q$. Notice that any occurrence of 01 in w must begin a block, and any occurrence of 10 must signify the end of a block. The balance equation at $(w, 0^{n-q}1^q)$ is

$$(qu + b_w + b_w t)p_{w,\lambda_0} = qp_{w,1^{q-1}2} + b_w t \sum_{\substack{w' \to w \\ 10 \to 01}} p_{w',\lambda_0} + b_w \sum_{\substack{w'' \to w \\ 01 \to 10}} p_{w'',\lambda_0}. \tag{2.2}$$

Since $\binom{q}{q-1,1} = 1$, we are left with

$$b_w(1+t)p_w = b_w t \sum_{\substack{w' \to w \\ 10 \to 01}} p_{w'} + b_w \sum_{\substack{w'' \to w \\ 01 \to 10}} p_{w''}$$

which will be satisfied if we set all p_w's to be equal.

For arbitrary partition $\lambda = 1^{m_1}2^{m_2}\cdots p^{m_p}$, the left hand side of the balance equation at (w,λ) is

$$((m_1 + \cdots + m_{p-1})u + m_2 + \cdots + m_p + b_w + b_w t)p_{w,\lambda}$$

These account for all the states that (w,λ) can transition to.

The right hand side of the balance equation at (w,λ) is

$$\sum_{i<p, \lambda \nearrow_i \lambda'} (m_{i+1}+1)p_{w,\lambda'} + \sum_{i>1, \lambda \searrow_i \lambda''} (m_{i-1}+1)up_{w,\lambda''} + b_w t \sum_{\substack{w' \to w \\ 10 \to 01}} p_{w',\lambda} + b_w \sum_{\substack{w'' \to w \\ 01 \to 10}} p_{w'',\lambda}$$

These account for all the states that can transition into (w,λ). Using Equation (2.1), the multinomial coefficients give

$$\frac{p_{w,\lambda'}}{p_{w,\lambda}} = \frac{m_i u}{m_{i+1}+1}, \ \lambda \nearrow_i \lambda' \implies m_i(\lambda') = m_i - 1, m_{i+1}(\lambda') = m_{i+1}+1, \forall i < p$$
$$\frac{p_{w,\lambda''}}{p_{w,\lambda}} = \frac{m_i}{(m_{i-1}+1)u}, \ \lambda \searrow_i \lambda'' \implies m_i(\lambda'') = m_i - 1, m_{i-1}(\lambda'') = m_{i-1}+1, \forall i > 1.$$

Then we have a term by term equality for each i where a corresponds to the first summation and b corresponds to the second. □

Proof of Theorem 1. This follows directly from Theorem 2 and Theorem 3. □

3 The colored Boolean process lumps to the restricted random growth model

In this section, we define the *restricted random growth model*, which is a Markov chain on the set of Young diagrams inside a rectangle. We show that the colored Boolean process lumps to the restricted random growth model.

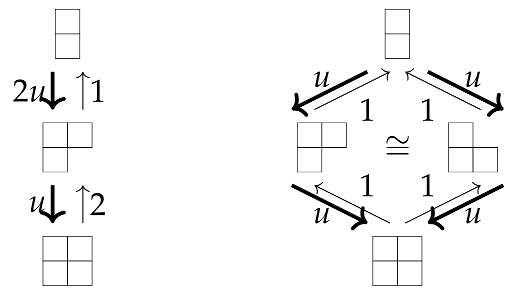

Figure 3: Transition diagram of the restricted random growth model on $\chi^{2,2}$. The Markov chain on the left can be viewed as a lumping of the Markov chain on the right by rearranging boxes into weakly decreasing order.

Definition 4. Define the *restricted random growth model* on the the set $\chi^{p,q} = \{\lambda : \lambda_1 \leq p, \ell(\lambda) = q\}$ of all partitions that fit inside a $q \times p$ rectangle but do not fit inside a shorter rectangle, with transition probabilities as follows:

- If $\nu \nearrow_i \lambda$, then $\Pr(\nu, \lambda) = \frac{m_i(\nu)u}{3n}$.
- If $\nu \searrow_i \lambda$, then $\Pr(\nu, \lambda) = \frac{m_i(\nu)}{3n}$.
- Otherwise if $\nu \neq \lambda$, then $\Pr(\nu, \lambda) = 0$ and $\Pr(\lambda, \lambda) = 1 - \sum_{\nu : \nu \neq \lambda} \Pr(\nu, \lambda)$.

We denote the unnormalized steady state probability of the restricted random growth model by π_{RRG}.

For two partitions λ and λ', if $\lambda \nearrow_i \lambda'$, then the Young diagram of λ' is obtained from the Young diagram of λ by adding a corner box to the topmost row of length i. If $\lambda \searrow_i \lambda'$, then we remove the corner box from the topmost row of length i. In other words, the restricted random growth model either adds or removes a box from a uniformly chosen part of the Young diagram of the partition (conditioned on staying in the $q \times p$ rectangle) as shown on the right hand side of Figure 3, then rearrange the parts in weakly decreasing order as shown on the left hand side of Figure 3. Random growth models are of independent interests and have been studied by many people [6].

Theorem 4. *The projection map on state spaces $\Omega_n^{p,q} \to \chi^{p,q}$ sending (w, λ) to λ (forgetting the positions of 0's) is a lumping of the colored Boolean process to the restricted random growth.*

It follows from Proposition 1 that

$$\pi_{\mathrm{RRG}}(\lambda) \propto \sum_{w \in \binom{[n]}{q}} \pi_{\mathrm{CBP}}(w, \lambda).$$

Proof. By Definition 3, we need to show that for any $\nu \neq \lambda$ and binary word w the following equation holds:

$$\Pr(\nu, \lambda) = \sum_{w'} Q((w, \nu), (w', \lambda)).$$

Then $Q((w, \nu), (w', \lambda)) \neq 0$ only if $w = w'$ by Definition 2, and this quantity is either $\frac{m_i(\nu)u}{3n}$ when $\nu \nearrow_i \lambda$ or $\frac{m_i(\nu)}{3n}$ when $\nu \searrow_i \lambda$. □

Theorem 5. *The steady state probabilities of the restricted random growth satisfy the following relations for all partitions $\nu, \lambda \in \chi^{p,q}$:*

$$\frac{\pi_{\mathrm{RRG}}(\lambda)}{\pi_{\mathrm{RRG}}(\nu)} = \frac{|S_n(\lambda)|}{|S_n(\nu)|} u^{|\lambda| - |\nu|}.$$

Proof. This follows from Theorem 4 and Theorem 3 and a computation on multinomial coefficients. □

Proof of Corollary 1. This follows from Theorem 4 and Theorem 5. □

Proof of Corollary 2. When $t = 1$, for any partition λ and for any two weak compositions $\mu, \nu \in S_n(\lambda)$, we have $\pi_{\mathrm{DASEP}}(\mu) = \pi_{\mathrm{DASEP}}(\nu)$. By Theorem 5 and the requirement for unnormalized steady state probabilities to be coprime, we see that the partition function of DSSEP is

$$\sum_{\substack{\lambda_1 \leq p, \\ \ell(\lambda) = q}} u^{|\lambda|} |S_n(\lambda)| = \binom{n}{q} \sum_{m_1 + \cdots + m_p = q} u^{m_1 + 2m_2 \cdots pm_p} \binom{q}{m_1, \ldots, m_p} = \binom{n}{q} (1 + u + \cdots + u^p)^q.$$

□

4 The stationary distribution of DASEP(n,2,2)

In this section, we give a complete description of the stationary distributions when there are two particles and two species, while the number of sites can be arbitrary.

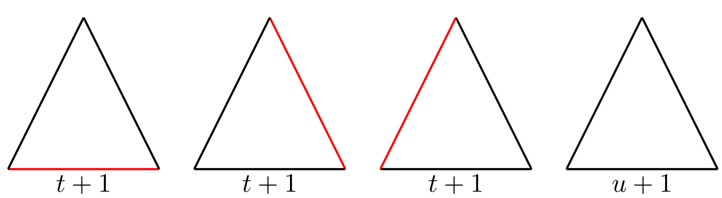

Figure 4: $a_1 = u + 3t + 4$

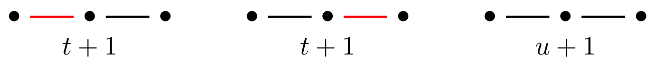

Figure 5: $b_1 = u + 2t + 3$

μ	$\pi_{\mathrm{DASEP}(2k+1,2,2)}(\mu)$
$S_n((1,1,0,\ldots,0))$	a_k
$0\ldots010^m20\ldots0$	$ua_k + u(t-1)(t+1)^m a_{k-m-1}, (0 \leq m < k)$
$0\ldots020^m10\ldots0$	$ua_k - u(t-1)(t+1)^m a_{k-m-1}, (0 \leq m < k)$
$S_n((2,2,0,\ldots,0))$	$u^2 a_k$

Table 2: The unnormalized steady state probabilities of DASEP$(2k+1,2,2)$.

μ	$\pi_{\mathrm{DASEP}(2k+2)}(\mu)$
$S_n((1,1,0,\ldots,0))$	b_k
$0\ldots010^m20\ldots0$	$ub_k + u(t-1)(t+1)^m b_{k-m-1}, (0 \leq m \leq k)$
$0\ldots020^m10\ldots0$	$ub_k - u(t-1)(t+1)^m b_{k-m-1}, (0 \leq m \leq k)$
$S_n((2,2,0,\ldots,0))$	$u^2 b_k$

Table 3: The unnormalized steady state probabilities of DASEP$(2k+2,2,2)$.

Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq -1}$ be polynomial sequences in u, t satisfying the recurrence relation

$$\begin{aligned} a_k &= (u+2t+3)a_{k-1} - (t+1)^2 a_{k-2} \\ b_k &= (u+2t+3)b_{k-1} - (t+1)^2 b_{k-2}. \end{aligned}$$

with initial conditions $b_{-1} = 0, a_0 = b_0 = 1, a_1 = u + 3t + 4$.

Theorem 6. *Consider matchings M in the cycle C_{2k+1} or the path L_{2k+1} with $(2k+1)$ vertices. Assign each matching M a weight of $(t+1)^{|M|}(u+1)^{k-|M|}$. Then the stationary distributions of* DASEP$(2k+1,2,2)$ *and* DASEP$(2k+2,2,2)$ *are given by Table 2 and Table 3 where a_k is the generating function of the matchings in C_{2k+1}, and b_k is the generating function of the matchings in L_{2k+1}, i.e.,*

$$\begin{aligned} a_k &= \sum_{M:C_{2k+1}} (t+1)^{|M|}(u+1)^{k-|M|} \\ b_k &= \sum_{M:L_{2k+1}} (t+1)^{|M|}(u+1)^{k-|M|}. \end{aligned}$$

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From coloured permutations to Hadamard products and zeta functions

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Abstract. We devise a constructive method for computing explicit combinatorial formulae for Hadamard products of certain rational generating functions. The latter arise naturally when studying so-called ask zeta functions of direct sums of modules of matrices or class- and orbit-counting zeta functions of direct products of groups. Our method relies on shuffle compatibility of coloured permutation statistics and coloured quasisymmetric functions, extending recent work of Gessel and Zhuang.

Keywords: Coloured permutations, permutation statistics, Hadamard products, shuffle compatibility, average sizes of kernels, zeta functions

1 Introduction

Permutation statistics are functions defined on permutations and their generalisations. Studying the behaviour of said functions on sets of permutations is a classical theme in algebraic and enumerative combinatorics. The origins of permutation statistics can be traced back to work of Euler and MacMahon. The past decades saw a flurry of further developments in the area; see e.g. [2, 32] and references therein. Recently, Gessel and Zhuang [17] developed an algebraic framework for systematically studying so-called shuffle-compatible permutation statistics by means of associated shuffle algebras. In their work, quasisymmetric functions and Hadamard products of rational generating functions played key roles.

Numerous types of zeta functions have been employed in the study of enumerative problems surrounding algebraic structures. L. Solomon [31] introduced zeta functions associated with integral representations and, in another influential paper, Grunewald, Segal, and Smith [18] initiated the study of zeta functions associated with (nilpotent and pro-p) groups. Following [18], a variety of methods have been developed and applied to

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predict the behaviour and study symmetries of zeta functions associated with algebraic structures, and to produce explicit formulae. Theoretical work of this type often employs a blend of combinatorics and p-adic integration; see [35] for a survey. On the practical side, a range of effective methods have been devised and used to symbolically compute zeta functions of algebraic structures; see [25] and the references therein.

A common feature of zeta functions $\zeta_G(s)$ attached to algebraic structures G (e.g. groups) in the literature is that they often admit an Euler factorisation $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$ into so-called local factors indexed by primes p. Deep results from p-adic integration often guarantee that these local factors are rational in p^{-s}, i.e. of the form $\zeta_{G,p}(s) = W_p(p^{-s})$ for some $W_p(Y) \in \mathbb{Q}(Y)$. A key theme is then to study how the $W_p(Y)$ vary with the prime p. In a surprising number of cases of interest, deep *uniformity results* ensure the existence of a *single* bivariate rational function $W(X, Y)$ such that $\zeta_{G,p}(s) = W(p, p^{-s})$ for all primes p (perhaps excluding a finite number of exceptions). In such situations, understanding our zeta function is tantamount to understanding $W(X, Y)$.

In this context, permutation statistics (and, more generally, combinatorial objects) have recently found spectacular applications, in particular when it comes to describing the numerators of the rational functions $W(X, Y)$ from above; see, for instance, [1, 12, 9, 8, 10, 34]. Conversely, the need for combinatorial descriptions of such zeta functions gave rise to new directions in the study of permutation statistics and, more generally, combinatorial objects; see, e.g. [4, 5, 11, 13, 16, 14, 33].

In the spirit of this line of research, in the present work we relate permutation statistics and *ask zeta functions*. Introduced in [26] and developed further in [27, 30, 7], ask zeta functions are generating functions encoding average sizes of kernels in suitable modules of matrices. One motivation for studying these functions comes from group theory. Indeed, for groups with a sufficiently powerful Lie theory, the enumeration of linear orbits and conjugacy classes boils down to determining average sizes of kernels within matrix Lie algebras—this is essentially the orbit-counting lemma.

Amidst a plethora of algebraically-defined zeta functions, ask zeta functions stand out as particularly amenable to combinatorial methods. Indeed, natural operations at the level of the modules (or groups) often translate into natural operations of corresponding rational generating functions. In particular, ask zeta functions of direct sums of modules are Hadamard products of the ask zeta function of the summands.

In this extended abstract (and forthcoming preprint [6]) we answer some of the questions from [30] and give a constructive algorithm (based on *coloured shuffle compatibility* of permutation statistics) to compute Hadamard products of certain ask zeta functions. Our results have corollaries pertaining to generating functions enumerating orbits of finite direct products of groups within various infinite families.

2 Hadamard products and coloured configurations

In this section, we provide a self-contained account of our main result pertaining to Hadamard products of suitable rational generating functions. Its proof relies on the coloured shuffle compatibility of certain permutation statistics and the structure of associated coloured shuffle algebras. We will describe the latter in Section 3.

Coloured permutations and descents. We consider coloured permutations with symbols taken from the poset $\Sigma = \{1 < 2 < \dots\}$ and colours taken from $\Gamma = \{0 > 1 > 2 > \cdots\}$. Let $\boldsymbol{a} = \sigma^\gamma = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ be a coloured permutation. We write $|\boldsymbol{a}| = n$ for the length of $\boldsymbol{a}$. We further write $\mathrm{sym}(\boldsymbol{a}) = \{\sigma_1, \dots, \sigma_n\}$, $\mathrm{pal}(\boldsymbol{a}) = \{\gamma_1, \dots, \gamma_n\}$, and $\mathrm{pal}^*(\boldsymbol{a}) = \mathrm{pal}(\boldsymbol{a}) \setminus \{0\}$. On the set of Γ-coloured positive integers, consider the total order

$$\cdots < 1^1 < 2^1 < \cdots < 1^0 < 2^0 < \dots;$$

that is, $\sigma_1^{\gamma_1} < \sigma_2^{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$ and $\sigma_1 < \sigma_2$, or if $\gamma_1 > \gamma_2$ in $\mathbb{Z}$ (equivalently: $\gamma_1 < \gamma_2$ in Γ). This is the usual *colour order*, corresponding to the left lexicographic order on $\Gamma \times \Sigma$. The *descent set* of $\boldsymbol{a}$ as above consists of all $i \in [n-1]$ such that $\sigma_i^{\gamma_i} > \sigma_{i+1}^{\gamma_{i+1}}$ together with 0 whenever $\gamma_1 \neq 0$. The *descent number* and *comajor index* are defined as always as functions of the descent set: $\mathrm{des}(\boldsymbol{a}) = |\mathrm{Des}(\boldsymbol{a})|$ and $\mathrm{comaj}(\boldsymbol{a}) = \sum_{i \in \mathrm{Des}(\boldsymbol{a})} (n - i)$.

Coloured configurations. Let $\mathcal{A}$ be the set of all coloured permutations, and let $\mathbb{Z}\mathcal{A}$ be the free abelian group with basis $\mathcal{A}$. We call elements of $\mathbb{Z}\mathcal{A}$ *coloured configurations*. These elements are of the form $f = \sum_{\boldsymbol{a} \in \mathcal{A}} f_{\boldsymbol{a}} \boldsymbol{a}$, where each $f_{\boldsymbol{a}}$ belongs to $\mathbb{Z}$ and almost all $f_{\boldsymbol{a}}$ are zero. Write $\mathrm{supp}(f) = \{\boldsymbol{a} \in \mathcal{A} : f_{\boldsymbol{a}} \neq 0\}$, $\mathrm{sym}(f) = \bigcup_{\boldsymbol{a} \in \mathrm{supp}(f)} \mathrm{sym}(\boldsymbol{a})$, and $\mathrm{pal}^*(f) = \bigcup_{\boldsymbol{a} \in \mathrm{supp}(f)} \mathrm{pal}^*(\boldsymbol{a})$. We call $f, g \in \mathbb{Z}\mathcal{A}$ *strongly disjoint* if $\mathrm{sym}(f) \cap \mathrm{sym}(g) = \varnothing = \mathrm{pal}^*(f) \cap \mathrm{pal}^*(g)$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$, let $\boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b} \in \mathbb{Z}\mathcal{A}$ be the sum over all shuffles of $\boldsymbol{a}$ and $\boldsymbol{b}$. We extend $\sqcup\!\sqcup$ to a bi-additive product on $\mathbb{Z}\mathcal{A}$.

Labels. Let $U = \{\pm X^k : k \in \mathbb{Z}\}$, viewed as a subgroup of the multiplicative group of the field $\mathbb{Q}(X)$. For $\alpha \colon \Gamma \to U$, write $\mathrm{supp}(\alpha) = \{c \in \Gamma : \alpha(c) \neq 1\}$ and, for $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ as above, let $\alpha(\boldsymbol{a}) = \prod_{i=1}^n \alpha(\gamma_i)$. A *labelled coloured configuration* is a pair (f, α), where $f \in \mathbb{Z}\mathcal{A}$ and $\alpha \colon \Gamma \to U$ satisfies $\mathrm{supp}(\alpha) \subseteq \mathrm{pal}^*(f)$. Given labelled coloured configurations (f, α) and (g, β) such that f and g are strongly disjoint, the pair $(f \sqcup\!\sqcup g, \alpha\beta)$ is a labelled coloured configuration too. (Here, $\alpha\beta$ denotes the pointwise product of α and β.)

Equivalence. Let (f, α) be a labelled coloured configuration. Let $\phi \colon \mathrm{sym}(f) \to S$ and $\psi \colon \mathrm{pal}^*(f) \to P$ be order-preserving bijections onto finite subsets of Σ and $\Gamma \setminus \{0\}$, respectively. Given ϕ and ψ, define a labelled coloured permutation (f', α') as follows. For $\boldsymbol{a} \in \mathrm{supp}(f)$, say $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$, write $\boldsymbol{a}' = \phi(\sigma_1)^{\psi(\gamma_1)} \cdots \phi(\sigma_n)^{\psi(\gamma_n)}$. Define $f' = \sum_{\boldsymbol{a} \in \mathrm{supp}(f)} f_{\boldsymbol{a}} \boldsymbol{a}'$. The support of α' is P and $\alpha'(\psi(c)) = \alpha(c)$ for $c \in \mathrm{pal}^*(f)$. We call (f, α) and each (f', α') (as ϕ and ψ range over possible choices) *equivalent*, written $(f, \alpha) \asymp (f', \alpha')$. This defines an equivalence relation on labelled coloured configurations.

Rational functions. Given a labelled coloured configuration (f, α) and $\varepsilon \in \mathbb{Z}$, we define a rational formal power series

$$W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f,\alpha}(X,Y) = \sum_{\boldsymbol{a} \in \mathrm{supp}(f)} f_{\boldsymbol{a}} \frac{\alpha(\boldsymbol{a}) X^{\varepsilon\,\mathrm{comaj}(\boldsymbol{a})} Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)(1-X^{\varepsilon}Y)\cdots(1-X^{\varepsilon|\boldsymbol{a}|}Y)} \in \mathbb{Q}(X)[\![Y]\!].$$

Note that, by construction, if $(f, \alpha) \asymp (f', \alpha')$, then $W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f',\alpha'}$ for all $\varepsilon \in \mathbb{Z}$.

Example 2.1. Let $f = 1^0 + 1^1$. Let $\alpha \colon \Gamma \to U$ with $\mathrm{supp}(\alpha) \subseteq \mathrm{pal}^*(f) = \{1\}$. Then

$$W^{\varepsilon}_{f,\alpha} = \frac{1 + \alpha(1) X^{\varepsilon} Y}{(1-Y)(1-X^{\varepsilon}Y)}.$$

Recall that the *Hadamard product* of two formal power series $A(Y) = \sum_{k=0}^{\infty} a_k Y^k$ and $B(Y) = \sum_{k=0}^{\infty} b_k Y^k$ is the power series $A(Y) *_Y B(Y) = \sum_{k=0}^{\infty} a_k b_k Y^k$.

Theorem 2.2. *Let (f, α) and (g, β) be labelled coloured configurations such that f and g are strongly disjoint. Then $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f \sqcup g,\, \alpha\beta}$ for each $\varepsilon \in \mathbb{Z}$.*

Example 2.3. Let $f = 1^0 + 1^1$ and $g = 2^0 + 2^2$. Then

$$\begin{aligned} f \sqcup g &= (1^0 + 1^1) \sqcup (2^0 + 2^2) \\ &= 1^0 \sqcup 2^0 + 1^0 \sqcup 2^2 + 1^1 \sqcup 2^0 + 1^1 \sqcup 2^2 \\ &= 1^0 2^0 + 2^0 1^0 + 1^0 2^2 + 2^2 1^0 + 1^1 2^0 + 2^0 1^1 + 1^1 2^2 + 2^2 1^1. \end{aligned}$$

Let α and β satisfy $\mathrm{supp}(\alpha) \subseteq \{1\}$ and $\mathrm{supp}(\beta) \subseteq \{2\}$. By Theorem 2.2,

$$\begin{aligned} W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} &= \frac{1 + \alpha(1) X^{\varepsilon} Y}{(1-Y)(1-X^{\varepsilon}Y)} *_Y \frac{1 + \beta(2) X^{\varepsilon} Y}{(1-Y)(1-X^{\varepsilon}Y)} \\ &= \frac{1 + (1 + \alpha(1) + \beta(2)) X^{\varepsilon} Y + (\alpha(1) + \beta(2) + \alpha(1)\beta(2)) X^{2\varepsilon} Y + \alpha(1)\beta(2) X^{3\varepsilon} Y^2}{(1-Y)(1-X^{\varepsilon}Y)(1-X^{2\varepsilon}Y)} \\ &= W^{\varepsilon}_{f \sqcup g,\, \alpha\beta}. \end{aligned}$$

Theorem 2.2 implies, in particular, that for each <u>fixed</u> $\varepsilon \in \mathbb{Z}$, the set

$$\left\{ W^{\varepsilon}_{f,\alpha} : (f, \alpha) \text{ is a coloured configuration} \right\}$$

is closed under Hadamard products in Y. Indeed, given coloured configurations (f, α) and (g, β), we can find (g', β') such that f and g' are strongly disjoint and $(g, \beta) \asymp (g', \beta')$. In that case, $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g',\beta'} = W^{\varepsilon}_{f \sqcup g',\, \alpha\beta'}$ is computed by the preceding theorem.

In Section 4, we will apply Theorem 2.2 to provide explicit combinatorial descriptions of Hadamard products of ask, class- and orbit-counting zeta functions.

3 Coloured shuffle compatibility

For technical reasons, in this section, we will only consider coloured permutations with colours drawn from $\{0 > 1 > \cdots > r-1\}$ (for sufficiently large r). For clarity, we occasionally refer to these as *r-coloured permutations*. A *coloured permutation statistic* is a function st defined on the set of coloured permutations such that given a coloured permutation σ^γ, if π is a permutation of the same length as σ and with the same relative order, then $\mathrm{st}(\sigma^\gamma) = \mathrm{st}(\pi^\gamma)$. Given coloured permutation statistics $\mathrm{st}_1, \ldots, \mathrm{st}_k$, we regard the tuple $(\mathrm{st}_1, \ldots, \mathrm{st}_k)$ as a coloured permutation statistic via $(\mathrm{st}_1, \ldots, \mathrm{st}_k)(\boldsymbol{a}) = (\mathrm{st}_1(\boldsymbol{a}), \ldots, \mathrm{st}_k(\boldsymbol{a}))$. Given a coloured permutation $\boldsymbol{a} = \sigma^\gamma = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$, let $\mathrm{col}_j(\boldsymbol{a}) := |\{i \in [n] : \gamma_i = j\}|$. The *colour vector* of a $\boldsymbol{a}$ is $\mathbf{col}(\boldsymbol{a}) = (\mathrm{col}_0(\boldsymbol{a}), \ldots, \mathrm{col}_{r-1}(\boldsymbol{a}))$; this is a weak composition of n. The functions $\mathbf{col}$, des, and comaj are coloured permutation statistics.

Recall from [17, 22] that a (coloured) permutation statistic st is *shuffle compatible* if for coloured permutations $\boldsymbol{a}$ and $\boldsymbol{b}$ on disjoint sets of symbols, the multiset $\{\{\mathrm{st}(\boldsymbol{c}) : \boldsymbol{c} \in \boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}\}\}$ only depends on $\mathrm{st}(\boldsymbol{a}), \mathrm{st}(\boldsymbol{b})$ and the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$. (Here, $\boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}$ denotes the set of all coloured permutations obtained as shuffles of $\boldsymbol{a}$ and $\boldsymbol{b}$.) Generalising [17, 22], we associate a shuffle algebra $\mathcal{A}_{\mathrm{st}}^{(r)}$ over $\mathbb{Q}$ to a shuffle-compatible coloured permutation statistic st as follows. First, st defines an equivalence relation $\sim_{\mathrm{st}}$ on r-coloured permutations via $\boldsymbol{a} \sim_{\mathrm{st}} \boldsymbol{b}$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same length and $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$; we refer to this as st-*equivalence*. We write $[\boldsymbol{a}]_{\mathrm{st}}$ to denote the st-equivalence class of $\boldsymbol{a}$. As a $\mathbb{Q}$-vector space $\mathcal{A}_{\mathrm{st}}^{(r)}$ has a basis given by the st-equivalence classes of r-coloured permutations. The multiplication is given by linearly extending the rule

$$[\boldsymbol{a}]_{\mathrm{st}}\,[\boldsymbol{b}]_{\mathrm{st}} = \sum_{\boldsymbol{c} \in \boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}} [\boldsymbol{c}]_{\mathrm{st}},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are r-coloured permutations on disjoint sets of symbols. (Thanks to the shuffle compatibility of st, this yields a well-defined multiplication on $\mathcal{A}_{\mathrm{st}}^{(r)}$.)

The main shuffle algebra of interest to us is the one attached to $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$. Let $p_0, \ldots, p_{r-1}, x, t$ be commuting variables over $\mathbb{Q}$; write $\boldsymbol{p} = (p_0, \ldots, p_{r-1})$. For a ring R, let $R[\![t*]\!]$ denote the ring $R[\![t]\!]$ with multiplication given by the Hadamard product in t.

Theorem 3.1.

(a) The tuple of statistics $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$ *is shuffle compatible.*

(b) The linear map on $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})}$ *defined by*

$$[\boldsymbol{a}]_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \mapsto \begin{cases} \frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\mathrm{comaj}(\boldsymbol{a})} t^{\mathrm{des}(\boldsymbol{a})+1}}{(1-t)(1-xt)\cdots(1-x^{|\boldsymbol{a}|}t)} z^{|\boldsymbol{a}|}, & \text{if } |\boldsymbol{a}| \geqslant 1 \\ \frac{1}{1-t}, & \text{if } |\boldsymbol{a}| = 0 \end{cases}$$

is an isomorphism from the (r-coloured) shuffle algebra of $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$ *onto the subalgebra of* $\mathbb{Q}[p_0, p_1, \ldots, p_{r-1}, z, x][\![t*]\!]$ *spanned by*

$$\left\{\frac{1}{1-t}\right\} \bigcup \left\{\frac{p_0^{c_0} \cdots p_{r-1}^{c_{r-1}}\, x^k\, t^{j+1}}{(1-t)(1-xt)\cdots(1-x^n t)} z^n\right\}_{n\geqslant 1,\, j\in[0,n],\, c_0,\ldots,c_{r-1}\in[0,r-1],\, k\in\left[\binom{j+1}{2}, nj-\binom{j}{2}\right]}.$$

While we omit the proof of the preceding theorem here, we should like to take this opportunity to provide a brief overview of its key steps and ingredients. The *coloured descent set* of a coloured permutation $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ is defined as

$$\mathrm{sDes}(\boldsymbol{a}) = \Big\{(i, \gamma_i) : i \in [n-1],\ \gamma_i \neq \gamma_{i+1} \text{ or } (\gamma_i = \gamma_{i+1} \text{ and } \sigma_i > \sigma_{i+1})\Big\} \cup \Big\{(n, \gamma_n)\Big\}.$$

This coloured permutation statistic, which was introduced by Mantaci and Reutenauer [21] while studying a coloured generalisation of Solomon's descent algebra, is shuffle compatible. Moreover, it refines the tuple $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$. As a consequence, the algebra $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})}$ is naturally a quotient of $\mathcal{A}^{(r)}_{\mathrm{sDes}}$.

Let $x_i^{(j)}$ for $i = 1, 2, \ldots$ and $j = 0, 1, \ldots, r-1$ be independent (commuting) variables. We write $\boldsymbol{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$. The *coloured quasisymmetric function* attached to an r-coloured permutation $\boldsymbol{a} = \sigma^\gamma$ of length n is

$$F_{\boldsymbol{a}}(\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(r-1)}) = \sum_{\substack{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_n \\ j\in\mathrm{Des}^*(\boldsymbol{a}) \Rightarrow i_j<i_{j+1}}} x_{i_1}^{(\gamma_1)} x_{i_2}^{(\gamma_2)} \cdots x_{i_n}^{(\gamma_n)},$$

where $\mathrm{Des}^*(\boldsymbol{a}) = \mathrm{Des}(\boldsymbol{a})\setminus\{0\}$. This is a (homogeneous) formal power series of degree n in the variables $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(r-1)}$. These functions were first introduced in [24]; see also [19, 22, 23]. The space $\mathrm{QSym}^{(r)}$ spanned by all such coloured quasisymmetric functions forms a $\mathbb{Q}$-algebra. It turns out that $\mathrm{QSym}^{(r)}$ and $\mathcal{A}^{(r)}_{\mathrm{sDes}}$ are canonically isomorphic. Our proof of Theorem 3.1 is based on a judicious choice of specialisations of coloured quasisymmetric functions; cf. [22, §4.4]. Theorem 2.2 is a consequence of Theorem 3.1.

4 Applications to zeta functions

4.1 Ask, class-counting, and orbit-counting zeta functions

The main purpose of the present section is to recall four families of zeta functions associated with algebraic structures (Examples 4.1–4.6). These will feature in our applications of Theorem 2.2 in Section 4.2. For further details, see [26, 27]. Rings will be assumed to be commutative and unital. In order to maintain consistency with the literature, we regard $d \times e$ matrices over a ring R as homomorphisms $R^d \to R^e$ acting by right multiplication.

Global ask zeta functions. Given a module $M \subseteq \mathrm{M}_{d\times e}(\mathbb{Z})$ of integral matrices, for each $n \geqslant 1$, let $M_n \subseteq \mathrm{M}_{d\times e}(\mathbb{Z}/n\mathbb{Z})$ denote the reduction of M modulo n. The *(global) ask zeta function* of M is the Dirichlet series $\zeta_M^{\mathrm{ask}}(s) = \sum_{n=1}^{\infty} a_n(M) n^{-s}$, where $a_n(M) \in \mathbb{Q}$ denotes the average size of the kernel of matrices in M_n. By the Chinese remainder theorem, $\zeta_M^{\mathrm{ask}}(s) = \prod_p \zeta_{M_p}^{\mathrm{ask}}(s)$ (Euler product), where the product is taken over all primes p and the local factor at p is given by $\zeta_{M_p}^{\mathrm{ask}}(s) = \sum_{k=0}^{\infty} a_{p^k}(M) p^{-ks}$, a power series in p^{-s}. Drawing upon deep results from p-adic integration and the theory of zeta functions of algebraic structures, it is known that each $\zeta_{M_p}^{\mathrm{ask}}(s)$ is rational in p^{-s}.

Local ask zeta functions. It is often advantageous to bypass global structures altogether and directly study variants of the local factors from above. Let $\mathfrak{O}$ be a compact discrete valuation ring. Let $\mathfrak{P}$ be the maximal ideal of $\mathfrak{O}$ and let q denote the size of the residue field $\mathfrak{O}/\mathfrak{P}$. Such rings $\mathfrak{O}$ are precisely the valuation rings of non-Archimedean local fields. Examples include the p-adic integers $\mathbb{Z}_p$ (in which case $\mathfrak{O}/\mathfrak{P} \cong \mathbb{F}_p$) and the ring $\mathbb{F}_q[\![z]\!]$ of formal power series over $\mathbb{F}_q$ (in which case $\mathfrak{P} = z\mathbb{F}_q[\![z]\!]$).

Given a module of matrices $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$, its associated *(local) ask zeta function* is the formal power series $\mathsf{Z}_M^{\mathrm{ask}}(Y) = \sum_{k=0}^{\infty} \alpha_k(M) Y^k$, where $\alpha_k(M)$ denotes the average size of the kernels within the reduction of M modulo $\mathfrak{P}^k$.

Example 4.1. $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y) = \frac{1-q^{-e}Y}{(1-Y)(1-q^{d-e}Y)}$; see [26, Prop. 1.5].

Example 4.2. Let $\mathfrak{O}$ have characteristic distinct from 2. Let $\mathfrak{so}_d(\mathfrak{O})$ be the module of antisymmetric $d \times d$ matrices over $\mathfrak{O}$. By [26, Prop. 5.11], $\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y) = \frac{1-q^{1-d}Y}{(1-Y)(1-qY)}$.

Example 4.3. Let $\mathfrak{n}_d(\mathfrak{O})$ be the module of strictly upper triangular $d \times d$ matrices over $\mathfrak{O}$. By [26, Prop. 5.15(i)], $\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{n}_d(\mathfrak{O})}(Y) = \frac{(1-Y)^{d-1}}{(1-qY)^d}$.

Class- and orbit-counting zeta functions. Let $\mathfrak{O}$ be a compact discrete valuation ring as above. Let G be a linear group scheme over $\mathfrak{O}$, with a given embedding into $d \times d$ matrices. The *orbit-counting zeta function* of G is the generating function $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} b_k(\mathsf{G}) Y^k$, where $b_k(\mathsf{G})$ denotes the number of orbits of the (finite) matrix group $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$ on its natural module $(\mathfrak{O}/\mathfrak{P}^k)^d$. The *class-counting zeta function* of G is the generating function $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} c_k(\mathsf{G}) Y^k$, where $c_k(\mathsf{G})$ denotes the number of conjugacy classes of $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$. Class-counting zeta functions go back to work of du Sautoy [15]. As shown in [26, 27], subject to restrictions on the residue field size q of $\mathfrak{O}$, class- and orbit-counting zeta functions of G are instances of ask zeta functions associated with modules of matrices over $\mathfrak{O}$. When passing between ask and class-counting zeta functions, one often needs to apply a transformation $Y \leftarrow q^m Y$ for a suitable integer m; see below for an example and cf. Lemma 4.7.

Example 4.4. Suppose that the residue field size q of $\mathfrak{O}$ is odd. By exponentiation, the free class-2-nilpotent Lie algebra on d generators over $\mathfrak{O}$ gives rise to a group scheme $\mathsf{F}_{2,d}$ over $\mathfrak{O}$. We may view $\mathsf{F}_{2,d}$ as an analogue of the free class-2-nilpotent group on d generators. Lins [20, Cor. 1.5] showed that

$$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \frac{1 - q^{\binom{d-1}{2}}Y}{\left(1-q^{\binom{d}{2}}Y\right)\left(1-q^{\binom{d}{2}+1}Y\right)}.$$

Looking back at Example 4.2, we observe that $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(q^{\binom{d}{2}}Y)$; this is no coincidence, see [27, Ex. 7.3].

Example 4.5. Let U_d be the group scheme of upper unitriangular $d \times d$ matrices over $\mathfrak{O}$. Suppose that $\gcd(q,(d-1)!) = 1$. By [26, Thm 1.7] (cf. [7, Prop. 4.12]) and Example 4.3, we have $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_d}(Y) = \frac{(1-Y)^{d-1}}{(1-qY)^d}$.

Graphs and graphical groups. Given a (finite, simple) graph Γ with distinct vertices $v_1,\dots,v_n$ and m edges, let M_Γ be the module of antisymmetric $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} = 0$ whenever v_i and v_j are non-adjacent. We write $\mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y)$ for $\mathsf{Z}^{\mathrm{ask}}_{M(\Gamma)}(Y)$. As shown in [30, Thm A], $\mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y)$ is a rational function in q and Y. In [30, §3.4], the graphical group scheme G_Γ associated with Γ is constructed; for an alternative but equivalent construction, see [28, §1.1]. By [30, Prop. 3.9], $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_\Gamma}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\Gamma}(q^m Y)$. Given graphs Γ_1 and Γ_2, let $\Gamma_1 \vee \Gamma_2$ denote their join, obtained from the disjoint union of Γ_1 and Γ_2 by adding edges connecting each vertex of Γ_1 to each vertex of Γ_2. Let K_n (resp. Δ_n) denote the complete (resp. edgeless) graph on n vertices.

Example 4.6. Consider $\Gamma = \Delta_n \vee \mathrm{K}_{n+1}$. Then Γ has $3\binom{n+1}{2}$ edges. It follows from [30, Thm 8.18] that

$$\mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y) = \frac{(1-q^{-n}Y)(1-q^{-n-1}Y)}{(1-q^{-1}Y)(1-Y)(1-qY)}.$$

Hadamard products and zeta functions. Let $\mathfrak{O}$ be as above. Elaborating further on what we wrote in the introduction, modules of matrices, (linear) group schemes, and graphs all admit natural operations which correspond to taking Hadamard products of zeta functions. In detail, given modules $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$ and $M' \subseteq \mathrm{M}_{d'\times e'}(\mathfrak{O})$, we regard $M \oplus M'$ as a submodule of $\mathrm{M}_{(d+d')\times(e+e')}(\mathfrak{O})$, embedded diagonally. Then $\mathsf{Z}^{\mathrm{ask}}_{M\oplus M'}(Y) = \mathsf{Z}^{\mathrm{ask}}_{M}(Y) *_Y \mathsf{Z}^{\mathrm{ask}}_{M'}(Y)$. Similarly, given (linear) group schemes G and G' over $\mathfrak{O}$, we obtain $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}\times\mathsf{G}'}(Y) = \mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) *_Y \mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}'}(Y)$. Finally, given graphs Γ and Γ', let $\Gamma \oplus \Gamma'$ denote their disjoint union. Then $\mathsf{Z}^{\mathrm{ask}}_{\Gamma\oplus\Gamma'}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y) *_Y \mathsf{Z}^{\mathrm{ask}}_{\Gamma'}(Y)$; moreover, $\mathsf{G}_{\Gamma\oplus\Gamma'} \cong \mathsf{G}_\Gamma \times \mathsf{G}_{\Gamma'}$.

4.2 Applications

It turns out that each zeta function from Examples 4.1–4.6 can be expressed in terms of the rational functions $W^{\epsilon}_{f,\alpha}(X,Y)$ attached to labelled coloured configurations as in Section 2. Omitting proofs, details of this are recorded in Table 1. This table is to be read as follows: for each row and each compact discrete valuation ring $\mathfrak{O}$ with residue field size q (possibly subject to further conditions on $\mathfrak{O}$ as in the examples above), the zeta function $\mathsf{Z}(Y)$ indicated in the leftmost column is obtained from the rational function in the rightmost column via $\mathsf{Z}(Y) = W^{\varepsilon}_{f,\alpha}(q, u(q)Y)$. In Table 1, we write $\underline{n}$ for the sum of all 2^n coloured permutations of the form $1^{\nu_1}\cdots n^{\nu_n}$ with $\nu_i \in \{0, i\}$. A "%" indicates that an entry coincides with the one immediately above it. We note that Table 1 does not constitute an exhaustive list of zeta functions expressible in terms of coloured configurations; we refer to our upcoming paper [6] for further examples and applications.

Zeta function	f	α	ε	$u(X)$	$W^{\varepsilon}_{f,\alpha}(X,Y)$
$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y)$	$\underline{1}$	$1 \leftarrow -X^{-d}$	$d-e$	1	$\frac{1-X^{-e}Y}{(1-Y)(1-X^{d-e}Y)}$
$\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y)$, $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(Y)$	%	%	1	%	$\frac{1-X^{1-d}Y}{(1-Y)(1-XY)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y)$	%	%	%	$X^{\binom{d}{2}}$	%
$\mathsf{Z}^{\mathrm{ask}}_{\Delta_n\vee\mathrm{K}_{n+1}}(Y)$	$\underline{2}$	$1,2 \leftarrow -X^{-n-1}$	1	X^{-1}	$\frac{(1-X^{1-n}Y)(1-X^{-n}Y)}{(1-Y)(1-XY)(1-X^2Y)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_{\Delta_n\vee\mathrm{K}_{n+1}}}(Y)$	%	%	%	$X^{3\binom{n+1}{2}-1}$	%
$\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d+1}}(Y)$	$\underline{d}$	$1,\dots,d \leftarrow -X^{-1}$	0	X	$\frac{(1-X^{-1}Y)^d}{(1-Y)^{d+1}}$

Table 1: Examples of zeta functions from labelled coloured configurations

We now explain how, subject to a compatibility condition, Theorem 2.2 can be used to explicitly compute Hadamard products of the zeta functions in Table 1. As explained in Section 4.1, we can interpret such Hadamard products as zeta functions associated with "products" of the objects under consideration. We first record an elementary fact.

Lemma 4.7. *Let R be a commutative ring. Let $A(Y) = \sum_{k=0}^{\infty} a_k Y^k$ and $B(Y) = \sum_{k=0}^{\infty} b_k Y^k$ be formal power series over R. Let $u, v \in R$. Then $A(uY) *_Y B(vY) = (A *_Y B)(uvY)$.* □

The compatibility condition that we alluded to above is that we require entries in the ε-column of Table 1 to agree for us to compute associated Hadamard products via Theorem 2.2 and Lemma 4.7. Thus, suppose that (f,α) and (g,β) are coloured configurations and let $u(X)$ and $v(X)$ each be of the form $\pm X^{\ell}$ for $\ell \in \mathbb{Z}$. Then

$$W^{\varepsilon}_{f,\alpha}\big(X,\, u(X)Y\big) *_Y W^{\varepsilon}_{g,\beta}\big(X,\, v(X)Y\big) = (W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta})\big(X,\, u(X)v(X)Y\big).$$

As explained in Section 2, by passing to equivalent labelled coloured configurations, we may assume that f and g are strongly disjoint. Theorem 2.2 then allows us to explicitly compute $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta}$. (Here it is crucial that a common value of ε is used in both factors.) All that remains to obtain our zeta function is to apply the specialisation $X \leftarrow q$.

To illustrate the scope of our method by means of, say, a group-theoretic application, first note that for *specific* choices of $d_1, \ldots, d_r$, a finite computation (using the algorithm in [30, §6], implemented in the software package Zeta [29]) can be used to determine $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_1} \times \cdots \times \mathsf{F}_{2,d_r}}(Y)$. Our results here go substantially further. Indeed, for any fixed r, an explicit finite computation produces a single formula for $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_1} \times \cdots \times \mathsf{F}_{2,d_r}}(Y) = \mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_1}}(Y) *_Y \cdots *_Y \mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_r}}(Y)$ as a *symbolic expression* involving variables $d_1, \ldots, d_r$.

Example 4.8. By combining Example 2.3 and Example 4.4, we find that $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d} \times \mathsf{F}_{2,d'}}(Y) = W\big(q, q^{\binom{d}{2}+\binom{d'}{2}} Y\big)$, where

$$W(X,Y) = \frac{1 + (1 - X^{-d} - X^{-d'})XY + (X^{-d-d'} - X^{-d} - X^{-d'})X^2Y + X^{-d-d'}X^3Y^2}{(1-Y)(1-XY)(1-X^2Y)}.$$

In the same spirit, using the data from Table 1, we can e.g. symbolically compute the orbit-counting zeta function of $\mathsf{U}_{d_1} \times \cdots \times \mathsf{U}_{d_r}$ and the ask zeta function of $\mathrm{M}_{d_1 \times e_1}(\mathfrak{O}) \oplus \cdots \oplus \mathrm{M}_{d_r \times e_r}(\mathfrak{O})$ when $d_1 - e_1 = \cdots = d_r - e_r$, all for fixed r but symbolic variables $d_1, \ldots, d_r$ and $e_1, \ldots, e_r$. In particular, the latter case provides an explicit description of the ask zeta function of $\mathrm{M}_{d_1}(\mathfrak{O}) \oplus \cdots \oplus \mathrm{M}_{d_r}(\mathfrak{O})$ in terms of coloured permutation statistics; cf. [30, Question 10.4(a)–(b)]. Such a description was previously only known for $d_1 = \cdots = d_r$; see [30, Prop. 10.3].

We conclude by noting that, to the best of our knowledge, the previously mentioned result [30, Prop. 10.3] and its close relative [26, Cor. 5.17] (both pertaining to direct powers of $\mathrm{M}_d(\mathfrak{O})$) were the only examples of Hadamard products of ask (as well as class- and orbit-counting) zeta functions expressed in terms of (coloured) permutation statistics. In the aforementioned results in the literature, the proofs rely on work of Brenti [3]. These proofs are based on the coincidence of the rational generating functions in question with those attached to so-called q-Eulerian polynomials of signed permutations. This preceded more recent machinery surrounding shuffle compatibility. This earlier work is now explained as part of the framework presented here.

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Murnaghan-Type Representations of the Elliptic Hall Algebra

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Abstract. We construct a new family of graded representations $\widetilde{W}_\lambda$ indexed by Young diagrams λ for the positive elliptic Hall algebra $\mathcal{E}^+$ which generalizes the standard $\mathcal{E}^+$ action on symmetric functions. These representations have homogeneous bases of eigenvectors for the action of the Macdonald element $P_{0,1} \in \mathcal{E}^+$ generalizing the symmetric Macdonald functions. We find an explicit combinatorial rule for the action of the multiplication operators $e_r[X]^\bullet$ generalizing the Pieri rule for symmetric Macdonald functions.

Keywords: Macdonald polynomials, elliptic Hall algebra, double affine Hecke algebra

1 Introduction

The space of symmetric functions, Λ, is a central object in algebraic combinatorics deeply connecting the fields of representation theory, geometry, and combinatorics. In his influential paper [11], Macdonald introduced a special basis $P_\lambda[X;q,t]$ for Λ over $\mathbb{Q}(q,t)$ simultaneously generalizing many other important and well-studied symmetric function bases like the Schur functions $s_\lambda[X]$. These symmetric functions $P_\lambda[X;q,t]$, called the symmetric Macdonald functions, exhibit many striking combinatorial properties and can be defined as the eigenvectors of a certain operator $\Delta : \Lambda \to \Lambda$, called the Macdonald operator, constructed using polynomial difference operators. It was discovered through the works of Bergeron, Garsia, Haiman, Tesler, and many others [10] [1] [2] that variants of the symmetric Macdonald functions called the modified Macdonald functions $\widetilde{H}_\lambda[X;q,t]$ have deep ties to the geometry of the Hilbert schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$. On the side of representation theory, it was shown first in full generality by Cherednik [4] that one can recover the symmetric Macdonald functions by considering the representation theory of certain algebras called the spherical double affine Hecke algebras (DAHAs) in type GL_n.

The positive elliptic Hall algebra (EHA), $\mathcal{E}^+$, was introduced by Burban and Schiffmann [3] as the positive subalgebra of the Hall algebra of the category of coherent sheaves on an elliptic curve over a finite field. This algebra has connections to many

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areas of mathematics including, most importantly for the present paper, to Macdonald theory. In [13], Schiffmann and Vasserot realize $\mathcal{E}^+$ as a stable limit of the positive spherical DAHAs in type GL_n. They show further that there is a natural action of $\mathcal{E}^+$ on Λ aligning with the spherical DAHA representations originally considered by Cherednik. In particular, the action of $P_{0,1} \in \mathcal{E}^+$ gives the Macdonald operator Δ. The action of $\mathcal{E}^+$ on Λ can be realized as the action of certain generalized convolution operators on the torus equivariant K-theory of the schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$.

Dunkl and Luque in [6] introduced symmetric and non-symmetric vector-valued (vv.) Macdonald polynomials. The term vector-valued here refers to polynomial-like objects of the form $\sum_\alpha c_\alpha X^\alpha \otimes v_\alpha$ for some scalars c_α, monomials X^α, and vectors v_α lying in some $\mathbb{Q}(q,t)$-vector space. The non-symmetric vv. Macdonald polynomials are distinguished bases for certain DAHA representations built from the irreducible representations of the finite Hecke algebras in type A. These DAHA representations are indexed by Young diagrams and exhibit interesting combinatorial properties relating to periodic Young tableaux. The symmetric vv. Macdonald polynomials are distinguished bases for the spherical (i.e. Hecke-invariant) subspaces of these DAHA representations. Naturally, the spherical DAHA acts on these spherical subspaces with the special element $Y_1 + \ldots + Y_n$ of spherical DAHA acting diagonally on the symmetric vv. Macdonald polynomials.

Dunkl and Luque in [6] (and in later work of Colmenarejo, Dunkl, and Luque [5] and Dunkl [7]) only consider the finite rank non-symmetric and symmetric vv. Macdonald polynomials. It is natural to ask if there is an infinite-rank stable-limit construction using the symmetric vv. Macdonald polynomials to give generalized symmetric Macdonald functions and associated representations of the positive elliptic Hall algebra $\mathcal{E}^+$. In this paper, we will describe such a construction (Thm. 2). We will obtain a new family of graded $\mathcal{E}^+$-representations $\widetilde{W}_\lambda$ indexed by Young diagrams λ and a natural generalization of the symmetric Macdonald functions $\mathfrak{P}_T$ indexed by certain labellings of infinite Young diagrams built as limits of the symmetric vv. Macdonald polynomials. For combinatorial reasons there is essentially a unique natural way to obtain this construction. For any λ we will consider the increasing chains of Young diagrams $\lambda^{(n)} = (n - |\lambda|, \lambda)$ for $n \geq |\lambda| + \lambda_1$ to build the representations $\widetilde{W}_\lambda$. These special sequences of Young diagrams are central to Murnaghan's Theorem [12] regarding the reduced Kronecker coefficients. As such we refer to the $\mathcal{E}^+$-representations $\widetilde{W}_\lambda$ as Murnaghan-type. For $\lambda = \emptyset$ we recover the $\mathcal{E}^+$ action on Λ and the symmetric Macdonald functions $P_\mu[X; q, t]$. We will show that these Murnaghan-type representations $\widetilde{W}_\lambda$ are mutually non-isomorphic. The existence of these representations of the elliptic Hall algebra raises many questions about possible new relations between Macdonald theory and geometry. Other authors have constructed families of $\mathcal{E}^+$-representations [8] [9]. Although there should exist a relationship between the Murnaghan-type representations $\widetilde{W}_\lambda$ and those of other authors, the construction in this paper appears to be distinct from prior $\mathcal{E}^+$-module constructions.

For technical reasons regarding the misalignment of the spectrum of the Cherednik operators Y_i we will need to restate many of the results of Dunkl and Luque in [6] using a re-oriented version of the Cherednik operators θ_i. This alternative choice of conventions greatly assists during the construction of the generalized Macdonald functions $\mathfrak{P}_T$. The combinatorics underpinning the non-symmetric vv. Macdonald polynomials originally defined by Dunkl and Luque will be reversed in the conventions appearing in this paper.

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2 Definitions and Notations

2.1 Some Combinatorics

We start with a description of many of the combinatorial objects which we will need for the remainder of this paper.

Definition 1. *A **partition** is a (possibly empty) sequence of weakly decreasing positive integers. Denote by $\mathbb{Y}$ the set of all partitions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ we set $\ell(\lambda) := r$ and $|\lambda| := \lambda_1 + \ldots + \lambda_r$. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Y}$ and $n \geq n_\lambda := |\lambda| + \lambda_1$ we set $\lambda^{(n)} := (n - |\lambda|, \lambda_1, \ldots, \lambda_r)$. We will identify partitions as defined above with **Young diagrams** of the corresponding shape in English notation i.e. justified up and to the left.*

Fix a partition λ with $|\lambda| = n$. We will require each of the following combinatorial constructions for types of labellings of the Young diagram λ. If a diagram λ appears as the domain of a labelling function then we are referring to the set of boxes of λ as the domain.

- *A non-negative **reverse Young tableau** $\mathrm{RYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \to \mathbb{Z}_{\geq 0}$ which is weakly decreasing along rows and columns.*
- *A non-negative **reverse semi-standard Young tableau** $\mathrm{RSSYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \to \mathbb{Z}_{\geq 0}$ which is weakly decreasing across rows and strictly decreasing down columns.*
- *A **standard Young tableau** $\mathrm{SYT}(\lambda)$ is a labelling $\tau : \lambda \to \{1, \ldots, n\}$ which is strictly increasing along rows and columns.*
- *A non-negative **periodic standard Young tableau** $\mathrm{PSYT}_{\geq 0}(\lambda)$ is a labelling $\tau : \lambda \to \{jq^b : 1 \leq j \leq n, b \geq 0\}$ in which each $1 \leq j \leq n$ occurs in exactly one box of λ and where the labelling is strictly increasing along rows and columns. Here we order the formal products jq^m by $jq^m < kq^\ell$ if $m > \ell$ or in the case that $m = \ell$ we have $j < k$. Note that $SYT(\lambda) \subset \mathrm{PSYT}_{\geq 0}(\lambda)$.*

Example 1. $$\begin{array}{|c|c|c|c|c|c|}\hline 17q^7 & 15q^5 & 16q^5 & 11q^3 & 7q^1 & 2q^0 \\ \hline 14q^6 & 12q^4 & 13q^4 & 9q^2 & 8q^0 \\ \cline{1-5} 10q^2 & 4q^1 & 5q^1 & 6q^1 \\ \cline{1-4} 3q^1 & 1q^0 \\ \cline{1-2} \end{array} \in \mathrm{PSYT}_{\geq 0}(6,5,4,2)$$

Definition 2. *Given a box, $\square$, in a Young diagram λ we define the content of $\square$ as $c(\square) := a - b$ where $\square$ is in row b and column a. Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ and $1 \leq i \leq n$. Whenever $\tau(\square) = iq^b$ for some box $\square \in \lambda$ we will write $c_\tau(i) := c(\square)$ and $w_\tau(i) := b$. Let $1 \leq j \leq n-1$ and suppose that for some boxes $\square_1, \square_2 \in \lambda$ that $\tau(\square_1) = jq^m$ and $\tau(\square_2) = (j+1)q^\ell$. Let τ' be the labelling defined by $\tau'(\square_1) = (j+1)q^m$, $\tau'(\square_2) = jq^\ell$, and $\tau'(\square) = \tau(\square)$ for $\square \in \lambda \setminus \{\square_1, \square_2\}$. If $\tau' \in \mathrm{PSYT}_{\geq 0}(\lambda)$ then we write $s_j(\tau) := \tau'$. Let $\Psi(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda)$ be the labelling defined by whenever $\tau(\square) = kq^a$ then either $\Psi(\tau)(\square) = (k-1)q^a$ when $k \geq 2$ or $\Psi(\tau)(\square) = nq^{a+1}$ when $k = 1$. We give the set $\mathrm{PSYT}_{\geq 0}(\lambda)$ a partial order defined by the following cover relations.*

- *For all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$, $\Psi(\tau) > \tau$.*
- *If $w_\tau(i) < w_\tau(i+1)$ then $s_i(\tau) > \tau$.*
- *If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then $s_i(\tau) > \tau$.*

Define the map $\mathfrak{p}_\lambda : \mathrm{PSYT}_{\geq 0}(\lambda) \to \mathrm{RYT}_{\geq 0}(\lambda)$ by $\mathfrak{p}_\lambda(\tau)(\square) = b$ whenever $\tau(\square) = iq^b$. We will write $\mathrm{PSYT}_{\geq 0}(\lambda; T)$ for the set of all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T \in \mathrm{RYT}_{\geq 0}(\lambda)$.

Example 2. $$\Psi\left(\begin{array}{|c|c|c|c|c|c|}\hline 1q^7 & 3q^5 & 5q^5 & 8q^2 & 12q^1 & 17q^0 \\ \hline 2q^6 & 4q^5 & 6q^5 & 14q^0 & 16q^0 \\ \cline{1-5} 7q^2 & 10q^1 & 11q^1 & 15q^0 \\ \cline{1-4} 9q^1 & 13q^0 \\ \cline{1-2} \end{array}\right) = \begin{array}{|c|c|c|c|c|c|}\hline 17q^8 & 2q^5 & 4q^5 & 7q^2 & 11q^1 & 16q^0 \\ \hline 1q^6 & 3q^5 & 5q^5 & 13q^0 & 15q^0 \\ \cline{1-5} 6q^2 & 9q^1 & 10q^1 & 14q^0 \\ \cline{1-4} 8q^1 & 12q^0 \\ \cline{1-2} \end{array}$$

Lemma 1. *Let $\lambda \in \mathbb{Y}$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. There exist $\min(T), \mathrm{top}(T) \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ such that for all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$, $\min(T) \leq \tau \leq \mathrm{top}(T)$.*

Example 3. *Given* $T = \begin{array}{|c|c|c|c|c|c|}\hline 7 & 5 & 5 & 2 & 1 & 0 \\ \hline 6 & 5 & 5 & 0 & 0 \\ \cline{1-5} 2 & 1 & 1 & 0 \\ \cline{1-4} 1 & 0 \\ \cline{1-2} \end{array} \in \mathrm{RYT}_{\geq 0}(6,5,4,2)$ *we have that*

$$\min(T) = \begin{array}{|c|c|c|c|c|c|} \hline 17q^7 & 12q^5 & 13q^5 & 10q^2 & 6q^1 & 1q^0 \\ \hline 16q^6 & 14q^5 & 15q^5 & 2q^0 & 3q^0 \\ \cline{1-5} 11q^2 & 7q^1 & 8q^1 & 4q^0 \\ \cline{1-4} 9q^1 & 5q^0 \\ \cline{1-2} \end{array} \quad \text{and } \operatorname{top}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1q^7 & 3q^5 & 5q^5 & 8q^2 & 12q^1 & 17q^0 \\ \hline 2q^6 & 4q^5 & 6q^5 & 14q^0 & 16q^0 \\ \cline{1-5} 7q^2 & 10q^1 & 11q^1 & 15q^0 \\ \cline{1-4} 9q^1 & 13q^0 \\ \cline{1-2} \end{array}.$$

Definition 3. *Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. Define $S(T) \in \mathrm{SYT}(\lambda)$ by ordering the boxes of λ according to $\square_1 \leq \square_2$ if and only if*

- *$T(\square_1) > T(\square_2)$ or*
- *$T(\square_1) = T(\square_2)$ and $\square_1$ comes before $\square_2$ in the column-standard labelling of λ.*

Define the composition $\mu(T)$ of n so that the Young subgroup $\mathfrak{S}_{\mu(T)}$ of $\mathfrak{S}_n$ is the group generated by the $s_i \in \mathfrak{S}_n$ such that the entries iq^a and $(i+1)q^b$ occur in the same row of $\min(T)$ *for some $a, b \geq 0$.*

Example 4. *For $T \in \mathrm{RYT}_{\geq 0}(6,5,4,2)$ as in Example 3 we have that* $S(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 5 & 8 & 12 & 17 \\ \hline 2 & 4 & 6 & 14 & 16 \\ \cline{1-5} 7 & 10 & 11 & 15 \\ \cline{1-4} 9 & 13 \\ \cline{1-2} \end{array}$.

Definition 4. *Let $\lambda \in \mathbb{Y}$, with $|\lambda| = n$ and $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$. An ordered pair of boxes $(\square_1, \square_2) \in \lambda \times \lambda$ is called an **inversion pair** of τ if $S(T)(\square_1) < S(T)(\square_2)$ and $i > j$ where $\tau(\square_1) = iq^a$, $\tau(\square_2) = jq^b$ for some $a, b \geq 0$. The set of all inversion pairs of τ will be denoted by* $\mathrm{Inv}(\tau)$.

Example 5. *In the labelling* $\begin{array}{|c|c|c|c|c|c|} \hline 17q^7 & 12q^5 & 13q^5 & 10q^2 & 6q^1 & 1q^0 \\ \hline 16q^6 & 14q^5 & 15q^5 & 2q^0 & 3q^0 \\ \cline{1-5} 11q^2 & 7q^1 & 8q^1 & 4q^0 \\ \cline{1-4} 9q^1 & 5q^0 \\ \cline{1-2} \end{array}$ *the pairs $(17q^7, 12q^5)$, $(14q^5, 13q^5)$, and $(5q^0, 4q^0)$ are examples of inversions. Here we have referred to boxes according to their labels.*

2.2 Positive Double Affine Hecke Algebra

Here we describe the positive double affine Hecke algebras in type GL_n.

Definition 5. *Define the **positive double affine Hecke algebra** $\mathcal{D}_n$ to be the $\mathbb{Q}(q,t)$-algebra generated by $T_1, \ldots, T_{n-1}$, $\theta_1, \ldots, \theta_n$, and $X_1, \ldots, X_n$ subject to the relations*

- $(T_i - 1)(T_i + t) = 0$
- $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$
- $T_iT_j = T_jT_i$, $|i-j| > 1$
- $\theta_i\theta_j = \theta_j\theta_i$
- $\theta_{i+1} = tT_i^{-1}\theta_iT_i^{-1}$
- $T_i\theta_j = \theta_jT_i$, $j \notin \{i, i+1\}$
- $X_iX_j = X_jX_i$
- $X_{i+1} = tT_i^{-1}X_iT_i^{-1}$
- $T_iX_j = X_jT_i$, $j \notin \{i, i+1\}$
- $\pi_nX_i = X_{i+1}\pi_n$
- $\pi_nX_n = qX_1\pi_n$.

where in the above $\pi_n := t^{n-1}\theta_1T_1^{-1}\cdots T_{n-1}^{-1}$. The ***finite Hecke algebra*** *$\mathcal{H}_n$ is the subalgebra of $\mathcal{D}_n$ generated by the elements $T_1, \ldots, T_{n-1}$ and the* ***positive affine Hecke algebra*** *$\mathcal{A}_n$ is the subalgebra of $\mathcal{D}_n$ generated by the elements $T_1, \ldots, T_{n-1}, \theta_1, \ldots, \theta_n$.*

Remark 1. *Note that $\mathcal{D}_n$ has a $\mathbb{Z}_{\geq 0}$-grading determined by $\deg(X_i) = 1$ and $\deg(\theta_i) = \deg(T_i) = 0$. We will sometimes write $\theta_i^{(n)}$ for $\theta_i \in \mathcal{A}_n$ to differentiate between $\theta_i \in \mathcal{A}_m$.*

Definition 6. *Let $\epsilon^{(n)} \in \mathcal{H}_n$ denote the (normalized) trivial idempotent given by*

$$\epsilon^{(n)} := \frac{1}{[n]_t!}\sum_{\sigma\in\mathfrak{S}_n} t^{\binom{n}{2}-\ell(\sigma)}T_\sigma$$

where $[n]_t! := \prod_{i=1}^{n}(\frac{1-t^i}{1-t})$. We will write $[\mu]_t! := [\mu_1]_t!\cdots[\mu_r]_t!$ for any composition $\mu = (\mu_1, \ldots, \mu_r)$. The ***positive spherical double affine Hecke algebra*** *$\mathcal{D}_n^{sph}$ is the (non-unital) subalgebra of $\mathcal{D}_n$ given by $\mathcal{D}_n^{sph} := \epsilon^{(n)}\,\mathcal{D}_n\,\epsilon^{(n)}$.*

Remark 2. *Given any $\mathcal{D}_n$-module V the space $\epsilon^{(n)}(V)$ is naturally a $\mathcal{D}_n^{sph}$-module. Note that although $1 \notin \mathcal{D}_n^{sph}$ the algebra $\mathcal{D}_n^{sph}$ is unital with unit $\epsilon^{(n)}$. Further, $\mathcal{D}_n^{sph}$ has a grading inherited from $\mathcal{D}_n$.*

2.3 Positive Elliptic Hall Algebra

Here we give a very brief description of the positive elliptic Hall algebra.

Definition 7. *For $\ell > 0$ define the special elements $P_{0,\ell}^{(n)}, P_{\ell,0}^{(n)} \in \mathcal{D}_n^{sph}$ by*

- $P_{0,\ell}^{(n)} = \epsilon^{(n)}\left(\sum_{i=1}^{n}\theta_i^\ell\right)\epsilon^{(n)}$
- $P_{\ell,0}^{(n)} = q^\ell\epsilon^{(n)}\left(\sum_{i=1}^{n}X_i^\ell\right)\epsilon^{(n)}$.

Theorem 1. *[13] There is a unique graded algebra surjection $\mathcal{D}_{n+1}^{sph} \to \mathcal{D}_n^{sph}$ determined for $\ell > 0$ by $P_{0,\ell}^{(n+1)} \to P_{0,\ell}^{(n)}$ and $P_{\ell,0}^{(n+1)} \to P_{\ell,0}^{(n)}$.*

The existence of the graded algebra surjections $\mathcal{D}_{n+1}^{\mathrm{sph}} \to \mathcal{D}_n^{\mathrm{sph}}$ allows for the following definition.

Definition 8. *[13] The* ***positive elliptic Hall algebra*** $\mathcal{E}^+$ *is the stable limit of the graded algebras* $\mathcal{D}_n^{sph}$ *with respect to the maps* $\mathcal{D}_{n+1}^{sph} \to \mathcal{D}_n^{sph}$. *For* $\ell > 0$ *define the special elements of* $\mathcal{E}^+$, $P_{0,\ell} := \lim_n P_{0,\ell}^{(n)}$ *and* $P_{\ell,0} := \lim_n P_{\ell,0}^{(n)}$.

Remark 3. *The elements* $P_{0,\ell}^{(n)}, P_{\ell,0}^{(n)}$ *for* $\ell > 0$ *generate* $\mathcal{D}_n^{sph}$ *and the elements* $P_{0,\ell}, P_{\ell,0}$ *for* $\ell > 0$ *generate* $\mathcal{E}^+$ *[13]. Further* $\mathcal{E}^+$ *has a* $\mathbb{Z}_{\geq 0}$*-grading determined by* $\deg(P_{0,\ell}) = 0$ *and* $\deg(P_{\ell,0}) = \ell$.

3 DAHA Modules from Young Diagrams

3.1 The $\mathcal{D}_n$-module V_λ

We begin by defining a collection of DAHA modules indexed by Young diagrams $\lambda \in \mathbb{Y}$. These modules are the same as those appearing in [6] but we take the approach of using induction from $\mathcal{A}_n$ to $\mathcal{D}_n$ for their definition.

Definition 9. *Let* $\lambda \in \mathbb{Y}$ *with* $|\lambda| = n$. *We will denote by* S_λ *the irreducible* $\mathcal{H}_n$ *module corresponding to the partition* λ. *Define the algebra homomorphism* $\rho_n : \mathcal{A}_n \to \mathcal{H}_n$ *by* $\rho_n(T_i) = T_i$ *and* $\rho_n(\theta_1) = 1$. *Let* $\rho_n^*(S_\lambda)$ *denote the* $\mathcal{A}_n$ *module determined by* $X(v) := \rho_n(X)(v)$ *for* $X \in \mathcal{A}_n$ *and* $v \in S_\lambda$. *Define the* $\mathcal{D}_n$*-module* V_λ *to be the induced module* $V_\lambda := \mathrm{Ind}_{\mathcal{A}_n}^{\mathcal{D}_n} \rho_n^*(S_\lambda)$.

We fix a distinguished basis $\{e_\tau | \tau \in \mathrm{SYT}(\lambda)\}$ for S_λ consisting of $\rho_n(\theta^{(n)})$-weight vectors uniquely normalized such that the next proposition holds. The defining relations for the $\mathcal{H}_n$ modules S_λ have been omitted but they can be inferred from the next proposition. The modules V_λ naturally have the basis given by $X^\alpha \otimes e_\tau$ where X^α is a monomial and $\tau \in \mathrm{SYT}(\lambda)$. Note that the action of π_n on V_λ is invertible so we may consider the action of π_n^{-1} although we have not formally included π_n^{-1} into the algebra $\mathcal{D}_n$.

Using the theory of intertwiners for DAHA and some combinatorics we are able to show the following structural results. The F_τ appearing below are the version of the non-symmetric vv. Macdonald polynomials following our conventions.

Proposition 1. *There exists a basis of* V_λ *consisting of* $\theta^{(n)}$*-weight vectors* F_τ *for* $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ *with distinct* $\theta^{(n)}$*-weights such that the following hold:*

- $\theta_i^{(n)}(F_\tau) = q^{w_\tau(i)} t^{c_\tau(i)} F_\tau$
- *If* $\tau \in \mathrm{SYT}(\lambda)$ *then* $F_\tau = 1 \otimes e_\tau$.

- *If* $s_i(\tau) > \tau$ *then* $F_{s_i(\tau)} = \left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right) F_\tau.$
- $F_{\Psi(\tau)} = q^{w_1(\tau)} X_n \pi_n^{-1} F_\tau.$

Example 6.

$$
\begin{aligned}
F_{\begin{array}{|c|c|}\hline 1q & 2q \\ \hline 3 \\ \cline{1-1}\end{array}} &= t^{-2} X_1 X_2 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} + t^{-2}\left(\frac{1-t}{1-qt^2}\right) X_2 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} \\
&+ \frac{t^{-2}}{1+t}\left(\frac{1-t}{1-qt^2}\right) X_2 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} - t^{-3}\left(\frac{1-t}{1-qt^2}\right) X_1 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} \\
&+ \frac{t^{-1}}{1+t}\left(\frac{1-t}{1-qt^2}\right) X_1 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}
\end{aligned}
$$

Using Mackey decomposition we obtain the following.

Proposition 2. *The $\mathcal{D}_n$-module V_λ has the following decomposition into $\mathcal{A}_n$-submodules:*

$$\operatorname{Res}^{\mathcal{D}_n}_{\mathcal{A}_n} V_\lambda = \bigoplus_{T \in \mathrm{RYT}_{\geq 0}(\lambda)} U_T$$

where $U_T := span_{\mathbb{Q}(q,t)}\{F_\tau : \mathfrak{p}_\lambda(\tau) = T\}$. Further, each $\mathcal{A}_n$-module U_T is irreducible.

3.2 Connecting Maps Between $V_{\lambda^{(n)}}$

In order to build the inverse systems which we will use to define Murnaghan-type modules for $\mathcal{E}^+$, we need to consider the following maps.

Definition 10. *Let $\lambda \in \mathbb{Y}$. For $n \geq n_\lambda$ define $\Phi_\lambda^{(n)} : V_{\lambda^{(n+1)}} \to V_{\lambda^{(n)}}$ as the $\mathbb{Q}(q,t)$-linear map determined by*

$$\Phi_\lambda^{(n)}(X^\alpha \otimes e_\tau) = \begin{cases} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes e_{\tau|_{\lambda^{(n)}}} & \alpha_{n+1} = 0, \tau(\square_0) = n+1 \\ 0 & \textit{otherwise} \end{cases}$$

where $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$.

The next proposition is a crucial step in proving the main theorem of this paper. Its proof relies heavily on the use of the re-oriented Cherednik operators θ_i and their spectral analysis as well as the existence of a triangular monomial expansion of the F_τ.

Proposition 3. *Let $T \in \mathrm{RYT}_{\geq 0}(\lambda^{(n)})$ and $T' \in \mathrm{RYT}_{\geq 0}(\lambda^{(n+1)})$ be such that $T(\square) = T'(\square)$ for $\square \in \lambda^{(n)}$ and $T'(\square_0) = 0$ for $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$. Then $\Phi_\lambda^{(n)}(F_{\mathrm{top}(T')}) = F_{\mathrm{top}(T)}$.*

The maps $\Phi_\lambda^{(n)}$ possess another remarkable stability property regarding the action of the elements $P_{(0,\ell)}^{(n)}$ for $\ell > 0$.

Proposition 4. *For all $\ell > 0$ and $n \geq n_\lambda$,*

$$\Phi_\lambda^{(n)}\left(P_{(0,\ell)}^{(n+1)} - \sum_{\square\in\lambda^{(n+1)}} t^{\ell c(\square)}\right) = \left(P_{(0,\ell)}^{(n)} - \sum_{\square\in\lambda^{(n)}} t^{\ell c(\square)}\right)\Phi_\lambda^{(n)}.$$

4 Positive EHA Representations from Young Diagrams

In this section we build $\mathcal{E}^+$-modules using the maps $\Phi_\lambda^{(n)}$ and the stability of the F_τ basis already described.

4.1 The $\mathcal{D}_n^{\mathbf{sph}}$-modules $W_{\lambda^{(n)}}$

Here we consider the spherical subspaces of the V_λ modules.

Definition 11. *For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ define the $\mathcal{D}_n^{sph}$-module $W_\lambda := \epsilon^{(n)}(V_\lambda)$.*

We will need the following combinatorial description of the AHA submodules of V_λ which contain a nonzero T_i-invariant vector.

Proposition 5. *For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$,*

$$\dim_{\mathbb{Q}(q,t)}\epsilon^{(n)}(U_T) = \begin{cases} 1 & T \in \mathrm{RSSYT}_{\geq 0}(\lambda) \\ 0 & T \notin \mathrm{RSSYT}_{\geq 0}(\lambda). \end{cases}$$

We define the symmetric vv. Macdonald polynomials in the following way. These will agree up to scalars with those in [6].

Definition 12. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. Define $P_T \in \epsilon^{(n)}(U_T)$ to be the unique element of the form*

$$P_T = F_{\mathrm{top}(T)} + \sum_{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\setminus\{\mathrm{top}(T)\}} \kappa_\tau F_\tau.$$

We can now use Prop. 1 and Prop. 3 to prove the following results for the P_T.

Proposition 6. *For all $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$,*

$$P_T = \sum_{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in\mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}}\right) F_\tau.$$

Example 7. $P_{\begin{smallmatrix}1&1\\0&\end{smallmatrix}} = F_{\begin{smallmatrix}1q&2q\\3&\end{smallmatrix}} + \left(\frac{qt^2-t^{-1}}{qt-t^{-1}}\right) F_{\begin{smallmatrix}1q&3q\\2&\end{smallmatrix}} + \left(\frac{qt^2-t^{-1}}{qt-t^{-1}}\right)\left(\frac{qt-t^{-1}}{q-t^{-1}}\right) F_{\begin{smallmatrix}2q&3q\\1&\end{smallmatrix}}$

Proposition 7. *The set $\{P_T : T \in \mathrm{RSSYT}_{\geq 0}(\lambda)\}$ is a $\mathbb{Q}(q,t)[\theta_1,\dots,\theta_n]^{\mathfrak{S}_n}$-weight basis for W_λ and $P_{0,\ell}^{(n)}(P_T) = \left(\sum_{\square\in\lambda} q^{\ell T(\square)} t^{\ell c(\square)}\right) P_T$.*

The following is an important stability result for the symmetric vv. Macdonald polynomials. Its proof relies on Prop. 3 and Prop. 7.

Corollary 1. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda^{(n)})$ and $T' \in \mathrm{RSSYT}_{\geq 0}(\lambda^{(n+1)})$ such that $T(\square) = T'(\square)$ for $\square \in \lambda^{(n)}$ and $T'(\square_0) = 0$ for $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$. Then $\Phi_\lambda^{(n)}(P_{T'}) = P_T$.*

4.2 Stable Limit of the $W_{\lambda^{(n)}}$

We now can define the stable-limit spaces $\widetilde{W}_\lambda$ and the generalized symmetric Macdonald functions.

Definition 13. *Let $\lambda \in \mathbb{Y}$. Define the infinite diagram $\lambda^{(\infty)} := \bigcup_{n\geq n_\lambda} \lambda^{(n)}$. Define $\Omega(\lambda)$ to be the set of all labellings $T : \lambda^{(\infty)} \to \mathbb{Z}_{\geq 0}$ such that $|\{\square \in \lambda^{(\infty)} : T(\square) \neq 0\}| < \infty$, T decreases weakly across rows, and T decreases strictly down columns.*

*Define the space $W_\lambda^{(\infty)}$ to be the inverse limit $\varprojlim W_{\lambda^{(n)}}$ with respect to the maps $\Phi_\lambda^{(n)}$. Let $\widetilde{W}_\lambda$ be the subspace of all bounded X-degree elements of $W_\lambda^{(\infty)}$. For any symmetric function $F \in \Lambda$ define $F[X]^\bullet$ to be the corresponding multiplication operator on $\widetilde{W}_\lambda$. Lastly, for $T \in \Omega(\lambda)$ define the **generalized symmetric Macdonald function** $\mathfrak{P}_T := \lim_n P_{T|_{\lambda^{(n)}}} \in \widetilde{W}_\lambda$.*

Remark 4. *Each $\mathfrak{P}_T$ is homogeneous of X-degree $\deg(\mathfrak{P}_T) = \sum_{\square\in\lambda^{(\infty)}} T(\square) < \infty$. The set of all $\mathfrak{P}_T$ for $T \in \Omega(\lambda)$ gives a $\mathbb{Q}(q,t)$-basis of $\widetilde{W}_\lambda$. Lastly, the multiplication operators $F[X]^\bullet$ are well-defined since $\Phi_\lambda^{(n)} X_{n+1} = 0$.*

Using Prop. 4 we can define the following operators on $\widetilde{W}_\lambda$ generalizing the Macdonald operators on the space of symmetric functions Λ.

Definition 14. *For $\ell > 0$ define the operator $\Delta_\ell : \widetilde{W}_\lambda \to \widetilde{W}_\lambda$ to be the stable-limit*
$$\Delta_\ell := \lim_n \left(P_{0,\ell}^{(n)} - \sum_{\square\in\lambda^{(n)}} t^{\ell c(\square)}\right).$$

4.3 $\mathcal{E}^+$ Action on $\widetilde{W}_\lambda$

Finally, we are ready to state the main result of this paper. This theorem follows by applying Prop. 4, Cor. 1, and an argument of Schiffmann-Vasserot (Lemma 1.3 in [13]).

Theorem 2 (Main Theorem). *For $\lambda \in \mathbb{Y}$, $\widetilde{W}_\lambda$ is a graded $\mathcal{E}^+$-module with action determined for $\ell > 0$ by $P_{\ell,0} \to q^\ell p_\ell[X]^\bullet$ and $P_{0,\ell} \to \Delta_\ell$. Further, $\widetilde{W}_\lambda$ is spanned by a basis of eigenvectors $\{\mathfrak{P}_T\}_{T\in\Omega(\lambda)}$ with distinct eigenvalues for the operator $\Delta = \Delta_1$ which we will refer to as the **Macdonald operator**.*

Remark 5. *For $\lambda = \emptyset$, $\widetilde{W}_\emptyset = \Lambda$ recovers the standard representation of $\mathcal{E}^+$. In this case, $\Omega(\emptyset) \equiv \mathbb{Y}$ and $\mathfrak{P}_\mu = P_\mu[X; q^{-1}, t]$ (up to nonzero scalar).*

By considering the grading of each module $\widetilde{W}_\lambda$ and the spectral theory of the Macdonald operator Δ we can prove the following.

Proposition 8. *For $\lambda, \mu \in \mathbb{Y}$ distinct, $\widetilde{W}_\lambda \ncong \widetilde{W}_\mu$ as graded $\mathcal{E}^+$-modules.*

5 Pieri Rule

In this section we give the description of a Pieri rule for the generalized symmetric Macdonald functions $\mathfrak{P}_T$. We need to consider the following q,t-rational function.

Definition 15. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ define*

$$K_T(q,t) := \frac{[\mu(T)]_t!}{[n]_t!} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\min(T))} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right).$$

Using Prop. 6 and some book-keeping we obtain the following finite-rank Pieri formula.

Theorem 3. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ and $1 \leq r \leq n$ we have the expansion*

$$e_r[X_1 + \ldots + X_n]P_T = \sum_S d^{(r)}_{S,T} P_S$$

where

$$\frac{d^{(r)}_{S,T}}{t^{\binom{r}{2}} e_r(1,\ldots,t^{n-1}) K_S(q,t)} = \sum_{\substack{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T) \\ s.t. \\ \Psi^r(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;S)}} t^{c_\tau(1)+\ldots+c_\tau(r)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right)$$
$$\times \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\Psi^r(\tau))} \left(\frac{q^{S(\square_1)}t^{c(\square_1)} - q^{S(\square_2)}t^{c(\square_2)}}{q^{S(\square_1)}t^{c(\square_1)} - q^{S(\square_2)}t^{c(\square_2)+1}} \right)$$

and S ranges over all $S \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ one can obtain from T by adding r 1's to the boxes of T with at most one 1 being added to each box.

Definition 16. *For $S, T \in \Omega(\lambda)$ and $r \geq 1$ define $\mathfrak{d}^{(r)}_{S,T}$ by $e_r[X]^{\bullet}(\mathfrak{P}_T) = \sum_{S \in \Omega(\lambda)} \mathfrak{d}^{(r)}_{S,T} \mathfrak{P}_S$. Define the **rank** $\mathrm{rk}(T)$ to be the minimal $n \geq n_\lambda$ such that $T|_{\lambda^{(\infty)}/\lambda^{(n)}} = 0$.*

We can use the stability from Cor. 1 to obtain a Pieri rule.

Corollary 2 (Pieri Rule)**.** *Let $S, T \in \Omega(\lambda)$ and $r \geq 1$. For all $n \geq \mathrm{rk}(T) + r$*

$$\mathfrak{d}^{(r)}_{S,T} = d^{(r)}_{S|_{\lambda^{(n)}}, T|_{\lambda^{(n)}}}.$$

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Plane partitions and rowmotion on rectangular and trapezoidal posets

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Abstract. We define a birational map between labelings of a rectangular poset and its associated trapezoidal poset. This map tropicalizes to a bijection between the plane partitions of these posets of fixed height, giving a new bijective proof of a result by Proctor. We also show that this map is equivariant with respect to birational rowmotion, resolving a conjecture of Williams and implying that birational rowmotion on trapezoidal posets has finite order.

Keywords: birational rowmotion, plane partitions, trapezoidal posets

1 Introduction

For a finite poset P, a *plane partition* of P (also known as a *P-partition*) is an order-preserving labeling of P with nonnegative integers. When P is the *rectangular poset* $R_{r,s}$, the Cartesian product of two chains of r and s elements, an elegant product formula for the number of plane partitions of P with maximum label at most ℓ was given by MacMahon [16]. Surprisingly, Proctor [19] showed that there is another poset, namely the *trapezoidal poset* $T_{r,s}$, that has the same number of plane partitions with maximum label at most ℓ for all ℓ. (See Figure 1 for a depiction of $R_{4,3}$ and $T_{4,3}$.)

Proctor's proof relies on a branching rule for Lie algebra representations and is not bijective. Partial bijections were later constructed by Stembridge [22] and Reiner [20] for $\ell = 1$, and Elizalde [5] for $\ell = 2$, but a full bijection for all ℓ was not given until work of Hamaker, Patrias, Pechenik, and Williams [10] using K-theoretic jeu de taquin.

Although the bijection given in [10] has many nice properties, it also has some shortcomings. First, it cannot be extended in a natural way to a continuous piecewise-linear map on real-valued labelings of the rectangle and trapezoid. As a result, it cannot be written using expressions in the tropical semiring (that is, using the operations addition, subtraction, and maximum). Second, it does not appear to be generally well-behaved with respect to a certain map on posets called *rowmotion*.

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(Combinatorial) rowmotion is a term coined by Striker and Williams [23] to describe a map first studied by Brouwer and Schrijver [1] that permutes the set of order ideals of a poset, sending an order ideal $I \subseteq P$ to the order ideal generated by the minimal elements of $P \setminus I$. It was shown in [1] that the action of rowmotion on order ideals of the rectangle $R_{r,s}$ has order exactly $r + s$. Einstein and Propp [4] observed that one can generalize combinatorial rowmotion to a piecewise-linear map or birational map. Results about birational rowmotion then descend to results for piecewise-linear rowmotion (via tropicalization) and further to combinatorial rowmotion. Birational rowmotion on rectangular posets is closely related to the *birational Robinson–Schensted–Knuth (RSK) correspondence*, also known as *tropical* or *geometric RSK*—see [18] for some discussion.

For rectangular posets, birational rowmotion maintains many of the important dynamical properties of combinatorial rowmotion. For example, Grinberg and Roby [7] showed that birational rowmotion on $R_{r,s}$ still has finite order $r + s$, which was observed by Glick and Grinberg [9, 17] to be equivalent to a phenomenon from discrete dynamics known as *type AA Zamolodchikov periodicity* (see Volkov [24]). However, the class of posets for which birational rowmotion is known to have finite order is very small [7, 8]. Grinberg and Roby conjecture that birational rowmotion on $T_{r,s}$ likewise has finite order $r + s$. (See also [6] for more on conjectured good behavior of the related *R-systems*.)

Given the apparent close relationship between the rectangular and trapezoidal posets, Williams conjectures (as noted in [7], based on work in [25]) that there should exist a birational map between labelings of $R_{r,s}$ and $T_{r,s}$ that intertwines with the action of rowmotion (see also Hopkins [11] for further discussion). In particular, such a map would prove that birational rowmotion on $T_{r,s}$ has finite order $r + s$. In work of Dao, Wellman, Yost-Wolff, and Zhang [3], it was shown that the bijection given in [10] does intertwine with combinatorial rowmotion on plane partitions of height 1, thereby showing that combinatorial rowmotion on $T_{r,s}$ has the correct order. However, they also note that it does not respect piecewise-linear or birational rowmotion, so it cannot be used to prove periodicity on $T_{r,s}$ in these cases.

Our main result is to settle these questions. We construct a birational map between labelings of $R_{r,s}$ and $T_{r,s}$ using birational toggles [2, 4]. We then show that this map:

- tropicalizes to a continuous, piecewise-linear map that restricts to a bijection between plane partitions of $R_{r,s}$ and $T_{r,s}$ of height at most ℓ, and
- is equivariant with respect to rowmotion on $R_{r,s}$ and $T_{r,s}$, implying that birational (as well as piecewise-linear and combinatorial) rowmotion on $T_{r,s}$ has order $r + s$.

We also generalize the *chain shifting lemma* proved by the current authors in [12] (see also the noncommutative conversion lemma by Grinberg–Roby [9]), which is closely related to Schützenberger promotion on semistandard Young tableaux [13]. In particular, we derive a new, simple proof of this lemma based on the duality of plane trees.

The full version of this paper, which includes proofs, can be found at [14].

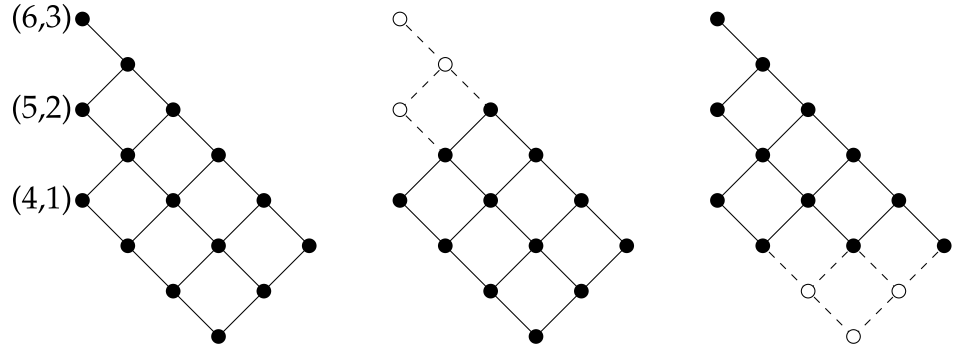

Figure 1: The right trapezoid $RT_{4,3}$, the rectangle $R_{4,3}$, and the trapezoid $T_{4,3}$.

2 Background

We begin with some background on posets and rowmotion. Fix positive integers $r \geq s$.

Definition 2.1. The *rectangle poset* $R_{r,s}$ is the Cartesian product of chains $[r] \times [s]$.
The *right trapezoid poset* $RT_{r,s}$ is the induced subposet $\{(i,j) \mid i - j < r\} \subseteq R_{r+s-1,s}$.
The *trapezoid poset* $T_{r,s}$ is the induced subposet $\{(i,j) \mid i + j > s\} \subseteq RT_{r,s}$.

See Figure 1. We draw our posets oriented in the plane so that the first coordinate increases to the northwest and the second coordinate increases to the northeast.

We can augment a poset P to a poset $\widehat{P}$ by adding minimum and maximum elements $\hat{0}$ and $\hat{1}$. Throughout, we will identify the labelings in $\mathbb{R}_+^P$ with the corresponding labelings in $\mathbb{R}_+^{\widehat{P}}$, where the labels at $\hat{0}$ and $\hat{1}$ are both 1.

Definition 2.2. For any $p \in P$, the *(birational) toggle* $t_p \colon \mathbb{R}_+^P \to \mathbb{R}_+^P$ is the map that acts on $y \in \mathbb{R}_+^P$ by fixing all coordinates except y_p and sending $y_p \mapsto \Big(\sum\limits_{q \gtrdot p} \frac{1}{y_q}\Big)^{-1} \Big(\sum\limits_{q \lessdot p} y_q\Big) \frac{1}{y_p}$.

The *rowmotion map* $\rho \colon \mathbb{R}_+^P \to \mathbb{R}_+^P$ is the composition $\rho = t_{L^{-1}(1)} \circ t_{L^{-1}(2)} \circ \cdots \circ t_{L^{-1}(n)}$ for any linear extension $L \colon P \to [n]$.

Definition 2.3. The *transfer map* $\psi^{-1} \colon \mathbb{R}_+^P \to \mathbb{R}_+^P$ is defined coordinatewise by $\psi^{-1}(y)_p = \frac{y_p}{\sum_{q \lessdot p} y_q}$. Its inverse ψ acts by $\psi(x)_p = \sum\limits_{\hat{0} \lessdot q_1 \lessdot \cdots \lessdot q_n = p} \prod\limits_{i=1}^{n} x_{q_i}$.

It will be convenient for us to work with the conjugate of rowmotion under the transfer map, $\tilde{\rho} = \psi^{-1} \circ \rho \circ \psi$. (This map was called *birational antichain rowmotion* or *barmotion* by Joseph and Roby [15].) An important property of $\tilde{\rho}$ is the following identity.

Proposition 2.4. *Let $x \in \mathbb{R}_+^P$, $y = \psi(x)$, and $z = \tilde{\rho}^{-1}(x)$. Then for $p \in P$,*

$$x_p^{-1} = \sum_{q \lessdot p} \frac{y_q}{y_p} \qquad \text{and} \qquad z_p^{-1} = \sum_{q \gtrdot p} \frac{y_p}{y_q}.$$

See Figure 2 for some example calculations involving these maps on $RT_{3,2}$.

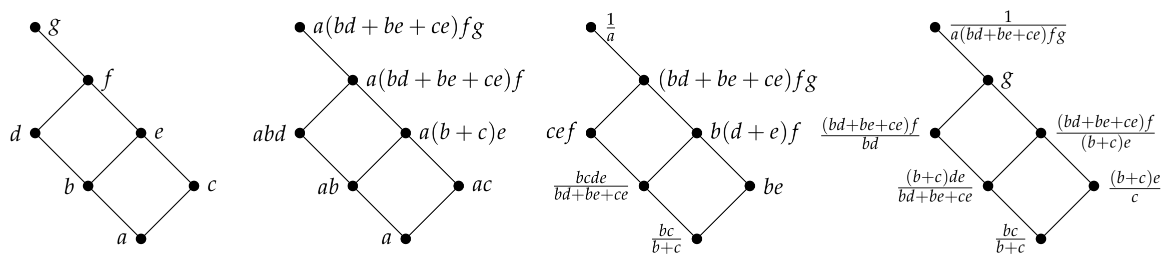

Figure 2: Labelings of $RT_{3,2}$: x, $y = \psi(x)$, $\rho^{-1}(y)$, and $z = \tilde{\rho}^{-1}(x) = \psi^{-1}(\rho^{-1}(y))$.

3 Chain Shifting in Skew Shapes

Let P be a poset and $x \in \mathbb{R}_+^P$. For a subset $S \subseteq P$, define the *weight* of S to be $w_S(x) = \prod_{p \in S} x_p$. We first show how to relate weights of certain subsets of P to weights of certain arborescences with respect to $y = \psi(x)$. (Recall that we set $y_{\hat{0}} = y_{\hat{1}} = 1$.)

Definition 3.1. An *upward arborescence of* P is a subgraph of $\widehat{P} \setminus \{\hat{1}\}$ such that every element of P has down degree 1. Similarly, a *downward arborescence of* P is a subgraph of $\widehat{P} \setminus \{\hat{0}\}$ such that every element of P has up degree 1. We denote the set of upward and downward arborescences of P by U_P and D_P, respectively.

Define the *weight* (with respect to y) of the edge e corresponding to the cover relation $p \lessdot q$ to be $\omega_e(y) = \frac{y_p}{y_q}$ and the *weight* of an arborescence T to be $\omega_T(y) = \prod_{e \in E(T)} \omega_e(y)$. (Note: The weight ω_T of an arborescence is different from the weight w_S of a subset.)

Example 3.2. The upward and downward arborescences of $RT_{3,2}$ are shown in Figure 3 with their weights. The weight of the first upward arborescence can be computed as

$$\frac{1}{y_{11}} \cdot \frac{y_{11}}{y_{21}} \cdot \frac{y_{11}}{y_{12}} \cdot \frac{y_{21}}{y_{31}} \cdot \frac{y_{21}}{y_{22}} \cdot \frac{y_{31}}{y_{32}} \cdot \frac{y_{32}}{y_{42}} = \frac{y_{11}y_{21}}{y_{12}y_{22}y_{42}}.$$

For any $y \in \mathbb{R}_+^P$, we can use $\omega_T(y)$ to define probability measures on U_P and D_P: for subsets $U \subseteq U_P$ and $D \subseteq D_P$, define

$$\mu_y(U) = \frac{\sum_{T \in U} \omega_T(y)}{\sum_{T \in U_P} \omega_T(y)}, \qquad \mu_y(D) = \frac{\sum_{T \in D} \omega_T(y)}{\sum_{T \in D_P} \omega_T(y)}.$$

Given a collection of saturated chains $\mathscr{C}$, let $U_P(\mathscr{C})$ and $D_P(\mathscr{C})$ denote the sets of all upward and downward arborescences that contain (the edges of) some chain in $\mathscr{C}$, and let $w_{\mathscr{C}}(x)$ denote the total weight of all chains in $\mathscr{C}$ (as subsets of P).

One can exploit the symmetry of Proposition 2.4 to compute the following result.

Corollary 3.3. *Let P be a poset, $x \in \mathbb{R}_+^P$, and $z = \tilde{\rho}^{-1}(x)$. Let $m, m', M, M' \in P$ such that m' is the unique element covered by m and M is the unique element covering M'. Let $\mathscr{C}$ be any collection of saturated chains from m to M, and similarly define $\mathscr{C}'$. Then* $\frac{\mu_y(U_P(\mathscr{C}))}{w_{\mathscr{C}}(x)} = \frac{\mu_y(D_P(\mathscr{C}'))}{w_{\mathscr{C}'}(z)}$.

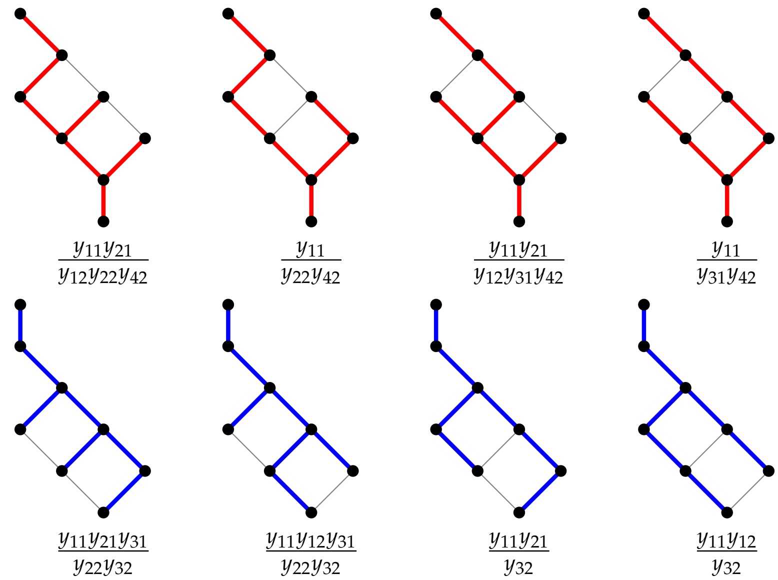

Figure 3: The four upward arborescences in $U_{RT_{3,2}}$ and the four downward arborescences in $D_{RT_{3,2}}$, together with their weights.

Let S be a *skew shape poset* as in Figure 4. Note that if $q \in S$ only covers a single element p, then any element of U_S must contain the edge $p \lessdot q$, so we call this edge *forced* for U_S. Likewise, the edge $p \lessdot q$ is *forced* for D_S if q is the only element covering p.

We define a bijection $\aleph\colon U_S \to D_S$ as follows. Translate $T \in U_S$ in the plane by the vector $(-\frac{1}{2}, -\frac{1}{2})$ (i.e., downward) to $\overline{T}$. Then form $\aleph(T)$ by taking all edges of S that do not intersect $\overline{T}$, together with all forced edges for D_S. See Figure 4 for an example.

We show that $\aleph$ affects the weight of each arborescence in a uniform way.

Lemma 3.4. *Let $T \in U_S$ and $y \in \mathbb{R}_+^S$. Then there exists a Laurent monomial $y^{\alpha(S)}$ depending only on S such that $\omega_{\aleph(T)}(y) = \omega_T(y) \cdot y^{\alpha(S)}$ for all $T \in U_S$.*

Corollary 3.5. *The bijection $\aleph\colon U_S \to D_S$ is measure-preserving: $\mu_y(U) = \mu_y(\aleph(U))$ for all $y \in \mathbb{R}_+^S$ and subsets $U \subseteq U_S$.*

Example 3.6. Consider again the arborescences for $RT_{3,2}$ in Figure 3. The bijection $\aleph$ sends each upward arborescence to the downward arborescence directly below it. In each case, $\aleph$ multiplies the weight by $y_{42} \cdot \frac{y_{12}y_{31}}{y_{32}}$, as predicted by Lemma 3.4.

We are now ready to prove a chain shifting lemma for skew shapes. Our first form is a generalization of the chain shifting lemma for rectangles proven by the current authors in [12] (and in the noncommutative setting by Grinberg and Roby [9]) to skew shapes S.

For $p \in S$, write $se(p) \neq \varnothing$ if p has a southeast neighbor and $se(p) = \varnothing$ otherwise. Given elements $p < q$ in S, let $\mathscr{C}_{p,q}$ denote the set of all saturated chains from p to q, and

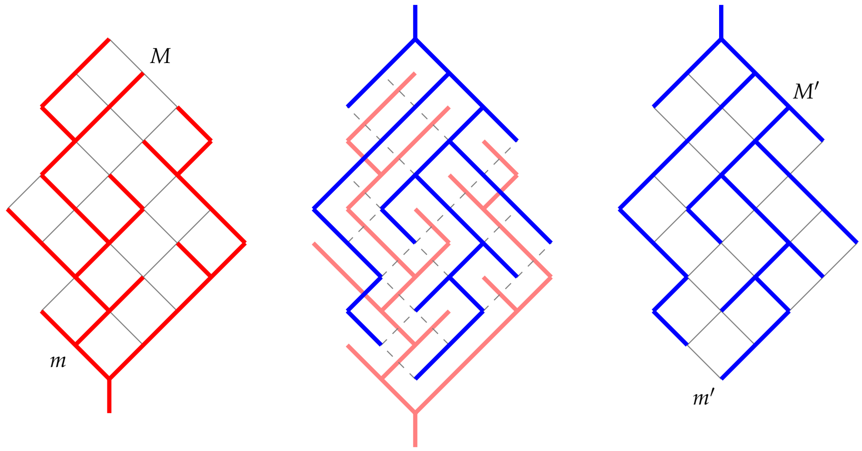

Figure 4: An upward arborescence T (in red) and its image $\aleph(T)$ (in blue).

let $\mathscr{C}^{se}_{p,q} \subseteq \mathscr{C}_{p,q}$ denote the subset consisting of those chains C for which $se(r) \neq \varnothing$ for all $r \in C$. (Also define the analogous notation for the directions sw, ne, and nw.)

Lemma 3.7. *Let $m' \lessdot m$ and $M' \lessdot M$ be elements of S such that $sw(m) = ne(M') = \varnothing$.*

(a) The bijection $\aleph$ restricts to a bijection from $U_S(\mathscr{C}^{se}_{m,M})$ to $D_S(\mathscr{C}^{nw}_{m',M'})$.

(b) Let $x \in \mathbb{R}^S_+$ and $z = \tilde{\rho}^{-1}(x)$. Then $w_{\mathscr{C}^{se}_{m,M}}(x) = w_{\mathscr{C}^{nw}_{m',M'}}(z)$.

Proof sketch. Let $T \in U_S(\mathscr{C}^{se}_{m,M})$, and let C be the chain in T from m to M. By the construction of $\aleph$, $\aleph(T) \in D_S$ contains a saturated chain upward from m' that does not cross C, so it must pass through M'. It follows that $\aleph(T) \in D_S(\mathscr{C}^{nw}_{m',M'})$. The reverse argument shows that $\aleph$ is a bijection, and (b) then follows from Corollaries 3.5 and 3.3. $\square$

Example 3.8. Let $S = RT_{32}$ and take $m = (2,1)$, $M = (3,2)$, $m' = (1,1)$, and $M' = (2,2)$. We can verify that Lemma 3.7(b) holds in this case using the labels in Figure 2:

$$z_{11}z_{21}z_{22} + z_{11}z_{12}z_{22} = \tfrac{bcdf}{b+c} + \tfrac{b(bd+be+ce)f}{b+c} = bdf + bef = x_{21}x_{31}x_{32} + x_{21}x_{22}x_{32}.$$

We can also verify Lemma 3.7(a) using Figure 3 by noting that $U_S(\mathscr{C}^{se}_{m,M})$ and $D_S(\mathscr{C}^{nw}_{m',M'})$ are the arborescences in the leftmost three columns, which are in bijection via $\aleph$.

The bijection $\aleph$ is a powerful tool for relating weights of subsets of P with respect to x and $z = \tilde{\rho}(x)$. The general strategy is simple: relate the quantities of interest to the weights of certain subsets of U_P and D_P, then show that these subsets are in bijection via $\aleph$. In this way, one can easily prove many previously established results about rowmotion on rectangles as well as further generalizations.

For another example of this, define the *left border* of the right trapezoid $RT_{r,s}$ to be the set of elements of the form $L = \{(\ell + r - 1, \ell) \mid 1 \leq \ell \leq s\}$. For $p, q \in RT_{r,s}$, let $\mathscr{C}^{L}_{p,q}$ be the subset of $\mathscr{C}_{p,q}$ consisting of chains that intersect L.

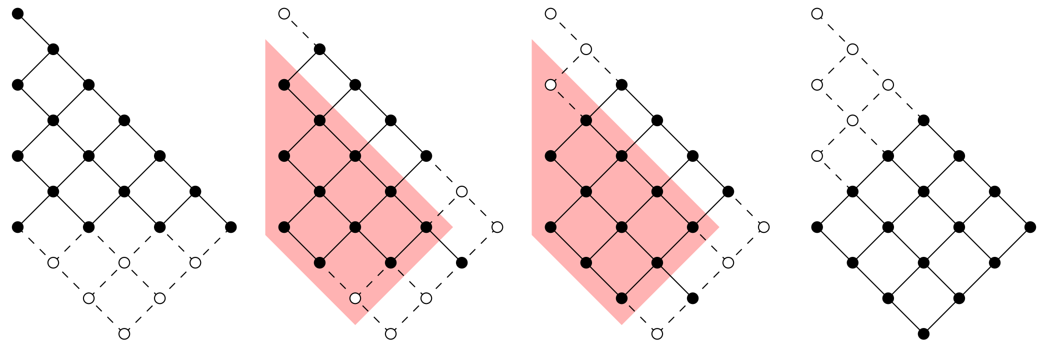

Figure 5: The four intermediate posets $I_4, \ldots, I_1$ lying inside of $RT_{4,4}$ with M_2 highlighted in red. The map ζ_2 acts on labelings of I_3 by applying $\tilde{\rho}_2^{-1}$ inside M_2 and shifting the other entries parallel to the sides.

Lemma 3.9. *Let $S = RT_{r,s}$, and let $m' \lessdot m$ and $M' \lessdot M$ such that $se(m) = ne(M') = \varnothing$.*

(a) The bijection $\aleph$ restricts to a bijection from $U_S(\mathscr{C}^L_{m,M})$ to $D_S(\mathscr{C}^L_{m',M'})$.

(b) Let $x \in \mathbb{R}^S_+$ and $z = \tilde{\rho}^{-1}(x)$. Then $w_{\mathscr{C}^L_{m,M}}(x) = w_{\mathscr{C}^L_{m',M'}}(z)$.

4 A map between the rectangle and trapezoid

In this section, we use the chain shifting lemmas (Lemmas 3.7 and 3.9) to define a birational map ζ between labelings of the rectangle $R_{r,s}$ and trapezoid $T_{r,s}$. In the tropical setting, this map will become a continuous, piecewise-linear, volume-preserving map between the chain polytopes of these two posets, which can be used to give a bijection between the plane partitions of $R_{r,s}$ and $T_{r,s}$ of height ℓ. To construct ζ, we need to construct certain intermediate posets as induced subposets of the right trapezoid $RT_{r,s}$.

Definition 4.1. Let $k \le s \le r$ be positive integers. The kth *intermediate poset* $I_k = I_{r,s,k}$ is the induced subposet of $RT_{r,s}$ on $T_{r,k} \cup [(k, k+1), (r+k-1, s)]$.

See Figure 5. Note that the leftmost minimal element of I_k is $(k, 1)$. One can easily verify that $I_1 = R_{r,s}$, $I_s = T_{r,s}$, and $|I_k| = rs$ for all k.

We now define maps $\zeta_k \colon \mathbb{R}^{I_{k+1}}_+ \to \mathbb{R}^{I_k}_+$ as follows. Consider the interval $M_k = [(k, 1), (r+k, k+1)] \subseteq RT_{r,s}$. For any $x \in \mathbb{R}^{I_{k+1}}_+$, let $\bar{x} \in \mathbb{R}^{M_k}_+$ be obtained by restricting x to $M_k \setminus \{(k,1)\} \subseteq I_{k+1}$ and setting $\bar{x}_{k,1}$ to be an arbitrary $a \in \mathbb{R}_+$ (say, 1). Let $\tilde{\rho}_k \colon \mathbb{R}^{M_k}_+ \to \mathbb{R}^{M_k}_+$ be the antichain rowmotion map on M_k. Then we define $\zeta_k(x) \in \mathbb{R}^{I_k}_+$ by

$$\zeta_k(x)_{ij} = \begin{cases} \tilde{\rho}_k^{-1}(\bar{x})_{ij} & \text{if } (i,j) \in I_k \cap M_k = M_k \setminus \{(r+k, k+1)\}, \\ x_{i+1,j} & \text{if } (i,j) \in I_k \setminus M_k \text{ and } j > k+1, \\ x_{i,j+1} & \text{if } (i,j) \in I_k \setminus M_k \text{ and } i < k. \end{cases}$$

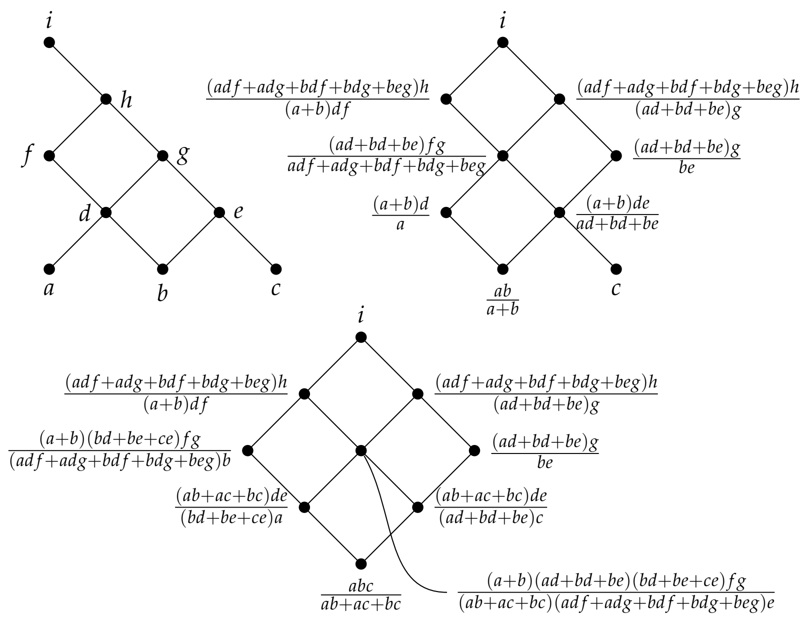

Figure 6: Applying ζ_2 and then ζ_1 to a labeling x of $T_{3,3}$, resulting in $\zeta(x)$.

See Figure 5. One can show that ζ_k does not depend on the choice of a.

We then define the birational map $\zeta = \zeta_1 \circ \zeta_2 \circ \cdots \circ \zeta_{s-1}$ from $\mathbb{R}_+^{T_{r,s}}$ to $\mathbb{R}_+^{R_{r,s}}$.

Example 4.2. Figure 6 shows the result of applying $\zeta = \zeta_1 \circ \zeta_2$ to a labeling $x \in \mathbb{R}_+^T$ when $T = T_{3,3}$. Note that $x_{13} = c = \zeta_2(x)_{12}$ as this label lies below M_2. Similarly $\zeta_2(x)_{j+1,3} = \zeta(x)_{j3}$ for $j = 1, 2, 3$ as these labels lie above M_1.

The key property of ζ that we will need to prove is that ζ preserves the total weight of all maximal chains. However, this does not hold for the maps ζ_k unless we restrict to a certain special class of polygonal chains.

Definition 4.3. A maximal chain $C \subseteq I_k \subseteq RT_{r,s}$ is *polygonal* if C intersects L (the left border of $RT_{r,s}$) or if $(k, 1) \in C$.

Note that all chains in the trapezoid and rectangle are polygonal (when $k = s$ or 1).

The following result relates the weights of polygonal chains under ζ_k. Since the only complicated part of ζ_k occurs inside M_k, it follows readily from Lemmas 3.7 and 3.9.

Proposition 4.4. *Let $\mathscr{P}_k$ be the collection of polygonal chains in I_k. For $x \in \mathbb{R}_+^{I_{k+1}}$, let $z = \zeta_k(x)$. Then $w_{\mathscr{P}_{k+1}}(x) = w_{\mathscr{P}_k}(z)$.*

It is now simple to deduce the following theorem.

Theorem 4.5. *Let $\mathscr{C}$ and $\mathscr{C}'$ be the sets of all maximal chains in $T_{r,s}$ and $R_{r,s}$, respectively. Then for all $x \in \mathbb{R}_+^{T_{r,s}}$, $w_{\mathscr{C}}(x) = w_{\mathscr{C}'}(\zeta(x))$.*

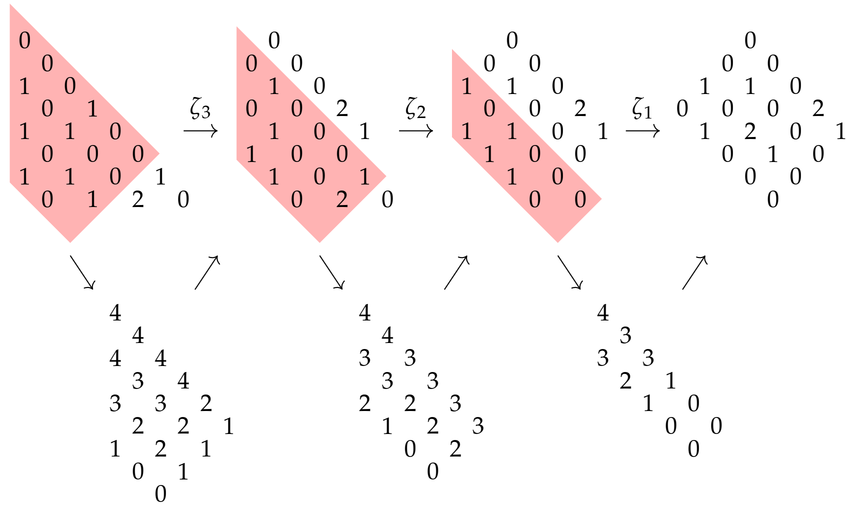

Figure 7: Example calculation of ζ for $r = 5$ and $s = 4$. The top row shows the labelings of the intermediate posets obtained from x when applying ζ_3, ζ_2, and ζ_1 with entries in M_k highlighted. The vertical maps show applications of (‡) and (†) on M_k.

4.1 Polytopes and plane partitions

We now examine the consequences of Proposition 4.4 and Theorem 4.5 in the piecewise-linear case. In this section, we will take all maps to be their piecewise-linear counterparts. In particular, the definition of ζ_k inside M_k utilizes the map $\tilde{\rho}^{-1}$ on M_k. By tropicalizing Proposition 2.4, we can compute $z = \tilde{\rho}^{-1}(x)$ via

$$z_p = -\max_{q \gtrdot p}\{y_p - y_q\} = \min_{q \gtrdot p}\{y_q\} - y_p, \quad \text{where} \tag{†}$$

$$y_p = \psi(x)_p = \max_{\hat{0} \lessdot q_1 \lessdot \cdots \lessdot q_n = p} \sum_i x_{q_i}. \tag{‡}$$

Example 4.6. An example calculation of ζ when $r = 5$ and $s = 4$ is given in Figure 7. Each ζ_k can be computed by applying (‡) and then (†) on M_k. (The minimum element of M_k is arbitrarily given the label 0, and the label of the maximum element of M_k is discarded after ζ_k is applied.) The entries outside of M_k are shifted downward appropriately.

To each intermediate poset $I_k \subseteq RT_{r,s}$, we can associate a *polygonal chain polytope*. This coincides with the chain polytope (as defined by Stanley [21]) when $k = 1$ or $k = s$.

Definition 4.7. The *polygonal chain polytope* $\widetilde{\mathcal{C}}(I_k) \subseteq \mathbb{R}^{I_k}$ is the set of all $\mathbb{R}$-labelings $x = (x_p)_{p \in I_k}$ such that $x_p \geq 0$ for all $p \in I_k$, and $\sum_{p \in C} x_p \leq 1$ for all polygonal chains $C \subseteq I_k$.

Although $\widetilde{\mathcal{C}}(I_k)$ is a lattice polytope when $k = 1$ or $k = s$ (when it is an ordinary chain polytope), this is not true in general. Nevertheless, for fixed r and s, these polytopes all

$$
\begin{array}{cccccccccccccccccc}
4 & & & & & & & & & & & & 4 & & & & \\
 & 4 & & & & & & & & & & 4 & & 4 & & & \\
4 & & 4 & & & & & & & & 4 & & 4 & & 3 & & \\
 & 3 & & 4 & & & & & & 1 & & 3 & & 3 & & 3 & \\
3 & & 3 & & 3 & & & & \xrightarrow{\psi\circ\zeta\circ\psi^{-1}} & & 1 & & 3 & & 1 & & 1 \\
 & 2 & & 2 & & 3 & & & & & & 0 & & 1 & & 0 & \\
1 & & 2 & & 2 & & 3 & & & & & & 0 & & 0 & & \\
 & 0 & & 1 & & 2 & & 0 & & & & & & 0 & & &
\end{array}
$$

Figure 8: Example of the bijection $\psi \circ \zeta \circ \psi^{-1}$ obtained by applying ψ to the labelings in Figure 7. Note that both plane partitions have the same height.

have the same volume and Ehrhart polynomial. In particular, if $\widetilde{\mathcal{C}}(I_k)$ is not a lattice polytope, then it exhibits *period collapse* of its Ehrhart quasi-polynomial.

Theorem 4.8. *The map $\zeta_k \colon \mathbb{R}^{I_{k+1}} \to \mathbb{R}^{I_k}$ defines a continuous, piecewise-linear, and lattice-preserving bijection from $\ell \cdot \widetilde{\mathcal{C}}(I_{k+1})$ to $\ell \cdot \widetilde{\mathcal{C}}(I_k)$ for all $\ell \in \mathbb{Z}_{\geq 0}$. Hence, for fixed r and s, the rational polytopes $\widetilde{\mathcal{C}}(I_k)$ share the same Ehrhart polynomial for all k.*

The following corollary then gives a bijective proof of the result of Proctor [19].

Corollary 4.9. *The continuous, piecewise-linear map $\psi \circ \zeta \circ \psi^{-1} \colon \mathbb{R}^{T_{r,s}} \to \mathbb{R}^{R_{r,s}}$ defines a bijection between plane partitions of $T_{r,s}$ and $R_{r,s}$ of height ℓ for all $\ell \in \mathbb{Z}_{\geq 0}$.*

Example 4.10. An example application of $\psi \circ \zeta \circ \psi^{-1}$ is given in Figure 8 (obtained by applying ψ to Figure 7). As required, both plane partitions have the same height.

5 Rowmotion equivariance

To prove that the map ζ defined in the previous section is equivariant with respect to the action of rowmotion $\tilde{\rho}$ (or, equivalently, that $\psi \circ \zeta \circ \psi^{-1}$ is equivariant with respect to ρ), we define a modified version of rowmotion on the intermediate posets I_k that is respected by the maps ζ_k.

As above, let $k \leq s \leq r$ and consider $I_k \subseteq RT_{r,s}$. Let $\mathscr{P}_k$ denote the set of polygonal chains in I_k, and let $\mathscr{P}_k(p)$ denote the subset of those chains that contain p.

Definition 5.1. For any $p \in I_k$, the *(birational) polygonal toggle* $\tau'_p \colon \mathbb{R}_+^{I_k} \to \mathbb{R}_+^{I_k}$ is the map that changes the p-coordinate of $x \in \mathbb{R}_+^{I_k}$ by $x_p \mapsto w_{\mathscr{P}_k(p)}(x)^{-1}$ while keeping all other coordinates fixed. The *(birational) polygonal rowmotion* map $\tilde{\varrho}_k : \mathbb{R}_+^{I_k} \to \mathbb{R}_+^{I_k}$ is the composition $\tilde{\varrho}_k = \tau'_{L^{-1}(|I_k|)} \circ \cdots \circ \tau'_{L^{-1}(1)}$ for any linear extension L of I_k.

When $k = 1$ or s (that is, on the rectangle or trapezoid), the set of polygonal chains is just the set of all maximal chains, and so $\tilde{\varrho}_k = \tilde{\rho}_k$ as shown by Joseph and Roby [15].

We then prove the following theorem.

Theorem 5.2. *The maps $\zeta_k \colon \mathbb{R}_+^{I_{k+1}} \to \mathbb{R}_+^{I_k}$ are equivariant with respect to polygonal rowmotion:*

$$\zeta_k \circ \tilde{\varrho}_{k+1} = \tilde{\varrho}_k \circ \zeta_k.$$

The proof involves categorizing the possible bottom and top parts of polygonal chains and expressing their weights using "partial transfer maps". We then use the bijection $\aleph$ to formulate and apply chain shifting results for these maps. (This proof requires the use of subtraction.) The following corollary follows immediately.

Corollary 5.3. *Let $T = T_{r,s}$ and $R = R_{r,s}$ be the rectangle and trapezoid poset. Then the map $\zeta \colon \mathbb{R}_+^T \to \mathbb{R}_+^R$ is equivariant with respect to birational (antichain) rowmotion:*

$$\zeta \circ \tilde{\rho}_T = \tilde{\rho}_R \circ \zeta.$$

In particular, birational rowmotion on the trapezoid ($\tilde{\rho}_T$ or ρ_T) has order $r + s$.

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Projective dimension of weakly chordal graphic arrangements

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Abstract. A graphic arrangement is a subarrangement of the braid arrangement whose set of hyperplanes is determined by an undirected graph. A classical result due to Stanley, Edelman and Reiner states that a graphic arrangement is free if and only if the corresponding graph is chordal, i.e., the graph has no chordless cycle with four or more vertices. In this article we extend this result by proving that the module of logarithmic derivations of a graphic arrangement has projective dimension at most one if and only if the corresponding graph is weakly chordal, i.e., the graph and its complement have no chordless cycle with five or more vertices.

Keywords: Hyperplane arrangements, graph theory, projective resolutions

1 Introduction

The principal algebraic invariant associated to a hyperplane arrangement $\mathcal{A}$ is its module of logarithmic vectors fields or derivation module $D(\mathcal{A})$. Such modules provide an interesting class of finitely generated graded modules over the coordinate ring of the ambient space of the arrangement. The chief problem is to relate the algebraic structure of $D(\mathcal{A})$ to the combinatorial structure of $\mathcal{A}$, i.e., whether it is free or more generally to determine its projective dimension or even graded Betti numbers. In general, this is notoriously difficult and still wide open, at its center is Terao's famous conjecture which states that over a fixed field of definition, the freeness of $D(\mathcal{A})$ is completely determined

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by combinatorial data. Conversely, one might ask which combinatorial properties of $\mathcal{A}$ are determined by the algebraic structure of $D(\mathcal{A})$.

It is natural to approach these very intricate questions by restricting attention to certain distinguished classes of arrangements.

A prominent and much studied class are the *graphic arrangements*, around which our present article revolves. They are defined as follows.

Definition 1.1. Let $V \cong \mathbb{Q}^\ell$ be an ℓ-dimensional $\mathbb{Q}$-vector space. Let $x_1, .., x_\ell$ be a basis for the dual space V^*. Given an undirected graph $G = (\mathcal{V}, E)$ with $\mathcal{V} = \{1, \ldots, \ell\}$, define an arrangement $\mathcal{A}(G)$ by

$$\mathcal{A}(G) := \{\ker(x_i - x_j) | \{i, j\} \in E\}.$$

Regarding the freeness of $D(\mathcal{A}(G))$, a nice complete answer is given by the following theorem, due to work by Stanley [15], and Edelman and Reiner [6].

Theorem 1.2 ([6, Thm. 3.3])**.** *The module $D(\mathcal{A}(G))$ is free if and only if the graph G is chordal, i.e., G does not contain a chordless cycle with four or more vertices.*

A recent refined result was established in [16] by Tran and Tsujie, who showed that the subclass of so-called strongly chordal graphs in the class of chordal graphs corresponds to the subclass of MAT-free arrangements, cf. [2], [4].

In this note, we will investigate the natural question raised by Kung and Schenck in [11] of whether it is possible to give a characterization of graphs G, similar to Theorem 1.2, for which the projective dimension of $D(\mathcal{A}(G))$ is bounded by a certain positive value. To this end, we consider the more general notion of *weakly chordal* graphs introduced by Hayward [9]:

Definition 1.3. A graph G is weakly chordal if G and its complement graph G^C do not contain a chordless cycle with five or more vertices.

It was subsequently discovered that many algorithmic questions that are intractable for arbitrary graphs become efficiently solvable within the class of weakly chordal graphs [10].

The main result of this paper is the following:

Theorem 1.4. *The projective dimension of $D(\mathcal{A}(G))$ is at most 1 if and only if the graph G is weakly chordal. Moreover, the projective dimension is exactly 1 if G is weakly chordal but not chordal.*

Along the way towards the preceding theorem, we will prove the following key result, yielding the more difficult implication of Theorem 1.4.

Theorem 1.5. *For $\ell \geq 6$, the projective dimension of $D(\mathcal{A}(C_\ell^C))$ is equal to 2, where C_ℓ^C is the complement of the cycle-graph with ℓ vertices, also called the (ℓ-)antihole.*

Moreover, we prove a refined result. Namely, in Theorem 5.10 we provide an explicit minimal free resolution of $D(\mathcal{A}(C_\ell^C))$.

Remark 1.6. This extended abstract corresponds to an article that is published as preprint on the arXiv ([3]).

2 Preliminaries – Graph Theory

In this section, we define objects of interest to us while studying graphic arrangements, notably specific graph classes and their attributes. The exposition is mostly based on [5]. We only consider simple, undirected graphs:

Definition 2.1. (i) A simple graph G on a set $\mathcal{V}$ is a tuple $(\mathcal{V}, E)$ with $E \subseteq \binom{\mathcal{V}}{2}$ the set of (undirected) edges connecting the vertices in $\mathcal{V}$.

(ii) The graph $G^C = \left(\mathcal{V}, \binom{\mathcal{V}}{2} \setminus E\right)$ is called the *complement graph* of G.

(iii) A graph $G' = (\mathcal{V}', E')$ with $\mathcal{V}' \subseteq \mathcal{V}, E' \subseteq E$ is called a *subgraph* of G. If E' is the set of all edges of E between vertices in $\mathcal{V}'$, i.e. $E' = \binom{\mathcal{V}'}{2} \cap E$, the graph G' is an *induced subgraph* of G.

Besides restricting the graph to a set of vertices, there are two basic operations we can perform on graphs, as described in [12]:

Definition 2.2. Let $G = (\mathcal{V}, E)$ be a graph and $e = \{i, j\} \in E$.

The graph $G' = (\mathcal{V}, E \setminus \{e\})$ is obtained from G through deletion of e and the graph $G'' = (\mathcal{V}'', E'')$ with V'' the vertex set obtained by identifying i and j and $E'' = \{\{\bar{p}, \bar{q}\} \mid \{p, q\} \in E'\}$ is obtained by contraction of G with respect to e.

We will define graph classes based on certain path or cycle properties:

Definition 2.3. 1. For $k \geq 2$, a path of length k is the graph $P_k = (\mathcal{V}, E)$ of the form

$$\mathcal{V} = \{v_0, \ldots, v_k\} \ , \ E = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$$

where all v_i are distinct.

2. If $P_k = (\mathcal{V}, E)$ is a path, and $k \geq 3$, then the graph $C_k = (\mathcal{V}, E \cup \{v_{k-1}, v_0\})$ is called a (k-)cycle.

For $k \geq 6$, we call C_k^C the *k-antihole*.

The main objects of interest in this article are graphs that satisfy a weaker condition than chordality and were introduced by Hayward in [9]:

Definition 2.4. A graph is called *weakly chordal* (or *weakly triangulated*) if it contains no induced k-cycle with $k \geq 5$ and no complement of such a cycle as an induced subgraph.

We prove the following:

Lemma 2.5. *For a weakly chordal graph $G = (\mathcal{V}, E)$, there exists a sequence of edges $e_1, .., e_k \notin E$, such that*

1. *$G_i = (\mathcal{V}, E \cup \{e_1, \ldots, e_i\})$ is weakly chordal for $i = 1, \ldots, k-1$,*
2. *the edge e_i is not part of an induced cycle C_4 in G_i for $i = 1, \ldots, k$ and*
3. *G_k is chordal.*

3 Preliminaries – Hyperplane Arrangements

In this section, we recall some fundamental notions form the theory of hyperplane arrangements. The standard reference is Orlik and Terao's book [12].

Definition 3.1. Let $\mathbb{K}$ be a field and let $V \cong \mathbb{K}^\ell$ be a $\mathbb{K}$-vector space of dimension ℓ. A hyperplane H in V is a linear subspace of dimension $\ell - 1$. A hyperplane arrangement $\mathcal{A} = (\mathcal{A}, V)$ is a finite set of hyperplanes in V.

Let V^* be the dual space of V and $S = S(V^*)$ be the symmetric algebra of V^*. Identify S with the polynomial algebra $S = \mathbb{K}[x_1, \ldots, x_\ell]$.

Definition 3.2. Let $\mathcal{A}$ be a hyperplane arrangement. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1 defined up to a constant. The product

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$$

is called a *defining polynomial* of $\mathcal{A}$.

Define the *rank* of $\mathcal{A}$ as $\mathrm{rk}(\mathcal{A}) := \mathrm{codim}_V(\cap_{H \in \mathcal{A}} H)$. If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then $(\mathcal{B}, V)$ is called a subarrangement. The *intersection lattice* $L(\mathcal{A})$ of the arrangement is the set of all non-empty intersections of elements of $\mathcal{A}$ (including V as the intersection over the empty set), with partial order by reverse inclusion. For $X \in L(\mathcal{A})$ define the *localization* at X as the subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

as well as the *restriction* $(\mathcal{A}^X, X)$ as an arrangement in X by

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \backslash \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

Define
$$L_k(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \mathrm{codim}_V(X) = k\}$$
and $L_{\geq k}(\mathcal{A}), L_{\leq k}(\mathcal{A})$ analogously.

Definition 3.3. Let $\mathcal{A}$ be a non-empty arrangement and let $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \backslash \{H_0\}$ and let $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a triple of arrangements with distinguished hyperplane H_0.

We can associate a special module to the hyperplane arrangement $\mathcal{A}$:

Definition 3.4. A $\mathbb{K}$-linear map $\theta : S \to S$ is a derivation if for $f, g \in S$:
$$\theta(f \cdot g) = f \cdot \theta(g) + g \cdot \theta(f).$$

Let $\mathrm{Der}_{\mathbb{K}}(S)$ be the S-module of derivations of S. This is a free S-module with basis the usual partial derivatives $\partial_{x_1}, \ldots, \partial_{x_\ell}$.

Define an S-submodule of $\mathrm{Der}_{\mathbb{K}}(S)$, called the module of $\mathcal{A}$-derivations, by
$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) | \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$
The arrangement $\mathcal{A}$ is called free if $D(\mathcal{A})$ is a free S-module.

The class of arrangements we are interested in are graphic arrangements:

Definition 3.5. Given a graph $G = (\mathcal{V}, E)$ with $\mathcal{V} = \{1, \ldots, \ell\}$, define an arrangement $\mathcal{A}(G)$ by
$$\mathcal{A}(G) := \{\ker(x_i - x_j) | \{i, j\} \in E\}.$$

Remark 3.6. Note that for a graphic arrangement $\mathcal{A}(G)$, localizations exactly correspond to disconnected unions of induced subgraphs of G.

For given derivations $\theta_1, \ldots, \theta_\ell \in \mathrm{Der}(S)$ we define the the *coefficient matrix*
$$M(\theta_1, \ldots, \theta_\ell) := \big(\theta_j(x_i)\big)_{1 \leq i,j \leq \ell},$$
i.e., the matrix of coefficients with respect to the standard basis $\partial_{x_1}, \ldots, \partial_{x_\ell}$ of $\mathrm{Der}(S)$. We recall Saito's useful criterion for the freeness of $D(\mathcal{A})$, cf. [12, Thm. 4.19].

Theorem 3.7. *For $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$, the following are equivalent:*

1. $\det(M(\theta_1, \ldots, \theta_\ell)) \in \mathbb{K}^\times Q(\mathcal{A})$,
2. $\theta_1, \ldots, \theta_\ell$ *is a basis of* $D(\mathcal{A})$.

3.1 Projective dimension

In this manuscript, we want to take a look at the non-free case of graphic arrangements and find a characterization for their different projective dimensions. For a comprehensive account of all the required homological and commutative algebra notions we refer to [17] respectively [7].

Definition 3.8. A *projective resolution* of a module M is a complex $P_\bullet$ with a map $\epsilon : P_0 \to M$, such that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

is exact and P_i is projective for all $i \in \mathbb{N}$.

We can define the notion of projective dimension:

Definition 3.9. Let M be an S-module. Its *projective dimension* $\mathrm{pd}(M)$ is the minimum integer n (if it exists), such that there is a resolution of M by projective S-modules

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

The projective dimension of an arrangement is the projective dimension of its derivation module and we simply write $\mathrm{pd}(\mathcal{A}) := \mathrm{pd}(D(\mathcal{A}))$. Note that since S is a polynomial ring, it follows from the Quillen-Suslin Theorem that in this case projective and free resolutions coincide. The following result is due to Terao, cf. [18, Lem. 2.1].

Proposition 3.10. *Let $X \in L(\mathcal{A})$. Then $\mathrm{pd}(\mathcal{A}_X) \leq \mathrm{pd}(\mathcal{A})$.*

An arrangement $\mathcal{A}$ is *generic*, if $|\mathcal{A}| > \mathrm{rk}(\mathcal{A})$ and for all $X \in L(\mathcal{A}) \setminus \{\cap_{H\in\mathcal{A}} H\}$ we have $|\mathcal{A}_X| = \mathrm{codim}_V(X)$. The next result, due to Rose and Terao [13], identifies generic arrangements as those with maximal projective dimension.

Theorem 3.11. *Let $\mathcal{A}$ be a generic arrangement. Then $\mathrm{pd}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}) - 2$.*

Important for our present investigations are the following examples of generic arrangements.

Example 3.12. Let C_ℓ be the cycle graph with ℓ vertices. Then, for $\ell \geq 3$, the graphic arrangement $\mathcal{A}(C_\ell)$ is generic. In particular, we have $\mathrm{pd}(\mathcal{A}(C_\ell)) = \mathrm{rk}(\mathcal{A}(C_\ell)) - 2 = \ell - 3$.

Since arrangements of induced subgraphs correspond to localizations, from Example 3.12 and Proposition 3.10 we obtain the following, first observed by Kung and Schenck [11, Cor. 2.4].

Corollary 3.13. *If G contains an induced cycle of length m, then $\mathrm{pd}(\mathcal{A}(G)) \geq m - 3$.*

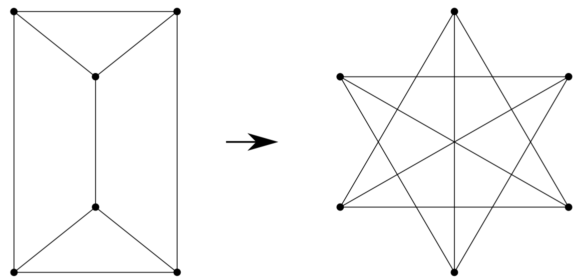

Figure 1: The triangular prism of [11] on the left is the same as ${C_6}^C$ on the right.

In [11], Kung and Schenck introduced a graph they called the "triangular prism" to serve as an example for a graphic arrangement $\mathcal{A}(G)$ whose projective dimension is strictly greater than $k-3$, k the length of the longest chordless cycle in G. Note that the graph they describe is the 6-antihole, see Figure 1. It does not have any cycle of length 5 or more, yet $\mathrm{pd}(\mathcal{A}(G)) = 2$ and it is not weakly chordal.

3.2 Terao's polynomial B

Let $\mathcal{A}$ be an arbitrary arrangement and H_0 a distinguished hyperplane. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the corresponding triple. Choose a map $\nu : \mathcal{A}'' \to \mathcal{A}'$ such that $\nu(X) \cap H_0 = X$ for all $X \in \mathcal{A}''$.

Terao defined the following polynomial

$$B(\mathcal{A}', H_0) = \frac{Q(\mathcal{A})}{\alpha_{H_0} \prod_{X \in \mathcal{A}''} \alpha_{\nu(X)}}.$$

The main properties of this polynomial can be summarized as follows:

Proposition 3.14. *[12, Lem. 4.39 and Prop. 4.41]*

1. $\deg B(\mathcal{A}', H_0) = |\mathcal{A}'| - |\mathcal{A}''|$.
2. *The ideal $(\alpha_{H_0}, B(\mathcal{A}', H_0))$ is independent of the choice of ν.*
3. *The polynomial $\theta(\alpha_{H_0})$ is contained in the ideal $(\alpha_{H_0}, B(\mathcal{A}', H_0))$ for all $\theta \in D(\mathcal{A}')$.*

In the following, we fix a hyperplane H_0 and simply write $B = B(\mathcal{A}', H_0)$ for Terao's polynomial.

By Proposition 3.14, we have an exact sequence:

$$0 \to D(\mathcal{A}) \hookrightarrow D(\mathcal{A}') \xrightarrow{\bar{\partial}'} \bar{S} \cdot \bar{B}, \tag{3.15}$$

where $\bar{S} = S/\alpha_{H_0}$, $\bar{B}$ is Terao's polynomial in $\bar{S}$ and $\overline{\partial'}(\theta) = \overline{\theta(\alpha_{H_0})}$.

The following new result regarding this sequence will be important in our subsequent proofs. It is a special case of "surjectivity theorems" for sequences of local functors recently obtained by the first author in [1].

Theorem 3.16. *Assume that* $\mathrm{pd}(\mathcal{A}_X) < \mathrm{codim}_V(X) - 2$ *for all* $X \in L_{\geq 2}(\mathcal{A}^{H_0})$. *Then the map* $\overline{\partial'}$ *in the sequence* (3.15) *is surjective. Hence, in this case, the sequence* (3.15) *is also right exact.*

Proof. This immediately follows from [1, Thm. 3.2, Thm. 3.3]. □

We record the following consequences of the preceding theorem.

Corollary 3.17. *Assume that* $\mathcal{A}_X$ *is free for all* $X \in L_2(\mathcal{A}^{H_0})$ *and* $\mathrm{pd}(\mathcal{A}) \leq 1$. *Then the sequence* (3.15) *is also right exact.*

Proof. This follows immediately from Theorem 3.16 and Proposition 3.10. □

Lemma 3.18. *Assume that* $\mathcal{A}_X$ *is free for all* $X \in L_2(\mathcal{A}^{H_0})$ *and* $\mathrm{pd}(\mathcal{A}) \leq 1$. *Then we also have* $\mathrm{pd}(\mathcal{A}') \leq 1$.

4 Weakly chordal graphic arrangements

The goal of this section is to show that a graphic arrangement of a weakly chordal graph has projective dimension at most 1, which gives one direction of our main Theorem 1.4.

Theorem 4.1. *Let* $G = (\mathcal{V}, E)$ *be a weakly chordal graph. Then* $\mathrm{pd}(\mathcal{A}(G)) \leq 1$.

Proof. Firstly, Lemma 2.5 implies that there exists a sequence of edges $e_1, \ldots, e_k$ such that $G_i = (\mathcal{V}, E \cup \{e_1, .., e_i\})$ is weakly chordal, the edge e_i is not the middle edge of any induced P_4 in G_i for $i = 1, .., k$, and G_k is chordal.

We prove that $\mathrm{pd}(\mathcal{A}(G_i)) \leq 1$ for all $i = 1, .., k$ by a descending induction. As G_k is chordal, the arrangement $\mathcal{A}(G_k)$ is free and hence $\mathrm{pd}(\mathcal{A}(G_k)) = 0$ by Theorem 1.2. So assume that $\mathrm{pd}(\mathcal{A}(G_j)) \leq 1$ for some $1 < j \leq k$. We will now argue that this implies $\mathrm{pd}(\mathcal{A}(G_{j-1})) \leq 1$ which finishes the proof.

Let H_0 be the hyperplane corresponding to the edge e_j in the arrangement $\mathcal{A}(G_j)$. We aim to apply Lemma 3.18 to $\mathcal{A}(G_j)$ and $\mathcal{A}(G_{j-1})$. To check the assumption of this result, we consider $X \in L_2(\mathcal{A}(G_j)^{H_0})$ and need to show that the arrangement $\mathcal{A}(G_j)_X$ is free.

Assume the contrary, i.e., that $\mathcal{A}(G_j)_X$ is not free. By definition of X, the arrangement $\mathcal{A}(G_j)_X$ is a graphic arrangement on an induced subgraph of G_j on four vertices containing the edge e_j. The assumption that this arrangement is not free implies that this induced subgraph is not chordal. As this subgraph only contains four vertices it must be the cycle C_4. This however contradicts condition (2) in Lemma 2.5 which states that the edge e_j cannot be an edge of an induced cycle C_4 in the graph G_j. Therefore, the arrangement $\mathcal{A}(G_j)_X$ is free for all $X \in L_2(\mathcal{A}(G_j)^{H_0})$.

Moreover, by the induction hypothesis, we have $\operatorname{pd}(\mathcal{A}(G_j)) \leq 1$. Thus, by Lemma 3.18, we also have $\operatorname{pd}(\mathcal{A}(G_{j-1})) \leq 1$ as desired. $\square$

Let us record the following result which immediately follows from the previous theorem and Theorem 1.2.

Corollary 4.2. *Let G be a weakly chordal but not chordal graph. Then* $\operatorname{pd}(\mathcal{A}(G)) = 1$.

5 Graphic arrangements of antiholes

The main result of this section yields the other direction of implications in Theorem 1.4. Recall that the graph C_ℓ^C is the complement graph of a cycle with ℓ vertices which is called the *ℓ-antihole*.

Theorem 5.1. *For all $\ell \geq 6$ it holds that*

$$\operatorname{pd}(\mathcal{A}(C_\ell^C)) = 2.$$

Let us first explain how this concludes the proof of Theorem 1.4.

Proof of Theorem 1.4, using Theorem 5.1. By Theorem 4.1, we have $\operatorname{pd}(\mathcal{A}(G)) \leq 1$ for a weakly chordal graph G and $\operatorname{pd}(\mathcal{A}(G)) = 1$ if G is not chordal by Corollary 4.2.

Conversely, assume that G is a graph such that $\operatorname{pd}(\mathcal{A}(G)) = 1$. In particular, by Theorem 1.2, the graph G is not chordal. Suppose G is also not weakly chordal. Then, by definition, there is either an $m \geq 5$ such that C_m is an induced subgraph or there is an $\ell \geq 6$ such that C_ℓ^C is an induced subgraph of G. In the first case, by Corollary 3.13, we have $\operatorname{pd}(\mathcal{A}(G)) \geq \ell - 3 \geq 2$; in the second case, by Proposition 3.10 and Theorem 5.1, we also have $\operatorname{pd}(\mathcal{A}(G)) \geq 2$. Both cases contradict our assumption. Hence, G is weakly chordal. $\square$

To prove Theorem 5.1, let us first introduce some notation for special derivations we will consider in this section. Let G be a graph with vertex set $\mathcal{V} = [\ell] := \{1, 2, \ldots, \ell\}$. Write $H_{ij} := \ker(x_i - x_j)$ for the hyperplane corresponding to the edge $\{i, j\}$ and let

$$\mathcal{A}_{\ell-1} := \{H_{ij} \mid 1 \leq i < j \leq \ell\}$$

be the graphic arrangement of the complete graph in $\mathbb{Q}^\ell$. We set

$$\theta_i := \sum_{j=1}^{\ell} x_j^i \partial_{x_j} \ (i \geq 0) \ \text{ and define } \ \varphi_i := \prod_{j \in [\ell] \setminus \{i-1, i, i+1\}} (x_i - x_j) \partial_{x_i}$$

for $i \neq 1, \ell$. Also define

$$\varphi_1 := \prod_{i=3}^{\ell-1} (x_1 - x_i) \partial_{x_1} \ \text{ and } \ \varphi_\ell := \prod_{i=2}^{\ell-2} (x_\ell - x_i) \partial_{x_\ell}.$$

In this section we always consider indices and vertices in $[\ell]$ in a cyclic way, i.e., we identify $i+\ell$ with i etc.

There is the following fundamental result due to K. Saito.

Theorem 5.2 ([14]). *$\mathcal{A}_{\ell-1}$ is free with basis $\theta_0, \dots, \theta_{\ell-1}$.*

With this, we can show the following.

Lemma 5.3. *Let*

$$\mathcal{B}_{i,j} := \mathcal{A}_{\ell-1} \setminus \{H_{s,s+1} \mid i \leq s \leq j\}.$$

Then $\mathcal{B}_{i,i+2}$ is free with basis

$$\theta_0, \dots, \theta_{\ell-3}, \varphi_{i+1}, \varphi_{i+2}.$$

With the same notation, we have:

Proposition 5.4. *If $i+2 \leq j$ and $(i,j) \neq (1,l)$, then $D(\mathcal{B}_{i,j})$ is generated by*

$$\theta_0, \dots, \theta_{\ell-3}, \varphi_{i+1}, \varphi_{i+2}, \dots, \varphi_j.$$

To obtain generators for $\mathcal{B}_{1,\ell}$ we need to modify the argument utilizing the polynomial B. For that purpose, we introduce the following new refined version of Proposition 3.14.

Theorem 5.5. *Let $\mathcal{A}$ be an arrangement, $H_1, H_2 \notin \mathcal{A}$ be distinct hyperplanes and let $\mathcal{A}_i := \mathcal{A} \cup \{H_i\}$. Assume that $H_1 = \ker(\alpha)$, $H_2 = \ker(\beta)$ and let B_i be the polynomial B with respect to $(\mathcal{A}, H_i)$. Assume that $\ker(\alpha+\beta) \in \mathcal{A}$, let b be the greatest common divisor of the reduction of B_1 and B_2 modulo (α, β) and let $b_2 b \equiv B_2$ modulo (α, β). Then for $\theta \in D(\mathcal{A})$ we have:*

$$\theta(\alpha) \in (\alpha, \beta B_1, b_2 B_1).$$

We can apply Theorem 5.5 to $\mathcal{B}_{1,\ell-1}$ and $\mathcal{B} := \mathcal{B}_{1,\ell} = \mathcal{A}_{\ell-1} \setminus \{H_{1,2}, \dots, H_{\ell-1,\ell}, H_{\ell,1}\}$. Namely, we can show the following:

Theorem 5.6. $D(\mathcal{B}) = \langle \theta_0, \dots, \theta_{\ell-3}, \varphi_1, \dots, \varphi_\ell \rangle_S$.

Note that $\psi_i := (x_{i-1} - x_i)\varphi_i - (x_{i+1} - x_{i+2})\varphi_{i+1} \in D(\mathcal{A}_{\ell-1}) = \langle \theta_0, \dots, \theta_{\ell-1} \rangle_S$ for $i = 1, \dots, \ell$, since

$$\psi_i(x_i - x_{i+1}) = -\prod_{j \in [\ell] \setminus \{i,i+1\}} (x_i - x_j) + \prod_{j \in [\ell] \setminus \{i,i+1\}} (x_{i+1} - x_j) \equiv 0 \mod (x_i - x_{i+1}).$$

Thus, there are f_{ij} such that

$$\psi_i - \sum_{j=0}^{\ell-3} f_{ij}\theta_j = -\theta_{\ell-2} \ (i = 1, 2, \dots, \ell). \tag{5.7}$$

So we have relations

$$\psi_i - \sum_{j=0}^{\ell-3} f_{ij}\theta_j = \psi_s - \sum_{j=0}^{\ell-3} f_{sj}\theta_s,$$

and they are generated by

$$\psi_1 - \sum_{j=0}^{\ell-3} f_{1j}\theta_j = \psi_i - \sum_{j=0}^{\ell-3} f_{ij}\theta_j \tag{5.8}$$

for $i = 2, \ldots, \ell$. We now prove that they indeed generate all the relations among the generators of $D(\mathcal{B})$.

Theorem 5.9. *All relations among the set of generators $\theta_0, \ldots, \theta_{\ell-3}, \varphi_1, \ldots, \varphi_\ell$ are generated by the ones given in Equations (5.8).*

Now we are ready to prove the following, which immediately implies Theorem 1.5.

Theorem 5.10. *The module $D(\mathcal{B})$ has the following minimal free resolution:*

$$0 \to S[-\ell+1] \to S[-\ell+2]^{\ell-1} \to \oplus_{i=0}^{\ell-4} S[-i] \oplus S[-\ell+3]^{\ell+1} \to D(\mathcal{B}) \to 0. \tag{5.11}$$

In particular, $\mathrm{pd}(\mathcal{B}) = 2$.

6 Remark on generalization of the result

A natural question arising from our Theorem 1.4 would be if this generalizes to the remaining projective dimensions, i.e. if $\mathcal{A}(G)$ has projective dimension $\leq k$ if and only if G and its complement graph do not contain a chordless cycle with $k+4$ or more vertices. This is however not the case, first note that in the case of projective dimension 0, it suffices for the graph itself to have no chordless cycle of length 4 or more and chordality is not closed under taking the complement (The complement of the 4-cycle for instance, is chordal, whereas the 4-cycle itself is not). Moreover, since the arrangement of the k-cycle is generic of rank $k-1$, it has maximal projective dimension $k-3$ (see Example 3.12) and by Theorem 1.5 its complement has projective dimension 2. Moreover, we found two counterexamples to the other direction of this conjecture in dimension 7; both graphs and their complements have no induced cycle of length more than 5, yet have projective dimension 3, which was also found by Hashimoto in [8].

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Realizing the s-Permutahedron via Flow Polytopes

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Abstract. In 2019, Ceballos and Pons introduced the s-weak order on s-decreasing trees, for any weak composition s. They proved its lattice structure and conjectured that it could be realized as the 1-skeleton of a polyhedral subdivision of a zonotope of dimension $n-1$. We answer their conjecture in the case where s is a (strict) composition by providing three geometric realizations of the s-permutahedron. The first one is the dual graph of a triangulation of a flow polytope of high dimension. The second, obtained using the Cayley trick, is the dual graph of a fine mixed subdivision of a sum of hypercubes that has the conjectured dimension. The third, obtained using tropical geometry, is the 1-skeleton of a polyhedral complex for which we can provide explicit coordinates of the vertices and whose support is a permutahedron as conjectured.

Keywords: s-decreasing tree, s-weak order, flow polytope, geometric realization, polyhedral subdivision, Cayley trick, tropical hypersurface.

1 Introduction

In [3, 4, 5], Ceballos and Pons introduced and studied the s-weak order, a lattice structure on s-decreasing trees parameterized by a weak composition $s = (s_1, \ldots, s_n)$. It generalizes the classical weak order on permutations of $[n] := \{1, \ldots, n\}$, that is recovered with $s = (1, \ldots, 1)$. Figure 1 shows the Hasse diagram of the $(1, 2, 1)$-weak order.

In the same way that the weak order on permutations is related to the Tamari order on Catalan objects, the s-weak order is related to the s-Tamari lattice which has received

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a lot of attention under various guises. It was first introduced by Préville–Ratelle and Viennot [14] on grid paths weakly above the path $\nu = NE^{s_n} \dots NE^{s_1}$. The Hasse diagram of the s-Tamari lattice was realized as the edge graph of a polyhedral complex by Ceballos et al. [2]. This complex is dual to a subdivision of a subpolytope of a product of simplices called $\mathcal{U}_{I,\bar{J}}$ and to a fine mixed subdivision of a generalized permutahedron. Bell et al. [1] showed that the s-Tamari lattice can also be realized as the graph dual to a triangulation of a flow polytope, by using a method of Danilov, Karzanov, and Koshevoy [6] for obtaining regular unimodular triangulations.

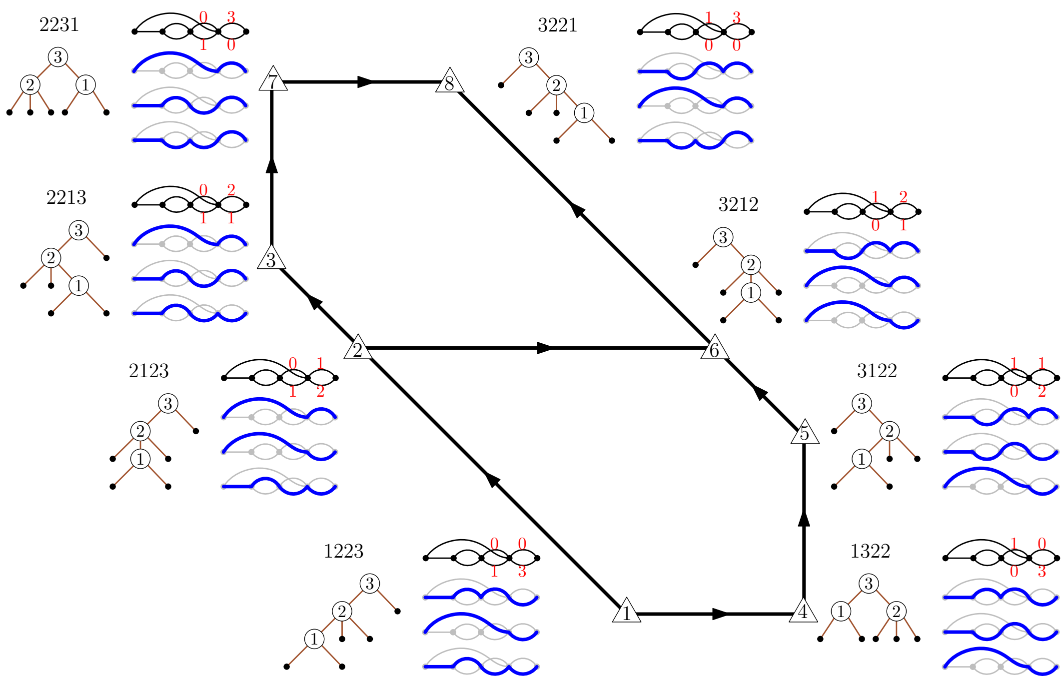


Figure 1: The s-permutahedron for $s = (1,2,1)$. The vertices are indexed by the following combinatorial objects: s-decreasing trees, Stirling s-permutations, maximal cliques of routes (omitting the all bumps or dips routes), and integer flows (in red on the topmost graph). The edges are oriented according to the s-weak order.

As notation for the rest of this article, let $s = (s_1, \dots, s_n)$ be a composition (*i.e.* a vector with positive integer entries). An *s-decreasing tree* is a planar rooted tree on n internal vertices (called nodes), labeled by $[n]$, such that the node labeled i has $s_i + 1$ children and any descendant j of i satisfies $j < i$. We denote by $T_0^i, \dots, T_{s_i}^i$ the subtrees of node i from left to right. The collection of s-decreasing trees is in bijection with 121-avoiding permutations of the word $1^{s_1}2^{s_2}\dots n^{s_n}$, called *Stirling s-permutations*. The bijection consists of reading labels along the in-order traversal of s-decreasing trees.

Let T be an s-decreasing tree and $1 \le x < y \le n$. We denote by $\mathrm{inv}(T)$ the multi-set of *tree-inversions* of T formed by pairs (y, x) with multiplicity (also called cardinality)

$$\#(y,x)_T = \begin{cases} 0, & \text{if } x \text{ is left of } y, \\ i, & \text{if } x \in T_i^y, \\ s_y, & \text{if } x \text{ is right of } y. \end{cases}$$

An *ascent* on an s-decreasing tree T is a pair (a, c) satisfying

- $a \in T_i^c$ for some $0 \leq i < s_c$,
- if $a < b < c$ and $a \in T_i^b$, then $i = s_b$, and
- if $s_a > 0$, then $T_{s_a}^a$ consists of only one leaf.

In [3] Ceballos and Pons introduced the *s-weak order* $\trianglelefteq$ on s-decreasing trees as follows. For s-decreasing trees R and T, we say that $R \trianglelefteq T$ if $\mathrm{inv}(R) \subseteq \mathrm{inv}(T)$.

If A is a subset of ascents of T, we denote by $T + A$ the s-decreasing tree whose inversion set is the smallest one that contains $\mathrm{inv}(T) \cup A$. Ceballos and Pons conjectured that the combinatorial complex whose faces are the intervals $[T, T + A]$, which they call the *s-permutahedron* $Perm_s$, has the following geometric structure.

Conjecture 1.1 ([3, Conj. 1], [5, Conj. 3.1.2]). *Let $s = (s_1, \ldots, s_n)$ be a weak composition. The s-permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to the zonotope $\sum_{1 \leq i < j \leq n} s_j \Delta_{ij}$, where $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathbb{R}^n$ and Δ_{ij} is the segment conv$\{e_i, e_j\}$.*

2 Three geometric realizations of the s-permutahedron

In the following subsections we provide background on the techniques we use and present our three realizations of the s-permutahedron, finally answering Conjecture 1.1 when s is a composition. The proofs are in the long version of this extended abstract [8].

Examples of the third realization are available on this webpage[1] and code can be found on this webpage[2]. Figure 1 shows the $(1, 2, 1)$-permutahedron together with the corresponding combinatorial objects used throughout this work.

2.1 Triangulations of flow polytopes

Let $G = (V, E)$ be a loopless connected oriented multigraph on vertices $V = \{v_0, \ldots, v_n\}$ with edges oriented from v_i to v_j if $i < j$ such that v_0 (resp. v_n) is the only source (resp. sink) of G. For any vertex v_i we denote by $\mathcal{I}_i$ its set of incoming edges and by $\mathcal{O}_i$ its set of outgoing edges.

[1]https://sites.google.com/view/danieltamayo22/gallery-of-s-permutahedra

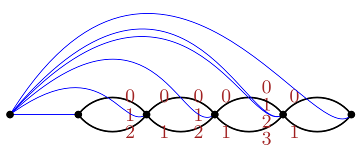
[2]https://cocalc.com/ahmorales/s-permutahedron-flows/demo-realizations

Given a vector $\mathbf{a} = (a_0, a_1 \ldots, a_{n-1}, a_n)$ such that $\sum_i a_i = 0$, a *flow* of G with *netflow* $\mathbf{a}$ is a vector $(f_e)_{e\in E} \in (\mathbb{R}_{\geq 0})^E$ such that $\sum_{e\in\mathcal{I}_i} f_e + a_i = \sum_{e\in\mathcal{O}_i} f_e$ for all $i \in [0, n]$. A flow $(f_e)_{e\in E}$ of G is called an *integer flow* if all f_e are integers. We denote by $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{a})$ the set of integer flows of G with netflow $\mathbf{a}$. A *route* of G is a path from v_0 to v_n *i.e.* a sequence of edges $((v_0, v_{k_1}), (v_{k_1}, v_{k_2}), \ldots, (v_{k_l}, v_n))$, with $0 < k_1 < k_2 < \ldots < k_l < n$. The *flow polytope* of G is

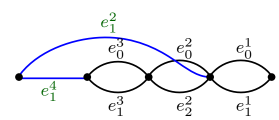
$$\mathcal{F}_G(\mathbf{a}) = \left\{ (f_e)_{e\in E} \text{ flow of } G \text{ with netflow } \mathbf{a} \right\} \subset \mathbb{R}^E.$$

It is a polytope of dimension $|E| - |V| + 1$. When it is not specified, the netflow is assumed to be $\mathbf{a} = (1, 0, \ldots, 0, -1)$. In this case, the vertices of $\mathcal{F}_G$ correspond to the routes of G.

Flow polytopes admit several nice subdivisions that can be understood via certain combinatorial properties of the graph G with respect to a framing. Let P be a route of G that contains vertices v_i and v_j. We denote by Pv_i the prefix of P that ends at v_i and v_iP the suffix of P that starts at v_i. A *framing* $\preceq$ of G is a choice of linear orders $\preceq_{\mathcal{I}_i}$ and $\preceq_{\mathcal{O}_i}$ on the sets of incoming and outgoing edges for each inner vertex v_i. This induces a total order on the set of partial routes from v_0 to v_i (resp. from v_i to v_n) by taking $Pv_i \preceq Qv_i$ if $e_P \preceq_{\mathcal{I}_j} e_Q$ where v_j is the first vertex after which the two partial routes coincide, and e_P, e_Q are the edges of P and Q that end at v_j. The definition of $v_iP \preceq v_iQ$ is similar using $\preceq_{\mathcal{O}_j}$. When G is endowed with such a framing $\preceq$, we say that G is *framed*. See Figure 2a for an example.



(a)



(b)

Figure 2: (a) The graph $\mathrm{oru}(s)$ for $s = (2, 3, 2, 2)$ with framing in red. (b) The graph $\mathrm{oru}(s)$ for $s = (1, 2, 1)$ with edge labels.

We say that routes P and Q of G are *in conflict* at a common path of inner vertices $[v_i, v_j]$ if the initial parts Pv_i and Qv_i are ordered differently than the final parts v_jP, v_jQ. Otherwise we say that P and Q are *coherent* at $[v_i, v_j]$. We say that P and Q are *coherent* if they are coherent at each common inner path.

Defining the sets of mutually coherent routes as the *cliques* of $(G, \preceq)$, we denote by $\mathit{MaxCliques}(G, \preceq)$ the set of maximal collections of cliques under inclusion. Given a set of routes C let Δ_C be the convex hull of the vertices of $\mathcal{F}_G$ corresponding to the routes in C.

Theorem 2.1 ([6, Sec. 1])**.** *The simplices $\{\Delta_C \mid C \in \mathit{MaxCliques}(G, \preceq)\}$ are the maximal cells of a regular triangulation of $\mathcal{F}_G$.*

The triangulation obtained this way is called the *DKK triangulation* of $\mathcal{F}_G$ with respect to the framing $\preceq$ and we denote it by $Triang_{DKK}(G, \preceq)$.

Another scheme to subdivide flow polytopes is a recursive procedure by Postnikov and Stanley (see [15]) based on subdividing $\mathcal{F}_G$ into two polytopes that are integrally equivalent to other flow polytopes. They used this to show that the volume of $\mathcal{F}_G$ equals the number of integer flows in $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{d})$, where $\mathbf{d} = (0, d_1, \ldots, d_{n-1}, -\sum_i d_i)$ and $d_i = \text{indeg}_G(v_i) - 1$. This recursive subdivision can be made compatible with DKK triangulations in what are called *framed Postnikov–Stanley triangulations* [13]. This allows for the following explicit bijection between the maximal cliques and the integer flows.

Theorem 2.2 ([13, Thm 7.8]). *Given a framed graph $(G, \preceq)$, the map*

$$\Omega_{G,\preceq} : \begin{cases} MaxCliques(G, \preceq) & \to \mathcal{F}_G^{\mathbb{Z}}(\mathbf{d}) \\ C & \mapsto (n_C(e) - 1)_{e \in E(G)} \end{cases},$$

where $n_C(v_i, v_j)$ is the number of times the edge (v_i, v_j) appears in the prefixes $\{Pv_j \mid P \in C\}$, is a bijection between the maximal cliques of $(G, \preceq)$ and the integer flows in $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{d})$.

We define a framed graph associated to the composition s such that the corresponding DKK triangulation encodes the combinatorial structure of the s-weak order.

Definition 2.3. Let $s = (s_1, \ldots, s_n)$ be a composition, and for convenience of notation set $s_{n+1} = 2$. The framed graph $(\text{oru}(s), \preceq)$ consists of vertices $\{v_{-1}, v_0, \ldots, v_n\}$ and

- for $i \in [n+1]$, there are $s_i - 1$ *source-edges* (v_{-1}, v_{n+1-i}) labeled $e_1^i, \ldots, e_{s_i-1}^i$,
- for $i \in [n]$, there are two edges $(v_{n+1-i-1}, v_{n+1-i})$ called *bump* and *dip* labeled e_0^i and $e_{s_i}^i$,
- the incoming edges of v_{n+1-i} are ordered $e_j^i \prec_{\mathcal{I}_{n+1-i}} e_k^i$ for $0 \le j < k \le s_i$,
- the outgoing edges of v_{n+1-i} are ordered $e_0^{i-1} \prec_{\mathcal{O}_{n+1-i}} e_{s_{i-1}}^{i-1}$.

We denote by $\text{oru}(s)$ the *s-oruga graph* and oru_n the *oruga graph* of length n which is the induced subgraph of $\text{oru}(s)$ with vertices $\{v_0, \ldots, v_n\}$. Figure 2a and Figure 2b show examples of our construction. The corresponding flow polytope $\mathcal{F}_{\text{oru}(s)}$ has dimension $|s| := \sum_{i=1}^n s_i$.

We describe the routes of $\text{oru}(s)$ intuitively as follows. Every route of $\text{oru}(s)$ starts from v_{-1}, lands in a vertex v_{n+1-k} via a source-edge labeled e_t^k and follows $k-1$ edges that are either bumps or dips denoted by a 01-vector δ. Formally, for $k \in [n+1]$, $t \in [s_k - 1]$, and $\delta = (\delta_1, \ldots, \delta_{k-1}) \in \{0,1\}^{k-1}$, we denote by $\text{R}(k, t, \delta)$ the sequence of edges $(e_{t_k}^k, e_{t_{k-1}}^{k-1}, \ldots, e_{t_1}^1)$ where $t_k = t$ and $t_j = \delta_j s_j$ for all $j \in [k-1]$.

Theorem 2.4. *The s-decreasing trees are in bijection with the maximal simplices of the DKK triangulation of $\mathcal{F}_{\text{oru}(s)}$ with respect to the framing $\preceq$.*

Proof. We describe a bijection between s-decreasing trees and integer flows of $\mathrm{oru}(s)$ with netflow $\mathbf{d} = (0, s_n, s_{n-1}, \ldots, s_1, -\sum_{i=1}^{n} s_i)$. The statement then follows from Theorems 2.2 and 2.1.

Given an integer $\mathbf{d}$-flow $(f_e)_e$ of $\mathrm{oru}(s)$ (note that it is enough to know the values $f_{e_0^i}$ for $i \in [n-1]$ to determine the entire integer flow), we build an s-decreasing tree inductively as follows. Start with the tree given by the node n and $s_n + 1$ leaves. At step i for $i \in [n-1]$, we have a partial s-decreasing tree with labeled nodes n to $n+1-i$, and $1 + \sum_{k=n+1-i}^{n} s_k$ leaves that we momentarily label from 0 to $\sum_{k=n+1-i}^{n} s_k$ along the counterclockwise traversal of the partial tree. Attach the next node $n-i$, with $s_{n-i}+1$ pending leaves, to the leaf of the partial tree labeled $f_{e_0^{n-i}}$. This procedure produces decreasing trees with the correct number of children at each node. Hence, after the n-th step we obtain an s-decreasing tree. Reciprocally, any s-decreasing tree can be built iteratively in this way, so it is associated to a choice of integers $f_{e_0^i} \in [0, \sum_{k=n+1-i}^{n} s_k]$ for all $i \in [n-1]$. □

We can now explicitly describe the DKK maximal cliques of coherent routes in terms of Stirling s-permutations.

Definition 2.5. Let s be a composition, and u a (possibly empty) prefix of a Stirling s-permutation. For all $a \in [n]$, we denote by t_a the number of occurrences of a in u, and we denote by c the smallest value in $[n]$ such that $0 < t_c < s_c$. If there is no such value, we set $c = n+1$ and $t_{n+1} = 1$. The definition of c implies that for all $a < c$, either $t_a = 0$ or $t_a = s_a$. Then we define $\mathrm{R}[u]$ to be the route $(e_{t_c}^{c}, e_{t_{c-1}}^{c-1}, \ldots, e_{t_1}^{1})$. For example, for the subword $u = 3372545$ of $w = 33725455716$ we have that $c = 5$, $t_5 = 2, t_4 = 1, t_3 = 2, t_2 = 1, t_1 = 0$ so $\mathrm{R}[u] = (e_2^5, e_1^4, e_2^3, e_1^2, e_0^1) = \mathrm{R}(5, 2, (1, 1, 1, 0))$.

Let w be a Stirling s-permutation. For $i \in [|s|]$, we denote by w_i the i-th letter of w, and for $i \in [0, |s|]$ we denote by $w_{[i]}$ the prefix of w of length i, with $w_{[0]} := \emptyset$. Let Δ_w be the set of routes $\{\mathrm{R}[w_{[i]}] \mid i \in [0, |s|]\}$ and identify it with the simplex whose vertices are the indicator vectors of these routes.

Note that each maximal clique always contains the routes $\mathrm{R}[w_{[0]}] = (e_1^{n+1}, e_0^n, \ldots, e_0^1) = \mathrm{R}(n+1, 1, (0)^n)$ and $\mathrm{R}[w_{[|s|]}] = (e_1^{n+1}, e_{s_n}^n, \ldots, e_{s_1}^1) = \mathrm{R}(n+1, 1, (1)^n)$. See Figure 3 for the example of Δ_w corresponding to the Stirling $(1,2,1)$-permutation $w = 3221$.

Lemma 2.6 ([8, Thm. 3.9])**.** *The maximal simplices of $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ are exactly the simplices Δ_w where w ranges over all Stirling s-permutations.*

The next theorem shows that the triangulation $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ encodes the combinatorics of the s-permutahedron.

Theorem 2.7 ([8, Thm. 3.18])**.** *The face poset of the s-permutahedron Perm_s is isomorphic (as a poset) to the set of interior simplices of $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ ordered by reverse inclusion.*

Figure 1 shows the graph dual to the DKK triangulation for $s = (1,2,1)$, which corresponds to the Hasse diagram of the $(1,2,1)$-weak order.

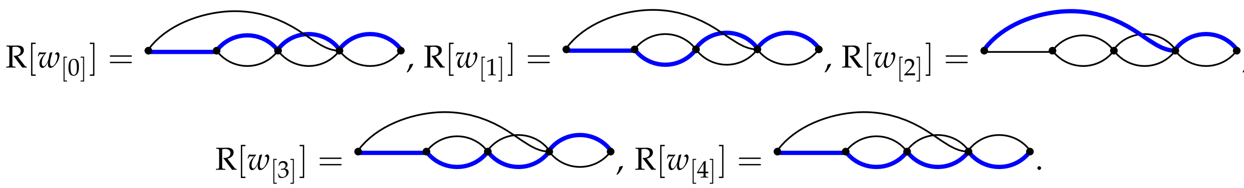


Figure 3: The maximal clique $\Delta_w = \{\mathrm{R}[w_{[0]}], \ldots, \mathrm{R}[w_{[|s|]}]\}$ corresponding to the Stirling $(1,2,1)$-permutation $w = 3221$.

2.2 Cayley trick and mixed subdivisions

The Cayley trick allows us to give another geometric realization of the s-permutahedron as the dual of a fine mixed subdivision of an $(n-1)$-dimensional polytope. This dimension coincides with the dimension of the polyhedral complex conjectured in 1.1.

For more details on the Cayley trick, see [7, Sec. 9.2] for a general introduction and [12, Sec. 7] for its application on flow polytopes. We slightly adapt the work of Mészáros–Morales for our special case of $\mathcal{F}_{\mathrm{oru}(s)}$.

Definition 2.8. For the polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^n$ their *Minkowski sum* is the polytope $P_1 + \ldots + P_k := \{\sum x_i \mid x_i \in P_i\}$. For the Minkowski sum of k copies of a polytope P we simply write kP. A *Minkowski cell* is a sum $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i. A *mixed subdivision* of a Minkowski sum is a subdivision of their convex hull such that all the cells of the subdivision are Minkowski cells (see [7, Def. 9.2.5]). A *fine mixed subdivision* is a minimal mixed subdivision via containment of its summands.

Let $e_1, \ldots, e_k$ be a basis of $\mathbb{R}^k$. We call the polytope $\mathcal{C}(P_1, \ldots, P_k) := conv(\{e_1\} \times P_1, \ldots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n$ the *Cayley embedding* of $P_1, \ldots, P_k$.

Proposition 2.9 (The Cayley trick [9])**.** *Let $P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^n$. The polytopal subdivisions (resp. triangulations) of $\mathcal{C}(P_1, \ldots, P_k)$ are in bijection with the mixed subdivisions (resp. fine mixed subdivisions) of $P_1 + \ldots + P_k$.*

To apply the Cayley trick to our triangulation $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ of the flow polytope $\mathcal{F}_{\mathrm{oru}(s)}$, we need to describe it as the Cayley embedding of some lower-dimensional polytopes. Recall that $\mathcal{F}_{\mathrm{oru}(s)}$ lives in the space of edges of the graph $\mathrm{oru}(s)$. We parameterize this space as $\mathbb{R}^p \times \mathbb{R}^{2n}$, where $p = 1 + \sum_{i=1}^{n}(s_i - 1)$ and $\mathbb{R}^p$ corresponds to the space of source-edges and $\mathbb{R}^{2n}$ to the space of bumps and dips (edges of oru_n, see Definition 2.3). Moreover, for all $i \in [n]$ and for any point in $\mathcal{F}_{\mathrm{oru}(s)}$, (*i.e.* a flow of $\mathrm{oru}(s)$), we have that the sum of its coordinates along edges e_0^i and $e_{s_i}^i$ is determined by the coordinates along the source-edges e_t^k for $k \in [i+1, n+1]$, $t \in [s_k - 1]$. Thus, $\mathcal{F}_{\mathrm{oru}(s)}$ is affinely equivalent to its projection on the space $\mathbb{R}^p \times \mathbb{R}^n$ where $\mathbb{R}^n$ corresponds to the space of edges e_0^i for $i \in [n]$.

With this parametrization, the indicator vector of the route of $\mathrm{oru}(s)$ denoted $\mathrm{R}(k, t, \delta)$ (as in the discussion after Def. 2.3) with $k \in [n+1]$, $t \in [s_k - 1]$ and $\delta \in \{0,1\}^{k-1}$ is

$$e_t^k \times \sum_{i \in [k-1],\, \delta_i = 0} e_0^i.$$

Thus, denoting by $\square_{k-1}$ these $(k-1)$-dimensional hypercubes with the set of vertices $\{0,1\}^{k-1} \times 0^{n-k+1}$ embedded in $\mathbb{R}^n$, we see that $\mathcal{F}_{\mathrm{oru}(s)}$ is the Cayley embedding of $\square_n$ and $\square_{k-1}$ repeated $s_k - 1$ times for $k \in [n]$. We denote by *Subdiv*$_\square(s)$ the fine mixed subdivision of the Minkowski sum of hypercubes $\square_n + \sum_{i=1}^n (s_i - 1)\square_{i-1} \subseteq \mathbb{R}^n$ obtained by intersecting the triangulation $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ with the subspace $\left\{\frac{1}{p}\right\}^p \times \mathbb{R}^n$.

The following theorem follows directly from the Cayley trick (Proposition 2.9), and the isomorphism between the face poset of Perm_s and the interior simplices of the DKK triangulation given in Theorem 2.7.

Theorem 2.10 ([8, Thm. 4.3])**.** *The face poset of the s-permutahedron Perm_s is isomorphic to the set of interior cells of $\mathrm{Subdiv}_\square(s)$ ordered by reverse inclusion. In particular, the s-decreasing trees are in bijection with the maximal cells of $\mathrm{Subdiv}_\square(s)$.*

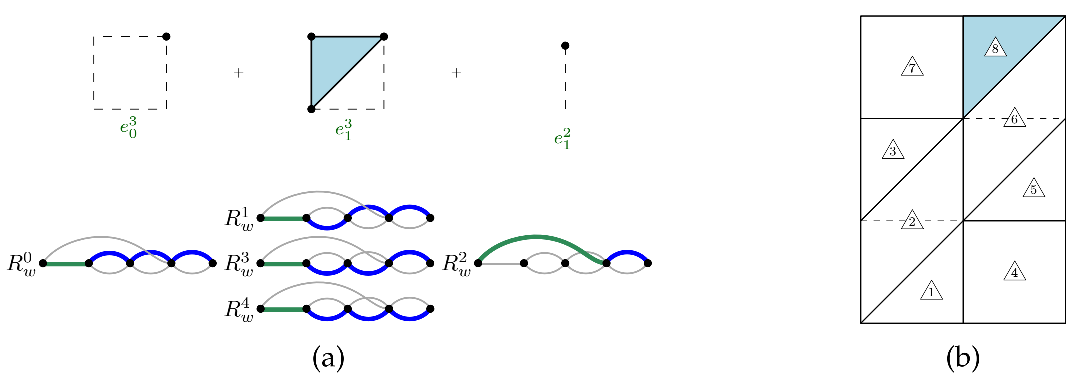


Figure 4: (a) Summands of the Minkowski cell corresponding to $w = 3221$ together with their corresponding routes in Δ_w. (b) Mixed subdivision of $2\square_2 + \square_1$ corresponding dually to the $(1,2,1)$-permutahedron. The cells are numbered according to Figure 1. The highlighted cell in blue corresponds to $w = 3221$ as obtained in Figure 4a.

Remark 2.11. We can use a different parameterization of the space where $\mathcal{F}_{\mathrm{oru}(s)}$ lives by considering the cube $\square_n$ as the Cayley embedding of two hypercubes $\square_{n-1}$, or equivalently intersect $\mathbb{R}^n$ with the hyperplane $x_n = \frac{1}{2}$. This allows us to lower the dimension and obtain a fine mixed subdivision of the Minkowski sum of hypercubes $(s_n+1)\square_{n-1} + \sum_{i=1}^{n-1}(s_i-1)\square_{i-1}$. We use this representation in our figures.

Figure 4a shows the mixed cell corresponding to the Stirling $(1,2,1)$-permutation $w = 3221$, obtained from the clique Δ_w with the Cayley trick. Figure 4b shows the entire mixed subdivision for the case $s = (1,2,1)$. Both figures are represented in the coordinate system (e_0^2, e_0^1).

2.3 Intersection of tropical hypersurfaces

In this section, we explain how to dualize our previous realizations in order to obtain our desired polytopal realization and fully answer the conjecture for strict compositions. Tropical geometry offers a convenient setting to dualize regular polyhedral subdivisions that interacts nicely with the Cayley trick.

This section is based on the work of Joswig in [10] and [11, Chap. 1]. Let $\mathcal{A} = \{\mathbf{a}^1, \dots, \mathbf{a}^m\}$ be a point configuration in $\mathbb{R}^d$ with integer coordinates, and $\mathcal{S}$ a subdivision of $\mathcal{A}$. The subdivision $\mathcal{S}$ is said to be *regular* if there is a function $\mathrm{h} : [m] \to \mathbb{R}, i \mapsto \mathrm{h}^i$ such that the faces of $\mathcal{S}$ are the images of the lower faces of the lift of $\mathcal{A}$ (the polytope with vertices $(\mathbf{a}^i, \mathrm{h}^i) \in \mathbb{R}^{d+1}$ for $i \in [m]$) by the projection that omits the last coordinate. In this case, the function h is called an *admissible height function* for $\mathcal{S}$.

Such a point configuration together with a height function h is associated to the *tropical polynomial* $F(\mathbf{x}) = \bigoplus_{i\in[m]} h^i \odot \mathbf{x}^{\mathbf{a}^i} = \min\left\{h^i + \langle \mathbf{a}^i, \mathbf{x}\rangle \mid i \in [m]\right\}$ in the min-plus algebra where $\mathbf{x} \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^d$. The *tropical hypersurface defined by F* is $\mathcal{T}(F) := \left\{\mathbf{x} \in \mathbb{R}^d \mid \text{the minimum of } F(\mathbf{x}) \text{ is attained at least twice}\right\}$ (see examples on Figure 5). This tropical hypersurface is the image of the codimension-2-skeleton of the *dome* $\mathcal{D}(F) = \left\{(\mathbf{x}, y) \in \mathbb{R}^{d+1} \mid \mathbf{x} \in \mathbb{R}^d, y \in \mathbb{R}, y \leq F(\mathbf{x})\right\}$ under the orthogonal projection that omits the last coordinate. The *cells* of $\mathcal{T}(F)$ are the projections of the faces of $\mathcal{D}(F)$ (here we include the regions of $\mathbb{R}^d$ delimited by $\mathcal{T}(F)$ as its d-dimensional cells). We say that $\mathcal{T}(F)$ is the *tropical dual* of the subdivision $\mathcal{S}$ with admissible function h since we have the following theorem.

Theorem 2.12 ([11, Thm. 1.13])**.** *There is a bijection between the k-dimensional cells of $\mathcal{S}$ and the $(d-k)$-dimensional cells of $\mathcal{T}(F)$ that reverses the inclusion order.*

We showed in [8, Lem. 5.2] that this bijection restricts to a bijection between the interior cells of $\mathcal{S}$ and the bounded cells of $\mathcal{T}(F)$.

In the case where $\mathcal{A}$ is a Cayley embedding, Joswig explains in [11, Cor. 4.9] how the Cayley trick allows us to describe the tropical dual of a regular mixed subdivision with an arrangement of tropical hypersurfaces. We consider $\mathcal{A}$ given by the vertices of the Cayley embedding $\mathcal{C}(P_1, \dots, P_k)$, with $P_j = \operatorname{conv}(\mathbf{a}^{j,1}, \dots, \mathbf{a}^{j,m_j})$ being a polytope in $\mathbb{R}^d$ with integer coordinate vertices, and consider a regular subdivision $\mathcal{S}$ given by the height $\mathrm{h} = (h^{1,1}, \ldots, h^{1,m_1}, \ldots, h^{k,m_k}) \in \mathbb{R}^{[m_1]\times\ldots\times[m_k]}$. After the Cayley trick we obtain the subdivision $\widetilde{\mathcal{S}}$ of the point configuration $\widetilde{\mathcal{A}}$ given by the points $\sum_{j=1}^{k} \mathbf{a}^{j,i_j}$ for $(i_1, \ldots, i_k) \in [m_1] \times \ldots \times [m_k]$ with height $h^{(i_1,\ldots,i_k)} = \sum_{j=1}^{k} h^{j,i_j}$.

Theorem 2.13 ([11, Cor. 4.9])**.** *The tropical dual of the mixed subdivision $\widetilde{\mathcal{S}}$ obtained after applying the Cayley trick to $\mathcal{S}$ is the polyhedral complex of cells induced by the arrangement of tropical hypersurfaces $\{\mathcal{T}(F_j) \mid j \in [m]\}$ where F_j is the tropical polynomial $F_j(\mathbf{x}) = \bigoplus_{i_j\in[m_j]} h^{j,i_j} \odot \mathbf{x}^{\mathbf{a}^{j,i_j}}$.*

For example, the arrangement on Figure 5 is dual to the mixed subdivision depicted on Figure 4b.

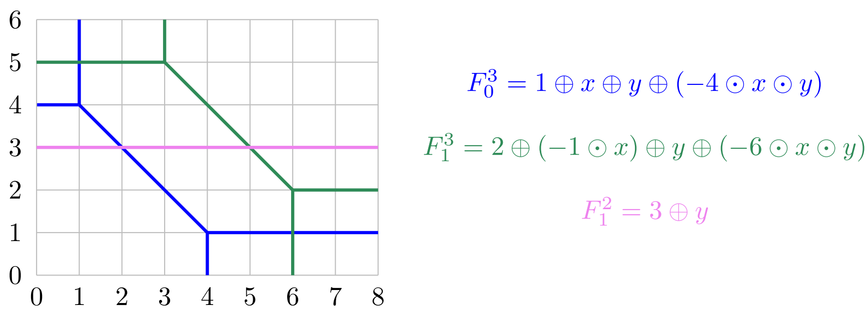


Figure 5: Arrangement of three tropical hypersurfaces, associated to the tropical polynomials on the right. The bounded cells of this arrangement give a realization of the $(1, 2, 1)$-permutahedron.

Danilov et al. provided explicit constructions of admissible height functions for the DKK triangulation ([6, Lem. 2 & 3]) that we can adapt to $\mathrm{oru}(s)$. We refined their results in [8, Lem. 5.5] to prove that the following height function is admissible.

Lemma 2.14 ([8, Lem. 5.6 and Prop. 5.7])**.** *Let s be a composition and $0 < \varepsilon < \frac{1}{n(1+\sum_{j=2}^{n}(2s_j+1))}$. Consider h_ε to be the function that associates to a route $\mathrm{R} := \mathrm{R}(k, t_k, \delta)$ of $\mathrm{oru}(s)$ the quantity $\mathrm{h}_\varepsilon(\mathrm{R}) = -\sum_{k\geq c>a\geq 1} \varepsilon^{c-a}(t_c + \delta_a)^2$, where $t_c = 0$ if $\delta_c = 0$ or $t_c = s_c$ if $\delta_c = 1$, for all $c \in [k-1]$. Then h_ε is an admissible height function for $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$.*

Since we defined in Subsection 2.2 the mixed subdivision $\mathrm{Subdiv}_\square(s)$ from the regular triangulation $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ via the Cayley trick, the following theorem directly follows from Theorem 2.13.

Theorem 2.15 ([8, Thm. 5.8])**.** *The tropical dual of $\mathrm{Subdiv}_\square(s)$ is the polyhedral complex induced by the arrangement of hypersurfaces $\mathcal{H}_s(\mathrm{h}) := \{\mathcal{T}(F_t^k) \mid k \in [2, n+1],\ t \in [s_k - 1]\}$, where h is an admissible height function for $\mathrm{Triang}_{DKK}(\mathrm{oru}(s), \preceq)$ and*

$$F_t^k(\mathbf{x}) = \bigoplus_{\delta\in\{0,1\}^{k-1}} \mathrm{h}(\mathrm{R}(k,t,\delta)) \odot \mathbf{x}^\delta = \min\left\{ \mathrm{h}(\mathrm{R}(k,t,\delta)) + \sum_{i\in[k-1]} \delta_i x_i \mid \delta \in \{0,1\}^{k-1} \right\}.$$

Definition 2.16. We denote by *$Perm_s(\mathrm{h})$* the polyhedral complex of bounded cells induced by the arrangement $\mathcal{H}_s(\mathrm{h})$.

Theorem 2.17 ([8, Thm. 5.10])**.** *The face poset of the geometric polyhedral complex $Perm_s(\mathrm{h})$ is isomorphic to the face poset of the combinatorial s-permutahedron $Perm_s$.*

Figure 6 shows some examples of such realizations of the s-permutahedron.

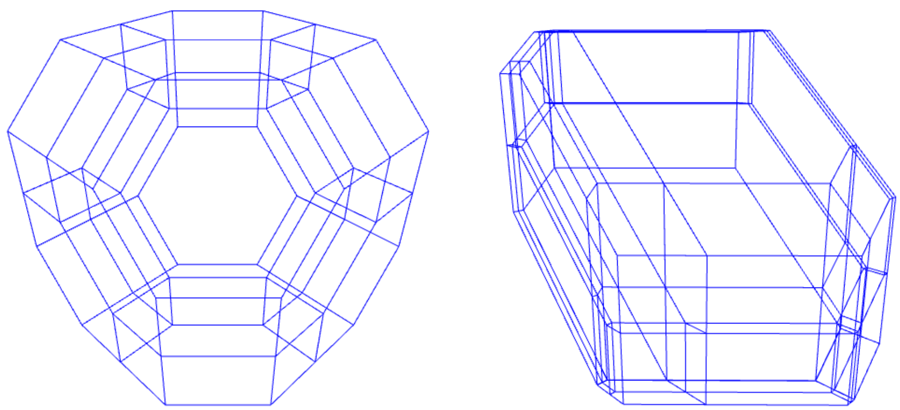


Figure 6: The $(1, 1, 1, 2)$-permutahedron (left) and the $(1, 2, 2, 2)$-permutahedron (right) via their tropical realization.

Moreover, we can describe the explicit coordinates of the vertices of $\mathrm{Perm}_s(\mathrm{h})$. For a Stirling s-permutation w, $a \in [n]$ and $t \in [s_a]$, we denote $i(a^t)$ the length of the prefix of w that precedes the t-th occurrence of a. As explained in the argument leading to Lemma 2.6, this prefix is associated to the route $\mathrm{R}[w_{[i(a^t)]}]$ in the clique Δ_w.

Theorem 2.18 ([8, Thm. 5.11])**.** *The vertex $\mathbf{v}(w) = (\mathbf{v}(w)_a)_{a\in[n]}$ of $Perm_s(\mathrm{h})$ associated to a Stirling s-permutation w has coordinates $\mathbf{v}(w)_a = \sum_{t=1}^{s_a} \Big(\mathrm{h}(\mathrm{R}[w_{[i(a^t)]}]) - \mathrm{h}(\mathrm{R}[w_{[i(a^t)+1]}])\Big)$.*

With these explicit coordinates, we obtain the directions of the edges of $\mathrm{Perm}_s(\mathrm{h})$ and show that its support, *i.e.* the union of faces of $\mathrm{Perm}_s(\mathrm{h})$, is a polytope combinatorially isomorphic to the $(n-1)$-dimensional permutahedron. This completely answers Conjecture 1.1 in the case where s is a composition, as then the zonotope $\sum_{1\le i<j\le n} s_j[\mathbf{e}_i, \mathbf{e}_j]$ is combinatorially isomorphic to the $(n-1)$-dimensional permutahedron.

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Monomial expansions for q-Whittaker and modified Hall-Littlewood polynomials

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Abstract. We consider the monomial expansion of the q-Whittaker polynomials given by the fermionic formula and via the *inv* and *quinv* statistics. We construct bijections between the parametrizing sets of these three models which preserve the x- and q-weights, and which are compatible with natural projection and branching maps. We apply this to the limit construction of local Weyl modules and obtain a new character formula for the basic representation of $\widehat{\mathfrak{sl}_n}$. Finally, we indicate how our main results generalize to the modified Hall-Littlewood case.

Keywords: q-Whittaker polynomial, modified Hall-Littlewood polynomial, local Weyl modules

1 Introduction

Let λ be a partition. For $n \geq 1$, let X_n denote the tuple of indeterminates $x_1, x_2, \cdots, x_n$. The q-Whittaker polynomial $W_\lambda(X_n; q)$ and the modified Hall-Littlewood polynomial $Q'_\lambda(X_n; q)$ are well-studied specializations of the modified Macdonald polynomial. Several different monomial expansions for these polynomials are known. In this article, our focus will be on three of these: the so-called *fermionic formulas* [13, (0.2), (0.3)] and the inv- and quinv-expansions arising from specializations of the formulas of Haglund-Haiman-Loehr [9] and Ayyer-Mandelshtam-Martin [1].

We recall that the Schur expansion of the $W_\lambda(X_n; q)$ (resp. $Q'_\lambda(X_n; q)$) has certain q-Kostka polynomials as coefficients [13]. In turn, this implies yet another monomial expansion, with the underlying indexing set involving pairs of semistandard Young tableaux of conjugate (resp. equal) shapes. This relates to the inv-expansion via the RSK correspondence [9].

The fermionic formula, expressed as a sum of products of q-binomials, is seemingly of a very different nature from all the other monomial expansions, and should probably viewed as a kind of compression of these formulas. Recently, Garbali-Wheeler [8]

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obtained a general formula of the fermionic kind for the full modified Macdonald polynomial $\widetilde{H}_\lambda(X_n; q, t)$.

The purpose of this article is to bijectively reconcile the fermionic formula with both the inv- and quinv-expansions. We construct bijections between the underlying sets of these three models which (i) preserve the x- and q-weights, and (ii) are compatible with natural projection and branching maps.

As a corollary, we obtain bijections between the inv- and quinv-models in the q-Whittaker and modified Hall-Littlewood specializations, partially answering a question of [1]. We find that the *inv*- and *quinv*-models are related by the simple *box-complementation* map of the fermionic model and that $inv + quinv$ is a constant on fibers of the natural projection. We also apply this to the limit construction for Weyl modules [7, 15] and obtain an apparently new character formula for the basic representation of the affine Lie algeba $\widehat{\mathfrak{sl}_n}$.

In this extended abstract, we describe the q-Whittaker polynomials in greater detail, contenting ourselves with brief remarks about the modified Hall-Littlewood case in §8 due to space limitations. Complete proofs will appear in [3].

2 Specializations of $\widetilde{H}_\lambda(X_n; q, t)$

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, we will draw its Young diagram $\mathrm{dg}(\lambda)$ following the English convention, as a left-up justified array of boxes, with λ_i boxes in the ith row from the top. The boxes are called the cells of $\mathrm{dg}(\lambda)$. We let $|\lambda| := \sum_i \lambda_i$. Fix $n \geq 1$ and let $\mathcal{F}(\lambda)$ denote the set of all maps ("fillings") $F : \mathrm{dg}(\lambda) \to [n]$ where $[n] = \{1, 2, \cdots, n\}$. If the values of F strictly increase (resp. weakly decrease) as we move down a column, we say F is a *column strict filling* (CSF) (resp. *weakly decreasing filling* (WDF)[1]), and denote the set of such fillings by $\mathrm{CSF}(\lambda)$ (resp. $\mathrm{WDF}(\lambda)$). The x-weight of a filling F is the monomial $x^F := \prod_{c \in \mathrm{dg}(\lambda)} x_{F(c)}$.

We recall that the modified Macdonald polynomial $\widetilde{H}_\lambda(X_n; q, t)$ is a symmetric polynomial in the x_i with $\mathbb{N}[q, t]$ coefficients. We expand this in powers of t; our interest lies in the coefficients of the lowest and highest powers [2, (3.1)]:

$$\widetilde{H}_\lambda(X_n; q, t) = \mathcal{H}_\lambda(X_n; q) t^0 + \cdots + W_\lambda(X_n; q) t^{\eta(\lambda)} \tag{2.1}$$

where $\eta(\lambda) = \sum_{j \geq 1} \binom{\lambda'_j}{2}$ where λ'_j denote the parts of the partition conjugate to λ. The $W_\lambda(X_n; q)$ is the q-Whittaker polynomial. The q-reversal (or reciprocal) polynomial of $\mathcal{H}_\lambda(X_n; q)$ coincides with the modified Hall-Littlewood polynomial $Q'_{\lambda'}(X_n; q)$ where λ'

[1]These latter ones may be easily transformed into the familiar *tabloids* by transposing rows and columns and replacing $i \mapsto n - i + 1$

is the partition conjugate to λ, i.e., $q^{\eta(\lambda')}\mathcal{H}_\lambda(X_n;q^{-1}) = Q'_{\lambda'}(X_n;q)$. These are further related to each other by $\omega W_\lambda(X_n;q) = Q'_{\lambda'}(X_n;q)$ where ω is the classical involution on the ring of symmetric polynomials.

Following Haglund-Haiman-Loehr [9] and Ayyer-Mandelshtam-Martin [1], there are statistics *inv, quinv* and *maj* on $\mathcal{F}(\lambda)$ such that

$$\widetilde{H}_\lambda(X_n;q,t) = \sum_{F\in\mathcal{F}(\lambda)} x^F q^{v(F)} t^{\mathrm{maj}(F)} \tag{2.2}$$

where $v \in \{\mathrm{inv}, \mathrm{quinv}\}$. The next lemma follows directly from the definition of maj [9]:

Lemma 1. *Let $F \in \mathcal{F}(\lambda)$. Then (i) $\mathrm{maj}(F) = \eta(\lambda)$ iff $F \in \mathrm{CSF}(\lambda)$, and (ii) $\mathrm{maj}(F) = 0$ iff $F \in \mathrm{WDF}(\lambda)$.*

Putting together (2.1), (2.2) and Lemma 1, we obtain for $v \in \{\mathrm{inv}, \mathrm{quinv}\}$:

$$W_\lambda(X_n;q) = \sum_{F\in\mathrm{CSF}(\lambda)} x^F q^{v(F)} \tag{2.3}$$

$$Q'_{\lambda'}(X_n;q) = \sum_{F\in\mathrm{WDF}(\lambda)} x^F q^{\eta(\lambda')-v(F)} \tag{2.4}$$

These are in fact symmetric in the x-variables and can be viewed as expansions in terms of the monomial symmetric functions in $x_1, x_2, \cdots, x_n$.

3 Fermionic formula for $W_\lambda(X_n;q)$

Let $n \geq 1$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n nonzero parts. Let $\mathrm{GT}(\lambda)$ denote the set of integral Gelfand-Tsetlin (GT) patterns with bounding row λ. Given $T \in \mathrm{GT}(\lambda)$, we denote its entries by T_i^j for $1 \leq i \leq j \leq n$ as in Figure 1. It will also be convenient to define $T_{j+1}^j = 0$ for all $1 \leq j \leq n$. We define the North-East and South-East differences of T by: $\mathrm{NE}_{ij}(T) = T_i^{j+1} - T_i^j$ and $\mathrm{SE}_{ij}(T) = T_i^j - T_{i+1}^{j+1}$ for $1 \leq i \leq (j+1) \leq n$. The GT inequalities ensure that these differences are non-negative.

We will interchangeably think of a GT pattern as a semistandard Young tableau (SSYT). In this perspective, $(T_1^j, T_2^j, \cdots, T_j^j)$ is the partition formed by the cells of the tableau which contain entries $\leq j$. It follows that $\mathrm{NE}_{ij}(T)$ is the number of cells in the i^{th} row of the tableau which contain the entry $j+1$. We let x^T denote the x-weight of the corresponding tableau. The following fermionic formula for the q-Whittaker polynomial appears in [10, 13] and follows readily from Macdonald's more general formula [14, Chap VI, (7.13)']:

$$W_\lambda(X_n;q) = \sum_{T\in\mathrm{GT}(\lambda)} x^T \prod_{1\leq i\leq j<n} \begin{bmatrix} NE_{ij}(T) + SE_{ij}(T) \\ NE_{ij}(T) \end{bmatrix}_q \tag{3.1}$$

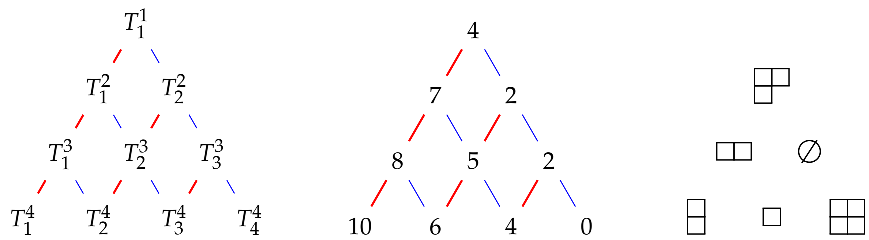


Figure 1: A GT pattern for $n = 4$. The NE and SE differences are those along the red and blue lines. On the right is a partition overlay compatible with this GT pattern.

Following [12], we define $\mathrm{wt}_q(T) = \prod_{1\le i\le j<n} \begin{bmatrix} NE_{ij}(T) + SE_{ij}(T) \\ NE_{ij}(T) \end{bmatrix}_q$.

3.1 Partition overlaid patterns

We recall that the q-binomial $\left[\begin{smallmatrix} k+\ell \\ k \end{smallmatrix}\right]_q$ is the generating function of partitions that fit into a $k \times \ell$ rectangle, i.e., $\left[\begin{smallmatrix} k+\ell \\ k \end{smallmatrix}\right]_q = \sum q^{|\gamma|}$ where $\gamma = (\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_k \ge 0)$ with $\ell \ge \gamma_1$. We also identify partitions of the above form with strictly decreasing k-tuples of integers between 0 and $k + \ell - 1$ via the bijection $\gamma \mapsto \overline{\gamma} = \gamma + \delta$ where $\delta = (k-1, k-2, \cdots, 0)$.

As shown in [15], the right-hand side of (3.1) can be interpreted in terms of the so-called *partition overlaid patterns* (POPs). A POP of shape λ is a pair (T, Λ) where $T \in \mathrm{GT}(\lambda)$ and $\Lambda = (\Lambda_{ij} : 1 \le i \le j < n)$ is a tuple of partitions such that each Λ_{ij} fits into a rectangle of size $NE_{ij}(T) \times SE_{ij}(T)$. For example, if T is the GT pattern of Figure 1, we could take $\Lambda_{11} = (2,1,0)$, $\Lambda_{12} = (2)$, $\Lambda_{13} = (1,1)$, $\Lambda_{22} = (0,0,0)$, $\Lambda_{23} = (1)$, $\Lambda_{33} = (2,2)$. We imagine the Λ_{ij} as being placed in a triangular array as in Figure 1. We let $\mathrm{POP}(\lambda)$ denote the set of POPs of shape λ. It is now clear from (3.1) that

$$W_\lambda(X_n; q) = \sum_{(T,\Lambda)\in \mathrm{POP}(\lambda)} x^T q^{|\Lambda|} \tag{3.2}$$

where $|\Lambda| = \sum_{i,j} |\Lambda_{ij}|$. We remark that $W_\lambda(X_n; q)$ is the character of the *local Weyl module* $W_{\mathrm{loc}}(\lambda)$ - a module for the current algebra $\mathfrak{sl}_n[t]$ [6, 5]. Further, POPs of shape λ index a special basis of this module with Gelfand-Tsetlin like properties [6, 15].

3.2 Projection and Branching for Partition overlaid patterns

Given $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$, we say that $\mu = (\mu_1, \mu_2, \cdots, \mu_{n-1})$ *interlaces* λ (and write $\mu \prec \lambda$) if $\lambda_i \ge \mu_i \ge \lambda_{i+1}$ for $1 \le i < n$. The q-Whittaker polynomials have the following important properties which readily follow from (3.2):

(*projection*) $W_\lambda(X_n; q = 0) = s_\lambda(X_n)$, the Schur polynomial, and

$$(\textit{branching})\ W_\lambda(x_1, x_2, \cdots, x_{n-1}, x_n = 1; q) = \sum_{\mu \prec \lambda} \prod_{1\le i<n} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix}_q \cdot W_\mu(X_{n-1}; q) \tag{3.3}$$

In fact, Chari-Loktev [6] lift (3.3) to the level of modules, showing that the local Weyl module $W_{\mathrm{loc}}(\lambda)$ when restricted to $\mathfrak{sl}_{n-1}[t]$ admits a filtration whose successive quotients are of the form $W_{\mathrm{loc}}(\mu)$ for $\mu \prec \lambda$; further their graded multiplicities are precisely given by the product of q-binomial coefficients that appear in (3.3).

The combinatorial shadow of projection is the map $\mathrm{pr} : \mathrm{POP}(\lambda) \to \mathrm{GT}(\lambda)$ given by $\mathrm{pr}(T, \Lambda) = T$. Likewise, we define *combinatorial branching* to be the map $\mathrm{br} : \mathrm{POP}(\lambda) \to \bigsqcup_{\mu \prec \lambda} \mathrm{POP}(\mu)$ defined by $\mathrm{br}(T, \Lambda) = (T^\dagger, \Lambda^\dagger)$ where $T^\dagger$ is obtained from T by deleting its bottom row, and $\Lambda^\dagger$ is obtained from Λ by deleting the overlays Λ_{ij} with $j = n - 1$.

3.3 Box complementation

In addition to pr and br, $\mathrm{POP}(\lambda)$ is endowed with another important map, which we term *box complementation*. Observe that given a partition $\pi = (\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k \geq 0)$ fitting into a $k \times \ell$ rectangle, i.e., with $\pi_1 \leq \ell$, we may consider its complement in this rectangle, defined by $\pi^c = (\ell - \pi_k \geq \ell - \pi_{k-1} \geq \cdots \geq \ell - \pi_1)$. Now, for $(T, \Lambda) \in \mathrm{POP}(\lambda)$, define $\mathrm{boxcomp}(T, \Lambda) = (T, \Lambda^c)$ where for each i, j, $(\Lambda^c)_{ij}$ is defined to be the complement of Λ_{ij} in its bounding rectangle of size $\mathrm{NE}_{ij}(T) \times \mathrm{SE}_{ij}(T)$.

We note that since $|\Lambda| \neq |\Lambda^c|$ in general, boxcomp preserves x-weights, but not q-weights. However $|\Lambda| + |\Lambda^c| = \sum_{i,j} \mathrm{NE}_{ij}(T)\, \mathrm{SE}_{ij}(T) =: \mathrm{area}(T)$ (in the terminology of [15]), which depends only on T.

4 Projection and branching for Column strict fillings

Our goal is to construct natural bijections between $\mathrm{CSF}(\lambda)$ and $\mathrm{POP}(\lambda)$ which explain the equality of (2.3) and (3.2) for $v = \mathrm{inv}, \mathrm{quinv}$. In addition to preserving x- and q-weights, we would like our bijections to be compatible with projection and branching. Towards this end, we first define these latter maps in the setting of $\mathrm{CSF}(\lambda)$.

4.1 Projection: rowsort

Given $F \in \mathrm{CSF}(\lambda)$, let $\mathrm{rsort}(F)$ denote the filling obtained from F by sorting entries of each row in ascending order. In light of the following easy lemma, we think of rsort as the projection map in the CSF setting.

Lemma 2. *If* $F \in \mathrm{CSF}(\lambda)$, *then* $\mathrm{rsort}(F) \in \mathrm{SSYT}(\lambda) \cong \mathrm{GT}(\lambda)$.

4.2 Branching: delete-and-splice

A strictly increasing sequence $a = (a_1 < a_2 < \cdots < a_m)$ of positive integers will also be termed a *column tuple* with $\mathrm{len}(a) = m \geq 0$. Let $\ell \geq 1$ and suppose $\sigma = (\sigma_1 < \sigma_2 <$

$\cdots < \sigma_{\ell-1})$ and $\tau = (\tau_1 < \tau_2 < \cdots < \tau_\ell)$ are column tuples of length $\ell - 1$ and ℓ respectively. We set $\sigma_0 = 0$ and let k denote the maximum element of the (non-empty) set $\{1 \leq i \leq \ell : \sigma_{i-1} < \tau_i\}$. Define $\mathrm{splice}(\sigma, \tau) = (\overline{\sigma}, \overline{\tau})$ where

$$\overline{\sigma}_i = \begin{cases} \sigma_i & 1 \leq i < k \\ \tau_i & k \leq i \leq \ell \end{cases} \qquad \text{and} \qquad \overline{\tau}_i = \begin{cases} \tau_i & 1 \leq i < k \\ \sigma_i & k \leq i < \ell \end{cases}$$

i.e., $\overline{\sigma}, \overline{\tau}$ are obtained by swapping certain suffix portions of σ, τ. The choice of k ensures that $\overline{\sigma}, \overline{\tau}$ are also column tuples; we also have $\mathrm{len}(\overline{\sigma}) = \mathrm{len}(\tau)$ and $\mathrm{len}(\overline{\tau}) = \mathrm{len}(\sigma)$. For instance, when $(\sigma, \tau) = \left(\begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \right)$, we get $(\overline{\sigma}, \overline{\tau}) = \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \end{array} \right)$.

We now define the delete-and-splice rectification ("dsplice") map on $F \in \mathrm{CSF}(\lambda)$ as follows: **(1)** delete all cells in F containing the entry n and let $F^\dagger$ denote the resulting filling. While its column entries remain strictly increasing, $F^\dagger$ may no longer be of partition shape. **(2)** Let $\sigma^{(j)}$ $(j \geq 1)$ denote the column tuple obtained by reading the j^{th} column of $F^\dagger$ from top to bottom. If $F^\dagger$ is not of partition shape, there exists $j \geq 1$ such that $\mathrm{len}(\sigma^{(j+1)}) = \mathrm{len}(\sigma^{(j)}) + 1$. Choose any such j and modify $F^\dagger$ by replacing the pair of columns $(\sigma^{(j)}, \sigma^{(j+1)})$ in $F^\dagger$ by $\mathrm{splice}(\sigma^{(j)}, \sigma^{(j+1)})$. This swaps the column lengths and brings the shape of $F^\dagger$ one step closer to being a partition. **(3)** If the shape of $F^\dagger$ is a partition, STOP. Else go back to step 2.

It is clear that this process terminates and finally produces a CSF of partition shape (filled by numbers between 1 and $n - 1$), which we denote $\mathrm{dsplice}(F)$. The following properties hold:

Proposition 1. *With notation as above: (i) $D := \mathrm{dsplice}(F)$ is independent of the intermediate choices of j made in step 2 of the procedure. (ii) $\mathrm{rsort}(D)$ is obtained from $\mathrm{rsort}(F)$ by deleting the cells containing the entry n. (iii) If μ and λ are the shapes of D and F respectively, then $\mu \prec \lambda$.*

We consider dsplice to be the combinatorial branching map in the CSF context. Its key property is its compatibility with the natural branching map br of the POP setting.

5 The main theorem

Theorem 1. *For any $n \geq 1$ and any partition $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ with at most n nonzero parts, there exist two bijections ψ_{inv} and ψ_{quinv} from $\mathrm{CSF}(\lambda)$ to $\mathrm{POP}(\lambda)$ with the following properties:*

1. If $\psi_v(F) = (T, \Lambda)$, then $x^F = x^T$ and $v(F) = |\Lambda|$, for $v = \mathrm{inv}$ or quinv.

2. The following diagrams commute ($v = \mathrm{inv}$ or quinv):

(A)

$$\begin{tikzcd} \mathrm{CSF}(\lambda) \arrow[rr, "\psi_v"] \arrow[dr, "\mathrm{rsort}"'] & & \mathrm{POP}(\lambda) \arrow[dl, "\mathrm{pr}"] \\ & \mathrm{GT}(\lambda) & \end{tikzcd}$$

(B)

$$\begin{tikzcd} \mathrm{CSF}(\lambda) \arrow[r, "\psi_v"] \arrow[d, "\mathrm{dsplice}"'] & \mathrm{POP}(\lambda) \arrow[d, "\mathrm{br}"] \\ \bigsqcup_{\mu \prec \lambda} \mathrm{CSF}(\mu) \arrow[r, "\psi_v"'] & \bigsqcup_{\mu \prec \lambda} \mathrm{POP}(\mu) \end{tikzcd}$$

3. The two bijections are related via the commutative diagram:

$$\begin{tikzcd} & \mathrm{CSF}(\lambda) \arrow[dl, "\psi_{\mathrm{inv}}"'] \arrow[dr, "\psi_{\mathrm{quinv}}"] & \\ \mathrm{POP}(\lambda) \arrow[rr, "\mathrm{boxcomp}"] & & \mathrm{POP}(\lambda) \end{tikzcd}$$

□

To summarize, ψ_{inv} and ψ_{quinv} acting on a CSF produce POPs with the same underlying GT pattern, but with complementary overlays. These bijections are compatible with the natural projection and branching maps, and preserve x- and appropriate q-weights (inv or quinv). Note the slight abuse of notation in part 2(B) above: for $\mu \prec \lambda$, $\mathrm{CSF}(\mu)$ denotes the set of column strict fillings $F : \mathrm{dg}(\mu) \to [n-1]$ (rather than $[n]$). Theorem 1, with the exception of part 2(B), can also be formulated in the setting of q-Whittaker functions in infinitely many variables. Next, we obtain the following corollaries:

Corollary 1. *Let $T \in \mathrm{GT}(\lambda)$ and let $\mathrm{rsort}^{-1}(T) = \{F \in \mathrm{CSF}(\lambda) : \mathrm{rsort}(F) = T\}$ be the fiber of rsort over T.*

1. $$\sum_{F \in \mathrm{rsort}^{-1}(T)} q^{\mathrm{inv}(F)} = \sum_{F \in \mathrm{rsort}^{-1}(T)} q^{\mathrm{quinv}(F)} = \mathrm{wt}_q(T).$$

2. $\mathrm{inv}(F) + \mathrm{quinv}(F) = \mathrm{area}(T)$ *is constant for* $F \in \mathrm{rsort}^{-1}(T)$.

An interpretation of $\mathrm{wt}_q(T)$ in terms of flags of subspaces compatible with nilpotent operators appears in [12, Theorem 5.8(i)]. In [1], the authors asked for an explicit bijection on $\mathcal{F}(\lambda)$ which interchanges the inv and quinv statistics. We describe this bijection on $\mathrm{CSF}(\lambda)$, thereby partially answering their question.

$$F = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&2&1&2&1&2&4&4&3\\\hline 2&2&3&3&3&4&&&&\\\hline 3&3&4&4&&&\multicolumn{4}{c}{}\\\cline{1-6} &&&&\multicolumn{6}{c}{}\\\cline{1-4}\end{array} \qquad \mathrm{zcount}(\cdot, F) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&1&0&2&1&1&2\\\hline 0&0&0&0&0&1&\multicolumn{4}{c}{}\\\cline{1-6} 0&0&2&2&\multicolumn{6}{c}{}\\\cline{1-4}\end{array}$$

Figure 2: Here $F \in \mathrm{CSF}(\lambda)$ for $\lambda = (10, 6, 4, 0)$ and $n = 4$. Cells of F are coloured according to their entries. The gray cells are the extra cells in the augmented diagram $\widehat{\mathrm{dg}}(\lambda)$. On the right are cellwise zcount values. Here $\mathrm{quinv}(F) = 12$.

Corollary 2. *The map* $\Omega : \psi_{inv}^{-1} \circ \psi_{\mathrm{quinv}} = \psi_{inv}^{-1} \circ \mathrm{boxcomp} \circ \psi_{\mathrm{inv}} : \mathrm{CSF}(\lambda) \to \mathrm{CSF}(\lambda)$ *is an involution satisfying* $\mathrm{inv}(\Omega(F)) = \mathrm{quinv}(F)$ *for all* $F \in \mathrm{CSF}(\lambda)$.

The explicit construction of the ψ_v and their inverses in the next section makes Ω effectively computable.

6 Proof sketch

For a partition λ, the augmented diagram $\widehat{\mathrm{dg}}(\lambda)$ is $\mathrm{dg}(\lambda)$ together with one additional cell below the last cell in each column (see Figure 2). Given $F \in \mathrm{CSF}(\lambda)$, a *quinv-triple* in F is a triple of cells (x, y, z) in $\widehat{\mathrm{dg}}(\lambda)$ such that (i) $x, z \in \mathrm{dg}(\lambda)$ and z is to the right of x in the same row, (ii) y is the cell immediately below x in its column, (iii) $F(x) < F(z) < F(y)$, where we set $F(y) = \infty$ if y lies outside $\mathrm{dg}(\lambda)$. It is easy to see that the quinv-triples considered in [1] for $F \in \mathcal{F}(\lambda)$ reduce to this description when F is a CSF rather than a general filling. Thus, $\mathrm{quinv}(F)$ as defined in [1] equals the number of quinv-triples in F (as defined above) for a CSF F.

Given $F \in \mathrm{CSF}(\lambda)$, we define a function *zcount* which tracks the contributions of individual cells of $\mathrm{dg}(\lambda)$ to $\mathrm{quinv}(F)$ as follows: for each cell $c \in \mathrm{dg}(\lambda)$, let $\mathrm{zcount}(c, F) =$ the number of quinv-triples (x, y, z) in F with $z = c$. Clearly

$$\sum_{c \in \mathrm{dg}(\lambda)} \mathrm{zcount}(c, F) = \mathrm{quinv}(F) \tag{6.1}$$

We next group cells of the filling F row-wise according to the entries they contain. More precisely, let $\mathrm{cells}(i, j, F) = \{c \in \mathrm{dg}(\lambda) : c \text{ is in the } i^{th} \text{ row and } F(c) = j + 1\}$ for $1 \leq i \leq j + 1 \leq n$. Figure 2 shows an example, with these groups colour-coded in each row. It readily follows from §3 that

$$|\,\mathrm{cells}(i, j, F)| = NE_{ij}(T), \text{ where } T = \mathrm{rsort}(F). \tag{6.2}$$

The next proposition brings the SE differences also into play [3]:

Proposition 2. *Let* $F \in \mathrm{CSF}(\lambda)$ *and* $T = \mathrm{rsort}(F)$. *Fix* $1 \leq i \leq j + 1 \leq n$. *(1) If* $c \in \mathrm{cells}(i, j, F)$, *then* $\mathrm{zcount}(c, F) \leq \mathrm{SE}_{ij}(T)$. *(2) If* $c, d \in \mathrm{cells}(i, j, F)$ *with* c *lying to the right of* d, *then* $\mathrm{zcount}(c, F) \geq \mathrm{zcount}(d, F)$. *(3) Further, equality holds in (1) for all* i, j *and all cells* $c \in \mathrm{cells}(i, j, F)$ *iff* $F = T$.

Figure 3: (left to right) Configuration of quinv, inv and refinv triples.

6.1 Definition of ψ_{quinv}

We now have all the ingredients in place to define ψ_{quinv}. Let $F \in \mathrm{CSF}(\lambda)$ and $T = \mathrm{rsort}(F)$. For each $1 \le i \le j+1 \le n$, consider the sequence

$$\Lambda_{ij} = (\mathrm{zcount}(c, F) : \ c \in \mathrm{cells}(i, j, F) \text{ traversed right to left in row } i). \tag{6.3}$$

In Figure 2, this amounts to reading the entries of a fixed colour from right to left in a given row of $\mathrm{zcount}(\cdot, F)$. By Proposition 2, this is a weakly decreasing sequence bounded above by $\mathrm{SE}_{ij}(T)$. Together with (6.2), this implies that Λ_{ij} may be viewed as a partition fitting into the $\mathrm{NE}_{ij}(T) \times \mathrm{SE}_{ij}(T)$ rectangle. Since $\mathrm{SE}_{ij} = 0$ for $i = j+1$, Λ_{ij} is the zero sequence in this case. We drop the pairs $(j+1, j)$ to conclude that if $\Lambda = (\Lambda_{ij} : 1 \le i \le j < n)$, then $(T, \Lambda) \in \mathrm{POP}(\lambda)$. We define $\psi_{quinv}(F) = (T, \Lambda)$. Clearly, $x^F = x^T$ and (6.1) implies $\mathrm{quinv}(F) = |\Lambda|$, establishing (1) of Theorem 1 for $v = \mathrm{quinv}$.

6.2 refinv triples

We now turn to the definition of ψ_{inv}. While we may anticipate doing this via a modification of the foregoing arguments, replacing quinv-triples with Haglund-Haiman-Loehr's inv-triples, that turns out not to work out-of-the-box. In place of the latter (see Figure 3), we consider triples (x, y, z) in $\widehat{\mathrm{dg}}(\lambda)$ where (i) $x, z \in \mathrm{dg}(\lambda)$ with z to the left of x in the same row, (ii) y is the cell immediately below x in its column. Given $F \in \mathrm{CSF}(\lambda)$, we call (x, y, z) a *refinv-triple* (or "reflected inv-triple") for F if in addition to (i) and (ii), we also have (iii) $F(x) < F(z) < F(y)$, where $F(y) := \infty$ if $y \notin \mathrm{dg}(\lambda)$. We have [3]:

Proposition 3. *For $F \in \mathrm{CSF}(\lambda)$, $\mathrm{inv}(F)$ equals the number of refinv-triples of F.*

Remarks. 1. We may in fact define a new statistic[2] *refinv* on all fillings $F \in \mathcal{F}(\lambda)$ as follows: $\mathrm{refinv}(F) = \mathrm{Inv}(F) - \sum_{u \in \mathrm{Des}\, F} \mathrm{coarm}(u)$, borrowing notation of [9, §2]. This replaces *arm* in HHL's definition by *coarm*. The content of Proposition 3 is that $\mathrm{refinv}(F) = \mathrm{inv}(F)$ for $F \in \mathrm{CSF}(\lambda)$. In fact, this equality holds more generally for all fillings F whose descent set is a union of rows of $\mathrm{dg}(\lambda)$.

2. The refinv triples for $F \in \mathrm{CSF}(\lambda)$ actually make an appearance in [13, §2.2], where they are attributed to Zelevinsky (and their total number denoted $\widetilde{ZEL}$). From this perspective, the content of Proposition 3 is that $\widetilde{ZEL}(F) = \mathrm{inv}(F)$.

[2]In fact, *refquinv* can also be likewise defined on all fillings, and agrees with *quinv* on CSFs. But rephrased in terms of refquinv-triples, this involves counting such triples with signs [3].

6.3 $\overline{\text{zcount}}$, zcount and the proof of the main theorem

Given $F \in \mathrm{CSF}(\lambda)$ and $c \in \mathrm{dg}(\lambda)$, define $\overline{\text{zcount}}(c, F) =$ the number of refinv-triples (x, y, z) in F with $z = c$. In light of Proposition 3, it is clear that

$$\sum_{c \in \mathrm{dg}(\lambda)} \overline{\text{zcount}}(c, F) = \mathrm{inv}(F) \tag{6.4}$$

We have the following relation between $\overline{\text{zcount}}$ and zcount [3]:

Proposition 4. *Let $F \in \mathrm{CSF}(\lambda)$ and $T = \mathrm{rsort}(F)$. Let $1 \le i \le j+1 \le n$ and $c \in \mathrm{cells}(i, j, F)$. Then $\mathrm{zcount}(c, F) + \overline{\text{zcount}}(c, F) = \mathrm{SE}_{ij}(T)$.*

We may now define ψ_{inv} following the template of ψ_{quinv}. Given $F \in \mathrm{CSF}(\lambda)$, let $T = \mathrm{rsort}(F)$. For each $1 \le i \le j < n$, consider the sequence:

$$\overline{\Lambda}_{ij} = (\overline{\text{zcount}}(c, F) : \ c \in \mathrm{cells}(i, j, F) \text{ traversed left to right in row } i)$$

Recall also the definition of the partition Λ_{ij} from (6.3). It follows from Propositions 2 and 4 that $\overline{\Lambda}_{ij}$ is the box-complement of Λ_{ij} in the $\mathrm{NE}_{ij}(T) \times \mathrm{SE}_{ij}(T)$ rectangle. Letting $\overline{\Lambda} = (\overline{\Lambda}_{ij} : 1 \le i \le j < n)$, we define $\psi_{\mathrm{inv}}(F) = (T, \overline{\Lambda})$. As in the case of quinv, we have $x^F = x^T$, and $\mathrm{inv}(F) = |\overline{\Lambda}|$ by (6.4). This proves part (1) of Theorem 1 for $v = \mathrm{inv}$.

Since by definition $\mathrm{pr}(\psi_v(F)) = T$ for $v = \mathrm{inv}, \mathrm{quinv}$, Part (2A) of Theorem 1 follows. Part (3) of Theorem 1 follows from the fact that Λ and $\overline{\Lambda}$ are box complements of each other in the appropriate rectangles. That the diagrams in part (2B) of Theorem 1 are commutative follows from an analysis of each elementary splice step of the dsplice map.

Finally, this leaves us with proving that the ψ_v are bijections. We sketch the construction of ψ_{inv}^{-1}. Given $(T, \Lambda) \in \mathrm{POP}(\lambda)$, construct the filling $F := \psi_{\mathrm{inv}}^{-1}(T, \Lambda) \in \mathrm{CSF}(\lambda)$ inductively row-by-row, from the bottom (n^{th}) row to the top as follows: (a) fill all cells of the n^{th} row (if nonempty) with n, (b) let $1 \le i \le j < n$; assuming that all rows of F strictly below row i have been completely determined and that the locations of entries $> (j+1)$ in row i have been determined, we now need to fill $\mathrm{NE}_{ij}(T)$ many cells of row i with the entry $j+1$. It turns out that the number of cells in row i in which we can potentially put a $j+1$ without violating the CSF condition thus far is exactly $k + \ell$ where $k = \mathrm{NE}_{ij}(T)$ and $\ell = \mathrm{SE}_{ij}(T)$. We label these cells $0, 1, \cdots, k + \ell - 1$ from right to left (left-to-right when defining $\psi_{\mathrm{quinv}}^{-1}$). We now use the identification from §3.1 of partitions fitting inside a $(k \times \ell)$-box with k-tuples of distinct integers in $0, 1, \cdots, k + \ell - 1$. Via this, the partition Λ_{ij} can be viewed as a k-tuple of candidate cells in row i; we put the entry $j+1$ into these, (c) fill the remaining cells of row i with the entry i. The rest of the argument is straightforward [3]. □

For example, let $n = 4$, $\lambda = (10, 6, 4, 0)$ and let T, Λ be the GT pattern and overlay depicted in Figure 1. Then $\psi_{\mathrm{quinv}}^{-1}(\mathcal{T}, \Lambda)$ is precisely the CSF F of Figure 2, while

$$\psi_{\mathrm{inv}}^{-1}(\mathcal{T}, \Lambda) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 2 & 1 & 1 & 1 & 3 & 2 & 1 & 4 & 4 & 2 \\ \hline 3 & 3 & 2 & 2 & 4 & 3 \\ \cline{1-6} 4 & 4 & 3 & 3 \\ \cline{1-4} \end{array}$$

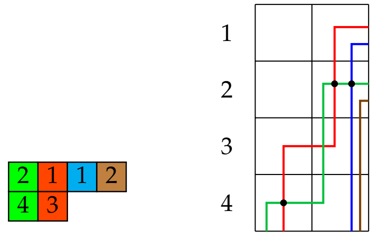


Figure 4: A CSF F with columns colour-coded to match its lattice path representation. The three marked intersections show that $\mathrm{inv}(F) = 3$.

7 Local Weyl modules and limit constructions

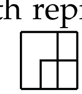
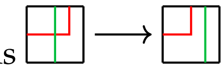
Finally, we can apply these ideas to the study of local Weyl modules, in particular to the *limit constructions* of [7, 15, 16]. Let $L(\Lambda_0)$ denote the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}_n}$ [11, Prop. 12.13]. Using Theorem 1 to replace POPs with CSFs as our model in [15, Corollary 5.13], we deduce [3]:

Proposition 5. *Fix $n \geq 2$ and consider the partition $\theta = (2, 1, 1, \cdots, 1, 0)$ with $n-1$ nonzero parts and $|\theta| = n$. For $k \geq 0$, let $\mathcal{C}_k$ denote the set of CSFs F of shape $k\theta$ and entries in $[n]$, with the property that either 1 occurs in the first column of F or 1 does not occur in its last column. Then $\sum_{k\geq 0} \sum_{F\in\mathcal{C}_k} x^F q^{k^2-\mathrm{inv}(F)}$ equals the character of $L(\Lambda_0)$.*

There is also a more general version with $\lambda + k\theta$ in place of $k\theta$ (for appropriate λ), mirroring [15, Corollary 5.13].

8 Concluding Remarks

For the modified Hall-Littlewood polynomials $Q'_{\lambda'}(X_n; q)$ of (2.4), the fermionic formula appears in [13, (0.2)]. Analogous to (3.2), this can now be recast as a *weighted sum* over *partition overlaid plane-partitions* (POPP) of shape λ. Theorem 1 takes the form of bijections from WDF(λ) to POPP(λ) (or equivalently, from tabloids to partition overlaid reverse-plane-partitions). The subtlety here is that POPPs need to be weighted with an additional power of q (which depends only on the underlying plane-partition, cf [13, (0.2)]). The refinv- or quinv-triples in this case also involve $\leq$ relations (rather than just $<$) and this extra q-power keeps track of certain equalities among the triples [3].

Secondly, the bijections of Theorem 1 (and those indicated above for the modified Hall-Littlewood case) have an attractive interpretation in terms of lattice-path diagrams [8, 4]. Figure 4 shows the lattice path representation of a CSF F; $\mathrm{inv}(F)$ is just the total number of intersections of the form  in the grid, and refining this further to each box of the grid produces the partition overlay as well [3]. Likewise $\mathrm{quinv}(F)$ counts non-intersections of the above form. The dsplice map of §4.2 translates into deletion of the last row of the grid followed by appropriate rectifications 

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Extremal weight crystals over affine Lie algebras of infinite rank

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Abstract. We explain extremal weight crystals over affine Lie algebras of infinite rank using combinatorial models: a spinor model due to Kwon, and an infinite rank analogue of Kashiwara–Nakashima tableaux due to Lecouvey. In particular, we show that the Lecouvey's tableau model combinatorially explains an extremal weight crystal structure of level zero. Using these combinatorial models, we explain an algebra structure of the Grothendieck ring for a category consisting of some extremal weight crystals.

Keywords: extremal weight crystals, affine Lie algebras of infinite rank, Jacobi–Trudi formula, Grothendieck ring

1 Introduction

Let $U_q(\mathfrak{g})$ be a quantum group associated with a Kac–Moody algebra $\mathfrak{g}$. For an integral weight λ, let $V(\lambda)$ be an extremal weight $U_q(\mathfrak{g})$-module of weight λ and $B(\lambda)$ be its associated crystal base (cf. [6]). It is significant to study extremal weight crystals because it is closely related to level-zero representations of quantum affine Lie algebras (of finite rank). For details, see [1, 2, 8, 17] and references therein. However, properties of extremal weight crystals over affine Lie algebras of infinite rank differ considerably from those over affine Lie algebras of finite rank. In this extended abstract, we discuss several properties of extremal weight crystals over affine Lie algebras of infinite rank.

An important observation by Naito and Sagaki [16] (see Proposition 3.2 also) is that for an integral weight λ of a nonnegative level, there exist $\lambda^0 \in E$ and $\lambda^+ \in P^+$ (and unique in some sense) such that

$$B(\lambda) \cong B(\lambda^0) \otimes B(\lambda^+). \tag{1.1}$$

This isomorphism suggests that a combinatorial model of $B(\lambda)$ ($\lambda \in P$) by combining that for $B(\lambda^0)$ ($\lambda^0 \in E$) and that for $B(\lambda^+)$ ($\lambda^+ \in P^+$).

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We associate $B(\lambda)$ ($\lambda \in E$) to a set $\mathbf{KN}^{\mathfrak{g}}(\lambda^{\dagger})$ ($\lambda^{\dagger} \in \mathcal{P}$) of $\mathfrak{g}_{\infty}$-type Kashiwara–Nakashima (simply KN) tableaux introduced by Lecouvey [15], which are an infinite rank analogue of KN tableaux. We define a $\mathfrak{g}_{\infty}$-crystal structure on $\mathbf{KN}^{\mathfrak{g}}(\lambda)$, and we construct an isomorphism between $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ and $B(\varpi_{\lambda})$ (Theorem 3.10). On the other hand, we associate $B(\lambda)$ for $\lambda \in P^{+}$ to a spinor model introduced by Kwon [12, 13]. Indeed, the crystal structure of a spinor model is already known and a spinor model is isomorphic to extremal weight crystals of dominant weights (see Theorem 3.6).

As an application, we characterize the Grothendieck ring $\mathcal{K}$ for a category $\mathcal{C}$ consisting of some extremal weight crystals. In particular, as similarly as (1.1), the set $\mathcal{K}$ has the tensor decomposition

$$\mathcal{K} = \mathcal{K}^{0} \otimes \mathcal{K}^{+}$$

where $\mathcal{K}^{0}$ and $\mathcal{K}^{+}$ are the subalgebra of $\mathcal{K}$ generated by $[B(\lambda)]$ for $\lambda \in E$ and $\lambda \in P^{+}_{\mathrm{int}}$, respectively. It is known that $\mathcal{K}^{0}$ is isomorphic to the ring of symmetric functions (Proposition 5.2) and $\mathcal{K}^{+}$ is isomorphic to the ring of formal power series (Theorem 5.4). One can find a full version of this extended abstract including proofs and details in [3].

2 Preliminaries

2.1 Notations

Let $\mathbb{Z}_{+}$ be the set of nonnegative integers. Let $\mathcal{P}$ be the set of partitions and, for $n \in \mathbb{Z}_{+}$, $\mathcal{P}_n = \{\, \lambda \in \mathcal{P} \mid \ell(\lambda) \le n \,\}$, where $\ell(\lambda)$ is the length of λ. Denote by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ the conjugate of λ.

For even $\ell \ge 2$, let G_{ℓ} be one of the algebraic groups: $\mathrm{Sp}_{\ell}, \mathrm{Pin}_{\ell}$, and O_{ℓ}. Let

$$\begin{aligned} &\mathscr{P}(\mathrm{Sp}_{\ell}) = \mathcal{P}_{\frac{\ell}{2}}, \qquad \mathscr{P}(\mathrm{Pin}_{\ell}) = \mathcal{P}_{\frac{\ell}{2}}, \\ &\mathscr{P}(\mathrm{O}_{\ell}) = \{\, \lambda \in \mathcal{P}_{\ell} \mid \lambda'_1 + \lambda'_2 \le \ell \,\}, \end{aligned}$$

and

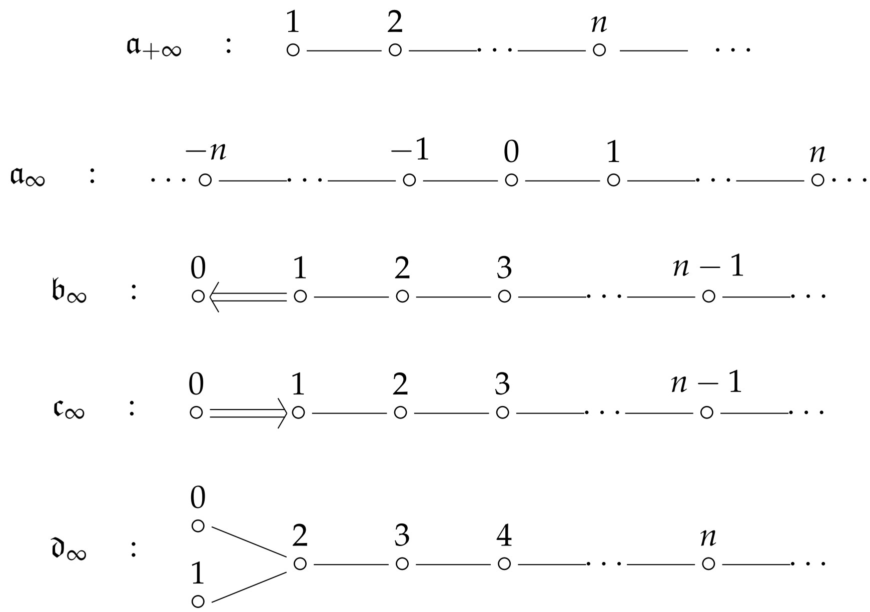
$$\mathscr{P}(G) = \{\, (\lambda, \ell) \mid \ell \in \mathbb{N}, \lambda \in \mathscr{P}(\mathrm{G}_{2\ell}) \,\}$$

for $G = \mathrm{Sp}, \mathrm{Pin}$, or O.

For an ordered set $\mathcal{A}$ and a skew shape λ/μ, denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of semistandard (or $\mathcal{A}$-semistandard) tableaux of shape λ/μ, that is, tableaux with letters in $\mathcal{A}$ such that entries in each row (resp. column) are weakly (resp. strictly) increasing. We omit a subscript $\mathcal{A}$ from $SST_{\mathcal{A}}(\lambda/\mu)$ if there is no confusion or it does not depend on the choice of $\mathcal{A}$.

2.2 Affine Lie algebras of infinite rank

A Lie algebra $\mathfrak{g}$ is of infinite rank if it is the Kac–Moody algebra associated with a generalized Cartan matrix of infinite rank. A Lie algebra of infinite rank is of affine type if every principal minor (of finite rank) of associated generalized Cartan matrix is positive. There are five (non-isomorphic) affine Lie algebras of infinite rank whose Dynkin diagram is connected and these are referred to Lie algebras $\mathfrak{a}_{+\infty}, \mathfrak{a}_\infty, \mathfrak{b}_\infty, \mathfrak{c}_\infty$, and $\mathfrak{d}_\infty$ (cf. [4]). The followings are Dynkin diagrams corresponding to stated affine Lie algebras of infinite rank.



In this extended abstract, we focus on providing results for $\mathfrak{g} = \mathfrak{b}_\infty, \mathfrak{c}_\infty$, or $\mathfrak{d}_\infty$. The corresponding results to ours can be found in [10] when $\mathfrak{g} = \mathfrak{a}_{+\infty}$ and in [11] when $\mathfrak{g} = \mathfrak{a}_\infty$. We use the following notations for affine Lie algebras $\mathfrak{g}_\infty$ of infinite rank.

- $I = \mathbb{Z}_+$: the index set
- $\{\alpha_i \,|\, i \in I\}$: the set of simple roots
- $\{\Lambda_i^{\mathfrak{g}} \,|\, i \in I\}$: the set of fundamental weights
- $P = \mathbb{Z}\Lambda_0^{\mathfrak{g}} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}\epsilon_i$: the weight lattice
- P^+ : the set of dominant weights, $\qquad E = \bigoplus_{i=1}^{\infty} \mathbb{Z}\epsilon_i \subseteq P$
- W : the Weyl group

In this paper, we take the simple roots α_i as below. Then we can derive the following equations on $\Lambda_i^{\mathfrak{g}}$.

$$\begin{array}{ll} \mathfrak{c}_\infty & \alpha_0 = -2\epsilon_1, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \geq 1) \\ & \Lambda_i^{\mathfrak{c}} = \Lambda_0^{\mathfrak{c}} + (\epsilon_1 + \cdots + \epsilon_i) \quad (i \geq 1) \\ \mathfrak{b}_\infty & \alpha_0 = -\epsilon_1, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \geq 1) \\ & \Lambda_i^{\mathfrak{b}} = 2\Lambda_0^{\mathfrak{b}} + (\epsilon_1 + \cdots + \epsilon_i) \quad (i \geq 1) \\ \mathfrak{d}_\infty & \alpha_0 = -\epsilon_1 - \epsilon_2, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \geq 1) \\ & \Lambda_1^{\mathfrak{d}} = \Lambda_0^{\mathfrak{d}} + \epsilon_1, \quad \Lambda_i^{\mathfrak{d}} = 2\Lambda_0^{\mathfrak{d}} + (\epsilon_1 + \cdots + \epsilon_i) \quad (i \geq 2) \end{array}$$

For an integer $n \geq 2$, let $\mathfrak{g}_n$ be the Lie subalgebra of $\mathfrak{g}_\infty$ generated by e_i, f_i $(i = 0, 1, \ldots, n-1)$. We write the expression $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ when we don't have to specify its rank.

For $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$, define

$$\varpi_\lambda = \sum_{i \geq 1} \lambda_i \epsilon_i \in E.$$

For simplicity, we write $\varpi_i = \varpi_{(1^i)}$ for $i \geq 1$. On the other hand, we suppose that a Lie algebra $\mathfrak{g}_\infty$ corresponds to an algebraic group G (and vice versa) as follows:

$$(\mathfrak{g}, G) \quad : \quad (\mathfrak{b}_\infty, \mathrm{Pin}), \ (\mathfrak{c}_\infty, \mathrm{Sp}), \ (\mathfrak{d}_\infty, \mathrm{O}) \tag{2.1}$$

Put

- $\Pi_i^{\mathfrak{c}} = \Lambda_i^{\mathfrak{c}} \quad (i \geq 0)$
- $\Pi_0^{\mathfrak{b}} = 2\Lambda_0^{\mathfrak{b}}, \ \Pi_i^{\mathfrak{b}} = \Lambda_i^{\mathfrak{b}} \quad (i \geq 1)$
- $\Pi_0^{\mathfrak{d}} = 2\Lambda_0^{\mathfrak{d}}, \ \overline{\Pi}_0^{\mathfrak{d}} = 2\Lambda_1^{\mathfrak{d}}, \ \Pi_1^{\mathfrak{d}} = \Lambda_0^{\mathfrak{d}} + \Lambda_1^{\mathfrak{d}}$, and $\Pi_i^{\mathfrak{d}} = \Lambda_i^{\mathfrak{d}} \quad (i \geq 2)$

and let

$$\Pi^{\mathfrak{g}}(\lambda, \ell) = \ell\, \Pi_0^{\mathfrak{g}} + \varpi_{\lambda'} \in P^+$$

for $(\lambda, \ell) \in \mathscr{P}(G)$. For $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) = t$, we have

$$\Pi^{\mathfrak{g}}(\lambda, \ell) = \begin{cases} \Pi_{\lambda_1}^{\mathfrak{g}} + \cdots + \Pi_{\lambda_\ell}^{\mathfrak{g}} & \text{if } t \leq \ell, \\ \Pi_{\lambda_1}^{\mathfrak{d}} + \cdots + \Pi_{\lambda_{2\ell-t}}^{\mathfrak{d}} + (t-\ell)\overline{\Pi}_0^{\mathfrak{d}} & \text{if } t > \ell. \end{cases}$$

The condition $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) > \ell$ holds only when $(\mathfrak{g}, G) = (\mathfrak{d}_\infty, \mathrm{O})$.

Let

$$P_{\mathrm{int}}^+ = \{\Pi^{\mathfrak{g}}(\lambda, \ell) \,|\, (\lambda, \ell) \in \mathscr{P}(G)\} \ \subseteq \ P^+.$$

Note that when $\mathfrak{g} = \mathfrak{c}_\infty$, we have $P_{\mathrm{int}}^+ = P^+$, and when $\mathfrak{g} = \mathfrak{b}_\infty$ or $\mathfrak{d}_\infty$, we have $P_{\mathrm{int}}^+ \subsetneq P^+$ and P_{int}^+ is the set of dominant weights with a positive even level. Recall that the level of $\lambda \in P$ is the value $\langle \lambda, K \rangle$, where K is the canonical central element of $\mathfrak{g}_\infty$ (cf. [4, Section 7.12]).

Remark 2.1. *The correspondence (2.1) between Lie algebras and algebraic groups originates from dual pairs due to Howe. For details, see [3, Remark 2.1].*

3 A combinatorial realization of extremal weight crystals

3.1 Extremal weight crystals

We recall the notion of an extremal weight crystal, which is introduced by Kashiwara (cf. [6, 8]). For $\lambda \in P$, let $V(\lambda)$ be an extremal weight module generated by an extremal weight vector. In particular, $V(\lambda)$ is an irreducible highest weight module when $\lambda \in P^+$ (cf. [5]). It is proved in [6] that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$, which provides a tool to interpret a given module in a combinatorial way. We simply say that $B(\lambda)$ is an extremal weight crystal.

When $\mathfrak{g}$ is a general Kac–Moody algebra, for $\lambda \in P$ and $w \in W$, there exists an isomorphism $B(\lambda) \cong B(w\lambda)$ of $\mathfrak{g}$-crystals [6]. Moreover, the converse of the above statement holds when $\mathfrak{g}$ is an affine Lie algebra of infinite rank.

Proposition 3.1 ([16, Proposition 3.9])**.** *When $\mathfrak{g}$ is an affine Lie algebra of infinite rank and $\lambda, \mu \in P$, we have $B(\lambda) \cong B(\mu)$ if and only if $\lambda \in W\mu$.*

From now on, we assume that all Lie algebras in this article are affine Lie algebras of infinite rank without otherwise stated. In particular, we use the notation $\mathfrak{g}_\infty$ to emphasize the infinite rank.

The key observation of this paper is that an extremal weight crystal $B(\lambda)$ ($\lambda \in P$) is decomposed into the tensor product of two extremal weight crystals.

Proposition 3.2 ([16, Section 4.2])**.** *For a nonnegative level $\lambda \in P$, there exist $\lambda^0 \in E$ and $\lambda^+ \in P^+$ such that*

$$B(\lambda) \;\cong\; B(\lambda^0) \otimes B(\lambda^+). \tag{3.1}$$

Moreover, for $\lambda^0, \mu^0 \in E$ and $\lambda^+, \mu^+ \in P^+$, we have

$$B(\lambda^0) \otimes B(\lambda^+) \cong B(\mu^0) \otimes B(\mu^+) \iff \lambda^+ = \mu^+,\ \lambda^0 \in W\mu^0.$$

By Proposition 3.2, we shift our focus to understand extremal weight crystals $B(\lambda)$ for $\lambda \in E$ or $\lambda \in P^+$. In particular, for given $\lambda \in E$, there exists unique $\mu \in W\lambda$ such that μ is of the form ϖ_α for some $\alpha \in \mathcal{P}$, and we denote by $\lambda^\dagger \in \mathcal{P}$ such a (unique) partition α. Since $B(\lambda) \cong B(\varpi_{\lambda^\dagger})$ by Proposition 3.1, we may assume that $\lambda \in E$ is of the form ϖ_α for some $\alpha \in \mathcal{P}$, indeed $\alpha = \lambda^\dagger$.

Remark 3.3. *For a nonpositive level $\lambda \in P$, we have an isomorphism*

$$B(\lambda) \;\cong\; B(\lambda^-) \otimes B(\lambda^0)$$

for some $\lambda^0 \in E$ and $\lambda^- \in -P^+$, which is obtained from (3.1) by applying dual crystals (cf. [7, Section 7.4]).

Example 3.4 ([3, Example 3.11]). *When $\mathfrak{g}_\infty = \mathfrak{c}_\infty$, consider $\lambda = 4\Pi_0^{\mathfrak{c}} + 2\epsilon_1 + 5\epsilon_3 - 3\epsilon_4 - \epsilon_5 + 4\epsilon_6 \in P$. Then we have $\nu = 4\Pi_0^{\mathfrak{c}} + 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_4 - 3\epsilon_5 \in W\lambda$ with*

$$\begin{aligned} \nu^+ &= 4\Pi_0^{\mathfrak{c}} + 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 = \Pi^{\mathfrak{c}}((3,3,2,1),4), \\ \nu^0 &= -\epsilon_4 - 3\epsilon_5. \end{aligned}$$

In this case, $(\nu^0)^\dagger (= (\lambda^0)^\dagger) = (3,1)$. Thus, we have $\lambda^0 = \varpi_{(3,1)}$ and $\lambda^+ = \Pi^{\mathfrak{c}}((3,3,2,1),4)$.

3.2 Spinor model

For $a,b,c \in \mathbb{Z}_+$, let $\lambda(a,b,c) = (2^{b+c},1^a)/(1^b)$ be a skew shape with two columns. Suppose that $T \in SST(\lambda(a,b,c))$ for some $a,b,c \in \mathbb{Z}_+$ and T' is the tableau obtained from T by sliding the right column of T by k positions down for $0 \le k \le \min\{a,b\}$. Set $\mathfrak{r}_T$ to be the maximal integer $k \ge 0$ such that $T' \in SST(\lambda(a-k,b-k,c+k))$.

For $a \in \mathbb{Z}_+$, let

$$\mathbf{T}^{\mathfrak{g}}(a) \;=\; \{\, T \in SST_{\mathbb{N}}(\lambda(a,b,c)) \,|\, (b,c) \in \mathcal{H}^{\mathfrak{g}},\ \mathfrak{r}_T \le r^{\mathfrak{g}} \,\},$$

where

$$\mathcal{H}^{\mathfrak{g}} = \begin{cases} \{0\} \times \mathbb{Z}_+ & \text{if } \mathfrak{g} = \mathfrak{c} \\ \mathbb{Z}_+ \times \mathbb{Z}_+ & \text{if } \mathfrak{g} = \mathfrak{b} \\ 2\mathbb{Z}_+ \times 2\mathbb{Z}_+ & \text{if } \mathfrak{g} = \mathfrak{d} \end{cases}, \qquad r^{\mathfrak{g}} = \begin{cases} 0 & \text{if } \mathfrak{g} = \mathfrak{b}, \mathfrak{c} \\ 1 & \text{if } \mathfrak{g} = \mathfrak{d} \end{cases},$$

and

$$\overline{\mathbf{T}}^{\mathfrak{d}}(0) = \bigsqcup_{(b,c)\in\mathcal{H}^{\mathfrak{d}}} SST_{\mathbb{N}}(\lambda(0,b,c+1)).$$

For $(\lambda,\ell) \in \mathscr{P}(G)$, put $t = \ell(\lambda)$ and

$$\widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda,\ell) = \begin{cases} \mathbf{T}^{\mathfrak{g}}(\lambda_\ell) \times \cdots \times \mathbf{T}^{\mathfrak{g}}(\lambda_1) & \text{if } t \le \ell, \\ \overline{\mathbf{T}}^{\mathfrak{d}}(0)^{t-\ell} \times \mathbf{T}^{\mathfrak{d}}(\lambda_{2\ell-t}) \times \cdots \times \mathbf{T}^{\mathfrak{d}}(\lambda_1) & \text{if } t > \ell. \end{cases}$$

Definition 3.5 ([12, 13]). *A spinor model $\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)$ of shape $(\lambda,\ell) \in \mathscr{P}(G)$ is the set of $(T_\ell,\ldots,T_1) \in \widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda,\ell)$ such that each pair (T_{i+1},T_i) satisfies the admissibility condition (cf. [12, Definition 6.7], [13, Definition 3.4]) for $1 \le i \le \ell-1$.*

Theorem 3.6 ([12, Theorem 7.4], [13, Theorem 4.4]). *For $(\lambda,\ell) \in \mathscr{P}(G)$, the set $\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)$ is a $\mathfrak{g}_\infty$-crystal and is isomorphic to $B(\Pi^{\mathfrak{g}}(\lambda,\ell))$ as $\mathfrak{g}_\infty$-crystals.*

Remark 3.7. *In this extended abstract, it is sufficient to describe $B(\lambda)$ for $\lambda \in P^+_{\mathrm{int}}$ (not P^+), and we intentionally omit some related notions; $\mathbf{T}^{\mathrm{sp}}$ in particular. The skipped ones can be found in [12, 13], which cover whole extremal (highest) weight crystals $B(\lambda)$ for $\lambda \in P^+$.*

The character of $\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)$ is defined to be

$$\mathrm{ch}\,\mathbf{T}^{\mathfrak{g}}(\lambda,\ell) \;=\; t^{\ell} \sum_{(T_\ell,\ldots,T_1)\in\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)} \prod_{i=1}^{\ell} \mathbf{x}^{T_i}$$

where t is a formal symbol and $\mathbf{x}^T = \prod_{i=1}^{\infty} x_i^{m_i}$, with m_i being the number of appearances of $i \geq 1$ in a semistandard tableau T. Indeed, we understand $x_i = e^{\epsilon_i}$ and $t = e^{\Pi_0^{\mathfrak{g}}}$ when we consider them as elements in the group algebra $\mathbb{Z}[P]$. An explicit formula of the character of a spinor model will be explained in Section 4.

3.3 Kashiwara–Nakashima tableaux

For $n \in \mathbb{Z}_+$, let $\mathcal{I}_n^{\mathfrak{g}}$ be the following ordered sets.

$$\begin{aligned}
\mathcal{I}_n^{\mathfrak{b}} &= \{\,\overline{n} < \cdots < \overline{1} < 0 < 1 < \cdots < n\,\} \\
\mathcal{I}_n^{\mathfrak{c}} &= \{\,\overline{n} < \cdots < \overline{1} < 1 < \cdots < n\,\} \\
\mathcal{I}_n^{\mathfrak{d}} &= \left\{\,\overline{n} < \cdots < \overline{2} < \begin{matrix}\overline{1}\\ 1\end{matrix} < 2 < \cdots < n\,\right\}
\end{aligned}$$

Here, $\mathcal{I}_n^{\mathfrak{d}}$ is a partially ordered set, and $(1, \overline{1})$ is the unique non-comparable pair in $\mathcal{I}_n^{\mathfrak{d}}$.

Definition 3.8 ([9]). *The ($\mathfrak{g}_n$-type) KN tableau of shape $\lambda \in \mathcal{P}$ is an $\mathcal{I}_n^{\mathfrak{g}}$-semistandard tableau T of shape λ such that each column of T is admissible and adjacent columns of T do not have certain (a,b)-configurations (cf. [9]). We denote by $\mathbf{KN}_n^{\mathfrak{g}}(\lambda)$ the set of KN tableaux of shape λ. Note that the condition for a tableau to be $\mathcal{I}_n^{\mathfrak{g}}$-semistandard is similar as the usual one with some exceptions (cf. [3, 9, 15]).*

For $\lambda \in \mathcal{P}$, we easily check that $\mathbf{KN}_n^{\mathfrak{g}}(\lambda) \subseteq \mathbf{KN}_{n+1}^{\mathfrak{g}}(\lambda)$ for $n \geq 1$. As a role of KN tableaux corresponding to the infinite rank, Lecouvey [15] introduces a tableau model, which we call a $\mathfrak{g}_\infty$-type KN tableau.

Definition 3.9 ([15]). *For $\lambda \in \mathcal{P}$, define*

$$\mathbf{KN}^{\mathfrak{g}}(\lambda) \;=\; \bigcup_{n\geq\ell(\lambda)} \mathbf{KN}_n^{\mathfrak{g}}(\lambda)$$

where the union is over $n > \ell(\lambda)$ when $\mathfrak{g} = \mathfrak{d}$. It is the set of $\mathcal{I}^{\mathfrak{g}}$-semistandard tableaux of shape λ satisfying the same (a,b)-configuration conditions as those in Definition 3.8, where $\mathcal{I}^{\mathfrak{g}}$ is the following ordered set.

$$\begin{aligned}
\mathcal{I}^{\mathfrak{b}} &= \{\cdots < \overline{n} < \cdots < \overline{1} < 0 < 1 < \cdots < n < \ldots\} \\
\mathcal{I}^{\mathfrak{c}} &= \{\cdots < \overline{n} < \cdots < \overline{1} < 1 < \cdots < n < \ldots\} \\
\mathcal{I}^{\mathfrak{d}} &= \left\{\cdots < \overline{n} < \cdots < \overline{2} < \begin{matrix}\overline{1}\\ 1\end{matrix} < 2 < \cdots < n < \ldots\right\}
\end{aligned}$$

Here, a pair $(1,\overline{1})$ in $\mathcal{I}^{\mathfrak{d}}$ is the unique non-comparable pair in $\mathcal{I}^{\mathfrak{d}}$.

It is known that $\mathbf{KN}^{\mathfrak{g}}_n(\lambda)$ is a $\mathfrak{g}_n$-crystal and is isomorphic to $B(\varpi_\lambda)$ as $\mathfrak{g}_n$-crystals [9]. Then we can extend this $\mathfrak{g}_n$-crystal structure to $\mathbf{KN}^{\mathfrak{g}}(\lambda)$. Moreover, we show that these extended $\mathfrak{g}_n$-crystal structures on $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ ranging over $n \geq \ell(\lambda)$ are compatible. From this observation, we induce a $\mathfrak{g}_\infty$-crystal structure on $\mathbf{KN}^{\mathfrak{g}}(\lambda)$. One of the main results is that this $\mathfrak{g}_\infty$-crystal $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ is isomorphic to the extremal weight crystal $B(\varpi_\lambda)$.

Theorem 3.10 ([3, Theorem 4.11])**.** *For $\lambda \in \mathcal{P}$, there exists an isomorphism of $\mathfrak{g}_\infty$-crystals.*

$$\mathbf{KN}^{\mathfrak{g}}(\lambda) \;\cong\; B(\varpi_\lambda)$$

4 Jacobi–Trudi type character formulas

For $r \in \mathbb{N}$, let $e_r(\mathbf{x})$ be the r-th elementary symmetric function in $\mathbf{x} = \{x_1, x_2, \dots, \}$, and set $e_0(\mathbf{x}) = 1$ and $e_r(\mathbf{x}) = 0$ for $r < 0$. For $r \in \mathbb{Z}$, define

$$\begin{aligned} &E_r(\mathbf{x}) = \sum_{i=0}^{\infty} e_i(\mathbf{x}) e_{r+i}(\mathbf{x}), \\ &E'_r(\mathbf{x}) = E_r(\mathbf{x}) - E_{r+2}(\mathbf{x}), \qquad E''_r(\mathbf{x}) = E_r(\mathbf{x}) + E_{r+1}(\mathbf{x}). \end{aligned}$$

We can easily check that $E_r(\mathbf{x}) = E_{-r}(\mathbf{x})$ for $r \in \mathbb{Z}$. In addition, we easily derive the following identities using $E^{\diamondsuit}_r(\mathbf{x})$ ($\diamondsuit \in \{\,\cdot\,, {}', {}''\}$).

Proposition 4.1 ([3, Proposition 5.2])**.** *For $a \in \mathbb{Z}_+$ ($a \in \mathbb{N}$ when $\mathfrak{g} = \mathfrak{d}$), the following equalities hold.*

$$\begin{aligned} &\operatorname{ch} \mathbf{T}^{\mathfrak{c}}(a) \;=\; tE'_a(\mathbf{x}), \qquad \operatorname{ch} \mathbf{T}^{\mathfrak{b}}(a) \;=\; tE''_a(\mathbf{x}), \qquad \operatorname{ch} \mathbf{T}^{\mathfrak{d}}(a) \;=\; tE_a(\mathbf{x}), \\ &\operatorname{ch} \mathbf{T}^{\mathfrak{d}}(0) + \operatorname{ch} \overline{\mathbf{T}}^{\mathfrak{d}}(0) \;=\; tE_0(\mathbf{x}), \quad \operatorname{ch} \mathbf{T}^{\mathfrak{d}}(0) - \operatorname{ch} \overline{\mathbf{T}}^{\mathfrak{d}}(0) \;=\; t\left(\sum_{i=0}^{\infty} e_i(\mathbf{x})\right)\left(\sum_{i=0}^{\infty} (-1)^i e_i(\mathbf{x})\right) \end{aligned}$$

In general, we explicitly write the (Jacobi–Trudi type) character formula of a spinor model in terms of $E^{\diamondsuit}_r(\mathbf{x})$ ($\diamondsuit \in \{\,\cdot\,, {}', {}''\}$).

Definition 4.2 ([14])**.** *For $\diamondsuit \in \{\,\cdot\,, {}', {}''\}$ and $(\lambda, \ell) \in \mathscr{P}(G)$, denote*

$$\begin{aligned} \Sigma^{\diamondsuit}_{(\lambda,\ell)}(\mathbf{x}) &= \det\left(E^{\diamondsuit}_{(\lambda_{\ell-i+1}+i-1)+(j-1)}(\mathbf{x}) + \delta(j \neq 1) E^{\diamondsuit}_{(\lambda_{\ell-i+1}+i-1)-(j-1)}(\mathbf{x})\right)_{i,j=1,\dots,\ell} \\ &= \begin{vmatrix} E^{\diamondsuit}_{\lambda_\ell} & E^{\diamondsuit}_{\lambda_\ell+1} + E^{\diamondsuit}_{\lambda_\ell-1} & \cdots & E^{\diamondsuit}_{\lambda_\ell+(\ell-1)} + E^{\diamondsuit}_{\lambda_\ell-(\ell-1)} \\ E^{\diamondsuit}_{\lambda_{\ell-1}+1} & E^{\diamondsuit}_{(\lambda_{\ell-1}+1)+1} + E^{\diamondsuit}_{(\lambda_{\ell-1}+1)-1} & \cdots & E^{\diamondsuit}_{(\lambda_{\ell-1}+1)+(\ell-1)} + E^{\diamondsuit}_{(\lambda_{\ell-1}+1)-(\ell-1)} \\ \vdots & \vdots & \ddots & \vdots \\ E^{\diamondsuit}_{\lambda_1+\ell-1} & E^{\diamondsuit}_{(\lambda_1+\ell-1)+1} + E^{\diamondsuit}_{(\lambda_1+\ell-1)-1} & \cdots & E^{\diamondsuit}_{(\lambda_1+\ell-1)+(\ell-1)} + E^{\diamondsuit}_{(\lambda_1+\ell-1)-(\ell-1)} \end{vmatrix} \end{aligned}$$

where $\delta(P) = 0$ if a statement P is false and $\delta(P) = 1$ otherwise. Also, define $S^{\mathfrak{g}}_{(\lambda,\ell)}(\mathbf{x})$ by

$$
\begin{aligned}
S^{\mathfrak{c}}_{(\lambda,\ell)}(\mathbf{x}) &= \Sigma'_{(\lambda,\ell)}(\mathbf{x}) \\
S^{\mathfrak{b}}_{(\lambda,\ell)}(\mathbf{x}) &= \Sigma''_{(\lambda,\ell)}(\mathbf{x}) \\
S^{\mathfrak{d}}_{(\lambda,\ell)}(\mathbf{x}) &= \begin{cases} \Sigma_{(\lambda,\ell)}(\mathbf{x}) & \text{if } t = \ell, \\ \dfrac{1}{2}\Sigma_{(\lambda,\ell)}(\mathbf{x}) + \dfrac{1}{2}\left(\displaystyle\sum_{i=0}^{\infty} e_i(\mathbf{x})\right)\left(\displaystyle\sum_{i=0}^{\infty}(-1)^i e_i(\mathbf{x})\right)\Sigma'_{(\lambda,\ell-1)}(\mathbf{x}) & \text{if } t < \ell, \\ \dfrac{1}{2}\Sigma_{(\mu,\ell)}(\mathbf{x}) - \dfrac{1}{2}\left(\displaystyle\sum_{i=0}^{\infty} e_i(\mathbf{x})\right)\left(\displaystyle\sum_{i=0}^{\infty}(-1)^i e_i(\mathbf{x})\right)\Sigma'_{(\mu,\ell-1)}(\mathbf{x}) & \text{if } t > \ell, \end{cases}
\end{aligned}
$$

where $t = \ell(\lambda)$ and $\mu = (\lambda_1, \ldots, \lambda_{2\ell-t})$. Note that the pair (μ, ℓ) appearing when $t > \ell$ satisfies that $(\mu,\ell) \in \mathscr{P}(G)$ and $\ell(\mu) < \ell$.

Proposition 4.3 ([3, Proposition 5.4]). *For $(\lambda, \ell) \in \mathscr{P}(G)$, the following holds.*

$$\operatorname{ch} \mathbf{T}^{\mathfrak{g}}(\lambda,\ell) \;=\; t^{\ell} S^{\mathfrak{g}}_{(\lambda,\ell)}(\mathbf{x})$$

5 The Grothendieck ring

Let $\mathcal{C}$ be the category of $\mathfrak{g}_\infty$-crystals whose object B has connected components isomorphic to $B(\lambda^0) \otimes B(\lambda^+)$ for some $\lambda^0 \in E$ and $\lambda^+ \in P^+_{\text{int}}$ with some finiteness conditions (see [3, Section 6.1]). We show that $\mathcal{C}$ is a monoidal category under the tensor product of crystals [3, Theorem 6.1]. Let $\mathcal{K} = \mathcal{K}(\mathcal{C})$ be the Grothendieck group of $\mathcal{C}$, i.e., the additive group of isomorphism classes $[B]$ for $B \in \mathcal{C}$. Define a multiplication on $\mathcal{K}$ by

$$[B] \cdot [B'] = [B \otimes B'].$$

Then we can show $\mathcal{K}$ forms an associative $\mathbb{Z}$-algebra. Note that we can find corresponding results for type A in [11].

We explain an algebra structure of $\mathcal{K}$ using the decomposition of tensor products of underlying crystals into connected components (cf. [16, Section 4]). Let $\mathcal{K}^0$ and $\mathcal{K}^+$ be the subgroups of $\mathcal{K}$ generated by $[B(\lambda)]$ for $\lambda \in E$ and $\lambda \in P^+_{\text{int}}$, respectively. It is clear $\mathcal{K} \subseteq \mathcal{K}^0 \otimes \mathcal{K}^+$ by definition. Conversely, for given $\lambda^0 \in E$ and $\lambda^+ \in P^+$, we have $B(\lambda^0) \otimes B(\lambda^+) \cong B(\lambda)$ for $\lambda \in P$ (cf. [16, Theorem 4.4]). Thus, we have $\mathcal{K} = \mathcal{K}^0 \otimes \mathcal{K}^+$.

To explain an algebra structure of $\mathcal{K}^0$, we consider the following decomposition.

Proposition 5.1 ([15]). *For $\lambda, \mu \in E$, we have*

$$B(\lambda) \otimes B(\mu) \;\cong\; \bigoplus_{\nu \in E} B(\nu)^{\oplus \mathtt{LR}^{\nu^\dagger}_{\lambda^\dagger \mu^\dagger}},$$

where $\mathtt{LR}^{\nu^\dagger}_{\lambda^\dagger \mu^\dagger}$ is the Littlewood-Richardson coefficient for partitions $\lambda^\dagger$, $\mu^\dagger$, and $\nu^\dagger$.

As a corollary, we know that $\mathcal{K}^0$ is a subalgebra of $\mathcal{K}$ and obtain an algebra isomorphism between $\mathcal{K}^0$ and the ring Sym of symmetric functions since their structure constants coincide.

Proposition 5.2 ([3, Proposition 6.5]). *There exists an algebra isomorphism*

$$\Psi^0 \,:\, \mathcal{K}^0 \longrightarrow \mathrm{Sym} \tag{5.1}$$

which, for $\lambda \in \mathcal{P}$, sends $[B(\varpi_\lambda)]$ to s_λ.

On the other hand, for $k \geq 0$ and $(\lambda, \ell) \in \mathscr{P}(G)$, put

$$H_k^{\mathfrak{g}} = [B(\Pi_k^{\mathfrak{g}})], \qquad H^{\mathfrak{g}}(\lambda, \ell) = [B(\Pi^{\mathfrak{g}}(\lambda, \ell))]$$

and $\overline{H}^{\mathfrak{d}}(0) = [B(\overline{\Pi}_0^{\mathfrak{d}})]$. By the semisimplicity result in [12, 13], we deduce that $[B] = [B']$ in $\mathcal{K}^+$ if and only if $\mathrm{ch}(B) = \mathrm{ch}(B')$. Thus, we can rewrite Proposition 4.3 as follows.

Proposition 5.3 ([3, Proposition 6.2]). *When $\mathfrak{g} = \mathfrak{d}$ and $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) < \ell$, the following identity holds in $\mathcal{K}^+$.*

$$\begin{aligned} H^{\mathfrak{d}}(\lambda, \ell) &= \frac{1}{2}\det(H^{\mathfrak{d}}_{(\lambda_{\ell-i+1}+i-1)+(j-1)} + \delta(j \neq 1) H^{\mathfrak{d}}_{(\lambda_{\ell-i+1}+i-1)-(j-1)})_{i,j=1,\ldots,\ell} \\ &\quad + \frac{1}{2}(H_0^{\mathfrak{d}} - \overline{H}_0^{\mathfrak{d}}) H^{\mathfrak{c}}(\lambda, \ell - 1) \end{aligned}$$

When $\mathfrak{g} = \mathfrak{d}$ and $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) > \ell$, the following identity holds in $\mathcal{K}^+$.

$$\begin{aligned} H^{\mathfrak{d}}(\lambda, \ell) &= \frac{1}{2}\det(H^{\mathfrak{d}}_{(\mu_{\ell-i+1}+i-1)+(j-1)} + \delta(j \neq 1) H^{\mathfrak{d}}_{(\mu_{\ell-i+1}+i-1)-(j-1)})_{i,j=1,\ldots,\ell} \\ &\quad - \frac{1}{2}(H_0^{\mathfrak{d}} - \overline{H}_0^{\mathfrak{d}}) H^{\mathfrak{c}}(\mu, \ell - 1), \end{aligned}$$

Here, $t = \ell(\lambda)$ and $\mu = (\lambda_1, \ldots, \lambda_{2\ell-t}, 0^{t-\ell})$. Otherwise, the following identity holds in $\mathcal{K}^+$.

$$H^{\mathfrak{g}}(\lambda, \ell) = \det(H^{\mathfrak{g}}_{(\lambda_{\ell-i+1}+i-1)+(j-1)} + \delta(j \neq 1) H^{\mathfrak{g}}_{(\lambda_{\ell-i+1}+i-1)-(j-1)})_{i,j\in[\ell]}$$

As a corollary, we know that $\mathcal{K}^+$ is a subalgebra of $\mathcal{K}$. Even though above two cases seem to contain different variables coming from $H^{\mathfrak{c}}$ (not $H^{\mathfrak{d}}$), the identity $E'_r = E_r - E_{r+2}$ implies that $H_k^{\mathfrak{c}}$ is a polynomial in $\{H_i^{\mathfrak{d}}\}$.

Let $\mathbf{h} = \{\mathtt{h}_k \,|\, k \in \mathbb{Z}_+\}$ ($\{\mathtt{h}_k \,|\, k \in \mathbb{Z}_+\} \cup \{\overline{\mathtt{h}}_0\}$ when $\mathfrak{g}_\infty = \mathfrak{d}_\infty$) be commuting formal variables, and $\mathbb{Z}[\![\mathbf{h}]\!]$ be the set of formal power series in $\mathbf{h}$. Using Proposition 5.3 and [3, Lemma 6.3], we can construct an isomorphism between $\mathcal{K}^+$ and $\mathbb{Z}[\![\mathbf{h}]\!]$.

Theorem 5.4 ([3, Theorem 6.4]). *Define $\Phi^+ : \mathbb{Z}[\![\mathbf{h}]\!] \longrightarrow \mathcal{K}^+$ by a $\mathbb{Z}$-algebra homomorphism sending $\mathtt{h}_k$ to $H_k^{\mathfrak{g}}$ (and $\overline{\mathtt{h}}_0$ to $\overline{H}_0^{\mathfrak{d}}$ when $\mathfrak{g} = \mathfrak{d}$). Then Φ^+ is an isomorphism of $\mathbb{Z}$-algebras.*

Finally, we can explicitly describe an algebra structure of $\mathcal{K}$. Based on the above results, we know that $\{[B(\varpi_i)]\,|\,i \geq 1\} \cup \{[B(\Pi^{\mathfrak{g}}_j)]\,|\,j \geq 0\}$ generates $\mathcal{K}$ as a $\mathbb{Z}$-algebra and hence it is sufficient to find a basis expansion of $[B(\Pi^{\mathfrak{g}}_a)] \cdot [B(\varpi_b)]$. A general result for the basis expansion of $[B(\Pi^{\mathfrak{g}}(\lambda, \ell))] \cdot [B(\varpi_\mu)]$ for $(\lambda, \ell) \in \mathscr{P}(G)$ and $\mu \in \mathcal{P}$ is given in [16, Section 4.3]. In particular, we obtain the basis expansion of $[B(\Pi^{\mathfrak{g}}_a)] \cdot [B(\varpi_b)]$ by applying the general result (see [3, Proposition 6.6]).

To characterize the algebra structure of $\mathcal{K}$, we introduce a set $\mathbf{z} = \{\mathtt{z}_k\,|\,k \in \mathbb{N}\}$ of (other) commuting formal variables. Define $\mathcal{A}_0 = \mathbb{Z}[\![\mathbf{h}]\!]$, $\mathcal{A}_n = \mathcal{A}_0[\mathtt{z}_1, \ldots, \mathtt{z}_n]$ for $n \in \mathbb{N}$, and $\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n$. We inductively define a $\mathbb{Z}$-algebra structure on $\mathcal{A}$ as follows.

- The multiplication on $\mathcal{A}_0$ is the usual multiplication.
- Suppose that the multiplication on $\mathcal{A}_{n-1}$ is defined. Define $a\mathtt{z}_n = \mathtt{z}_n a + \delta_n(a)$ for $a \in \mathcal{A}_{n-1}$, where δ_n is a derivation on $\mathcal{A}_{n-1}$ such that

$$\mathfrak{c}_\infty \qquad \begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leq k \leq n-1) \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{\min\{a, n-i\}} \mathtt{z}_i \mathtt{h}_{a+n-i-2j} & (a \in \mathbb{Z}_+) \end{cases}$$

$$\mathfrak{b}_\infty \qquad \begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leq k \leq n-1) \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \mathtt{z}_i \left(\sum_{j=0}^{\min\{a, b-i\}} \mathtt{h}_{a+b-i-2j} + \delta(b-i > a) \sum_{k=1}^{b-i-a} \mathtt{h}_{b-i-a-k} \right) & (a \in \mathbb{Z}_+) \end{cases}$$

$$\mathfrak{d}_\infty \qquad \begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leq k \leq n-1) \\ \delta_n(\mathtt{h}_0) = \displaystyle\sum_{i=0}^{n-1} \bigoplus_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \mathtt{z}_i \mathtt{h}_{b-i-2j}, \quad \delta_n(\overline{\mathtt{h}}_0) = \sum_{i=0}^{n-1} \bigoplus_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \mathtt{z}_i \overline{\mathtt{h}}_{b-i-2j} \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \mathtt{z}_i \left(\sum_{j=0}^{\min\{\lfloor \frac{a+b-i}{2} \rfloor, b-i\}} \mathtt{h}_{a+b-i-2j} + \delta(b-i \geq a) \sum_{k=0}^{\lfloor \frac{b-i-a}{2} \rfloor} \overline{\mathtt{h}}_{b-i-a-2k} \right) \ (a \in \mathbb{N}) \end{cases}$$

where $\overline{\mathtt{h}}_a = \mathtt{h}_a$ for $a \geq 1$. Now, we obtain an algebra isomorphism between $\mathcal{K}$ and $\mathcal{A}$.

Theorem 5.5 ([3, Theorem 6.8])**.** *The assignment sending $[B(\Pi^{\mathfrak{g}}_a)]$ to $\mathtt{h}_a$ ($[B(\overline{\Pi}^{\mathfrak{d}}_0)]$ to $\overline{\mathtt{h}}_0$) and $[B(\varpi_b)]$ to $\mathtt{z}_b$ defines a $\mathbb{Z}$-algebra isomorphism $\Psi : \mathcal{K} \to \mathcal{A}$. Indeed, we have $\Psi = \Psi^0 \otimes \Psi^+$, where Ψ^+ is the inverse map of Φ^+ given in Theorem 5.4.*

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Lascoux polynomials and Gelfand–Zetlin patterns

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Abstract. We give a new combinatorial description for Lascoux polynomials and for symmetric Grothendieck polynomials in terms of cellular decompositions of Gelfand–Zetlin polytopes. This generalizes a similar result on key polynomials by Kiritchenko, Smirnov, and Timorin.

Keywords: Lascoux polynomials, key polynomials, Gelfand–Zetlin polytopes

1 Introduction

In this paper, we provide a new combinatorial description of Lascoux polynomials in terms of subdivisions of Gelfand–Zetlin polytopes and certain collections of their faces. Lascoux polynomials, denoted by $\mathscr{L}_\alpha^{(\beta)}$, form a basis for $\mathbb{Z}[\beta][x_1, x_2, \ldots]$, where α runs over the set of weak compositions (i.e., infinite sequences of nonnegative integers with finitely many positive entries). They simultaneously generalize key polynomials and Grassmannian Grothendieck polynomials; the latter family represents classes of structure sheaves of Schubert varieties in the connective K-theory of a Grassmannian, as shown by A. Buch [2]. Both of these families are superfamilies of Schur polynomials.

Lascoux polynomials were defined by A. Lascoux [6] in terms of homogeneous divided difference operators; just as many other families of polynomials defined using these operators, they have nonnegative coefficients. Although Lascoux polynomials do not have a description in geometric or representation-theoretic terms, they admit several combinatorial descriptions: for example, T. Yu [9] provides a description of Lascoux polynomials in terms of set-valued tableaux, generalizing simultaneously Buch's description of symmetric Grothendieck polynomials in terms of set-valued Young tableaux and A. Lascoux and M.-P. Schützenberger's tableau formula for key polynomials ([7]).

Lascoux polynomials $\mathscr{L}_\alpha^{(\beta)}$ specialized at $\beta = 0$ are equal to key polynomials. Suppose $w \in S_n$ is a permutation such that $\alpha = (\alpha_1, \ldots, \alpha_n) = w(\lambda)$ for a suitable partition

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$\lambda = (\lambda_1, \dots, \lambda_n)$. The key polynomials $\kappa_\alpha = \kappa_{w,\lambda}$ are defined as the characters of Demazure modules $D_{w,\lambda}$, i.e. B-submodules in the irreducible $\mathrm{GL}(n)$-representation V_λ with the highest weight λ. The module $D_{w,\lambda}$ is defined as the smallest B-submodule containing the extremal vector $wv_\lambda \in V_\lambda$, where $B \subset \mathrm{GL}(n)$ is a fixed Borel subgroup. A character formula for Demazure modules was stated in [3] and proved by H. H. Andersen in [1] (the original proof by M. Demazure contained a gap). The first combinatorial description of these characters was given in [7].

In [5], V. Kiritchenko, E. Smirnov, and V. Timorin provide a formula for key polynomials in terms of integer points in Gelfand–Zetlin polytopes. Let λ be a strictly dominant weight for $\mathrm{GL}(n)$; then it defines an integer convex polytope $GZ(\lambda) \subset \mathbb{R}^{n(n-1)/2}$, called the Gelfand–Zetlin polytope. This polytope admits a projection $\pi\colon GZ(\lambda) \to \mathrm{wt}(\lambda)$ into the weight polytope of V_λ. For each permutation $w \in S_n$, one can construct a collection of faces $F_{w,\lambda}$ of $GZ(\lambda)$, such that $\kappa_{w,\lambda} = \sum \exp(\pi(z))$, where z ranges over the set of integer points in $F_{w,\lambda}$ (see [5, Corollary 5.2]).

The main purpose of this paper is to generalize this result, constructing a combinatorial description of symmetric Grothendieck and Lascoux polynomials in terms of subdivisions of Gelfand–Zetlin polytopes. For this we construct a cellular decomposition $\mathcal{C}$ of $GZ(\lambda)$ whose 0-cells coincide with the integer points in $GZ(\lambda)$. Now, to each i-dimensional cell C_i we assign a monomial $m(C_i)$ in $x_1, \dots, x_n$; for a 0-cell $z \in GZ(\lambda)$ we have $m(C_i) = \exp(\pi(z))$. Some cells correspond to the zero monomial. Our main result is as follows:

$$\mathscr{L}^{(\beta)}_{w,\lambda} = \sum_{C_i \in \mathcal{C} \cap F_{w,\lambda}} \beta^i m(C_i),$$

where the sum is taken over all cells situated inside the collection of faces $F_{w,\lambda}$.

Informally, the Lascoux polynomial $\mathscr{L}^{(\beta)}_{w,\lambda}$ can be viewed as a "weighted Euler characteristic" of the subdivision $\mathcal{C} \cap F_{w,\lambda}$ for the collection of faces $F_{w,\lambda}$. Namely, i-dimensional cells of this subdivision correspond to monomials of degree $i + \ell(w)$ with coefficient β^i in front of them.

It would be very interesting to establish a bijection of our construction of cells indexing monomials in Lascoux polynomials with T. Yu's description in terms of set-valued tableaux. In particular, we expect the crystal operations on set-valued tableaux (see [9]) to have a nice description in terms of Gelfand–Zetlin polynomials. However, we do not address these questions in this paper, leaving them as a subject of subsequent work.

An extended exposition of the results presented in this note, with all proofs and further discussion, can be found in [8].

2 Preliminaries

2.1 Lascoux polynomials

To define Lascoux polynomials, we need two families of operators: *divided difference operators* ∂_i, with $1 \leq i \leq n-1$, acting on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ and *Demazure–Lascoux operators* $\pi_i^{(\beta)}$, again with $1 \leq i \leq n-1$, acting on the ring $\mathbb{Z}[\beta, x_1, \dots, x_n]$ equipped with a formal parameter β.

Definition 2.1. The i-th *divided difference operator* ∂_i acts on polynomial $f = f(x_1, x_2, \dots)$ in the following way:

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}},$$

where $s_i f$ is obtained from f by permuting variables x_i and x_{i+1}.

We consider operators $\pi_i^{(\beta)}$, that are modifications of divided differences operators.

Definition 2.2. *The i^{th} Demazure–Lascoux operator* $\pi_i^{(\beta)}$ acts on $f \in \mathbb{Z}[\beta][x_1, x_2, \dots]$ in the following way:

$$\pi_i^{(\beta)}(f) = \partial_i(x_i f + \beta x_i x_{i+1} f).$$

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be an infinite sequence of nonnegative integers with finitely many positive entries.

Definition 2.3. *The Lascoux polynomial* $\mathscr{L}_\alpha^{(\beta)} \in \mathbb{Z}[\beta][x_1, x_2, \dots]$ *associated with* α is defined by:

$$\mathscr{L}_\alpha^{(\beta)} = \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition: } \alpha_1 \geq \alpha_2 \geq \dots \\ \pi_i^{(\beta)}(\mathscr{L}_{s_i\alpha}^{(\beta)}) & \text{otherwise, where } \alpha_i < \alpha_{i+1} \end{cases}$$

Since the Demazure–Lascoux operators satisfy the braid relations, we can associate a Lascoux polynomial to partition λ and permutation $w \in S_n$ in the following way:

$$\mathscr{L}_{w,\lambda}^{(\beta)} = \pi_{i_k}^{(\beta)} \dots \pi_{i_2}^{(\beta)} \pi_{i_1}^{(\beta)}(x^\lambda),$$

where $(s_{i_k}, \dots, s_{i_1})$ is a reduced word for permutation $w = s_{i_1} \dots s_{i_k}$.

It is well-known (cf., for instance, [9]) that specializations of Lascoux polynomials provide other nice families of polynomials. Namely, taking $\beta = 0$ gives key polynomials $\kappa_{w,\lambda} = \mathscr{L}_{w,\lambda}^{(\beta)} |_{\beta=0}$. If we take the Lascoux polynomial of the longest permutation, we get a symmetric Grothendieck polynomial $G_\lambda^{(\beta)} = \mathscr{L}_{w_0,\lambda}^{(\beta)} = \pi_{w_0}^{(\beta)}(x^\lambda)$. Finally, taking these two specializations simultaneously gives us Schur polynomials: $S_\lambda = \kappa_{w_0,\lambda} = \pi_{w_0}^{(\beta)}(x^\lambda)|_{\beta=0}$.

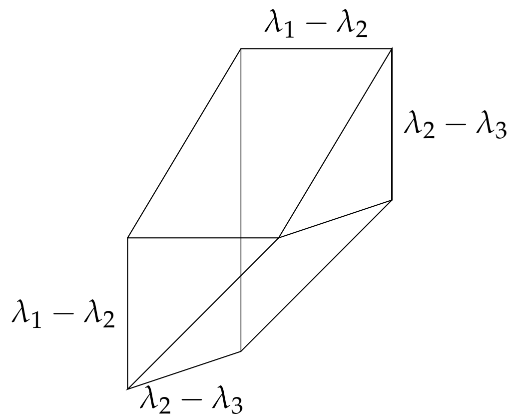


Figure 1: Gelfand–Zetlin polytope

2.2 Gelfand–Zetlin patterns

Let λ be a partition, i.e. a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Consider the space $\mathbb{R}^d$, where $d = \frac{n(n-1)}{2}$, with coordinates y_{ij} indexed by pairs (i, j) of positive integers satisfying $i + j \leq n$. The following triangular tableau

$$\begin{array}{ccccccccc}
\lambda_n & & \lambda_{n-1} & & \lambda_{n-2} & & \ldots & & \lambda_1 \\
& y_{11} & & y_{12} & & \ldots & & y_{1,n-1} & \\
& & y_{21} & & \ldots & & y_{2,n-2} & & \\
& & & \ddots & \vdots & \iddots & & & \\
& & & & y_{n-1,1} & & & &
\end{array} \tag{2.1}$$

is called a *Gelfand–Zetlin pattern*, if all y_{ij} are integers, and every small triangle in this tableau satisfies inequalities $y_{i-1,j} \leq y_{i,j} \leq y_{i-1,j+1}$. Here we formally set $y_{0j} = \lambda_{n+1-j}$.

Gelfand–Zetlin patterns parametrize elements of the *Gelfand–Zetlin basis* in the $\mathrm{GL}(n)$-module $V(\lambda)$ with highest weight λ (see [4]). The number of such patterns for a fixed top row λ can be computed using Weyl's dimension formula:

$$\dim V(\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j - i + j}{j - i}.$$

2.3 Gelfand–Zetlin polytopes

Gelfand–Zetlin patterns can be viewed as integer points in $\mathbb{R}^{\frac{n(n-1)}{2}}$. The convex hull of these points is called a *Gelfand–Zetlin polytope* and denoted $GZ(\lambda)$. It is easy to see that the set of integer points in $GZ(\lambda)$ gives us exactly the set of Gelfand–Zetlin patterns.

Example 2.4. For $n = 3$, the Gelfand–Zetlin polytope $GZ(\lambda)$ is defined in $\mathbb{R}^3$ by the following inequalities: $\lambda_3 \le x \le \lambda_2$, $\lambda_2 \le y \le \lambda_1$, $x \le z \le y$. If all λ_i are distinct, it is three-dimensional, as shown on Fig. 1.

3 Enhanced Gelfand–Zetlin patterns

3.1 Construction of enhanced Gelfand–Zetlin patterns

In this section we define *enhanced Gelfand–Zetlin patterns*, i.e. Gelfand–Zetlin patterns with some additional data, which we will call *enhancement*. These data are of two kinds: first, some elements in a pattern may be encircled, and second, some pairs of neighbor elements in consecutive rows can be joined by an edge.

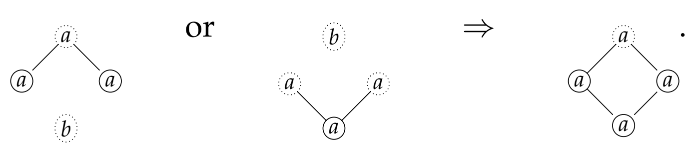
Informally, the pattern without enhancement stands for the "maximal" point of the closure of the corresponding cell, i.e. the point with the largest sum of coordinates.

Definition 3.1. A Gelfand–Zetlin pattern with the top row $(\lambda_n, \dots, \lambda_1)$ with some entries marked by circles and with edges between certain neighboring entries is said to be an *enhanced Gelfand–Zetlin patterns*, if these elements satisfy the following conditions:

1. The numbers in the first row are encircled.

2. The two entries joined by an edge must be equal, and the bottom entry should be encircled. The converse does not have to be true: two equal neighboring entries are not necessarily joined by an edge.

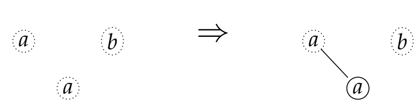
3. If two neighboring entries in a row are joined by edges with an entry above them, they must also be joined with the entry below them, and vice versa. Pictorially:

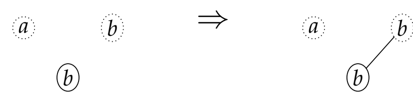
   (a dotted circle around an entry means that it may be either encircled or not).

4. If two entries in the topmost row are equal, then the entry below them (which is equal to both of them) is encircled and connected to both of them by edges.

5. If $a < b$ and the pattern contains the following triangle: $\begin{smallmatrix} a & & b \\ & a & \end{smallmatrix}$, then there is an edge between the two a's. Pictorially:

6. If $a < b$ and the pattern contains the following triangle: $\begin{smallmatrix} a & & b \\ & b & \end{smallmatrix}$ with the bottom entry encircled, then there is an edge between the two b's:

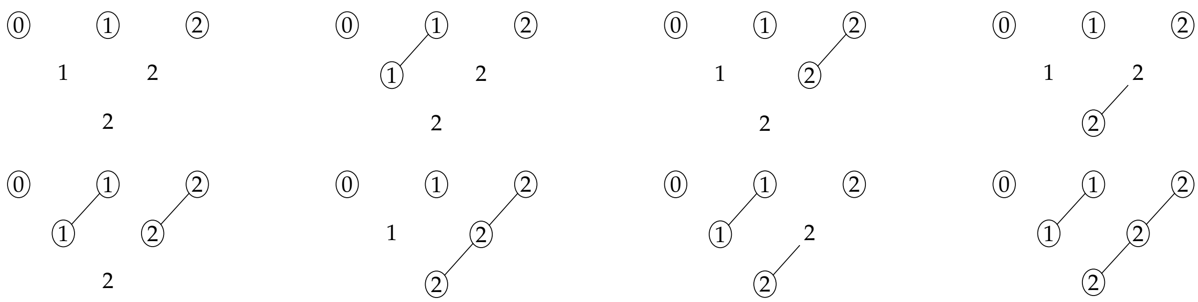


7. For a triangle $\begin{smallmatrix} a & & a \\ & a & \end{smallmatrix}$: if the two top entries can be connected by a path of edges, the bottom entry should be encircled and connected with them.

8. If in a triangle $\begin{smallmatrix} a & & a \\ & a & \end{smallmatrix}$ the bottom entry is encircled, then it should be connected with at least one of them by an edge.

We denote the set of all enhanced patterns with the first row λ by $\mathcal{P}(\lambda)$.

Example 3.2. The pattern $\begin{smallmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \\ & & 2 & & \end{smallmatrix}$ has eight enhancements.



Example 3.3. The pattern $\begin{smallmatrix} 0 & & 1 & & 2 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}$ has four enhancements.

Note that according to Definition 3.1 (4), the last entry in the second row must be encircled and connected to the middle entry in the first row.

An enhanced pattern can be viewed as a graph (with marked vertices). Consider the connected components of this graph.

Lemma 3.4. *The connected components of an enhanced Gelfand–Zetlin pattern satisfy the following:*

1. *the entries in the topmost row belong to the same connected component if and only if they are equal;*

2. *each connected component either has a unique highest vertex or contains one or more entries from the topmost row;*

3. *all vertices in a connected component, possibly except the highest one, are encircled. In particular, the number of connected components is not less than the number of distinct λ_i's plus the number of entries without circles.*

Proof. This follows immediately from Definition 3.1. □

Definition 3.5. The *rank* $\operatorname{rk} P$ of an enhanced pattern P is the number of entries without circles.

Now introduce the notion of a *reduced enhanced pattern*. Let us index the positions that may contain a NE-SW edge by simple reflections from S_n as shown on Fig. 2. On each NE-SW edge joining $y_{i,j}$ with $y_{i-1,j+1}$ in our pattern we write the corresponding simple reflection if the entries joined by this edge are equal to $y_{0,i+j} = \lambda_{n+1-i-j}$ (that is, are maximal possible on this diagonal). Then take the word formed by the letters on the edges read from bottom to top, from right to left. If this word is reduced, then the corresponding pattern P is said to be reduced. Then denote the product by w^-. Given a reduced pattern P, define the permutation corresponding to P as $w(P) = w_0 w^-$.

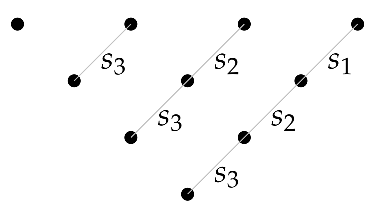


Figure 2: Assigning permutation to an enhanced pattern

Example 3.6. All enhanced patterns from Example 3.2 except the seventh one are reduced. They correspond to the following permutations: $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$, $s_2 s_1$, $s_1 s_2$, $s_2 s_1$, s_1, s_2, Id.

Finally, given a permutation $w \in W$, denote by $\mathcal{P}(w, \lambda)$ the set of all reduced enhanced patterns P from $\mathcal{P}(\lambda)$ such that $w(P) \leq w$ in the Bruhat order. We will use this set of patterns later in Theorem 4.3.

3.2 Efficient enhanced patterns

Definition 3.7. A enhanced pattern P is said to be *inefficient* if it contains a triangle of the form $^{a}\,{}_{a}\,^{a}$ such that its bottom entry is not connected with the right one by an edge, and *efficient* otherwise. The set of all efficient reduced enhanced patterns with the first row λ is denoted by $\mathcal{P}^+(\lambda)$. Like in the previous subsection, for a fixed $w \in S_n$ denote by $\mathcal{P}^+(w, \lambda)$ the set of all efficient reduced enhanced patterns P satisfying $w(P) \leq w$ in the Bruhat order.

Proposition 3.8. *Every enhanced pattern of rank zero is efficient.*

Proof. Take an inefficient enhanced pattern P. This means that it contains a triangle of the form ${}^{a}{}_{a}{}^{a}$ such that there is no edge between the bottom and the right entries. Definition 3.1 implies that these two entries are contained in different connected components, both marked with the same number a. This means that at least one of these components contains a vertex without circle, so the rank of P cannot be zero. □

Moreover, it turns out that for an efficient enhanced pattern, the edges provide redundant data. Namely, we have the following lemma.

Lemma 3.9. *The edges in an efficient enhanced pattern are uniquely determined by positions of encircled vertices.*

Proof. The conditions listed in Definition 3.1 imply that positions of edges are defined by positions of encircled vertices in all cases except for case (8). In the latter case there are two possibilities of joining the bottom vertex in the triangle ${}^{a}{}_{a}{}^{a}$ with one of its neighbors in the upper row, and only one of them defines an efficient pattern. □

For a reduced efficient enhanced GZ-pattern P, we assign to it a monomial x^P in the following way. Let $S_i(P)$ be the sum of numbers in the i-th row of the pattern P, with $S_0(P) = \lambda_1 + \cdots + \lambda_n$, and let $D_i(P)$ stand for the number of entries without circles in the i-th row of P. Denote $d_{n+1-i} = d_{n+1-i}(P) = S_{i-1}(P) - S_i(P) + D_i(P)$. Then

$$x^P = \beta^{\operatorname{rk} P} x_1^{d_1} \dots x_n^{d_n}.$$

Example 3.10. All enhanced GZ-patterns patterns in Example 3.2 are efficient; the corresponding monomials are

$$\beta^3 x_1^2 x_2^2 x_3^2, \quad \beta^2 x_1^2 x_2^2 x_3, \quad \beta^2 x_1^2 x_2^2 x_3, \quad \beta^2 x_1^2 x_2 x_3^2, \quad \beta x_1^2 x_2^2, \quad \beta x_1^2 x_2 x_3, \quad \beta x_1^2 x_2 x_3, \quad x_1^2 x_2.$$

In Example 3.3, the first two patterns are inefficient, and the second two correspond to $\beta x_1 x_2 x_3^2$ and $x_1 x_2 x_3$, respectively.

4 Main results

In this section we give the main results of this paper. We start with constructing a cellular decomposition for $GZ(\lambda)$. The cells are indexed by enhanced Gelfand–Zetlin patterns, and the set of 0-dimensional cells is exactly the set of integer points in $GZ(\lambda)$. The second main result is as follows: Lascoux polynomial $\mathscr{L}_{w,\lambda}^{(\beta)}$ is equal to the sum of monomials corresponding to all efficient reduced enhanced patterns $P \in \mathcal{P}^+(\lambda)$ such that $w(P) \leq w$ in the Bruhat order.

4.1 Cellular decomposition of Gelfand–Zetlin polytopes

Let $GZ(\lambda) \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ be a Gelfand–Zetlin polytope. In this section we construct its cellular decomposition, with cells indexed by enhanced Gelfand–Zetlin patterns.

Construction 4.1. Let P be an enhanced pattern with entries a_{ij}, and let $y \in GZ(\lambda)$ be a point with coordinates y_{ij}. To each coordinate y_{ij} we assign an equality or a double inequality as follows:

1. if there is an edge going up from a_{ij} to $a_{i-1,j}$ or $a_{i-1,j+1}$ (or both), then $y_{ij} = y_{i-1,j}$ or $y_{ij} = y_{i-1,j+1}$, respectively;
2. if there are no edges going up from a_{ij}, and this entry is encircled, then $y_{ij} = a_{ij}$;
3. if there are no edges going up from a_{ij} and this entry is not encircled, we impose a double inequality on y_{ij} as follows:
   (a) If the entry $a_{i-1,j}$ satisfies $a_{ij} - a_{i-1,j} \geq 2$, then $a_{ij} - 1 < y_{ij}$; otherwise, $y_{i-1,j} < y_{ij}$;
   (b) If $a_{i-1,j+1}$ is equal to a_{ij}, we set $y_{ij} < y_{i-1,j+1}$; otherwise, $y_{ij} < a_{ij}$.

Denote the set defined by these equalities and inequalities by $\widehat{C_P}$. This is "almost" the required cell corresponding to P; however, it does not necessarily lie in $GZ(\lambda)$. To get an actual cell, take the affine span L of $\widehat{C_P}$ and intersect $\widehat{C_P}$ with the relative interior of $GZ(\lambda) \cap L$ in L:

$$C_P = \widehat{C_P} \cap (GZ(\lambda) \cap L)^0.$$

This set is convex and open in L.

Informally, the relation between an enhanced pattern P and the corresponding set C_P is as follows. For each connected component in P containing only encircled entries with the same numbers, all the corresponding coordinates of points in C_P are equal to this number. On the other hand, if a connected component has a non-encircled vertex, the corresponding coordinate can take values in an interval determined by the condition (4) of Definition 3.1; note that the length of this interval does not exceed $i - 1$, where i is the row number. All the remaining coordinates in the same connected component (corresponding to encircled entries) are equal to this coordinate.

The first main result of this paper states that this is indeed a cellular decomposition of $GZ(\lambda)$.

Theorem 4.2. *For each $P \in \mathcal{P}(\lambda)$, the set $C_P \subset GZ(\lambda)$ is homeomorphic to an open ball of dimension* $\operatorname{rk} P$. *These balls C_P form a cellular decomposition of $GZ(\lambda)$ whose zero-dimensional cells coincide with $GZ(\lambda) \cap \mathbb{Z}^{\frac{n(n-1)}{2}}$.*

Moreover, this cellular decomposition is compatible with the Bruhat order on S_n. Namely, in [5] the authors define a collection of special faces (called *dual Kogan faces*) of $GZ(\lambda)$ for each $w \in S_n$. The following result holds.

Theorem 4.3. *Let F_w be the set of dual Kogan faces of $GZ(\lambda)$ corresponding to w in the sense of [5, Theorem 4.3]. Then we have*

$$F_w = \bigcup_{P \in \mathcal{P}(w,\lambda)} \overline{C}_P.$$

4.2 Lascoux polynomials as sums over efficient enhanced patterns

The following is the second main result of this paper.

Theorem 4.4. *Let $w \in S_n$ be a permutation and λ be a partition. Then the Lascoux polynomial $\mathscr{L}^{(\beta)}_{w,\lambda}$ is equal to*

$$\mathscr{L}^{(\beta)}_{w,\lambda} = \sum_{P \in \mathcal{P}^+(w,\lambda)} x^P.$$

The sum is taken over all efficient reduced enhanced patterns P with $w(P) \leq w$. In the case $w = w_0$ we get an expression for the symmetric Grothendieck polynomial:

Corollary 4.5. *Let λ be a partition. Then the symmetric Grothendieck polynomial $G^{(\beta)}_\lambda(x_1, \ldots, x_n)$ is equal to*

$$G^{(\beta)}_\lambda(x_1, \ldots, x_n) = \mathscr{L}^{(\beta)}_{w_0,\lambda} = \sum_{P \in \mathcal{P}^+(\lambda)} x^P.$$

The specialization of the equality from Theorem 4.4 gives an expression for key polynomials, obtained in [5]:

Theorem 4.6 ([5, Theorem 5.1])**.** *Let $w \in S_n$ be a permutation and λ be a partition. Then the key polynomial $\kappa_{w,\lambda}$ is equal to*

$$\kappa_{w,\lambda} = \sum_{P \in \mathcal{P}^+(w,\lambda)} x^P,$$

where the sum is taken over efficient reduced enhanced patterns P of rank 0.

Another immediate corollary from Theorem 4.4, to the best of our knowledge, did not appear in the literature before.

Corollary 4.7. *Let λ be a partition and $u, w \in S_n$ be permutations such that $u \leq w$ in the Bruhat order on S_n. Then the polynomial $\mathscr{L}^{(\beta)}_{w,\lambda} - \mathscr{L}^{(\beta)}_{u,\lambda}$ has nonnegative coefficients.*

4.3 Example: $GZ(3,2,0)$

Let $\lambda = (3,2,0)$. The Gelfand–Zetlin polytope $GZ(\lambda)$ with its cellular decomposition defined in Theorem 4.2 is shown in Figure 3 below. All cells except the two purple ones (one one-dimensional and one two-dimensional) are efficient.

Now let us establish the correspondence between permutations from S_3 and combinations of faces of this polytope. The identity permutation id corresponds to the vertex with the highest sum of coordinates (it is marked by a larger black dot in Figure 3). The simple transpositions s_1 and s_2 correspond to the vertical and horizontal edges adjacent to this vertex, respectively. The cellular decompositions of these edges are shown in Figure 4.

The permutation s_1s_2 corresponds to the back trapezoid face (shown by blue color in Figure 5), while the permutation s_2s_1 corresponds to two faces, a triangular and a rectangular one, highlighted in green. Now, for each of these sets of faces, we need to take its cellular decomposition and compute the sum of all the monomials corresponding to the cells occurring in it; this would give us the Lascoux polynomials. The figures are self-explanatory; the picture for the symmetric Grothendieck polynomial $\mathscr{L}_{s_1s_2s_1,\lambda}$ is too bulky, so we do not provide it here.

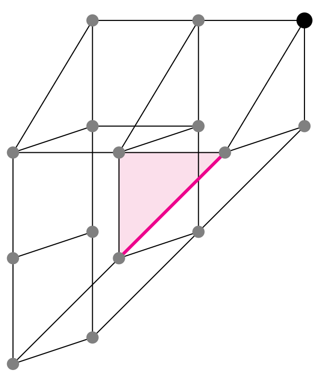


Figure 3: Cellular decomposition of $GZ(3,2,0)$

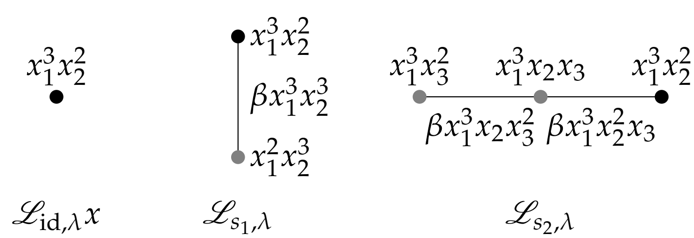


Figure 4: Lascoux polynomials for id, s_1, and s_2

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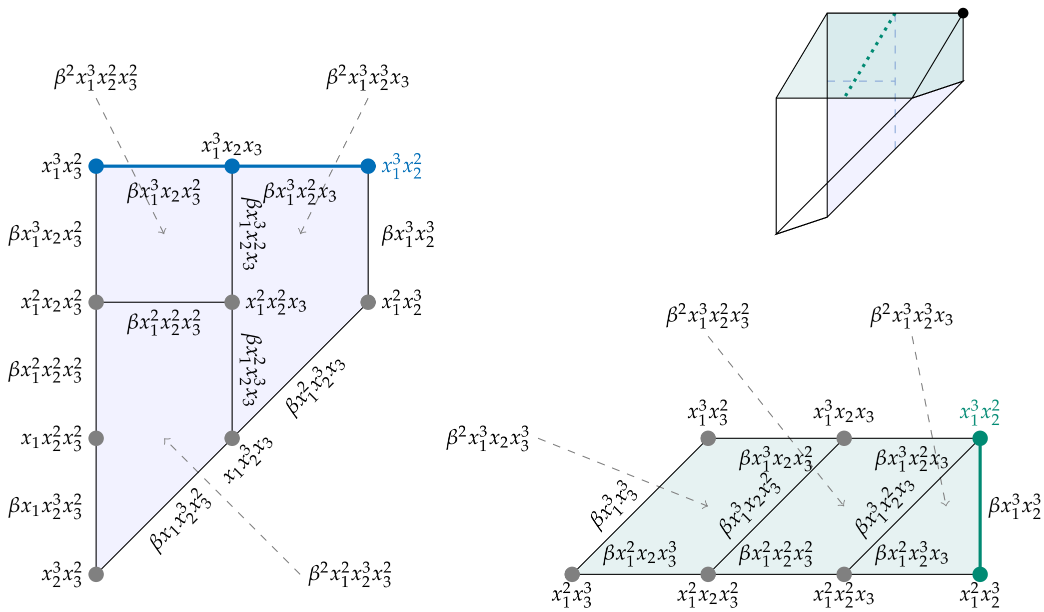

Figure 5: Lascoux polynomials for $s_1 s_2$ and $s_2 s_1$

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Chain polynomials of generalized paving matroids

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Abstract. We prove that the chain polynomial of the lattice of flats of a paving matroid is real-rooted, and we define a class of matroids called *generalized paving matroids*. Generalized paving matroids associated to subspace lattices are shown to have real-rooted chain polynomials, by a study of a q-analog of the subdivision operator. We finish by studying single element extensions, and prove that the chain polynomials of the lattice of flats of single element extensions of U_n^n and U_n^{n-1} are real-rooted.

Keywords: matroid, chain polynomial, geometric lattice, real-rootedness

1 Introduction

The chain polynomial of a finite poset P is defined as

$$c_P(t) := \sum_{k \geq 0} c_k(P) t^k, \tag{1.1}$$

where $c_k(P)$ is the number of k-element chains in P. The chain polynomials of posets in several important classes have been proven to be real-rooted. For example face lattices of simplicial [8] and cubical polytopes [1], $(3+1)$-free posets [14, Corollary 2.9], and for some classes of distributive lattices [7, 18], but not all [16]. In [2] the authors asked for which posets the chain polynomial is real-rooted. In particular the following conjecture was formulated.

Conjecture 1.1. *[2, Conjecture 1.2] The chain polynomial $c_{\mathcal{L}}(t)$ is real-rooted for every geometric lattice $\mathcal{L}$.*

This conjecture can bee seen as a member of a family of recent conjectures about the real-rootedness of Kazhdan-Lusztig polynomials [9, 11] and Chow ring Poincaré polynomials [10, 17] associated matroids arising in the emerging Hodge theory of matroids [12].

In [2], Athanasiadis and Kalampogia-Evangelinou proved Conjecture 1.1 for the subspace lattices $\mathcal{L}_n(q)$, for the partition lattices Π_n and Π_n^B, and for the lattice of flats of near-pencils and uniform matroids.

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The purpose of this paper is to verify Conjecture 1.1 for further classes of geometric lattices. We prove Conjecture 1.1 for the lattice of flats of paving matroids, a class of matroids which is conjectured to correspond to almost all finite matroids. We define a new class of matroids, called *generalized paving matroids*, that includes the class of paving matroids and we prove Conjecture 1.1 for generalized paving matroids associated to subspace lattices. In the process we extend the well studied *subdivision operator* $\mathcal{E}$ (see [6, 8]) to subspace lattices, and prove several real-rootedness results concerning these generalized subdivision operators. We finish the paper by studying single-element extensions and proving that the chain polynomials of the lattice of flats of single-element extensions of some uniform matroids are real-rooted.

2 Interlacing polynomials

In this section we collect a few results and terminology that will be needed in subsequent sections, for proofs we refer to [6].

Let $f, g \in \mathbb{R}[t]$ be real-rooted polynomials with nonnegative coefficients and of degrees r and s, respectively. Let $x_r \leq \cdots \leq x_2 \leq x_1$ be the zeros of f, and let $y_s \leq \cdots \leq y_2 \leq y_1$ be the zeros of g. We say that g *interlaces* f (written $g \preceq f$) if either $r = s$ and

$$y_s \leq x_r \leq \cdots \leq y_2 \leq x_2 \leq y_1 \leq x_1$$

or $r = s + 1$ and

$$x_r \leq y_s \leq x_{r-1} \leq \cdots \leq y_2 \leq x_2 \leq y_1 \leq x_1.$$

We say that a sequence of polynomials $f_1, f_2, \ldots, f_m \in \mathbb{R}[t]$ is *interlacing* if $f_i \preceq f_j$ whenever $i < j$.

Proposition 2.1. *Let $f, g, h \in \mathbb{R}[t]$.*

1. *If $f \preceq g$ and $f \preceq h$, then $f \preceq ag + bh$ for all $a, b \geq 0$.*
2. *If $g \preceq f$ and $h \preceq f$, then $ag + bh \preceq f$ for all $a, b \geq 0$.*

Proposition 2.2. *Let $f_0, f_1, \ldots, f_m \in \mathbb{R}[t]$ be an interlacing sequence of real-rooted polynomials with positive leading coefficients.*

1. *Every nonnegative linear combination f of $f_0, f_1, \ldots, f_m$ is real-rooted, and $f_0 \preceq f \preceq f_m$;*
2. *If we define*
$$g_k := t\sum_{i=0}^{k-1} f_i + \sum_{i=k}^{m} f_i,$$
for $k = 0, 1, \ldots, m$, then $\{g_i\}_{i=0}^{m}$ is interlacing.

Proposition 2.3. *Let $f_1, \ldots, f_m \in \mathbb{R}[t]$. If $f_1 \preceq f_2 \preceq \cdots \preceq f_m$ and $f_1 \preceq f_m$, then $f_i \preceq f_j$ for all $i \leq j$.*

Lemma 2.4. *Suppose $f_0, f_1, \ldots, f_n$ is an interlacing sequence of polynomials of degree d, such that for each $0 \leq j \leq n$, the polynomial f_j has nonnegative leading coefficient and all zeros in the interval $[-1, 0]$. Then the sequence $g_0, g_1, \ldots, g_{n+1}$ defined by*

$$g_k = t \sum_{j=0}^{k-1} f_j + (1+t) \sum_{j=k}^{n} f_j,$$

is interlacing.

Proof. Let $h_j(t) = (1-t)^d f_j(t/(1-t))$ and $r_j(t) = (1-t)^{d+1} g_j(t/(1-t))$. Then $\{h_j\}_{j=0}^n$ is an interlacing sequence of polynomials with nonnegative coefficients. Moreover,

$$r_j = t \sum_{j=0}^{k-1} h_j + \sum_{j=k}^{n} h_j,$$

and hence $\{r_j\}_{j=0}^{n+1}$ is interlacing by Proposition 2.2. Since $g_j = (1+t)^{d+1} r_j(t/(1+t))$, the lemma follows. □

3 Generalized subdivision operators

Let P be a locally finite and graded poset with a least element $\hat{0}$, such that $[\hat{0}, x]$ is isomorphic to $[\hat{0}, y]$ whenever x and y have the same rank. Define a linear operator $\mathcal{E}_P : \mathbb{R}[t] \to \mathbb{R}[t]$ by $\mathcal{E}_P(1) = 1$, and

$$\mathcal{E}_P(t^n) = \sum_{j=1}^{n} |\{\hat{0} < x_1 < \cdots < x_j = x\}| \cdot t^j = \frac{t}{(1+t)^2} \cdot c_{[\hat{0},x]}(t),$$

where x is any element in P of rank n. Let further $\mathcal{R}_P : \mathbb{R}[t] \to \mathbb{R}[t]$ be the linear operator defined by

$$\mathcal{R}_P(t^n) = \sum_{k=0}^{n} r_{n,k} t^k,$$

where $r_{n,k}$ is the number of elements in $[\hat{0}, x]$ of rank k, where x is any element in P of rank n. Hence, for $n \geq 1$,

$$\mathcal{E}_P(t^n) = t\mathcal{E}_P\big(\mathcal{R}_P(t^n) - t^n\big) \quad \text{and} \quad (1+t)\mathcal{E}_P(t^n) = t\mathcal{E}_P(\mathcal{R}_P(t^n)). \tag{3.1}$$

If P is a Boolean lattice, then $\mathcal{E}_P$ is the *subdivision operator* $\mathcal{E}$, which has the property that

$$\mathcal{E}(f_\Delta(t)) = f_{\mathrm{sd}(\Delta)}(t),$$

for any simplicial complex Δ, where f_Δ is the f-polynomial of Δ and $\mathrm{sd}(\Delta)$ is the barycentric subdivision of Δ, see [6, 8]. The subdivision operator is important in proving real-rootedness for polynomials associated to simplicial complexes, posets or Ehrhart theory [6]. In this section we will generalize and refine the following result to subspace lattices $\mathcal{L}_n(q)$.

Proposition 3.1. *[5, Section 4] The sequence $\{\mathcal{E}\left(t^i(t+1)^{d-i}\right)\}_{i=0}^d$ is interlacing.*

Let $\mathcal{L}(q)$ be the inverse limit, as $n \to \infty$, of the subspace lattices $\mathcal{L}_n(q)$ of all subspaces of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is a finite field with q elements. Denote by $\mathcal{E}_q$, the linear operator $\mathcal{E}_{\mathcal{L}(q)}$. Hence

$$\mathcal{E}_q(t^n) = t\mathcal{E}_q\big(G_n(t) - t^n\big), \tag{3.2}$$

where $G_n(t) = \sum_{k=0}^n \binom{n}{k}_q t^k$, and $\binom{n}{k}_q$, $0 \le k \le n$, are the *Gaussian polynomials* which may be defined recursively by $\binom{n}{0}_q = 1$, and

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q, \tag{3.3}$$

see [15, Section 1.7]. Henceforth we let q be any real number greater or equal to 1.

Lemma 3.2. *Let n be a nonnegative integer. Then*

$$\mathcal{E}_q(t^k G_{n+1-k}(t)) = t\sum_{j=0}^{k-1} \mathcal{E}_q(t^j G_{n-j}(qt)) + (1+t)\sum_{j=k}^{n} \mathcal{E}_q(t^j G_{n-j}(qt)), \quad 0 \le k \le n+1, \tag{3.4}$$

and

$$\mathcal{E}_q(t^k G_{n-k}(qt)) = \mathcal{E}_q(t^k G_{n-k}(t)) + (q^{n-k}-1)\mathcal{E}_q(t^{k+1} G_{n-(k+1)}(t)), \quad 0 \le k \le n. \tag{3.5}$$

Proof. The identity (3.3) implies

$$t^k G_{n+1-k}(t) - t^{n+1} = \sum_{j=k}^{n} t^j G_{n-j}(qt). \tag{3.6}$$

By (3.2) and (3.6),

$$\mathcal{E}_q(t^{n+1}) = t\mathcal{E}_q(G_{n+1}(t) - t^{n+1}) = t\sum_{j=0}^{n} \mathcal{E}_q(t^j G_{n-j}(qt)),$$

which combined with (3.6) gives (3.4).
Similarly, the identity

$$q^k \binom{n}{k}_q = \binom{n}{k}_q + (q^n - 1)\binom{n-1}{k-1}_q \tag{3.7}$$

implies (3.5). □

The following theorem generalizes Theorem 3.1 to any $q \geq 1$.

Theorem 3.3. *Let n be a nonnegative integer. The sequence of polynomials $\{\mathcal{E}_q(t^k G_{n-k}(t))\}_{k=0}^n$ is interlacing. Moreover all zeros of $\mathcal{E}_q(t^k G_{n-k}(t))$ lie in the interval $[-1,0]$.*

Proof. The proof is by induction over n, the case $n = 0$ being trivial.

Suppose true for $n-1 \geq 0$. Since $\{\mathcal{E}_q(t^k G_{n-k}(t))\}_{k=0}^n$ is interlacing, we have by (3.5) and [6, Corollary 8.6] that $\{\mathcal{E}_q(t^k G_{n-k}(qt))\}_{k=0}^n$ is interlacing. Moreover (3.5) implies

$$\mathcal{E}_q(t^k G_{n-k}(t)) \prec \mathcal{E}_q(t^k G_{n-k}(qt)) \prec \mathcal{E}_q(t^{k+1} G_{n-k-1}(t)),$$

so that all zeros of $\mathcal{E}_q(t^k G_{n-k}(qt))$ are in the interval $[-1,0]$. The lemma now follows by induction from (3.4) and Lemma 2.4. □

Corollary 3.4. *Suppose $f = \sum_{k=0}^d h_k t^k G_{d-k}$, where $h_k \geq 0$ for all $0 \leq k \leq d$. Then $\mathcal{E}_q(f)$ is real-rooted and $\mathcal{E}_q(G_d) \prec \mathcal{E}_q(f) \prec \mathcal{E}_q(t^d)$.*

Proof. Follows immediately from Proposition 2.2 and Theorem 3.3. □

For $d \leq n$, let $G_{n,k}^d$ be the polynomial obtained from $t^k G_{n-k}(t)$ by removing all terms t^j, where $j > d$.

Lemma 3.5. *Let d be a nonnegative integer.*

(a) If $n \geq d$, then $\{\mathcal{E}_q(G_{n,k}^d)\}_{k=0}^d$ is interlacing.

(b) If $0 \leq k \leq d$, then $\{\mathcal{E}_q(G_{n,k}^d)\}_{n=d}^\infty$ is interlacing.

Proof. We first prove (a) by induction over $n \geq d$, the case $n = d$ being Theorem 3.3. Assume true for n. Equations (3.6) and (3.7) imply

$$t^k G_{n+1-k} - t^{n+1} = t^k G_{n-k} + \sum_{j=k+1}^{n} q^{n+1-j} t^j G_{n-j},$$

and thus

$$\mathcal{E}_q(G_{n+1,k}^d) = \mathcal{E}_q(G_{n,k}^d) + \sum_{j=k+1}^{d} q^{n+1-j} \mathcal{E}_q(G_{n,j}^d),$$

which by [6, Corollary 8.6] proves that $\{\mathcal{E}_q(G_{n+1,k}^d)\}_{k=0}^d$ is interlacing, and that $\mathcal{E}_q(G_{n,k}^d) \prec \mathcal{E}_q(G_{n+1,k}^d)$. This proves (a) by induction.
Notice that

$$\lim_{n\to\infty} \frac{\mathcal{E}_q(G_{n,k}^d)}{\binom{n-k}{d}_q} = \mathcal{E}_q(t^d),$$

and that $\mathcal{E}_q(G_{d,k}^d) \prec \mathcal{E}_q(t^d)$ by Theorem 3.3. Hence (b) now follows from Proposition 2.3. □

The next theorem generalizes a recent result [3] of Athanasiadis and Kalampogia-Evangelinou from Boolean lattices to subspace lattices. Suppose P is a graded poset, and $S = \{0 = s_0 < s_1 < s_2 < \cdots\} \subseteq \mathbb{N}$. Consider the rank selected poset $P_S := \{x \in P : \rho(x) \in S\}$. Define a linear operator $T_S : \mathbb{R}[t] \to \mathbb{R}[t]$ by

$$T_S(t^k) = \begin{cases} 0 & \text{if } k \notin S, \\ t^i & \text{if } k = s_i. \end{cases}$$

Theorem 3.6. *Let n be a positive integer, and let S be a subset of $\mathbb{N}$ containing 0. The sequence$\{\mathcal{E}_P(T_S(t^k G_{n-k}))\}_{k=0}^n$ is interlacing, where $P = \mathcal{L}(q)_S$. In particular the chain polynomial of $\mathcal{L}_n(q)_S$ is real-rooted.*

Proof. The proof is omitted in this extended abstract. □

4 Generalized paving matroids

Recall that a geometric lattice $\mathcal{L}$ of rank $d+1$ is the lattice of flats of a paving matroid on E if and only if

- the set $\mathcal{H}$ of hyperplanes of $\mathcal{L}$ form a *d-partition*, i.e., $|H| \geq d$ for each $H \in \mathcal{H}$, and for each set S of size d there exists a unique $H \in \mathcal{H}$ such that $S \subseteq H$;
- the flats of rank $k \leq d-1$ are the sets of size k of the Boolean lattice on E.

For example, if $\mathcal{P} = \binom{[n]}{d}$, then $\mathcal{P}$ is a d-partition of $[n]$ and, hence, there is a paving matroid whose set of hyperplanes is $\mathcal{H}_1 = \binom{[n]}{d}$. Another example, a 2-partition of [7], is $\mathcal{H}_2 = \{\{1,2,4\},\{1,3,7\},\{1,5,6\},\{3,4,6\},\{2,6,7\},\{4,5,7\},\{2,3\},\{2,5\},\{2,3\},\{3,5\}\}$.

We will now generalize this construction to any geometric lattice, and prove that the chain polynomials of the lattice of flats of generalized paving matroids associated to subspace lattices and Boolean lattices are real-rooted.

Let $\mathcal{L}$ be a geometric lattice with rank function ρ on a ground set E. Suppose $d \geq 1$, and suppose $\mathcal{H} \subset \mathcal{L}$ satisfies

(a) $\rho(H) \geq d$ for each $H \in \mathcal{H}$,

(b) for each $F \in \mathcal{L}$ with $\rho(F) = d$, there exists a unique $H \in \mathcal{H}$ such that $F \leq H$.

Let $\mathcal{L}(\mathcal{H})$ be the graded meet semi-lattice of rank $d+1$,

$$\mathcal{L}(\mathcal{H}) = \{F \in \mathcal{L}(\mathcal{H}) : \rho(F) \leq d-1\} \cup \mathcal{H} \cup \{E\}.$$

Lemma 4.1. *$\mathcal{L}(\mathcal{H})$ is a geometric lattice with set of hyperplanes $\mathcal{H}$.*

Proof. Let ρ' be the rank function of $\mathcal{L}(\mathcal{H})$, and let $\vee$ and $\vee'$ (respectively, $\wedge$ and $\wedge'$) be the joins (respectively, the meets) in $\mathcal{L}$ and $\mathcal{L}(\mathcal{H})$, respectively. We will prove that $\mathcal{L}(\mathcal{H})$ is (1) a lattice, (2) atomistic and (3) semimodular:

1. $\mathcal{L}(\mathcal{H})$ **is a lattice -** Since $\mathcal{L}(\mathcal{H})$ is a finite meet semi-lattice with a largest element, then, by [15, Proposition 3.3.1], it is a lattice;

2. $\mathcal{L}(\mathcal{H})$ **is atomistic -** Let $F \in \mathcal{L}(\mathcal{H})$. If $\rho'(F) \leq d-1$, then, by definition of $\mathcal{L}(\mathcal{H})$, if $F = \bigvee_i F_i$, then $F = \bigvee'_i F_i$. If $\rho'(F) = d$, then there exist a unique $G \in \mathcal{L}$ such that $\rho(G) = d$ and $G \leq F$. Since $G = \bigvee_i G_i$ for atoms $G_i \in \mathcal{L}$, then $F = \bigvee'_i G_i$. Finally, if $F = E$, then F is the join of two elements in $\mathcal{H}$ and hence a join of atoms, by the above;

3. $\mathcal{L}(\mathcal{H})$ **is semimodular -** Let $F, G \in \mathcal{L}(\mathcal{H})$. We want to prove that

$$\rho'(F) + \rho'(G) \geq \rho'(F \vee' G) + \rho'(F \wedge' G). \tag{4.1}$$

   There are three different cases to deal with:

   - F **and** G **are in** $\mathcal{H}$. Then $\rho'(F) = \rho'(G) = d$, $\rho'(F \vee' G) = d+1$ and $\rho'(F \wedge' G) \leq d-1$, which implies (4.1);
   - F **is not in** $\mathcal{H}$ **and** G **is in** $\mathcal{H}$. If $F \leq' G$, then there is nothing to prove. Otherwise, $\rho'(F \wedge' G) \leq \rho'(F) - 1$ and $F \vee' G = E$, and (4.1) holds;
   - F **and** G **are not in** $\mathcal{H}$. We may assume F and G are smaller than E. Then

$$\begin{aligned}\rho'(F) + \rho'(G) &= \rho(F) + \rho(G) \geq \rho(F \vee G) + \rho(F \wedge G)\\ &= \rho(F \vee G) + \rho'(F \wedge' G).\end{aligned}$$

     Hence it remains to prove

$$\rho'(F \vee' G) \leq \rho(F \vee G). \tag{4.2}$$

     If $\rho(F \vee G) \leq d-1$, then $F \vee G = F \vee' G$ and so (4.2) holds. If $\rho(F \vee G) = d$, then there exists $H \in \mathcal{H}$ such that $F \vee G \leq H$, and hence $\rho'(F \vee' G) \leq d$. If $\rho(F \vee G) \geq d+1$, then (4.2) holds since $\rho'(E) = d+1$.

□

If $\mathcal{L}$ is the Boolean algebra, then the lattices $\mathcal{L}(\mathcal{H})$ are precisely the lattices of flats of paving matroids.

Notice that

$$c_{\mathcal{L}(\mathcal{H})}(t) = c_{\mathcal{L}^d}(t) + t \sum_{H \in \mathcal{H}} c_{[\hat{0},H]^d}(t), \tag{4.3}$$

where $\mathcal{L}^d = \{F \in \mathcal{L} : \rho(F) \leq d-1\} \cup \{E\}$ is the truncation of $\mathcal{L}$ to rank d.

The next theorem verifies Conjecture 1.1 for the lattice of flats of paving matroids and generalized paving matroids associated to subspace lattices.

Theorem 4.2. *Suppose $\mathcal{L}$ is a subspace lattice $\mathcal{L}_n(q)$ or a Boolean lattice, and that $\mathcal{H}$ satisfies (a) and (b). Then $c_{\mathcal{L}(\mathcal{H})}(t)$ is real-rooted and $c_{\mathcal{L}^d}(t) \prec c_{\mathcal{L}(\mathcal{H})}(t)$.*

Proof. Clearly,

$$c_{\mathcal{L}^d}(t) = (1+t) \cdot \mathcal{E}_q(G^{d-1}_{n,d-1}(t)) \quad \text{and} \quad c_{[\hat{0},H]^d}(t) = (1+t) \cdot \mathcal{E}_q(G^{d-1}_{m,d-1}(t)),$$

where $m \leq n$, and $q = 1$ for the Boolean case. The theorem now follows from Proposition 2.1, Lemma 3.5 and (4.3).

□

5 Single-element extensions

Let N be a matroid with ground set E. Recall that, if $T \subset E$, then the *deletion of T in N* is $N \backslash T$, the matroid on $E \backslash T$ with independent sets given by

$$\{I : I \text{ is independent in } N \text{ and } I \cap T = \varnothing\}.$$

In this case, N is called an *extension* of $N \backslash T$. For the particular case when $|T| = 1$, we call N a *single-element extension* of $N \backslash T$.

A *modular cut* $\mathcal{M}$ of a matroid M is a set of flats of M that satisfies the following:

(i) if $F \in \mathcal{M}$ and F' is a flat of M containing F, then $F' \in \mathcal{M}$;

(ii) if $F_1, F_2 \in \mathcal{M}$ and $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{M}$.

There is a one-to-one correspondence between the modular cuts of a matroid M and single-element extensions of M, see [13, Chapter 7.2]. Hence, for each modular cut $\mathcal{M}$ of M we can associate a single-element extension $M +_{\mathcal{M}} e$ of M, where e is an element not in the ground set of M, whose lattice of flats fall into the three following disjoint classes (see [13, Corollary 7.2.]):

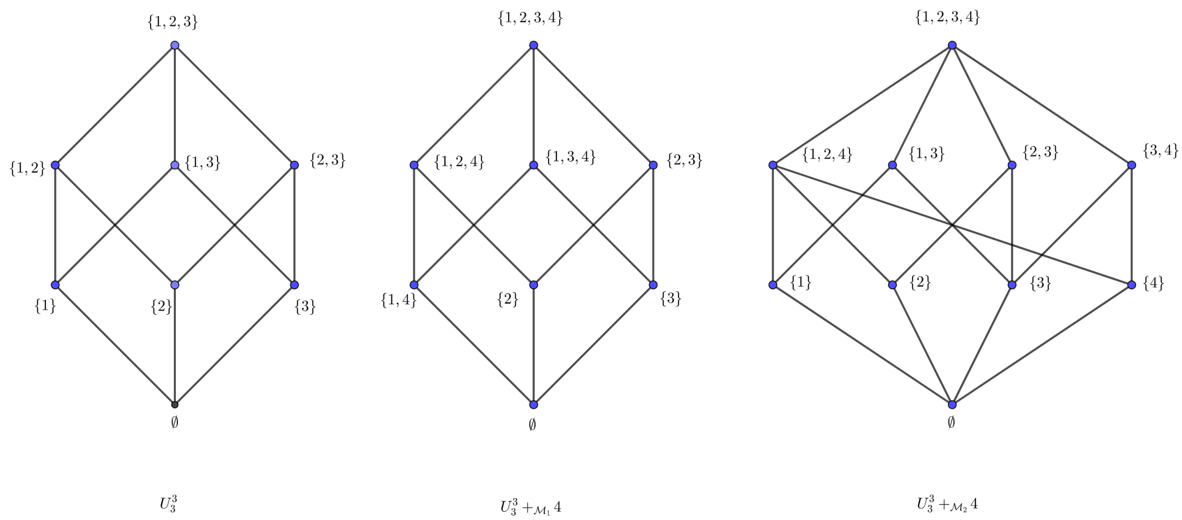
(i) flats F of M that are not in $\mathcal{M}$;

(ii) sets $F \cup e$, where F is a flat of M that is in $\mathcal{M}$;

(iii) sets $F \cup e$, where F is a flat of M that is not in $\mathcal{M}$ and F is not contained in a member F' of $\mathcal{M}$ of rank $r(F) + 1$.

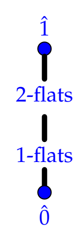
Moreover, we can use this construction to determine all matroids: any matroid M is obtained from a uniform matroid U^n_n by a sequence of single-element extensions.

For example, consider the uniform matroid U^3_3. Then, $\mathcal{M}_1 = [[1], [3]]$ and $\mathcal{M}_2 = [[2], [3]]$ are modular cuts. The lattices of flats of U^3_3, $U^3_3 +_{\mathcal{M}_1} 4$ and $U^3_3 +_{\mathcal{M}_2} 4$ are given as follows:



Lemma 5.1. *Conjecture 1.1 is true for all matroids of ranks 1, 2 or 3.*

Proof. The result is trivial for matroids of ranks 1 and 2. For matroids of rank 3, its lattice of flats is given by



and, hence, its chain polynomial is $c_{\mathcal{L}}(t) = [1 + (m_1 + m_2)t + et^2](1 + t)^2$, where m_i is the number of i-flats and e is the number of edges between 1-flats and 2-flats. Since $e \leq m_1 m_2$, then $c_{\mathcal{L}}(t)$ is real-rooted. □

Now, consider uniform matroids U_n^n, $n \geq 4$. The lattice of flats of $\mathcal{M}$ is B_n. So, if $\mathcal{M}$ is a modular cut of U_n^n (and, in general, a modular cut of U_n^r), then $\mathcal{M} = \emptyset$ or $\mathcal{M} = [X, [n]]$, where $X \subseteq [n]$. Hence, every flat of the lattice of flats of $U_n^n +_{\mathcal{M}} \{n+1\}$ fall into one of the following disjoint classes:

(i) $F \subset \{1, \ldots, n, n+1\}$ such that $\{1, \ldots, m, n+1\}$ is not a subset of F. In this case, the rank of F is $|F|$;

(ii) $F \subset \{1, \ldots, n, n+1\}$ such that $\{1, \ldots, m, n+1\} \subseteq F$. In this case, the rank of F is $|F| - 1$.

It follows that $U_n^n +_{\mathcal{M}} \{n+1\}$ is isomorphic to the direct product $U_{m+1}^m \times U_{n-m}^{n-m}$.

Lemma 5.2. *[6, Theorem 7.6] If all zeros of $\mathcal{E}(f)$ and $\mathcal{E}(g)$ lie in the interval $[-1,0]$, then so does $\mathcal{E}(fg)$.*

Lemma 5.3. *Let P and Q be two posets with a least and a greatest element such that $|P|, |Q| \geq 2$, and define*

$$\hat{c}_P(t) = \frac{t}{(1+t)^2} \cdot c_P(t).$$

Then

$$\hat{c}_{P\times Q} = \hat{c}_P \diamond \hat{c}_Q := \mathcal{E}(\mathcal{E}^{-1}(\hat{c}_P)\mathcal{E}^{-1}(\hat{c}_Q)).$$

Proof. Omitted in the long abstract. □

Lemma 5.4 ([5]). *If*

$$f(x) = \sum_{k=0}^{d} h_k x_k (1+x)^{d-k}$$

has $h_k \geq 0$ for all $0 \leq k \leq d$, then all zeros of $\mathcal{E}(f)$ are real, simple and located in $[-1,0]$. In particular, the h-polynomial of a Cohen-Macaulay poset is real-rooted.

Now, we can prove the following:

Theorem 5.5. *If the h-polynomials of the order complexes of the posets P and Q have nonnegative coefficients, then the chain polynomial of $P \times Q$ is real-rooted.*

Proof. It follows directly from Lemmas 5.4, 5.2 and 5.3. □

Corollary 5.6. *The chain polynomial of $U_n^n +_{\mathcal{M}} \{n+1\}$ is real-rooted for any modular cut of U_n^n.*

Proof. As mentioned before, $U_n^n +_{\mathcal{M}} \{n+1\}$ is isomorphic to $U_{m+1}^m \times U_{n-m}^{n-m}$. So, by Lemma 5.3,

$$\hat{c}_{\mathcal{L}(U_n^n +_{\mathcal{M}} \{n+1\})} = \hat{c}_{\mathcal{L}(U_{m+1}^m)} \diamond \hat{c}_{\mathcal{L}(U_{n-m}^{n-m})}.$$

By Theorem 3.3, $c_{\mathcal{L}(U_{m+1}^m)}$ and $c_{\mathcal{L}(U_{n-m}^{n-m})}$ are real-rooted. So, by Lemma 5.2, $c_{\mathcal{L}(U_n^n +_{\mathcal{M}} \{n+1\})}$ is real-rooted. □

Corollary 5.7. *The chain polynomial of $U_n^{n-1} +_{\mathcal{M}} \{n+1\}$ is real-rooted for any modular cut of U_n^{n-1}.*

Proof. First, observe that U_n^{n-1} is a truncation of U_n^n and that modular cuts of U_n^{n-1} are also intervals. Let $\hat{\mathcal{M}} = [[m],[n]]$ be a modular cut of U_n^{n-1} and $\mathcal{M} = [[m],[n]]$ be a modular cut of U_n^n. So $U_n^{n-1} +_{\hat{\mathcal{M}}} \{n+1\}$ is a truncation of $U_n^n +_{\mathcal{M}} \{n+1\}$. Hence, if $\mathcal{H}$ is the set of hyperplanes of $U_n^n +_{\mathcal{M}} \{n+1\} = U_{m+1}^m \times U_{n-m}^{n-m}$, then $H \in \mathcal{H}$ if and only if

$H = A \times [n-m]$ or $H = [m+1] \times B$, where $A \in \mathcal{A}$ is a hyperplane of U^m_{m+1} and $B \in \mathcal{B}$ is a hyperplane of U^{n-m}_{n-m}. So,

$$\begin{aligned}
\hat{c}_{\mathcal{L}(U^{n-1}_n +_{\hat{\mathcal{M}}} \{n+1\})}(t) &= \hat{c}_{\mathcal{L}(U^n_n +_{\mathcal{M}} \{n+1\})}(t) - t \sum_{H_i \in \mathcal{H}} \hat{c}_{[\emptyset, H_i]}(t) \\
&= \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t) - t \sum_{A_i \in \mathcal{A}} \hat{c}_{[\emptyset, A_i \times [n-m]]}(t) \\
&- t \sum_{B_j \in \mathcal{B}} \hat{c}_{[\emptyset, [m+1] \times B_j]}(t) \\
&= \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t) - t \sum_{A_i \in \mathcal{A}} \hat{c}_{[\emptyset, A_i]}(t) \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t) \\
&- t \sum_{B_j \in \mathcal{B}} \hat{c}_{\mathcal{L}(U^m_{m+1})}(t) \diamond \hat{c}_{[\emptyset, B_j]}(t).
\end{aligned}$$

Since $\hat{c}_{[\emptyset, A_i]}(t) \preceq \hat{c}_{\mathcal{L}(U^m_{m+1})}(t)$ for all $A_i \in \mathcal{A}$ and $\hat{c}_{[\emptyset, B_j]}(t) \preceq \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t)$ for all $B_j \in \mathcal{B}$,

$$\hat{c}_{[\emptyset, A_i \times [n-m]]}(t) \preceq \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t), \text{ for all } A_i \in \mathcal{A}$$

and

$$\hat{c}_{[\emptyset, [m+1] \times B_j]}(t) \preceq \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t), \text{ for all } B_j \in \mathcal{B}.$$

by [4, Theorem 3]. Hence $\hat{c}_{\mathcal{L}(U^{n-1}_n +_{\hat{\mathcal{M}}} \{n+1\})}(t)$ is real-rooted. □

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A Galois structure on the orbit of large steps walks in the quadrant

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Abstract. The enumeration of weighted walks in the quarter plane reduces to studying a functional equation with two catalytic variables. When the steps of the walk are small, Bousquet-Mélou and Mishna defined a group called *the group of the walk* which turned out to be crucial in the classification of the small steps models. In particular, its action on the catalytic variables provides a convenient set of changes of variables in the functional equation. This particular set called the *orbit* has been generalized to models with arbitrary large steps by Bostan, Bousquet-Mélou and Melczer (BBMM). However, the orbit had till now no underlying group.

In this article, we endow the orbit with the action of a Galois group, which extends the notion of the group of the walk to models with large steps. As an application, we look into a general strategy to prove the algebraicity of models with small backwards steps, which uses the fundamental objects that are *invariants* and *decoupling*. The group action on the orbit allows us to develop a Galoisian approach to these two notions. Up to the knowledge of the finiteness of the orbit, this gives systematic procedures to test their existence and construct them. Our constructions lead to the first proofs of algebraicity of weighted models with large steps, proving in particular a conjecture of BBMM, and allowing to find new algebraic models with large steps.

Keywords: functional equations, Galois theory, quadrant walks

1 Introduction and preliminaries

1.1 Walks in the quarter plane

A weighted walk in the quarter plane is defined as follows. Consider a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}$. To each *step* s of $\mathcal{S}$ we attach its *weight* w_s which is a nonzero complex number. The tuple $\mathcal{W} = (\mathcal{S}, (w_s)_{s \in \mathcal{S}})$ is called a *weighted model of walks*. A *weighted walk in the quarter plane* of length n on the model $\mathcal{W}$ is then a sequence of points $P_0, \ldots, P_n$ in $\mathbb{N} \times \mathbb{N}$ such that for all i there exists $s_i \in \mathcal{S}$ satisfying $s_i + P_i = P_{i+1}$. The *weight* of the walk is the product $w_{s_0} w_{s_1} \ldots w_{s_{n-1}}$ of the weights of the steps taken by the walk.

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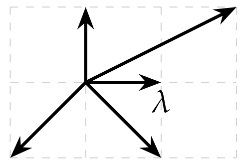


The weighted model $\mathcal{G}_\lambda$ (for which we take any nonzero λ in $\mathbb{C}$) along with an example of a walk on $\mathcal{G}_\lambda$, of length 8, ending at $(3,0)$ and of weight λ^2.

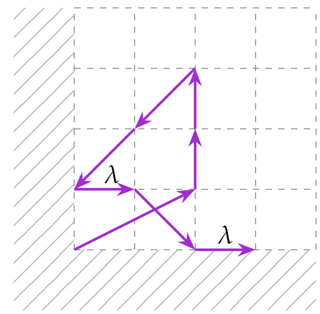


Figure 1: An example of weighted model and walk

The enumeration of weighted walks in the quarter plane has attracted a lot of attention over the past 20 years. Indeed, these objects are general enough to encode many objects in combinatorics (families of permutations, trees, maps), probability theory (stochastic processes, games of chance, sums of discrete random variables) or statistics (non-parametric tests). The attraction for this topic comes from the fact that the solution of this problem requires many different techniques and points of view, from combinatorics of course, but also from probability theory, computer algebra, differential Galois theory, complex analysis, geometry...

1.2 Generating function and classification

Given a model $\mathcal{W}$, denote by $q_{i,j,n}$ the sum of the weights of walks in the quadrant on $\mathcal{W}$ of length n starting at P_0 (taken as $(0,0)$ unless stated otherwise) and terminating at (i,j). The generating function for these walks is defined as

$$Q(X,Y,t) = \sum_{i,j,n\geq 0} q_{i,j,n} X^i Y^j t^n.$$

The weighted model is encoded as the *Laurent polynomial of the model* defined as $S(X,Y) = \sum_{s\in\mathcal{S}} w_s X^{s_x} Y^{s_y}$. From any weighted model $\mathcal{W}$, it is quite easy to form a functional equation for $Q(X,Y,t)$, as we demonstrate in the following example.

Example 1. Let $\mathcal{G}_\lambda = \{(-1,-1),(0,1),((1,0),\lambda),(2,1),(1,-1)\}$ as in Figure 1 (Example 2.1 in [2], see also Remark 2.2 for alternate weightings). Its Laurent polynomial is $S(X,Y) = \frac{1}{XY} + Y + \lambda X + \frac{X}{Y} + X^2 Y$. We now construct a recurrence on the walks: a walk terminating at coordinates (i,j) can be completed by a step s of $\mathcal{S}$ as long as $(i,j)+s$ is in $\mathbb{N}\times\mathbb{N}$. This translates into the following functional equation:

$$\begin{aligned}Q(X,Y,t) &= 1 + tX^2YQ(X,Y,t) + t\lambda XQ(X,Y,t) + tYQ(X,Y,t)\\ &+ t\frac{X}{Y}\left(Q(X,Y,t) - Q(X,0,t)\right) + t\frac{1}{XY}\left(Q(X,Y,t) - Q(X,0,t) - Q(0,Y,t) + Q(0,0,t)\right).\end{aligned}$$

Such an equation is then usually put in the following normal form:

$$\widetilde{K}(X,Y,t)Q(X,Y,t) = XY - t(X^2+1)Q(X,0,t) - tQ(0,Y,t) + tQ(0,0,t), \tag{1.1}$$

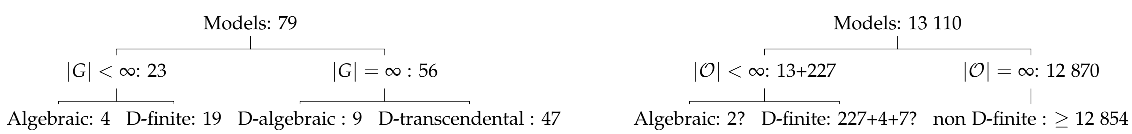
with $\widetilde{K}(X,Y,t)$ the *kernel polynomial of the walk* being equal here to $XY\,(1 - tS(X,Y))$.

Given a class of combinatorial objects, a natural question is to determine where its generating function fits in the classical hierarchy of power series of $\mathbb{C}(X,Y)[[t]]$

$$\textit{rational} \subset \textit{algebraic} \subset \textit{D-finite} \subset \textit{D-algebraic},$$

where algebraic series satisfy polynomial equations; D-finite series satisfy one linear differential equation in each variable X, Y, t; and D-algebraic series satisfy polynomial differential equations, all the coefficients being taken in the polynomial ring $\mathbb{C}[X,Y,t]$.

For walks, this hierarchy measures the complexity of a model: the lower its generating function in this hierarchy, the simpler the walks. The *catalytic variables equations* like (1.1) do not immediately allow to conclude to the position of their solutions in this hierarchy, hence the question of classifying the complexity of a model of walks in the quarter plane is highly nontrivial.



(a) $\mathcal{S} \subset \{-1,0,1\}^2$ (b) $\mathcal{S} \subset \{-2,-1,0,1\}^2$

Figure 2: The partial classifications for two families of unweighted models

For instance, consider the restriction of the problem to unweighted models with steps contained in $\{-1,0,1\}^2$ (these models are commonly called *small steps models*). The classification completed in 2018 and summarized in Figure 2a shows the numerous different behaviours that arise. Following the success of this first classification, the set of considered models has been extended to models with steps in $\{-2,1,0,-1\}^2$ in [3]. It is summarized in Figure 2b, and is currently incomplete. One of the reasons is that not all the tools used in the classification of small steps models extend to large steps.

For small steps models, [9, Chapter 4] proposes a complete strategy with Galois theoretic tools to classify solutions of functional equation of the form (1.1) when the kernel polynomial is biquadratic and the orbit is finite. Unfortunately, theses tools rely heavily on an elliptic uniformization of the algebraic curve associated with the kernel polynomial. We propose here a an extension of these tools with in sight a particular strategy to prove algebraicity, which goes beyond the elliptic framework since it deals with kernel polynomials of arbitrary degree. It has the advantage to stay within the realm of Laurent power series and is almost algorithmic until an algebraic characterization of certain invariants. Moreover, we hope that the geometric framework hidden behind our constructions will allow to adapt entirely the strategy of [1] to large steps models.

1.3 The group and the orbit

A fundamental object which arises in the study of models with small steps is the *group of the walk*, introduced by Bousquet-Mélou and Mishna in [5], following [9]. It is defined as follows. For a small steps model, we can write its Laurent polynomial in two ways:

$$\begin{aligned} S(X,Y) &= A_{-1}(X)/Y + A_0(X) + A_1(X)Y \\ &= B_{-1}(Y)/X + B_0(Y) + B_1(Y)X. \end{aligned}$$

Assume $A_{-1}(X)$, $A_1(X)$, $B_{-1}(Y)$ and $B_1(Y)$ to be nonzero. The polynomial $S(x,y)$ is left unchanged by the two *birational transformations* (that are involutions) of $\mathbb{C} \times \mathbb{C}$ defined as

$$\Phi\colon (u,v) \mapsto \left(\frac{B_{-1}(v)}{uB_1(v)}, v\right) \qquad \Psi\colon (u,v) \mapsto \left(u, \frac{A_{-1}(u)}{vA_1(u)}\right).$$

The *group of the walk* is then defined as $\langle \Phi, \Psi \rangle$, the subgroup of birational transformations of $\mathbb{C} \times \mathbb{C}$ generated by Φ and Ψ. This group turned out to be a crucial algebraic invariant of a model with small steps. For instance, an unweighted model with small steps has a D-finite generating function if and only if the group is finite (see the introduction of [1]).

The group also acts on pairs of catalytic variables. The orbit of its action on the pair (x,y) has a graph structure: the vertices are the pairs (u,v) of the orbit of (x,y), and two pairs are adjacent if one can be obtained from the other by applying Φ or Ψ to it.

If the model contains a large step, the equation $S(x,y) = S(x,y')$ may have non-rational solutions in x and y because of the higher degree of the polynomials, therefore in that case this group cannot be defined as a group of birational transformations of $\mathbb{C} \times \mathbb{C}$. Nonetheless, Bostan, Bousquet-Mélou and Melczer noted in [3] that the graph could be defined independently from the group, and called it the *orbit of the walk*. It is defined as follows. Denote by $\mathbb{K} = \overline{\mathbb{C}(x,y)}$ the algebraic closure of $\mathbb{C}(x,y)$ for two indeterminates x and y.

Definition 2 (Definition 3.1 in [3])**.** Let (u,v) and (u',v') be in $\mathbb{K} \times \mathbb{K}$. Then (u,v) and (u',v') are called *x-adjacent* if $S(u,v) = S(u',v')$ and $u = u'$. Similarly, they are called *y-adjacent* if $S(u,v) = S(u',v')$ and $v = v'$. We denote these two equivalence relations by $\sim^x$ and $\sim^y$. Two pairs are then adjacent if they are either x-adjacent or y-adjacent, and this relation is denoted by $\sim$. We denote by $\sim^*$ the transitive closure of $\sim$. The *orbit of the walk* is the set of pairs (u,v) such that $(u,v) \sim^* (x,y)$, and is denoted $\mathcal{O}$.

Example 3. For the small steps model $S(X,Y) = 1/X + 1/Y + XY$, the orbit is the cycle

$$(x,y) \overset{\Phi}{\longleftrightarrow} \left(\frac{1}{xy}, y\right) \overset{\Psi}{\longleftrightarrow} \left(\frac{1}{xy}, x\right) \overset{\Phi}{\longleftrightarrow} (y,x) \overset{\Psi}{\longleftrightarrow} \left(y, \frac{1}{xy}\right) \overset{\Phi}{\longleftrightarrow} \left(x, \frac{1}{xy}\right) \overset{\Psi}{\longleftrightarrow} (x,y)$$

Example 4. For $\mathcal{G}_\lambda$, the equation $S(x,y) = S(x',y)$ has three solutions x, x_1 and x_2 with x_1 algebraic of degree two over $\mathbb{C}(x,y)$. Continuing the construction, it turns out that the orbit $\mathcal{O}$ is finite of size 12 with coordinates in $\mathbb{C}(x,y,x_1)$ (see Figure 3).

In Section 2, we introduce a Galois framework to study the orbit. It allows us to generalize the group of the walk to weighted models with arbitrarily large steps.

1.4 A strategy based on invariants and decoupling

In their article [1], Bernardi, Bousquet-Mélou and Raschel introduced a general strategy for proving algebraicity of a small steps model in the quadrant, later adapted to models of the three-quadrant cone in [6]. This method relies critically on objects called *invariants*, of which we use one flavor in the realm of formal power series, the *t-invariants*.

Define the subring $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$ of power series of $\mathbb{C}(X,Y)[[t]]$ whose coefficients in the t-expansion are of the form $\frac{A_n(X,Y)}{B_n(X)C_n(Y)}$ for A_n, B_n and C_n polynomials over $\mathbb{C}$. A power series in $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$ is said to have *poles of bounded order at 0* if there is a bound on the order of the poles at $X=0$ and $Y=0$ of its coefficients (Definition 2.1 in [6]).

Definition 5. Let $F(X,Y,t)$ and $G(X,Y,t)$ be two power series of $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$. They are *t-equivalent* (with respect to $\widetilde{K}$) if the power series $\frac{F(X,Y,t)-G(X,Y,t)}{\widetilde{K}(X,Y,t)} \in \mathbb{C}_{\text{mul}}(X,Y)[[t]]$ has poles of bounded order at 0. This equivalence is denoted by $F(X,Y,t) \equiv G(X,Y,t)$.

The t-equivalence relation is compatible with ring operations, and is used to define the notions of *t-invariants* and *t-decoupling*.

Definition 6 (Invariants, Def. 2.3 in [6])**.** Let $F(X,t), G(Y,t) \in \mathbb{C}_{\text{mul}}(X,Y)[[t]]$. The pair $(F(X,t), G(Y,t))$ is called a *pair of t-invariants* if $F(X,t) \equiv G(Y,t)$.

Definition 7 (Decoupling)**.** Let $H(X,Y,t)$ be a series of $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$. Then $H(X,Y,t)$ admits a *t-decoupling* if there exist $F(X,t)$ in $\mathbb{C}_{\text{mul}}(X)[[t]]$ and $G(Y,t)$ in $\mathbb{C}_{\text{mul}}(Y)[[t]]$ such that $H(X,Y,t) \equiv F(X,t) + G(Y,t)$.

Example 8. Consider the model defined by $S(X,Y) = XY + \frac{1}{X} + \frac{1}{Y}$ (the same as Example 3). The fraction XY admits the obvious decoupling $XY \equiv \frac{1}{t} - \frac{1}{X} - \frac{1}{Y}$. Moreover, the following identity induces a pair of rational invariants: $X + \frac{1}{tX} - \frac{1}{X^2} \equiv Y + \frac{1}{tY} - \frac{1}{Y^2}$.

When all the components of a pair of t-invariants or a t-decoupling are rational fractions, we speak of *rational* t-invariants or t-decoupling. This notion of invariants intervenes in the following result, on which the strategy of [1, 6] relies crucially.

Lemma 9 (Lemma 2.6 in [6])**.** *Let $(F(X,t), G(Y,t))$ be a pair of t-invariants. If the coefficients of the power series $\frac{F(X,t)-G(Y,t)}{\widetilde{K}(X,Y,t)} \in \mathbb{C}_{\text{mul}}(X,Y)[[t]]$ have no pole at $X=0$ nor $Y=0$, then there exists a series $A(t)$ in $\mathbb{C}[[t]]$ such that $F(X,t) = G(Y,t) = A(t)$.*

The strategy of [1, 6] applies verbatim to large steps models with small backward steps, and goes as follows. Using a *rational t-decoupling* of XY (or more generally

$X^{k+1}Y^{l+1}$ for other starting points) and the special shape of the equation (e.g. Equation (1.1)), we construct a first pair of t-invariants. Next, we combine it with a pair of *non-constant rational t-invariants* using ring operations, to eventually obtain a third pair of invariants that satisfy the conditions of Lemma 9. As this pair involves $Q(X,0,t)$ and $Q(0,Y,t)$, the lemma gives two equations with one catalytic variable on these series. By a result of Bousquet-Mélou and Jehanne in [4], they must be algebraic, so is $Q(X,Y,t)$.

The existence of non-constant rational t-invariants and decoupling is crucial to conduct this strategy. In sections 3.2 and 3.3, we give a Galois approach to these two notions, which exploits the notion of the group of the walk introduced in Section 2, providing a *systematic* construction of these objects up to their existence and the finiteness of the orbit. This is an alternative approach to the one developed in [8] where the authors search for a polynomial decoupling (see the discussion of Example 3.19 in [2]).

Using our systematic approach, we were able to conduct the strategy on the model $\mathcal{G}_\lambda$, proving a conjecture of Bostan, Bousquet-Mélou and Melczer in [3]. We detail the proof in Section 4 as an illustration of the strategy. Moreover, for a family of models with large steps $(\mathcal{H}_n)_n$ whose orbits are conjectured to be finite, we were able to conjecture that $X^{i+1}Y^{j+1}$ (which appears in place of XY in equations of the form (1.1) when considering a starting point (i,j) other than $(0,0)$) admits a decoupling for several (i,j) (namely, $(n-1,0)$ and $((n+1)k-1,k-1)$ for every k). We successfully proved the algebraicity for several of these starting points for $n \leq 4$, hinting a possibly infinite family of algebraic models with arbitrarily large steps (see Appendix E of [2]).

2 A Galois structure on the orbit

The proofs and constructions in Sections 2, 3 and 4 are detailed in our upcoming paper [2]. We consider a weighted model $\mathcal{W}$ with a non-univariate step polynomial. We denote by k the field $\mathbb{C}(S(x,y))$. Recall also that $\mathbb{K} = \overline{\mathbb{C}(x,y)}$, and that if $M|L$ is a subextension of $\mathbb{K}|L$, a L-algebra automorphism $\sigma\colon M \to M$ is a ring homomorphism such that $\sigma_{|L} = \mathrm{id}_L$. We denote by $\mathrm{Aut}(M|L)$ the group of L-algebra automorphisms of M.

We first endow the orbit with a group action as follows. If $\sigma : \mathbb{K} \to \mathbb{K}$ is a $\mathbb{C}$-algebra automorphism, define its action on a pair $(u,v) \in \mathbb{K} \times \mathbb{K}$ by $\sigma \cdot (u,v) = (\sigma \cdot u, \sigma \cdot v)$.

Lemma 10 (Lemmas 3.7 and 3.8 in [2]). *The orbit is stable under the action of $k(x)$ and $k(y)$-algebra automorphisms of $\mathbb{K}$, which all preserve the relations $\sim^x$ and $\sim^y$.*

This lemma has a field theoretic counterpart: define $k(\mathcal{O})$ to be the subextension of $\mathbb{K}|k$ generated by the coordinates of the pairs of $\mathcal{O}$.

Theorem 11 (Theorem 3.9 in [2]). *The field extensions $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$ are Galois.*

We denote by $G_x = \mathrm{Aut}(k(\mathcal{O})|k(x))$ and $G_y = \mathrm{Aut}(k(\mathcal{O})|k(y))$ their respective Galois groups, and by G_{xy} their intersection $G_x \cap G_y$ (which is the Galois group of the extension

$k(\mathcal{O})|k(x,y))$. We recall that the algebraic extension $k(\mathcal{O})|k(x)$ is *Galois* if $k(x)$ coincides with the subfield of $k(\mathcal{O})$ formed by the elements fixed by every automorphism in G_x.

Definition 12 (Group of the walk)**.** We define the *group of the walk* $G = \langle G_x, G_y \rangle$ to be the subgroup of k-algebra automorphisms of $k(\mathcal{O})$ generated by G_x and G_y.

It is easy to see from its definition that G acts by graph automorphisms on the graph of $\mathcal{O}$, and that its action is faithful. Moreover, while the group G is *a priori* not finitely generated, the left cosets $G_x/G_{x,y}$ and $G_y/G_{x,y}$ are of finite cardinal, respectively $d_x = \deg_X \widetilde{K}$ and $d_y = \deg_Y \widetilde{K}$ (Lemma 3.14 in [2]). We then fix $I_x = \{\mathrm{id}, \iota_1^x, \dots, \iota_{d_x-1}^x\}$ and $I_y = \{\mathrm{id}, \iota_1^y, \dots, \iota_{d_y-1}^y\}$ two respective sets of representatives for these two cosets.

Theorem 13 (Theorem 3.16 in [2])**.** *The subgroup $\langle I_x, I_y \rangle$ of G acts transitively on $\mathcal{O}$.*

Thus, the orbit $\mathcal{O}$ is realized as the action of a finite set of automorphisms on the pair (x,y), completing the analogy with the small steps setting.

Example 14 (Examples 3.10 and 3.18 in [2])**.** For $\mathcal{W}$ a model with small steps that has both positive and negative steps in each direction, $k(\mathcal{O}) = \mathbb{C}(x,y)$. Therefore, $G_{xy} = 1$, of index two in G_x and G_y. Hence, $G_x = \langle \psi \rangle$ and $G_y = \langle \phi \rangle$ with $\psi^2 = \phi^2 = 1$, and we find $G = \langle \phi, \psi \rangle$. The identities $\psi(h(x,y)) = h(\Psi(x,y))$ and $\phi(h(x,y)) = h(\Phi(x,y))$ yield an isomorphism between G and the group of small steps (§ 1.3).

Example 15 (continuing Example 4)**.** For $\mathcal{G}_\lambda$, we saw that $k(\mathcal{O}) = \mathbb{C}(x,y,x_1)$, with x_1 algebraic of degree 2 over $\mathbb{C}(x,y)$. Hence, $G_{xy} = \langle \tau \rangle$ with $\tau^2 = 1$, and after some computation we find $G_x = \langle \tau, \tau' \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $G_y = \langle \tau, \sigma \rangle \simeq S_3$ for $\tau'^2 = 1$ and $\sigma^3 = 1$. In the end, $G = \langle \tau, \tau', \sigma \rangle \simeq S_4$, which in this particular case coincides with the full group of graph automorphisms of $\mathcal{O}$.

3 Construction of invariants and decoupling

3.1 Fractions as elements of $k(\mathcal{O})$

To apply the Galois framework of Section 2 to the construction of rational invariants and decoupling, we define an evaluation of some fractions of $\mathbb{C}(X,Y,t)$ into $k(\mathcal{O})$. Its definition relies crucially on the fact that the kernel polynomial $\widetilde{K}(X,Y.t)$ is irreducible in $\mathbb{C}[X,Y,t]$ (Lemma 3.10 in [2]).

Definition 16. We call a fraction $H(X,Y,t)$ of $\mathbb{C}(X,Y,t)$ *regular* if the denominator of H is not divisible by $\widetilde{K}(X,Y,t)$.

Note that fractions of $\mathbb{C}(X,Y)$, $\mathbb{C}(X,t)$ and $\mathbb{C}(Y,t)$ are automatically regular because $\widetilde{K}(X,Y,t)$ is irreducible and trivariate by assumption on $\mathcal{W}$.

Definition 17. If (u,v) is a pair of the orbit and $H(X,Y,t)$ is a regular fraction of $\mathbb{C}(X,Y,t)$, define its evaluation on (u,v) to be $H_{(u,v)} = H(u,v,1/S(x,y)) \in k(\mathcal{O})$.

The evaluation on a pair of the orbit naturally extends to $\mathbb{C}$-linear combinations of pairs of the orbit (called 0-*chains*):

$$\text{for } c = \sum_{(u,v)\in\mathcal{O}} c_{u,v}(u,v), \quad \text{define} \quad H_c = \sum_{(u,v)\in\mathcal{O}} c_{u,v} H_{(u,v)}$$

Proposition 18 (Proposition 3.23 in [2])**.** *The evaluation homomorphism sending a regular fraction H to its evaluation $H_{(x,y)}$ maps bijectively $\mathbb{C}(X,t)$ to $k(x)$, $\mathbb{C}(Y,t)$ to $k(y)$ and $\mathbb{C}(X,Y)$ to $k(x,y)$. A regular fraction evaluates to 0 if and only if its numerator is divisible by $\widetilde{K}(X,Y,t)$.*

Thus, we can consider regular fractions as elements of the field $k(\mathcal{O})$, so as to benefit from our Galois-theoretic formalism. The homomorphism induces a relation on regular fractions: two regular fractions of $\mathbb{C}(X,Y,t)$ are called *Galois-equivalent* if their evaluations induce the same element in $k(\mathcal{O})$. Like the t-equivalence (Definition 5), this equivalence relation induces notions of invariants and decoupling:

- A pair of regular fractions $(I(X,t), J(Y,t))$ that are Galois equivalent is called a pair of *Galois invariants*. By Proposition 18, the evaluation homomorphism gives a correspondence between pairs of Galois invariants and elements of the subfield $k(x) \cap k(y)$ of $k(\mathcal{O})$, which we denote by k_{inv}, whose elements are fixed by G.

- Likewise, we say that a regular fraction $H(X,Y,t)$ admits a *Galois decoupling* pair $(F(X.t), G(Y,t))$ if H is Galois-equivalent to $F+G$. As above, a fraction H admitting a Galois decoupling corresponds through the evaluation homomorphism to a fraction h in $k(x,y)$ that writes as $h = f + g$ for some $f \in k(x)$ (fixed by G_x) and $g \in k(y)$ (fixed by G_y).

Proposition 3.23 in [2] implies that t-equivalent regular fractions are Galois equivalent. Therefore, the existence of rational t-invariants or t-decoupling of a fraction H is conditioned to the existence of their Galois counterparts (of which we give a complete treatment in the next two subsections). Once we have obtained non-constant Galois invariants or decoupling, we simply check if the Galois-equivalences involved are also t-equivalences, so that we obtain non-constant t-invariants and t-decoupling. If one of these two steps fails, then we know that non-constant rational t-invariants or t-decoupling of H do not exist.

3.2 Galois invariants

A pair of constant invariants $(F(t), F(t))$ is mapped by the evaluation homomorphism at (x,y) to an element of k. Therefore, the existence of non-constant Galois invariants is reduced to the field-theoretic question of whether the inclusion $k \subset k_{\text{inv}}$ is proper or not. This question is answered through the following result, which is a special instance of a theorem proved in the more general context of finite algebraic correspondences by Fried in [10], which we translate in the context of walks. This extends Theorem 4.6 in [1] and Corollary 4.6.11 in [9] to the large steps case.

Theorem 19 (Theorem 4.3 in [2])**.** *The following statements are equivalent:*

1. *the orbit $\mathcal{O}$ is finite,*
2. *G is a finite group,*
3. *there exists a pair of non-constant Galois invariants.*

When finite, the orbit is described by two explicit polynomials that cancel the left and right coordinates. In this case, the extension $k_{\text{inv}}|k$ is purely transcendental of transcendence degree 1, and any nonconstant coefficient of these polynomials is a generator, making the construction of rational invariants systematic (e.g. Equation (4.3) for $\mathcal{G}_\lambda$). Finally, as $k(\mathcal{O})^G = k_{\text{inv}}$, then $k(\mathcal{O})|k_{\text{inv}}$ is a finite Galois extension with Galois group G.

3.3 Galois decoupling

We now assume that the orbit is finite. Given a regular fraction $H(X, Y, t)$, we want to find a criterion for whether it admits a Galois decoupling, and if it does, to compute it. To this end, we define a notion of decoupling in the orbit.

Definition 20 (Definition 5.7 in [2])**.** Let $(\gamma_x, \gamma_y, \alpha)$ be a tuple of 0-chains such that $(x, y) = \gamma_x + \gamma_y + \alpha$ (with (x, y) being considered as a vertex in the orbit). This is called a *decoupling of (x, y) in the orbit* if for every regular $H(X, Y, t)$ the following conditions hold: (1) $H_{\gamma_x} \in k(x)$, $H_{\gamma_y} \in k(y)$ and (2) $H_\alpha = 0$ when H admits a Galois decoupling.

In the proof of Theorem 4.11 in [1], the authors construct an explicit decoupling of (x, y) for the cyclic orbits of small steps models. We extend their result to an arbitrary finite orbit using our Galois-theoretic framework and graph homology.

Theorem 21 (Theorem 5.10 in [2])**.** *If the orbit is finite, the pair (x, y) always admits a decoupling $(\gamma_x, \gamma_y, \alpha)$ in the orbit (in the sense of Definition 20).*

Thanks to this result, the question of the existence of the Galois decoupling of a regular fraction may be decided through an evaluation:

Proposition 22. *If $(\gamma_x, \gamma_y, \alpha)$ is a such a tuple, then a regular fraction $H(X, Y, t)$ admits a Galois decoupling if and only if $H_\alpha = 0$, and the decoupling is given by $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y}$.*

Note however that for an arbitrary 0-chain c, it is not always convenient to compute the evaluation H_c, because the coordinates of elements of the orbit are random algebraic elements. A friendlier family for computer algebra is composed of 0-chains of the form

$$c = \sum_{(u,v)\in\mathcal{O}, P(u)=0} (u,v) \quad \text{or} \quad c = \sum_{(u,v)\in\mathcal{O}, P(v)=0} (u,v)$$

with P a polynomial over $\mathbb{C}(x,y)$, which we call *symmetric chains*. It is easy to compute the evaluation of regular fractions over symmetric chains via *Newton's identities*. Some symmetric chains are presented as *level lines* for a well chosen distance in the graph of the orbit. They are denoted by X_i and Y_i in Figure 3.

Assuming a *distance transitivity property* on the graph of the orbit, we refine Theorem 21 by showing that the pair (x,y) admits a decoupling $(\gamma_x, \gamma_y, \alpha)$ with γ_x and γ_y composed of level lines, whose expression is explicit (Theorem 5.34 in [2]). This assumption was verified for all finite orbits arising from weighted models with steps in $\{-1,0,1,2\}$ (Figure 10 in [3]) which includes the one of Figure 3. Other families of orbits have been checked such as the ones arising from *Hadamard* (Section 11 of [3]) and *Tandem* models (Section 3.2 of [7]).

Example 23. For the orbits of the same type as in Figure 3, the pair (x,y) admits a decoupling in terms of the symmetric chains X_i and Y_i. It reads

$$(x,y) = \left(\frac{X_0}{2} - \frac{X_1}{8} + \frac{X_2}{8}\right) + \left(\frac{Y_0}{4} - \frac{Y_1}{4}\right) + \alpha.$$

We evaluated the fraction XY on it to obtain its Galois decoupling (4.2).

4 An example: the model $\mathcal{G}_\lambda$ is algebraic

We illustrate here with the model $\mathcal{G}_\lambda$ the generic strategy for proving algebraicity of large steps models with small backward steps described in Section 1.4. First, recall the equation found for the generating function of walks on $\mathcal{G}_\lambda$ in Example 1:

$$\widetilde{K}(X,Y,t)Q(X,Y,t) = XY - t(X^2+1)Q(X,0,t) - tQ(0,Y,t) + tQ(0,0,t). \qquad (1.1)$$

Note that the generating function $Q(X,Y,t)$ has polynomial coefficients, hence the left hand-side of the equation is t-equivalent to 0, so the functional equation translates into

$$XY \equiv \Big(t(X^2+1)Q(X,0,t) - tQ(0,0)\Big) + tQ(0,Y,t). \qquad (4.1)$$

Moreover, using the decoupling of (x,y) in the orbit of $\mathcal{G}_\lambda$ (Example 23), Proposition 22 gives a Galois decoupling of the fraction XY, which is checked to be the t-decoupling

$$XY \equiv -\frac{3\lambda X^2 t - \lambda t - 4X}{4t(X^2+1)} + \frac{-\lambda Y - 4}{4Y}. \qquad (4.2)$$

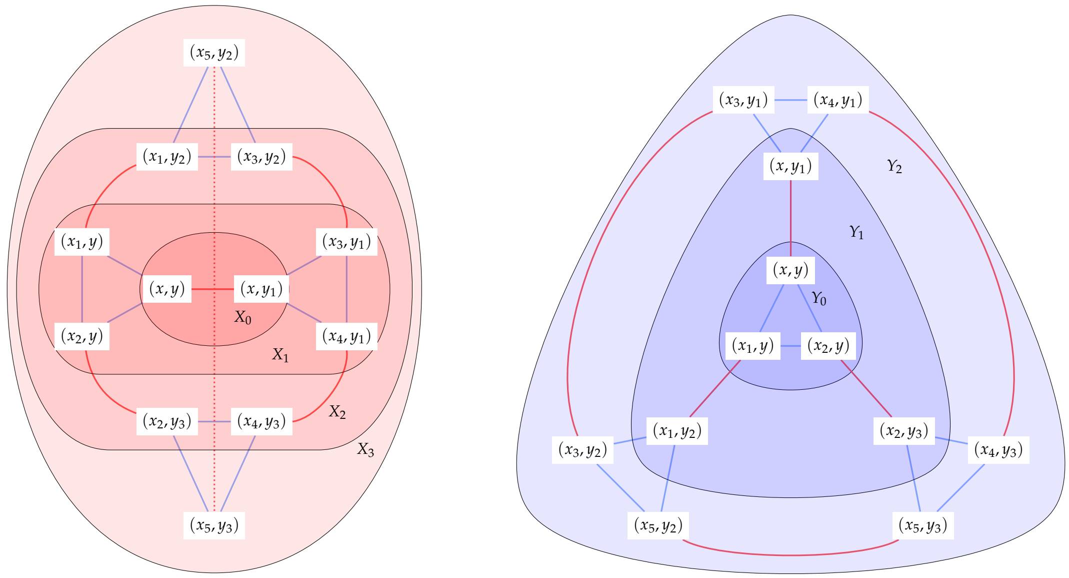


Figure 3: The orbit $\mathcal{O}_{12}$ of $\mathcal{G}_\lambda$ in two perspectives, illustrating a distance transitivity property (the 0-chains X_i and Y_i are the sums of vertices in their respective regions).

Combining Equations (4.1) and (4.2), we obtain the t-equivalence

$$\Big(t(X^2+1)Q(X,0,t) - tQ(0,0,t)\Big) + \frac{3\lambda X^2 t - \lambda t - 4X}{4t(X^2+1)} \equiv \frac{-\lambda Y - 4}{4Y} - tQ(0,Y,t),$$

which gives a first pair of invariants $P_1 = (I_1(X,t), J_1(Y,t))$. Note that $J_1(Y,t)$ has a pole at $Y = 0$, so this pair does not satisfy the conditions of Lemma 9.

The orbit of the model $\mathcal{G}_\lambda$ being finite, we also obtain automatically the following pair $P_2 = (I_2(X,t), J_2(Y,t))$ of Galois invariants, which we check to be t-invariants:

$$\left(\frac{\left(-\lambda^2 X^3 - 1\, X^4 - X^6 + X^2 + 1\right)t^2 - X^2 \lambda \left(X^2 - 1\right)t + X^3}{t^2 X (X^2+1)^2}, \frac{-t\, Y^4 + \lambda t Y + Y^3 + t}{Y^2 t} \right). \tag{4.3}$$

In order to find a pair of invariants satisfying the conditions of Lemma 9, the heuristic is to combine the pairs of invariants P_1 and P_2 using ring operations in order to remove their poles both in X and Y, by examining their Taylor expansions in their respective variables. Unlike the previous steps, the pole elimination is not systematic and requires a case by case treatment. This leads us to define $P_3 = (I_3(X,t), J_3(Y,t))$ to be

$$P_2\left(P_1 - \tfrac{\lambda}{4}\right) - P_1^3 + \left(2tQ(0,0) - \tfrac{\lambda}{4}\right)P_1^2 + \left(2t\tfrac{\partial Q}{\partial y}(0,0) - t^2 Q(0,0)^2 + \tfrac{5\lambda^2}{16}\right)P_1.$$

Using the functional equation, we are now able to check that this pair of t-invariants indeed satisfies the conditions of Lemma 9. Therefore, there exists a power series $A(t)$

in $\mathbb{C}[[t]]$ such that $I_3(X,t) = J_3(Y,t) = A(t)$. These are equations with one catalytic variable for $Q(X,0,t)$ and $Q(0,Y,t)$ that satisfy the assumptions of Theorem 3 in [4]. This allows us to conclude that these series are algebraic over $\mathbb{C}(X,Y,t)$, so that the same holds for the generating function $Q(X,Y,t)$ of the model $\mathcal{G}_\lambda$. Following the method of [4], we found an explicit minimal polynomial for the series $Q(0,0,t)$ of degree 32 with coefficients in $\mathbb{Q}(\lambda,t)$, proving in particular the algebraicity conjecture on the excursion series of the two models in [3] (lines 2 and 3 in Table 4, which are the reversed models of $\mathcal{G}_0$ and $\mathcal{G}_1$ but sharing the same excursion series).

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Locally Invariant Vectors in Representations of Symmetric Groups

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Abstract. For each permutation w in S_n and each irreducible representation $(\rho_\lambda, V_\lambda)$ of S_n, we determine when $\rho_\lambda(w)$ admits a non-zero invariant vector in V_λ. We find that non-zero invariant vectors exist in most cases, with very few exceptions.

Keywords: symmetric group, locally invariant vector, cyclic permutation representation, global conjugacy class

1 Introduction

This extended abstract is based on the results of [8]. The main results of this article are motivated by various problems in representation theory which we survey in the first four subsections of this introduction. The main results are stated in Section 1.5. Section 2 contains an outline of the proof of our main theorem. Details can be found in [8]. In Section 3, we list some interesting questions for further study.

1.1 Locally Invariant Vectors

Let G be a finite group and let V be a complex representation of G. A G-invariant vector in V is a vector $v \in V$ such that $g \cdot v = v$ for all $g \in G$. The representation V admits a G-invariant vector if and only if it contains the trivial representation of G as a subrepresentation.

In this article, we will be concerned with *locally* G-invariant vectors. Fixing an element $g \in G$, we ask if there exists a non-zero vector $v \in V$ such that $g \cdot v = v$. It is easy to see that the existence of such a vector depends on $g \in G$ only through its conjugacy class.

Let $C(G)$ denote the set of conjugacy classes of G. Let $\mathrm{Irr}(G)$ denote the set of irreducible complex representations of G up to isomorphism.

Question 1. Given a finite group G, for which pairs $(C, V) \in C(G) \times \mathrm{Irr}(G)$ does there exist a non-zero vector $v \in V$ such that $g \cdot v = v$ for some element $g \in C$?

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1.2 Cyclic Permutation Representations

A *cyclic representation* of G is a representation that is induced from a multiplicative character of a cyclic subgroup of G. Artin [1] proved that every complex representation of any finite group can be expressed as a virtual rational linear combination of cyclic representations. This allowed him to show that some integer power of the Artin L-function associated to any representation of G extends to a meromorphic function on the complex plane.

Brauer [3] showed that every representation of G is a virtual *integer* linear combination of representations induced from linear characters of (not necessarily Abelian) subgroups of G. Brauer showed that the subgroups of G can all be taken to be *elementary* (product of a p-group with a cyclic group of order coprime to p for some prime p). This variation on Artin's theorem was used to improve Artin's result on L-functions, concluding that Artin L-functions extend to meromorphic functions on the complex plane.

A *permutation representation* is a representation induced from the trivial representation of a subgroup of G.

Definition 2 (Cyclic permutation representation)**.** A *cyclic permutation representation* of a finite group G is a representation that is induced from the trivial representation of a cyclic subgroup of G.

In general (even for symmetric groups), it is not true that every representation of G is a virtual rational linear combination of cyclic permutation representations.

Given $g \in G$, let $V_g = \mathrm{Ind}_{\langle g \rangle}^{S_n} 1$ denote the representation of G induced from the trivial representation of the cyclic group $\langle g \rangle$ generated by g. The isomorphism class of V_g depends only on the conjugacy class of g in G. By Frobenius reciprocity, given $V \in \mathrm{Irr}(G)$, g admits a non-zero invariant vector in V if and only if V occurs in the decomposition of V_g into irreducibles. Thus Question 1 can be reformulated in terms of cyclic permutation representations as follows:

Question 3. Given a finite group G, for which pairs $(C, V) \in C(G) \times \mathrm{Irr}(G)$ does V occur in V_g?

Let $Z_G(g)$ denote the centralizer of g in G. The following definition is due to Heide and Zalessky [4].

Definition 4 (Global conjugacy class)**.** Let G be a finite group. The conjugacy class of an element $g \in G$ is said to be a *global conjugacy class* if every irreducible representation of G occurs in the permutation representation $\mathrm{Ind}_{Z_G(g)}^{G} 1$.

The group algebra $\mathbf{C}[G]$ of G can be thought of as a representation of G via the action of G on itself by conjugation, called the *adjoint representation* of G. Heide, Saxl, Tiep, and Zalessky [5] showed that the adjoint representation of G contains every irreducible representation of G for every finite simple group G except $G = SU(n, q^2)$ when n is odd and coprime to $q+1$. Since $\mathbf{C}[G]$ is a direct sum of the representations $\mathrm{Ind}_{Z_G(g)}^G 1$ as g runs over the conjugacy classes of G, if G admits a global conjugacy class then the adjoint representation of G contains every irreducible representation of G. Heide and Zalessky [4, Conjecture 1.5] conjectured that the converse is true: if every irreducible representation of a finite simple group G occurs in its adjoint representation then G admits a global conjugacy class. They proved this conjecture for alternating groups A_n, $n > 4$, and for all sporadic simple groups.

Sheila Sundaram [10, Theorem 5.1] characterized global conjugacy classes for all symmetric groups (see Theorem 7). She showed [10, Theorem 1.1] that a symmetric group S_n admits a global conjugacy class if and only if $n = 6$ or $n \geq 8$.

For every element $g \in G$, the cyclic group $\langle g \rangle$ generated by g is a subgroup of the centralizer group $Z_G(g)$. It follows that $\mathrm{Ind}_{Z_G(g)}^G 1$ is a subrepresentation of $\mathrm{Ind}_{\langle g \rangle}^G 1$. Thus, if the conjugacy class of G is a global class, then every irreducible representation V of G admits a non-zero vector $v \in V$ such that $g \cdot v = v$.

Definition 5. Let G be a finite group, and let C be a conjugacy class in G. We say that C is a *cyclically global* class if $\mathrm{Ind}_{\langle g \rangle} 1$ contains every irreducible representation of G.

A complete answer to the equivalent Questions 1 or 3 will result in the characterization of cyclically global conjugacy classes in G.

1.3 Immersion of Representations

Prasad and Raghunathan [9] proposed a partial order on automorphic representations called immersion. Adapted to finite groups, it may be defined as follows.

Definition 6. Given representations (ρ, V) and (σ, W) of G, say that V *is immersed in* W, denoted $V \preccurlyeq W$, if for every $g \in G$ and every $\lambda \in \mathbf{C}$, the multiplicity of λ as an eigenvalue of $\rho(g)$ does not exceed the multiplicity of λ as an eigenvalue of $\sigma(g)$.

In particular, if V is a subrepresentation of W, then $V \preccurlyeq W$.

Let 1 denote the trivial representation of G. Then $1 \preccurlyeq V$ if and only if, for every $g \in G$, there exists a non-zero vector $v \in V$ such that $g \cdot v = v$.

1.4 Results for Symmetric Groups

In this section, we follow standard notation from the theory of symmetric functions. See e.g., Macdonald [7].

Let S_n denote the nth symmetric group. The conjugacy class of $w \in S_n$ is completely determined by the cycle type of w which is a partition $\mu \vdash n$. For each $\mu \vdash n$, let w_μ denote a permutation with cycle type μ.

Following Schur, irreducible representations of S_n are elegantly characterized by associated symmetric functions. If the representation V of S_n has character $\chi : S_n \to \mathbf{C}$, its Frobenius characteristic is defined as the symmetric function

$$\mathrm{ch}_n \chi = \sum_{\mu \vdash n} \frac{\chi(w_\mu)}{z_\mu} p_\mu,$$

where p_μ denotes the power sum symmetric function associated to the partition μ and z_μ denotes the number of permutations in S_n that commute with w_μ.

For every partition $\lambda \vdash n$, there is a unique irreducible representation V_λ of S_n whose character χ^λ satisfies

$$\mathrm{ch}_n \chi^\lambda = s_\lambda,$$

where s_λ is the Schur function associated to $\lambda \vdash n$. The representations $\{V_\lambda \mid \lambda \vdash n\}$ are the irreducible representations of S_n.

Sundaram's characterization of global conjugacy classes for symmetric groups is the following.

Theorem 7 (Sundaram [10, Theorem 5.1])**.** *Let $n \neq 4, 8$. A partition of n is the cycle type of a global conjugacy class in S_n if and only if it has at least two parts, and all its parts are odd and distinct.*

When $\mu = (n)$, w_μ is an n-cycle in S_n and $Z_{S_n}(w_{(n)}) = \langle w_{(n)} \rangle$. The decomposition of the cyclic permutation representation of S_n induced from $\langle w_{(n)} \rangle$ into irreducible representations has a nice combinatorial interpretation.

Theorem 8 (Kraśkiewicz and Weyman [6])**.** *Let χ_r denote the character of $\langle w_{(n)} \rangle$ which takes $w_{(n)}$ to $e^{2\pi i r/n}$. For any $\lambda \vdash n$, the multiplicity of V_λ in $\mathrm{Ind}_{\langle w_{(n)} \rangle}^{S_n} \chi_r$ is given by the number $a_{\lambda,r}$ of standard tableaux of shape λ whose major index is congruent to r modulo n.*

However, it is not easy to say when $a_{\lambda,r}$ is positive. This question was resolved by Swanson [12, Theorem 1.5]. When $r = 0$, his results prove a conjecture of Sundaram [11, Remark 4.8].

Theorem 9. *For $\lambda \vdash n$, V_λ occurs in $\mathrm{Ind}_{\langle w_{(n)} \rangle}^{S_n} 1$ unless λ is one of*

1. $(n-1, 1)$,
2. $(2, 1^{n-2})$ *with n odd,*
3. (1^n) *with n even.*

1.5 Our Main Results

Let A_n denote the alternating group, a subgroup of index 2 in S_n.

Main Theorem. *The only pairs of partitions (λ, μ) of a given integer n such that w_μ does not admit a nonzero invariant vector in V_λ are the following:*

1. $\lambda = (1^n)$, *μ is any partition of n for which $w_\mu \notin A_n$,*
2. $\lambda = (n-1, 1)$, $\mu = (n)$, $n \geq 2$,
3. $\lambda = (2, 1^{n-2})$, $\mu = (n)$, *$n \geq 3$ is odd,*
4. $\lambda = (2^2, 1^{n-4})$, $\mu = (n-2, 2)$, *$n \geq 5$ is odd,*
5. $\lambda = (2, 2)$, $\mu = (3, 1)$,
6. $\lambda = (2^3)$, $\mu = (3, 2, 1)$,
7. $\lambda = (2^4)$, $\mu = (5, 3)$,
8. $\lambda = (4, 4)$, $\mu = (5, 3)$,
9. $\lambda = (2^5)$, $\mu = (5, 3, 2)$.

It follows that most irreducible representations of S_n admit w-invariant vectors for every permutation w. In terms of the notion of immersion (Definition 6), we have

Theorem 10. *Given a partition $\lambda \vdash n$, $V_{(n)} \preccurlyeq V_\lambda$ if and only if λ is not one of*

1. (1^n),
2. $(n-1, 1)$ *for $n \geq 2$,*
3. $(2, 1^{n-2})$ *when $n \geq 3$ is odd,*
4. $(2^2, 1^{n-4})$, *when $n \geq 5$ is odd,*
5. $(2, 2)$, (2^3), (2^4), (4^2) *and* (2^5).

Because the sign representation does not admit any non-zero invariant vector for a permutation that does not lie in A_n, conjugacy classes of S_n that are not contained in A_n cannot be cyclically global. We find that most conjugacy classes of S_n which are contained in A_n are cyclically global.

Theorem 11. *Given a partition $\mu \vdash n$ the conjugacy class in S_n consisting of permutations with cycle type μ is cyclically global if and only if it is contained in A_n and μ is not one of*

1. (n) *for* $n \geq 2$*,*
2. $(n-2,2)$ *for* $n \geq 5$ *odd,*
3. $(3,1)$*,* $(5,3)$*.*

While conjugacy classes that are not contained in A_n cannot be cyclically global, for most of them, the only obstruction to being cyclically global is the sign representation.

Definition 12 (Persistent class)**.** A permutation $w \in S_n$ is said to be persistent if $\mathrm{Ind}_{\langle w \rangle}^{S_n} 1$ contains V_λ for every $\lambda \vdash n$ with the possible exception of $\lambda = (1^n)$. If w is persistent then every permutation in its conjugacy class is persistent.

By Frobenius reciprocity, w is persistent if there exists a non-zero $v \in V_\lambda$ such that $w \cdot v = v$ for all $\lambda \vdash n$ such that $\lambda \neq (1^n)$. It turns out that for most partitions μ, w_μ is persistent.

Theorem 13. *Given* $\mu \vdash n$*,* w_μ *is persistent unless* μ *is one of the following:*

1. (n) *when* $n \geq 2$*,*
2. $(n-2,2)$*, when* $n \geq 5$ *is odd,*
3. $(3,1)$*,* $(3,2,1)$*,* $(5,3)$*,* $(5,3,2)$*.*

2 Proof of the Main Theorem

The main theorem is proved using Swanson's theorem (Theorem 9) and the Littlewood-Richardson rule. We outline the main steps in the proof in this section.

2.1 Reformulation in terms of Symmetric Functions

Definition 14. Given symmetric functions f and g with integer coefficients, say that $f \geq g$ if $f - g$ is a non-negative integer combination of Schur functions.

Define

$$f_\mu = \mathrm{ch}_n \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1.$$

Then V_λ occurs in $\mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1$ if and only if

$$f_\mu \geq s_\lambda \tag{2.1}$$

If $\mu = (\mu_1, \dots, \mu_k)$ let $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_k}$ be the Young subgroup corresponding to the cycles of w_μ. Let D_μ be the subgroup of S_μ generated by the cycles of w_μ. Thus D_μ is a product of cyclic groups of orders $\mu_1, \mu_2, \dots, \mu_k$. Using induction in stages,

$$f_\mu = \operatorname{ch} \operatorname{Ind}_{S_\mu}^{S_n} \operatorname{Ind}_{D_\mu}^{S_\mu} \operatorname{Ind}_{C_\mu}^{D_\mu} 1.$$

Therefore

$$f_\mu \geq \operatorname{ch} \operatorname{Ind}_{S_\mu}^{S_n} \operatorname{Ind}_{D_\mu}^{S_\mu} 1 = \prod_{i=1}^{k} f_{(\mu_i)}. \tag{2.2}$$

Swanson's theorem (Theorem 9) tells us that $f_{(n)} \geq s_\lambda$ for most partitions λ of n. We will use this fact, together with the inequality (2.2), to establish (2.1) in most cases using the Littlewood-Richardson rule. Recall that the Littlewood-Richardson coefficients $c^\lambda_{\alpha\beta}$ are defined by

$$s_\alpha s_\beta = \sum_\lambda c^\lambda_{\alpha\beta} s_\lambda.$$

The Littlewood-Richardson rule [7, Section I.9] asserts that $c^\lambda_{\alpha\beta}$ is the number of LR-tableaux of shape λ/α and weight β. Recall that an LR-tableau is a semistandard skew-tableau whose reverse row reading word is a lattice permutation.

2.2 The Basic Lemmas

Our proof of the main will make frequent use of the following lemmas.

Lemma 15. *For every partition λ of $p+q$, and every partition α of p that is contained in λ, there exists a partition β of q such that $s_\alpha s_\beta \geq s_\lambda$.*

Proof. Let $T_{\lambda\alpha}$ denote the skew-tableau obtained by putting i in the ith cell of each column of λ/α. Let β be the weight of $T_{\lambda\alpha}$. For example, if $\lambda = (5,4,4,1)$ and $\alpha = (3,2,1)$ then

$$T_{\lambda\alpha} = \begin{array}{|c|c|c|c|c|} \hline \; & \; & \; & 1 & 1 \\ \hline \; & \; & 1 & 2 \\ \cline{1-4} \; & 1 & 2 & 3 \\ \cline{1-4} 1 \\ \cline{1-1} \end{array},$$

and β is $(5,2,1)$. Since every $i+1$ occurs below an i, $T_{\lambda\alpha}$ is an LR-tableau. The Littlewood-Richardson rule implies that $s_\alpha s_\beta \geq s_\lambda$. □

Lemma 15 is nothing more than the well-known statement that the skew-Schur function $s_{\lambda/\alpha}$ is non-zero whenever $\lambda \supset \alpha$. However, the method of constructing β in the proof is also used in our proof of the main theorem.

Lemma 16. *Given integers $p \geq 2$, $q \geq 1$, and a partition $\lambda \vdash (p+q)$ different from $(1^{(p+q)})$, there exists a partition $\beta \vdash q$ such that $f_{(q)} \geq s_\beta$ and $\beta \subset \lambda$.*

Proof. We consider the following cases:

Case 1: $\lambda \supset (q-1, 1)$

Since $p \geq 2$, the skew shape $\lambda/(q-1,1)$ has at least two cells. If at least one of these cells lies in the first row of λ, then choose $\beta = (q)$. If at least one of these cells lies in the first column of λ, then choose $\beta = (q-2,1,1)$ If neither of the above happens, then $\lambda/(q-1,1)$ has at least two cells in its second row. In this case $q-1 \geq 3$. Choose $\beta = (q-2,2)$. The possible placements of the cells of $\lambda/(q-1,1)$ are shown in Figure 1.

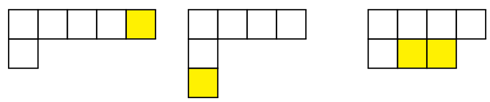

Figure 1: Possible placements of two cells of $\lambda/(q-1,1)$

In all these cases, Theorem 9 implies that $f_{(q)} \geq s_\beta$.

Case 2: $\lambda \supset (1^q)$ **and** q **is even**

Since $\lambda \neq (1^{p+q})$, the skew-shape $\lambda/(1^q)$ must contain at least one cell in the first row. Take $\beta = (2, 1^{q-2})$. By Theorem 9, $f_{(q)} \geq s_\beta$, since q is even.

Case 3: $\lambda \supset (2, 1^{q-2})$ **and** q **is odd**

If $\lambda/(2,1^{q-2})$ has a cell in its first row, take $\beta = (3, 1^{q-3})$. If $\lambda/(2,1^{q-2})$ has a cell in its first column, take $\beta = (1^q)$. By Theorem 9, $f_{(q)} \geq s_\beta$, since q is odd. Otherwise the second column of $\lambda/(2,1^{q-2})$ must have at least two cells in its second column. In this case $q \geq 4$. Take $\beta = (2,2,1^{q-4})$. The possible placements of the cells of $\lambda/(2,1^{q-2})$ are

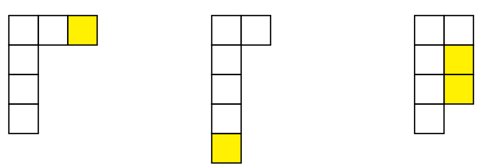

Figure 2: Possible placements of cells of $\lambda/(2,1^{q-2})$.

shown in Figure 2.

All remaining λ**:**

Take β to be any partition of q that is contained in λ. Since λ does not contain any of the exceptions of Theorem 9, $f_{(q)} \geq s_\beta$. □

2.3 The Main Steps of the Proof

For convenience, we will say that a partition μ is persistent if w_μ is persistent in the sense of Definition 12. The goal is to prove that, for $n \geq 11$, every partition $\mu \vdash n$ is persistent, except for the partitions $\mu = (n)$, and $\mu = (n-2, 2)$ when n is odd. The cases $n \leq 10$ are easily solved by computer calculation, using, for example, the Sage Mathematical Software [13]. The following lemma deals with most partitions that have two parts.

Lemma 17. *If $\mu = (p, q)$ where $p \geq q \geq 4$, then μ is persistent.*

For the details of the proof, we refer the reader to [8]. We only outline the main idea here.

We wish to show that $f_{(p,q)} \geq s_\lambda$ for every $\lambda \vdash (p+q)$ different from (1^{p+q}). By (2.2), $f_{(p,q)} \geq f_{(p)} f_{(q)}$. Hence, in order to show that $f_{(p,q)} \geq s_\lambda$, it suffices to find $\alpha \vdash p$ and $\beta \vdash q$ such that

$$f_{(p)} \geq s_\alpha, \; f_{(q)} \geq s_\beta, \text{ and } s_\alpha s_\beta \geq s_\lambda. \tag{2.3}$$

Lemma 16 allows us to choose $\beta \vdash q$ such that $\beta \subset \lambda$ and $f_{(q)} \geq s_\beta$. Using Lemma 15 with the roles of α and β reversed, we may choose $\alpha \vdash p$ such that $s_\alpha s_\beta \geq s_\lambda$. If $f_{(p)} \geq s_\alpha$ we are done. Otherwise, α must be one of the partitions occurring in Swanson's theorem (Theorem 9). Most of these cases are dealt with by prescribing a replacement for α and β so that (2.3) holds.

Lemma 17 can be leveraged to deal with most partitions with more than two parts using the following lemma.

Lemma 18. *A partition $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ with $k \geq 2$ is persistent if the partition $\tilde{\mu}$ obtained by removing a part μ_i from μ is persistent and $n - \mu_i \geq 4$.*

In order to prove the lemma, we wish to show that $f_\mu \geq s_\lambda$ for every $\lambda \vdash n$ except $\lambda = (1^n)$. Suppose $\mu = (\mu_1, \ldots, \mu_k)$. Noting that $C_{\tilde{\mu}} \times C_{(\mu_i)} \subset C_\mu$ and $D_{\tilde{\mu}} \times D_{(\mu_i)} = D_\mu$, we have

$$\mathrm{Ind}_{C_\mu}^{D_\mu} 1 \geq \mathrm{Ind}_{C_{\tilde{\mu}}}^{D_{\tilde{\mu}}} 1 \otimes \mathrm{Ind}_{C_{(\mu_i)}}^{D_{(\mu_i)}} 1$$

Inducing to $S_{p_1} \times \cdots \times S_{p_k}$, and then to $S_{p_1 + \cdots + p_k}$ gives

$$f_\mu \geq f_{\tilde{\mu}} f_{(\mu_i)}.$$

Hence it suffices to show that $f_{\tilde{\mu}} f_{(\mu_i)} \geq s_\lambda$ for all $\lambda \vdash n$ except $\lambda = (1^n)$. As before, it suffince to find $\alpha \vdash n - \mu_i$ and $\beta \vdash \mu_i$ such that $\alpha \neq (1^n)$, $f_{(\mu_i)} \geq \beta$ and $s_\alpha s_\beta \geq s_\lambda$. Again, using Lemma 16, we may choose β such that $\beta \subset \lambda$ and $f_{(\mu_i)} \geq s_\beta$. Again, using Lemma 15 with the roles of α and β reversed, we may choose $\alpha \vdash n - \mu_i$ such that $s_\alpha s_\beta \geq s_\lambda$. In this case, there is a way to replace α and β with another pair which have the required properties. The details of the proof are found in [8].

Lemmas 17 and 18 take care of most cases of partitions. To complete the proof, they need to be carefully put together with a few more cases, for which we refer the reader to [8].

3 Futher Questions

We conclude this extended abstract by enumerating a few interesting open questions.

Question 19. Classify the global conjugacy classes of the alternating group A_n.

Heide and Zalesski [4] proved the existence of at least one such class for each n and gave an algorithm to find it.

Question 20. Find "effective" versions of Artin and Brauer induction theorem (discussed in Section 3) for symmetric groups.

For the Artin induction theorem, this would mean finding, for each positive integer n, a set of pairs (μ, χ) where $\mu \vdash n$ and $\chi : \langle w_\mu \rangle \to \mathbf{C}$ is a multiplicative character such that the representations $\mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} \chi$ form a basis for the space of class functions on S_n. For the Brauer induction theorem, this would mean finding, for each positive integer n and each prime p, a set of pairs (A, χ) where A is an elementary subgroup of S_n and $\chi : A \to \mathbf{C}$ is a multiplicative character such that the characters of the representations $\mathrm{Ind}_A^{S_n} \chi$ form a basis for the space of class functions on S_n. Techniques developed by Boltje, Snaith and Symonds [2] may be useful in this context.

Consider the representation $U_\mu^\chi = \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} \chi$, where $\chi : \langle w_\mu \rangle \to \mathbf{C}$ is a primitive multiplicative character.

Question 21. Determine the set of triples (λ, μ, χ) such that U_μ^χ contains V_λ.

Since Swanson [12] solves this problem for $\mu = (n)$, this problem could also be amenable to the methods of [8].

Question 22. Find subgroups H of S_n that are maximal among subgroups for which V_λ occurs in $\mathrm{Ind}_H^{S_n} 1$ for every $\lambda \vdash n$.

Theorem 11 shows that there are many cyclic subgroups with this property, and hence there should be a large class of such maximal subgroups.

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Excedance quotients, Quasisymmetric Varieties, and Temperley–Lieb algebras

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Abstract. Let $R_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in n variables and consider the ideal $\langle \mathrm{QSym}_n^+ \rangle \subseteq R_n$ generated by quasisymmetric polynomials without constant term. It was shown by Aval–Bergeron–Bergeron that $\dim\left(R_n/\langle \mathrm{QSym}_n^+ \rangle\right) = C_n$ the nth Catalan number. We explain here this phenomenon by defining a set of permutations QSV_n with the following properties: first, QSV_n is a basis of the Temperley–Lieb algebra $\mathsf{TL}_n(2)$, and second, when considering QSV_n as a collection of points in $\mathbb{Q}^n$, the top-degree homogeneous component of the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$ is $\langle \mathrm{QSym}_n^+ \rangle$. Our construction has a few byproducts which are independently noteworthy.

Résumé. Soit $R_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ l'anneau des polynômes en n variables, et considérez l'idéal $\langle \mathrm{QSym}_n^+ \rangle \subseteq R_n$ engendré par les polynômes quasisymétriques sans terme constant. Il a été démontré par Ava–Bergeron–Bergeron que $\dim\left(R_n/\langle \mathrm{QSym}_n^+ \rangle\right) = C_n$ le n-ième nombre de Catalan. Nous expliquons ici ce phénomène en construisant un ensemble de permutations QSV_n ayant les propriétés suivantes: premièrement, QSV_n est une base de l'algèbre de Temperley–Lieb $\mathsf{TL}_n(2)$, et deuxièmement, en considérant QSV_n comme une collection de points dans $\mathbb{Q}^n$, la composante homogène de degré supérieur de l'idéal $\mathbf{I}(\mathrm{QSV}_n)$ est $\langle \mathrm{QSym}_n^+ \rangle$. Notre construction a quelques sous-produits qui sont indépendamment dignes d'intérêt.

Keywords: Quasisymmetric Polynomials, Bruhat order, Excedance, Temperley–Lieb

1 Introduction

Quasisymmetric functions originate in the work of Stanley [18], where they appear as enumeration series for P-partitions. Later, Gessel [8] gave a more algebraic treatment of the ring QSym spanned by all quasisymmetric functions, establishing a beautiful analogy with the classical ring of symmetric functions Sym. The importance of QSym has continued to increase: [1] established QSym as a universal setting for enumerative combinatorial invariants, and in recent years quasisymmetric functions have been at the center of a number of research programs (many examples can be found in [11, 15, 16] and references therein).

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In this abstract, based on the paper [4], we explore the striking similarity between quasisymmetric functions and the invariant theory of finite reflection groups. Chevalley's theorem states that each finite reflection group W acts naturally on a polynomial ring R, and the quotient of R by the ideal $\langle R_+^W \rangle$ generated by positive degree invariants is isomorphic to the regular module of W; see [13, Chapter 3]. Hivert [12] shows that the quasisymmetric polynomials QSym_n in $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ are likewise the invariants of an action of the Temperley–Lieb algebra $\mathsf{TL}_n(2)$ on R_n. Writing $\langle \mathrm{QSym}_n^+ \rangle$ for the ideal generated by the positive degree quasisymmetric polynomials, [2, 3] show that the dimension of the coinvariant space $R_n/\langle \mathrm{QSym}_n^+ \rangle$ and $\mathsf{TL}_n(2)$ agree: both are the nth Catalan number C_n. Since $\mathsf{TL}_n(2)$ shares many nice properties with reflection groups, one might expect a Chevalley-type theorem from this coincidence, but there is no obvious $\mathsf{TL}_n(2)$-action on $R_n/\langle \mathrm{QSym}_n^+ \rangle$: Hivert's action is not multiplicative and $\langle \mathrm{QSym}_n^+ \rangle$ is not a $\mathsf{TL}_n(2)$-submodule.

Motivated by the discussion above, we revisit two modules which afford the left regular representation of the symmetric group S_n:

(1) the quotient $R_n/\langle \mathrm{Sym}_n^+ \rangle$ of the polynomial ring $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ by the ideal generated by positive-degree symmetric polynomials Sym_n^+, and
(2) the coordinate ring $R_n/\mathbf{I}(S_n)$ for the vertices of the regular permutohedron S_n in $\mathbb{Q}^n$, which are the points $(\sigma_1, \ldots, \sigma_n)$ for each permutation σ on n letters.

Module (1) is a famous case of Chevalley's theorem: the S_n-invariants of R_n are the symmetric polynomials, and $R_n/\langle \mathrm{Sym}_n^+ \rangle$ is the S_n coinvariant ring. On the other hand, module (2) comes from the left multiplicative action of S_n on the permutohedron realized on the coordinate ring $R_n/\mathbf{I}(S_n)$ where $\mathbf{I}(S_n)$ is the vanishing ideal. However, as seen in the work of Garsia and Procesi [7] and reference therein, a careful inspection reveals that these modules determine one another! Consider the ideal

$$I_n = \langle f(x_1, \ldots, x_n) - f(1, \ldots, n) \mid f \in \mathrm{Sym}_n^+ \rangle \subseteq \mathbf{I}(S_n).$$

For each $f \in R_n$, let $\mathsf{h}(f)$ denote the top-degree homogeneous component of f, and for any ideal I in R_n write $\mathsf{gr}(I) = \langle \mathsf{h}(f) \mid f \in I \rangle$. Then $\mathsf{gr}(I_n) \supseteq \langle \mathrm{Sym}_n^+ \rangle$, and Gröbner basis theory gives a linear isomorphism $R_n/\mathsf{gr}(I_n) \cong R_n/I_n$. We therefore have

$$|S_n| = \dim\left(R_n/\langle \mathrm{Sym}_n^+ \rangle\right) \geqslant \dim\left(R_n/\mathsf{gr}(I_n)\right) = \dim\left(R_n/I_n\right) \geqslant \dim(R_n/\mathbf{I}(S_n)) = |S_n|,$$

so that $I_n = \mathbf{I}(S_n)$ and $\mathsf{gr}(I_n) = \langle \mathrm{Sym}_n^+ \rangle$, and $R_n/\langle \mathrm{Sym}_n^+ \rangle \cong R_n/\mathbf{I}(S_n)$ as vector spaces. This isomorphism respects the S_n-action on each quotient: both $\mathbf{I}(S_n)$ and $\langle \mathrm{Sym}_n^+ \rangle$ are fixed spaces for the standard S_n-action on R_n, and this action coindices with the action on points for $R_n/\mathbf{I}(S_n)$. Thus, we have an S_n-module isomorphism $R_n/\langle \mathrm{Sym}_n^+ \rangle \cong R_n/\mathbf{I}(S_n)$, though the left hand side has a natural grading and the right hand side does not.

Our work in [4] applies this approach to quasisymmetric functions and Temperley–Lieb algebras. It is known that $\langle \mathrm{Sym}_n^+ \rangle \subseteq \langle \mathrm{QSym}_n^+ \rangle$, and that there is a surjective algebra homomorphism $\phi : \mathbb{C}S_n \to \mathsf{TL}_n(2)$. Guided by these relationships, we searched for a subset $\mathrm{QSV}_n \subseteq S_n \subseteq \mathbb{Q}^n$ which satisfies:

(i) $|\mathrm{QSV}_n| = C_n$,
(ii) the image $\phi(\mathrm{QSV}_n)$ is a basis of $\mathsf{TL}_n(2)$, and
(iii) considering the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$, we have $\mathsf{gr}\big(\mathbf{I}(\mathrm{QSV}_n)\big) = \langle \mathrm{QSym}_n^+ \rangle$.

Assuming such a set exists, one can define an action of $\mathsf{TL}_n(2)$ on the space $R_n/\langle \mathrm{QSym}_n^+ \rangle$ using Gröbner basis theory and the multiplication constants for the basis obtained from QSV_n. However, QSV_n is not readily found: it took several years of computer exploration to find a list of candidates for small values of n. We have now found it, along with a number of remarkable properties that should be of interest to the wider community.

The set $\mathrm{QSV}_n \subseteq S_n$ is defined in Section 3. After discovering it, we noticed that the cycle structure of permutations in QSV_n determine a noncrossing partition, tying them to a more general story of Coxeter–Catalan combinatorics for the symmetric groups [5] (see also [17]). For example, writing Q_λ to denote the element of QSV_n indexed by the partition λ,

$$\lambda = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \qquad \text{corresponds to} \qquad Q_\lambda = (1)(72)(653)(4).$$

Through this connection, [9, 10] and [20] have studied bases of general Temperley–Lieb algebras which specialize to $\phi(\mathrm{QSV}_n)$ for $\mathsf{TL}_n(2)$, so only condition (iii) remains.

Our initial attempts to prove condition (ii) also led us to an exciting discovery about how QSV_n sits in S_n. In Section 4 we define an equivalence relation $\sim$ on S_n using the weak excedance set of a permutation and its inverse. We call the equivalence classes of $S_n/\sim$ *excedance classes*, and show that each noncrossing partition λ bijectively determines an excedance class $\mathcal{C}_\lambda$. Surprisingly, the Bruhat order induces a well-defined quotient order on excedance classes. In the following, $\preceq$ denotes the order on noncrossing partitions which is dual to Young's lattice, described further in Section 3.

Theorem 4.2. *Writing $\leqslant$ for the relation on excedance classes $S_n/\sim$ induced by the Bruhat order, $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ if and only if $\lambda \preceq \mu$.*

This exhibits a duality between sub- and quotient orders of the Bruhat poset: a parallel result is given by [10] for the set QSV_n as a sub-poset of the Bruhat order (see Section 3). The result of [10] also simplifies the proof of Theorem 4.2 we give in [4].

Corollary 4.3. *Each excedance class $\mathcal{C}_\lambda$ is an interval in the Bruhat order, with upper bound $Q_\lambda \in \mathrm{QSV}_n$ and lower bound given by a* 321*-avoiding permutation.*

The combinatorics of excedance classes are very rich, and there is much left to explore. In Section 5, we use excedance classes of S_n to produce bases of $\mathsf{TL}_n(2)$. Using results of [10] and [20], our Theorem 5.1 restates the fact that QSV_n satisfies condition (ii) above. However, our technique is more general, and produces many (often novel) bases of $\mathsf{TL}_n(2)$ coming from the surjection $\phi : \mathbb{C}S_n \to \mathsf{TL}_n(2)$.

Theorem 5.2. *Let $n \geqslant 0$ and for each noncrossing partition λ of size n, fix an element $w_\lambda \in \mathcal{C}_\lambda$. Then the set $\{\phi(w_\lambda) \mid \text{noncrossing partitions } \lambda\}$ is a basis of $\mathsf{TL}_n(2)$.*

Finally, in Section 6 we outline our approach to proving that the set QSV_n satisfies condition (iii) above. The space of positive-degree quasisymmetric polynomials QSym_n has a homogeneous basis of monomial quasisymmetric functions M_α indexed by the compositions $\alpha \vDash d$ of positive integers $d > 0$ with length $\ell(\alpha) \leqslant n$. For each such composition α, we construct a nonhomogeneous polynomial $P_\alpha \in R_n$ for which $\mathsf{h}(P_\alpha) = M_\alpha$ and show the following.

Theorem 6.3. *The ideal $\langle P_\alpha \mid \alpha \vDash d \text{ with } d > 0 \text{ and } \ell(\alpha) \leqslant n\rangle \subseteq R_n$ is the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$ and $\langle \mathrm{QSym}_n^+\rangle = \mathsf{gr}\big(\mathbf{I}(\mathrm{QSV}_n)\big)$.*

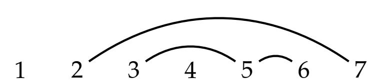
From this, we obtain a linear isomorphism $R_n/\mathbf{I}(\mathrm{QSV}_n) \cong R_n/\langle \mathrm{QSym}_n^+\rangle$.

2 Noncrossing partitions and Bruhat order

Noncrossing partitions: Let n be a nonnegative integer. A *noncrossing partition* of size n is a diagram λ consisting of:

1. the positive integers $1, \ldots, n$, placed from left to right along a horizontal axis; and
2. a set of left-to-right arcs $i \frown j = (i, j)$, $i < j$ drawn above the axis with no intersections or coterminal points: λ contains no pair $i \frown k$, $j \frown l$ with $i \leqslant j < k \leqslant l$.

For example,

$\lambda =$  (2.1)

is a noncrossing partition of size 7 containing three arcs: $2 \frown 7$, $3 \frown 5$, and $5 \frown 6$.

Considering a noncrossing partition λ as an (undirected) graph, the connected components of λ give a partition of the set $[n] = \{1, \ldots, n\}$, which is the origin of the term. For example, the noncrossing partition shown in Equation (2.1) corresponds to the set partition $\big\{\{1\}, \{2, 7\}, \{3, 5, 6\}, \{4\}\big\}$. Let

$$\mathrm{NCP}_n = \{\text{noncrossing partitions of size } n\}.$$

The size of NCP_n is the nth Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$ [19].

Given an arc $i \frown j \in \lambda$, say that i is the *left endpoint* and j is the *right endpoint*, and let

$$\lambda^+ = \{\text{left endpoints in } \lambda\} \qquad \text{and} \qquad \lambda^- = \{\text{right endpoints in } \lambda\}.$$

For example, with the noncrossing partition λ in (2.1), $\lambda^+ = \{2, 3, 5\}$ and $\lambda^- = \{5, 6, 7\}$. The arcs in λ give a bijection between the sets λ^+ and λ^-, so that $|\lambda^+| = |\lambda^-|$.

Permutations and the Bruhat order: Let S_n denote the group of permutations of $[n]$. We represent elements of S_n either by using the standard one- and two-line notations or as a product of cycles. We also write ℓ for the length function, so that for $w \in S_n$, $\ell(w)$ is the number of inversions of w: $\ell(w) = |\{(i, j) \mid 1 \leqslant i < j \leqslant n \text{ and } w_i > w_j\}|$.

The *Bruhat order* on S_n is the partial order generated by the relation

$$v < w \qquad \text{if and only if} \qquad wv^{-1} \text{ is a transposition } (i\,j) \text{ and } \ell(v) < \ell(w).$$

This order is ubiquitous in the study of S_n and related objects (for examples, see [6]).

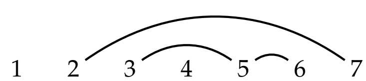
3 The set QSV_n

Let λ be a noncrossing partition of size n. Define a permutation $Q_\lambda \in S_n$ by

$$Q_\lambda(j) = \begin{cases} i & \text{if } j \in \lambda^- \text{ and } i \frown j \in \lambda \\ k & \text{if } j \notin \lambda^- \text{ and } k \text{ is the largest element connected to } j \text{ in } \lambda \end{cases}$$

Thus, Q_λ sends each $j \in [n]$ to its leftward neighbor in λ, if such a neighbor exists, and otherwise sends j to the rightmost element of its connected component.

The cycles of Q_λ correspond to the connected components of λ, for example, with

$\lambda =$  we have $\quad Q_\lambda = (1)(72)(653)(4) = 1764352.$

Let $\mathrm{QSV}_n = \{Q_\lambda \mid \lambda \in \mathrm{NCP}_n\}$. For example, the elements of QSV_3 are:

$$Q_{1 \frown 3,\ 2} = 321, \qquad Q_{1 \frown 2 \frown 3} = 312, \qquad Q_{1 \frown 2,\ 3} = 213,$$

$$Q_{1,\ 2 \frown 3} = 132, \qquad \text{and} \qquad Q_{1,\ 2,\ 3} = 123.$$

Remark 3.1. Given any n-cycle $c \in S_n$, [5] gives a bijection between NCP_n and the interval between the identity and c in the absolute order on S_n. Our construction of the permutations Q_λ realize this bijection for the n-cycle $c = (n \cdots 21)$.

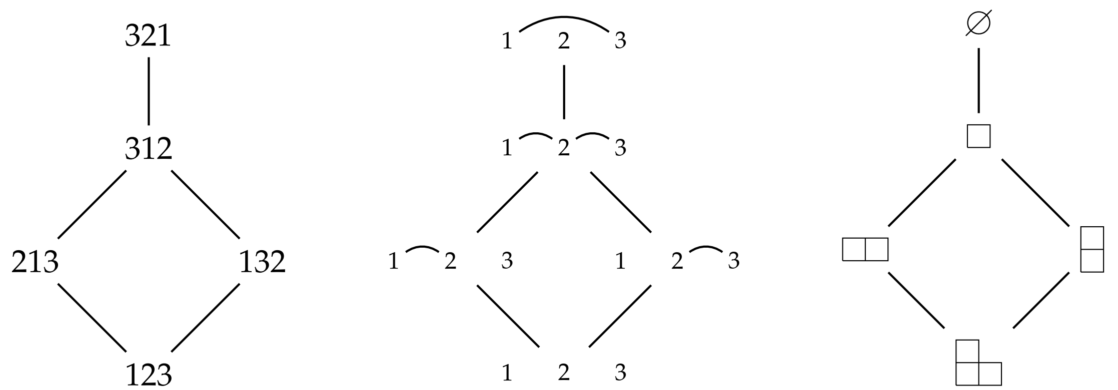


Figure 1: From left to right, the Hasse diagrams of: QSV_3 with the Bruhat order; NCP_3 with $\preceq$; and the dual interval in the Young's lattice.

The Bruhat order on QSV_n: The Bruhat order on S_n described in Section 2 restricts to a partial order on the set QSV_n. This order turns out to be very natural, as is described in the paper [10], and we recall the description for use in later sections.

Define a partial order $\preceq$ on the set NCP_n of noncrossing partitions as the extension of the covering relation: λ is covered by μ if and only if λ is obtained from μ in one of the following ways:

1. removing an arc of the form $i \frown i{+}1$ from μ, or
2. replacing any arc $i \frown k$ in μ with two arcs $i \frown j$ and $j \frown k$ for some $i < j < k$ which do not intersect or share a left or right endpoint with any other arc in μ.

Proposition 3.2 ([10, Theorem 1.1 and Corollary 7.5])**.** *Let λ and μ be noncrossing partitions of size n. The following are equivalent:*

1. $\lambda \preceq \mu$,

2. $Q_\lambda \leqslant Q_\mu$ in the Bruhat order.

Moreover, the partial orders on NCP_n and QSV_n are each dual to the interval between the empty diagram and the staircase in Young's lattice; see Figure 1.

Remark 3.3. In fact, [10] describes the Bruhat order on the set $\{\omega_0 w \omega_0^{-1} \mid w \in \mathrm{QSV}_n\}$, where ω_0 is the longest element of S_n. Vis-a-vis Remark 3.1, these are the non-crossing partitions associated with the cycle $(12\ldots n)$ instead of $(n\ldots 21)$. Since conjugation by ω_0 is an automorphism of the Bruhat order, this result is equivalent to Proposition 3.2.

4 The excedance quotient of the Bruhat order

In this section we describe a novel equivalence relation $\sim$ on S_n and show that it induces a quotient of the Bruhat order. This equivalence relation is defined in a simple way using the weak excedances of a permutation. We have discovered a number of nice properties of the equivalence classes in $S_n/\sim$, which we summarize after our initial definition.

Given a permutation $w \in S_n$, a *weak excedance* of w is a pair (i, w_i) for which $i \leqslant w_i$. We define the *excedance values* $E_{val}(w)$ and *excedance positions* $E_{pos}(w)$ to be the sets

$$E_{val}(w) = \{w_i \mid (i, w_i) \text{ is a weak excedance of } w\}, \text{ and}$$

$$E_{pos}(w) = \{i \mid (i, w_i) \text{ is a weak excedance of } w\}.$$

The sets $E_{val}(w)$ and $E_{pos}(w)$ are most easily seen using two-line notation for permutations. For example, marking the non-excedances of a permutation in red,

$$w = \overset{12345678}{35142658}, \qquad E_{pos}(w) = \{1,2,4,6,8\}, \qquad \text{and} \qquad E_{val}(w) = \{3,4,5,6,8\}.$$

We define the *excedance relation* $\sim$ on S_n by:

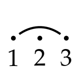
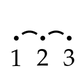
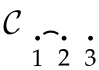
$$v \sim w \qquad \text{if and only if} \qquad E_{val}(v) = E_{val}(w) \text{ and } E_{pos}(v) = E_{pos}(w), \tag{4.1}$$

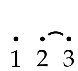
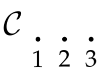
and say that each equivalence class of $S_n/\sim$ is an *excedance class*.

We now summarize our main results on excedance classes. Each noncrossing partition λ of size n determines an excedance class:

$$\mathcal{C}_\lambda = \{w \in S_n \mid E_{val}(w) = [n] - \lambda^+ \text{ and } E_{pos}(w) = [n] - \lambda^-\}.$$

This construction is bijective, so that the excedance classes are counted by the Catalan numbers. For example, the five excedance classes of S_3 are:

$\mathcal{C}$  $= \{\overset{123}{321}, \overset{123}{231}\}$, $\mathcal{C}$  $= \{\overset{123}{312}\}$, $\mathcal{C}$  $= \{\overset{123}{213}\}$,

$\mathcal{C}$  $= \{\overset{123}{132}\}$, and $\mathcal{C}$  $= \{\overset{123}{123}\}$.

The Bruhat order induces a relation on $S_n/\sim$. Recall the order $\preceq$ from Section 3.

Theorem 4.2. *Writing $\leqslant$ for the relation on excedance classes $S_n/\sim$ induced by the Bruhat order, $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ if and only if $\lambda \preceq \mu$.*

Our proof Theorem 4.2 in [4] includes the intermediate result that each excedance class $\mathcal{C}_\lambda$ contains unique Bruhat-minimal and Bruhat-maximal elements, and moreover these are respectively a 321-avoiding permutation and the element $Q_\lambda \in \mathrm{QSV}_n$. Combined with Theorem 4.2, this implies the following corollary.

Corollary 4.3. *Each excedance class $\mathcal{C}_\lambda$ is an interval in the Bruhat order, with maximum $Q_\lambda \in \mathrm{QSV}_n$ and minimum given by a 321-avoiding permutation.*

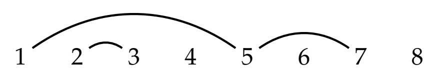
We now identify the minimal element of each excedance class. For a noncrossing partition λ of size n, enumerate the sets λ^+, λ^-, $[n]-\lambda^+$, and $[n]-\lambda^-$ in increasing order as

$$\lambda^+ = \{a_1 < a_2 < \cdots < a_s\}, \qquad \lambda^- = \{b_1 < b_2 < \cdots < b_s\},$$
$$[n]-\lambda^+ = \{x_1 < x_2 < \cdots < x_{n-s}\}, \qquad \text{and} \qquad [n]-\lambda^- = \{y_1 < y_2 < \cdots < y_{n-s}\}.$$

Let $T_\lambda \in S_n$ be the permutation with

$$T_\lambda(i) = \begin{cases} a_r & \text{if } i \in \lambda^- \text{ and } i = b_r \\ x_r & \text{if } i \notin \lambda^- \text{ and } i = y_r. \end{cases}$$

Thus, the two-line notation for T_λ can be obtained by placing the elements of λ^+ in increasing left-to-right order below the elements of λ^-, and placing the elements of $[n]-\lambda^+$ below the elements of $[n]-\lambda^-$ in the same manner. For example, with $n = 8$ and

$\lambda =$ 

we have $\lambda^+ = \{1,2,5\}$ and $\lambda^- = \{3,5,7\}$, $[8]-\lambda^+ = \{3,4,6,7,8\}$, and $[8]-\lambda^- = \{1,2,4,6,8\}$, and consequently

$$T_\lambda = \overset{12345678}{34162758},$$

where non-excedances are marked in red, as at the beginning of Section 4.

Proposition 4.4. *For all noncrossing partitions λ, $T_\lambda \in \mathcal{C}_\lambda$, is the Bruhat-minimum element of $\mathcal{C}_\lambda$, and is 321-avoiding.*

Remark 4.5. Proposition 4.4 implicitly defines a bijection between 321-avoiding permutations and noncrossing partitions. This bijection is equivalent to one used by Zinno in [20] and Gobet in [9].

5 Bases for the Temperley–Lieb Algebra $\mathsf{TL}_n(2)$

The Temperley–Lieb algebra $\mathsf{TL}_n(2)$ is the $\mathbb{C}$-algebra generated by elements $e_1, \ldots, e_{n-1}$ subject to the following relations for each $1 \leqslant i, j \leqslant n$

$$e_i^2 = 2e_i; \qquad e_ie_j = e_je_i \text{ if } |i-j| > 1; \qquad e_ie_je_i = e_i \text{ if } |i-j| = 1.$$

There is a surjective algebra morphism from the symmetric group algebra $\mathbb{C}S_n$ to $\mathsf{TL}_n(2)$ given by $\phi : \mathbb{C}S_n \to \mathsf{TL}_n(2)$ where $\phi(s_i) = 1 - e_i$. In particular $\mathsf{TL}_n(2) \cong S_n/\ker(\phi)$.

It is well-known that the images of all 321-avoiding permutations under ϕ forms a basis for $\mathsf{TL}_n(2)$. Gobet [9] shows that the set QSV_n has a similar property.

Theorem 5.1 ([9, Theorem 7.21]). *For all $n \geqslant 0$, the set $\phi(\mathrm{QSV}_n)$ is a basis for $\mathsf{TL}_n(2)$.*

In our investigation of excedance classes we found an application of their structure the problem of computing sets of permutations which give bases of $\mathsf{TL}_n(2)$ under ϕ. We include it here as it is a nice result of our current investigation.

Theorem 5.2. *Let $n \geqslant 0$ and for each noncrossing partition λ of size n, fix an element $w_\lambda \in \mathcal{C}_\lambda$. Then the set $\{\phi(w_\lambda) \mid$ noncrossing partitions $\lambda\}$ is a basis of $\mathsf{TL}_n(2)$.*

Here, we discuss its implications: taking $w_\lambda = Q_\lambda$ in the theorem gives yet another proof of Theorem 5.1, confirming the results of [10] and [20]. In general, however, many bases obtained via Theorem 5.2 are novel. The smallest novel example can be found with $n = 4$: the set

$$\{\overset{1234}{4312}, \overset{1234}{4231}, \overset{1234}{4213}, \overset{1234}{3142}, \overset{1234}{1432}, \overset{1234}{4123}, \overset{1234}{3214}, \overset{1234}{3124}, \overset{1234}{2143}, \overset{1234}{1323}, \overset{1234}{2134}, \overset{1234}{1324}, \overset{1234}{1243}, \overset{1234}{1234}\}$$

meets the criteria of Theorem 5.2, and accordingly maps to a basis of $\mathsf{TL}_n(2)$ under ϕ. This set is neither QSV_4 nor the set of 321-avoiding permutations ($4312 \notin \mathrm{QSV}_4$ and is not 321-avoiding). Moreover, the set above is not described in [10, 20]: each subset of S_4 in these sources which is not QSV_4 contains more than one element from certain excedance classes and none from others.

6 The quasisymmetric variety

In this section, we summarize Theorem 6.3 and its proof, which is given in full in our paper [4]. As in the introduction, let QSym_n denote the quasisymmetric polynomials in $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ and write M_α for the monomial quasisymmetric function indexed by the composition α. In Section 6.1, we define a family of non-homogeneous polynomials P_α which are also indexed by compositions and we show that

$$P_\alpha = M_\alpha + \text{lower degree terms}. \tag{6.1}$$

For a permutation $\sigma \in S_n$, we write $P_\alpha(\sigma)$ for the evaluation of P_α at $x_1 = \sigma_1$, $x_2 = \sigma_2$, and so on. Recall the set QSV_n defined in Section 3.

Theorem 6.2. *For each non-empty integer composition α with at most n parts and any $\sigma \in \mathrm{QSV}_n$ we have $P_\alpha(\sigma) = 0$.*

Our proof of Theorem 6.2 in [4] uses the noncrossing cycle structure of each element $\sigma \in \mathrm{QSV}_n$, as well as a sign-reversing involution to establish desired vanishing property.

Now recall that for any $f \in R_n$, $\mathsf{h}(f)$ denotes the homogeneous top-degree component of f, and that for any ideal $I \subseteq R_n$, we write $\mathsf{gr}(I) = \langle \mathsf{h}(f) \mid f \in I \rangle$. Standard results in Gröbner basis theory give a linear isomorphism $R_n/I \cong R_n/\mathsf{gr}(I)$. With Theorem 6.2 and the dimension considerations set out in the introduction, this proves of our main result.

Theorem 6.3. *The ideal $\langle P_\alpha \mid$ non-empty compositions α of length $\ell(\alpha) \leqslant n\rangle \subseteq R_n$ is the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$ and*

$$\langle \mathrm{QSym}_n^+ \rangle = \mathsf{gr}\big(\mathbf{I}(\mathrm{QSV}_n)\big),$$

where QSym_n^+ denotes the set of positive-degree quasisymmetric functions.

Using Gröbner basis theory again, we obtain the following corollary.

Corollary 6.4. *We have $R_n/\langle \mathrm{QSym}_n^+\rangle \cong R_n/\mathbf{I}(\mathrm{QSV}_n)$ as vector spaces.*

Remark 6.5. Remarks 3.1 and 3.3 describe the combinatorics of the sets $\{w\sigma w \mid \sigma \in \mathrm{QSV}_n\}$, each of which corresponds to a unique n-cycle $c \in S_n$. It is natural to consider how Theorems 6.2 and 6.3 generalize to these sets as well, and we explain this below.

1. For the set $\{\omega_0\sigma\omega_0 \mid \sigma \in \mathrm{QSV}_n\}$ corresponding to the Coxeter element $c = (1\,2\ldots n)$, our results generalize completely. In particular, the modified polynomials

$$\omega_0 P_\alpha \omega_0 = P_\alpha(-x_n+n+1,\ldots,-x_2+n+1,-x_1+n+1)$$

vanish on every permutation $\omega_0\sigma\omega_0$ for $\sigma \in \mathrm{QSV}_n$. Moreover,

$$\mathsf{h}\big(\omega_0 P_\alpha \omega_0\big) = M_\alpha(-x_n,\ldots,-x_2,-x_1) = (-1)^{|\alpha|} M_{\overleftarrow{\alpha}},$$

where for a composition $\alpha = (\alpha_1,\ldots,\alpha_k)$, $M_{\overleftarrow{\alpha}}$ denotes the monomial quasisymmetric function corresponding to the reverse $\overleftarrow{\alpha} = (\alpha_k,\ldots,\alpha_1)$. This is closely related to the automorphisms of the ring of quasisymmetric functions (see, for example [14]).

2. For the sets corresponding to n-cycles other than $(1\,2\ldots n)$ and $(n\ldots 2\,1)$, the vanishing ideal does not have top-degree homogeneous component $\langle \mathrm{QSym}_n^+\rangle$.

6.1 The vanishing polynomial P_α

In this section we define the polynomials P_α and prove Theorem 6.2. We begin with a short review of compositions and the refinement order as they relate to QSym.

A *composition* is a sequence of positive integers $\alpha = (\alpha_1,\ldots,\alpha_k)$. We refer to k as the *length* of α and to $d = \sum_{i=1}^k \alpha_i$ as the *size* of α. Compositions are partially ordered by refinement: the composition α refines another composition $\beta = (\beta_1,\ldots,\beta_\ell)$ if there exists a sequence $1 = f_1 < f_2 < \cdots < f_{\ell+1} = k+1$ for which $\beta_i = \alpha_{f_i} + \alpha_{f_i+1} + \cdots + \alpha_{f_{i+1}-1}$, and in this case we write $\beta \succeq \alpha$. Whenever we have a refinement relation $\beta \succeq \alpha$, we will use the notation $f_1, f_2, \ldots, f_{\ell+1}$ to refer to the sequence of indices in the definition.

For each composition of length $k \geqslant 1$, the monomial quasisymmetric function $M_\alpha \in R_n$ is defined by

$$M_\alpha = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k},$$

where the sum is over subsets $\{i_1, \ldots, i_k\}$ of $[n]$, enumerated in increasing order. Using the same convention we define the vanishing polynomial $P_\alpha \in R_n$ to be

$$P_\alpha = \sum_{\beta \geq \alpha} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n} \prod_{j=1}^{\ell} \Big((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) \prod_{s=f_j+1}^{f_{j+1}-1} (-i_j)^{\alpha_s} \Big).$$

While this formula appears to be quite dense, expanding it reveals an intuitive combinatorial structure. We compute one example in its entirety for the sake of exposition:

$$\begin{aligned} P_{(1,2,1)}(x_1, \ldots, x_4) =& (x_1 - 1)(x_2^2 - 2^2)(x_3 - 3) + (x_1 - 1)(x_2^2 - 2^2)(x_4 - 4) \\ &+ (x_1 - 1)(x_3^2 - 3^2)(x_4 - 4) + (x_2 - 2)(x_3^2 - 3^2)(x_4 - 4) \\ &\quad - (x_1 - 1)(x_2^2 - 2^2)2 - (x_1 - 1)(x_3^2 - 3^2)3 - (x_1 - 1)(x_4^2 - 4^2)4 \\ &\quad - (x_2 - 2)(x_3^2 - 3^3)3 - (x_2 - 2)(x_4^2 - 4^2)4 - (x_3 - 3)(x_4^2 - 4^2)4 \\ &\qquad - (x_1 - 1)1^2(x_2 - 2) - (x_1 - 1)1^2(x_3 - 3) - (x_1 - 1)1^2(x_4 - 4) \\ &\qquad - (x_2 - 2)2^2(x_3 - 3) - (x_2 - 2)2^2(x_4 - 4) - (x_3 - 3)3^2(x_4 - 4) \\ &\qquad\quad + (x_1 - 1)1^3 + (x_2 - 2)2^3 + (x_3 - 3)3^3 + (x_4 - 4)4^3, \end{aligned}$$

where summands corresponding to the same index $\beta \succeq (1,2,1)$ are grouped horizontally and by alignment. These values of β are respectively $(1,2,1)$, $(1,3)$, $(3,1)$, and (4).

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Combinatorial properties of triangular partitions

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Abstract. A *triangular partition* is a partition whose Ferrers diagram can be separated from its complement (as a subset of $\mathbb{N}^2$) by a straight line. Having their origins in number theory and computer vision, triangular partitions have been studied from a combinatorial perspective by Corteel et al. under the name *plane corner cuts*, and more recently by Bergeron and Mazin in the context of algebraic combinatorics. Here we derive new enumerative, geometric, and algorithmic properties of such partitions.

We give a new characterization of triangular partitions and the cells that can be added or removed while preserving the triangular condition, and use it to describe the Möbius function of the restriction of Young's lattice to triangular partitions. We obtain a formula for the number of triangular partitions whose Young diagram fits inside a square, deriving a new proof of Lipatov's enumeration theorem for balanced words. Finally, we present an algorithm that generates all the triangular partitions of a given size, which is significantly more efficient than previous ones and allows us to compute the number of triangular partitions of size up to 10^5.

Keywords: triangular partition, corner cut, balanced word, Young's lattice

1 Introduction

An integer partition is said to be triangular if its Ferrers diagram can be separated from its complement by a straight line. Triangular partitions and their higher-dimensional generalizations have been studied from several perspectives during the last five decades. They first appeared in the context of combinatorial number theory [5], where they were called *almost linear sequences*. Later, the closely related notion of *digital straight lines* became relevant in the field of computer vision [6]. From a combinatorial perspective, triangular partitions were first studied by Onn and Sturmfels [12], who defined them in any dimension and called them *corner cuts*. Soon after, Corteel et al. [7] found an expression for the generating function for the number of plane corner cuts.

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Renewed interest in triangular partitions has recently come from algebraic combinatorics; specifically, from the study of generalizations of the *shuffle theorem* and, more broadly, of the ubiquitous connections between Dyck paths, parking functions, diagonal coinvariant spaces, and Macdonald polynomials. In generalizing Dyck paths to Fuss-Catalan paths, then rational Dyck paths, and then rectangular Dyck paths, a natural next step is to consider lattice paths (with unit south and east steps) that stay weakly below the line segment from $(0,s)$ to $(r,0)$, where r and s are any positive real numbers. These paths arise in recent work of Blasiak et al. [4] generalizing the shuffle theorem. Motivated by this result, Bergeron and Mazin [2] coined the terms *triangular partitions*, *triangular Dyck paths*, and *triangular parking functions*, and studied some of their combinatorial and algebraic properties.

In this abstract we obtain further enumerative, geometric, poset-theoretic, and algorithmic properties of triangular partitions. In Section 2 we give basic definitions and summarize some of the work from [7, 2]. In Section 3 we give a simple alternative characterization of triangular partitions, as those for which the convex hull of the Ferrers diagram and that of its complement (as a subset of $\mathbb{N}^2$) have an empty intersection. We also characterize which cells can be added to or removed from the Young diagram while preserving triangularity.

In Section 4 we study the restriction of Young's lattice to triangular partitions. It was shown in [2] that this poset is a lattice. Here we completely describe its Möbius function, and we provide an explicit construction of the join and meet of two triangular partitions.

In Section 5, we introduce a new encoding of triangular partitions in terms of balanced words, and use it to implement an algorithm which computes the number of triangular partitions of each size up to N in time $\mathcal{O}(N^{5/2})$. This allows us to produce the first 10^5 terms of this sequence, compared to the 39 terms that were known previously.

In Section 6, refining the approach from [7], we obtain generating functions for triangular partitions with a given number of removable and addable cells. In Section 7, we provide a formula for the number of triangular partitions whose Young diagram fits inside a square (or equivalently, inside a staircase), which involves Euler's totient function. As a byproduct, we obtain a new combinatorial proof of a formula of Lipatov [8] for the number of balanced words.

Due to space constraints, proofs are omitted from this extended abstract.

2 Background

A *partition* λ is a weakly decreasing sequence of positive integers, often called the *parts* of λ. We will write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, or $\lambda = \lambda_1\lambda_2\ldots\lambda_k$ when there is no confusion. We call $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ the *size* of λ. If $|\lambda| = n$, we say that λ is a partition of n.

Let $\mathbb{N}$ denote the set of positive integers. The *Ferrers diagram* of λ is the set of lattice

points
$$\{(a,b) \in \mathbb{N}^2 \mid 1 \leq b \leq k,\ 1 \leq a \leq \lambda_b\}.$$
The *Young diagram* of λ is the set of unit squares (called *cells*) whose north-east corners are the points in the Ferrers diagram. We identify each cell with its north-east corner, so we also use the term cell to refer to points in the Ferrers diagram. In particular, we say that a cell lies above, below or on a line when the north-east corner does. We will often identify λ with its Ferrers and Young diagrams, and use notation such as $c = (a,b) \in \lambda$.

For a partition $\lambda = \lambda_1\lambda_2\ldots\lambda_k$, we call λ_1 its *width*, and k its *height*. Let $\sigma^k = (k, k-1, \ldots, 2, 1)$ denote the *staircase partition* of height k. The *conjugate* of λ, obtained by reflecting its Ferrers diagram along the diagonal $y = x$, will be denoted by λ'. Identifying λ with its Ferrers diagram, we define its *complement* to be the set $\mathbb{N}^2 \setminus \lambda$.

Definition 1. A partition $\tau = \tau_1\tau_2\ldots\tau_k$ is *triangular* if there exist positive real numbers r and s such that
$$\tau_j = \lfloor r - jr/s \rfloor,$$
for $1 \leq j \leq k$, and $k = \lfloor s - s/r \rfloor$.

In other words, τ is triangular if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below the line that passes through $(0,s)$ and $(r,0)$ for some $r, s \in \mathbb{R}_{>0}$. See Figure 1 for an example. This line is called a *cutting line* of τ. Unlike in the definition given in [2], here we do not allow τ to have parts equal to 0, hence the condition on k. We often use τ to denote a triangular partition.

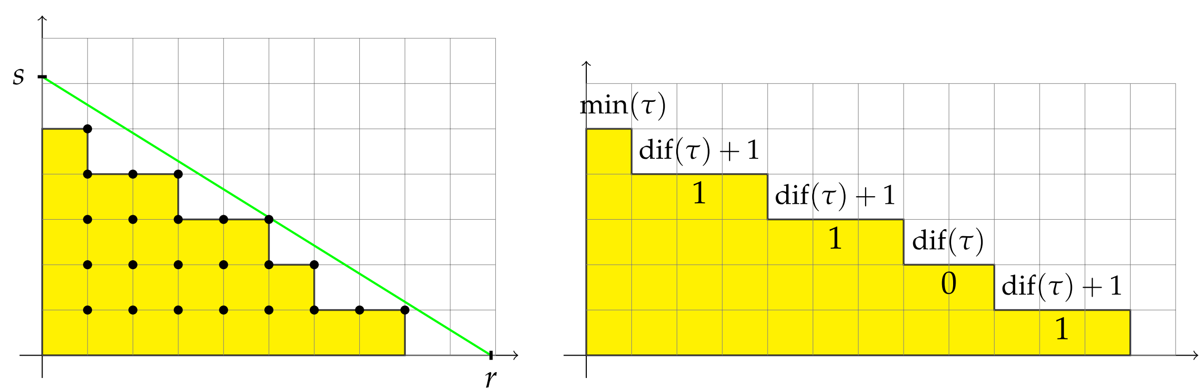

Figure 1: Left: A cutting line for the triangular partition $(8,6,5,3,1)$. Right: Applying the bijection from Theorem 19 to $\tau = (12,9,7,4,1)$ gives $\chi(\tau) = (1,2,1011)$.

Denote by $\Delta(n)$ the set of triangular partitions of n, and by $\Delta = \bigcup_{n\geq 0} \Delta(n)$ the set of all triangular partitions. The following two results are due to Corteel et al. [7].

Theorem 2 ([7]). *The generating function for triangular partitions can be expressed as*

$$G_\Delta(z) = \sum_{n\geq 0} |\Delta(n)| z^n = \frac{1}{1-z} + \sum_{\gcd(a,b)=1} \sum_{\substack{0\leq j<a \\ 0\leq i<b}} \sum_{1\leq m<k} z^{N_\Delta(a,b,k,m,i,j)},$$

where

$$\begin{aligned} N_\Delta(a,b,k,m,i,j) = (k-1)\left(\frac{(a+1)(b+1)}{2} - 1\right) + \binom{k-1}{2} ab + ij \\ + i(k-1)a + j(k-1)b + T(a,b,j) + T(b,a,i) + m, \end{aligned} \tag{2.1}$$

and $T(a,b,j) = \sum_{r=1}^{j}(\lfloor rb/a \rfloor + 1)$.

Theorem 3 ([7]). *There exist positive constants c and c' such that, for all $n > 1$,*

$$cn\log n < |\Delta(n)| < c'n\log n.$$

Let $\lambda = \lambda_1 \ldots \lambda_k$ be a partition, and let $c = (i,j)$ be a cell of its Young diagram. Define the *arm length* and the *leg length* of c to be $a(c) = \lambda_j - i$ and $\ell(c) = \lambda'_i - j$, that is, the number of cells to the right of c in its row, and above c in its column, respectively. Bergeron and Mazin [2] give the following characterization of triangular partitions.

Lemma 4 ([2, Lemma 1.2]). *A partition λ is triangular if and only if $t^-_\lambda < t^+_\lambda$, where*

$$t^-_\lambda = \max_{c\in\lambda} \frac{\ell(c)}{a(c)+\ell(c)+1}, \quad \text{and} \quad t^+_\lambda = \min_{c\in\lambda} \frac{\ell(c)+1}{a(c)+\ell(c)+1}.$$

Definition 5. A cell of $\tau \in \Delta$ is *removable* if removing it from τ yields a triangular partition. A cell of the complement $\mathbb{N}^2 \setminus \tau$ is *addable* if adding it to τ yields a triangular partition.

Lemma 6 ([2, Lemma 4.5]). *Every nonempty triangular partition has either one removable cell and two addable cells, two removable cells and one addable cell, or two removable cells and two addable cells.*

Let $\mathbb{Y}_\Delta$ be the poset of triangular partitions ordered by containment of their Young diagrams; equivalently, the restriction of Young's lattice to the subset of triangular partitions. The covering relations in $\mathbb{Y}_\Delta$ can be described as follows.

Lemma 7 ([2, Lemma 4.2]). *Let $\tau, \nu \in \mathbb{Y}_\Delta$ such that $\tau < \nu$. Then, $\tau \lessdot \nu$ if and only if τ is obtained from ν by removing exactly one cell. In particular, $\mathbb{Y}_\Delta$ is ranked by the size of the partitions.*

Lemma 8 ([2, Corollary 4.1, Lemma 4.4]). *The poset $\mathbb{Y}_\Delta$ has a planar Hasse diagram, and it is a lattice.*

3 Characterizations of triangular partitions

Bergeron and Mazin's [2] characterization of triangular partitions, given in Lemma 4 above, requires computing some quotients of arm and leg lengths for all the cells in the partition. In this section, we introduce an alternative and arguably simpler characterization of triangular partitions in terms of convex hulls, along with various ways to identify removable and addable cells. We then use these to describe an algorithm which determines if an integer partition is triangular and finds its removable and addable cells. The convex hull of a set $S \subseteq \mathbb{N}^2$ will be denoted by $\mathrm{Conv}(S)$.

Proposition 9. *A partition λ is triangular if and only if $\mathrm{Conv}(\lambda) \cap \mathrm{Conv}(\mathbb{N}^2 \setminus \lambda) = \emptyset$.*

We will use the term *vertex* in the sense of a 0-dimensional face of a polygon; in particular, $\mathrm{Conv}(\tau)$ may have lattice points in its boundary that are not vertices.

Proposition 10. *Two cells in $\tau \in \Delta$ are removable if and only if they are consecutive vertices of $\mathrm{Conv}(\tau)$ and the line passing through them does not intersect $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$. Similarly, two cells in $\mathbb{N} \setminus \tau$ are addable if and only if they are consecutive vertices of $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$ and the line passing through them does not intersect $\mathrm{Conv}(\tau)$.*

An immediate consequence is that a triangular partition can have no more than two removable cells and no more than two addable cells, as we knew from Lemma 6.

Proposition 11. *A cell $c = (a, b) \neq (1, 1)$ in $\tau \in \Delta$ is its only removable cell if and only if it is a vertex of $\mathrm{Conv}(\tau)$ and both of the following hold:*

- *if $a > 1$, the line containing the edge of $\mathrm{Conv}(\tau)$ adjacent to c from the left intersects $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$ to the right of c;*
- *if $b > 1$, the line containing the edge of $\mathrm{Conv}(\tau)$ adjacent to c from below intersects $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$ above c.*

The characterization for a single addable cell is analogous.

The above characterizations can be used to describe an algorithm that determines whether a partition λ of n into k parts is triangular, and if it is, it finds its removable and addable cells. The algorithm first finds the vertices of $\mathrm{Conv}(\lambda)$ and $\mathrm{Conv}(\mathbb{N}^2 \setminus \lambda)$, and then it searches for a segment of the boundary of one of these convex hulls such that the line containing it does not intersect the opposite convex hull. By Proposition 10, such a segment joins two removable or addable cells. This algorithm has complexity $\mathcal{O}(k)$ for the initialization and $\mathcal{O}(\min\{k, \sqrt{n}\})$ for the rest of its steps, whereas an algorithm based on Bergeron and Mazin's Lemma 4 would take time $\mathcal{O}(n)$.

4 The triangular Young poset

Bergeron and Mazin [2] introduced the poset $\mathbb{Y}_\Delta$ of triangular partitions ordered by containment of their Young diagrams. They showed that it has a planar Hasse diagram, and deduced from this property that $\mathbb{Y}_\Delta$ is a lattice, and it is ranked by the size of each partition. Here we describe the Möbius function of $\mathbb{Y}_\Delta$, and we give explicit constructions for the meet and the join of any two elements.

Our first result confirms Bergeron's conjecture (personal communication, 2022) that the Möbius function only takes values in $\{-1, 0, 1\}$.

Theorem 12. *Let $\tau, \nu \in \mathbb{Y}_\Delta$ such that $\tau \le \nu$. The value of the Möbius function is:*

$$\mu(\tau,\nu) = \begin{cases} 1 & \text{if either } \tau = \nu \text{ or there exist } \zeta^1 \neq \zeta^2 \text{ such that } \nu = \zeta^1 \vee \zeta^2 \text{ and } \tau \lessdot \zeta^1, \zeta^2, \\ -1 & \text{if } \tau \lessdot \nu, \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in [2] that the faces of the Hasse diagram of $\mathbb{Y}_\Delta$ are polygons with an even number of sides. We can interpret Theorem 12 as stating that, if $\tau < \nu$ and ν does not cover τ, then $\mu(\tau, \nu)$ equals 1 if $[\tau, \nu]$ is one of the polygonal faces, and 0 otherwise.

The next result explicitly characterizes the join and meet of two elements of $\mathbb{Y}_\Delta$. A similar formula works for the join and the meet of any number of elements.

Proposition 13. *The join and the meet of $\tau, \nu \in \mathbb{Y}_\Delta$ are given by*

$$\tau \vee \nu = \mathbb{N}^2 \cap \mathrm{Conv}(\tau \cup \nu) \quad \text{and} \quad \tau \wedge \nu = \mathbb{N}^2 \setminus \Big(\mathbb{N}^2 \cap \mathrm{Conv}\left(\mathbb{N}^2 \setminus (\tau \cap \nu)\right)\Big).$$

5 Bijections to balanced words and efficient generation

In this section we present two different interpretations of triangular partitions in terms of factors of Sturmian words. The first interpretation, which is hinted at in [2], is quite natural, and it will allow us to prove some enumeration formulas in Section 7. The second interpretation encodes families of triangular partitions by one single balanced word, along with two other parameters, and it will be used in Section 5.4 to implement efficient algorithms to count triangular partitions by their size.

5.1 Balanced words

Recall that a factor of a word is a consecutive subword. An infinite binary word s is *Sturmian* if, for every $n \ge 1$, the number of factors of s of length n equals $n+1$. Sturmian words have applications in combinatorics, number theory, and dynamical systems; see [9, Chapter 2] for a thorough study.

It is known that a finite binary word $w = w_1 \dots w_\ell$ is a factor of some Sturmian word if and only if it is *balanced*, that is, for any $h \leq \ell$ and $i, j \leq \ell - h + 1$, we have

$$\left|(w_i + w_{i+1} + \cdots + w_{i+k-1}) - (w_j + w_{j+1} + \cdots + w_{j+k-1})\right| \leq 1.$$

This condition says that for any two factors of w of the same length, the number of ones in these factors differs by at most 1. We denote by $\mathcal{B}$ the set of all balanced words, and by $\mathcal{B}_\ell$ the set of those of length ℓ.

The following enumeration formula for balanced words is due to Lipatov [8]. We use φ to denote Euler's totient function.

Theorem 14 ([8]). *The number of balanced words of length ℓ is*

$$|\mathcal{B}_\ell| = 1 + \sum_{i=1}^{\ell} (\ell - i + 1)\varphi(i).$$

5.2 First Sturmian interpretation

Definition 15. A triangular partition is *wide* (respectively *tall*) if it admits a cutting line $x/r + y/s = 1$ with $r > s$ (respectively $r < s$).

It can be shown that every triangular partition must be wide, tall, or both. Additionally, a triangular partition τ is wide if and only if its conjugate τ' is tall.

Lemma 16. *For any triangular partition $\tau = \tau_1 \dots \tau_k$, we have*

$$\tau \textit{ is wide } \Leftrightarrow \tau_1 \geq k \Leftrightarrow \textit{ the parts of } \tau \textit{ are distinct},$$

$$\tau \textit{ is wide and tall } \Leftrightarrow \tau_1 = k \Leftrightarrow \tau = \sigma^k.$$

Given a wide triangular partition $\tau = \tau_1 \dots \tau_k$, define the binary word

$$\omega(\tau) = 10^{\tau_1 - \tau_2 - 1} 10^{\tau_2 - \tau_3 - 1} \dots 10^{\tau_{k-1} - \tau_k - 1} 10^{\tau_k - 1}. \tag{5.1}$$

The fact that all the parts of τ are distinct guarantees that the exponents are nonnegative. For example, $\omega(86531) = 10110101$.

Proposition 17. *For every $k, \ell \geq 1$, the map ω is a bijection between the set of wide triangular partitions with k parts and first part equal to ℓ, and the set of balanced words of length ℓ with k ones that start with* 1.

5.3 Second Sturmian interpretation

Our second encoding of triangular partitions using balanced words appears to be new. Let ϵ denote the empty partition, and let $\mathcal{W}$ be the set of wide triangular partitions with at least two parts. Let $\mathcal{B}^0$ denote the set of balanced words that contain at least one 0.

First we describe the possible sets that can be obtained by taking the differences of consecutive parts in a wide triangular partition. For $\tau = \tau_1 \dots \tau_k \in \mathcal{W}$, define

$$\mathcal{D}(\tau) = \{\tau_1 - \tau_2,\ \tau_2 - \tau_3,\ \dots,\ \tau_{k-1} - \tau_k\}.$$

Lemma 18. *For any $\tau = \tau_1 \dots \tau_k \in \mathcal{W}$, either $\mathcal{D}(\tau) = \{d\}$ or $\mathcal{D}(\tau) = \{d, d+1\}$ for some $d \geq 1$ such that $\tau_k \leq d+1$.*

Define also $\min(\tau) = \tau_k$, $\operatorname{dif}(\tau) = \min \mathcal{D}(\tau)$, and $\operatorname{wrd}(\tau) = w_1 \dots w_{k-1}$ where, for $i \in [k-1]$, we let $w_i = \tau_i - \tau_{i+1} - \operatorname{dif}(\tau)$. Lemma 18 guarantees that $w_i \in \{0,1\}$ for all i.

Theorem 19. *The map $\chi = (\min, \operatorname{dif}, \operatorname{wrd})$ is a bijection between $\mathcal{W}$ and the set*

$$\mathcal{T} = \{(m, d, w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}^0 \mid m \leq d+1;\ w1 \in \mathcal{B}^0 \text{ if } m = d+1\}.$$

Its inverse is given by the map

$$\xi(m, d, w_1 \dots w_{k-1}) = \tau_1 \dots \tau_k, \quad \text{where } \tau_i = m + \sum_{j=i}^{k-1} (w_j + d) \text{ for } i \in [k].$$

Additionally, given $\tau \in \mathcal{W}$ with image $\chi(\tau) = (m, d, w)$, its number of parts equals the length of w plus one, and its size is

$$|\tau| = km + \binom{k}{2} d + \sum_{i=1}^{k-1} i w_i. \tag{5.2}$$

5.4 Efficient generation

At the time of writing this abstract, the entry of the OEIS [11, A352882] for the number triangular partitions of n only includes values for $n \leq 39$. These are the terms that appear in [7], where they were obtained using the generating function in Theorem 2. Computing more terms using this generating function is impractical for large n.

Theorem 19 can be used to implement a much more efficient algorithm that can quickly compute the first 10^5 terms of the sequence. On input N, our algorithm to compute $|\Delta(n)|$ for $1 \leq n \leq N$ performs a depth first search through the tree of balanced words of length up to $\lfloor \sqrt{2N} \rfloor$. The parent of a nonempty balanced word in this tree is the balanced word obtained by removing its last letter. For each $w \in \mathcal{B}_\ell$, our algorithm can quickly determine whether $w0$ and $w1$ are balanced by keeping a vector that records, for each $h \leq \ell$, whether all the factors of length h have the same number of ones, or otherwise, whether the rightmost factor of w has more or less ones than other factors.

For each $w \in \mathcal{B}_\ell$ with $\ell \leq \sqrt{2N}$, the algorithm finds all the values $m, d \in \mathbb{N}$ such that $(m, d, w) \in \mathcal{T}$, as defined in Theorem 19, and such that the size function given in equation (5.2) is at most N. Each triplet (m, d, w) accounts for two triangular partitions, namely $\tau = \chi(m, d, w)$ and its conjugate, except when $w = 0^{k-1}$ (for some $k \geq 2$) and $m = d$, in which case it accounts for only one partition, the staircase σ^k.

A C++ implementation of this algorithm is available at [1]. In a standard laptop computer, this algorithm yields the first 10^3 terms of the sequence $|\Delta(n)|$ in under one second, the first 10^4 terms in one minute, and the first 10^5 terms in about one hour.

Proposition 20. *The above algorithm finds $|\Delta(n)|$ for $1 \leq n \leq N$ in time $\mathcal{O}(N^{5/2})$. Additionally, it can be modified to generate all (resp., all wide) triangular partitions of size at most N in time $\mathcal{O}(N^3 \log N)$ (resp., $\mathcal{O}(N^{5/2} \log N)$).*

The first 10^5 terms of the sequence $|\Delta(n)|/(n \log n)$ are plotted on the left of Figure 2. The plot suggests that, for large n, this sequence oscillates between two decreasing functions that differ by about 0.05.

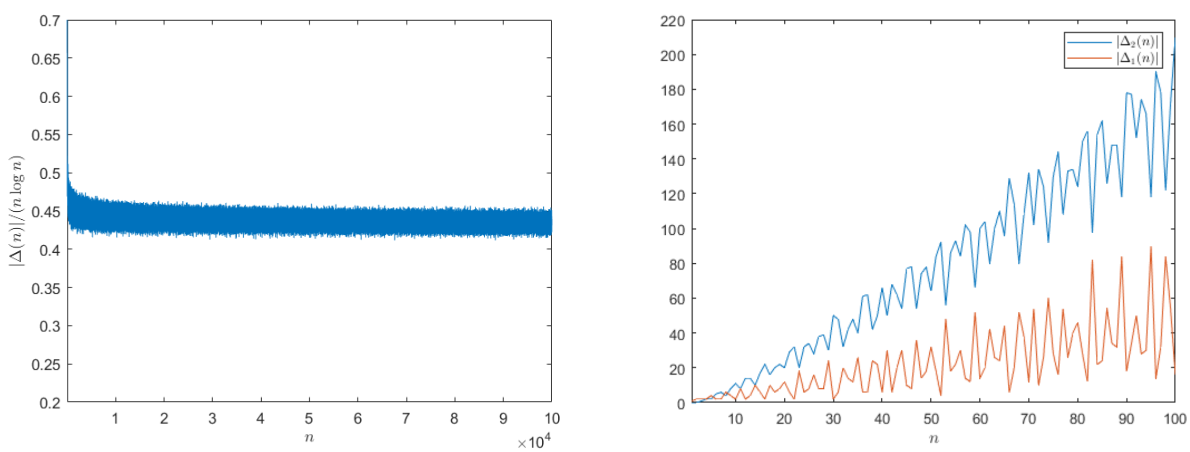

Figure 2: Left: The first 10^5 terms of the sequence $|\Delta(n)|/(n \log n)$. Right: Plot of $|\Delta_2(n)|$ and $|\Delta_1(n)|$ for $1 \leq n \leq 100$.

6 Generating functions for subsets of triangular partitions

Let Δ_1 and Δ_2 denote the subsets of triangular partitions with one removable cell and with two removable cells, respectively. Let Δ^1 and Δ^2 denote the subsets of triangular partitions with one addable cell and with two addable cells, respectively. Let $\Delta_2^2 = \Delta_2 \cap$

Δ^2. Denote partitions of size n in each subset by $\Delta_1(n)$, $\Delta_2(n)$, $\Delta^1(n)$, $\Delta^2(n)$ and $\Delta_2^2(n)$. In this section we obtain generating functions for each of these sets, refining Theorem 2. In our following result, $N_\Delta(a,b,k,m,i,j)$ is the function defined in equation (2.1).

Proposition 21. *The generating function for triangular partitions with two removable cells can be expressed as*

$$G_{\Delta_2}(z) = \sum_{n\geq 0} |\Delta_2(n)| z^n = \sum_{\gcd(a,b)=1} \sum_{\substack{0\leq j<a \\ 0\leq i<b}} \sum_{k\geq 2} z^{N_\Delta(a,b,k,k,i,j)}.$$

Proposition 22. *The generating functions for partitions in Δ_1, Δ^2, Δ^1, Δ_2^2 can be written in terms of $G_\Delta(z)$ (given in Theorem 2) and $G_{\Delta_2}(z)$ (given in Proposition 21) as follows:*

$$G_{\Delta_1}(z) = G_\Delta(z) - G_{\Delta_2}(z) - 1, \qquad G_{\Delta^2}(z) = \frac{1-z}{z} G_\Delta(z) + \frac{1}{z} G_{\Delta_2}(z) - \frac{1}{z},$$
$$G_{\Delta^1}(z) = \frac{2z-1}{z} G_\Delta(z) - \frac{1}{z} G_{\Delta_2}(z) + \frac{1}{z}, \quad G_{\Delta_2^2}(z) = \frac{1-2z}{z} G_\Delta(z) + \frac{1+z}{z} G_{\Delta_2}(z) - \frac{1}{z}.$$

We can use the expression for G_{Δ_2} given in Proposition 21 to write an algorithm to find $|\Delta_2(n)|$. We have computed the first 100 terms of this sequence using a MATLAB implementation of this algorithm, which is available at [1]. The initial terms of the sequences $|\Delta_1(n)|$ and $|\Delta_2(n)|$, plotted on the right of Figure 2, suggest that $|\Delta_2(n)| > |\Delta_1(n)|$ for all $n \geq 9$, although we do not have a proof of this. It is interesting to note that both the local maxima of $|\Delta_1(n)|$ and the local minima of $|\Delta_2(n)|$ seem to occur precisely when $n \equiv 2 \pmod 3$. On the other hand, $|\Delta(n)|$ does not exhibit such periodic extrema.

7 Triangular subpartitions and a combinatorial proof of Lipatov's formula for balanced words

For $\tau \in \Delta$, let $I(\tau) = |\{\nu \in \Delta : \nu \subseteq \tau\}|$ denote the number of triangular subpartitions of τ. We start by giving a recurrence for this number. In some particular cases, we will be able to obtain explicit formulas for $I(\tau)$. In this section we will also derive a new proof of Theorem 14.

Let c^- and c^+ be the removable cells of τ. Following [2], denote by τ° the triangular partition that is obtained from τ by removing all the cells in the segment joining c^- and c^+. If τ has only one removable cell, then $c^- = c^+$, and τ° is simply the partition obtained by removing this cell.

Lemma 23. *For any $\tau \in \Delta(n)$ with $n \geq 1$,*

$$I(\tau) = I(\tau \setminus \{c^-\}) + I(\tau \setminus \{c^+\}) - I(\tau^\circ) + 1.$$

This recurrence relation, along with the base case $I(\epsilon) = 1$, allows us to compute $I(\tau)$ for any $\tau \in \Delta$, although not very efficiently. For example, for the staircase, the first few terms of the sequence $I(\sigma^\ell)$ for $\ell \geq 0$ are $1, 2, 5, 12, 25, 48, 83, \ldots$.

We use the terms *height* and *width* of a partition τ to refer to the number of parts and the largest part of τ, respectively. In order to find explicit formulas for $I(\tau)$ in some cases, let us consider the closely related problem of counting triangular partitions whose width is at most ℓ and whose height is at most h; equivalently, those whose Young diagram fits inside an $h \times \ell$ rectangle. We denote by $\Delta^{h\times\ell}$ the set of such partitions.

Lemma 24. *Let $h, \ell \geq 1$, and let $\nu \in \Delta$. Then $\nu \in \Delta^{h\times\ell}$ if and only if $\nu \subseteq \tau$, where $\tau = \tau_1 \ldots \tau_h$ is the triangular partition given by $\tau_i = \left\lfloor \ell + 1 - \frac{\ell(i-1)+1}{h} \right\rfloor$, for $1 \leq i \leq h$.*

Our next goal is to give a formula for $I(\sigma^\ell)$, which, by Lemma 24, equals the number of triangular partitions that fit inside an $\ell \times \ell$ square, that is, $\left|\Delta^{\ell\times\ell}\right|$. The proof of the following lemma uses the bijection ω from equation (5.1).

Lemma 25. *For $\ell \geq 1$, the number of triangular partitions of width exactly ℓ and height at most ℓ is $|\mathcal{B}_\ell|/2$, and*

$$\left|\Delta^{\ell\times\ell} \setminus \Delta^{(\ell-1)\times(\ell-1)}\right| = I(\sigma^\ell) - I(\sigma^{\ell-1}) = |\mathcal{B}_\ell| - 1.$$

Combining the above lemma with Lipatov's enumeration formula for balanced words (Theorem 14), we deduce the following result.

Theorem 26. *For any $\ell \geq 0$,*

$$\left|\Delta^{\ell\times\ell}\right| = I(\sigma^\ell) = 1 + \sum_{i=1}^{\ell} \binom{\ell - i + 2}{2} \varphi(i).$$

Unfortunately, the proof of Theorem 26 that relies on Lipatov's formula does not give a conceptual understanding of why the terms $\binom{\ell-i+2}{2}$ and $\varphi(i)$ appear.

Instead, we have been able to find a direct, combinatorial proof of Theorem 26 that explains why these terms appear. While this proof does not fit in this extended abstract, we briefly describe its main ideas. First we give a bijection ϕ between triangular partitions (except those that have all parts equal to one) and the set $\{(a, b, d, e) \in \mathbb{N}^4 \mid d < a,\ \gcd(d, e) = 1\}$, and characterize the set $\phi(\Delta^{\ell\times\ell})$. Then we show that, for fixed $d < e$ with $\gcd(d, e) = 1$, by combining the points (a, b) for which $(a, b, d, e) \in \phi(\Delta^{\ell\times\ell})$, with (a certain linear transformation of) the points (a, b) for which $(a, b, e, e - d) \in \phi(\Delta^{\ell\times\ell})$, one obtains precisely the set of lattice points in a certain triangle, which are counted by $\binom{\ell-e+2}{2}$. Summing over all pairs $d < e$ with $\gcd(d, e) = 1$ gives our formula for $\left|\Delta^{\ell\times\ell}\right|$.

As an added benefit, our argument also provides a new proof of Lipatov's formula (Theorem 14), which is fundamentally different from the existing proofs that have appeared over the years, all of which are quite technical; see e.g. [10, 3].

Similar formulas for the number of triangular subpartitions in other rectangles can be derived from Theorem 26.

Corollary 27. *For $\ell \geq 2$,*

$$\left|\Delta^{\ell\times(\ell-1)}\right| = \frac{1}{2} + \sum_{i=1}^{\ell} \frac{(\ell-i+1)^2}{2}\varphi(i), \quad \left|\Delta^{\ell\times(\ell-2)}\right| = 1-\ell+\sum_{i=1}^{\ell} \frac{(\ell-i+1)(\ell-i)+1}{2}\varphi(i).$$

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Asymptotics of Bounded Lecture-Hall Tableaux

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Abstract. We study the asymptotics of bounded lecture hall tableaux. Limit shapes form when the bounds of the lecture hall tableaux go to infinity linearly in the lengths of the partitions describing the large-scale shapes of these tableaux. We prove Conjecture 6.1 in [8], stating that the slopes of the rescaled height functions in the scaling limit satisfy a complex Burgers equation. We also show that the fluctuations of the unrescaled height functions converge to the Gaussian free field. The proof is based on new construction and analysis of Schur generating functions for the lecture hall tableaux, whose corresponding particle configurations do not form a Gelfand-Tsetlin scheme; and the corresponding dimer models are not doubly periodic.

Résumé. Nous étudions l'asymptotique des tableaux de la salle de cours bornés. Les formes limites se forment lorsque les bornes des tableaux de la salle de cours tendent vers l'infini linéairement par rapport aux longueurs des partitions décrivant les formes à grande échelle de ces tableaux. Nous démontrons la Conjecture 6.1 dans [8], affirmant que les pentes des fonctions de hauteur mises à l'échelle dans la limite d'échelle satisfont une équation de Burgers complexe. Nous montrons également que les fluctuations des fonctions de hauteur non mises à l'échelle convergent vers le champ libre gaussien. La preuve repose sur une nouvelle construction et une analyse des fonctions génératrices de Schur pour les tableaux de la salle de cours, dont les configurations de particules correspondantes ne forment pas un schéma de Gelfand-Tsetlin; et les modèles de dimères correspondants ne sont pas doublement périodiques.

Keywords: lecture hall tableaux, limit shape, Gaussian free field

1 Introduction

Lecture hall tableaux were introduced in [10] as fillings of Young tableaux satisfying certain conditions, which generalize both lecture hall partitions ([2, 3]) and anti-lecture

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hall compositions ([11]), and also contain reverse semistandard Young tableaux as a limit case. Lecture hall partitions and anti-lecture hall compositions have attracted considerable interest among combinatorists in the last two decades; see the recent survey [21] and references therein.

We now define the lecture hall tableaux. Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$. Each integer λ_i is called a part of λ. The length $l(\lambda)$ of λ is the number of parts. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ can be identified with its Young diagram, which consists of unit squares (cells) with integer coordinates (i, j) satisfying $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$. For two partitions λ and μ we write $\mu \subset \lambda$ to mean that the Young diagram of μ is contained in that of λ as a set. In this case, a skew shape λ/μ is defined to be the set-theoretic difference λ/μ of their Young diagrams. We denote by $|\lambda/\mu|$ the number of cells in λ/μ. A partition λ is also considered as a skew shape by $\lambda/\emptyset$; where $\emptyset$ represents the empty partition.

A tableau of shape λ/μ is a filling of the cells in λ/μ with nonnegative integers. In other words, a tableau is a map $T : \lambda/\mu \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers.

An n-lecture hall tableau of shape λ/μ is a tableau L of shape λ/μ satisfying the following conditions

$$\frac{L(i,j)}{n + c(i,j)} \geq \frac{L(i,j+1)}{n + c(i,j+1)}, \qquad \frac{L(i,j)}{n + c(i,j)} > \frac{L(i+1,j)}{n + c(i+1,j)}.$$

where $c(i, j) = j - i$ is the content of the cell (i, j). The set of n-lecture hall tableaux is denoted by $LHT_n(\lambda/\mu)$. For $L \in LHT_n(\lambda/\mu)$, let $\lfloor L \rfloor$ be the tableaux of shape λ/μ whose (i, j)th entry is $\lfloor \frac{L(i,j)}{(n-i+j)} \rfloor$.

See the left graph of Figure 1 for an example of a lecture hall tableaux.

We shall study lecture hall tableaux with an extra condition as follows:

$$L(i,j) < t(n + j - i)$$

We say these tableaux are bounded by $t > 0$. These tableaux are called bounded lecture hall tableaux and are enumerated in [9].

The main aim to study the asymptotics of bounded n-lecture hall tableaux as $n \to \infty$. We shall first recall a bijection between lecture hall tableaux and non-intersecting path configurations in [9], and then investigate the asymptotics (limit shape and height fluctuations) of the corresponding non-intersecting path configurations. Now we define the graph on which the non-intersecting path configurations correspond to the lecture hall tableaux.

1. Given a positive integer t, the lecture hall graph is a graph $\mathcal{G}_t = (V_t, E_t)$. This graph can be described through an embedding in the plane with vertex set V_t given by

- $\left(i, \frac{j}{i+1}\right)$ for $i \geq 0$ and $0 \leq j < t(i+1)$.

and the directed edges given by

- from $\left(i, k + \frac{r}{i+1}\right)$ to $\left(i+1, k + \frac{r}{i+2}\right)$ for $i \geq 0$, $0 \leq r \leq i$ and $0 \leq k < t$
- from $\left(i, k + \frac{r+1}{i+1}\right)$ to $\left(i, k + \frac{r}{i+1}\right)$ for $i \geq 0$ and $0 \leq r \leq i$ and $0 \leq k < t-1$ or for $i \geq 0$ and $0 \leq r < i$ and $k = t-1$.

2. Given a positive integer t and a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, a non-intersecting path configuration is a system of n paths on the graph $\mathcal{G}_t$. For each integer i satisfying $1 \leq i \leq n$, the ith path starts at $\left(n-i, t - \frac{1}{n-i+1}\right)$, ends at $(n - i + \lambda_i, 0)$ and moves only downwards and rightwards. The paths are said to be not intersecting if they do not share a vertex.

See the middle graph of 1 for an example of $\mathcal{G}_3$ and a configuration of non-intersecting paths on $\mathcal{G}_3$.

Given a positive integer t and a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, the non-intersecting path system is a system of n paths on the graph $\mathcal{G}_t$. The ith path starts at $\left(n-i, t - \frac{1}{n-i+1}\right)$ and ends at $(\lambda_i + n - i, 0)$. The paths are called non-intersection if they do not share a vertex.

Theorem 1. *([9])There is a bijection between the bounded lecture hall tableaux of shape λ and bounded by t and non-intersecting paths on $\mathcal{G}_t$ starting at $\left(n-i, t - \frac{1}{n-i+1}\right)$ and ending at $(n - i + \lambda_i, 0)$ for $i = 1, 2, \ldots, n$.*

More precisely, there are exactly $|\lambda|$ non-vertical edges present in the non-intersecting path configuration in $\mathcal{G}_t$ corresponding to a lecture-hall tableaux of shape λ. These edges have left endpoints located at $\left(n+j-i-1, \frac{L(i,j)}{n+j-i}\right)$. The non-intersecting path configuration corresponding to the lecture hall tableaux is the unique non-intersecting path configuration joining $\left(n-i, t - \frac{1}{n-i+1}\right)$ and $(n-i+\lambda_i, 0)$ for $i = 1, 2, \ldots, n$ obtained by adding only vertical edges to these present non-vertical edges.

One can see that for an n-lecture hall tableaux bounded by t, t is also the height of the corresponding lecture hall graph $\mathcal{G}_t$, and n is also the total number of paths in the corresponding non-intersecting path configuration on $\mathcal{G}_t$. See Figure 1 for an example of such a correspondence.

We shall investigate the asymptotics of bounded lecture hall tableaux as $n, t \to \infty$ by studying the asymptotics of the corresponding non-intersecting paths. These asymptotics were studied in [8] using the (not fully rigorous) tangent method; here we attack this problem by analyzing Schur polynomials. The tangent method gives the frozen

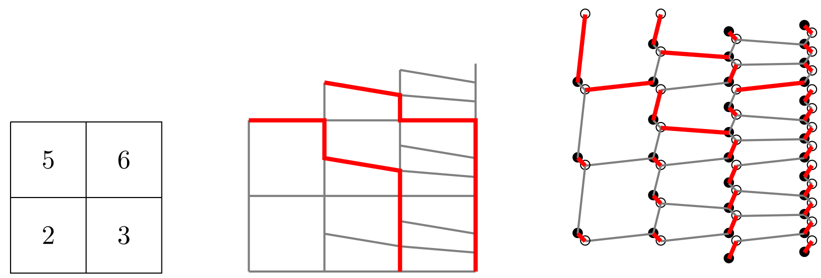

Figure 1: Tableau, non-intersecting paths, and dimers (Figure 1 in [8]). The left graph represents a lecture hall tableaux L of shape $\lambda = (2,2)$ with $L(1,1) = 5$, $L(1,2) = 6$, $L(2,1) = 2$, $L(2,2) = 3$ and $n = 2$. Then $\frac{L(1,1)}{n+1-1} = \frac{5}{2}$; $\frac{L(2,1)}{n+1-2}$=2;$\frac{L(1,2)}{n+2-1} = 2$;$\frac{L(2,2)}{n+2-2} = \frac{3}{2}$. The lecture hall tableaux is bounded by $t = 3$. The middle graph represents the corresponding non-intersecting path configuration. The right graph represents a dimer configuration on a graph which is not doubly-periodic.

boundary without the full limit shape; instead Conjecture 6.1 were made in [8], indicating that the slopes of the rescaled height functions in the scaling limit satisfy the complex Burgers equation. The complex Burgers equation was proved to be the governing equation of height functions in the scaling limit for uniform lozenge tilings and for other doubly periodic dimer models [13]. This equation naturally arises through a variational problem, we refer to [1] for a detailed study of the variational problem. Here we note that for lecture hall tableaux no variational principle has been established and although lecture hall tableaux naturally corresponds to non-interacting paths configurations and dimer configurations on a hexagon-octagon lattice ([8]), the corresponding hexagon-octagon lattice in this case is not doubly periodic as in the setting in [13]; see the right graph of Figure 1.

The Schur generating function approach was applied to study uniform dimer model on a hexagonal lattice in a trapezoid domain in [5, 6], and for uniform dimer model on a rectangular square grid in [7]. A generalized version of the Schur generating function was defined to study the non-uniform dimer model on rail-yard graphs in [4, 16, 15, 17, 19]. Schur processes are specializations of the Macdonald processes when $q = t$, hence the asymptotics of Schur processes can also be obtained by investigating the more general Macdonald processes; see [20, 18]. All the existing Schur-generating functions seem to be defined in the setting of the Gelfand-Tsetlin scheme; however the lecture hall tableaux are novel in the sense that on a skew shape they cannot be computed by skew Schur functions; and the corresponding particle configurations induced by the non-intersecting path configurations of the lecture hall tableaux do not satisfy the interlacing conditions required by the Gelfand-Tsetlin scheme; see Figure 2 for an example.

By constructing a novel Schur generating function specifically for the lecture hall tableaux and analyzing its asymptotics, in this paper we obtain a full description of the limit shape, including the moment formulas for the counting measures and the complex Burgers equation; resolving Conjecture 6.1 in [8].

The Gaussian free field, as a high dimensional time analog of the Brownian motion, was proved to be the rule of height fluctuations for dimer models on a large class of graphs ([12, 14]). In this paper we show that the unrescaled height fluctuations of the lecture hall tableaux converge to the Gaussian free field when t goes to infinity linearly as n goes to infinity.

The main results (with exact statements given in later sections after a number of precise definitions) are as follows.

- In Section 2, we discuss the moment formula for the limit counting when $n \to \infty$, $t \to \infty$ and $\frac{t}{n} \to \alpha \in (0, \infty)$ (Theorem 2); the equation of the boundary curve separating different phases (Theorem 3) and that the slopes of the (rescaled) height function in the scaling limit satisfy the complex Burgers equation; confirming Conjecture 6.1 in [8] (Theorem 4).

- In Section 3, we prove the convergence of the (unrescaled) height fluctuation to the Gaussian free field (GFF) $n \to \infty$, $t \to \infty$ and $\frac{t}{n} \to \alpha \in (0, \infty)$ (Theorem 5)

2 Limit Shape when $t \to \infty$ and Complex Burger's Equation

Let $\mathcal{M}$ be a random non-intersecting path configuration on $\mathcal{G} = \mathcal{G}_t$. Let n be the total number of non-intersecting paths. Let $\kappa \geq 0$ be an integer. Let $\epsilon > 0$ be sufficiently small such that the region $y \in (\kappa, \kappa + \epsilon]$ does not intersect any non-vertical edge of $\mathcal{G}$. We associate a partition $\lambda^{(\kappa)}$ as follows:

- $\lambda_1^{(\kappa)}$ is the number of absent vertical edges of $\mathcal{M}$ intersecting $y = \kappa + \epsilon$ to the left of the rightmost vertical edges present in $\mathcal{M}$.

- for $j \geq 2$, $\lambda_j^{(\kappa)}$ is the number of absent vertical edges of $\mathcal{M}$ intersecting $y = \kappa + \epsilon$ to the left of the jth rightmost vertical edges present in $\mathcal{M}$.

See Figure 2 for an example.

For $\mathbf{x} = (x_0, x_1, \ldots)$ Let $s_{\lambda/\mu}(\mathbf{x})$ be the skew Schur function. For any tableaux T of shape λ/μ, let

$$\mathbf{x}^T = \prod_{(i,j) \in \lambda/\mu} x_{T(i,j)};$$

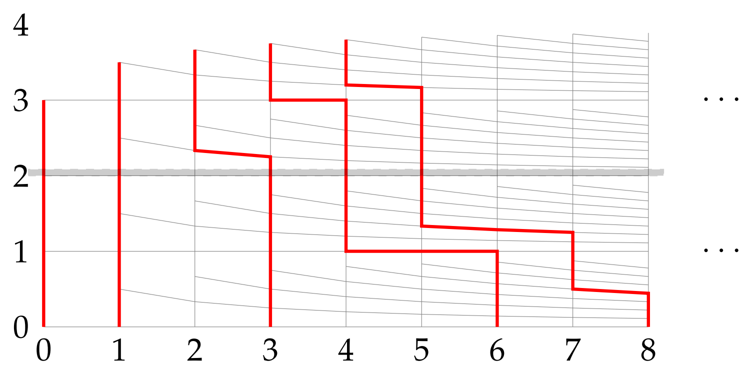

Figure 2: Non-intersecting lattice paths on $\mathcal{G}_4$ for $n = 5$. We have $\lambda^{(3)} = (1,0,0,0,0)$, $\lambda^{(2)} = (1,1,1,0,0)$, $\lambda^{(1)} = (3,3,1,0,0)$ and $\lambda^{(0)} = (4,3,1,0,0)$. The sequence of partitions $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ do not form a Gelfand-Tsetlin scheme.

we define

$$L^n_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in LHT_n(\lambda/\mu)} \mathbf{x}^{\lfloor T \rfloor}$$

Let ρ_κ be the probability distribution of $\lambda^{(\kappa)}$. Define the Schur generating function for ρ_κ as follows:

$$\mathcal{S}_{\rho_\kappa}(|\mathbf{x}|, \mathbf{u}) = \sum_{\lambda \in \mathbb{Y}} \rho_\kappa(\lambda) \frac{s_\lambda(|\mathbf{x}| + \mathbf{u})}{s_\lambda(|\mathbf{x}|)} \tag{2.1}$$

where

$$\mathbf{u} = (u_1, u_2, \ldots, u_n); \qquad \mathbf{x} = (x_1, x_2, \ldots, x_t); \qquad |\mathbf{x}| = x_1 + x_2 + \ldots + x_t$$

and

$$s_\lambda(|\mathbf{x}| + \mathbf{u}) := s_\lambda(|\mathbf{x}| + u_1, |\mathbf{x}| + u_2, \ldots, |\mathbf{x}| + u_n) \tag{2.2}$$
$$s_\lambda(|\mathbf{x}|) := s_\lambda(|\mathbf{x}|, \ldots, |\mathbf{x}|) \tag{2.3}$$

Let λ be a length-N partition. We define the counting measure $m(\lambda)$ as a probability measure on $\mathbb{R}$ as follows:

$$m(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right).$$

If λ is random, then we can define the corresponding random counting measure.

let $S_{\mathbf{m}}(z) = z + \sum_{k=1}^{\infty} M_k(\mathbf{m}) z^{k+1}$ be the moment generating function of the measure $\mathbf{m}$, where $M_k(\mathbf{m}) = \int x^k d\mathbf{m}(x)$, and $S_{\mathbf{m}}^{(-1)}$ be its inverse for the composition. Let $R_{\mathbf{m}}(z)$ be the *Voiculescu R-transform* of $\mathbf{m}$ defined as

$$R_{\mathbf{m}}(z) = \frac{1}{S_{\mathbf{m}}^{(-1)}(z)} - \frac{1}{z}.$$

Then

$$H_{\mathbf{m}}(u) = \int_0^{\ln u} R_{\mathbf{m}}(t)dt + \ln\left(\frac{\ln u}{u-1}\right). \tag{2.4}$$

In particular, $H_{\mathbf{m}}(1) = 0$, and

$$H'_{\mathbf{m}}(u) = \frac{1}{uS_{\mathbf{m}}^{(-1)}(\ln u)} - \frac{1}{u-1}. \tag{2.5}$$

Assume as $n \to \infty$, the rescaled graph $\frac{1}{n}\mathcal{G}$ approximate a bounded simply-connected region $\mathcal{R} \subset \mathbb{R}^2$. Let $\mathcal{L}$ be the set of (χ, y) inside $\mathcal{R}$ such that the density $d\mathbf{m}_y(\frac{\chi}{1-y})$ is not equal to 0 or 1. Then $\mathcal{L}$ is called the liquid region. Its boundary $\partial\mathcal{L}$ is called the frozen boundary. Let

$$\widetilde{\mathcal{L}} := \{(\chi, s) : (\chi, y) \in \mathcal{L}\}$$

where s, y are given as Theorem 2.

Theorem 2. *Let n be the the total number of non-interacting paths in $\mathcal{G}$, and let t be the height of $\mathcal{G}$. Let $\rho_\kappa(n)$ be the probability distribution of $\lambda^{(\kappa)}$. Assume*

$$y := \lim_{n\to\infty} \frac{\kappa}{n}; \qquad s := \lim_{n\to\infty} \frac{|\mathbf{x}_\kappa|}{|\mathbf{x}|}; \qquad \alpha := \lim_{n\to\infty} \frac{t}{n}; \tag{2.6}$$

such that

$$s \in (0, 1); \qquad y \in (0, \alpha).$$

Then random measures $\mathbf{m}_{\rho_\kappa(n)}$ converge as $n \to \infty$ in probability, in the sense of moments to a deterministic measure $\mathbf{m}_y$ on $\mathbb{R}$, whose moments are given by

$$\int_{\mathbb{R}} x^j \mathbf{m}_y(dx) = \frac{1}{2(j+1)\pi\mathbf{i}} \oint_1 \frac{dz}{z-1+s} \left((z-1+s)H'_{\mathbf{m}_0}(z) + \frac{z-1+s}{z-1}\right)^{j+1}$$

Here $\mathbf{m}_0$ is the limit counting measure for the boundary partition $\lambda^{(0)} \in \mathbb{Y}_n$ as $n \to \infty$, and $H_{\mathbf{m}_0}$ is defined as in (2.4).

The main idea to prove Theorem 2 is to use a differential operator acting on the Schur generating function defined by (2.1), which gives the moments of $\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}$; by proving that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}\right]^2 = \lim_{n\to\infty}\left[\mathbb{E}\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}\right]^2;$$

it follows that the limit counting measure is deterministic. The explicit integral formula for $\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}$ follows from the Residue theorem.

Theorem 3. *Let*

$$U_y(z) := (z-1+s)H'_{\mathbf{m}_0}(z) + \frac{z-1+s}{z-1} \tag{2.7}$$

Assume the liquid region is nonempty, and assume that for any $x \in \mathbb{R}$, the equation $U_y(z) = x$ has at most one pair of complex conjugate roots. Then for any point (χ, y) lying on the frozen boundary, the equation $U_y(z) = \chi$ has double roots.

The main idea to prove Theorem 3 is to compute the density of the measure $d\mathbf{m}_y(x)$ by the Stieljes transform

$$\frac{d\mathbf{m}_y(x)}{dx} = -\lim_{\epsilon\to 0+}\frac{1}{\pi}\Im(\mathrm{St}_{\mathbf{m}_y}(x+\mathbf{i}\epsilon)) \tag{2.8}$$

where $\Im(\cdot)$ represents the imaginary part of a complex number and $\mathrm{St}_{\mathbf{m}_y}$ is the Stieljes transform of the measure $\mathbf{m}_y$; and then find the boundary of the region where the density is 0 or 1 (frozen region).

Example 1. *Assume the bottom boundary partition is given by*

$$\lambda^{(0)}(n) := ((p-1)n, (p-1)(n-1), \dots, p-1) \in \mathbb{Y}_n$$

where p, n are positive integers. We have

$$U_y(z) = \frac{pz^{p-1}(z-1+s)}{z^p - 1}$$

Assume $p = 3$. then for each $\chi \in \mathbb{R}$ the equation $U_y(z) = \chi$ has at most one pair of nonreal conjugate roots. The condition that $U_y(z) = \chi$ has double roots gives

$$\begin{cases} U_y(z) = \chi. \\ U'_y(z) = 0 \end{cases}$$

which gives the parametric equation for (x, s) as follows.

$$\begin{cases} \chi = \frac{3z^3}{z^3+2} \\ s = \frac{z^3-3z+2}{z^3+2} \end{cases}$$

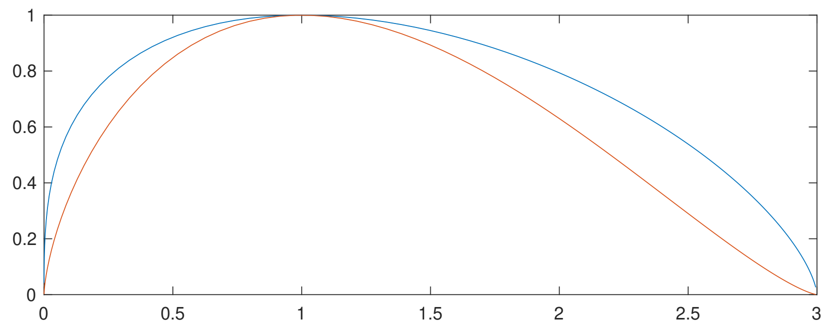

Figure 3: Frozen boundary for the scaling limit of weighted non-interaction paths. The blue curve is for the uniform weight; the red curve is when the limit weight function s satisfies $y = (1-s)^2$.

1. *When $x_1 = x_2 = \ldots = x_n$, and $\alpha = 1$, we have $s = 1 - y$. The frozen boundary is given by the blue curve of Figure 3.*
2. *When $\alpha = 1$, and $y = (1-s)^2$. The frozen boundary is given by the red curve of Figure 3.*

On the lecture hall graph $\mathcal{G}$, define a random height function h associated to a random non-intersecting path configuration as follows. The height at the lower left corner is 0, and the height increases by 1 whenever crossing a path from the left to the right. Define the rescaled height function by

$$h_n(\chi, y) := \frac{1}{n} h(n\chi, ny)$$

Following similar computations before Lemma 8.1 of [4], we obtain that when (χ, y) is in the liquid region,

$$\lim_{n\to\infty} \frac{dh_n(\chi, y)}{d\chi} = \frac{1}{\pi}\mathrm{Arg}(\mathbf{z}_+(\chi, y) - 1 + s).$$

where $\mathbf{z}_+(\chi, y)$ is the unique root in the upper half plane of the equation $U_y(z) = \chi$.

Theorem 4. *Assume $\mathcal{G}$ is uniformly weighted such that $s = 1 - y$. Suppose that the assumptions of Theorem 3 holds. Let*

$$u = \frac{1}{\mathbf{z}_+(\chi, y) S_{\mathbf{m}_0}^{(-1)}(\ln \mathbf{z}_+(\chi, y))}$$

Then

$$\frac{\partial h}{\partial x} = \frac{1}{\pi}\left(2 - \mathrm{Arg}(u)\right); \qquad \frac{\partial h}{\partial y} = \frac{1}{\pi}\Im u \tag{2.9}$$

where $\operatorname{Arg}(\cdot)$ *is the branch of the argument function taking values in* $[0, 2\pi)$. *Moreover,* u *satisfies the complex Burgers equation*

$$u_x - uu_y = 0. \tag{2.10}$$

3 Height Fluctuations and Gaussian Free Field

Let C_0^∞ be the space of smooth real-valued functions with compact support in the upper half plane $\mathbb{H}$. The **Gaussian free field** (GFF) Ξ on $\mathbb{H}$ with the zero boundary condition is a collection of Gaussian random variables $\{\xi_f\}_{f \in C_0^\infty}$ indexed by functions in C_0^∞, such that the covariance of two Gaussian random variables ξ_{f_1}, ξ_{f_2} is given by

$$\operatorname{Cov}(\xi_{f_1}, \xi_{f_2}) = \int_{\mathbb{H}} \int_{\mathbb{H}} f_1(z) f_2(w) G_{\mathbb{H}}(z, w) dz d\bar{z} dw d\overline{w},$$

where

$$G_{\mathbb{H}}(z, w) := -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \overline{w}} \right|, \qquad z, w \in \mathbb{H}$$

is the Green's function of the Dirichlet Laplacian operator on $\mathbb{H}$. The Gaussian free field Ξ can also be considered as a random distribution on C_0^∞ of $\mathbb{H}$, such that for any $f \in C_0^\infty$, we have

$$\Xi(f) = \int_{\mathbb{H}} f(z) \Xi(z) dz := \xi_f;$$

where $\Xi(z)$ is the generalized function corresponding to the linear functional Ξ. Note that GFF is conformally invariant; in the sense that for any simply-connected domain $\mathcal{D} \subsetneq \mathbb{C}$, and let $\phi : \mathcal{D} \to \mathbb{H}$ be a conformal map from $\mathcal{D}$ to $\mathbb{H}$. Then the GFF on $\mathcal{D}$ is

$$\Xi_{\mathcal{D}}(z) := \Xi(\phi(z))$$

See [22] for more about GFF.

Let f be a function of r variables. Define the symmetrization of f as follows

$$\operatorname{Sym}_{x_1, \ldots, x_r} f(x_1, \ldots, x_r) := \frac{1}{r!} \sum_{\sigma \in S_r} f(x_{\sigma(1)}, \ldots, x_{\sigma(r)}); \tag{3.1}$$

Assumption 1. *Let* l *be a fixed positive integer. Assume there exists*

$$0 = a_1 < b_1 < a_2 < b_2 < \ldots < a_l < b_l$$

such that $\mathbf{m}_0$*, the limit counting measure corresponding to the partition on the bottom boundary satisfies*

$$\frac{d\mathbf{m}_0}{dx} = \begin{cases} 1 & \text{if } a_i < x < b_i \\ 0 & \text{if } b_j < x < a_{j+1} \end{cases}$$

where $i \in [l]$ *and* $j \in [l-1]$.

Theorem 5. *Suppose that Assumption 1 holds. For each* $z \in \mathbb{H}$*, let*

$$\mathbf{\Delta}_n(z) := \Delta_n(n\chi_{\widetilde{\mathcal{L}}}(z), ns_{\widetilde{\mathcal{L}}}(z)) := \sqrt{\pi}\left|\{g \in [n] : \lambda_g^{(n-ny(s_{\widetilde{\mathcal{L}}}(z)))} - n + g \geq n\chi_{\widetilde{\mathcal{L}}(z)}\}\right|$$

Under the assumption of Theorem 2, $\mathbf{\Delta}_n(z) - \mathbb{E}\mathbf{\Delta}_n(z)$ *converge to GFF in the upper half plane in the sense that for each* $s \in (0,1)$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\chi^j\left(\Delta_n(n\chi, ns) - \mathbb{E}\Delta_n(n\chi, ns)\right)d\chi = \int_{z\in\mathbb{H}:s_{\widetilde{\mathcal{L}}}(z)=s}\chi_{\widetilde{\mathcal{L}}}^j(z)\frac{d\chi_{\widetilde{\mathcal{L}}}(z)}{dz}\Xi(z)dz$$

The main idea to prove Theorem 5 is to first show that a collection of certain observables converge to a Gaussian vector in the scaling limit by applying the Wick's moment formula; then find an explicit diffeomorphism from the liquid region to the upper half plane, which gives the convergence of the observables to the pull-back of GFF in $\mathbb{H}$.

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Shuffle theorems and sandpiles

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Abstract. We provide an explicit description of the recurrent configurations of the sandpile model on a family of graphs $\widehat{G}_{\mu,\nu}$, which we call *clique-independent* graphs, indexed by two compositions μ and ν. Moreover, we define a *delay* statistic on these configurations, and we show that, together with the usual *level* statistic, it can be used to provide a new combinatorial interpretation of the celebrated *shuffle theorem* of Carlsson and Mellit. More precisely, we will see how to interpret the polynomials $\langle \nabla e_n, e_\mu h_\nu \rangle$ in terms of these configurations.

Keywords: Shuffle theorem, sandpile model, recurrent configurations

1 Introduction

1.1 Shuffle theorem

The *shuffle theorem* of Carlsson and Mellit [4] is a recent breakthrough that provided a positive solution to a long-standing conjecture about a combinatorial formula for the Frobenius characteristic of the so-called diagonal harmonics. More precisely, this theorem provides the monomial expansion of the symmetric function ∇e_n, where e_n is the elementary symmetric function of degree n in the variables $x_1, x_2, \dots$, and ∇ is the famous *nabla* operator introduced by Bergeron and Garsia in the 90's. In this formula, to each *labelled Dyck path* of size n corresponds a monomial, where the variables $x_1, x_2, \dots$ keep track of the labels, while the variables q and t keep track of the bistatistic (dinv, area).

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In [14] Loehr and Remmel provided an alternative combinatorial interpretation of the same symmetric function in terms of the same objects, but using the bistatistic (area, pmaj). In particular, they showed bijectively that the two combinatorial formulas coincide. In the present article we show that this last combinatorial formula has a natural interpretation in terms of the sandpile model.

1.2 Sandpile model

The *(abelian) sandpile model* is a combinatorial dynamical system on graphs first introduced by Bak, Tang and Wiesenfeld [3] in the context of "self-organized criticality" in statistical mechanics. The sandpile model (and variants of it) have found applications in a wide variety of mathematical contexts including enumerative combinatorics, tropical geometry, and Brill–Noether theory, among others: see [13] for a nice introductory monograph. *In the present article we only consider the sandpile model with a sink.*

A well-known link between the combinatorics of this dynamical system and the one of the underlying graph is given by the so-called *recurrent configurations* (see Definition 9). For example, the recurrent configurations of the sandpile model are in bijection with the spanning trees of the graph (see e.g. [7]).

If the underlying graph presents some symmetries, then it is natural to look at the recurrent configurations "modulo" those symmetries. For example, for the complete graph we can identify recurrent configurations that are the same up to a permutation of the vertices (not moving the sink): perhaps not surprisingly, we still get an interesting combinatorics, as in this case we find Catalan many such "sorted" configurations.

More formally, consider the sandpile model on a graph G, and let $\mathsf{Aut}(G)$ be the automorphism group of G. Consider a subgroup Γ of the stabilizer of the sink. Now Γ acts naturally on the set $\mathsf{Rec}(G)$ of recurrent configurations: we are interested in the orbits of this action, that we will call *sorted recurrent configurations*.

1.3 Main result

We will consider an explicit family of graphs $\widehat{G}_{\mu,\nu}$ indexed by pairs of compositions μ and ν. For such a graph $\widehat{G}_{\mu,\nu}$ we will look at a subgroup Γ of its automorphism group that will be isomorphic to the Young subgroup $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ of the symmetric group $\mathfrak{S}_n$, where $n = |\mu| + |\nu|$. We denote by $\mathsf{SortRec}(\mu,\nu)$ the set of the corresponding sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$.

For every recurrent configuration κ of $\widehat{G}_{\mu,\nu}$, we will define a new statistic, called the *delay* of κ (denoted $\mathsf{delay}(\kappa)$), which we will couple with the usual *level* statistic (denoted $\mathsf{level}(\kappa)$). To state our main result, we need a few more definitions.

Given a composition $\mu = (\mu_1, \mu_2, \dots)$, we denote by e_μ the product $e_{\mu_1} e_{\mu_2} \cdots$, and similarly $h_\mu = h_{\mu_1} h_{\mu_2} \cdots$, where h_n is the complete homogeneous symmetric function of degree n. Finally, we denote by $\langle -, - \rangle$ the Hall scalar product on symmetric functions.

Theorem 1. *For every pair of compositions μ, ν such that $n = |\mu| + |\nu|$ we have*

$$\langle \nabla e_n, e_\mu h_\nu \rangle = \sum_{\kappa \in \mathsf{SortRec}(\mu,\nu)} q^{\mathsf{level}(\kappa)} t^{\mathsf{delay}(\kappa)}.$$

Notice that for $\mu = \varnothing$, the coefficient $\langle \nabla e_n, h_\nu \rangle$ is simply the coefficient of $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots$ in ∇e_n, hence this formula gives in particular a new combinatorial interpretation of the monomial expansion of the symmetric function ∇e_n in terms of the sandpile model.

The idea of the proof is to show that the sorted recurrent configurations with the bistatistic (level, delay) correspond bijectively to the labelled Dyck paths predicted by the shuffle theorem with the bistatistic (area, pmaj).

1.4 Comments

Theorem 1 extends several previous results in the literature: the case $\widehat{G}_{\varnothing,(n)}$ was already worked out in [8], (a slight modification of) the case $\widehat{G}_{(m,n-m),\varnothing}$ already appears in [1, 11], while the case $\widehat{G}_{(m),(n-m)}$ is dealt with in [10].

Other articles in which sorted recurrent configurations make their appearance are for example [2] and [9]. It should be noticed that the works [8] and [9] inspired the results in [6] and [5] respectively, which belong to tropical geometry and Brill-Noether theory.

We hope that the findings in the present article motivate further investigation of sorted recurrent configurations, and their relation to other parts of mathematics.

2 Combinatorics of the shuffle theorem

For every $n \in \mathbb{N}$, we set $[n] := \{1, 2, \ldots, n\}$.

The pmaj statistic was first introduced in [14]. The area statistic is classical.

Definition 1. A *Dyck path* of size n is a lattice path going from $(0,0)$ to (n,n), using only north and east steps and staying weakly above the line $x = y$ (also called the *main diagonal*). A *labelled Dyck path* is a Dyck path whose vertical steps are labelled with (not necessarily distinct) positive integers such that, when placing the labels in the square to the right of its step, the labels appearing in each column are strictly increasing from bottom to top. For us, a *parking function*[1] of size n is a labelled Dyck path of size n whose labels are precisely the elements of $[n]$. See Figure 1 for an example.

The set of all parking functions of size n is denoted by $\mathsf{PF}(n)$.

[1] These are in bijection with the functions $f\colon [n] \to [n]$ such that $\#\{1 \leq j \leq n \mid f(j) \geq i\} \leq n + 1 - i$, by defining $f(i)$ to be the column of the label i.

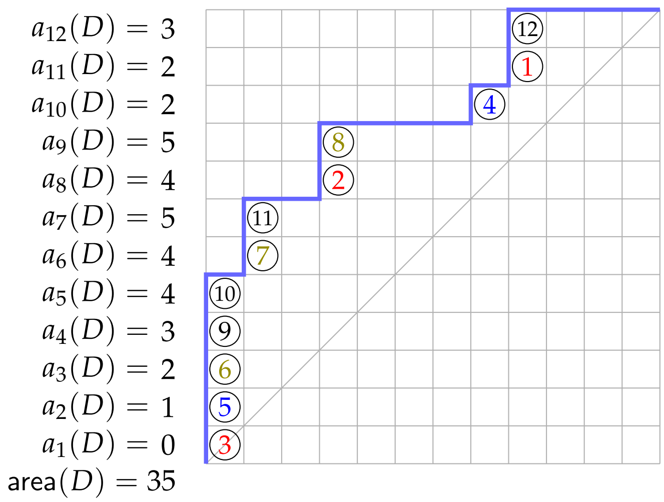

Figure 1: An element D of $\mathsf{PF}((4,3);(3,2))$.

Definition 2. Given $D \in \mathsf{PF}(n)$, we define its *area word* to be the string of integers $a(D) = a_1(D) \cdots a_n(D)$ where $a_i(D)$ is the number of whole squares in the i-th row (from the bottom) between the path and the main diagonal. We define the *area* of D as

$$\mathsf{area}(D) := \sum_{i=1}^{n} a_i(D).$$

Example 1. The area word of the path in Figure 1 is 012344545223 and its area is 35.

To introduce the other statistic, we need a couple of definitions.

Definition 3. Let $a_1 a_2 \cdots a_k$ be a string of integers. We define its *descent set*

$$\mathsf{Des}(a_1 a_2 \cdots a_k) := \{1 \leq i \leq k-1 \mid a_i > a_{i+1}\}$$

and its *major index* $\mathsf{maj}(a_1 a_2 \cdots a_k)$ as the sum of the elements of the descent set.

Definition 4. Let $D \in \mathsf{PF}(n)$. We define its *parking word* $p(D)$ as follows.

Let C_1 be the multiset containing the labels appearing in the first column of D, and let $p_1(D) := \max C_1$. At step i, let C_i be the multiset obtained from C_{i-1} by removing $p_{i-1}(D)$ and adding all the labels in the i-th column of the D; let

$$p_i(D) := \max \{x \in C_i \mid x \leq p_{i-1}(D)\}$$

if this last set is non-empty, and

$$p_i(D) := \max C_i$$

otherwise. We finally define the parking word of D as $p(D) := p_1(D) \cdots p_n(D)$.

Definition 5. We define the statistic *pmaj* on $D \in \mathsf{PF}(n)$ as

$$\mathsf{pmaj}(D) := \mathsf{maj}(p_n(D) \cdots p_1(D)).$$

Example 2. For example, the parking word of the parking function D in Figure 1 is[2] $\overline{10}976532\overline{11}841\overline{12}$. In fact, we have $C_1 = \{3,4,5,6,9,10\}$, $C_2 = \{3,4,5,6,9,7,11\}$, $C_3 = \{3,4,5,6,7,11\}$, and so on. The descent set of the reverse is $\{1,5\}$, so $\mathsf{pmaj}(D) = 6$.

Definition 6. For $D \in \mathsf{PF}(n)$ we set $l_i(D)$ to be the label of the i-th vertical step. Then the *pmaj reading word* of D is the sequence $l_1(D) \cdots l_n(D)$, i.e. the sequence of the labels read bottom to top.

For example, the labelled Dyck path in Figure 1 has pmaj reading word $3569\overline{10}7\overline{11}2841\overline{12}$.

Given two compositions $\mu = (\mu_1, \mu_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$ with $|\mu| + |\nu| = n$, let $K_{\mu_1} = \{n, n-1, \dots, n-\mu_1+1\}$, $K_{\mu_2} = \{n-\mu_1, n-\mu_1-1, \dots, n-\mu_1-\mu_2+1\}$, and so on, and let $I_{\nu_1} = \{1, 2, \dots, \nu_1\}$, $I_{\nu_2} = \{\nu_1+1, \nu_1+2, \dots, \nu_1+\nu_2\}$, and so on. Notice that the sets $K_{\mu_1}, K_{\mu_2}, \dots, I_{\nu_1}, I_{\nu_2}, \dots$ form a partition of $[n]$.

Let now $\uparrow K_{\mu_i}$ be the word consisting of the elements of K_{μ_i} in increasing order: for example $\uparrow K_{\mu_1} = (n-\mu_1+1)(n-\mu_1+2)\cdots(n-1)n$. Similarly, let $\downarrow I_{\nu_j}$ be the word consisting of the elements of I_{ν_j} in decreasing order: for example $\downarrow I_{\nu_1} = \nu_1(\nu_1-1)\cdots 21$.

Consider the shuffle

$$\mathsf{W}(\mu;\nu) := \uparrow K_{\mu_1} \sqcup\!\sqcup \uparrow K_{\mu_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \uparrow K_{\mu_{\ell(\mu)}} \sqcup\!\sqcup \downarrow I_{\nu_1} \sqcup\!\sqcup \downarrow I_{\nu_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \downarrow I_{\nu_{\ell(\nu)}},$$

which we can think of as a set of permutations in $\mathfrak{S}_n$ in one-line notation. Let $\mathsf{PF}(\mu;\nu)$ be the set of parking functions whose pmaj reading word is in $\mathsf{W}(\mu;\nu)$.

For example[2], $\mathsf{W}((4,3);(3,2)) = 9\overline{10}\,\overline{11}\,\overline{12} \sqcup\!\sqcup 678 \sqcup\!\sqcup 54 \sqcup\!\sqcup 321$, and the pmaj reading word of the parking function D in Figure 1 belongs to it, so that $D \in \mathsf{W}((4,3);(3,2))$.

We can now state the shuffle theorem in the form that is suitable for our purposes: this is a combination of the main results in [4] and [14] combined with *superization*: see [12, Chapter 6].

Theorem 2. *For every pair of compositions μ and ν with $|\mu| + |\nu| = n$ we have*

$$\langle \nabla e_n, e_\mu h_\nu \rangle = \sum_{D \in \mathsf{PF}(\mu;\nu)} q^{\mathsf{area}(D)} t^{\mathsf{pmaj}(D)}.$$

3 The clique-independent graphs $\widehat{G}_{\mu,\nu}$

Definition 7. Let μ, ν be two compositions (i.e. tuples of positive integers). Set $n = |\mu| + |\nu|$. We define a graph $G_{\mu,\nu}$ with set of vertices $[n] := \{1, 2, \dots, n\}$ consisting of the following *components*[3]:

[2] We put a bar on the two-digit numbers not to confuse them.

[3] Notice that the notation is coherent with the one used in Section 2.

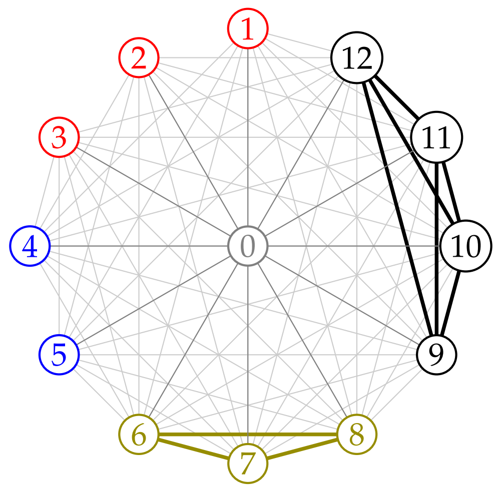

Figure 2: The graph $\widehat{G}_{(4,3),(3,2)}$.

- $\ell(\mu)$ *clique components*, i.e. complete graphs, $K_{\mu_1}, K_{\mu_2}, \dots$, on $\mu_1, \mu_k, \dots$ vertices respectively. The vertices of K_{μ_1} are $n, n-1, \dots, n-\mu_1+1$; the vertices of K_{μ_2} are $n-\mu_1, n-\mu_1-1, \dots, n-\mu_1-\mu_2+1$; and so on.
- $\ell(\nu)$ *independent components*, i.e. graphs without edges, $I_{\nu_1}, I_{\nu_2}, \dots$, on $\nu_1, \nu_2, \dots$ vertices respectively; the vertices of I_{ν_1} are $1, 2, \dots, \nu_1$; the vertices of I_{ν_2} are $\nu_1+1, \nu_1+2, \dots, \nu_1+\nu_2$; and so on.

Finally, two vertices in distinct components are always connected by an edge.

Example 3. If $\mu = \varnothing$, then $G_{\varnothing,\nu}$ is the complete multipartite graph $K_{\nu_1,\nu_2,\dots}$. If $\nu = \varnothing$, then $G_{\mu,\varnothing}$ is isomorphic to the complete graph $K_{|\mu|}$; however, for our purposes we will distinguish between $G_{(|\mu|),\varnothing}$ and $G_{(\mu_1,\mu_2,\dots),\varnothing}$, as we will consider the action of different groups of automorphisms, which will lead to different sorted configurations.

Given one of our labelled graphs $G_{\mu,\nu}$, we define the graph $\widehat{G}_{\mu,\nu}$ simply as $G_{\mu,\nu}$ to which we add a vertex 0, and we connect it with every other vertex. We will consider the sandpile on $\widehat{G}_{\mu,\nu}$, where 0 is the sink. Figure 2 is an illustration of the graph $\widehat{G}_{(4,3),(3,2)}$.

4 Basics of the sandpile model

In the present work with a *graph* we will always mean a simple graph, i.e. a graph with no loops and no multiple edges.

Definition 8. Let G be a finite, undirected graph on the vertex set $\{0, 1, \dots, n\}$.

A *configuration* of the *sandpile (model) on* G is a map $\kappa : [n] \cup \{0\} \to \mathbb{Z}$ that assigns a (integer) number of "grains of sand" to each vertex of G.

If $0 \leq \kappa(v) \leq \deg(v)$, we say that v is *stable*, and otherwise it is *unstable*. Any vertex can *topple* (or *fire*), and “donate a single grain” to each of its neighbors: the result is a new configuration κ' in which $\kappa'(v) = \kappa(v) - \deg(v)$ and for any $w \neq v$

$$\kappa'(w) = \begin{cases} \kappa(w) + 1, & \text{if } (v, w) \text{ is an edge} \\ \kappa(w), & \text{otherwise.} \end{cases}$$

For any $v \in \{0, \dots, n\}$ we write ϕ_v for the *toppling operator* at vertex v. That is $\phi_v(\kappa)$ is a new configuration obtained from κ by toppling the vertex v.

The vertex 0 is special in this model, and we call it the *sink*, while we call all the others *nonsink* vertices. We say that a configuration κ is *non-negative* if all of its nonsink vertices are non-negative, *stable* if all of its nonsink vertices are stable, and *unstable* if at least one of its nonsink vertices is unstable.

Remark 1. Notice that the notion of stable configuration has no dependency on the value on the sink. Therefore, as it is customary, we will ignore the value of a configuration on the sink, and consider the configurations as restricted on the nonsink vertices. Moreover, we will identify every configuration κ with the word $\kappa(n)\kappa(n-1)\cdots\kappa(2)\kappa(1)$.

Example 4. Consider the graph $\widehat{G}_{(4,3),(3,2)}$ (see Figure 2), whose vertices are $\{0\} \cup [12]$, and let 0 be the sink. The configuration κ given by[2] $3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a stable configuration. We compute a few topplings:

$$\begin{aligned}
\phi_0(\kappa) &= 4\overline{11}\,\overline{12}\,\overline{12}9\overline{11}\,\overline{12}\,\overline{11}5\overline{10}84, \\
(\phi_{\overline{10}} \circ \phi_0)(\kappa) &= 5\overline{12}0\overline{13}\,\overline{10}\,\overline{12}\,\overline{13}\,\overline{12}6\overline{11}95, \\
(\phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 6\overline{13}\,11\overline{11}\,\overline{13}\,\overline{14}\,\overline{13}7\overline{12}\,\overline{10}6, \\
(\phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 7\overline{14}\,22\overline{12}1\overline{15}\,\overline{14}8\overline{13}\,\overline{11}7, \\
(\phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 8\overline{15}\,33\overline{13}23\overline{15}9\overline{14}\,\overline{12}8, \\
(\phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 9\overline{16}\,44\overline{14}3449\overline{15}\,\overline{13}9, \\
(\phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= \overline{10}\,\overline{17}\,55\overline{15}455\overline{10}5\overline{13}9, \\
(\phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= \overline{11}\,\overline{18}\,66\overline{16}566\overline{11}539.
\end{aligned}$$

Definition 9. Let κ be a stable configuration, and consider the configuration $\phi_0(\kappa)$. We say that κ is *recurrent*[4] if there is an order of all the nonsink vertices such that toppling the vertices in that order we always stay non-negative. Of course at the end of this sequence of topplings we will be back to κ. More precisely, a configuration κ is recurrent if there is a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ such that

$$\phi_0(\kappa), (\phi_{\sigma(1)} \circ \phi_0)(\kappa), (\phi_{\sigma(2)} \circ \phi_{\sigma(1)} \circ \phi_0)(\kappa), \dots, (\phi_{\sigma(n)} \circ \cdots \circ \phi_{\sigma(1)} \circ \phi_0)(\kappa) = \kappa$$

[4]In the literature “recurrent” is sometimes used in a broader sense than in this paper. Configurations that are recurrent in our sense are called *critical* in these settings.

are all non-negative configurations. In this case, σ is the *toppling word* of this sequence of topplings, and we say that this sequence *verifies the recurrence* of κ.

Example 5. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a recurrent configuration for $\widehat{G}_{(4,3),(3,2)}$: indeed it is easy to check that $\sigma = \overline{10}976532\overline{11}841\overline{12}$ verifies the recurrence of κ (cf. Example 2).

Remark 2. It is well known (see e.g. [2, Theorem 2.4]) that the condition for κ to be recurrent is equivalent to say that starting from $\phi_0(\kappa)$ there is no proper (possibly empty) subset A of $[n]$ such that toppling all the vertices of A brings $\phi_0(\kappa)$ to a stable configuration.

Definition 10. Given a recurrent configuration κ of G, we define its *level* as

$$\mathsf{level}(\kappa) := -|E_s(G)| + \sum_{i=1}^{n} \kappa(i)$$

where $E_s(G)$ is the set of edges of G that are not incident to the sink.

It is well-known that $\mathsf{level}(\kappa) \geq 0$, and there exists a recurrent configuration of level 0 if G is connected [15].

Remark 3. For $\widehat{G}_{\mu,\nu}$ with $|\mu| + |\nu| = n$ we have

$$|E_s(\widehat{G}_{\mu,\nu})| = \binom{n}{2} - \sum_{i \geq 0} \binom{\nu_i}{2}.$$

Example 6. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ for $\widehat{G}_{(4,3),(3,2)}$ has level

$$\mathsf{level}(\kappa) = -\binom{12}{2} + \binom{3}{2} + \binom{2}{2} + 97 = 35.$$

5 Toppling algorithm and delay

Consider the sandpile on a graph G with vertices $\{0\} \cup [n]$, where 0 is the sink. Let κ be a recurrent configuration of G. Consider Algorithm 1.
Before discussing the algorithm, let us look at an example.

Example 7. Consider again the configuration κ from Example 4: in that example we actually computed the sequence of toppling given by the first iteration of the **for** loop of Algorithm 1 applied to κ. We compute the second iteration of the **for** loop:

$$(\phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{12}677\overline{17}677\overline{12}64\overline{10},$$
$$(\phi_8 \circ \phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{13}7885788\overline{13}75\overline{11},$$
$$(\phi_4 \circ \phi_8 \circ \phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{14}8996898286\overline{12},$$
$$(\phi_1 \circ \phi_4 \circ \phi_8 \circ \phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{15}9\overline{10}\,\overline{10}79\overline{10}93862,$$

and finally the third and last iteration of the **for** loop:

$$(\phi_{\overline{12}}\circ\phi_1\circ\phi_4\circ\phi_8\circ\phi_{\overline{11}}\circ\phi_2\circ\phi_3\circ\phi_5\circ\phi_6\circ\phi_7\circ\phi_9\circ\phi_{\overline{10}}\circ\phi_0)(\kappa) = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973 = \kappa.$$

Hence, the output of Algorithm 1 applied to κ is the word $\overline{10}976532\overline{11}841\overline{12}$ (cf. Example 2 and Example 5).

Algorithm 1 Toppling algorithm

Input: A graph G and a recurrent configuration κ
Output: The word of nonsink vertices in the order they have been toppled
  Topple the sink, i.e. compute $\phi_0(\kappa)$
  Initialize the output word as empty
  while there are nonsink vertices that are untoppled **do**
    for i going from n to 1 (in decreasing order) **do**
      if vertex i is unstable **then**
        Topple vertex i
        Append i to the output word
      end if
    end for
  end while

Observe that by construction the algorithm terminates: since κ is recurrent, $\phi_0(\kappa)$ is non-negative and at least one of the vertices adjacent to the sink is unstable; then every time we topple we stay non-negative, and since κ is recurrent the process must go through all the nonsink vertices (otherwise we found a subset A of nonsink vertices such that after we topple its vertices we are in a stable configuration, cf. Remark 2).

By construction the algorithm outputs a toppling sequence that verifies the recurrence of κ. We can now define our new statistic on recurrent configurations.

Definition 11. Let κ be a recurrent configuration of G. For every $i \in [n]$, let $r_i(\kappa)$ be the number of **for** loop iterations in Algorithm 1 that occurred before the one in which the vertex i is toppled (so if i is toppled in the first iteration, then $r_i(\kappa) = 0$). Then we define the *delay* of κ as

$$\mathsf{delay}(\kappa) := \sum_{i=1}^{n} r_i(\kappa).$$

Remark 4. If σ is the output of Algorithm 1 applied to κ, then clearly

$$\mathsf{delay}(\kappa) = \mathsf{maj}(\sigma_n\sigma_{n-1}\cdots\sigma_1).$$

Example 8. For the configuration κ of Example 4, we got in Example 7 that Algorithm 1 gives $\sigma = \overline{10}976532\overline{11}841\overline{12}$, so that $\mathsf{delay}(\kappa) = \mathsf{maj}(\overline{12}148\overline{11}235679\overline{10}) = 1 + 5 = 6$. Indeed, looking at the computation of the algorithm, we find that the word $r_1(\kappa)r_2(\kappa)\cdots$ in this case is indeed 100100010012, whose letters add up to 6 (cf. Example 2).

6 Sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$

Consider the Young subgroup $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ of the symmetric group $\mathfrak{S}_n$ consisting of the permutations that preserve the components of $G_{\mu,\nu}$. We want to consider configurations "modulo" the natural action of $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ on the set of configurations. More precisely, a *sorted configuration*[5] of the sandpile on $\widehat{G}_{\mu,\nu}$ is a configuration κ that is weakly decreasing inside each clique component of $\widehat{G}_{\mu,\nu}$ and weakly increasing inside each independent component of $\widehat{G}_{\mu,\nu}$: if $i,j \in K_{\mu_r}$ and $i<j$, then $\kappa(i) \leq \kappa(j)$; if $i,j \in I_{\nu_s}$ and $i<j$, then $\kappa(i) \geq \kappa(j)$.

Example 9. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a sorted recurrent configuration for $\widehat{G}_{(4,3),(3,2)}$ (recall that in our notation $\kappa = \kappa(n)\kappa(n-1)\cdots\kappa(1)$).

Let κ be a sorted recurrent configuration of $\widehat{G}_{\mu,\nu}$. Let $\sigma \in \mathfrak{S}_n$ be the toppling word produced by Algorithm 1 applied to κ.

For every independent component I_{ν_s} of $G_{\mu,\nu}$, we order its vertices in decreasing order, and if $v_j^{(s)}$ is the j-th vertex of I_{ν_s}, we set

$$\widetilde{\kappa}(v_j^{(s)}) := \kappa(v_j^{(s)}) + \nu_s - j.$$

For every vertex v in a clique component K_{μ_r} we set

$$\widetilde{\kappa}(v) := \kappa(v).$$

For every $i \in [n]$, we set

$$u_{\sigma^{-1}(i)} := \sigma^{-1}(i) + \widetilde{\kappa}(i) - n.$$

Example 10. For the configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ in Example 9, we found in Example 7 that Algorithm 1 gives $\sigma = \overline{10}976532\overline{11}841\overline{12}$. Hence $\widetilde{\kappa} = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{11}\,\overline{11}\,\overline{11}4\overline{11}83$, and the word $u := u_1u_2\cdots$ is 011345365223.

Given a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, we add $\sigma(0) := 0$ in front of it, and we define its *runs* as its maximal consecutive decreasing substrings. Now for every $i \in [n]$, we define $w_{\sigma^{-1}(i)} = w_{\sigma^{-1}(i)}(\sigma)$ as

$$\begin{aligned} w_{\sigma^{-1}(i)}(\sigma) &:= \#\{\text{numbers in the same run of } i \text{ and larger than } i\} \\ &+ \#\{\text{numbers smaller than } i \text{ in the run immediately to the left of the one containing } i\}. \end{aligned}$$

Example 11. The runs of $\sigma = \overline{10}976532\overline{11}841\overline{12}$ are separated by bars: $0|\overline{10}976532|\overline{11}841|\overline{12}$, so that the word $w = w_1(\sigma)w_2(\sigma)\cdots$ is 123456776434.

[5]The relation with the general definition of *sorted configuration* given in Section 1.2 is simply that we are picking a specific convenient element in each orbit.

The following propositions characterize the sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$.

Proposition 1. *Let κ be a sorted recurrent configuration of $\widehat{G}_{\mu,\nu}$. Let $\sigma \in \mathfrak{S}_n$ be the toppling word produced by Algorithm 1 applied to κ. Then for every $i \in [n]$*

$$0 \leq u_{\sigma^{-1}(i)} < w_{\sigma^{-1}(i)}.$$

Proposition 2. *Let κ be a sorted stable configuration of $\widehat{G}_{\mu,\nu}$, and let $\sigma \in \mathfrak{S}_n$ be such that for every $i \in [n]$*

$$0 \leq u_{\sigma^{-1}(i)} < w_{\sigma^{-1}(i)}.$$

Then κ is recurrent and σ is the toppling word given by Algorithm 1 applied to κ.

We omit the proofs, but an instance can be checked by comparing Examples 10 and 11.

7 Bijection with parking functions

We now provide a bijection between recurrent sorted configurations of $\widehat{G}_{\mu,\nu}$ and the parking functions in $\mathsf{PF}(\mu;\nu)$.

Let $\mathsf{SortRec}(\mu,\nu)$ be the set of sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$. Define the function $\Phi : \mathsf{SortRec}(\mu,\nu) \to \mathsf{PF}(\mu,\nu)$ in the following way: given $\kappa \in \mathsf{SortRec}(\mu,\nu)$, in the notation of Section 6, we set $\Phi(\kappa)$ to be the (unique) parking function of size $n = |\mu| + |\nu|$ such that the label i occurs in column $n - \widetilde{\kappa}(i)$ (we number the columns increasingly from left to right) for every $i \in [n]$.

Example 12. The parking function $D \in \mathsf{PF}((4,3);(3,2))$ in Figure 1 is the image $\Phi(\kappa)$ of the configuration κ in Example 4 ($\widetilde{\kappa}$ is computed in Example 10).

We can finally state the main result of our article.

Theorem 3. *The map Φ is a well-defined bijection such that $\mathsf{area}(\Phi(\kappa)) = \mathsf{level}(\kappa)$ and such that the σ obtained from the Algorithm 1 applied to κ equals the pmaj word of $\Phi(\kappa)$, so that $\mathsf{pmaj}(\Phi(\kappa)) = \mathsf{delay}(\kappa)$.*

Now Theorem 1 is an immediate consequence of this result combined with Theorem 2. It can be checked in the instance of Example 12 (cf. Examples 7, 8, 2, 1 and 6).

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Real matroid Schubert varieties, zonotopes, and virtual Weyl groups

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Abstract. We show that the matroid Schubert variety of a real hyperplane arrangement is homeomorphic to the zonotope of the arrangement with parallel faces identified. Using this explicit model, we compute the homology and fundamental group of the matroid Schubert variety in terms of combinatorial data of the underlying oriented matroid. When the hyperplane arrangement is a Coxeter arrangement, we show that the equivariant fundamental group is a virtual analogue of the associated Weyl group.

Keywords: matroid Schubert varieties, zonotopes, virtual Weyl groups

1 Introduction

Several recent breakthroughs in matroid theory have come from understanding the topology of certain algebraic varieties associated to hyperplane arrangements. One such variety is the *matroid Schubert variety*, which compactifies the ambient vector space of a central essential hyperplane arrangement. Originally studied by Ardila–Boocher [1] and Li [8], it has most notably found applications to the Dowling–Wilson top-heavy conjecture for representable matroids [5].

We study the topology of matroid Schubert varieties $Y_{\mathcal{A}}$ associated with real hyperplane arrangements $\mathcal{A}$. The main result is an explicit homeomorphism from $Y_{\mathcal{A}}$ to a natural quotient of the zonotope associated with $\mathcal{A}$. As a consequence, we obtain presentations for the homology and fundamental group of $Y_{\mathcal{A}}$ that depend only on the oriented matroid data of $\mathcal{A}$. When $\mathcal{A}$ is a Coxeter arrangement, we also show that the equivariant fundamental groups are of independent interest. We call them virtual Weyl groups, since they are quotients of virtual Artin groups [3] (which themselves generalise the virtual braid group in type A).

Full details of the results in this extended abstract will be presented as part of the forthcoming paper [7].

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2 Setup

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{A} = (\alpha_e)_{e \in E} \in (V^*)^E$ be a representation of an $\mathbb{F}$-linear matroid M with (finite) ground set E. The *rank* of $S \subseteq E$ is $\operatorname{rk} S = \dim \operatorname{span}\{\alpha_e \colon e \in S\}$, and a *flat* of M is a subset $F \subseteq E$ that is not strictly contained in another subset of the same rank. Flats of M are partially ordered by inclusion, and this poset is in fact a geometric lattice $\mathcal{L}(M)$ called the *lattice of flats* of M.

Remark 1. The α_e should be thought of as defining hyperplanes $H_e = \ker \alpha_e \subseteq V$. When M is simple, the α_e are all nonzero and no two are parallel. In this case, M is the matroid associated to the central hyperplane arrangement determined by the H_e.

For every $F \in \mathcal{L}(M)$, define the subspace $V_F = \bigcap_{e \in F} \ker \alpha_e \subseteq V$. The *localisation* M^F is the matroid on ground set F with flats $\{G \subseteq F \colon G \in \mathcal{L}(M)\}$. It has a representation $\mathcal{A}^F = (\alpha_e)_{e \in F} \in ((V/V_F)^*)^F$. The *contraction* M_F is the matroid on ground set $E \setminus F$ with flats $\{G \setminus F \colon F \subseteq G \in \mathcal{L}(M)\}$. It has a representation $\mathcal{A}_F = (\alpha_e|_{V_F})_{e \in E \setminus F} \in (V_F^*)^{E \setminus F}$.

Without loss of generality, assume that the α_e span V^*. (If M is simple, this would correspond to the hyperplane arrangement defined in Remark 1 being essential.) In this case, the choice of $\mathcal{A}$ defines an embedding $V \to \mathbb{F}^E$ by $v \mapsto (\alpha_e(v))_{e \in E}$. Considering $\mathbb{F}^E$ as the subset of $(\mathbb{P}^1)^E = (\mathbb{F} \cup \{\infty\})^E$ with all coordinates finite, the closure of V in $(\mathbb{P}^1)^E$ (in the Zariski topology) is the *matroid Schubert variety* $Y_\mathcal{A}$ of $\mathcal{A}$. A key property of matroid Schubert varieties is the existence of an affine paving.

Proposition 2 ([9, Lemmas 7.5 and 7.6]). *The matroid Schubert variety $Y_\mathcal{A}$ has a stratification $Y_\mathcal{A} = \bigsqcup_{F \in \mathcal{L}(M)} Y_\mathcal{A}^F$, where*

$$Y_\mathcal{A}^F = \{(y_e)_{e \in E} \in Y_\mathcal{A} \colon y_e = \infty \text{ if and only if } e \notin F\} \cong V/V_F \cong \mathbb{F}^{\operatorname{rk} F}.$$

Further, $\overline{Y_\mathcal{A}^G} = \bigsqcup_{F \subseteq G} Y_\mathcal{A}^F \cong Y_{\mathcal{A}^G}$ for every $G \in \mathcal{L}(M)$.

Henceforth we fix $\mathbb{F} = \mathbb{R}$. In this case much of the combinatorics of $\mathcal{A}$ (and hence M) can be encoded in the geometry of a convex polytope.

Definition 3. The *zonotope* associated to $\mathcal{A}$ is the Minkowski sum

$$\mathcal{Z}_\mathcal{A} = \sum_{e \in E} [-1, 1]\alpha_e = \left\{ \sum_{e \in E} c_e \alpha_e \colon -1 \leqslant c_e \leqslant 1 \text{ for all } e \in E \right\} \subset V^*.$$

Equivalently, it is the image of the cube $[-1, 1]^E$ under the projection $(c_e)_{e \in E} \mapsto \sum_{e \in E} c_e \alpha_e$.

The face structure of $\mathcal{Z}_\mathcal{A}$ can be understood explicitly using the oriented matroid structure of $\mathcal{A}$ in terms of *covectors*. Define a map $V \to \{+, -, 0\}^E$ by sending $v \in V$ to $C = (C_e)_{e \in E}$, where $C_e = +$ if $\alpha_e(v) > 0$, $-$ if $\alpha_e(v) < 0$, and 0 if $\alpha_e(v) = 0$. The image of this map is the set of covectors of $\mathcal{A}$. Each covector C gives a decomposition

$E = C_+ \sqcup C_- \sqcup C_0$, where $C_+ = \{e \in E\colon C_e = +\}$, $C_- = \{e \in E\colon C_e = -\}$, and $C_0 = \{e \in E\colon C_e = 0\}$. Observe that the flats of $\mathcal{A}$ are exactly the zero sets C_0 of covectors of $\mathcal{A}$. Further, the set of covectors has a partial order induced from the product order on $\{+,-,0\}^E$ defined by $0 < +,-$ on each coordinate. Adjoining a top element gives the *face lattice* of $\mathcal{A}$.

We can associate a face of $\mathcal{Z}_\mathcal{A}$ to each covector C as follows:

$$C \mapsto \sum_{e\in C_+} \alpha_e - \sum_{e\in C_-} \alpha_e + \sum_{e\in C_0} [-1,1]\alpha_e. \tag{2.1}$$

It follows immediately that the face associated to a covector C is (isometric to) $\mathcal{Z}_{\mathcal{A}^F}$, where F is the zero set of C.

Proposition 4 ([4, Proposition 2.2.2])**.** *The map* (2.1) *is an order-reversing bijection between covectors of $\mathcal{A}$ and faces of $\mathcal{Z}_\mathcal{A}$ (under inclusion).*

Example 5. In Figure 1 we visualise the rank 2 braid arrangement. Concretely, in the above notation we have $V^* = \mathbb{R}^3/\mathbb{R}(1,1,1)$, $E = \{1,2,12\}$, and $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,-1,1)$, and $\alpha_{12} = (1,0,-1) = \alpha_1 + \alpha_2$. By fixing an isomorphism $V \cong V^*$ using the dot product, we draw both the hyperplanes in V (in black) and the zonotope in V^* (in blue) in the same plane. Finally, we label the regions of V by their covectors.

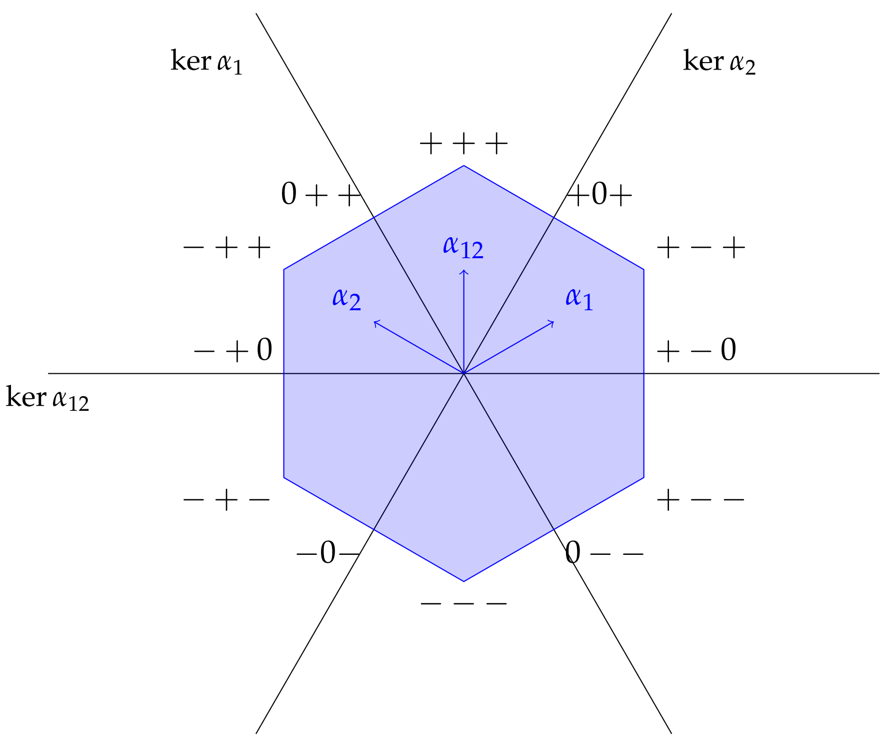

Figure 1: The rank 2 braid arrangement

3 The combinatorial model

Observe from the explicit equations (2.1) that faces of $\mathcal{Z}_{\mathcal{A}}$ whose covectors correspond to the same flat are translates of each other. Let $\mathcal{Z}_{\mathcal{A}}/\!\sim$ be the quotient of $\mathcal{Z}_{\mathcal{A}}$ obtained by identifying two points if one is moved to the other by such a translation. Intuitively, $\mathcal{Z}_{\mathcal{A}}/\!\sim$ is the result of identifying parallel faces in $\mathcal{Z}_{\mathcal{A}}$.

The stratification of $\mathcal{Z}_{\mathcal{A}}$ by (relatively open) faces descends to a stratification of $\mathcal{Z}_{\mathcal{A}}/\!\sim$ indexed by $\mathcal{L}(M)$. Denote by $\mathcal{Z}_{\mathcal{A}}^F$ the stratum corresponding to $F \in \mathcal{L}(M)$.

Theorem 6. *The real matroid Schubert variety $Y_{\mathcal{A}}$ (with the analytic topology) is homeomorphic to $\mathcal{Z}_{\mathcal{A}}/\!\sim$.*

Proof sketch. We claim that every homeomorphism $f\colon \mathbb{R} \to (-1,1)$ determines a homeomorphism $\phi\colon Y_{\mathcal{A}} \to \mathcal{Z}_{\mathcal{A}}/\!\sim$. First observe that there is a map $V \to \mathbb{R}^E \to (-1,1)^E \to \mathcal{Z}_{\mathcal{A}}^E$ defined by $v \mapsto (\alpha_e(v)) \mapsto (f(\alpha_e(v))) \mapsto \sum_{e\in E} f(\alpha_e(v))\alpha_e$. Unfortunately this composition does not obviously extend to the desired map, as the projection $[-1,1]^E \to \mathcal{Z}_{\mathcal{A}}$ does not descend to a well-defined map after identifying parallel faces. Nevertheless, we claim that a suitable interpretation of the formula $(y_e)_{e\in E} \mapsto \sum_{e\in E} f(y_e)\alpha_e$ extends the map $V \to \mathcal{Z}_{\mathcal{A}}^E$ to the required continuous ϕ. In fact, if $y = (y_e)_{e\in E} \in Y_{\mathcal{A}}^F$, there are several possible values for $(f(y_e))_{e\in E} \in [-1,1]^E$ allowed by continuity, but they correspond to different covectors with the same zero set F and hence $\sum_{e\in E} f(y_e)\alpha_e$ is well-defined in the quotient $\mathcal{Z}_{\mathcal{A}}/\!\sim$.

Since $Y_{\mathcal{A}}$ is compact and $\mathcal{Z}_{\mathcal{A}}/\!\sim$ is Hausdorff, the continuous map ϕ is a homeomorphism if it is a bijection. By construction it maps $Y_{\mathcal{A}}^F$ to $\mathcal{Z}_{\mathcal{A}}^F$, so it is enough to verify bijectivity on each stratum separately. Further, it is enough to check the open stratum $Y_{\mathcal{A}}^E \cong V$, since each $Y_{\mathcal{A}}^F$ is the open stratum in $Y_{\mathcal{A}^F}$.

For injectivity, let $v, w \in V$ and consider $(d_e) = (f(\alpha_e(v)) - f(\alpha_e(w))) \in (-1,1)^E$. Observe that f must be (strictly) increasing or decreasing; without loss of generality, assume that f is increasing. The sign of d_e is then the same as the sign of $\alpha_e(v-w)$. So $\sum_{e\in E} d_e\alpha_e(v-w)$ is non-negative, and it is zero if and only if $d_e = \alpha_e(v-w) = 0$ for every $e \in E$. But if v and w map to the same point in $\mathcal{Z}_{\mathcal{A}}^E$, then $\sum_{e\in E} d_e\alpha_e = 0$. It follows that $\alpha_e(v-w) = 0$ for every $e \in E$, and thus $v = w$ as the α_e span V^*.

To show surjectivity, consider the quotients of $Y_{\mathcal{A}}$ and $\mathcal{Z}_{\mathcal{A}}/\!\sim$ identifying all boundary strata to a point ∞. Both quotients are homeomorphic to spheres, with induced cell decompositions $V \sqcup \{\infty\}$ and $\mathcal{Z}_{\mathcal{A}}^E \sqcup \{\infty\}$ respectively. Since ϕ sends strata to strata, it descends to a continuous cellular map $\overline{\phi}$ between the quotients. If ϕ were not surjective, then the image of $\overline{\phi}$ would be contained in the sphere minus one point and hence be homeomorphic to (a subset of) V. By the Borsuk–Ulam theorem $\overline{\phi}$ would not be injective. In particular, cellularity of $\overline{\phi}$ implies that $\overline{\phi}|_V = \phi|_V$ would not be injective, contradicting what was shown above. □

Remark 7. In the case of Coxeter arrangements, Theorem 6 was proved in [6, Appendix A] using somewhat involved root system combinatorics.

Example 8. If $\dim V = \dim V^* = 2$, then $\mathcal{Z}_{\mathcal{A}}$ is a $2n$-gon (where $n \geqslant 2$ is the number of rank 1 flats). Identifying parallel edges of $\mathcal{Z}_{\mathcal{A}}$ gives a connected compact orientable surface without boundary. The resulting cell structure on the surface has one 0-cell if n is even and two 0-cells if n is odd. By an Euler characteristic computation and the classification of surfaces, it follows that $Y_{\mathcal{A}}$ is homeomorphic to Σ_g (if $n = 2g$ is even) or Σ_g with two (distinct) points identified (if $n = 2g+1$ is odd). For example, the matroid Schubert variety corresponding to the rank 2 braid arrangement of Example 5 is homeomorphic to the torus with two points identified.

4 Computations of invariants

The combinatorial model of Theorem 6 allows for easy computation of some topological invariants of $Y_{\mathcal{A}}$.

Homology

There is a cellular chain complex for $\mathcal{Z}_{\mathcal{A}}/{\sim}$ with cells given by the strata $\mathcal{Z}_{\mathcal{A}}^F$. The boundary map in this complex is necessarily zero, since in the computation for any cell $\mathcal{Z}_{\mathcal{A}}^F$ opposite faces of $\mathcal{Z}_{\mathcal{A}}^F$ both occur and with opposite sign. Their contributions cancel, as the opposite faces are identified in $\mathcal{Z}_{\mathcal{A}}/{\sim}$. Hence the homology of $Y_{\mathcal{A}}$ is easy to compute.

Proposition 9. $H_\bullet(Y_{\mathcal{A}}, \mathbb{Z}) \cong H_\bullet(\mathcal{Z}_{\mathcal{A}}/{\sim}, \mathbb{Z}) \cong \bigoplus_{F \in \mathcal{L}(M)} \mathbb{Z} x_F$, *where* $\deg x_F = \operatorname{rk} F$.

Remark 10. It is interesting to note that the cellular boundary maps for the analogous cell structures on complex matroid Schubert varieties are also zero, but for the different reason that the cells are concentrated in even (real) dimension.

Fundamental group

The fundamental group of a cell complex depends only on the 2-skeleton. For $\mathcal{Z}_{\mathcal{A}}/{\sim}$, the cells of dimension k correspond to the flats of rank k. We take the 0-cell corresponding to the unique rank 0 flat to be the basepoint. There is then a presentation of $\pi_1(Y_{\mathcal{A}})$ with generators x_F indexed by rank 1 flats and relations indexed by rank 2 flats.

To compute the relations, it is helpful to work with an acyclic reorientation of $\mathcal{A}$ (this does not change the zonotope $\mathcal{Z}_{\mathcal{A}}$). Let G be a rank 2 flat. The rank 1 flats contained in G can be ordered as follows. If G contains n rank 1 flats, the zonotope $\mathcal{Z}_{\mathcal{A}^G}$ is a $2n$-gon. One vertex of this $2n$-gon has a covector without any $+$ coordinates (this follows from the choice of acyclic orientation). A length n sequence of edges from this vertex to its opposite vertex defines a total order $F_1 < \ldots < F_n$ on the rank 1 flats contained in G. There are two such sequences, giving opposite orders. The relation corresponding to G then says that the two paths $x_{F_1} \cdots x_{F_n}$ and $x_{F_n} \cdots x_{F_1}$ determined by these sequences are equal.

Theorem 11. *The fundamental group $\pi_1(Y_{\mathcal{A}})$ has a presentation with generators $\{x_F \colon F \in \mathcal{L}(M), \operatorname{rk} F = 1\}$ and relations $x_{F_1} \cdots x_{F_n} x_{F_1}^{-1} \cdots x_{F_n}^{-1}$ for every rank 2 flat G, where $F_1, \ldots, F_n$ are the rank 1 flats contained in G ordered as above.*

Example 12. Continuing Example 5, the fundamental group of the matroid Schubert variety in this case has a presentation $\langle x_1, x_2, x_{12} \mid x_1 x_{12} x_2 x_1^{-1} x_{12}^{-1} x_2^{-1} \rangle$.

Remark 13. When $\mathcal{Z}_{\mathcal{A}}$ is the (type A) permutohedron of dimension n, the homology and fundamental group were computed in this way in [2, Proposition 8.3] and [2, Theorem 8.1], generalising the $n = 2$ computation in Example 12. In particular, the fundamental group was shown to be isomorphic to the triangular group $\mathbf{Tr}_{n+1}$, also known as the pure flat braid group.

Remark 14. Every rank 2 oriented matroid is representable over $\mathbb{R}$ [4, Corollary 8.3.3], so the above presentation of $\pi_1(Y_{\mathcal{A}})$ can be used to define a group for any oriented matroid.

Homotopy groups

In certain cases, the higher homotopy groups $\pi_n(Y_{\mathcal{A}})$ are also known.

Theorem 15 ([2, Theorem 8.1])**.** *If $\mathcal{Z}_{\mathcal{A}}$ is the (type A) permutohedron, then $\mathcal{Z}_{\mathcal{A}}/\sim$ is a classifying space and hence $\pi_n(Y_{\mathcal{A}}) = \pi_n(\mathcal{Z}_{\mathcal{A}}/\sim)$ is trivial for all $n > 1$.*

The proof uses the theory of non-positively curved polyhedral complexes. We expect that the same result holds more generally, at least for nice enough choices of $\mathcal{A}$.

5 Coxeter arrangements

We first fix some notation. Let Φ be a root system with simple roots Π and positive roots Φ^+. Each (positive) root defines a hyperplane in the dual space, and the corresponding hyperplane arrangement is called a *Coxeter arrangement*. By abuse of notation, we also use Φ^+ for the matroid representation with coordinates given by the positive roots.

Further, let $(m_{\alpha,\beta})_{\alpha,\beta\in\Pi}$ be the Coxeter matrix associated to the root system Φ, and let $\Sigma = \{\sigma_\alpha \colon \alpha \in \Pi\}$ and $S = \{s_\alpha \colon \alpha \in \Pi\}$ be abstract sets indexed by Π. The Artin group A has a presentation with generators Σ and relations $\operatorname{Prod}(\sigma_\alpha, \sigma_\beta, m_{\alpha,\beta}) = \operatorname{Prod}(\sigma_\beta, \sigma_\alpha, m_{\alpha,\beta})$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $m_{\alpha,\beta} \neq \infty$. Here $\operatorname{Prod}(a, b, m)$ is the word $aba\ldots$ of length

m. Similarly, the Weyl group W has a presentation with generators S and relations $s_\alpha^2 = 1$ for all $\alpha \in \Pi$ and $\mathrm{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta}) = \mathrm{Prod}(s_\beta, s_\alpha, m_{\alpha,\beta})$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $m_{\alpha,\beta} \neq \infty$.

Bellingeri–Paris–Thiel [3] have recently defined the *virtual Artin group* VA as the free product of W and A modulo some "mixed relations" coming from the action of W on Φ. Their definition unifies the Coxeter-theoretic and knot-theoretic generalisations of the classical braid group to Artin groups and virtual braid groups respectively. We are interested in a quotient of their group that can be considered as a virtual analogue of the corresponding Weyl group.

Definition 16 ([3])**.** The *virtual Artin group* VA is the free product of W and A modulo relations $\mathrm{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta} - 1)\sigma_\alpha = \sigma_\gamma \mathrm{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta} - 1)$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $m_{\alpha,\beta} \neq \infty$. In these relations, the positive root γ is defined as α if $m_{\alpha,\beta}$ is even and β if $m_{\alpha,\beta}$ is odd.

Definition 17. The *virtual Weyl group* VW is the quotient of VA by the relations $\sigma_\alpha^2 = 1$ for all $\alpha \in \Pi$.

There is a surjective group homomorphism $\mathrm{VW} \to W$ defined on generators by $\sigma_\alpha, s_\alpha \mapsto s_\alpha$ for all $\alpha \in \Pi$, and we call its kernel the *pure virtual Weyl group* PVW. The map $\pi_P \colon \mathrm{VA} \to W$ [3, Section 2] is the composition of this map with the quotient $\mathrm{VA} \to \mathrm{VW}$, and its kernel is the *pure virtual Artin group* PVA.

Proposition 18. *The fundamental group $\pi_1(Y_{\Phi^+})$ is isomorphic to* PVW*.*

Proof sketch. There is a presentation of the pure virtual Artin group with generators $\{\zeta_\beta \colon \beta \in \Phi\}$ and certain relations [3, Theorem 2.6]. As PVW is the image of PVA under the quotient map $\mathrm{VA} \to \mathrm{VW}$, we can obtain a presentation of PVW by imposing (consequences of) the relations $\sigma_\alpha^2 = 1$ to the presentation of PVA.

In fact, the generator ζ_β is the element $ws_\alpha\sigma_\alpha w^{-1} \in \mathrm{VA}$ for some $w \in W$ and $\alpha \in \Pi$ such that $w(\alpha) = \beta$ [3, p. 197]. This definition is independent of the choices of w and α [3, Lemma 2.2]. Then $-\beta = ws_\alpha(\alpha)$, so $\zeta_{-\beta} = (ws_\alpha)s_\alpha\sigma_\alpha(s_\alpha w^{-1}) = w\sigma_\alpha s_\alpha w^{-1}$ and $\zeta_\beta\zeta_{-\beta} = ws_\alpha\sigma_\alpha^2 s_\alpha w^{-1}$. But this is the identity if and only if $\sigma_\alpha^2 = 1$, so PVW has a presentation with generators $\{\zeta_\beta \colon \beta \in \Phi^+\}$ and the same relations as PVA.

A root subsystem $\Phi' \subset \Phi$ is *parabolic* if $\Phi' \cap \Phi^+$ corresponds to a flat of the Coxeter arrangement. The relations in the above presentation of PVA correspond to choices of simple roots for rank 2 parabolic root subsystems of Φ. One can compute the relations and show that, after accounting for the extra relations $\zeta_\beta\zeta_{-\beta} = 1$, the relations for pairs of simple roots depend only on the parabolic root subsystem, and that they are the same as the relations in Theorem 11 coming from the rank 2 flats. Hence the presentations of PVW and $\pi_1(Y_{\Phi^+})$ define the same group. □

The Weyl group W acts on V, and hence on Y_{Φ^+} and $\pi_1(Y_{\Phi^+})$. We can therefore consider the W-equivariant fundamental group $\pi_1^W(Y_{\Phi^+})$ [6, Definition 11.1]. As the unique 0-cell is fixed by the action of W, taking it as the basepoint gives a semidirect product decomposition $\pi_1^W(Y_{\Phi^+}) \cong W \ltimes \pi_1(Y_{\Phi^+})$. The homomorphism $W \to \mathrm{Aut}(\pi_1(Y_{\Phi^+}))$ defining the semidirect product is exactly the W-action indicated above. Explicitly, an element $w \in W$ acts on generators of $\pi_1(Y_{\Phi^+})$ by $\zeta_\beta \mapsto \zeta_{w(\beta)}$.

Theorem 19. *The W-equivariant fundamental group $\pi_1^W(Y_{\Phi^+})$ is isomorphic to the virtual Weyl group* VW*.*

Proof sketch. The virtual Weyl group also has a semidirect product decomposition $W \ltimes$ PVW descending from the semidirect product decomposition of the virtual Artin group [3, Proposition 2.1]. As $\pi_1(Y_{\Phi^+}) \cong$ PVW (Proposition 18) and the action of W on PVW [3, p. 203] agrees with the action of W on $\pi_1(Y_{\Phi^+})$, the semidirect products $\pi_1^W(Y_{\Phi^+})$ and VW must be isomorphic. □

Remark 20. In type A, the virtual Weyl group is known as the *flat (virtual) braid group*. It was called the virtual symmetric group in [6], where it was realised as the equivariant fundamental group $\pi_1^{S_n}(Y_{\Phi^+})$ of the corresponding matroid Schubert variety [6, Lemma 11.6].

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Chromatic functions, interval orders, and increasing forests

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Abstract. The chromatic quasisymmetric functions (csf) of Shareshian and Wachs associated to unit interval orders have attracted a lot of interest since their introduction in 2016, both in combinatorics and geometry, because of their relation to the famous Stanley-Stembridge conjecture (1993) and to the topology of Hessenberg varieties, respectively. In the present work we study the csf associated to the larger class of interval orders with no restriction on the length of the intervals. Inspired by an article of Abreu and Nigro, we show that these csf are weighted sums of certain quasisymmetric functions associated to the increasing spanning forests of the associated incomparability graphs. Furthermore, we define quasisymmetric functions that include the unicellular LLT symmetric functions and generalize an identity due to Carlsson and Mellit. Finally we conjecture a formula giving their expansion in the type 1 power sum quasisymmetric functions which should extend a theorem of Athanasiadis.

Keywords: Chromatic quasisymmetric functions, LLT quasisymmetric functions, increasing spanning forests

1 Introduction

In [15] Shareshian and Wachs introduced the *chromatic quasisymmetric function* $\chi_G[X;q]$ associated to every graph G whose vertices are totally ordered, as a sum over proper colorings of G of suitable monomials. At $q=1$ the series $\chi_G[X;q]$ reduces to the well-known chromatic symmetric function $\chi_G[X;1]=\chi_G(x)$ introduced by Stanley in [17]. A famous conjecture of Stanley and Stembridge ([17, Conjecture 5.1], [18, Conjecture 5.5]) states that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then $\chi_G[X;1]$ is e-positive, i.e. its expansion in the elementary symmetric functions has coefficients in $\mathbb{N}$. Shareshian and Wachs showed (cf. [15, Theorem 4.5]) that if G is the incomparability

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graph of a poset that is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free, then $\chi_G[X;q]$ is a symmetric function, and they conjecture that it is e-positive, i.e. its expansion in the elementary symmetric functions has coefficients in $\mathbb{N}[q]$. Thanks to a result of Guay-Paquet [9], it is known that the Shareshian-Wachs conjecture implies the Stanley-Stembridge conjecture. The former problem attracted a lot of attention recently: see e.g. [10, 2, 16, 7, 12, 8].

The posets that are $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free are precisely the *unit interval orders* (see [14]), whose elements are intervals in $\mathbb{R}$ of the same length, and an interval a is smaller than an interval b if all the points of a are strictly smaller than all the points of b. If in such a poset we order the intervals increasingly according to their left endpoints, then we get a total order on them, and now the incomparability graphs of these posets will inherit this total order on the vertices, giving the labelled graphs G involved in the Shareshian-Wachs conjecture. In our article we call these labelled graphs *Dyck graphs*, as they are in a natural bijection with Dyck paths.

If in the definition of unit interval orders we drop the condition on the intervals to have all the same length, then we get the *interval orders*. The incomparability graphs of these posets will be called *interval graphs* in our article, and their chromatic quasisymmetric functions $\chi_G[X;q]$ are the object of our study.

Inspired by the work of Abreu and Nigro [1], given an interval graph G, for every increasing spanning forest F of G we will define a quasisymmetric function $\mathcal{Q}_F^{(G)}$ so that the following formula holds (the statistic $\mathsf{wt}_G(F)$ is essentially the one in [1], while $\mathsf{ISF}(G)$ is the set of increasing spanning forests of G).

Theorem 4.1. *Given an interval graph G on n vertices, we have*

$$\chi_G[X;q] = \sum_{F\in\mathsf{ISF}(G)} q^{\mathsf{wt}_G(F)}\mathcal{Q}_F^{(G)}. \tag{1.1}$$

For every simple graph G with totally ordered vertices we introduce the quasisymmetric function $\mathrm{LLT}_G[X;q]$, analogous to $\chi_G[X;q]$ but defined as a sum over all (not necessarily proper) colorings of G of suitable monomials.

The main result of this article is the following theorem, stated in plethystic notation (ρ and ψ are well-known involutions of the algebra QSym of quasisymmetric functions).

Theorem 5.1. *Given G an interval graph on n vertices, we have*

$$(1-q)^n\rho\left(\psi\chi_G\left[X\frac{1}{1-q}\right]\right) = \mathrm{LLT}_G[X;q].$$

This result extends the identity in [6, Proposition 3.5] proved by Carlsson and Mellit when G is a Dyck graph.

In [5] the authors study a family of quasisymmetric functions that they call *type 1 quasisymmetric power sums*, denoted Ψ_α. Actually $\{\Psi_\alpha \mid \alpha \text{ composition}\}$ is a basis of QSym that refines the power symmetric function basis.

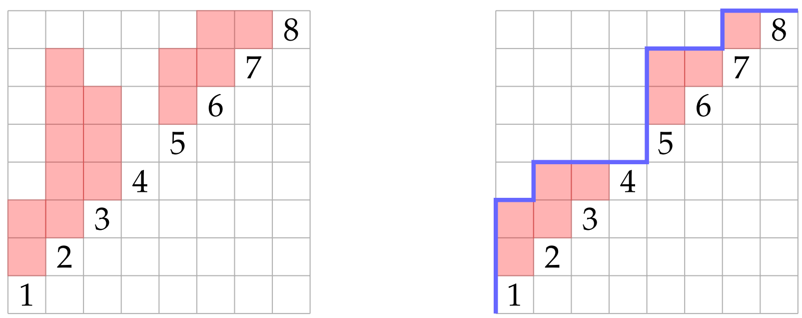

Figure 1: The interval graph $G = ([8], \{(1,2), (1,3), (2,3), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (5,6), (5,7), (6,7), (6,8), (7,8)\})$, on the left. On the right, the Dyck graph $G_2 = ([8], \{(1,2), (1,3), (2,3), (2,4), (3,4), (5,6), (5,7), (6,7), (7,8)\})$.

We state the following conjecture which is supposed to provide an extension of the formula proved by Athanasiadis in [4].

Conjecture 6.1. *For any interval graph G on n vertices we have*

$$\rho\psi\chi_G[X;q] = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \sum_{\sigma \in \mathcal{N}_{G,\alpha}} q^{\widetilde{\mathsf{inv}}_G(\sigma)}.$$

2 Preliminaries

For every $n \in \mathbb{Z}_{>0} := \{1, 2, 3, \dots\}$ we will use the notation $[n] := \{1, 2, \dots, n\}$.

2.1 Interval graphs

In this abstract a graph will always be simple, i.e. no loops and no multiple edges.

In our work a (*labelled*) *graph* $G = ([n], E)$ will be called *interval* if whenever $\{i, j\} \in E$ and $i < j$, then $\{i, k\} \in E$ for every $i < k \leq j$. We will call $\mathcal{IG}_n$ the set of all interval graphs with vertex set $[n]$.

We can represent an interval graph $G = ([n], E)$ in the following way: in a $n \times n$ square grid we order the columns from left to right with numbers $1, 2, \dots, n$ and similarly the rows from bottom to top; then we color the cells $\{i, j\} \in E$ with $i < j$. See Figure 1, on the left, for an example[1].

Notice that in these pictures we simply obtain a bunch of (possibly empty) colored columns, starting just above the diagonal cells. Hence clearly there are $n!$ interval graphs on n vertices.

[1]Sometimes we denote by (i, j) an edge $\{i, j\} \in E$ with $i < j$, like in the caption of Figure 1.

Given an interval graph G on n vertices, we can consider its *flipped*, obtained from G by replacing each edge $\{i,j\}$ with an edge $\{n+1-i, n+1-j\}$: in terms of pictures, this corresponds to flip the picture of G around the line $y=-x$.

An interval graph G on n vertices such that its flipped is still an interval graph is called a *Dyck graph*. The explanation of the name is obvious, since the picture of a Dyck graph determines a Dyck path: see the graph G_2 in Figure 1, on the right (the Dyck path is the thicker one).

It turns out that the interval graphs are the incomparability graphs of certain posets called *interval orders* (hence their name).

Given a (naturally labelled) poset $P = ([n], <_P)$, its *incomparability (labelled) graph* $\mathrm{Inc(P)} = ([n], E_P)$ is defined by setting $\{i,j\} \in E_P$ if and only if i and j are incomparable in P.

Let $\mathcal{I}$ be the set of all bounded closed intervals of $\mathbb{R}$, and given $I=[a,b]$ and $J=[c,d]$ we set $I \prec J$ if and only if $b<c$. Clearly $(\mathcal{I}, \prec)$ is a poset. Any subposet of $(\mathcal{I}, \prec)$ is called an *interval order*.

2.2 Symmetric and quasisymmetric functions

In this section we recall a few basic facts of symmetric and quasisymmetric functions, mainly to fix the notation.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of $n \in \mathbb{N}$ (denoted $\alpha \vDash n$), we denote its *size* by $|\alpha| = \sum_i \alpha_i = n$ and its *length* by $\ell(\alpha) = k$. For brevity, sometimes we will use the *exponential notation*, so that for example we will write (1^4) for $(1,1,1,1)$, or $(1^3, 2^2, 1, 3)$ for $(1,1,1,2,2,1,3)$.

To a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n we associate a set $\mathsf{set}(\alpha) = \mathsf{set}_n(\alpha) \subseteq [n-1]$:

$$\mathsf{set}(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

Viceversa, to a subset $S \subseteq [n-1]$ whose elements are $i_1 < i_2 < \cdots < i_k$ we associate the composition

$$\mathsf{comp}(S) = \mathsf{comp}_n(S) = (i_1, i_2 - i_1, i_3 - i_2, \dots, i_k - i_{k-1}, n - i_k) \vDash n.$$

Notice that the functions set_n and comp_n are inverse of each others.

Given a composition $\alpha \vDash n$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, its *reversal* is $\alpha^r = (\alpha_k, \alpha_{k-1}, \dots, \alpha_1)$, its *complement* is $\alpha^c = \mathsf{comp}([n-1] \setminus \mathsf{set}(\alpha))$, and its *transpose* is $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

We denote by QSym the algebra of quasisymmetric functions in the variables $x_1, x_2, \dots$ and coefficients in $\mathbb{Q}(q)$, where q is a variable.

Given $n \in \mathbb{N}$ and $S \subseteq [n-1]$, we define the *fundamental (Gessel) quasisymmetric function* $L_{n,S}$ as

$$L_{n,S} := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in S \Rightarrow i_j \neq i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

and for every $\alpha \vDash n$, we define $L_\alpha := L_{n,\mathsf{set}(\alpha)}$.

It is well known that $\{L_\alpha \mid \alpha \text{ composition}\}$ is a basis of QSym.

We have the following three involutions of QSym: $\psi : \mathrm{QSym} \to \mathrm{QSym}$, defined by $\psi(L_\alpha) := L_{\alpha^c}$, $\rho : \mathrm{QSym} \to \mathrm{QSym}$ defined by $\rho(L_\alpha) = L_{\alpha^r}$, and $\omega : \mathrm{QSym} \to \mathrm{QSym}$ defined by $\omega(L_\alpha) = L_{\alpha^t}$.

We will use the *plethysm* of quasisymmetric functions: cf. [11].

2.3 Colorings and (co)inversions

Given $n \in \mathbb{Z}_{>0}$, let $G = ([n], E)$ be a (simple) graph.

A *coloring* of G is simply a function $\kappa : [n] \to \mathbb{Z}_{>0}$. We call $\mathsf{C}(G)$ the set of colorings of G. We can and will identify a coloring $\kappa \in \mathsf{C}(G)$ with the word $\kappa(1)\kappa(2)\cdots\kappa(n)$ in the alphabet $\mathbb{Z}_{>0}$.

A *coloring* of G is called *proper* if $\{i,j\} \in E$ implies $\kappa(i) \neq \kappa(j)$. We call $\mathsf{PC}(G)$ the set of proper colorings of G. Notice that with the above identifications we always have that the symmetric group $\mathfrak{S}_n$ is a subset of $\mathsf{PC}(G)$.

Given $\kappa \in \mathsf{C}(G)$ a *G-inversion* of κ is a pair (i,j) with $\{i,j\} \in E$, $i<j$ and $\kappa(i) > \kappa(j)$. Similarly, a *G-coinversion* of κ is a pair (i,j) with $\{i,j\} \in E$, $i<j$ and $\kappa(i) < \kappa(j)$. We denote by $\mathsf{Inv}_G(\kappa)$, respectively $\mathsf{CoInv}_G(\kappa)$, the set of G-inversions, respectively G-coinversions, of κ. Let us denote by $\mathsf{Inv}(G)$ the (finite) set of possible sets of G-inversions of a coloring of G: in other words $\mathsf{Inv}(G) := \{\mathsf{Inv}_G(\sigma) \mid \sigma \in \mathfrak{S}_n\}$. Similarly, we set $\mathsf{CoInv}(G) := \{\mathsf{CoInv}_G(\sigma) \mid \sigma \in \mathfrak{S}_n\}$.

We can now set for every $\kappa \in \mathsf{C}(G)$

$$\mathsf{inv}_G(\kappa) := |\mathsf{Inv}_G(\kappa)| \quad \text{and} \quad \mathsf{coinv}_G(\kappa) := |\mathsf{CoInv}_G(\kappa)|.$$

Example 2.1. Consider the graph G in Figure 1, and $\sigma = 31852647 \in \mathfrak{S}_8 \subseteq \mathsf{PC}(G)$. Then

$$\begin{aligned}\mathsf{Inv}_G(\sigma) &= \{(1,2),(3,4),(3,5),(3,6),(6,7)\} \text{ and}\\ \mathsf{CoInv}_G(\sigma) &= \{(1,3),(2,3),(2,4),(2,5),(2,6),(2,7),(5,6),(5,7),(6,8),(7,8)\},\end{aligned}$$

so that $\mathsf{inv}_G(\sigma) = 5$ and $\mathsf{coinv}_G(\sigma) = 10$.

Let $\phi : \mathsf{C}(G) \to \mathfrak{S}_n$ be the *standardization* from left to right: given $\kappa(1)\kappa(2)\cdots\kappa(n)$, if $c_1 < c_2 < \cdots < c_k$ is the ordered set of values $\kappa(i)$, then $\phi(\kappa)$ is the permutation obtained by replacing the d_1 occurrences of c_1 with the numbers $1,2,\dots,d_1$ from left to right, then the d_2 occurrences of c_2 with the numbers $d_1+1, d_1+2, \dots, d_1+d_2$ from left to right, and so on. For example $\phi(3253353) = 2163475$.

Remark 2.2. Observe that for any $\kappa \in \mathsf{C}(G)$,

$$\mathsf{CoInv}_G(\kappa) \subseteq \mathsf{CoInv}_G(\phi(\kappa)) \quad \text{and} \quad \mathsf{Inv}_G(\kappa) = \mathsf{Inv}_G(\phi(\kappa)).$$

The asymmetry is due to the fact that the standardization ϕ is from left to right. But observe that if $\kappa \in \mathsf{PC}(G)$, then in fact $\mathsf{CoInv}_G(\kappa) = \mathsf{CoInv}_G(\phi(\kappa))$ as well.

2.4 Interval graphs and (co)inversions

Given $n \in \mathbb{Z}_{>0}$, let $G = ([n], E)$ be an interval graph.

Given $\tau \in \mathfrak{S}_n$, set

$$\mathsf{Des}_G(\tau) := \{i \in [n-1] \mid \tau(i) > \tau(i+1) \text{ or } \{\tau(i), \tau(i+1)\} \in E\} \subseteq [n-1].$$

The next proposition is sort of implicit in the work of Shareshian and Wachs [15].

Proposition 2.3. *Given $G = ([n], E)$ an interval graph, for every $S \in \mathsf{Inv}(G)$ we have*

$$\sum_{\substack{\kappa \in \mathsf{PC}(G) \\ \mathsf{Inv}_G(\kappa) = S}} q^{\mathsf{inv}_G(\kappa)} x_\kappa = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{Inv}_G(\sigma) = S}} q^{\mathsf{inv}_G(\sigma)} L_{n, \mathsf{Des}_G(\sigma^{-1})} = q^{|S|} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{Inv}_G(\sigma) = S}} L_{n, \mathsf{Des}_G(\sigma^{-1})},$$

and for every $S \in \mathsf{CoInv}(G)$ we have

$$\sum_{\substack{\kappa \in \mathsf{PC}(G) \\ \mathsf{CoInv}_G(\kappa) = S}} q^{\mathsf{coinv}_G(\kappa)} x_\kappa = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{CoInv}_G(\sigma) = S}} q^{\mathsf{coinv}_G(\sigma)} L_{n, \mathsf{Des}_G(\sigma^{-1})} = q^{|S|} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{CoInv}_G(\sigma) = S}} L_{n, \mathsf{Des}_G(\sigma^{-1})}.$$

3 Increasing spanning forests and quasisymmetric functions

Given a graph $G = ([n], E)$, we say that a subgraph $F \subseteq G$ is a *spanning forest* if F is a forest on the vertices $[n]$. In this case, the connected components are labelled trees, with the vertex set contained in $[n]$. Given such a tree T, we call $root(T)$ its minimal vertex. Then T is called *increasing* if in the paths stemming from $root(T)$ the other vertices appear in increasing order.

A spanning forest F of a graph $G = ([n], E)$ is called *increasing* if all its connected components are increasing trees. In this case, we think of F as the ordered collection $F = (T_1, T_2, \dots, T_k)$, where the T_i are its connected components, ordered so that

$$root(T_1) < root(T_2) < \cdots < root(T_k).$$

E.g. the forest $F = (T_1, T_2)$ in Figure 2, with[1] $T_1 = (V(T_1), E(T_1)) = (\{1,3\}, \{(1,3)\})$ and $T_2 = (V(T_2), E(T_2)) = (\{2,4,5,6,7,8\}, \{(2,4),(2,5),(2,6),(5,7),(6,8)\})$, is an increasing spanning forest of the graph G in Figure 1, .

We denote by $\mathsf{ISF}(G)$ the set of increasing spanning forests of G.

Given a graph $G = ([n], E)$ and an $F \in \mathsf{ISF}(G)$, $F = (T_1, T_2, \dots, T_k)$, we say that a pair (u, v) with $u, v \in [n]$ is a *G-inversion* of F if $u \in V(T_i)$, $v \in V(T_j)$, $i > j$ and $(u, v) \in E$ (so that $u < v$). Given an edge $(u, v) \in E(T_i)$ of T_i we define its *weight* in G,

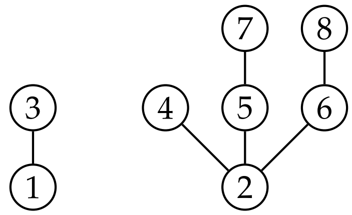

Figure 2: An example of increasing spanning forest of the graph G in Figure 1.

denoted $\mathsf{wt}_G((u,v))$, to be the number of $w \in V(T_i)$ vertex of T_i such that $u \leq w < v$ and $(w,v) \in E(G)$. So for every tree T_i we define its *weight* in G as

$$\mathsf{wt}_G(T_i) = \sum_{(u,v)\in E(T_i)} \mathsf{wt}_G((u,v))$$

and finally the *weight* of F (in G) as

$$\mathsf{wt}_G(F) := \#\{G\text{-inversions of } F\} + \sum_{i=1}^{k} \mathsf{wt}_G(T_i).$$

Example 3.1. The forest $F = (T_1, T_2)$ in Figure 2 is an increasing spanning forest of the graph G in Figure 1: we observe that its only G-inversion is $(2,3)$ (as 3 occurs in T_1, 2 occurs in T_2 and $(2,3) \in E$), $\mathsf{wt}_G(T_1) = \mathsf{wt}_G((1,3)) = 1$, and

$$\begin{aligned}\mathsf{wt}_G(T_2) &= \mathsf{wt}_G((2,4)) + \mathsf{wt}_G((2,5)) + \mathsf{wt}_G((2,6)) + \mathsf{wt}_G((5,7)) + \mathsf{wt}_G((6,8)) \\ &= 1+1+2+2+2 = 8,\end{aligned}$$

so that $\mathsf{wt}_G(F) = 1+1+8 = 10$.

Let $G = ([n], E)$ be an interval graph, i.e. $G \in \mathcal{IG}_n$. We define a function $\Phi_G : \mathsf{PC}(G) \to \mathsf{ISF}(G)$ via Algorithm 1 and Algorithm 2.

Algorithm 1 Algorithm defining the function $\mathsf{getW}(G, v, S, \kappa)$

Input: A graph $G = ([n], E)$, $S \subset [n]$, $v \in [n] \setminus S$, and $\kappa \in \mathsf{PC}(G)$
Output: W ▷ It will be $W \subseteq S \cup \{v\}$

```
W ← {v}
for w ∈ S do
    if {u ∈ W | u < w, (u,w) ∈ E and κ(u) < κ(w)} ≠ ∅ then
        W ← W ∪ {w}
    end if
end for
```

Algorithm 2 The algorithm defining the function $\Phi_G(\kappa)$

Input: A graph $G = ([n], E)$ and $\kappa \in \mathsf{PC}(G)$
Output: $F = (T_1, T_2, \dots)$ ▷ It will be $F \in \mathsf{ISF}(G)$

```
S ← [n]
F ← ( )                                             ▷ Empty list
while S ≠ ∅ do
    v ← min(S)
    S ← S \ {v}
    T = (V(T), E(T)) ← ({v}, ∅)                     ▷ The tree we are going to build
    W ← getW(G, v, S, κ)                            ▷ Defined in Algorithm 1
    for i ∈ {2, ..., #W} do
        L ← {u ∈ V(T) | u < W_i and (u, W_i) ∈ E}   ▷ W = {W_1 < W_2 < ··· < W_#W}
        r ← #{u ∈ L | (u, W_i) ∈ E and κ(u) < κ(W_i)}
        T ← (V(T) ∪ {W_i}, E(T) ∪ {(L_{#L−r+1}, W_i)})   ▷ L = {L_1 < L_2 < ··· < L_#L}
        S ← S \ {W_i}
    end for
    Append T to the right of F
end while
```

Proposition 3.2. *Given a graph $G = ([n], E)$, the Algorithm 2 defines a function $\Phi_G : \mathsf{PC}(G) \to \mathsf{ISF}(G)$.*

The first nontrivial property of the function Φ_G is its surjectivity.

Theorem 3.3. *Let $G = ([n], E)$ a graph. There exists an explicit function $f_G : \mathsf{ISF}(G) \to \mathfrak{S}_n \subset \mathsf{PC}(G)$ such that $\Phi_G \circ f_G(F) = F$ for every $F \in \mathsf{ISF}(G)$. In particular Φ_G is surjective, f_G is injective, and $\mathsf{ISF}(G) = \{\Phi_G(\sigma) \mid \sigma \in \mathfrak{S}_n\} = \{\Phi_G(\sigma) \mid \sigma \in f_G(\mathsf{ISF}(G))\}$.*

When $G = ([n], E)$ is an interval graph, Φ_G has also the following property.

Proposition 3.4. *Given an interval graph $G = ([n], E)$, the function $\Phi_G : \mathsf{PC}(G) \to \mathsf{ISF}(G)$ defined by Algorithm 2 is such that for every $\kappa, \kappa' \in \mathsf{PC}(G)$, $\Phi_G(\kappa) = \Phi_G(\kappa')$ if and only if $\mathsf{CoInv}_G(\kappa) = \mathsf{CoInv}_G(\kappa')$. Moreover $\mathsf{wt}_G(\Phi_G(\kappa)) = \mathsf{coinv}_G(\kappa)$ for every $\kappa \in \mathsf{PC}(G)$.*

We are now ready to define quasisymmetric functions associated to increasing spanning forests of interval graphs.

Definition 3.5. Given an interval graph $G = ([n], E)$, and given $F \in \mathsf{ISF}(G)$, we define the formal power series

$$\mathcal{Q}_F^{(G)} = \mathcal{Q}_F^{(G)}[X] := \sum_{\substack{\kappa \in \mathsf{PC}(G) \\ \Phi_G(\kappa) = F}} x_\kappa.$$

We have the following fundamental formula.

Theorem 3.6. *Given an interval graph $G = ([n], E) \in \mathcal{IG}_n$, and given $F \in \mathsf{ISF}(G)$, we have*

$$\mathcal{Q}_F^{(G)} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{CoInv}_G(\sigma) = \mathsf{CoInv}_G(F)}} L_{n, \mathsf{Des}_G(\sigma^{-1})}, \tag{3.1}$$

where

$$\mathsf{CoInv}_G(F) := \mathsf{CoInv}_G(f_G(F)).$$

4 Interval orders, chromatic functions and LLT

Given any simple graph $G = ([n], E)$, Shareshian and Wachs defined in [15] its *chromatic quasisymmetric function* as

$$\chi_G[X; q] := \sum_{\kappa \in \mathsf{PC}(G)} q^{\mathsf{coinv}_G(\kappa)} x_\kappa.$$

The following theorem is a direct consequence of Proposition 3.4 and Theorem 3.3.

Theorem 4.1. *Given an interval graph $G = ([n], E)$, we have*

$$\chi_G[X; q] = \sum_{F \in \mathsf{ISF}(G)} q^{\mathsf{wt}_G(F)} \mathcal{Q}_F^{(G)}. \tag{4.1}$$

Example 4.2. For $G = ([3], \{(1,2), (1,3)\})$, the increasing spanning forests of G are

$$F_1 = (([3], \{(1,2), (1,3)\})), \quad F_2 = ((\{1,3\}, \{(1,3)\}), (\{2\}, \varnothing)),$$
$$F_3 = ((\{1,2\}, \{(1,2)\}), (\{3\}, \varnothing)), \quad F_4 = ((\{1\}, \varnothing), (\{2\}, \varnothing), (\{3\}, \varnothing)),$$

and we compute

$$\mathsf{wt}_G(F_1) = 2, \quad \mathsf{wt}_G(F_2) = \mathsf{wt}_G(F_3) = 1, \quad \mathsf{wt}_G(F_4) = 0,$$

$$\mathcal{Q}_{F_1}^{(G)} = L_{(1,2)} + L_{(1^3)}, \quad \mathcal{Q}_{F_2}^{(G)} = \mathcal{Q}_{F_3}^{(G)} = L_{(1^3)}, \quad \mathcal{Q}_{F_4}^{(G)} = L_{(2,1)} + L_{(1^3)},$$

hence finally

$$\chi_G[X; q] = L_{(2,1)} + q^2 L_{(1,2)} + (1 + 2q + q^2) L_{(1^3)}.$$

The following corollary is a reformulation of [15, Theorem 3.1] in our cases, and it follows immediately from Theorems 4.1 and 3.6.

Corollary 4.3. *Given an interval graph G on n vertices, we have*

$$\chi_G[X; q] = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{coinv}_G(\sigma)} L_{n, \mathsf{Des}_G(\sigma^{-1})}. \tag{4.2}$$

Given any simple graph $G = ([n], E) \in \mathcal{IG}_n$, we define its *LLT quasisymmetric function* as

$$\mathrm{LLT}_G[X;q] := \sum_{\kappa \in \mathsf{C}(G)} q^{\mathsf{inv}_G(\kappa)} x_\kappa.$$

The following formula is an immediate consequence of Proposition 2.3.

Theorem 4.4. *Given any interval graph $G = ([n], E)$, we have*

$$\mathrm{LLT}_G[X;q] = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{inv}_G(\sigma)} L_{n,\mathsf{Des}(\sigma^{-1})}.$$

The name LLT of these quasisymmetric functions comes from the following well-known facts: when G is a Dyck graph, $\mathrm{LLT}_G[X;q]$ is a symmetric function, and in fact it is a so called *unicellular LLT symmetric functions* (see e.g. [3, Section 3]). In this case the formula in Theorem 4.4 is well known (e.g. it can be deduced from [13, Theorem 8.6]).

5 The main identity

Recall from Section 2.2 the involutions ψ and ρ. We use the plethysm of quasisymmetric functions: cf. [11].

Theorem 5.1. *Let $G = ([n], E)$ be an interval graph. Then*

$$(1-q)^n \rho\left(\psi \chi_G\left[X\frac{1}{1-q}\right]\right) = \mathrm{LLT}_G[X;q]. \tag{5.1}$$

Remark 5.2. This is really an extension of [6, Proposition 3.5]. Indeed, when G is a Dyck graph, $\chi_G[X;q]$ is symmetric (by [15, Theorem 4.5]), the plethysm reduces to the usual plethysm of symmetric functions (cf. [11]), ρ fixes the symmetric functions while ψ gives the usual ω involution of symmetric functions, and $\mathrm{LLT}_G[X;q]$ is precisely the unicellular LLT symmetric function corresponding to the Dyck graph G, so, our (5.1) is just a rewriting of [6, Proposition 3.5].

6 Expansions in the Ψ_α

In [5] the authors study a family of quasisymmetric functions that they call *type 1 quasisymmetric power sums*, and they denote Ψ_α. Actually $\{\Psi_\alpha \mid \alpha \text{ composition}\}$ is a basis of QSym, and these quasisymmetric functions refine the power symmetric functions, i.e. for any partition $\lambda \vdash n$

$$\sum_{\substack{\alpha \vDash n \\ \lambda(\alpha)=\lambda}} \Psi_\alpha = p_\lambda \ , \tag{6.1}$$

where $\lambda(\alpha)$ is the unique partition obtained by rearranging in weakly decreasing order the parts of α, and the $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ are the usual *power symmetric functions*.

Given $G = ([n], E)$ a graph and $\sigma \in \mathfrak{S}_n$ a permutation, we say that $r \in [n]$ is a *left-to-right G-maximum* if for every $s \in [r-1]$ we have $\sigma(s) < \sigma(r)$ and $\{\sigma(s), \sigma(r)\} \notin E$. Notice that 1 is always a left-to-right G-maximum, that we call *trivial*. We set

$$\widetilde{\mathsf{inv}}_G(\sigma) := \{\{\sigma(i), \sigma(j)\} \in E \mid i < j \text{ and } \sigma(i) > \sigma(i+1)\},$$

and

$$\widetilde{\mathsf{Des}}_G(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1) \text{ and } \{\sigma(i), \sigma(i+1)\} \notin E\}.$$

We say that $i \in [n-1]$ is a *G-descent* if $i \in \widetilde{\mathsf{Des}}_G(\sigma)$.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vDash n$, let $\mathcal{N}_{G,\alpha}$ be the set of $\sigma \in \mathfrak{S}_n$ such that if we break $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ into contiguous segments of lengths $\alpha_1, \alpha_2, \dots, \alpha_k$, each contiguous segment has neither a G-descent nor a nontrivial left-to-right G-maximum.

Given a composition α, define $z_\alpha := z_{\lambda(\alpha)}$, where, as usual, for every partition $\lambda \vdash n$, if m_i denotes the number of parts of λ equal to i, then $z_\lambda := \prod_{i=1}^n m_i! \cdot i^{m_i}$.

Finally, recall the involution $\omega : \mathrm{QSym} \to \mathrm{QSym}$ from Section 2.2.

We state our conjecture.

Conjecture 6.1. *For any interval graph $G = ([n], E)$ we have*

$$\omega \chi_G[X; q] = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \sum_{\sigma \in \mathcal{N}_{G,\alpha}} q^{\widetilde{\mathsf{inv}}_G(\sigma)}.$$

This conjecture should generalize the following formula, proposed by Shareshian and Wachs [15, Conjecture 7.6] and later proved by Athanasiadis [4].

Theorem 6.2. *For any Dyck graph $G = ([n], E)$ we have*

$$\omega \chi_G[X; q] = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \sum_{\sigma \in \mathcal{N}_{G,\lambda}} q^{\widetilde{\mathsf{inv}}_G(\sigma)}.$$

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Crystals for variations of decomposition tableaux

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Abstract. Our previous work introduced a category of extended queer crystals, whose connected normal objects have unique highest weight elements and characters that are Schur Q-polynomials. Our initial models for such crystals were based on semistandard shifted tableaux. Here, we introduce a simpler construction using certain "primed" decomposition tableaux, which slightly generalize the decomposition tableaux used in work of Grantcharov et al. This leads to a new, much shorter proof of the highest weight properties of the normal subcategory of extended queer crystals. We also describe a natural crystal structure on set-valued decomposition tableaux. Our results give the first crystal constructions for shifted set-valued tableaux, and lead to partial progress on a conjectural formula of Cho and Ikeda for K-theoretic Schur P-functions.

Keywords: Crystals, K-theoretic Schur P-functions, queer Lie superalgebras, decomposition tableaux, set-valued tableaux

1 Introduction

Crystals are an abstraction for the crystal bases of quantum group representations, and can be viewed as acyclic directed graphs with labeled edges and weighted vertices, satisfying certain axioms. Crystals for $\mathfrak{gl}_n$ and other classical Lie algebras were first studied by Kashiwara [9, 10] and Lusztig [12, 13] in the 1990s. More recent work by Grantcharov et al. [3, 4] introduced crystals for the queer Lie superalgebra $\mathfrak{q}_n$.

Our previous work [14] defined a slightly modified category of $\mathfrak{q}_n^+$-crystals, which share many nice features with $\mathfrak{gl}_n$-crystals and $\mathfrak{q}_n$-crystals. For example, $\mathfrak{q}_n^+$-crystals have a natural tensor product and a standard crystal corresponding to the vector representation of the quantum group $U_q(\mathfrak{q}_n)$. This lets one define a subcategory of *normal crystals*, consisting of crystals whose connected components can each be embedded in some tensor power of the standard crystal.

In [14], our primary models for normal $\mathfrak{q}_n^+$-crystals were derived from *semistandard shifted tableaux*, using crystal operators with very technical formulas. One of the main results of this note is to introduce a much simpler model for normal $\mathfrak{q}_n^+$-crystal based on a "primed" generalization of *decomposition tableaux*. The latter tableaux served as the original model for normal (non-extended) $\mathfrak{q}_n$-crystals in [3].

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After formally defining primed decomposition tableaux, we equip them with a natural family of $\mathfrak{q}_n^+$-crystal operators, identify their highest weight elements, and construct a primed generalization of a useful "insertion scheme" from [3]. As an application, we give a short, alternate proof that normal $\mathfrak{q}_n^+$-crystals are determined up to isomorphism by their characters (which range over all Schur Q-positive symmetric polynomials in n variables), and also by by their multisets of highest weights (which range over all strict partitions with at most n parts).

Our other main results concern a new crystal structure on a "set-valued" generalization of decomposition tableaux. Several authors (for example, [5, 18, 20]) have recently studied $\mathfrak{gl}_n$-crystal structures on unshifted set-valued tableaux. The characters of these crystals give K-theoretic symmetric functions of independent interest. It has been an open problem to extend such constructions to shifted tableaux.

Addressing this open problem, we show that a certain natural family of *set-valued decomposition tableaux* has a normal $\mathfrak{gl}_n$-crystal structure. This structure is formally similar to the one in [18] for unshifted set-valued tableaux, though somewhat more technical. Cho and Ikeda [6] has conjectured that the weight generating function for set-valued decomposition tableaux recovers the K-theoretic Schur P-function GP_λ. As partial progress on this conjecture, our results imply that this generating function is at least symmetric and equal to GP_λ plus a (possibly infinite) $\mathbb{Z}$-linear combination of GP_μ's with $|\mu| > |\lambda|$.

2 Abstract crystals

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{P} = \{1, 2, 3, \dots\}$. Fix $n \in \mathbb{N}$ and let $[n] = \{1, 2, \dots, n\}$. Let $\mathcal{B}$ be a set with maps $\mathrm{wt}\colon \mathcal{B} \to \mathbb{N}^n$ and $e_i, f_i\colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$, where $0 \notin \mathcal{B}$. We assume that if $b, c \in \mathcal{B}$ then $f_i(b) = c$ if and only if $e_i(c) = b$. This means that the maps e_i and f_i encode a directed graph with vertex set $\mathcal{B}$, to be called the *crystal graph*, with an edge $b \xrightarrow{i} c$ if $f_i(b) = c$. The *string lengths* $\varepsilon_i, \varphi_i\colon \mathcal{B} \to \{0, 1, 2, \dots\} \sqcup \{\infty\}$ are

$$\varepsilon_i(b) := \sup\left\{k \geq 0 \mid e_i^k(b) \neq 0\right\} \text{ and } \varphi_i(b) := \sup\left\{k \geq 0 : f_i^k(b) \neq 0\right\}. \tag{2.1}$$

We assume that $\varepsilon_i(b)$ and $\varphi_i(b)$ are always finite. If the set $\mathcal{B}$ is finite then its *character* is the polynomial $\mathrm{ch}(\mathcal{B}) := \sum_{b \in \mathcal{B}} \prod_{i \in [n]} x_i^{\mathrm{wt}(b)_i} \in \mathbb{N}[x_1, x_2, \dots, x_n]$. Finally, let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{Z}^n$ be the standard basis.

Definition 2.1. The set $\mathcal{B}$ is a *$\mathfrak{gl}_n$-crystal* if for all $i \in [n-1]$ and $b \in \mathcal{B}$ we have (a) $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \mathbf{e}_i - \mathbf{e}_{i+1}$ if $e_i(b) \neq 0$, and (b) $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}$.

We refer to wt as the *weight map* and to each e_i as a *raising operator*. Each connected component of the crystal graph of $\mathcal{B}$ may be viewed as a $\mathfrak{gl}_n$-crystal by restricting the weight map and crystal operators; these objects are called *full subcrystals*. A *crystal isomorphism* is a weight-preserving bijection that induces an isomorphism of crystal graphs.

Example 2.2. The *standard $\mathfrak{gl}_n$-crystal* $\mathbb{B}_n = \left\{\boxed{i} : i \in [n]\right\}$ has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n} \quad \text{with } \mathrm{wt}(\boxed{i}) := \mathbf{e}_i.$$

The set of formal tensors $\mathcal{B} \otimes \mathcal{C} := \{b \otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$ has a unique $\mathfrak{gl}_n$-crystal structure with $\mathrm{wt}(b \otimes c) := \mathrm{wt}(b) + \mathrm{wt}(c)$ and with

$$e_i(b \otimes c) := \begin{cases} b \otimes e_i(c) & \text{if } \varepsilon_i(b) \le \varphi_i(c) \\ e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \end{cases} \tag{2.2}$$

for $i \in [n-1]$, where it is understood that $b \otimes 0 = 0 \otimes c = 0$ [1, §2.3]. This follows the "anti-Kashiwara convention," which reverses the tensor product order in [3, 4].

3 Queer crystals

The general linear Lie algebra $\mathfrak{gl}_n$ has two super-analogues, one of which is the *queer Lie superalgebra* $\mathfrak{q}_n$. Grantcharov et al. developed a theory of crystals for $\mathfrak{q}_n$ in [3, 4], which we review here. Assume $n \ge 2$. Let $\mathcal{B}$ be a $\mathfrak{gl}_n$-crystal with maps $e_{\overline{1}}, f_{\overline{1}} : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ satisfying $f_{\overline{1}}(b) = c$ if and only if $b = e_{\overline{1}}(c)$ when $b, c \in \mathcal{B}$. Define $\varepsilon_{\overline{1}}, \varphi_{\overline{1}} : \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ as in (2.1) but with $i = \overline{1}$. Below, we say that one map $\phi : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ *preserves* another map $\eta : \mathcal{B} \to \mathcal{X}$ if $\eta(\phi(b)) = \eta(b)$ whenever $\phi(b) \ne 0$.

Definition 3.1. The $\mathfrak{gl}_n$-crystal $\mathcal{B}$ is a *$\mathfrak{q}_n$-crystal* if for all $b \in \mathcal{B}$:

(a) $\mathrm{wt}(e_{\overline{1}}(b)) = \mathrm{wt}(b) + \mathbf{e}_1 - \mathbf{e}_2$ whenever $e_{\overline{1}}(b) \ne 0$,

(b) $\varphi_{\overline{1}}(b) + \varepsilon_{\overline{1}}(b)$ is 0 if $\mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0$ and 1 otherwise, and

(c) $e_{\overline{1}}$ and $f_{\overline{1}}$ commute with e_i, f_i while preserving ε_i, φ_i for all $3 \le i \le n-1$.

Assume $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal. The corresponding *$\mathfrak{q}_n$-crystal graph* has vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} c$ whenever $f_i(b) = c$ for any $i \in \{\overline{1}, 1, 2, \ldots, n-1\}$.

Example 3.2. The *standard $\mathfrak{q}_n$-crystal* $\mathbb{B}_n = \left\{\boxed{i} : i \in [n]\right\}$ has crystal graph

$$\boxed{1} \overset{\overline{1}}{\underset{1}{\rightrightarrows}} \boxed{2} \xrightarrow[2]{} \boxed{3} \xrightarrow[3]{} \cdots \xrightarrow[n-1]{} \boxed{n} \quad \text{with } \mathrm{wt}(\boxed{i}) := \mathbf{e}_i.$$

Suppose $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{q}_n$-crystals. The set $\mathcal{B} \otimes \mathcal{C}$ already has a $\mathfrak{gl}_n$-crystal structure. There is a unique way of viewing this object as a $\mathfrak{q}_n$-crystal [3, Thm. 1.8] with

$$e_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes e_{\overline{1}}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ e_{\overline{1}}(b) \otimes c & \text{otherwise.} \end{cases} \tag{3.1}$$

4 Extended crystals

We continue to assume $n \geq 2$. The following theory of *extended* $\mathfrak{q}_n$*-crystals* (abbreviated as $\mathfrak{q}_n^+$*-crystals* from now on) was introduced in our previous work [14]. Suppose $\mathcal{B}$ is a $\mathfrak{q}_n$-crystal with additional maps $e_0, f_0 : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ satisfying $f_0(b) = c$ if and only if $b = e_0(c)$ when $b, c \in \mathcal{B}$. Define $\varepsilon_0, \varphi_0 : \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ by the formula (2.1) with $i = 0$.

Definition 4.1. The $\mathfrak{q}_n$-crystal $\mathcal{B}$ is a $\mathfrak{q}_n^+$*-crystal* if for all $b \in \mathcal{B}$:

(a) $\mathrm{wt}(e_0(b)) = \mathrm{wt}(b)$ if $e_0(b) \neq 0$,

(b) $\varphi_0(b) + \varepsilon_0(b)$ is 0 if $\mathrm{wt}(b)_1 = 0$ and 1 otherwise, and

(c) e_0 and f_0 commute with e_i, f_i while preserving ε_i, φ_i for all $2 \leq i \leq n-1$.

Assume $\mathcal{B}$ is a $\mathfrak{q}_n^+$-crystal. The corresponding $\mathfrak{q}_n^+$*-crystal graph* has vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} c$ whenever $f_i(b) = c$ for any $i \in \{\overline{1}, 0, 1, 2, \dots, n-1\}$.

Example 4.2. The *standard* $\mathfrak{q}_n^+$*-crystal* $\mathbb{B}_n^+$ has crystal graph

$$\begin{tikzcd}
\boxed{1} \arrow[r, dashed, "\overline{1}"] \arrow[r, swap, "1"] \arrow[d, dotted, swap, "0"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n} \\
\boxed{1'} \arrow[r, dashed, "\overline{1}"] \arrow[r, swap, "1"] & \boxed{2'} \arrow[r, "2"] & \boxed{3'} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n'}
\end{tikzcd}
\qquad \text{with } \mathrm{wt}(\boxed{i}) = \mathrm{wt}(\boxed{i'}) := \mathbf{e}_i.$$

If $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{q}_n^+$-crystals then the $\mathfrak{gl}_n$-crystal $\mathcal{B} \otimes \mathcal{C}$ has a $\mathfrak{q}_n^+$-crystal structure with

$$e_0(b \otimes c) := \begin{cases} e_0(b) \otimes c & \text{if } \mathrm{wt}(b)_1 \neq 0 \\ b \otimes e_0(c) & \text{if } \mathrm{wt}(b)_1 = 0 \end{cases} \tag{4.1}$$

and

$$e_{\overline{1}}(b \otimes c) := \begin{cases} b \otimes e_{\overline{1}}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ f_0 e_{\overline{1}}(b) \otimes e_0(c) & \text{if } \mathrm{wt}(b)_1 = 0,\ f_0 e_{\overline{1}}(b) \neq 0, \text{ and } e_0(c) \neq 0 \\ e_0 e_{\overline{1}}(b) \otimes f_0(c) & \text{if } \mathrm{wt}(b)_1 = 0,\ e_0 e_{\overline{1}}(b) \neq 0, \text{ and } f_0(c) \neq 0 \\ e_{\overline{1}}(b) \otimes c & \text{otherwise} \end{cases} \tag{4.2}$$

where it is again understood that $b \otimes 0 = 0 \otimes c = 0$ [14, Thm. 3.14].

Remark 4.3. For $i \in \mathbb{Z}$ define $i' := i - \frac{1}{2} \in \mathbb{Z}' := \mathbb{Z} - \frac{1}{2}$. We refer to elements of $\mathbb{Z} \sqcup \mathbb{Z}'$ as *primed numbers*. A *primed word* is a finite sequence of primed numbers. We identify each primed word $w = w_1 w_2 \cdots w_m$ with $w_i \in \{1' < 1 < \cdots < n' < n\}$ with the formal tensor $w_1 \otimes w_2 \otimes \cdots \otimes w_m \in (\mathbb{B}_n^+)^{\otimes m}$. This allows us to evaluate $\mathrm{wt}(w)$, $e_i(w)$, and $f_i(w)$ for $i \in [n-1]$ using the definition of the $\mathfrak{q}_n^+$-crystal $(\mathbb{B}_n^+)^{\otimes m}$. For example, the weight of w becomes the vector whose ith component is the number of letters equal to i or i'.

There are well-known, explicit signature rules to compute the crystals operators on tensor powers of the standard $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, and $\mathfrak{q}_n^+$-crystals (and hence on primed words). We omit this background material in this extended abstract; see [2, 14] for the full details.

5 Decomposition tableaux

Assume $\lambda = (\lambda_1 > \lambda_2 > \cdots > 0)$ is a *strict* integer partition. Let $\ell(\lambda)$ be the number of nonzero parts of λ. The *shifted diagram* of λ is the $\mathsf{SD}_\lambda := \{(i, i+j-1) : i \in [\ell(\lambda)] \text{ and } j \in [\lambda_i]\}$. We often refer to the pairs $(i,j) \in \mathsf{SD}_\lambda$ as boxes. A box $(i,j) \in \mathsf{SD}_\lambda$ is on the *diagonal* if $i = j$. A *shifted tableau* of shape λ is an assignment of numbers to the boxes in SD_λ.

A *hook word* is a sequence of positive integers $w = w_1 w_2 \cdots w_n$ such that $w_1 \geq w_2 \geq \cdots \geq w_m < w_{m+1} < w_{m+2} < \cdots < w_n$ for some $m \in [n]$. The *weakly decreasing part* of such a hook word w is the (always nonempty) subword $w_1 w_2 \cdots w_m$, while the *increasing part* of w is the (possibly empty) subword $w_{m+1} w_{m+2} \cdots w_n$.

Following [3], we define a *(semistandard) decomposition tableau* of shape λ to be a shifted tableau T of shape λ such that if ρ_i denotes row i of T, then (1) each ρ_i is a hook word and (2) ρ_i is a hook subword of maximal length in $\rho_{i+1}\rho_i$ for each $i \in [\ell(\lambda) - 1]$. Note that this definition is different from Serrano's definition in [19], which uses the opposite weak/strict inequality convention for hook words. Let $\mathsf{DecTab}_n(\lambda)$ be the set of decomposition tableaux of shape λ with all entries in $[n]$.

Example 5.1. We draw tableaux in French notation, so that row indices increase from bottom to top and column indices increase from left to right. Then $\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 1 \\ \hline \end{array} \in \mathsf{DecTab}_2(\lambda)$ for $\lambda = (3,2)$, but $T = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 1 \\ \hline 2 & 2 & 3 \\ \hline \end{array}$ is a not a decomposition tableau even though its rows are hook words, as $\rho_2\rho_1 = 21223$ contains the hook subword 2223, which is longer than 223.

Remark 5.2. The maximal hook subword condition in the definition of a decomposition tableau is equivalent to a set of inequalities that must hold for certain triples of entries. Concretely, a shifted tableau is a decomposition tableau if and only none of the following patterns with $a \leq b \leq c$ and $x < y < z$ occur in consecutive rows [3, Prop. 2.3]:

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \cdots & b \\ \hline a & \cdots & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|}\cline{2-5} \multicolumn{1}{c|}{} & \cdots & c & \cdots & b \\ \hline & \cdots & a & \cdots & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \cdots & x \\ \hline y & \cdots & z \\ \hline \end{array}, \quad \text{or} \quad \begin{array}{|c|c|c|c|c|}\cline{2-5} \multicolumn{1}{c|}{} & \cdots & & \cdots & x \\ \hline & \cdots & y & \cdots & z \\ \hline \end{array}.$$

Here, the leftmost boxes are on the main diagonal and the ellipses "$\cdots$" indicate sequences of zero or more columns.

Define the *middle element* of a hook word w to be the last letter in the weakly decreasing subword $w\downarrow$. Suppose T is a decomposition tableau of strict partition shape λ. We call any tableau formed by adding primes to the middle elements in a subset of rows in T a *primed decomposition tableau* of shape λ. Let $\mathsf{DecTab}_n^+(\lambda)$ denote the set of such tableaux with all entries in $\{1' < 1 < \cdots < n' < n\}$.

Example 5.3. $\begin{array}{ccc} & \boxed{1} & \\ \boxed{2} & \boxed{1} & \boxed{2} \end{array}$, $\begin{array}{ccc} & \boxed{1'} & \\ \boxed{2} & \boxed{1} & \boxed{2} \end{array}$, $\begin{array}{ccc} & \boxed{1} & \\ \boxed{2} & \boxed{1'} & \boxed{2} \end{array}$, and $\begin{array}{ccc} & \boxed{1'} & \\ \boxed{2} & \boxed{1'} & \boxed{2} \end{array}$ are all in $\mathsf{DecTab}_2^+(\lambda)$ for $\lambda = (3,1)$.

The *row reading word* of a shifted tableau T is the word $\mathsf{row}(T)$ formed by reading the rows from left to right, but starting with last row. The *reverse reading word* of T is the reversal of $\mathsf{row}(T)$; we denote this by $\mathsf{revrow}(T)$. For example, $\mathsf{revrow}\left(\begin{array}{ccc} & \boxed{1} & \\ \boxed{2} & \boxed{1} & \boxed{1'} \end{array}\right) = 1'121$.

A *crystal embedding* is a weight-preserving injective map $\phi : \mathcal{B} \to \mathcal{C}$ between crystals that commutes with all crystal operators, in the sense that $\phi(e_i(b)) = e_i(\phi(b))$ and $\phi(f_i(b)) = f_i(\phi(b))$ for all $b \in \mathcal{B}$ when we set $\phi(0) = 0$. Our first new result is the following theorem, which extends [3, Thm. 2.5(a)] from $\mathfrak{q}_n$-crystals to $\mathfrak{q}_n^+$-crystals.

Theorem 5.4. There is a unique $\mathfrak{q}_n^+$-crystal structure on $\mathsf{DecTab}_n^+(\lambda)$ that makes $\mathsf{revrow} : \mathsf{DecTab}_n^+(\lambda) \to (\mathbb{B}_n^+)^{\otimes|\lambda|}$ into a $\mathfrak{q}_n^+$-crystal embedding. This structure restricts to a $\mathfrak{q}_n$-crystal on $\mathsf{DecTab}_n(\lambda)$, for which $\mathsf{revrow} : \mathsf{DecTab}_n(\lambda) \to \mathbb{B}_n^{\otimes|\lambda|}$ is a $\mathfrak{q}_n$-crystal embedding. Finally, the characters of these crystals are the symmetric polynomials

$$\mathrm{ch}(\mathsf{DecTab}_n(\lambda)) = P_\lambda(x_1, x_2, \dots, x_n) \quad \text{and} \quad \mathrm{ch}(\mathsf{DecTab}_n^+(\lambda)) = Q_\lambda(x_1, x_2, \dots, x_n)$$

where P_λ and Q_λ are the *Schur P- and Q-functions* of λ.

An important property of many crystals is the existence of unique *highest weight elements*. For $\mathfrak{gl}_n$-crystals, such elements are exactly the sources in the crystal graph. The precise definitions of highest weight elements for $\mathfrak{q}_n$ and $\mathfrak{q}_n^+$-crystals from [3, 14] are more technical, and given as follows.

Assume $\mathcal{B}$ is a $\mathfrak{gl}_n$-crystal. An *i-string* in $\mathcal{B}$ is a connected component in the subgraph of the crystal graph retaining only the $\xrightarrow{i}$ arrows. Let $\sigma_i : \mathcal{B} \to \mathcal{B}$ be the involution that reverses each i-string, so that the first and last elements are swapped, the second and second-to-last elements are swapped, and so on.

Definition 5.5. An element b in a $\mathfrak{q}_n$-crystal $\mathcal{B}$ is *$\mathfrak{q}_n$-highest weight* if $e_i(b) = e_{\bar{i}}(b) = 0$ for $i \in [n-1]$, where $e_{\bar{i}} := (\sigma_{i-1}\sigma_i)\cdots(\sigma_2\sigma_3)(\sigma_1\sigma_2)e_{\bar{1}}(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_i\sigma_{i-1})$ for $2 \le i < n$.

Definition 5.6. An element b in a $\mathfrak{q}_n^+$-crystal $\mathcal{B}$ is *$\mathfrak{q}_n^+$-highest weight* if it is $\mathfrak{q}_n$-highest weight with $\sigma_{i-1}\cdots\sigma_2\sigma_1 e_0 \sigma_1\sigma_2\cdots\sigma_{i-1}(b) = 0$ for all $i \in [n]$.

Let λ be a strict partition with $\ell(\lambda) = k$. The *first border strip* of a shifted shape SD_λ is the minimal subset S containing $(1, \lambda_1)$ such that if $(i,j) \in S$ and $i \neq j$, then either $(i+1, j) \in S$, or $(i, j-1) \in S$ when $(i+1, j) \notin \mathsf{SD}_\lambda$.

Let $\mathsf{SD}_\lambda^{(1)}$ be the first border strip of SD_λ. The set difference $\mathsf{SD}_\lambda - \mathsf{SD}_\lambda^{(1)}$ is either empty when $k = 1$ or equal to SD_μ for a strict partition μ with $\ell(\mu) = k - 1$. For

$i \in [k-1]$ let $\mathsf{SD}_\lambda^{(i+1)}$ be the first border strip of $\mathsf{SD}_\lambda - (\mathsf{SD}_\lambda^{(1)} \sqcup \cdots \sqcup \mathsf{SD}_\lambda^{(i)})$. Finally, let $T_\lambda^{\mathsf{highest}}$ be the shifted tableau of shape λ with all i entries in $\mathsf{SD}_\lambda^{(i)}$.

Example 5.7. If $\lambda = (6,4,2,1)$ then the boxes with $\bullet$ below make up the first border strip

$$\begin{array}{cccccc} & & & \bullet & & \\ & & \square & \bullet & & \\ & \square & \square & \bullet & \bullet & \\ \square & \square & \square & \square & \bullet & \bullet \end{array} \quad \text{and we have} \quad T_\lambda^{\mathsf{highest}} = \begin{array}{cccccc} & & & 1 & & \\ & & 2 & 1 & & \\ & 3 & 2 & 1 & 1 & \\ 4 & 3 & 2 & 2 & 1 & 1 \end{array}.$$

The $\mathfrak{q}_n$-part of the following is [3, Thm. 2.5(b)], while the $\mathfrak{q}_n^+$-extension is new:

Theorem 5.8. The shifted tableau $T_\lambda^{\mathsf{highest}}$ is the unique $\mathfrak{q}_n$-highest weight element of $\mathsf{DecTab}_n(\lambda)$ and also the unique $\mathfrak{q}_n^+$-highest weight element of $\mathsf{DecTab}_n^+(\lambda)$.

6 Decomposition insertion

This section introduces a "primed" extension of Grantcharov et al.'s insertion scheme from [3, §3]. Suppose T is a primed decomposition tableau and $x \in \mathbb{Z} \sqcup \mathbb{Z}'$. We will form another primed decomposition tableau $x \xrightarrow{\mathsf{dec}} T$ by the following insertion procedure. On step i of this algorithm, a number a_i is inserted into row i of T, starting with $a_1 := x$.

To compute the insertion on step i, set $a = \lceil a_i \rceil$ and remove any prime from middle element m_i of row i (if the row is nonempty). The (unprimed) number a is added to the end of the (now unprimed) row if this creates a hook word; otherwise, a replaces the leftmost entry b from the increasing part of the row with $b \geq a$, then b replaces the leftmost entry c from the weakly decreasing part of the row with $c < b$.

Now we must decide the value a_{i+1} and whether to add back a prime to the middle element of the row. There are two cases:

(A) Suppose the row was initially empty, or the location of the middle element has moved (necessarily to the right). If $a_i \in \mathbb{Z}'$ then we add a prime to the new middle element. If no entries were bumped from the row, then the algorithm halts at this step and we say the insertion is *even* if $m_i \in \mathbb{Z}$ and *odd* if $m_i \in \mathbb{Z}'$. Otherwise, we set $a_{i+1} = c$ when $m_i \in \mathbb{Z}$ and $a_{i+1} = c'$ when $m_i \in \mathbb{Z}'$. For example:

$$\boxed{4}\boxed{2}\boxed{2}\boxed{1^\circ}\boxed{3} \leftarrow 1^\bullet = a_i \quad \rightsquigarrow \quad a_{i+1} = 2^\circ \leftarrow \boxed{4}\boxed{3}\boxed{2}\boxed{1}\boxed{1^\bullet}.$$

Here $\circ$ and $\bullet$ indicate arbitrary, unspecified choice of primes.

(B) Suppose instead that the location of the row's middle element has not changed. If $m_i \in \mathbb{Z}'$ then we add back a prime to the middle element. If no entries were bumped from the row, then the algorithm halts at this step and we say the insertion is *even* if $a_i \in \mathbb{Z}$ and *odd* if $a_i \in \mathbb{Z}'$. Otherwise, we set $a_{i+1} = c$ when $a_i \in \mathbb{Z}$ and $a_{i+1} = c'$ when $a_i \in \mathbb{Z}'$. For example:

$$\boxed{4}\boxed{2}\boxed{2}\boxed{1^\circ}\boxed{3} \leftarrow 3^\bullet = a_i \quad \rightsquigarrow \quad a_{i+1} = 2^\bullet \leftarrow \boxed{4}\boxed{3}\boxed{2}\boxed{1^\circ}\boxed{3}.$$

Definition 6.1. Given any primed word $w = w_m \cdots w_2 w_1$, form

$$P_{\mathsf{dec}}(w) := w_m \xrightarrow{\mathsf{dec}} (\cdots \xrightarrow{\mathsf{dec}} (w_2 \xrightarrow{\mathsf{dec}} (w_1 \xrightarrow{\mathsf{dec}} \varnothing))\cdots)$$

by inserting the letters of w into the empty tableau $\varnothing$. Let $Q_{\mathsf{dec}}(w)$ be the tableau with the same shape as $P_{\mathsf{dec}}(w)$ that has i (respectively, i') in the box added by $w_i \xrightarrow{\mathsf{dec}}$ if this insertion is even (respectively, odd).

Example 6.2. For $w = 4'4332'3'32'1'$, $P_{\mathsf{dec}}(w) = \begin{array}{ccccc} & & 1' & & \\ & 2 & 2' & 3 & \\ 4 & 3 & 3 & 3 & 4 \end{array}$ and $Q_{\mathsf{dec}}(w) = \begin{array}{ccccc} & & 7 & & \\ & 4 & 5' & 9' & \\ 1 & 2' & 3 & 6 & 8 \end{array}$.

A shifted tableau with n boxes is *standard* if its rows and columns are increasing and it has exactly one entry equal to i' or i for each $i \in [n]$.

Theorem 6.3. The map $w \mapsto (P_{\mathsf{dec}}(w), Q_{\mathsf{dec}}(w))$ is a bijection from the set of all words with letters in $\{1' < 1 < 2' < 2 < \dots\}$ to the set of pairs (P, Q) of shifted tableaux with the same shape such that P is a primed decomposition tableau and Q is a standard shifted tableau with no primed diagonal entries.

Let $w^{\mathbf{r}}$ be the reverse of w. On unprimed words, the map $w \mapsto (P_{\mathsf{dec}}(w^{\mathbf{r}}), Q_{\mathsf{dec}}(w^{\mathbf{r}}))$ is [3, Def. 4.1] and gives a bijection to pairs (P, Q) where P is an (unprimed) decomposition tableau and Q is a standard shifted tableau of the same shape with no primed entries.

A map $\phi : \mathcal{B} \to \mathcal{C}$ between ($\mathfrak{gl}_n$, $\mathfrak{q}_n$, or $\mathfrak{q}_n^+$) crystals is a *quasi-isomorphism* if for each full subcrystal $\mathcal{B}' \subseteq \mathcal{B}$ there is a full subcrystal $\mathcal{C}' \subseteq \mathcal{C}$ such that $\phi|_{\mathcal{B}'}$ is an isomorphism $\mathcal{B}' \to \mathcal{C}'$. The $\mathfrak{q}_n$ part of the following more substantial result is [3, Thm. 4.5].

Theorem 6.4. The map P_{dec} defines $\mathfrak{q}_n$ and $\mathfrak{q}_n^+$ crystal quasi-isomorphisms

$$\mathbb{B}_n^{\otimes m} \to \bigsqcup_{\substack{\text{strict } \lambda \vdash m \\ \ell(\lambda) \le n}} \mathsf{DecTab}_n(\lambda) \quad \text{and} \quad (\mathbb{B}_n^+)^{\otimes m} \to \bigsqcup_{\substack{\text{strict } \lambda \vdash m \\ \ell(\lambda) \le n}} \mathsf{DecTab}_n^+(\lambda).$$

Moreover, and the full $\mathfrak{q}_n$-subcrystals of $\mathbb{B}_n^{\otimes m}$ and the full $\mathfrak{q}_n^+$-subcrystals of $(\mathbb{B}_n^+)^{\otimes m}$ are the subsets on which Q_{dec} is constant.

7 Applications to normal crystals

A ($\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, or $\mathfrak{q}_n^+$-) crystal is *normal* if each of its full subcrystals is isomorphic to a full subcrystal of a tensor power of the relevant standard crystal. Normal crystals are automatically preserved by disjoint unions and tensor products.

One motivation for the new results in this article was to provide a simpler and more intuitive proof of the following theorem, which was our main result in [14]. One application of this theorem is a new Littlewood-Richardson rule for multiplying Schur Q-functions [14, Cor. 1.7].

Theorem 7.1. The following properties hold for normal $\mathfrak{q}_n^+$-crystals:

(a) Suppose $\mathcal{B}$ is a connected normal $\mathfrak{q}_n^+$-crystal. Then $\mathcal{B}$ has a unique $\mathfrak{q}_n^+$-highest weight element, whose weight λ is a strict partition with at most n parts, and it holds that $\mathcal{B} \cong \mathsf{DecTab}_n^+(\lambda)$ and $\mathrm{ch}(\mathcal{B}) = Q_\lambda(x_1, x_2, \dots, x_n)$.

(b) For each strict partition λ with at most n parts, there is a connected normal $\mathfrak{q}_n^+$-crystal with highest weight λ.

(c) Finite normal $\mathfrak{q}_n^+$-crystals are isomorphic if and only if they have the same characters, which range over all Schur Q-positive symmetric polynomials in $x_1, x_2, \dots, x_n$.

Proof. If $\mathcal{B}$ is a connected normal $\mathfrak{q}_n^+$-crystal then $\mathcal{B} \cong \mathsf{DecTab}_n^+(\lambda)$ for some strict partition λ with $\ell(\lambda) \leq n$ by Theorem 6.4. Theorem 5.8 implies that $\mathcal{B}$ has a unique $\mathfrak{q}_n^+$-highest weight element of weight λ. This proves part (a). Part (b) follows from Theorems 5.4 and 5.8. Part (c) holds since Schur Q-polynomials are linearly independent. □

The crux of this proof is Theorem 6.4 regarding decomposition insertion. Proving Theorem 6.4 is not a trivial exercise, but this is significantly easier than for the analogous result used in [14], which involves a more technical insertion algorithm defined in [15].

By essentially the same proof, one can derive a $\mathfrak{q}_n$-version of this theorem (involving Schur P-polynomials in place of Schur Q-polynomials); this proof strategy is similar to what appears in [3]. There is also a classical version of Theorem 7.1 for normal $\mathfrak{gl}_n$-crystals (see [1, Thms. 3.2 and 8.6] or [14, Thm. 1.1]) which implies that the character of every finite normal $\mathfrak{gl}_n$-crystal is Schur positive.

8 Set-valued tableaux

Let $\mathbb{M} = \{1' < 1 < 2' < 2 < \dots\}$ and define $\mathsf{Set}(\mathbb{M})$ to be the set of finite, nonempty subsets of $\mathbb{M}$. For $S, T \in \mathsf{Set}(\mathbb{M})$ write $S \prec T$ if $\max(S) < \min(T)$ and $S \preceq T$ is $\max(S) \leq \min(T)$. Finally, for $S \in \mathsf{Set}(\mathbb{M})$ let $x^S = \prod_{i \in S} x_{\mathsf{unprime}(i)}$.

Fix a strict partition λ. A *set-valued shifted tableau* of shape λ is a filling T of the shifted diagram SD_λ by elements of $\mathsf{Set}(\mathbb{M})$. For such tableau T define $x^T = \prod_{(i,j) \in T} x^{T_{ij}}$ where T_{ij} is the entry of T in box (i,j). A set-valued shifted tableau T is *semistandard* if it has all of the following properties: (1) no unprimed number appears twice in the same row, (2) no primed number appears twice in the same columns, and (3) rows and columns are weakly increasing in the sense that $T_{ij} \preceq T_{i,j+1}$ and $T_{ij} \preceq T_{i+1,j}$ for all relevant positions.

Let $\mathsf{SetShTab}^+(\lambda)$ be the set of all semistandard set-valued shifted tableaux of shape λ, and let $\mathsf{SetShTab}^+_n(\lambda)$ be the subset with all entries at most n. Let $\mathsf{SetShTab}(\lambda) \subseteq \mathsf{SetShTab}^+(\lambda)$ and $\mathsf{SetShTab}_n(\lambda) \subseteq \mathsf{SetShTab}^+_n(\lambda)$ be the subsets of tableaux with no primed numbers in any diagonal boxes. The *K-theoretic Schur P- and Q-functions* of λ, as introduced by Ikeda and Naruse [7], are the power series

$$GP_\lambda = \sum_{T \in \mathsf{SetShTab}(\lambda)} x^T \in \mathbb{N}[\![x_1, x_2, \dots]\!] \quad\text{and}\quad GQ_\lambda = \sum_{T \in \mathsf{SetShTab}^+(\lambda)} x^T \in \mathbb{N}[\![x_1, x_2, \dots]\!].$$

Often the definitions of these power series involve a bookkeeping parameter β. Here, for simplicity, we have set $\beta = 1$.

Remark 8.1. It turns out that GP_λ and GQ_λ are both Schur positive symmetric functions, though of unbounded degree [17, Thms. 3.27 and 3.40]. Specializations of GP_λ and GQ_λ give equivariant K-theory representatives for Schubert varities in the maximal isotropic Grassmannians of orthogonal and symplectic types [7, Cor. 8.1]. These symmetric functions have a number of remarkable positivity properties; see [8, 11, 16].

A *distribution* of a tableau with set-valued entries is a tableau of the same shape formed by replacing every set-valued entry by one of its elements. A semistandard set-valued shifted tableau is just a set-valued tableau whose distributions are all semistandard shifted tableaux. Analogously, define a *(semistandard) set-valued decomposition tableau* of strict partition shape λ to be a set-valued shifted tableau whose distributions are each (semistandard) decomposition tableaux of shape λ. Let $\mathsf{SetDecTab}(\lambda)$ be the set of all such tableaux and let $\mathsf{SetDecTab}_n(\lambda)$ be the subset with all entries at most n.

Conjecture 8.2 (Cho–Ikeda [6])**.** It holds that $GP_\lambda = \sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$.

Remark 8.3. It would be natural to define $\mathsf{SetDecTab}^+(\lambda)$ as the set of set-valued shifted tableaux with entries from $\mathrm{Set}(\mathbb{M})$ whose distributions are each primed decomposition tableaux of shape λ. But in general $GQ_\lambda \neq \sum_{T \in \mathsf{SetDecTab}^+(\lambda)} x^T$ and it remains an open problem to find even a conjectural decomposition tableau formula for GQ-functions.

Crystals for $\mathfrak{gl}_n$ have been identified on unshifted (semistandard) set-valued tableaux (see, e.g., [5, 18, 20]), and it is a natural open problem to find similar structures on shifted tableaux. We have identified one such crystal structure on set-valued decomposition tableaux, which implies a weaker form of Conjecture 8.2.

Fix a strict partition λ and $T \in \mathsf{SetDecTab}(\lambda)$. The *reverse reading word* of T is the word $\mathsf{revrow}(T)$ formed by iterating over the boxes of T in the reverse reading word order (starting with the last box of the first row and proceeding row by row, reading each row right to left), and listing the entries of each box in decreasing order. Define $\mathrm{wt}(T) = \mathrm{wt}(\mathsf{revrow}(T))$. For example, if $T = \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 123 & 234 \\ \hline \end{array}$ then $\mathsf{revrow}(T) = 432\,321\,4\,3\,1$ and $\mathrm{wt}(T) = (2, 2, 3, 2)$.

Fix $i \in [n-1]$. Mark each i in $\mathsf{revrow}(T)$ by a right parenthesis ")" and each $i+1$ by a left parenthesis "(". A letter in $\mathsf{revrow}(T)$ is *i-unpaired* if it is equal to i or $i+1$ but does not belong to a matching pair of parentheses.

Definition 8.4. Given $i \in \mathbb{P}$ and a set-valued decomposition tableau T, construct $e_i(T)$ in the following way. Define $e_i(T) = 0$ if there are no i-unpaired letters equal to $i+1$. Otherwise, suppose the first i-unpaired $i+1$ in $\mathsf{revrow}(T)$ occurs in box (x,y) of T.

(a) Form $e_i(T)$ from T by changing the $i+1$ in box (x,y) to i if this yields a set-valued decomposition tableau. For example, $e_2 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 3 & 13 & 123 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 3 & 12 & 123 \\ \hline \end{array}$.

(b) Otherwise, some box (a,b) preceding (x,y) in the reverse row reading word order has $\{i, i+1\} \subseteq T_{ab}$. If (a,b) is the last such box, then form $e_i(T)$ by removing $i+1$ from T_{ab} and adding i to T_{xy}. One can show that the box (a,b) must either have $a = x$ and $b > y$, or $a = x-1$ and $b < y$, as in the examples

$$e_3 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 4 & 1 & 34 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 34 & 1 & 3 \\ \hline \end{array} \quad \text{and} \quad e_2 : \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 4 & 123 & 234 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 23 \\ \hline 4 & 12 & 234 \\ \hline \end{array}.$$

Theorem 8.5. For each strict partition λ with at most n parts, $\mathsf{SetDecTab}_n(\lambda)$ has a $\mathfrak{gl}_n$-crystal structure for the raising operators $e_1, e_2, \ldots, e_{n-1}$ given in Definition 8.4.

Corollary 8.6. The power series $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ is symmetric.

We can slightly extend this partial progress on Ikeda's conjecture. A power series $f \in \mathbb{Z}[\![x_1, x_2, \ldots]\!]$ satisfies the *K-theoretic Q-cancelation property* if for all $1 \leq i < j$ the power series $f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{j-1}, \frac{-t}{1+t}, x_{j+1}, \ldots)$ does not depend on t, that is, belongs to $\mathbb{Z}[\![x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots]\!]$. The symmetric functions satisfying the K-theoretic Q-cancelation property are exactly the ones that may be (uniquely) expressed as formal (i.e., possibly infinite) $\mathbb{Z}$-linear combinations of GP-functions [7, Prop. 3.4].

Proposition 8.7. The symmetric power series $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ has the K-theoretic Q-cancelation property and lowest degree term P_λ, so is equal to GP_λ plus a (possibly infinite) $\mathbb{Z}$-linear combination of GP_μ's with $|\mu| > |\lambda|$.

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Triangular (q, t)-Schröder Polynomials and Khovanov-Rozansky Homology

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Abstract. We define generalized Schröder polynomials $S_\lambda(q, t, a)$ for triangular partitions and prove that these polynomials recover the triangular (q, t)-Catalan polynomials of [2] at $a = 0$. Moreover, we show that the Poincaré polynomials of the reduced Khovanov-Rozansky homology of Coxeter knots of these partitions are given by $S_\lambda(q, t, a)$. Finally, combined with recent results in [8], we compute the Poincaré polynomial of the $(d, dnm + 1)$-cable of the (n, m)-torus knot, thus proving a special case of the Oblomkov-Rassmusen-Shende conjecture [16, 18] for generic unibranched planar curves with two Puiseux pairs.

1 Introduction

A fundamental pursuit in knot theory for the last century has been the classification of knots and links. One particularly effective method has been the study of certain homology theories that realize the knot as certain chain complexes, whose Poincaré polynomials, which compute the graded dimensions of the homology groups, are then used as knot invariants. One such especially celebrated homology theory is (reduced) *Khovanov-Rozansky homology*. This tri-graded theory associates to each link L a polynomial in three variable $P_L^{KR}(q, t, a)$. It turns out that computing these polynomials explicitly is very difficult, and the pursuit of a closed form for them has spurned a remarkable volume of deep and surprising results bridging combinatorics with low dimensional topology and algebraic geometry. One of the only cases where these polynomials are explicitly known is the case of *torus knots and links*, where by transforming these knots to certain binary sequences and defining a family of recursions, Hogancamp and Mellit were able to compute explicit solutions [11, 15]. The connections between these polynomials and (q, t)-Catalan combinatorics have been deeply established [5, 12, 10], with the $a = 0$

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specialization of $P_L^{KR}(q,t,a)$ for a torus knot recovering the *(q,t)-Catalan polynomial* and higher a powers equaling *(q,t)-Schröder polynomials*.

One of our first main results is the generalization of (q,t)-Schröder polynomials to the context of *triangular partitions*, which are defined as maximal partitions that fit under a line of arbitrary slope (i.e. certain Dyck paths under lines with non-integer intercepts.). These partitions were thoroughly studied by Bergeron and Mazin [1]. In particular, our triangular Schröder polynomial recovers at $a = 0$ the triangular (q,t)-Catalan polynomials studied in [1] that appeared in the generalized shuffle theorem under any line [2]. Our construction relies on certain recursions introduced by Gorsky-Mazin-Vazirani [9] using so called (m,n)*-invariant subsets*, which allow us to produce certain binary sequences from the triangular partitions and compute them using the recursions.

Recently, Oblomkov and Rozansky considered *Coxeter links*, which contain torus knots and links as special cases, and identified their homology with certain sections on the flag Hilbert scheme [17]. Even more recently, Galashin and Lam introduced a family of knots that arise directly from certain monotone paths on an $m \times n$ grid and proved that all *monotone links* are Coxeter. Thus, since the Gorsky-Mazin-Vazirani recursions agree with the Hogancamp-Mellit recursions, using the results above we prove that the Poincaré polynomial of the reduced Khovanov-Rozansky homology of Coxeter knots arising from triangular partitions is precisely our triangular Schröder polynomial.

The next natural family of links to try to understand are cabled torus knots. Informally, a *cabled knot* is a knot within a knot, so that the string that makes up the knot locally carries a smaller knot on it. A highly nontrivial and celebrated conjecture due to Oblomkov-Rasmussen-Shende, relates the Khovanov-Rozansky homology of algebraic links to certain plane curve singularities on the Hilbert scheme of points. Combining results in [8] with our previous results above, we compute the Poincaré polynomial for a certain family of cabled knots, proving a special case of the *Oblomkov-Rasmussen-Shende conjecture* for unibranched planar curves with two Puiseux pairs.

2 Background and Definitions

2.1 Recursions for the Poincaré Series of Link Homology

In [11] the third author and Anton Mellit introduced a recursive method for computing the Poincaré series of the reduced Khovanov-Rozansky homology of torus links. Given two finite binary sequences $\mathbf{u}$ and $\mathbf{v}$ with the same number of 1's, they introduced the power series $R_{\mathbf{u},\mathbf{v}}(q,t,a)$ via the following recursive relations:

$$R_{0\mathbf{u},0\mathbf{v}} = t^{-|\mathbf{u}|}R_{\mathbf{u}1,\mathbf{v}1} + qt^{-|\mathbf{u}|}R_{\mathbf{u}0,\mathbf{v}0}, \qquad R_{1\mathbf{u},0\mathbf{v}} = R_{\mathbf{u}1,\mathbf{v}}, \qquad R_{\emptyset,0^n} = \left(\frac{1+a}{1-q}\right)^n,$$

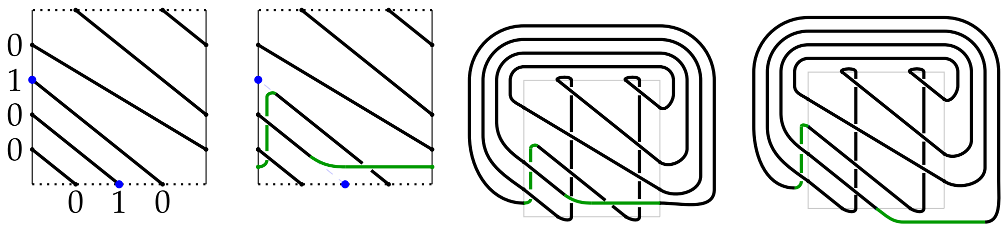


Figure 1: (Left) Steps 1-4 in constructing the link $L_{\mathbf{u},\mathbf{v}}$ for $\mathbf{u} = \mathbf{0010}$ and $\mathbf{v} = \mathbf{010}$. (Right) Two diagrams for the knot $L_{0010,010}$. Middle right: the closure of the diagram to its left (step 5). Far right: an equivalent closure considered in [11].

$$R_{1\mathbf{u},1\mathbf{v}} = (t^{|\mathbf{u}|} + a)R_{\mathbf{u},\mathbf{v}}, \qquad R_{0\mathbf{u},1\mathbf{v}} = R_{\mathbf{u},\mathbf{v}1}, \qquad R_{0^m,\emptyset} = \left(\frac{1+a}{1-q}\right)^m,$$

where $|\mathbf{u}|$ is the number of 1's in $\mathbf{u}$ and $R_{\emptyset,\emptyset} = 1$. Let $l(\mathbf{u})$ denote the length of $\mathbf{u}$.

Theorem 1 ([11])**.** *Let (n, m) be any positive integers. The Poincaré series of the reduced Khovanov-Rozansky homology of the (n, m)-torus link is given by*

$$P^{KR}_{L_{n,m}}(q,t,a) = (1-q)R_{0^n,0^m} = R_{0^{n-1}1,0^{m-1}1}.$$

Furthermore, it follows from their construction that for $|\mathbf{u}| = |\mathbf{v}| = 1$, the recurrence applied to $R_{\mathbf{u},\mathbf{v}}(q,t,a)$ will terminate and compute the Poincaré series of the reduced Khovanov-Rozansky homology for the link $L_{\mathbf{u},\mathbf{v}}$constructed as follows:
Step 1: Mark $\ell(\mathbf{v})$ points on the bottom edge and $\ell(\mathbf{u})$ points on the left edge of $[0,1]^2$, labeled with the sequences starting from the bottom left corner. Mark also the points on the top and right edges directly across from marked points labeled by 0.
Step 2: Starting with the lowest point on the left edge and leftmost point on the bottom edge, connect the dots with diagonal non-intersecting lines until all points are matched.
Step 3: Erase the tail of the line connected to the point labeled 1 on the bottom wall and connect it to right side of $[0,1]^2$, going *above* all other strands in the process.
Step 4: Erase the tail of the line connected to the point labeled 1 on the left wall, pass it *underneath* all other strands beneath it, and connect it once again to the left wall, but now in the first position, directly across the new marked point created in Step 3.
Step 5: Close the diagram by identifying the edges in the usual way for a torus.

Example 2. Consider $\mathbf{u} = \mathbf{0010}$ and $\mathbf{v} = \mathbf{010}$ with lengths $\ell(\mathbf{u}) = \mathbf{4}$, $\ell(\mathbf{v}) = \mathbf{3}$, and with $|\mathbf{u}| = |\mathbf{v}| = \mathbf{1}$. Steps 1 and 2 followed by 3 and 4 will yield the left two diagrams in Figure 1. It's closure, Step 5, is the third diagram. Iteratively applying the recurrence we see that the Poincare series of the KhR-homology of $L_{\mathbf{u},\mathbf{v}}$ is then equal to:

$$\begin{aligned}R_{0010,010} &= t^{-1}R_{0101,101} + qt^{-1}R_{0100,100} = \dots \\ &= t^{-1}(t+a)(a)R_{\emptyset,\emptyset} + (qt^{-1})^2(a)R_{\emptyset,\emptyset} + (qt^{-1})(t^{-1})(t+a)(a)R_{0,\emptyset} \\ &= a(t^{-1}(t+a) + (qt^{-1})^2) + (qt^{-1})(t^{-1})(t+a)(a)\frac{(1+a)}{1-q}.\end{aligned}$$

2.2 Invariant Subsets and Dyck Paths

Given positive integers m and n, an (n,m)-***Dyck path*** is a lattice path from $(m,0)$ to $(0,n)$ consisting exclusively of north and west steps that stays weakly below the diagonal line $y = n - \frac{n}{m}x$. Indexing each cell by its top right lattice point, for any such choice of (n,m) we define an ***Anderson filling*** on each of the cells of the lattice via the function $\gamma : \mathbb{Z}^2 \to \mathbb{Z}$ by $\gamma(x,y) = mn - nx - my$.

Definition 3. A subset $\Delta \subset \mathbb{Z}_{\geq 0}$ is called (n,m)-***invariant*** if $\Delta + n \subset \Delta$ and $\Delta + m \subset \Delta$. Let $I_{n,m}$ denote the set of all (n,m)-invariant subsets. In addition, an (n,m)-invariant subset Δ is called 0-***normalized*** if $0 \in \Delta$. We will use the notation $I^0_{n,m}$ for the set of 0-normalized (n,m)-invariant subsets.

If n and m are relatively prime then the set of (n,m)-Dyck paths is in a natural bijection with the set of 0-normalized (n,m)-invariant subsets. Namely, given an (n,m)-Dyck path D let $\mathrm{Gaps}(D)$ be the set of positive Anderson labels corresponding to the cells above D (positivity of a label is equivalent to the cell fitting under the diagonal). The corresponding 0-normalized (n,m)-invariant subset is given by $\Delta(D) = \mathbb{Z}_{\geq 0} \setminus \mathrm{Gaps}(D)$. It is not hard to see that this defines a bijection.

The rational (q,t)-Catalan polynomials and the rational Schröder polynomials are usually defined in terms of the (n,m)-Dyck and Schröder paths. However, it is more suitable for us to follow [6, 7] and define these polynomials in terms of the invariant subsets. The two approaches are equivalent due to the bijection described above. To do so, we need the following statistics. Let $\Delta \in I_{n,m}$ and define:

- the ***area*** to be the number of gaps in Δ, $\mathrm{area}(\Delta) := \sharp(\mathbb{Z}_{\geq 0} \setminus \Delta)$.
- the n-***generators*** of Δ as the set $\mathtt{ngen}(\Delta) := \Delta \setminus (\Delta + n) = \{a \in \Delta : a - n \notin \Delta\}$.
- the ***codinv*** as the number of gaps in length m intervals beginning at n-generators:

$$\mathrm{codinv}(\Delta) := \sum_{a \in \mathtt{ngen}(\Delta)} \sharp\{a \leq g < a + m : g \notin \Delta\}, \tag{2.1}$$

$$\mathrm{dinv}(\Delta) := \delta(n,m) - \mathrm{codinv}(\Delta), \quad \delta(n,m) := \frac{(n-1)(m-1)}{2}.$$

Definition 4. For each coprime pair (m,n), the ***rational*** (q,t)***-Catalan polynomial***, denote $C_{n,m}(q,t)$, is given by:

$$C_{n,m}(q,t) := \sum_{\Delta\in I^0_{n,m}} q^{\operatorname{area}(\Delta)} t^{\operatorname{dinv}(\Delta)} = (1-q) \sum_{\Delta\in I_{n,m}} q^{\operatorname{area}(\Delta)} t^{\operatorname{dinv}(\Delta)}.$$

In order to define Schröder polynomials, we will need a couple more ingredients.

- Let $\texttt{Cogen}(\Delta) := \{a \in \mathbb{Z} : a \notin \Delta, a+n \in \Delta, a+m \in \Delta\}$ be the set of ***double cogenerators*** of Δ.
- Let $k \in \mathbb{Z}$. Set $\lambda_k(\Delta) := \sharp\{a \in \texttt{ngen}(\Delta) : k+n+1 \leq a \leq k+n+m\}$.

Definition 5. For each coprime pair (m,n), the ***Schröder polynomial*** $S_{n,m}(q,t,a)$ is:

$$S_{n,m}(q,t,a) := \sum_{\Delta\in I^0_{n,m}} q^{\operatorname{area}(\Delta)} t^{\operatorname{dinv}(\Delta)} \prod_{k\in\texttt{Cogen}(\Delta)} \left(1 + at^{-\lambda_k(\Delta)}\right).$$

Example 6. In Figure 2, the bijection between $(3,4)$-Dyck paths and the 0-normalized $(3,4)$-invariant subsets is illustrated, complemented with a computation of the area and dinv statistics, as well as the factors necessary for the Schröder polynomial, for two out of five invariant subsets in $I^0_{3,4}$. The rest are computed similarly. Summing all together, one obtains the Schröder polynomial:

$$\begin{aligned} S_{3,4}(q,t,a) =& t^3(1+a)(1+at^{-1})(1+at^{-2}) + qt^2(1+a)(1+at^{-1}) + qt(1+a)(1+at^{-1}) \\ &+ q^2t(1+a)(1+at^{-1}) + q^3(1+a) \\ =& q^3 + q^2t + qt^2 + t^3 + qt + a(q^3 + q^2t + qt^2 + t^3 + q^2 + 2qt + t^2 + q + t) \\ &+ a^2(q^2 + qt + t^2 + q + t + 1) + a^3. \end{aligned}$$

2.3 Recursions for Invariant Subsets

In [9] the fourth author together with Gorsky and Vazirani introduced a recursion computing the rational (q,t)-Catalan series and showed that their recursion is equivalent to the Hogancamp-Mellit recursion in the case of the torus link. Hence, in the relatively prime case the Gorsky-Mazin-Vazirani recursion recovers the (q,t)-Catalan polynomials.

Let (m,n) be a pair of positive relatively prime integers. In order to define the recursion, one needs to consider subfamilies in the set of invariant subsets $I_{n,m}$ given by fixing the intersection of the subsets with the interval $[0, n+m-1]$. Let $\mathbf{w} \in \{0,1\}^{n+m}$ be a binary sequence of length $n+m$.

Definition 7. Set $I_{\mathbf{w}} := \{\Delta \in I_{n,m} : \forall 0 \leq i < n+m, i \in \Delta \Leftrightarrow w_i = 1\}$ and define:

$$P_{\mathbf{w}}(q,t,a) := \sum_{\Delta\in I_{\mathbf{w}}} q^{\operatorname{area}(\Delta)} t^{\operatorname{codinv}(\Delta)} \prod_{k\in\texttt{Cogen}(\Delta)\cap\mathbb{Z}_{\geq 0}} (1 + at^{\lambda_k(\Delta)}). \tag{2.2}$$

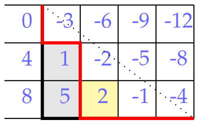
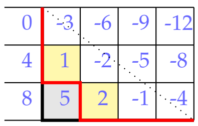
	Gaps $=\{2\}$ $\Delta=\mathbb{Z}_{\geq 0}\setminus\{2\}$ $\mathrm{area}(\Delta)=1$	3-gen $=\{0,1,5\}$ $\mathrm{codinv}(\Delta)=2$ $\mathrm{dinv}(\Delta)=1$	$\mathtt{Cogen}=\{-3,2\}$ $(1+a)(1+at^{-1})$
	Gaps $=\{1,2\}$ $\Delta=\mathbb{Z}_{\geq 0}\setminus\{1,2\}$ $\mathrm{area}(\Delta)=2$	3-gen $=\{0,4,5\}$ $\mathrm{codinv}(\Delta)=2$ $\mathrm{dinv}(\Delta)=1$	$\mathtt{Cogen}=\{1,2\}$ $(1+a)(1+at^{-1})$

Figure 2: Three out of five $(3,4)$-Dyck paths are on the left, with the cells corresponding to the gaps in yellow. The corresponding $(3,4)$-invariant subsets are in the second column, together with the area, 3-generators, codinv, and dinv in the third column, and the corresponding Schröder factor is in the fourth.

Then, the Schröder polynomial can be obtained from (2.2),

$$S_{n,m}(q,t,a)=\frac{(1-q)t^{\delta(n,m)}}{q^{n+m}}P_{0^{n+m}}(q,t^{-1},a)=\frac{t^{\delta(n,m)}}{q^{n+m-1}}P_{0^{n+m-1}1}(q,t^{-1},a).$$

The polynomials $P_{\mathbf{w}}$ satisfy a recursion, however, in order to match this recursion to the Hogancamp-Mellit recursion, certain adjustments are required.

First, we need to replace the sequence $\mathbf{w}$ of length $n+m$ by two sequences $(\mathbf{x},\mathbf{y})$ in the alphabet $\{0,\bullet,1\}$ of lengths m and n respectively. The sequence $\mathbf{x}$ records gaps (encoded by 0), n-generators (encoded by 1), and the rest of the elements of Δ (encoded by $\bullet$) on the interval $[n,n+m-1]$. Similarly, the sequence $\mathbf{y}$ records gaps, m-generators, and the rest of the elements of Δ on the interval $[m,n+m-1]$. In other words,

Definition 8. Let $\mathbf{x},\mathbf{y}$ be sequences as above. Let $I_{\mathbf{x},\mathbf{y}}$ be the set of $\Delta\in I_{n,m}$ such that:

$$\forall 0\leq k<m\begin{cases}x_k=0\Leftrightarrow k+n\notin\Delta,\\ x_k=1\Leftrightarrow k+n\in\mathtt{ngen}(\Delta),\\ x_k=\bullet\Leftrightarrow k\in\Delta,\end{cases}\quad\forall 0\leq k<n\begin{cases}y_k=0\Leftrightarrow k+m\notin\Delta,\\ y_k=1\Leftrightarrow k+m\in\mathtt{mgen}(\Delta),\\ y_k=\bullet\Leftrightarrow k\in\Delta.\end{cases}$$

Example 9. Let $(n,m)=(4,7)$ and $\Delta=\mathbb{Z}_{\geq 0}\setminus\{0,1,2,3,4,6,7,8,10\}$. Then, the associated binary sequence $w=00000100010$ yields the ternary sequences $\mathbf{x}=(01000\bullet 0)$ and $\mathbf{y}=(0010)$, since the only 4-generator in $[4,10]$ is 5 and the only 7-generator in $[7,10]$ is 9. In particular, 9 is not a 4-generator since $9-4=5\in\Delta$. Thus, $\Delta\in I_{00000100010}=I_{01000\bullet 0,0010}$.

The statistics on $I_{n,m}$ are modified as follows. Set:

$$\mathrm{area}'(\Delta)=\sharp\{k\in\mathbb{Z}_{\geq n+m}:k\notin\Delta\}=\sharp\left(\mathrm{Gaps}(\Delta)\cap\mathbb{Z}_{\geq n+m}\right),$$

$$\mathrm{codinv}'(\Delta)=\sum_{a\in\mathtt{ngen}(\Delta)}\sharp\{k\in\mathbb{Z}_{\geq n+m}:a\leq k<a+m,\ k\notin\Delta\}-\frac{\lambda(\Delta)(\lambda(\Delta)-1)}{2},$$

where $\lambda(\Delta) := \lambda_{-1}(\Delta) = \sharp\{a \in \mathtt{ngen}(\Delta) : n \leq a < n+m\}$.

Definition 10. Given sequences $\mathbf{x}$ and $\mathbf{y}$ as above, let

$$Q_{\mathbf{x},\mathbf{y}}(q,t,a) := \sum_{\Delta \in I_{\mathbf{x},\mathbf{y}}} q^{\mathrm{area}'(\Delta)} t^{-\mathrm{codinv}'(\Delta)} \prod_{k \in \mathtt{Cogen}(\Delta) \cap \mathbb{Z}_{\geq 0}} (1 + at^{-\lambda_k(\Delta)}).$$

Note that for any $\Delta \in I_{0^{n+m}}$ one gets $\mathrm{area}'(\Delta) = -n - m + \mathrm{area}(\Delta)$, $\mathrm{codinv}'(\Delta) = \mathrm{codinv}(\Delta)$, and all the double co-generators are non-negative. Therefore,

$$Q_{0^m,0^n}(q,t,a) = q^{-n-m} P_{0^{n+m}}(q,t^{-1},a).$$

Theorem 11 ([9])**.** *The following recursions hold:*

$$\begin{aligned} &Q_{0\mathbf{u},0\mathbf{v}} = t^{-|\mathbf{u}|} Q_{\mathbf{u}1,\mathbf{v}1} + qt^{-|\mathbf{u}|} Q_{\mathbf{u}0,\mathbf{v}0}, && Q_{1\mathbf{u},0\mathbf{v}} = Q_{\mathbf{u}1,\mathbf{v}\bullet}, && Q_{\bullet\mathbf{u},\bullet\mathbf{v}} = Q_{\mathbf{u}\bullet,\mathbf{v}\bullet}, \\ &Q_{1\mathbf{u},1\mathbf{v}} = (t^{|\mathbf{u}|} + a) Q_{\mathbf{u}\bullet,\mathbf{v}\bullet}, && Q_{0\mathbf{u},1\mathbf{v}} = Q_{\mathbf{u}\bullet,\mathbf{v}1}, && Q_{\varnothing,\varnothing} = 1. \end{aligned}$$

Finally, notice that in the recursion for Q one can completely ignore all the $\bullet$'s. Also, it follows that the recursion always terminates in $Q_{\varnothing,\varnothing}$, so one doesn't need the normalization conditions for $Q_{\varnothing,0^n}$ and $Q_{0^m,\varnothing}$.

Theorem 12 ([9])**.** *Let $(\mathbf{u},\mathbf{v})$ be the sequences obtained from the sequences $(\mathbf{x},\mathbf{y})$ by ignoring all $\bullet$'s. Then*

$$R_{\mathbf{u},\mathbf{v}}(q,t,a) = Q_{\mathbf{x},\mathbf{y}}(q,t,a).$$

Corollary 13 ([15, 9])**.** *The Poincaré polynomial of the reduced Khovanov-Rozansky homology of the (n,m)-torus knot is given by*

$$R_{0^{m-1}1,0^{n-1}1}(q,t,a) = Q_{0^{m-1}1,0^{n-1}1}(q,t,a) = \frac{P_{0^{n+m-1}1}(q,t^{-1},a)}{q^{n+m-1}} = \frac{S_{n,m}(q,t,a)}{t^{\delta(n,m)}}.$$

Remark 14. *The last formula was first proven by Anton Mellit in [15]. We follow notations from [11] and [9], where the result was generalized to torus links.*

2.4 Monotone and Coxeter Links

In [4] Galashin-Lam study a family of links, called *monotone* that arise from certain curves on the plane. They define a new invariant, the *elliptic Hall algebra superpolynomial*, which they prove recovers the HOMFLY polynomial of L_C and conjecture agrees with the Poincaré series of the Khovanov-Rozansky homology of any algebraic link L_C.

Definition 15. Let C denote a curve from $(0,n)$ to $(m,0)$. A ***monotone link*** L_C is a projection onto $\mathbb{R}^2/\mathbb{Z}^2$ of a curve C such that the x- and y-coordinates of C are monotone increasing and decreasing, respectively[1]. Trace the projection of C starting from the left top corner, crossing the earlier strand on top.

[1]This differs slightly from the definition in [4] by a flip sending $x \mapsto -x$.

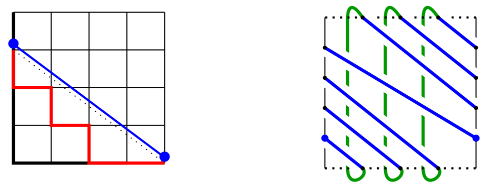


Figure 3: The curve C on the left and its projection onto $[0,1]^2$ and annular closure β_C, on the right. The associated triangular partition is displayed in red, so that $\mu = (3,2,1,0,0)$ with $\mathbf{b} = (0,1,1,1)$ and $\mathbf{e} = (1,1,1)$. It is straightforward to verify that the braid $\beta^{cox}_{\mathbf{b},\mathbf{e}} = JM_2^1 JM_3^1 JM_4^1 \sigma_1\sigma_2\sigma_3$ is isotopic to β_C.

It is well known [11] that if C is the straight diagonal line, then L_C is the (m,n) torus link, with the special case of m,n relatively prime yielding a knot.

Let $\mathbb{A}$ denote the annulus. Given a curve C, we can construct its annular closure $\beta_C \in \mathbb{A} \times [0,1]$ as follows. Consider the projection of C onto $[0,1]^2$. Now, identify the top and bottom boundaries so that for each point $x \in (0,1)$ for which $(x,0)$ and $(x,1)$ are in C, the line connecting them lies *underneath* all other strands. Denote the resulting braid in $\mathbb{A} \times [0,1]$ by β_C (see Figure 3).

Denote by $\sigma_i \in B_n$, with B_n the braid group, the positive crossing of the i^{th} and $i+1^{st}$ strands, i.e. the i^{th} strand is above the $i+1^{st}$ strand.

Definition 16. Given sequences $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}^m_{\geq 0}$ and $\mathbf{e} = (\epsilon_1, \ldots, \epsilon_{m-1}) \in \{0,1\}^{m-1}$, the ***Coxeter braid*** $\beta^{cox}_{\mathbf{b},\mathbf{e}}$ is given by:

$$\beta^{cox}_{\mathbf{b},\mathbf{e}} := JM_1^{b_1} \ldots JM_m^{b_m} \sigma_1^{\epsilon_1} \ldots \sigma_{m-1}^{\epsilon_{m-1}}, \tag{2.3}$$

where $JM_i := \sigma_i \ldots \sigma_{m-1}\sigma_{m-1} \ldots \sigma_i$ for each $1 \leq i \leq m$.

To any curve C, we can assign a Coxeter braid β^{cox}_C in the following way. Let $\mu = (n, \mu_1, \ldots, \mu_m)$ be such that $(\mu_1, \ldots, \mu_m)$ is the transpose of the triangular partition corresponding to the curve C. Set $b_{m-i+1} = \mu_{i-1} - \mu_i$ (with $\mu_0 = n$) and for each $1 \leq i \leq m-1$, set $\epsilon_i = 0$ if C passes through the lattice point (i,j) for some $j \in \mathbb{Z}$, with $\epsilon_i = 1$ otherwise. Then for $\mathbf{e}_C = (\epsilon_1, \ldots, \epsilon_{m-1})$ and $\mathbf{b_C} = (b_1 \ldots, b_m)$, let $\beta^{cox}_C := \beta^{cox}_{\mathbf{b}_C,\mathbf{e}_C}$.

Theorem 17 ([4])**.** *The braid β^{cox}_C is conjugate to the annular closure β_C of C. In particular, all monotone links in $\mathbb{A} \times [0,1]$ are Coxeter links, and all Coxeter links arise in this way.*

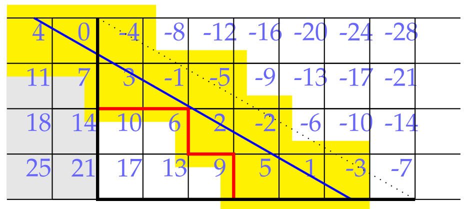


Figure 4: The triangular partition $\lambda = (2,3) = \tau_{3.2,5.6}$ with $(m,n) = (4,7)$ and the Anderson labels denoted in blue.

3 Main Results

3.1 Schröder Polynomials for Triangular Partitions

Definition 18. A partition λ is called ***triangular*** if there exist two not necessarily integral points $(0,s)$ and $(r,0)$ such that λ consists of all the boxes below the line connecting these points, in which case we denote λ by $\tau_{r,s}$.

Evidently, for any given triangular partition λ, there exist many choices for positive real numbers r and s such that $\lambda = \tau_{r,s}$ (see [1] for details). In particular, one can always choose r and s in such a way that $\lambda = \tau_{r,s}$, and $r/s = n/m$, where (n,m) are positive relatively prime integers with $r \leq n$ (equivalently, the line connecting $(r,0)$ to $(0,s)$ is below the line connecting $(n,0)$ to $(0,m)$). Generalized (q,t)-Catalan polynomials corresponding to triangular partitions appeared in the generalized shuffle theorem [2].

We claim that to any triangular partition $\tau_{r,s}$ one can associate a pair of binary sequences $\mathbf{u}(s,t)$ and $\mathbf{v}(s,t)$ [3], which we explain how to construct in Example 20 below. With this in hand, we extend the Schröder polynomial to the triangular setting.

Definition 19. Let $\lambda = \tau_{r,s}$ and (n,m) be as above with associated sequences $\mathbf{u}(r,s)$ and $\mathbf{v}(r,s)$ as in Example 20. The ***(q,t)-Schröder polynomial*** for triangular partitions is defined as $S_\lambda(q,t,a) := t^{|\lambda|} R_{\mathbf{u}(r,s),\mathbf{v}(r,s)}(q,t,a)$.

Example 20. Let $\lambda = \tau_{3.2,5.6} = (2,3)$ as in Figure 4 and observe that the line connecting $(0,3.2)$ and $(5.6,0)$ has the same slope as the diagonal line connecting $(0,4)$ and $(7,0)$. We call the line connecting $(0,3.2)$ and $(5.6,0)$ the ***shifted diagonal***, with $(n,m) = (4,7)$ denoting the closest line above it with equal slope and integer x and y -intercepts.

Let $W = \{-5,-4\ldots,5\} = [-5,5]$ be the window of labels of all cells intersected by the shifted diagonal (shaded yellow in Figure 4). The subdiagrams of λ are in bijection with the subfamily $I^0_{3.2,5.6} \subset I^0_{4,7}$ consisting of subsets Δ, such that $\{1,2,3,5\} \cap \Delta = \emptyset$, where $\{1,2,3,5\} = \mathrm{Gaps}(\lambda)$. This is equivalent to saying that $\Delta \cap W = \{0,4\}$. Hence,

$$\begin{aligned}I^0_{3.2,5.6} =& \{\Delta \in I^0_{4,7} : \{1,2,3,5\} \cap \Delta = \emptyset\}\\ =& \{\Delta \in I^0_{4,7} : \Delta \cap W = \{0,4\}\} = \{\Delta \in I^0_{4,7} : \Delta + 5 \in I_{00000100010}\}.\end{aligned}$$

$\mathrm{Gaps} \cap \mathbb{Z}_{\geq 6} = \{6,9,13\}$ area $= 3$	4-gen $= \{0,7,10,17\}$ codinv $= 4$, dinv $= 1$	$\mathtt{Cogen} = \{3,13\}$ $(1+a)(1+at^{-1})$
$\mathrm{Gaps} \cap \mathbb{Z}_{\geq 6} = \{6,9\}$ area $= 2$	4-gen $= \{0,7,10,13\}$ codinv $= 2$, dinv $= 3$	$\mathtt{Cogen} = \{3,6,9\}$ $(1+a)(1+at^{-1})(1+at^{-2})$
$\mathrm{Gaps} \cap \mathbb{Z}_{\geq 6} = \{6,10\}$ area $= 2$	4-gen $= \{0,7,9,14\}$ codinv $= 3$, dinv $= 2$	$\mathtt{Cogen} = \{5,10\}$ $(1+a)(1+at^{-1})$

Figure 5: For three of the nine $\Delta \in I^0_{3.2,5.6}$ we record the gaps that are greater than 5, since only those contribute to area and codinv (respectively, area$'$ and codinv$'$ on $I_{01000\bullet 0,0010}$). There cannot be any double co-generators below the interval $W = [0, n+m-1] - 5$, therefore all co-generators are used for the Schröder factors.

That is, the family $I^0_{3.2,5.6}$ is simply $I_{00000100010} = I_{01000\bullet 0,0010}$ from Example 9 shifted down by 5. So setting $\mathbf{u}(3.2,5.6) = 010000$ and $\mathbf{v}(3.2,5.6) = 0010$, we obtain $S_\lambda(q,t,a) = t^5 R_{010000,0010}(q,t,a)$. In Figure 5 we illustrate the computation of the contributions towards $S_\lambda(q,t,a)$ of three of the nine invariant subsets in $I^0_{3.2,5.6}$. All together:

$$\begin{aligned} S_\lambda(q,t,a) =& q^5(1+a) + q^4 t(1+a)(1+at^{-1}) + q^3t^2(1+a)(1+at^{-1}) \\ &+ q^3t(1+a)(1+at^{-1}) + q^2t^3(1+a)(1+at^{-1})(1+at^{-2}) \\ &+ q^2t^2(1+a)(1+at^{-1}) + qt^3(1+a)(1+at^{-1}) \\ &+ qt^4(1+a)(1+at^{-1})(1+at^{-2}) + t^5(1+a)(1+at^{-1})(1+at^{-2}). \end{aligned}$$

Note that plugging in $a = 0$ we recover the corresponding (q,t)-Catalan polynomial:

$$C_\lambda(q,t) = S_\lambda(q,t,0) = q^5 + q^4t + q^3t^2 + q^2t^3 + qt^4 + t^5 + q^3t + q^2t^2 + qt^3.$$

With the definition of the Schröder polynomial established for any triangular partition, we can now state our first main theorem.

Theorem 21. [3] *The triangular Schröder polynomial $S_\lambda(q,t,a)$ at $a = 0$ specializes to the triangular (q,t)-Catalan polynomial of* [1] *and* [2]*. Hence, for $\lambda = \tau_{r,s}$ as before, we obtain that*

$$C_\lambda(q,t) = S_\lambda(q,t,0) = t^{|\lambda|} R_{\mathbf{u}(r,s),\mathbf{v}(r,s)}(q,t,0).$$

By construction, the polynomial $S_\lambda(q,t,a)$ depends on a choice of a shifted diagonal. At $a = 0$ this corresponds to choosing an appropriate slope in the definition of the dinv statistic [1, 2]. The Catalan polynomial doesn't depend on that choice (this follows from the shuffle theorem of [2], see also [13, 14]). The shuffle theorem argument can be generalized to show that the full Schröder polynomial also depends only on the partition. Nonetheless, this will also follow from our results below.

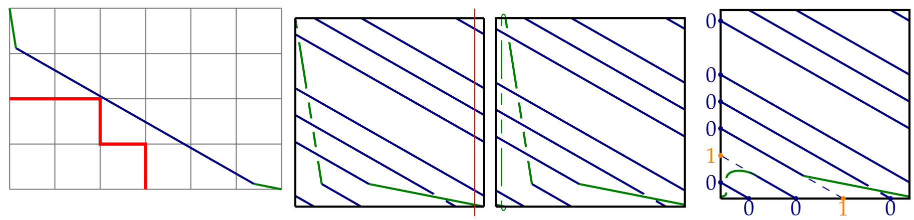


Figure 6: Left: $\lambda = (3,2)$ together with the monotone curve C obtained by augmenting the shifted diagonal. Second: the monotone knot $K_{r,s}$ drawn on a torus. We cut a vertical strip on the right (the red line) and reattach it on the left for the third picture. We also close the vertical green strand. Finally, we pull the green strand from under the blue ones to obtain the picture on the right, which is the knot $L_{010000,0010}$.

3.2 The Monotone Knot of a Triangular Partition

Let $\lambda = \tau_{r,s}$ be a triangular partition, and (r,s) be as in the previous section: $r/s = n/m$, where $n, m \in \mathbf{Z}_{>0}$ are relatively prime and $r \le n$. The monotone curve C from $(0, \lceil r \rceil)$ to $(\lceil s \rceil, 0)$ is constructed by augmenting the shifted diagonal connecting $(0,r)$ to $(s,0)$ by adding an almost vertical segment at the top and an almost horizontal segment at the bottom (see Figure 6). Let $K_{r,s}$ be the closure of the corresponding monotone braid β_C (see Section 2.4). It follows from [4, Prop. 7.5] that $K_{r,s}$ is isotopic to the closure of the Coxeter braid β_C^{cox}, which only depends on the partition λ and not on the choice of the shifted diagonal. We will call it the ***Coxeter knot*** of the partition λ and denote it K_λ.

Theorem 22 ([3])**.** *The monotone knot $K_{r,s}$ is isotopic to the knot $L_{\mathbf{u}(r,s),\mathbf{v}(r,s)}$ (see Section 2.1 for a definition). In particular, the Poincaré polynomial of the reduced Khovanov-Rozansky homology of the Coxeter knot K_λ is given by $P^{KR}_{K_\lambda}(q,t,a) = R_{\mathbf{u}(r,s),\mathbf{v}(r,s)}(q,t,a) = t^{-|\lambda|}S_\lambda(q,t,a)$.*

Remark 23. *Theorem 22 implies that the Schröder polynomial $S_\lambda(q,t,a)$ does not depend on the choice of the shifted diagonal $(0,r)-(s,0)$, but only on the triangular partition $\lambda = \tau_{r,s}$.*

Example 24. In Figure 6 we illustrate the construction of the monotone knot $K_{r,s}$ and its isotopy to $L_{\mathbf{u}(r,s),\mathbf{v}(r,s)}$ for $(r,s) = (3.2, 5.6)$, continuing Example 20.

In the case when $(r,s) = (dn, dm)$, where d, n, m are integers and n and m are relatively prime, Galashin and Lam in [4, Lem. 8.1] proved that the monotone knot $K_{\tau_{dn,dm}}$ is the $(d, dnm+1)$-cable of the (n,m)-torus knot, which is an algebraic knot: it can be obtained as the intersection of the planar curve $(x = t^{dn}, y = t^{dm} + t^{dm+1})$ with a small 3D sphere around the origin in $\mathbb{C}^2$. Such curves were studied by the fourth author, Gorsky, and Oblomkov in [8], were they showed that the Poincaré polynomial $P_{\overline{JC}}(t)$ of the Compactified Jacobian of such a curve is a specialization of the (dn, dm) (q,t)-Catalan polynomial. Combining this with our Theorem 22, we obtain

$$P_{\overline{JC}}(t) = t^{2\delta} C_{nd,md}(1, t^{-2}) = t^{2\delta} S_{\tau_{dn,dm}}(1, t^{-2}, 0) = P^{KR}_{K_{\tau_{dn,dm}}}(1, t^{-2}, 0),$$

which is a special case of the Oblomkov-Rasmussen-Shende conjecture for such curves.

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qtRSK*: A probabilistic dual RSK correspondence for Macdonald polynomials

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Abstract. We introduce a probabilistic generalization of the dual Robinson–Schensted–Knuth correspondence, called qtRSK*, depending on two parameters q and t. This correspondence extends the qRSt correspondence, recently introduced by the authors, and allows the first tableaux-theoretic proof of the dual Cauchy identity for Macdonald polynomials. By specializing q and t, one recovers the row and column insertion version of the classical dual RSK correspondence as well as of q- and t-deformations thereof which are connected to q-Whittaker and Hall–Littlewood polynomials, but also a novel correspondence for Jack polynomials.

Keywords: dual RSK, growth diagrams, Macdonald polynomials, Jack polynomials

1 Introduction

The Robinson–Schensted–Knuth (RSK) correspondence is a bijection between matrices of nonnegative integers with finite support and pairs of semistandard Young tableaux of the same shape and has significant applications in combinatorics, representation theory, probability theory and algebraic geometry. It was introduced by Knuth [8] and generalizes the Robinson–Schensted correspondence (RS) introduced by Robinson [14] for permutations and independently by Schensted [15] for words. A closely related bijection is the dual RSK correspondence (RSK*) introduced by Knuth [8] which yields a bijective proof of the dual Cauchy identity

$$\sum_{\lambda} s_\lambda(\mathbf{x}) s_{\lambda'}(\mathbf{y}) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (1 + x_i y_j), \tag{1.1}$$

where the sum is over all partitions λ, s_λ denotes the Schur polynomial in the variables $\mathbf{x} = (x_1, \dots, x_m)$ or $\mathbf{y} = (y_1, \dots, y_n)$ respectively, and λ' is the conjugate of λ.

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All of the above mentioned correspondences have been extended in various directions throughout the last few decades. Among others several randomized generalizations of RS, RSK and RSK* were introduced in [2, 3, 4, 10, 11, 12, 13]. These generalizations associate to each permutation or nonnegative integer matrix respectively a distribution on pairs of (dual) (semi)standard Young tableaux depending on a parameter q or t and thereby giving a proof of the (dual) Cauchy identity for q-Whittaker or Hall–Littlewood symmetric functions. Similar to the classical RSK algorithm, these randomized generalizations have many applications to probabilistic models, compare for example with [2, 3, 4, 10].

In a previous paper [1], the authors introduced a randomized generalization for RS called qRSt depending on two parameters q and t. This generalization was designed to prove the squarefree part of the Cauchy identity for the Macdonald symmetric functions $P_\lambda(\mathbf{x}; q, t)$ and $Q_\lambda(\mathbf{x}; q, t)$. Analogously to Macdonald symmetric functions, which specialize to q-Whittaker, Hall–Littlewood and Schur symmetric polynomials, the qRSt correspondence specializes to the corresponding randomized variations of RS, and to the row and column insertion versions of RS itself for $q = t = 0$ or $q = t \to \infty$ respectively.

In this abstract we present a unifying generalization of both qRSt and RSK*, called qtRSK*, and thereby give the first tableaux-theoretic proof of the dual Cauchy identity for Macdonald polynomials

$$\sum_\lambda P_\lambda(\mathbf{x}; q, t) P_{\lambda'}(\mathbf{y}; t, q) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (1 + x_i y_j). \tag{1.2}$$

Our map specializes to known randomized generalizations of RSK* by specializing q or t respectively, and to (q, t)-variations of RS for words or RS, i.e., qRSt, by restricting the input matrices. In particular we obtain a novel correspondence for Jack polynomials with an intriguing property when restricting to words.

This extended abstract is organized as follows. In §2 we present the notion of an insertion algorithm by using local growth rules. In §3 we review Macdonald polynomials and introduce (q, t)-analogue of up and dual down operators. In §4 we define the forward and backward probabilities which are the building block of qtRSK* which is introduced in §5. In §6 we discuss the properties of qtRSK*. For further details, including the proofs, we refer the reader to our paper [7].

Notation

We assume the reader is familiar with (skew) Young diagrams, semistandard Young tableaux (abbreviated SSYT), and Schur polynomials, as defined, e.g., in [16, Ch. 7]. We draw Young diagrams in French notation and starting from §4 in *Quebecois notation*, in which the boxes are right-justified instead of left-justified. We write SSYT(λ) (resp.,

$\mathrm{SSYT}^*(\lambda)$) for the set of SSYTs (resp., dual SSYTs) of shape λ, where a dual SSYT is a filling of the cells of λ with strictly increasing rows and weakly increasing columns. If T is a (dual) SSYT, we denote by $T^{(i)}$ the shape of the subtableau consisting of entries at most i.

2 Insertion algorithms and local dual growth rules

Young's lattice is the partial order $(\mathbb{Y}, \subseteq)$ on partitions defined by the inclusion of Young diagrams; its meet and join are given by $\cap$ and $\cup$, respectively. We say that λ/μ is a *horizontal strip* (resp., vertical strip) if no two cells of λ/μ are in the same column (resp., row), where we use the notation $\mu \prec \lambda$ (resp., $\mu \prec' \lambda$). We define the *up operator* U_x and *dual down operator* D^*_y as $\mathbb{Q}(x,y)$-linear maps on the $\mathbb{Q}(x,y)$-vector space $\mathbb{Q}(x,y)\mathbb{Y}$ with basis $\mathbb{Y}$ via

$$U_x\lambda = \sum_{\nu \succ \lambda} x^{|\nu/\lambda|}\nu, \qquad D^*_y\lambda = \sum_{\mu \prec' \lambda} y^{|\lambda/\mu|}\mu.$$

The up and dual down operator satisfy the commutation relation

$$D^*_y U_x = (1+xy)U_x D^*_y. \tag{2.1}$$

The commutation relation immediately implies the dual Cauchy identity (1.1). Indeed by rewriting the Schur polynomials as

$$s_\lambda(\mathbf{x}) = \langle U_{x_m} \cdots U_{x_1}\varnothing, \lambda\rangle, \qquad s_{\lambda'}(\mathbf{y}) = \left\langle D^*_{y_1} \cdots D^*_{y_n}\lambda, \varnothing\right\rangle, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined by $\langle \lambda, \mu \rangle = \delta_{\lambda,\mu}$, for all $\lambda, \mu \in \mathbb{Y}$, the dual Cauchy identity follows by a straight forward induction using the commutation relation, see for example [7, §2.2]. Define the sets $\mathcal{U}^k(\lambda,\rho) := \{\nu : \lambda \prec' \nu \succ \rho, |\nu/(\lambda \cup \rho)| = k\}$ and $\mathcal{D}^k(\lambda,\rho) := \{\mu : \lambda \succ \mu \prec' \rho, |(\lambda \cap \rho)/\mu| = k\}$. The equation (2.1) can be reformulated as the set of equations

$$|\mathcal{U}^k(\lambda,\rho)| = |\mathcal{D}^k(\lambda,\rho)| + |\mathcal{D}^{k-1}(\lambda,\rho)|. \tag{2.3}$$

for all partitions λ, ρ and non-negative integers k. It turns out to be quite fruitful to prove these equations bijectively.

An *inner corner* of a partition λ is a cell $c \in \lambda$ such that $\lambda/\mu = \{c\}$ for a partition $\mu \subseteq \lambda$. An *outer corner* of λ is a cell $c \notin \lambda$ such that $\nu/\lambda = \{c\}$ for a partition ν with $\lambda \subseteq \nu$. We call an inner corner c of $\lambda \cap \rho$ *removable* with respect to (λ, ρ) if λ/μ is a horizontal strip and ρ/μ is a vertical strip, where $(\lambda \cap \rho)/\mu = \{c\}$. Analogously we call an outer corner c of $\lambda \cup \rho$ *addable* with respect to (λ, ρ) if ν/λ is a vertical strip and ν/ρ is a horizontal strip, where $\nu/(\lambda \cup \rho) = \{c\}$. For both removable and addable corners we omit referring to (λ, ρ) whenever the partitions λ, ρ are clear from context. For an example see Figure 1.

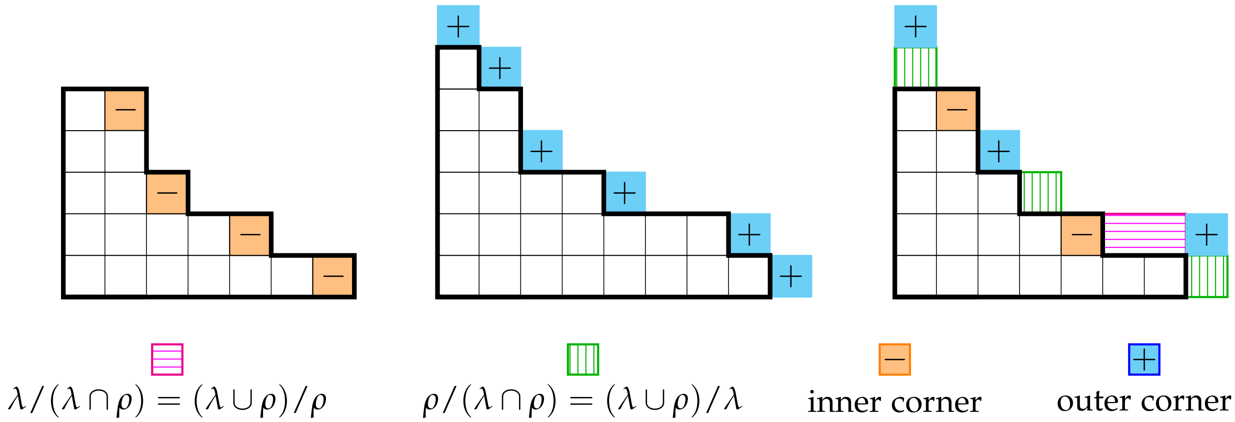

Figure 1: For $\lambda = (7,7,3,2,2)$ and $\rho = (8,5,4,2,2,1)$: (left) the partition $\lambda\cap\rho$ together with all inner corners, (middle) the partition $\lambda\cup\rho$ with all outer corners, and (right) $\lambda\cap\rho$ with all *removable* inner corners of $\lambda\cap\rho$ and *addable* outer corners of $\lambda\cup\rho$. At the bottom we show the color and shading code for cells in certain skew shapes.

Each partition ν in $\mathcal{U}^k(\lambda,\rho)$ corresponds to a k-subset of the addable outer corners of $\lambda\cup\rho$ and each partition μ in $\mathcal{D}^k(\lambda,\rho)$ corresponds to a k-subset of the removable inner corners of $\lambda\cap\rho$. We call a collection $F_\bullet = \{F_{\lambda,\rho,k} : \lambda,\rho \in \mathbb{Y}, k \in \mathbb{N}\}$ of bijections

$$F_{\lambda,\rho,k} : \mathcal{D}^{k-1}(\lambda,\rho) \cup \mathcal{D}^k(\lambda,\rho) \to \mathcal{U}^k(\lambda,\rho),$$

a set of *local dual growth rules*. Two of the many possible bijections $F_{\lambda,\rho,k}$ are very natural: the *dual row insertion bijection* $F^{\text{row}}_{\lambda,\rho,k}$ and the *dual column insertion bijection* $F^{\text{col}}_{\lambda,\rho,k}$. For $k=1$ the dual row (resp., column) insertion bijection maps a removable inner corner to the next addable outer corner in a row above (resp., column to the right) and sends the empty set to the lowest (resp., left-most) addable outer corner. Figure 2 illustrates this case. For $k>1$ both maps are defined recursively by

$$F^\bullet_{\lambda,\rho,k}(X) = \begin{cases} \bigcup\limits_{x\in X} F^\bullet_{\lambda,\rho,1}(\{x\}) & |X| = k, \\ F^\bullet_{\lambda,\rho,1}(\emptyset) \cup \bigcup\limits_{x\in X} F^\bullet_{\lambda,\rho,1}(\{x\}) & |X| = k-1, \end{cases} \tag{2.4}$$

where $F^\bullet_{\lambda,\rho,k}$ stands for $F^{\text{row}}_{\lambda,\rho,k}$ or $F^{\text{col}}_{\lambda,\rho,k}$ respectively.

Each set of local growth rules $F_\bullet$ determines a bijection

$$\mathrm{RSK}^*_{F_\bullet} : \{0,1\}^{m\times n} \to \bigcup_\lambda \mathrm{SSYT}(\lambda) \times \mathrm{SSYT}^*(\lambda), \qquad A \mapsto (P,Q).$$

While the bijection $\mathrm{RSK}^*_{F_\bullet}$ is best understood by using Fomin's growth diagrams [6], we describe it as an insertion algorithm in order to save space and refer the reader to [7, §2.3] for more details.

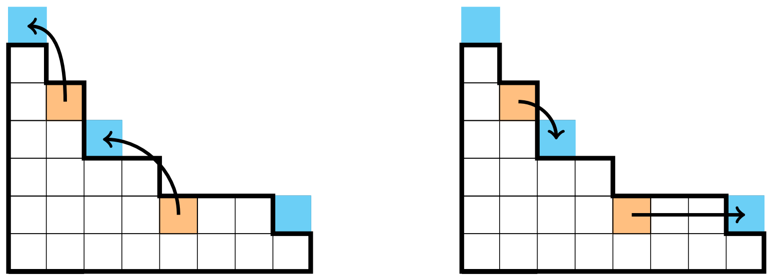

Figure 2: The two maps $F^{\text{row}}_{\lambda,\rho,1}$ (left) and $F^{\text{col}}_{\lambda,\rho,1}$ (right) for $\lambda = (7,7,3,2,2)$ and $\rho = (8,5,4,2,2,1)$. The removable inner corners (colored in orange) and the addable outer corners (colored in blue) are obtained in Figure 1.

Definition 2.1. *Let $F_\bullet$ be a set of local dual growth rules, T an SSYT and $i_1 < \cdots < i_r$ positive integers. We define the $F_\bullet$-insertion of $i_1, \ldots, i_r$ into T as the SSYT $\widehat{T}$ obtained as follows. Call the (multi-)*[1] *set $\{i_1, \ldots, i_r\}$ the insertion queue. Let i be the smallest integer of the insertion queue. Denote by C the set of cells of $F_{\lambda,\rho,k}(\mu)/(\lambda \cup \rho)$ where $\lambda = T^{(i)}$, $\rho = \widehat{T}^{(i-1)}$, $\mu = T^{(i-1)}$ and k is the multiplicity of i in the insertion queue. Place i into each cell of C, delete all i's from the insertion queue and add all entries which have been replaced (bumped) in the current step to the insertion queue. Repeat the previous step until the insertion queue is empty.*

For a $m \times n$ $\{0,1\}$-matrix A denote by $i_1^{(j)} < \cdots < i_{r_j}^{(j)}$ the rows for which A has a 1 entry in the j-th column. The *insertion tableau* P is obtained by the successive $F_\bullet$-insertion of $i_1^{(j)}, \ldots, i_{r_j}^{(j)}$, starting with $j = 1$, into the empty tableau. The *recording tableau* Q is the dual SSYT such that $Q^{(j)}$ has the same shape as P after the j-th insertion process.

3 Macdonald polynomials

We review certain basic properties of Macdonald polynomials, following [9, Ch. VI]. The *Macdonald symmetric functions* $P_\lambda(\mathbf{x}; q, t)$ are symmetric functions in an infinite set of variables $\mathbf{x} = (x_1, x_2, \ldots)$ with coefficients in the field $\mathbb{Q}(q,t)$ of rational functions in two additional variables q and t. While they are originally defined indirectly by a linear algebra criterion, we take the somewhat unusual perspective to define them combinatorially using their monomial expansion via SSYTs.

We define for a cell $c = (x,y) \in \lambda$ its *arm-length* $a_\lambda(c)$ and its *leg-length* $\ell_\lambda(c)$ by

$$a_\lambda(c) = \lambda_y - x, \qquad \ell_\lambda(c) = \lambda'_x - y.$$

[1] Note that at the initial step this is just an ordinary set. The same is true for Definition 5.1.

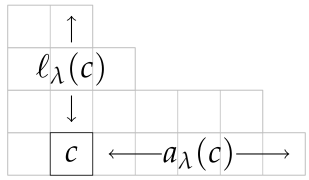

Figure 3: The Young diagram of the partition $\lambda = (7,6,3,2)$ for which the cell $c = (2,1)$ is marked.

The *hook-length* of c is defined as $h_\lambda(c) = a_\lambda(c) + \ell_\lambda(c) + 1$. The cell c as in Figure 3 has arm-length $a_\lambda(c) = 5$, leg-length $\ell_\lambda(c) = 3$, and hook-length $h_\lambda(c) = 9$. We define the *(q,t)-hook-lengths* $h^\ell_\lambda(c) = 1 - q^{a_\lambda(c)} t^{\ell_\lambda(c)+1}$ and $h^a_\lambda(c) = 1 - q^{a_\lambda(c)+1} t^{\ell_\lambda(c)}$ for $c \in \lambda$, and $h^\ell_\lambda(c) = h^a_\lambda(c) = 1$ if $c \notin \lambda$. Further we need their ratio which is denoted by $b_\lambda(c)$

$$b_\lambda(c) = \frac{h^\ell_\lambda(c)}{h^a_\lambda(c)}.$$

For $\mu \subseteq \lambda$, define[2]

$$\psi_{\lambda/\mu}(q,t) = \prod_{c \in \mathcal{R}_{\lambda/\mu} - \mathcal{C}_{\lambda/\mu}} \frac{b_\mu(c)}{b_\lambda(c)}, \qquad \varphi^*_{\lambda/\mu}(q,t) = \prod_{c \in \mathcal{C}_{\lambda/\mu} - \mathcal{R}_{\lambda/\mu}} \frac{b_\lambda(c)}{b_\mu(c)},$$

where $\mathcal{R}_{\lambda/\mu}$ (resp., $\mathcal{C}_{\lambda/\mu}$) is the set of all cells in λ which are in the same row (resp., column) as a cell of λ/μ. For a semistandard Young tableau T and a dual semistandard Young tableau T^*, define the rational functions $\psi_T(q,t), \varphi^*_{T^*}(q,t)$ by

$$\psi_T(q,t) = \prod_{i \geq 1} \psi_{T^{(i)}/T^{(i-1)}}(q,t), \qquad \varphi^*_{T^*}(q,t) = \prod_{i \geq 1} \varphi_{T^{*(i)}/T^{*(i-1)}}(q,t).$$

Macdonald [9, Ch. VI (7.13)] showed the following monomial expansions over semistandard Young tableaux of shape λ

$$P_\lambda(\mathbf{x};q,t) = \sum_{T \in \mathrm{SSYT}(\lambda)} \psi_T(q,t)\mathbf{x}^T, \qquad P_{\lambda'}(\mathbf{x};t,q) = \sum_{T^* \in \mathrm{SSYT}^*(\lambda)} \varphi^*_{T^*}(q,t)\mathbf{x}^{T^*}. \tag{3.1}$$

By using the linear algebraic definition, Macdonald proved the following generalization of the dual Cauchy identity.

Theorem 3.1 ([9, Ch. VI (5.4)]). *Let $\boldsymbol{x} = (x_1, x_2, \dots)$ and $\boldsymbol{y} = (y_1, y_2, \dots)$ be two sets of variables. Then*

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda P_\lambda(\mathbf{x};q,t) P_{\lambda'}(\mathbf{y};t,q). \tag{3.2}$$

[2]Contrary to Macdonald [9, Ch. VI (6.24)] we use the symbol φ^* instead of ψ'.

In this abstract our goal is to provide a tableaux-theoretic proof of this theorem by starting with the monomial expansion of Macdonald polynomials. We define the *(q,t)-up operator* and *(q,t)-dual down operator* as

$$U_x(q,t)\lambda = \sum_{\nu \succ \lambda} x^{|\nu/\lambda|}\psi_{\nu/\lambda}(q,t)\ \nu, \qquad D_y^*(q,t)\lambda = \sum_{\mu \prec' \lambda} y^{|\lambda/\mu|}\varphi^*_{\lambda/\mu}(q,t)\ \mu.$$

Theorem 3.2. *The (q,t)-up and (q,t)-dual down operators satisfy the commutation relation*

$$D_y^*(q,t)U_x(q,t) = (1+xy)U_x(q,t)D_y^*(q,t). \tag{3.3}$$

Note that the commutation relation (3.3) is actually equivalent to the skew version of the dual Cauchy identity, compare to [9, Ch. VI, Ex 6(c)]. It is immediate by the definition of the (q,t)-up and (q,t)-dual down operator and the monomial expansions of P_λ and $P_{\lambda'}$ in (3.1) that

$$P_\lambda(\mathbf{x};q,t) = \langle U_{x_m}(q,t)\cdots U_{x_1}(q,t)\varnothing, \lambda\rangle, \qquad P_{\lambda'}(\mathbf{y};t,q) = \langle D^*_{y_1}(q,t)\cdots D^*_{y_n}(q,t)\lambda, \varnothing\rangle,$$

when restricting to $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_n)$. The Cauchy identity (3.2) follows algebraically by the same standard argument as in the Schur case.

4 The qtRSK* correspondence

Definition 4.1. *Let X and Y be finite sets equipped with weight functions $\omega : X \to A$, $\overline{\omega} : Y \to A$, where A is an algebra. A* probabilistic bijection *from (X,ω) to $(Y,\overline{\omega})$ is a pair of maps $\mathcal{P}, \overline{\mathcal{P}} : X \times Y \to A$ satisfying*

1. *for each $x \in X$, $\sum_{y\in Y} \mathcal{P}(x,y) = 1$, and for each $y \in Y$, $\sum_{x \in X} \overline{\mathcal{P}}(x,y) = 1$,*
2. *for each $x \in X$ and $y \in Y$, $\omega(x)\mathcal{P}(x,y) = \overline{\mathcal{P}}(x,y)\overline{\omega}(y)$.*

For the remainder of the abstract we write $\mathcal{P}(x \to y)$ for $\mathcal{P}(x,y)$ and $\overline{\mathcal{P}}(x \leftarrow y)$ for $\overline{\mathcal{P}}(x,y)$, and think of $\mathcal{P}(x \to y)$ as the "probability" of mapping x to y, called *forward probability*, and of $\overline{\mathcal{P}}(x \leftarrow y)$ as the "probability" of mapping y back to x, called the *backward probability*. We put "probability" in quotes because we do not require $\mathcal{P}(x \to y), \overline{\mathcal{P}}(x \leftarrow y) \in [0,1]$ (they need not even be real-valued). We refer to (2) as the *compatibility condition*. It is immediate that a probabilistic bijection from (X,ω) to $(Y,\overline{\omega})$ implies the identity

$$\sum_{x\in X}\omega(x) = \sum_{y \in Y}\overline{\omega}(y).$$

We want to point out, that there is an easy connection between the concept of probabilistic bijections and joint distributions, compare for example with [1, Remark 4.1.4].

For partitions λ, ρ, μ, ν satisfying $\mu \prec \lambda \prec' \nu$ and $\mu \prec' \rho \prec \nu$ we define the weights

$$\omega_{\lambda,\rho}(\mu) = \psi_{\lambda/\mu}(q,t)\varphi^*_{\rho/\mu}(q,t), \qquad \overline{\omega}_{\lambda,\rho}(\nu) = \psi_{\nu/\rho}(q,t)\varphi^*_{\nu/\lambda}(q,t).$$

Analogously to (2.3), the commutation relation (3.3) is equivalent to the family of equations

$$\sum_{\mu \in \mathcal{D}^k(\lambda,\rho) \cup \mathcal{D}^{k-1}(\lambda,\rho)} \omega_{\lambda,\rho}(\mu) = \sum_{\nu \in \mathcal{U}^k(\lambda,\rho)} \overline{\omega}_{\lambda,\rho}(\nu). \tag{4.1}$$

In the remainder of this section we define the forward probabilities $\mathcal{P}_{\lambda,\rho}(\mu \to \nu)$ and the backward probabilities $\overline{\mathcal{P}}_{\lambda,\rho}(\mu \leftarrow \nu)$ which form a probabilistic bijection and thereby prove this equation. Before we can define these probabilities, we need to introduce some notations.

Denote by d the number of removable inner corners of $\lambda \cap \rho$. For a subset $\mathbf{R} \subseteq [d] = \{1, 2, \dots d\}$ we define $\mu^{(\mathbf{R})}$ as the partition obtained by removing from $\lambda \cap \rho$ the i-th removable inner corner, counted from bottom to top, for all $i \in \mathbf{R}$. For a subset $\mathbf{S} \subseteq [0, d] = \{0, 1, \dots, d\}$ we define $\nu^{(\mathbf{S})}$ as the partition obtained by adding to $\lambda \cup \rho$ the i-th addable ("supplementable") outer corner, where we count the addable outer corners again from bottom to top but starting with 0.

As we see in a moment, it turns out to be convenient to draw Young diagrams using *Quebecois notation* in which the boxes are right-justified instead of left-justified, i.e., one obtains this new convention by reflecting diagrams in French convention vertically, see Figure 4. We define R_i (resp., $\overline{R}_i$) to be the lower right (resp., upper left) corner of the i-th removable inner corner of $\lambda \cap \rho$, S_i (resp., $\overline{S}_i$) to be the lower right (resp., upper left) corner of the i-th addable outer corner of $\lambda \cup \rho$, and set $I_i = \overline{R}_i$ and $O_i = S_i$. For an example see Figure 4. For the rest of the abstract we identify a point with coordinates (x, y) with the monomial $q^x t^y$. Since the expressions we are interested in are homogeneous rational functions of degree 0 in the above defined points, these expressions are invariant under translation of the points and hence well-defined.

For $\mathbf{R} \subseteq [d]$ and $\mathbf{S} \subseteq [0, d]$, we define the probabilities

$$\mathcal{P}_{\lambda,\rho}(\mu^{(\mathbf{R})} \to \nu^{(\mathbf{S})}) = \prod_{s \in \mathbf{S}} \frac{\prod_{i \in [d] \setminus \mathbf{R}} (S_s - I_i)}{\prod_{j \in [0,d] \setminus \mathbf{S}} (S_s - O_j)} \prod_{r \in \mathbf{R}} \frac{\prod_{j \in [0,d] \setminus \mathbf{S}} (R_r - O_j)}{\prod_{i \in [d] \setminus \mathbf{R}} (R_r - I_i)}, \tag{4.2}$$

$$\overline{\mathcal{P}}_{\lambda,\rho}(\mu^{(\mathbf{R})} \leftarrow \nu^{(\mathbf{S})}) = \prod_{s \in \mathbf{S}} \frac{\prod_{i \in [d] \setminus \mathbf{R}} (\overline{S}_s - I_i)}{\prod_{j \in [0,d] \setminus \mathbf{S}} (\overline{S}_s - O_j)} \prod_{r \in \mathbf{R}} \frac{\prod_{j \in [0,d] \setminus \mathbf{S}} (\overline{R}_r - O_j)}{\prod_{i \in [d] \setminus \mathbf{R}} (\overline{R}_r - I_i)}. \tag{4.3}$$

For an integer $k \geq 0$ and a set $\mathbf{S}$, we denote by $\binom{\mathbf{S}}{k}$ the set of k-element subsets of $\mathbf{S}$.

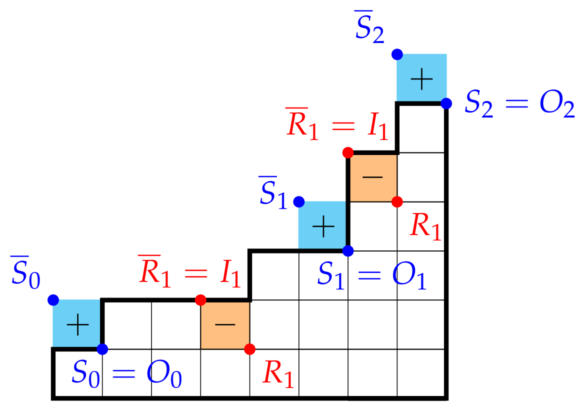

Figure 4: The partition $\lambda \cup \rho$ together with the points $I_i, O_j, R_i, \overline{R}_i, S_j$ and $\overline{S}_j$ for $\lambda = (7,7,3,2,2)$ and $\rho = (8,5,4,2,2,1)$ as in Figure 1.

Theorem 4.2. *Let λ, ρ be partitions, d the number of removable inner corners of $\lambda \cap \rho$, and $k \in [d+1]$. The probabilities defined in (4.2) and (4.3) satisfy*

$$\sum_{\mathbf{S} \in \binom{[0,d]}{k}} \mathcal{P}_{\lambda,\rho}(\mu^{(\mathbf{R})} \to \nu^{(\mathbf{S})}) = 1 \qquad \text{for each } \mathbf{R} \in \binom{[d]}{k-1} \cup \binom{[d]}{k}, \tag{4.4}$$

$$\sum_{\mathbf{R} \in \binom{[d]}{k-1} \cup \binom{[d]}{k}} \overline{\mathcal{P}}_{\lambda,\rho}(\mu^{(\mathbf{R})} \leftarrow \nu^{(\mathbf{S})}) = 1 \qquad \text{for each } \mathbf{S} \in \binom{[0,d]}{k}, \tag{4.5}$$

$$\frac{\omega_{\lambda,\rho}(\mu^{(\mathbf{R})})}{\overline{\omega}_{\lambda,\rho}(\nu^{(\mathbf{S})})} = \frac{\overline{\mathcal{P}}_{\lambda,\rho}(\mu^{(\mathbf{R})} \leftarrow \nu^{(\mathbf{S})})}{\mathcal{P}_{\lambda,\rho}(\mu^{(\mathbf{R})} \to \nu^{(\mathbf{S})})} \qquad \text{for each } \mathbf{R} \in \binom{[d]}{k-1} \cup \binom{[d]}{k}, \mathbf{S} \in \binom{[0,d]}{k}. \tag{4.6}$$

The above theorem shows that our probabilities define a probabilistic bijection. The proof of (4.4) and (4.5) uses an extension of Lagrange interpolation for symmetric polynomials by Chen and Louck [5]. The proof of (4.6) is based on a careful analysis of the involved terms and alternative representations of the probabilities. We refer the reader to [7, §5] for more details.

5 The qtRSK* correspondence

We view the probabilities $\mathcal{P}_{\lambda,\rho}$ as a set of "probabilistic" local dual growth rules and define the qtRSK* correspondence analogously to the insertion algorithm RSK$^*_{F_\bullet}$ in §2.

Definition 5.1. *Let T be a semistandard Young tableau and $i_1 < \cdots < i_r$ be positive integers.*

The qtRSK*-insertion of $i_1, \ldots, i_r$ into T, denoted

$$(i_1, \ldots, i_r) \xrightarrow{qt\mathrm{RSK}^*} T = \widehat{T},$$

is the probability distribution computed as follows. Call the (multi-) set $\{i_1, \ldots, i_r\}$ the insertion queue. Let i be the smallest integer of the insertion queue and denote by k the multiplicity of i in the insertion queue. For each $\nu \in \mathcal{U}^k(\lambda, \rho)$, place i in each cell of $\nu / (\lambda \cup \rho)$ with probability $\mathcal{P}_{\lambda,\rho}(\mu \to \nu)$, where $\lambda = T^{(i)}$, $\rho = \widehat{T}^{(i-1)}$, $\mu = T^{(i-1)}$. Delete all i's from the insertion queue and add all entries which have been replaced (bumped) by an i to the insertion queue. Repeat the previous step until the insertion queue is empty.

For an $m \times n$ $\{0,1\}$-matrix A denote by $i_1^{(j)} < \cdots < i_{r_j}^{(j)}$ the rows for which A has a 1 entry in the j-th column. The qtRSK*-correspondence associates to A a probability distribution $\mathcal{P}(A \to P, Q)$ on pairs (P, Q) of an SSYT P and a dual SSYT Q of the same shape, where the probability $\mathcal{P}(A \to P, Q)$ is the sum of the forward probabilities of all ways to obtain the insertion tableau P by successively qtRSK*-inserting $i_1^{(j)}, \ldots, i_{r_j}^{(j)}$, starting with $j = 1$, into the empty tableau and Q as the recording tableau by the analogous construction as for RSK$^*_{F_\bullet}$. Note that the backward probabilities $\overline{\mathcal{P}}(A \leftarrow P, Q)$ are defined analogously by summing over the backward probabilities instead of the forward probabilities. By using the perspective of Fomin's growth diagrams, it is not difficult to prove that qtRSK* defines a probabilistic bijection between the weighted sets of $m \times n$ $\{0,1\}$-matrices with weight ω and $\bigsqcup_{\lambda \subseteq (m^n)} \mathrm{SSYT}(\lambda) \times \mathrm{SSYT}^*(\lambda)$ with weight $\overline{\omega}$, where

$$\omega(A) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (x_i y_j)^{A_{i,j}} \qquad \text{and} \qquad \overline{\omega}(P, Q) = \psi_P(q,t) \varphi^*_Q(q,t) \mathbf{x}^P \mathbf{y}^Q.$$

See [7, §4.4] for more details.

Example 5.2. *The insertion* $(2,3) \xrightarrow{qt\mathrm{RSK}^*} \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$ *produces*

$$\begin{array}{|c|c|c|} \hline 3 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 2 \\ \hline \end{array} \quad \textit{with probability} \quad = \mathcal{P}_{(2),(1)}((1) \to (3)) = \frac{1 - qt}{1 - q^2 t},$$

$$\begin{array}{|c|c|c|} \hline 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} \quad \textit{with probability} \quad \begin{aligned} &= \mathcal{P}_{(2),(1)}((1) \to (2,1)) \mathcal{P}_{(2,1),(2,1)}((2) \to (3,2)) \\ &= q^2 t \frac{(1-q)^2 (1-t)^2}{(1-qt)(1-q^2)(1-q^2 t)(1-q^2 t^2)}, \end{aligned}$$

$$\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \quad \textit{with probability} \quad \begin{aligned} &= \mathcal{P}_{(2),(1)}((1) \to (2,1)) \mathcal{P}_{(2,1),(2,1)}((2) \to (2,2,1)) \\ &= q t^2 \frac{(1-q)^2 (1-q^2 t)}{(1-qt)(1-q^2)(1-q^2 t^2)}, \end{aligned}$$

$$\begin{array}{|c|cc}\hline 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \textit{with probability} \quad \begin{aligned} &= \mathcal{P}_{(2),(1)}((1) \to (2,1))\mathcal{P}_{(2,1),(2,1)}((2) \to (3,1,1)) \\ &= qt\frac{(1-q)^2(1-t)}{(1-q^2)(1-qt)^2}. \end{aligned}$$

6 Properties of qtRSK*

Our randomized qtRSK* correspondence can be specialized in two different ways: one can specialize the parameter q, t or restrict the correspondence to a smaller family of matrices.

For $q, t \in [0, 1)$ or $q, t \in (1, \infty)$ the probabilities $\mathcal{P}(A \to P, Q)$ and $\overline{\mathcal{P}}(A \leftarrow P, Q)$ take values in $[0, 1]$, i.e., they become actual probabilities. The qtRSK*-insertion specializes to the q-Whittaker dual row insertion ($t = 0$) first described by Matveev and Petrov [10, §5.1] and to a Hall–Littlewood dual-row insertion ($q = 0$), first described Matveev and Petrov in [10, §5.4] as a q-Whittaker dual column insertion. Finally for $q = t = 0$ or $q = t \to \infty$ we obtain the row or column insertion version of RSK*.

By restricting the input of qtRSK* to $\{0, 1\}$-matrices with at most one entry equal to 1 in each column, we obtain a (q, t)-deformation of RS for words. By further restricting to permutation matrices we obtain the qRSt correspondence. The restriction of qtRSK* to words is in particular interesting when further specializing to Jack polynomials, i.e., by setting $q = t^\alpha$ and taking the limit $t \to 1$. We prove in [7, Thm 6.5] that in the Jack limit of qtRSK* restricted to words, interchanging adjacent columns of the input matrix does not affect the distribution of the P-tableau. Note that this can not be extended to all $\{0, 1\}$-matrices.

Similarly to the classical dual RSK, the qtRSK* correspondence also yields a tableaux-theoretic proof of the dual Pieri rule for Macdonald polynomials. This can be obtained by considering a growth diagram with one column for which the number of 1 entries is fixed to k; this corresponds to multiplying by e_k.

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An extended generalization of RSK via the combinatorics of type A quiver representations

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Abstract. The classical Robinson–Schensted–Knuth correspondence is a bijection from nonnegative integer matrices to pairs of semi-standard Young tableaux. Based on the work of, among others, Burge, Hillman, Grassl, Knuth and Gansner, it is known that a version of this correspondence gives, for any nonzero integer partition λ, a bijection from arbitrary fillings of λ to reverse plane partitions of shape λ, via Greene–Kleitman invariants. By bringing out the combinatorial aspects of our recent results on quiver representations, we construct a family of bijections from fillings of λ to reverse plane partitions of shape λ parametrized by a choice of Coxeter element in a suitable symmetric group. We recover the above version of the Robinson–Schensted–Knuth correspondence for a particular choice of Coxeter element depending on λ.

Résumé. La correspondance Robinson–Schensted–Knuth classique est une bijection partant des matrices à coefficients des entiers naturels vers les paires de tableaux de Young semi-standards. Basé sur les travaux, entre autres, de Burge, Hillman, Grassl, Knuth et Gansner, on sait qu'une version de cette correspondance donne, pour toute partage d'un entier non nulle λ, une bijection allant des remplissages arbitraires de λ vers les partitions planes renversées de forme λ, via les invariants de Greene–Kleitman. En faisant ressortir les aspects combinatoires de nos récents résultats sur les représentations de carquois, nous construisons une famille de bijections partant des remplissages de λ vers les partitions planes renversées de forme λ, paramétrées par un choix d'élément de Coxeter dans un groupe symétrique approprié. Nous récupérons la version de la correspondance Robinson–Schensted–Knuth ci-dessus pour un choix particulier d'élément de Coxeter dépendant de λ.

Keywords: Quiver representations, Robinson–Schensted–Knuth, Reverse plane partitions.

1 Introduction

The Robinson–Schensted–Knuth (RSK) correspondence is a fundamental bijection from nonnegative integer matrices to pairs of semi-standard Young tableaux of the same shape. For further details, the reader may consult the following references: [16], [6].

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Based on observations of various works of Burge [3], Hillman–Grassl [12] and Knuth [13], Gansner [7, 9] constructed a generalized version of this correspondence, via Greene–Kleitman invariants, which gives a bijection from arbitrary fillings to reverse plane partitions of the same shape.

Our paper [4] studies a representation-theoretic setting in which a version of RSK exists. In the present paper, we present an explicit, combinatorial form of the results from [4]. Given a fixed nonzero integer partition λ, we present the construction of a family of maps $(\mathrm{RSK}_{\lambda,c})_c$ from fillings of λ to reverse plane partitions of shape λ parametrized by c a Coxeter element of the symmetric group $\mathfrak{S}_n$ where $n-1$ is the hook-length of the box $(1,1)$ in λ. We can state the following result from [4].

Theorem 1. *The map* $\mathrm{RSK}_{\lambda,c}$ *gives a one-to-one correspondence from fillings of shape* λ *to reverse plane partitions of shape* λ*. Moreover, for any* λ*, there exists a unique (up to inverse) choice of* c *such that* $\mathrm{RSK}_{\lambda,c}$ *coincides with the usual* RSK.

No knowledge in quiver representation is required to read this abstract, except for Section 5 in which we discuss the connection with quiver representations. For more details on this work, we refer the reader to [5].

2 Gansner's Ferrers Diagram RSK

In this section, we describe Gansner's correspondence explicitly.

2.1 Some vocabulary

An *integer partition* is a weakly decreasing nonnegative integer sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}^*}$ with finitely many nonzero terms. The *length* of λ is the minimal $k \in \mathbb{N}$ such that $\lambda_{k+1} = 0$. We endow $(\mathbb{N}^*)^2$ with the Cartesian product order $\trianglelefteq$. The *Ferrers diagram of* λ $\mathrm{Fer}(\lambda)$ is the subset of $(\mathbb{N}^*)^2$ given by pairs (i,j) such that $i \leqslant \lambda_j$. We call any map $f : \mathrm{Fer}(\lambda) \longrightarrow \mathbb{N}$ a *filling of shape* λ . Such a filling f is a *reverse plane partition* whenever f weakly increases with respect to $\trianglelefteq$. We give an example of a reverse plane partition of shape $(5,3,3,2)$ in Figure 1.

0	3	5	5	7
1	5	5		
4	6	9		
4	10			

Figure 1: A reverse plane partitions of shape $\lambda = (5,3,3,2)$.

2.2 Greene–Kleitman invariants

Let $G = (G_0, G_1)$ be a finite directed graph, where G_0 is the set of vertices of G, and $G_1 \subset (G_0)^2$ is the set of arrows of G. Assume that G has no multi-arrows.

We see a path γ in G as a finite sequence of vertices $(v_0, \ldots, v_k)$ such that $(v_i, v_{i+1}) \in G_1$. Denote by $s(\gamma) = v_0$ its source and by $t(\gamma) = v_k$ its target. Write $\mathrm{Supp}(\gamma) = \{v_0, \ldots, v_k\}$ to denote the support of γ. For $\ell \geqslant 1$, we extend the notion of support to ℓ-tuples of paths $\underline{\gamma} = (\gamma_1, \ldots, \gamma_\ell)$ as $\mathrm{Supp}(\underline{\gamma}) = \bigcup_{i=1}^{\ell} \mathrm{Supp}(\gamma_i)$. For $\ell \geqslant 1$, write $\Pi_\ell(G)$ the set of ℓ-tuples of paths in G.

From now on, assume that G is acyclic, meaning there is no nontrivial path γ in G such that $s(\gamma) = t(\gamma)$. An *antichain* of G is any subset of vertices $\{w_1, \ldots, w_r\} \subset G_0$ such that there is no path γ in G with $s(\gamma) = w_i$ and $t(\gamma) = w_j$ for all $1 \leqslant i, j \leqslant r$ with $i \neq j$.

A *filling* of G is a map $f : G_0 \longrightarrow \mathbb{N}$. We assign to any ℓ-tuple of paths $\underline{\gamma}$ of G a *f-weight* defined by

$$\mathsf{wt}_f(\underline{\gamma}) = \sum_{v \in \mathrm{Supp}(\underline{\gamma})} f(v).$$

Set $M_0^G(f) = 0$, and for all integers $\ell \geqslant 1$, $M_\ell^G(f) = \max_{\underline{\gamma} \in \Pi_\ell(G)} \mathsf{wt}_f(\underline{\gamma})$. We define the *Greene–Kleitman invariant* of f in G as

$$\mathrm{GK}_G(f) = \left(M_\ell^G(f) - M_{\ell-1}^G(f) \right)_{\ell \geqslant 1}.$$

See Figure 2 for an explicit computation example.

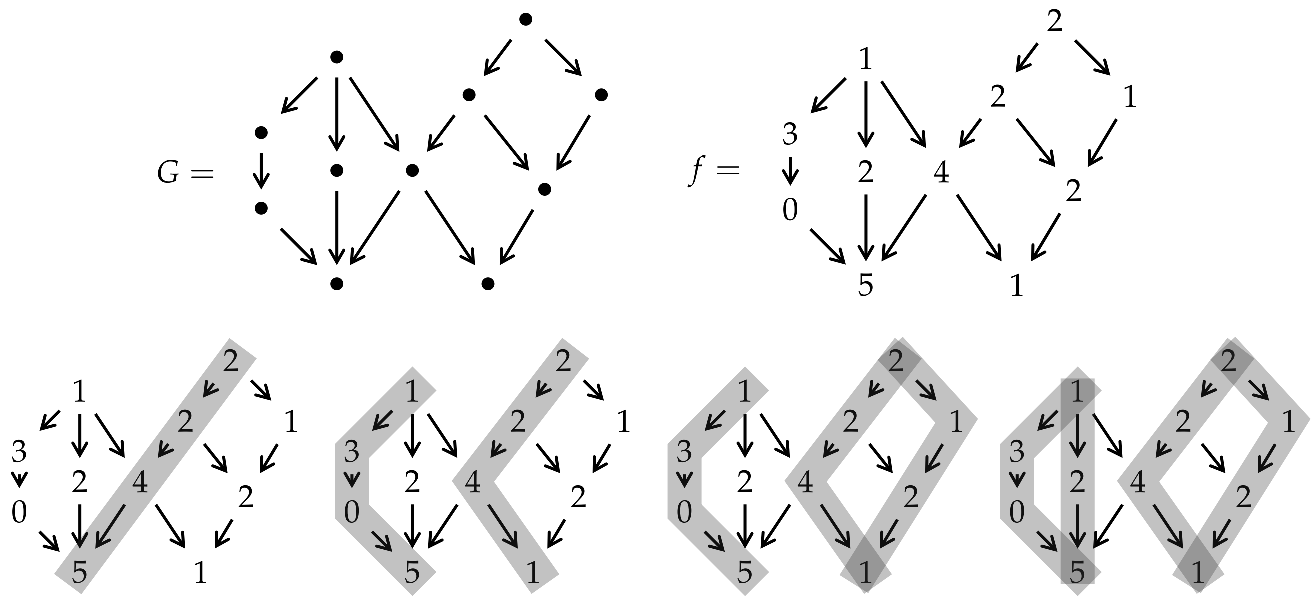
Proposition 2 (Greene–Kleitman [11])**.** *Let G be a finite direct acyclic graph and f be a filling of G. The integer sequence $\mathrm{GK}_G(f)$ is an integer partition of length the maximal cardinality of an antichain in G.*

2.3 Ferrers diagram RSK

Throughout this section, we highlight Gansner's generalized version of the RSK correspondence, which gives, for any nonzero integer partition λ, a bijection from fillings of shape λ to reverse plane partitions of shape λ.

Fix a nonzero integer partition λ. Let G_λ be the oriented acyclic graph such that:

- its vertices are the elements of $\mathrm{Fer}(\lambda)$;
- its arrows are given by:
  - $(i, j) \longrightarrow (i+1, j)$ whenever $(i, j), (i+1, j) \in \mathrm{Fer}(\lambda)$;
  - $(i, j) \longrightarrow (i, j+1)$ whenever $(i, j), (i, j+1) \in \mathrm{Fer}(\lambda)$.



$$\mathrm{GK}_G(f) = (13, 5, 3, 2)$$

Figure 2: An example of the computation of GK_G.

For all $m \in \mathbb{Z}$, write $D_m(\lambda) = \{(i, j) \in \mathrm{Fer}(\lambda) \mid i - j + \lambda_1 = m\}$ for the mth diagonal of λ. Note that $D_m(\lambda) \neq \varnothing$ for $1 \leqslant m \leqslant h_\lambda(1, 1)$, where $h_\lambda(1, 1) = \#\{(i, j) \in \mathrm{Fer}(\lambda) \mid i = 1 \text{ or } j = 1\}$ denotes the *hook length of the box* $(1, 1)$ *in* λ.

For each value $1 \leqslant m \leqslant h_\lambda(1, 1)$, consider (u_m, v_m) the maximal element of $D_m(\lambda)$. Write $G_\lambda(m)$ for the full subgraph of G_λ given by the poset ideal generated by (u_m, v_m). Note that $G_\lambda(m)$ admits only one source $(1, 1)$, and only one sink (u_m, v_m).

We define $g = \mathrm{RSK}_\lambda(f)$ to be the filling of shape λ defined by

$$\forall m \in \{1, \ldots, h_\lambda(1, 1)\},\ \forall (i, j) \in D_m(\lambda), \quad g(i, j) = \mathrm{GK}_{G_\lambda(m)}(f)_{u_m - i + 1}.$$

See Figure 3 for an explicit calculation of $\mathrm{RSK}_\lambda(f)$ for a given filling of $\lambda = (5, 3, 3, 2)$.

Theorem 3 (Gansner [9])**.** *Let λ be a nonzero integer partition. The map* RSK_λ *is a bijection from fillings of shape λ to reverse plane partitions of shape λ.*

Remark. If λ is a rectangle, we can recover the classical RSK. See [11] and [10, Section 6] for more details.

Moreover, a parallel can be made with Britz and Fomin's version of the RSK algorithm [2], where we compute sequences of integer partitions for an $n \times n$ nonnegative integer matrix as growth diagrams. A generalized version of RSK was also exploited by Krattenthaler [14] on polyominos. From a given filling f of shape λ, the integer partitions we can read on diagonals $D_m(\lambda)$ of $\mathrm{RSK}_\lambda(f)$ correspond precisely to the results

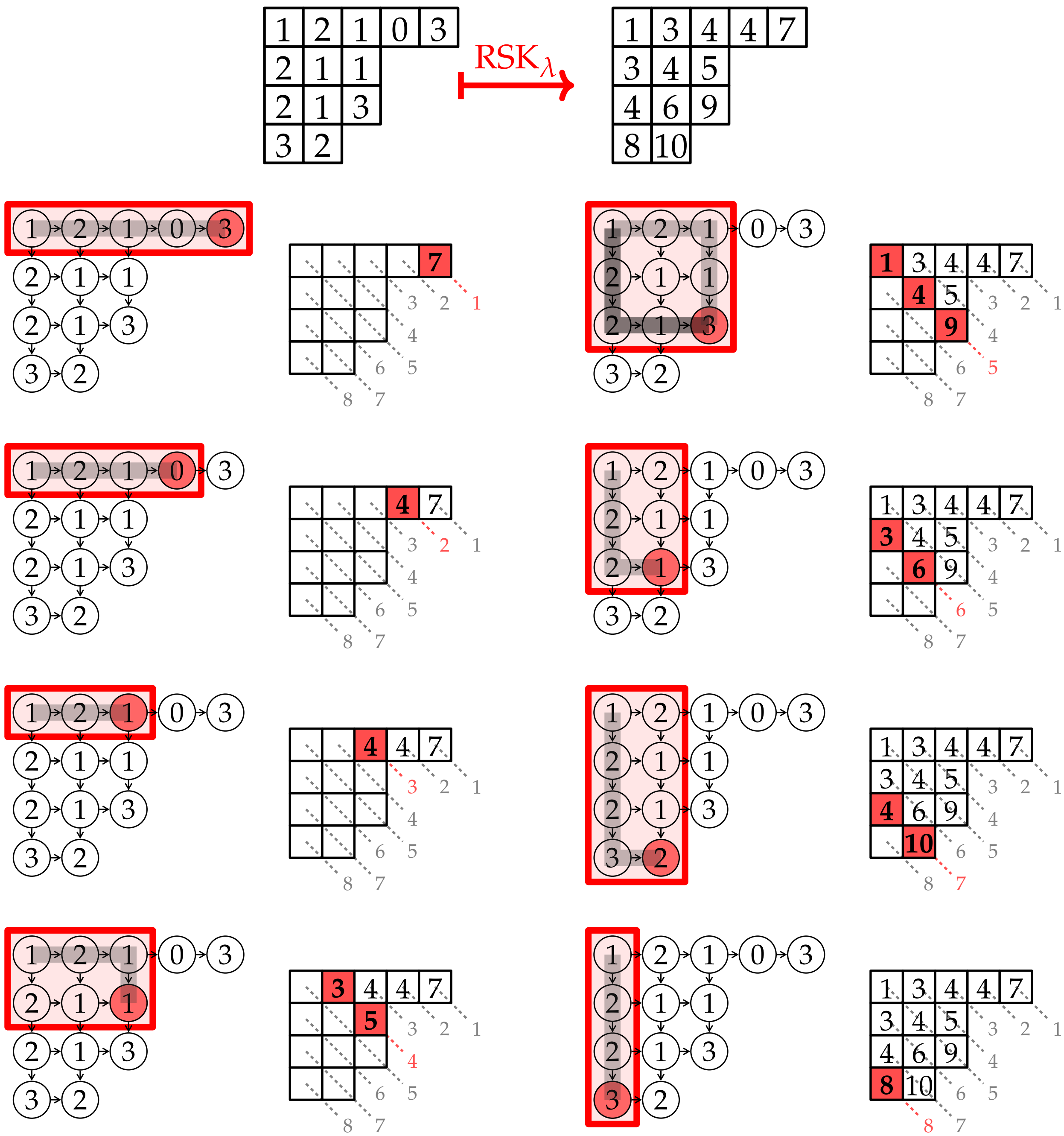


Figure 3: Explicit calculations of $\mathrm{RSK}_\lambda(f)$ for a given filling f of shape $\lambda = (5,3,3,2)$. For $1 \leqslant m \leqslant 8$, each framed subgraph corresponds to the subgraph $G_\lambda(m)$, and each filled diagonal colored in red corresponds to $\mathrm{GK}_{G_\lambda(m)}(f)$.

obtained at the end of each line by using the Krattenthaler growth diagram algorithm version.

3 Some tools

In this section, we give the definition of some combinatorial objects that will be useful to present our generalized version of Gansner's RSK correspondence.

3.1 Interval bipartitions

An *interval bipartition* is a pair $(\mathbf{B}, \mathbf{E}) \in \mathcal{P}(\mathbb{N}^*)^2$ such that $\{\mathbf{B}, \mathbf{E}\}$ is a set partition of $\{i, \ldots, j\}$ for some $1 \leqslant i \leqslant j$. Call it *elementary* whenever $1 \in \mathbf{B}$ and $\max(\mathbf{B} \cup \mathbf{E}) \in \mathbf{E}$.

Fix $(\mathbf{B}, \mathbf{E})$ as an interval bipartition. Write $\mathbf{B} = \{b_1 < b_2 < \ldots < b_p\}$. We define the integer partition $\lambda(\mathbf{B}, \mathbf{E})$ by $\lambda(\mathbf{B}, \mathbf{E})_i = \#\{e \in \mathbf{E} \mid b_i < e\}$. If we also write $\mathbf{E} = \{e_1 < \ldots < e_q\}$, we can also describe $\lambda(\mathbf{B}, \mathbf{E})$ by its Ferrers diagram: we have $(i, j) \in \mathrm{Fer}(\lambda(\mathbf{B}, \mathbf{E}))$ whenever $b_i < e_{q-j+1}$. It allows us to label the ith row of $\mathrm{Fer}(\lambda(\mathbf{B}, \mathbf{E}))$ by b_i and the jth row by e_{q-j+1}. See Figure 4 for an example of such an object.

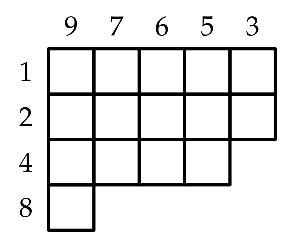


Figure 4: The (labelled) integer partition $\lambda(\mathbf{B}, \mathbf{E})$ with $\mathbf{B} = \{1, 2, 4, 8\}$ and $\mathbf{E} = \{3, 5, 6, 7, 9\}$.

Proposition 4. *For any integer partition λ, there exists an interval bipartition $(\mathbf{B}, \mathbf{E})$ such that $\lambda(\mathbf{B}, \mathbf{E}) = \lambda$. Moreover, if λ is a nonzero integer partition, there exists a unique elementary interval bipartition satisfying this property.*

3.2 (Type A) Coxeter elements

For any $n \geqslant 2$, let $\mathfrak{S}_n$ be the symmetric group on n letters. For $1 \leqslant i < j \leqslant n$, write (i, j) for the transposition exchanging i and j. For $1 \leqslant i < n$, let s_i be the adjacent transposition $(i, i+1)$. Let S be the set of the adjacent transpositions.

For any $w \in \mathfrak{S}_n$, an expression of w is a way to write w as a product of adjacent transpositions in S. The length $\ell(w)$ of w is the minimal number of adjacent transpositions in S needed to express w. Whenever, for some $1 \leqslant i < n$, $\ell(s_i w) < \ell(w)$, we say that s_i is initial in w. Similarly, we call s_i final in w whenever $\ell(w s_i) < \ell(w)$.

A *Coxeter element (of $\mathfrak{S}_n$)* is an element $c \in \mathfrak{S}_n$ which can be written as a product of all the adjacent transpositions, in some order, where each of them appears exactly once. For example, $c = s_2s_1s_3s_6s_5s_4s_8s_7 = (1,3,4,7,9,8,6,5,2)$ is a Coxeter element of $\mathfrak{S}_9$.

Lemma 5. *An element $c \in \mathfrak{S}_n$ is a Coxeter element if and only if c is a long cycle which can be written as follows*

$$c = (c_1, c_2, \ldots, c_m, c_{m+1}, \ldots, c_n)$$

where $c_1 = 1 < c_2 < \ldots < c_m = n > c_{m+1} > \ldots > c_n > c_1 = 1$.

3.3 Auslander–Reiten quivers

Let $c \in \mathfrak{S}_n$ be a Coxeter element. The *Auslander–Reiten quiver of c*, denoted $\mathrm{AR}(c)$, is the oriented graph satisfying the following conditions:

- The vertices of $\mathrm{AR}(c)$ are the transpositions (i,j), with $i < j$, in $\mathfrak{S}_n$;
- The arrows of $\mathsf{AR}(c)$ are given, for all $i < j$, by
  - $(i,j) \longrightarrow (i, c(j))$ whenever $i < c(j)$;
  - $(i,j) \longrightarrow (c(i), j)$ whenever $c(i) < j$.

To construct recursively such a graph, we can first find the initial adjacent transpositions of c, which are all the sources, and step by step, using the second rule, construct the arrows and the vertices of $\mathrm{AR}(c)$ until we reach all the transpositions of $\mathfrak{S}_n$. Note that the sinks of $\mathrm{AR}(c)$ are given by the final adjacent transpositions of c. See Figure 5 for an explicit example.

Remark. The Auslander–Reiten quiver of a Coxeter element has a representation-theoretic meaning: briefly it corresponds to the oriented graph whose vertices are the indecomposable representations of a certain type A quiver, and whose arrows are the irreducible morphisms between them.

To see further details about Auslander-Reiten quivers of type A quivers in particular, we refer the reader to [15, Section 3.1]. To learn more about quiver representation theory, and for more in-depth knowledge on the notion of Auslander–Reiten quivers, we invite the reader to look at [1].

4 An extended generalized Ferrers diagram RSK

In the following, we describe a generalized version of RSK using (type A) Coxeter elements, and state the main result.

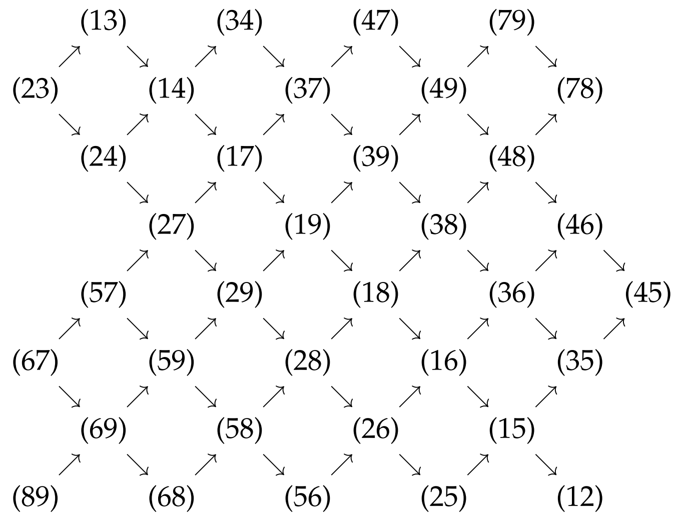


Figure 5: The Auslander–Reiten quiver of $c = (1,3,4,7,9,8,6,5,2) = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$.

Let λ be a nonzero integer partition and consider $(\mathbf{B}, \mathbf{E})$ the unique elementary interval bipartition such that $\lambda(\mathbf{B}, \mathbf{E}) = \lambda$. Set $n = h_\lambda(1,1) + 1$. Let $c \in \mathfrak{S}_n$ and consider $\mathrm{AR}(c)$ its Auslander–Reiten quiver.

Recall that if $\mathbf{B} = \{b_1 < \ldots < b_p\}$ and $\mathbf{E} = \{e_1 < \ldots < e_q\}$, then $(i,j) \in \mathrm{Fer}(\lambda)$ if and only if $b_i < e_{q-j+1}$. It allows us to label each box (i,j) by a transposition $(b_i, e_{q-j+1}) \in \mathfrak{S}_n$. Thus it allows us to construct a one-to-one correspondence from fillings of shape λ to fillings of the Auslander–Reiten quiver $\mathrm{AR}(c)$ which are supported on vertices $(b,e) \in \mathbf{B} \times \mathbf{E}$ such that $b < e$. Explicitly, for any filling f of shape λ, we define $\overline{f}$ be the filling of $\mathrm{AR}(c)$ defined by $\overline{f}(b_i, e_{q-j+1}) = f(i,j)$ whenever $(i,j) \in \mathrm{Fer}(\lambda)$ and $\overline{f}(x,y) = 0$ otherwise.

As in Section 2, for $m \in \{1, \ldots, n-1\}$, let (u_m, v_m) be the maximal pair with respect of $\trianglelefteq$ in $D_m(\lambda)$. The boxes in the ideal generated by (u_m, v_m) correspond to pairs (i,j) such that $b_i \leqslant m < e_{q-j+1}$, and therefore (u_m, v_m) is the maximal pair satisfying this condition.

For each $m \in \{1, \ldots, n-1\}$, we consider the subgraph $\mathrm{AR}_m(c)$ of $\mathrm{AR}(c)$ where the vertices are the transpositions (i,j) with $i \leqslant m < j$. This subgraph has only one source and only one sink.

We define $g = \mathrm{RSK}_{\lambda,c}(f)$ to be the fillings of shape λ defined for $m \in \{1, \ldots, n-1\}$ by

$$\forall (i,j) \in D_m(\lambda), \quad g(i,j) = \mathrm{GK}_{\mathrm{AR}_m(c)}(f)_{u_m - i + 1}.$$

See Figure 6 for an explicit example.

Our main result is the following.

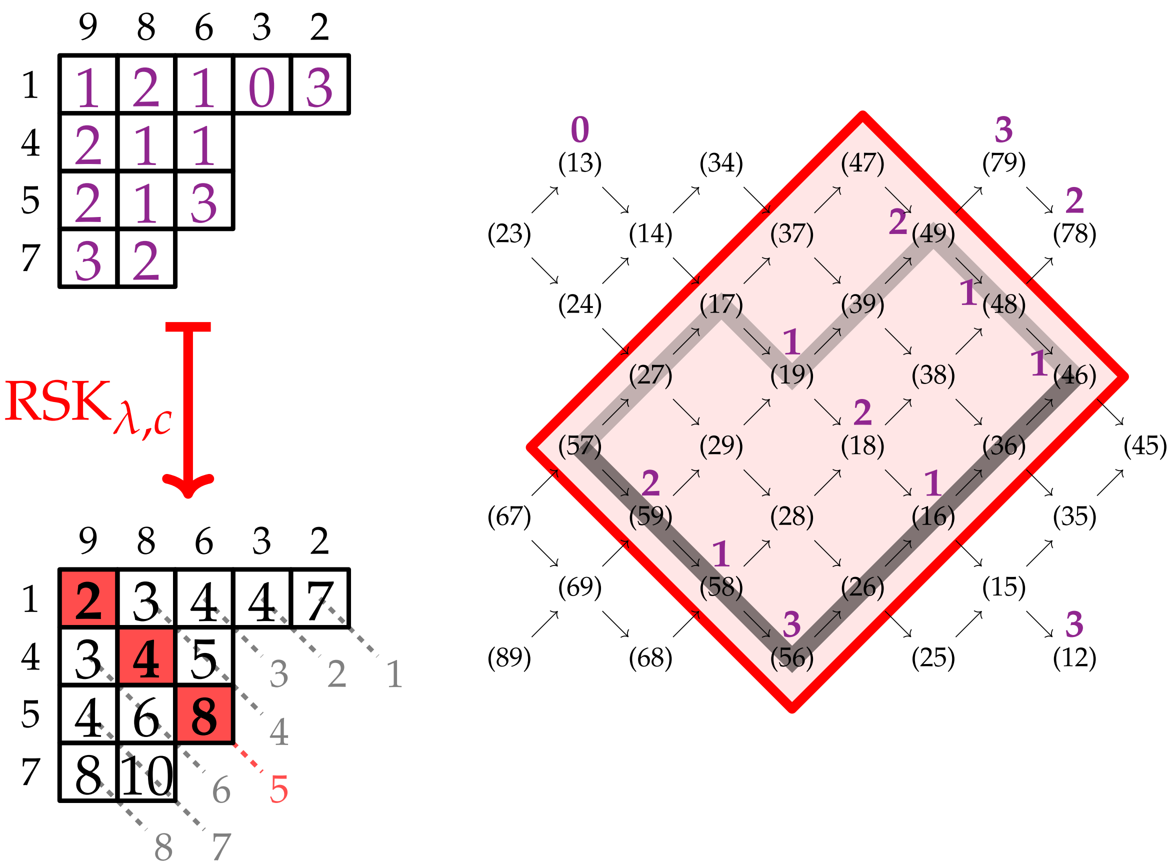


Figure 6: Explicit calculation of $\mathrm{RSK}_{\lambda,c}(f)$ for the boxes in $D_5(\lambda)$ from a filling of $\lambda = (5,3,3,2)$, with $c = (1,3,4,7,9,8,6,5,2)$

Theorem 6. *Let λ be a nonzero integer partition. Consider $n = h_\lambda(1,1) + 1$. Let $c \in \mathfrak{S}_n$ be a Coxeter element. The map $\mathrm{RSK}_{\lambda,c}$ gives a one-to-one correspondence from fillings of shape λ to reverse plane partitions of shape λ.*

The following result shows that we extended the RSK correspondence.

Proposition 7. *Let λ be a nonzero integer partition. Consider $n = h_\lambda(1,1) + 1$ and $(\mathbf{B}, \mathbf{E})$ be the only elementary interval bipartition such that $\lambda(\mathbf{B}, \mathbf{E}) = \lambda$. Let $c \in \mathfrak{S}_n$ be the Coxeter element such that*

- *for $i \in \{1, \ldots, n-1\}$, $(i, i+1)$ is final in c if and only if $i \in \mathbf{B}$ and $i+1 \in \mathbf{E}$;*
- *for $i \in \{2, \ldots, n-2\}$, $(i, i+1)$ is initial in c if and only if $i \in \mathbf{E}$ and $i+1 \in \mathbf{B}$.*

Then $\mathrm{RSK}_{\lambda,c} = \mathrm{RSK}_{\lambda}$. *Moreover, c and* c^{-1} *are the unique Coxeter element of* $\mathfrak{S}_n$ *satisfying this property.*

Remark. Gansner's RSK for a fixed integer partition λ admits a local description in terms of toggles on G_λ. Based on the proof given in [4], for $c = (1, 2, \dots, n)$, we can give a local description in terms of toggles on $\mathrm{AR}(c)$. However, more works need to be done for a general choice of c, as this local description does not extend naturally.

5 Some words about quiver representation theory

This section aims to give a dictionary to link the result from [4] with Theorem 6.

Fix $Q = (Q_0, Q_1)$ a type A quiver. A *finite dimensional representation E of Q over $\mathbb{C}$* is an assignment of a finite dimensional $\mathbb{C}$-vector space E_q to each vertex q of Q, and an assignment of a $\mathbb{C}$-linear transformation $E_\alpha : E_i \longrightarrow E_j$ to each arrow $\alpha : i \to j$ of Q. For two representations E and F, a morphism $\phi : E \longrightarrow F$ is the data of a $\mathbb{C}$-linear map ϕ_q for each vertex q of Q such that for any arrow $\alpha : i \to j$, $\phi_j E_\alpha = F_\alpha \phi_i$. Denote by $\mathrm{rep}_{\mathbb{K}}(Q)$ the category consisting of the representations of Q.

Any representation E of Q can be uniquely decomposed into a direct sum of copies of indecomposable representations up to isomorphism. Thus, we can consider the invariant which counts the number of indecomposable summands of each isomorphism class in E. Write it $\mathrm{Mult}(E)$.

In [10], A. Garver, R. Patrias and H. Thomas introduce a new invariant of quiver representations, called the generic Jordan form data. For any representation E of Q, write $\mathrm{GenJF}(E)$ for the generic Jordan form data of E. This data encodes the generic behavior of a nilpotent endomorphism $N = (N_q)_{q \in Q_0}$ of the representation via the size of the Jordan blocks of each N_q. In some subcategories, the representation can be recovered from this invariant up to isomorphism.

They also show that the map from Mult to GenJF generalizes the RSK correspondence for type A quivers, using Gansner's previous work [8].

As this map is bijective, if we restrict it to the representation in some subcategories $\mathscr{C}$, one can be interested to get an explicit way to invert it. An algebraic method developed in [10] asks the subcategory $\mathscr{C}$ to satisfy the following property. For any $E \in \mathscr{C}$, there exists a dense open set Ω (in the Zariski topology) in the set of representations admitting a nilpotent endomorphism with Jordan forms encoded by $\mathrm{GenJF}(E)$ such that any $F \in \Omega$ is isomorphic to E. Such a subcategory is said to be *canonically Jordan recoverable (CJR)*.

More recently, in [4], we gave a combinatorial characterization of all the CJR subcategories of representations of Q, substancially enlarging the family of subcategories for which GenJF is a complete invariant given by [10]. The maximal such subcategories can be described thanks to the elementary interval partitions $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n+1\}$.

The following table compares the representation-theoretic tools used in [4] and the combinatorial tools used to describe our generalized RSK.

Combinatorial tools	**Representation-theoretic tools**
Coxeter element of $\mathfrak{S}_n$	Orientation of an A_{n-1} type quiver Q
Transposition in $\mathfrak{S}_n$	Indecomposable representation in $\operatorname{rep}_{\mathbb{C}}(Q)$
AR quiver of c	AR quiver of $\operatorname{rep}_{\mathbb{C}}(Q)$
Integer partition λ with $h_\lambda(1,1) = n-1$	CJR subcategory $\mathscr{C}$ of $\operatorname{rep}_{\mathbb{C}}(Q)$
Filling of λ	$\operatorname{Mult}(E)$ for some $E \in \mathscr{C}$
Reverse plane partition of λ	$\operatorname{GenJF}(E)$ for E in $\mathscr{C}$.

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Pattern-avoiding polytopes and Cambrian lattices

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Abstract. For each Coxeter element c in the symmetric group, we define a pattern-avoiding Birkhoff subpolytope whose vertices are the c-singletons. We show that the normalized volume of our polytope is equal to the number of longest chains in a corresponding type A Cambrian lattice. Our work extends a result of Davis and Sagan which states that the normalized volume of the convex hull of the 132 and 312 avoiding permutation matrices is the number of longest chains in the Tamari lattice, a special case of a type A Cambrian lattice. Furthermore, we prove that each of our polytopes is unimodularly equivalent to the order polytope of the heap of the c-sorting word of the longest permutation. This gives an affirmative answer to a generalization of a question posed by Davis and Sagan.

Keywords: order polytopes, heap, Birkhoff polytopes, Cambrian lattices, permutations

1 Introduction

The sequence [6, A003121] counts shifted standard tableaux of staircase shape and longest chains in the Tamari lattice; it also counts the number of reduced words in a certain commutation class of the longest permutation. More recently, it was shown by Davis and Sagan in [2] that this sequence gives the normalized volume of a certain "pattern-avoiding polytope," a subpolytope of the Birkhoff polytope whose vertices are 132 and 312 avoiding permutations. Since these permutations form a distributive sublattice of the right weak order, Davis and Sagan asked whether their polytope might be unimodularly equivalent to the order polytope of the poset of join irreducibles of the 132 and 312 avoiding permutations.

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In the same paper, Davis and Sagan pointed out that the 132 and 312 avoiding permutations are known to be the c-singletons for the symmetric group for a specific "Tamari" Coxeter element c and proposed that it would be interesting to define similar pattern-avoiding polytopes for other Coxeter groups and study them from the perspective of c-singletons. The c-singletons are the spine of an important lattice called the c-Cambrian lattice [7], and they form a distributive sublattice of the right weak order [4].

In this article, we associate a pattern-avoiding polytope to each Coxeter element c in the symmetric group. We define this polytope to be the convex hull of the permutation matrices of the c-singletons (see Sections 3.2 and 4.1). We prove that our polytope is indeed unimodularly equivalent to the order polytope of the poset of join irreducibles of the c-singletons (see Section 5). In particular, for the Tamari Coxeter element, our result answers Davis and Sagan's question in the affirmative.

2 Background and notation

Denote the symmetric group on $n+1$ elements by A_n. We can represent a permutation $w \in A_n$ in *one-line notation* as $w = w(1)w(2)\cdots w(n+1)$. For each $i \in \{1,\ldots,n\}$, we write $s_i \in A_n$ to denote the *simple reflection* (or *adjacent transposition*) that swaps i and $i+1$ and fixes all other letters. Every permutation can be expressed as a product of simple reflections. Given $w \in A_n$, the minimum number of simple reflections among all such expressions for w is called the *(Coxeter) length* of w, and is denoted by $\ell(w)$. A *reduced decomposition* of w is an expression $w = s_{i_1}\cdots s_{i_{\ell(w)}}$ realizing the Coxeter length of w. To simplify notation, we refer to such a decomposition via its *reduced word* $\left[i_1\cdots i_{\ell(w)}\right]$. For example, consider $w = 51342 \in A_4$. One of its reduced decompositions is $s_4s_2s_3s_2s_4s_1$ with [423241] as the corresponding reduced word, and $\ell(w) = 6$.

A *Coxeter element* c in A_n is a product of all n simple reflections in any order, where each reflection appears exactly once. The longest permutation of A_n is the permutation $w_0 = (n+1)n\ldots321$ and $\ell(w_0) = \binom{n+1}{2}$.

Simple reflections satisfy *commutation relations* of the form $s_is_j = s_js_i$ for $|i-j| > 1$. An application of a commutation relation to a product of simple reflections is called a *commutation move*. When referring to reduced words, we will say adjacent letters i and j in a reduced word *commute* when $|i-j| > 1$. Given a reduced word $[u]$ of a permutation, the equivalence class consisting of all words that can be obtained from $[u]$ by a sequence of commutation moves is the *commutation class* of $[u]$.

2.1 Heaps

We review the classical theory of heaps, which was used in [12] to study fully commutative elements of a Coxeter group. Heaps also appeared as "the natural partial orders" in [3, Definition 6] and [5, Definition 1] and they were used to study certain acyclic domains. For a detailed list of attributions on the theory of heaps, see [10, Solutions to Exercise 3.123(ab)].

Definition 2.1. *Given a reduced word $[u] = [u_1 \cdots u_\ell]$ of a permutation, consider the partial order $\preccurlyeq$ on the set $\{1, \ldots, \ell\}$ obtained via the transitive closure of the relations*

$$x \prec y$$

for $x < y$ such that $|u_x - u_y| \leq 1$ (that is, u_x and u_y do not commute). For each $1 \leq x \leq \ell$, the label *of the poset element x is u_x. This labeled poset is called the* heap *for $[u]$, denoted* $\mathsf{Heap}([u])$. *The Hasse diagram for this poset with elements $\{1, \ldots, \ell\}$ replaced by their labels is called the* heap diagram *for $[u]$. The labels in the heap diagram are drawn in increasing order from left to right.*

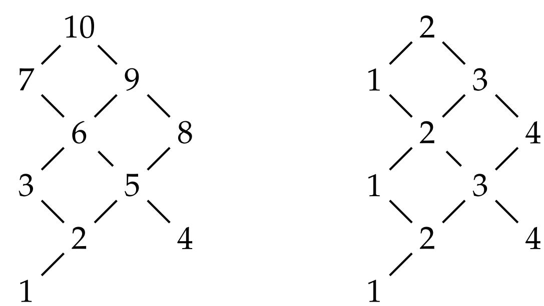

Figure 1: Hasse diagram of the underlying poset (left) and the heap diagram (right) of a commutation class of w_0 given in Example 2.2.

Example 2.2. *Consider a reduced word $[u] = [u_1 \ldots u_{10}] = [1214321432]$ of the longest element w_0 in A_4.*

1. *Figure 1 (left) shows a Hasse diagram of the underlying unlabeled poset* $\mathsf{Heap}([u])$. *Here $\ell = 10$ and so the the elements of the heap poset* $\mathsf{Heap}([u])$ *are $\{1, 2, \ldots, 10\}$.*

2. *Figure 1 (right) shows the heap diagram for* $\mathsf{Heap}([u])$. *The possible labels of the poset elements are $\{1, 2, 3, 4\}$.*

Linear extensions of $\mathsf{Heap}([u])$ relate to the commutation class of $[u]$.

Definition 2.3. *A* linear extension *$\pi = \pi(1) \cdots \pi(\ell)$ of a partial order $\preccurlyeq$ on $\{1, \ldots, \ell\}$ is a total order on the poset elements that is consistent with the structure of the poset. That is, $x \prec y$ implies $\pi(x) < \pi(y)$. A* labeled linear extension *of the heap of a reduced word $[u] = [u_1 \cdots u_\ell]$ is a word $\left[u_{\pi(1)} \cdots u_{\pi(\ell)}\right]$, where $\pi = \pi(1) \cdots \pi(\ell)$ is a linear extension of the heap.*

Proposition 2.4 ([12, Proof of Proposition 2.2] and [10, Solutions to Exercise 3.123(ab)])**.** *Given a reduced word $[u]$, the set of labeled linear extensions of the heap for $[u]$ is the commutation class of $[u]$.*

Example 2.5. *Three of labeled linear extensions of* $\mathsf{Heap}([u])$ *from Example 2.2 are* $[u]$ *itself,* $[1243124312]$*, and* $[4123412312]$*. Notice that these reduced words all belong to the same commutation class, due to Proposition 2.4.*

2.2 Order polytopes

In this section, we review *order polytopes*, following Stanley's paper [11]. Given a finite poset P, the order polytope of P is given by

$$\mathcal{O}(P) := \{x \in \mathbb{R}^P : 0 \leqslant x_t \leqslant 1 \text{ for all } t \in P \text{ and } x_t \leqslant x_s \text{ when } t \leqslant_P s\}.$$

Many basic properties of an order polytope are answered by the combinatorial structure of the poset. Below are some properties that are relevant to us.

1. The dimension of $\mathcal{O}(P)$ is given by the number of elements in P.
2. The volume of $\mathcal{O}(P)$ can be computed from the number of linear extensions of P.
3. The vertices of $\mathcal{O}(P)$ are exactly the indicator vectors of order ideals of P.

In this paper, we will be focusing on order polytopes for certain heap posets.

3 c-singletons

3.1 c-sorting words and c-sortable permutations

In this section, we review c-sorting words and c-sortable elements, which were introduced in [8]. Given a Coxeter element c and reduced word $[a_1 a_2 \dots a_n]$, define an infinite word

$$c^\infty := a_1 a_2 \dots a_n \mid a_1 a_2 \dots a_n \mid \cdots$$

consisting of repeated copies of the given reduced word for c. The symbols "|" are "dividers" which facilitate the definition of sortable elements. The c-sorting word of $w \in A_n$ is the lexicographically first (as a sequence of positions in c^∞) subword of c^∞ that is a reduced word for w. We denote this word by $\mathsf{sort_c}(w)$.

We say that the identity permutation is *c-sortable*. If w is not the identity permutation, we can think of $\mathsf{sort_c}(w)$ as a sequence of nonempty subsets of $\{a_1, \dots, a_n\}$. The subsets $K_1, K_2, \dots, K_p$ in this sequence are the sets of letters of c that occur between two adjacent dividers, so we have $x \in K_j$ if x is in the jth copy of c inside c^∞. We say that a permutation w is *c-sortable* if $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p$. The set of c-sortable permutations does not depend on the choice of reduced word for c.

Example 3.1. *Consider the Coxeter element $c = s_1 s_2 s_3 s_4 = [1234]$ of A_4. Then the c-sorting word of the permutation* 42351 *is* $[1234 \mid 2 \mid 1]$. *Our subsets are $K_1 = \{1, 2, 3, 4\}$, $K_2 = \{2\}$, and $K_3 = \{1\}$. Since $K_2 \not\supseteq K_3$, these sets do not form a nested sequence and therefore* 42351 *is not c-sortable. On the other hand, the permutation* 43215 *has c-sorting word* $[123 \mid 12 \mid 1]$ *and is c-sortable.*

Reading showed in [9] that the restriction of the right weak order to c-sortable elements is a lattice which is isomorphic to an important quotient of the right weak order called the c-Cambrian lattice [7]. For the Coxeter element $c = s_1 s_2 \ldots s_n$, the c-sortable elements form the Tamari lattice. For this reason, we refer to this Coxeter element as the "Tamari" Coxeter element of A_n. Cambrian lattices and c-sortable elements have strong connections to cluster algebras, representation theory, and many areas of combinatorics, and they are widely studied. We will be interested in a subclass of c-sortable elements, called c-singletons, which we describe next.

3.2 c-singleton permutations

There is an order-preserving projection $\pi_\downarrow^c : A_n \to A_n$ which sends an element w to the largest c-sortable element that is weakly below w in the right weak order [9, Proposition 3.2]. In [4], Hohlweg, Lange, and Thomas used this map to introduce an important subclass of c-sortable elements: A c-sortable w is called a *c-singleton* if the preimage of $\{w\}$ under $\pi_\downarrow^c$ is the singleton $\{w\}$ itself. We will use the following characterization of c-singletons.

Theorem 3.2 ([4, Theorem 2.2]). *A permutation w is a c-singleton if and only if some reduced word of w is a prefix of a word in the commutation class of $\mathsf{sort}_\mathsf{c}(w_0)$, the c-sorting word of the longest permutation w_0.*

The set of c-singletons form a distributive sublattice of the right weak order due to [4, Proposition 2.5]. We denote this lattice by $\mathcal{L}$(c-singletons). By [5, Proposition 3], $\mathcal{L}$(c-singletons) is isomorphic to the lattice of order ideals of $\mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w_0))$, which we denote by $J(\mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w_0)))$.

Proposition 3.3. *The following map is a poset isomorphism*

$$\begin{aligned} f : \mathcal{L}(\textit{c-singletons}) &\to J(\mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w_0))) \\ w &\mapsto \mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w)) \end{aligned}$$

between the c-singletons and the order ideals of the heap poset $\mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w_0))$.

As we noted in Section 2.2, the vertices of the order polytope $\mathcal{O}(P)$ of a poset P correspond to the order ideals of P. As a consequence, the c-singletons are in bijection with the vertices of $\mathcal{O}(\mathsf{Heap}(\mathsf{sort}_c(w_0)))$.

4 c-Birkhoff polytopes

The *Birkhoff polytope* is the convex hull of all permutation matrices. Davis and Sagan [2] studied a "pattern-avoiding" subpolytope of the Birkhoff polytope whose vertices correspond to the permutations avoiding the pattern 132 and 312. As noted in [2, Remark 3.6], for the "Tamari" Coxeter element $c = s_1 s_2 \ldots s_n$, the c-singletons are precisely the permutations which avoid these same patterns 132 and 312.

4.1 A pattern-avoidance criterion for c-singletons

There is a similar classification of c-singletons for other c (see Proposition 4.1). In this section, we generalize Davis and Sagan's pattern-avoiding polytope coming from the "Tamari" Coxeter element c to all Coxeter elements c in A_n.

Let c be a Coxeter element in A_n. There is exactly one commutation class for c, so every reduced word for c has the same heap. By abuse of notation we write $\mathsf{Heap}(c)$ to denote this heap. Then $\mathsf{Heap}(c)$ is the partial order on $\{1, \ldots, n\}$ obtained via the transitive closure of the cover relations $i - 1 \prec i$ if $i - 1$ appears to the left of i in every reduced word of c, and $i - 1 \succ i$ otherwise.

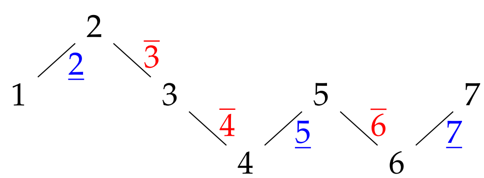

Figure 2: The heap diagram for the Coxeter element $c = s_1 s_4 s_3 s_2 s_6 s_5 s_7$ which corresponds to lower-barred numbers $\underline{2}$, $\underline{5}$, $\underline{7}$ and upper-barred numbers $\overline{3}$, $\overline{4}$, $\overline{6}$.

As described in [7, Chapter 6], we partition the integers in $[2, n]$ into lower-barred and upper-barred numbers $\underline{[2, n]}$ and $\overline{[2, n]}$, respectively. If $i - 1 \prec i$, define i to be a lower-barred number $\underline{i} \in \underline{[2, n]}$; If $i - 1 \succ i$, define i to be a upper-barred number $\bar{i} \in \overline{[2, n]}$. For example, see Figure 2.

We say that a permutation w avoids the pattern $31\underline{2}$ if w contains no 312-pattern such that the last entry "2" in the pattern is a lower-barred number. Similarly, a permutation w avoids the pattern $\overline{2}31$ if the one-line notation w contains no 231-pattern such that the first entry "2" in the pattern is an upper-barred number.

The following result of [7, Proposition 5.7] characterizes c-sortable and c-singleton permutations using pattern-avoidance.

Proposition 4.1. *A permutation $w \in A_n$ is c-sortable if and only if the one-line notation of w avoids the patterns $31\underline{2}$ and $\overline{2}31$. Furthermore, a c-sortable permutation w is a c-singleton if and only if w avoids the patterns $13\underline{2}$ and $\overline{2}13$.*

Definition 4.2. *For a Coxeter element c in A_n, we define the* c-singleton Birkhoff polytope*, or* c-Birkhoff polytope *for short, to be the convex hull of the permutation matrices corresponding to the c-singletons, that is, the permutations avoiding the four patterns listed in Proposition 4.1. We denote the c-Birkhoff polytope as* $\mathsf{Birk}(c)$.

Note that our convention is that the permutation matrix of the permutation $w = w(1)\ldots w(n+1)$ has 1's in entries $(i, w(i))$.

Remark 4.3. *Davis and Sagan suggested in [2, Remark 3.6] that it would be interesting to define pattern-avoiding polytopes for other Coxeter groups. Theorem 4.12 of [8] characterizes type B and D c-sortable elements as signed permutations satisfying certain pattern avoidance conditions. This can be used to give us an analog of Proposition 4.1 for type B and D c-singletons.*

4.2 Relations for c-Birkhoff polytopes

The classical Birkhoff polytope of A_n lives in a $(n+1)^2$-dimensional ambient space. Since each of its row and column sum up to one, it is a n^2-dimensional polytope. The row and column relations for the classical Birkhoff polytope also hold in the c-Birkhoff polytope, since the vertices still come from permutation matrices. Our goal in this section is to exhibit $\binom{n}{2}$ additional relations which any point in $\mathsf{Birk}(c)$ satisfies, so that it is in fact an $\binom{n+1}{2}$-dimensional polytope.

From the pattern avoidance criteria in Proposition 4.1, we see that if w is a c-singleton and $i \in \underline{[2,n]}$, all numbers less than i or all numbers greater than i must appear after i in the one-line notation w and similarly for upper-barred numbers. In particular, if $m = \max(i+1, n-i+2)$, then i cannot appear in any of the last m spots of w.

We will consider points in $\mathsf{Birk}(c)$ as matrices $(x_{i,j})$ for $1 \leq i, j \leq n+1$.

Proposition 4.4. *Let c be a Coxeter element in A_n, and let $(x_{i,j})$ be a point in the c-Birkhoff polytope. For $2 \leq i \leq n$, if $i \in \underline{[2,n]}$ and $m = \max(i+1, n+2-i)$, we have*

$$x_{m,i} = x_{m+1,i} = \cdots = x_{n+1,i} = 0.$$

Otherwise, $i \in \overline{[2,n]}$ and if $r = \min(i-1, n+2-i)$, then

$$x_{1,i} = x_{2,i} = \cdots = x_{r,i} = 0.$$

Using the pattern avoidance repeatedly puts restrictions on which numbers can appear together in the first u spots of a c-singleton for some values of u.

Theorem 4.5. *Let c be a Coxeter element in A_n. For each $1 \leq i \leq \frac{n-1}{2}$ and $i+1 \leq u \leq n-i$, there exists a sequence $i = v_0 < v_1 < \cdots < v_d$, where $d \geq 1$, such that*

$$\sum_{j=0}^{d} \sum_{i=1}^{u} x_{i,v_j}$$

is equal to either 1 or d (depending on i and u) for all points in the c-Birkhoff polytope.

Example 4.6. *Let $c = s_1 s_2 s_4 s_3$. Then the c-Birkhoff polytope has $\binom{4}{2} = 6$ additional relations. Proposition 4.4 gives us four relations*

$$x_{5,2} = x_{4,3} = x_{5,3} = x_{1,4} = 0.$$

Theorem 4.5 gives us two more relations. For $i = 1$, $u = 2$, the sequence is $v_0 = 1, v_1 = 3, v_2 = 4$, and we have the relation

$$\sum_{j=0}^{2} \sum_{i=1}^{2} x_{i,v_j} = \sum_{j \in \{1,3,4\}} \sum_{i=1}^{2} x_{i,j} = x_{1,1} + x_{2,1} + x_{1,3} + x_{2,3} + x_{1,4} + x_{2,4} = 1.$$

For $i = 1$, $u = 3$, the sequence is $v_0 = 1$, $v_1 = 5$ and we have the relation

$$\sum_{j=0}^{1} \sum_{i=1}^{3} x_{i,v_j} = \sum_{j \in \{1,5\}} \sum_{i=1}^{3} x_{i,j} = x_{1,1} + x_{2,1} + x_{3,1} + x_{1,5} + x_{2,5} + x_{3,5} = 1.$$

5 $\mathsf{Birk}(c)$ and $\mathcal{O}(\mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w_0)))$

In this section, we will prove $\mathsf{Birk}(c)$ is unimodularly equivalent to the order polytope $\mathcal{O}(\mathsf{Heap}(\mathsf{sort}_\mathsf{c}(w_0)))$. We will achieve this by first explicitly constructing a lattice-preserving projection Π_c on $\mathsf{Birk}(c)$, and then show the existence of a unimodular transformation $\mathcal{U}_c$.

5.1 A lattice-preserving projection

Let c be a Coxeter element of A_n. We define a projection Π_c on $(n+1) \times (n+1)$-matrices which reads $\binom{n+1}{2}$ of the entries in a specific order. We describe the reverse order by reading entries in the matrix.

Let $1 < \underline{p_1} < \cdots < \underline{p_r} < n+1$ be the set of lower-barred numbers and $\overline{q_1} < \ldots \overline{q_s} < n+1$ be the set of upper-barred numbers for c. Let σ be the permutation

$(n+1)\, p_r\, p_{r-1} \dots p_1\, 1\, q_1 q_2 \, \dots \, q_s$ written in one-line notation. The first entries we will read are

$$\begin{aligned}
&(p_1 - 1, p_1), (p_1 - 2, p_1), \dots, (1, p_1), \\
&\dots \\
&(p_r - 1, p_r), (p_r - 2, p_r), \dots, (1, p_r), \\
&(n, n+1), (n-1, n+1), \dots, (1, n+1).
\end{aligned}$$

The remaining entries come from $q_s, \dots, q_1$. For each q_i, take $q_i - 1$ entries as follows:

- Let $m = \min(q_i - 1, n+1-q_i)$. Let $\sigma_1^i, \sigma_2^i, \dots, \sigma_m^i$ be the m numbers of σ (in one-line notation) immediately before q_i, from right to left.
- First take the m entries $(n+1, \sigma_1^i), (n, \sigma_2^i), \dots, (n+2-m, \sigma_i^m)$.
- Then take the additional $q_i - 1 - m$ entries $(q_i - 1, q_i), (q_i - 2, q_i), \dots, (m+1, q_i)$.

Example 5.1. *Let $c = s_1 s_4 s_3 s_2 s_6 s_5 s_7$ be the Coxeter element whose Heap diagram and corresponding upper- and lower-barred numbers are illustrated in Figure 2. Then $\underline{p_1}, \underline{p_2}, \underline{p_3} = \underline{2}, \underline{5}, \underline{7}$ and $\overline{q_1}, \overline{q_2}, \overline{q_3} = \overline{3}, \overline{4}, \overline{6}$. We have $\sigma = 87521346$. We compute the projection Π_c in Figure 3 (left).*

	28	X	X	24	X	18	11
		X	X	25	X	19	12
			X	26	6	20	13
				27	7	21	14
					8	22	15
	3			X		23	16
4	1	9		X			17
2	X	5	10	X		X	

0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0
1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

Figure 3: Left: The projection Π_c of Example 5.1. Right: permutation matrix for b_4 of Example 5.5.

Theorem 5.2. *Π_c is a lattice-preserving projection on the c-Birkhoff polytope.*

5.2 A diagonal reading word

Let c be a Coxeter element of A_n and let $\mathcal{R}_c$ denote the labeled linear extension of $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ which is lexicographically first, in the sense of lexicographic order on heap labels. Observe that $\mathcal{R}_c$ is the word formed by concatenating the diagonals of

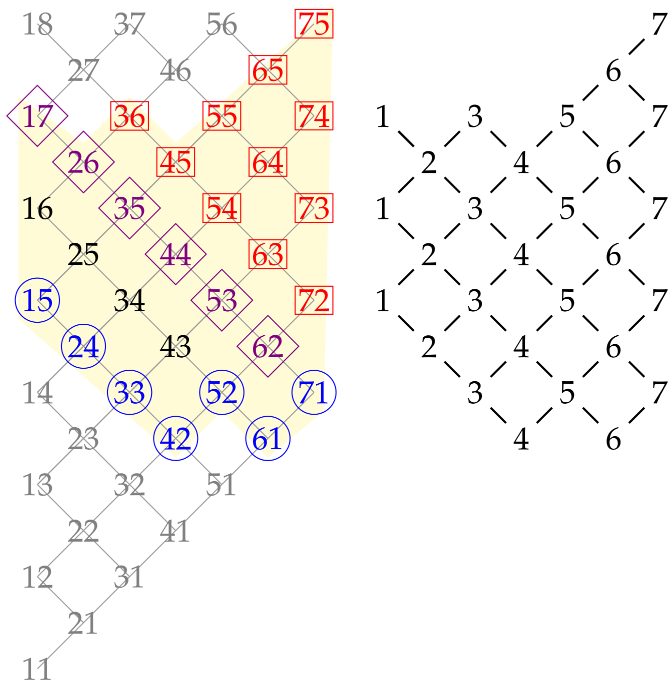

Figure 4: Left: Algorithm for constructing the heap $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ for $[4321657]$-sorting word of the longest element w_0 in A_7. Right: The heap diagram for $\mathsf{Heap}(\mathsf{sort}_c(w_0))$.

$\mathsf{Heap}(\mathsf{sort}_c(w_0))$ from left to right; within each diagonal, read from southeast to northwest. For this reason, we refer to the reduced word $\mathcal{R}_c$ as the *diagonal reading word* of $\mathsf{Heap}(\mathsf{sort}_c(w_0))$.

Thanks to [1, 5], we can give a nice algorithmic construction of $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ using the upper- and lower-barred numbers corresponding to c. For example, Figure 4 shows this construction for $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ where $c = [4321657]$ with lower-barred numbers $\underline{5}, \underline{7}$ and upper-barred numbers $\overline{2}, \overline{3}, \overline{4}, \overline{6}$. This algorithm gives us the following lemma.

Lemma 5.3. *Let c be any Coxeter element in A_n with s upper-barred numbers $1 < \overline{q_1} < \cdots < \overline{q_s} < n+1$ and r lower-barred numbers $1 < \underline{p_1} < \cdots < \underline{p_r} < n+1$. Then*

$$\mathcal{R}_c = \Big[(\underline{p_1}-1)...1\Big] \ldots \Big[(\underline{p_r}-1)...1\Big]\, [n...1]\, [n...(n-\overline{q_s}+2)] \ldots [n...(n-\overline{q_1}+2)]$$

is a concatenation of n factors, where each factor is a decreasing sequence of consecutive integers.

Example 5.4. *For example, the reduced word $[u] = [1]\,[21]\,[4321]\,[432]$ given in Example 2.2 is the diagonal reading word $\mathcal{R}_{[1243]}$ of the heap diagram in Figure 1. The diagonal reading word of the heap diagram given in Figure 4 is*

$$\mathcal{R}_{[4321657]} = [4321]\,[654321]\,[7654321]\,[76543]\,[765]\,[76]\,[7]\,.$$

5.3 Unimodular equivalence

Note that $\mathcal{R}_c$ is of length $\binom{n+1}{2} = \ell(w_0)$, and write $\mathcal{R}_c = \left[r_1 \ldots r_{\ell(w_0)}\right]$. For each $1 \leqslant i \leqslant \ell(w_0)$, define b_i to be the length-i prefix of $\mathcal{R}_c$, that is, $b_i = [r_1 \ldots r_i]$. Since $\mathcal{R}_c$ is a labeled linear extension of $\mathsf{Heap}(\mathsf{sort_c}(w_0))$, Proposition 2.4 tells us that it is in the commutation class of $\mathsf{sort_c}(w_0)$, and thus Theorem 3.2 tells us that each b_i is a c-singleton.

Given a c-singleton w, let $f(w)$ be the corresponding order ideal of $\mathsf{Heap}(\mathsf{sort_c}(w_0))$, where f is as defined in Proposition 3.3. Consider the vector in $\mathbb{R}^{\ell(w_0)}$ defined by the indicator function of $f(w)$, following the linear extension given by $\mathcal{R}_c$. Let $o(w)$ denote this vector in reverse order. In particular, note that $o(b_i)$ is the vector whose last i entries are 1s and whose all other entries are 0s.

Example 5.5. *Let c be as in Example 5.1 with $\underline{p_1}, \underline{p_2}, \underline{p_3} = \underline{2}, \underline{5}, \underline{7}$ and $\overline{q_1}, \overline{q_2}, \overline{q_3} = \overline{3}, \overline{4}, \overline{6}$. We have $\mathcal{R}_c = [1\ 4321\ 654321\ 7654321\ 76543\ 765\ 76]$. Therefore $b_4 = s_1 s_4 s_3 s_2 = 25134678$ and its permutation matrix is in Figure 3 (right). We can then compute*

$$\Pi_c(b_4) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1)\,, \text{ and}$$
$$o(b_4) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1)\,.$$

Lemma 5.6. *Let c be a Coxeter element of A_n. Then the $\left(\binom{n+1}{2} - i + 1\right)^{th}$ entry of the vector $\Pi_c(b_i)$ is 1, and all earlier entries of $\Pi_c(b_i)$ are zero. That is, the matrix whose columns are $\Pi_c(b_i)$ is a lower antidiagonal triangular matrix with 1's along the antidiagonal.*

Theorem 5.7. *Fix a Coxeter element c in A_n. There exists a $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix $\mathcal{U}_c$ with 1's on the main diagonal such that $\mathcal{U}_c \circ \Pi_c(b_i) = o(b_i)$ for all $1 \leqslant i \leqslant \binom{n+1}{2}$. Furthermore, we have $\mathcal{U}_c \circ \Pi_c(w) = o(w)$ for any c-singleton w.*

Corollary 5.8. *The c-Birkhoff polytope $\mathsf{Birk}(c)$ is unimodularly equivalent to the order polytope $\mathcal{O}(\mathsf{Heap}(\mathsf{sort_c}(w_0)))$.*

Proof. This follows from the facts that the projection Π_c preserves lattice points (Theorem 5.2) and that the linear transformation $\mathcal{U}_c$ has determinant 1 (Theorem 5.7). □

Corollary 5.9. *The normalized volume of the c-Birkhoff polyope is equal to the number of longest chains in the corresponding Cambrian lattice.*

Corollary 5.9 recovers, and generalizes, a result of Davis and Sagan in [2].

Remark 5.10. *One might ask whether our result generalizes as follows: If $w \in A_n$ and $[u]$ is a reduced word for w then the order polytope of $\mathsf{Heap}[u]$ is unimodularly equivalent to the convex hull of the permutations corresponding to order ideals of $\mathsf{Heap}[u]$. This is not true in general; for A_4 the reduced words $[2123243212]$ and $[3432312343]$ are counterexamples. It would be interesting to determine when the above statement holds.*

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A Consistent Sandpile Torsor Algorithm for Regular Matroids

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Abstract. Every regular matroid is associated with a *sandpile group*, which acts simply transitively on the set of bases in various ways. Ganguly and the second author introduced the notion of *consistency* to describe classes of actions that respect deletion-contraction in a precise sense, and proved the consistency of rotor-routing torsors (and uniqueness thereof) for plane graphs.

In this work, we prove that the class of actions introduced by Backman, Baker, and the fourth author, is consistent for regular matroids. This generalizes the above existence assertion, as well as makes progress on the goal of classifying all consistent actions.

Keywords: regular matroid, sandpile group, sandpile torsor, fourientation

1 Introduction

For over a century, mathematicians have been interested in enumerative properties of the *spanning trees* of graphs. A remarkable and relatively recent observation is that the set of spanning trees of a graph (and more generally, the *bases* of a *regular matroid*) admit interesting group actions, which bestow on these sets a group-like structure. We are curious about this mysterious algebraic structure, especially in cases where it is surprisingly canonical.

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To be more precise, the *sandpile group* (also called the critical group, Jacobian, etc.) $S(G)$ of a graph G is a finite abelian group whose size is equal to the number of spanning trees of G. The algebraic structure discussed in the previous paragraph is given by a *simply transitive action* of $S(G)$ on the spanning trees of G. Loosely speaking, we call such an action a *sandpile torsor*. To define sandpile torsors in a systematic way, it is necessary to work on graphs with some auxiliary data (see [19, Theorem 8.1]).

One possible setup is to work with *embedded* graphs (called *ribbon graphs* or *maps*). There are at least two known ways to associate each *rooted* embedded graph with a sandpile torsor: the *rotor-routing model* (see [13]) and the *Bernardi bijections* (see [5]). While these approaches have been shown to give different actions in general [8, 18], the situation changes dramatically when restricting to *plane graphs* (i.e., *planar* embedded graphs): both actions are independent of the root chosen ([7, Theorem 2], [5, Theorem 5.1]), and they in fact produce the same *sandpile torsor algorithm (on plane graphs)* [5, Theorem 7.1] (see [14] for an alternate definition of this action). This lead Klivans to conjecture that this algorithm was in some sense canonical, and all "nice" sandpile torsor algorithms on plane graphs must have the same structure [15, Conjecture 4.7.17]. This conjecture was made precise and proven by Ganguly and the second author [10].

The first challenge to resolve this conjecture was to give a suitable definition for a "nice" sandpile torsor algorithm. To do this, the authors introduce the notion of *consistency*. In general, sandpile groups do not behave well with respect to contraction and deletion: the additive relation $|S(G)| = |S(G \setminus e)| + |S(G/e)|$ implies that it is almost impossible to relate these groups directly in an algebraically meaningful way. Nevertheless, for specific combinations of group elements and spanning trees, there is a way to make sense of the contraction and deletion operations. A *consistent* sandpile torsor algorithm is essentially one which respects these operations.

A bit more precisely, fix a class of graphs G's (e.g., planar graphs) and a class of auxiliary structures α's for these graphs (e.g., planar embeddings) with a notion of deletion and contraction. Moreover, suppose both classes are minor closed. To any pair (G, α), a sandpile torsor algorithm associates a simply transitive action $\cdot$ of the sandpile group on the spanning trees. Note that the sandpile group is generated by equivalence classes corresponding to individual arcs (directed edges). On graphs, these arcs indicate a single chip on a vertex and a single negative chip on an adjacent vertex.

We say that the sandpile torsor algorithm is consistent if, given (G, α) that induces $\cdot$, and any arc f and spanning tree T of G such that $f \cdot T = T'$, we have:

- for any edge $e \notin T \cup T' \cup f$, the pair $(G \setminus e, \alpha \setminus e)$ induces $\cdot'$ with $f \cdot' T = T'$,
- for any edge $e \in T \cap T' \setminus f$, the pair $(G/e, \alpha/e)$ induces $\cdot''$ with $f \cdot'' (T \setminus e) = T' \setminus e$,
- the action of f does not modify the part of a spanning tree falling into a different biconnected component than f.

The two main results of [10] were proving the existence and the uniqueness of a consistent sandpile torsor algorithm on plane graphs.[1] The main theorem of our paper is generalizing the existence result to a regular matroid context, proving [10, Conjecture 6.11].

Building off work from Bacher, de la Harpe, and Nagnibeda [1], Merino defined the sandpile group of a regular matroid [16]. For regular matroids, *bases* play the role that spanning trees played for graphs. In particular, the regular matroid version of a sandpile torsor algorithm is a map from any regular matroid M (with some auxiliary data) to a simply transitive action of the sandpile group of M on the set of bases of M.

Using the auxiliary data of *acyclic circuit-cocircuit signatures*, Backman, Baker, and the fourth author defined a sandpile torsor algorithm for regular matroids which was motivated from polyhedral geometry [2]. We call this the *BBY algorithm*.[2] Later, the first author [9], and the fourth author with Backman and Santos [4] independently showed that the same definition works also for the broader class of *triangulating circuit-cocircuit signatures*. We will continue to refer to this more general setting as the BBY algorithm.

The notion of consistency can be defined analogously for matroids. Moreover, deletion and contraction can be defined for triangulating signatures. Our main result is the following. (For a more formal statement, see Theorem 2.22.)

Theorem 1.1. *The BBY algorithm (that associates a sandpile action to a regular matroid equipped with a triangulating circuit-cocircuit signature) is consistent.*

We note that since rotor-routing torsors on plane graphs are special cases of BBY torsors, this theorem also implies the "existence" part from [10], i.e., that the rotor-routing algorithm is consistent, see Section 4.2. We conjecture that a converse also holds, namely, that for the auxiliary structure of triangulating signatures, the BBY action is the unique consistent sandpile action. This is an modified version of [10, Conjecture 6.14] for triangulating signatures instead of acyclic signatures.

The arguments to prove our theorem are fundamentally different from those of [10], as [10] frequently uses the vertices of the graph in its arguments, which do not have a matroidal analogue. Instead, we apply a framework introduced by the first author [9] that gives an alternate definition of the BBY algorithm using *fourientations*, an object that was first defined by Backman and Hopkins [3]. Our proof essentially comes down to classifying ways that consistency could be violated and then showing that each of these potential possibilities leads to a contradiction.

[1] More precisely, there is a unique collection of four sandpile torsor algorithms on plane graphs that are all closely related.

[2] The (implicit) original name of the corresponding bijections was *geometric bijections*, which the fourth author prefers.

2 Background and Notation

For a set X, we write $X \setminus x$ for $X \setminus \{x\}$, and use similar notation for other operations with a singleton. We call elements of $\mathbb{Z}^E$ *(integral) 1-chains*, where E is an index set. For a 1-chain $\overrightarrow{P}$, and some $e \in E$, we write $\overrightarrow{P}\langle e\rangle$ for the coefficient of e in $\overrightarrow{P}$. We also write $P := \{e \in E : \overrightarrow{P}\langle e\rangle \neq 0\}$ for the *support* of $\overrightarrow{P}$. For $e \in E$, denote by $\overrightarrow{P} \setminus e$ the 1-chain in $\mathbb{Z}^{E\setminus e}$ obtained by restricting $\overrightarrow{P}$ to $E \setminus e$. A 1-chain is *simple* if every coefficient is in $\{-1, 0, 1\}$. An *arc* is a simple 1-chain whose support has only one element. We write arcs in the form $\overrightarrow{e}$, where $e \in E$.

2.1 Oriented Matroids and Regular Matroids

We assume standard background on matroid and oriented matroid theory from the reader. Some standard references are [17] and [6]. Let A be an $r \times m$ *totally unimodular* matrix of full row rank, i.e., a matrix over the reals in which the determinant of every square submatrix is either $-1, 1$, or 0. Let E be a set that indexes the columns of A. Then A represents a *regular matroid* $M := M(A)$ whose ground set is E and of rank r. Denote by $\mathbf{B}(M), \mathbf{C}(M), \mathbf{C}^*(M)$ the set of bases, circuits, and cocircuits of M, respectively. We fix such A and M for the rest of this paper.

We call a simple 1-chain whose support is a circuit (resp. cocircuit) a *signed circuit* (resp. *signed cocircuit*) of M. The collections of signed circuits and signed cocircuits of M are denoted by $\overrightarrow{\mathbf{C}}(M)$ and $\overrightarrow{\mathbf{C}^*}(M)$, respectively. The sets $\overrightarrow{\mathbf{C}}(M)$ and $\overrightarrow{\mathbf{C}^*}(M)$ give M the structure of an oriented matroid, and it is shown in [6, Corollary 7.9.4] that all oriented matroid structures on M are equivalent up to reorientation.

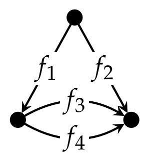
An *orientation* is a map from E to $\{-, +\}$. Adopting the usual convention, we write $\mathcal{O}\langle x\rangle$ for the value of $\mathcal{O}$ at x. Denote by $\mathbf{O}(M)$ the set of all orientations of M.

Definition 2.1. Let $\mathcal{O}$ be an orientation and P be a subset of E. Then we write ${}_P\mathcal{O}$ for the orientation obtained by reversing the elements of P. In other words, for $x \in E$, we have

$$ {}_P\mathcal{O}\langle x\rangle = \begin{cases} -\mathcal{O}\langle x\rangle & \text{if } x \in P, \\ \mathcal{O}\langle x\rangle & \text{if } x \notin P. \end{cases} $$

Definition 2.2. Let $\mathcal{O}$ be an orientation and $\overrightarrow{P}$ be a simple 1-chain. We say that $\overrightarrow{P}$ is *compatible with* $\mathcal{O}$ if for all $f \in P$, the sign of $\overrightarrow{P}\langle f\rangle$ matches the sign of $\mathcal{O}\langle f\rangle$. We denote compatibility by writing $\overrightarrow{P} \sim \mathcal{O}$.

Let $\mathcal{O} \in \mathbf{O}(M)$ and $\overrightarrow{C}$ be a signed circuit that is compatible with $\mathcal{O}$. We say that ${}_C\mathcal{O}$ is a *circuit reversal* of $\mathcal{O}$. Define *cocircuit reversals* analogously. Two orientations $\mathcal{O}_1$ and $\mathcal{O}_2$ differ by *circuit-cocircuit reversals* if $\mathcal{O}_1$ can be sent to $\mathcal{O}_2$ by a sequence of circuit and/or cocircuit reversals. It is easy to show that this is an equivalence relation on $\mathbf{O}(M)$.



$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure 1: A graph (graphic matroid) and its corresponding representing matrix.

Example 2.3. Take the graphic matroid in Figure 1, and take the orientation $\mathcal{O}$ with $\mathcal{O}\langle f_1\rangle = +$, $\mathcal{O}\langle f_2\rangle = -$, $\mathcal{O}\langle f_3\rangle = +$ and $\mathcal{O}\langle f_4\rangle = +$ (in short, $(+,-,+,+)$; we will use this shorthand throughout). The signed circuit $C = \overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_3}$ is compatible with $\mathcal{O}$. By reversing C, we get the orientation $(-,+,-,+)$.

Definition 2.4. The *circuit-cocircuit equivalence classes* of M are the orientations of M modulo the equivalence relation defined in the previous paragraph. The set of these equivalence classes is denoted $\mathbf{G}(M)$. For any element $\mathcal{O} \in \mathbf{O}(M)$, we write $[\mathcal{O}]$ for the equivalence class of $\mathbf{G}(M)$ containing $\mathcal{O}$.

The set $\mathbf{G}(M)$ was first explored by Gioan [11, 12], and it serves as an intermediate object to define the BBY action because of the natural torsor structure described in the next section; in particular, we have the following enumerative fact.

Theorem 2.5. *[12] For a regular matroid M, we have $|\mathbf{G}(M)| = |\mathbf{B}(M)|$.*

2.2 The sandpile group and its canonical action on $\mathbf{G}(M)$

Definition 2.6. Let $\Lambda(M) \subset \mathbb{Z}^E$ be the lattice generated $\overrightarrow{\mathbf{C}}(M)$ and $\Lambda^*(M) \subset \mathbb{Z}^E$ be the lattice generated by $\overrightarrow{\mathbf{C}^*}(M)$. The *sandpile group* of M is defined by:

$$S(M) := \frac{\mathbb{Z}^E}{\Lambda(M) \oplus \Lambda^*(M)}.$$

For a 1-chain $\overrightarrow{P}$, we write $[\overrightarrow{P}]$ for the equivalence class of $S(M)$ containing $\overrightarrow{P}$. Note that the sandpile group $S(M)$ is generated by elements $\{[\overrightarrow{f}] \mid \overrightarrow{f}$ is an arc of $M\}$. In [2], the authors define a natural group action of $S(M)$ on the set $\mathbf{G}(M)$, which generalizes the additive action in the more classical graphical case where elements of $S(M)$ and $\mathbf{G}(M)$ are represented as "chip configurations". For details on the "chip" perspective, see [15]. This natural action is called the canonical action.

Definition 2.7. [2] The *canonical action* of $S(M)$ on $\mathbf{G}(M)$ is defined by linearly extending the following action of each generator $[\overrightarrow{f}]$ on circuit-cocircuit reversal classes. Given $[\mathcal{O}]$, one can prove that there exists some orientation $\mathcal{O}' \in \mathbf{O}(M)$ such that $-\overrightarrow{f} \sim \mathcal{O}'$ and $[\mathcal{O}'] = [\mathcal{O}]$. Define the action by $[\overrightarrow{f}] \cdot [\mathcal{O}] = [{}_f\mathcal{O}']$.

Lemma 2.8. *[2, Theorem 4.3.1.] The canonical action is well-defined and simply transitive.*

Example 2.9. Take the graphic matroid on Figure 1, and take the orientation $\mathcal{O} = (-,-,+,+)$. Since $-\overrightarrow{f_1} \sim \mathcal{O}$, we have $[\overrightarrow{f_1}] \cdot [\mathcal{O}] = [(+,-,+,+)]$.

As a more interesting example, take $[\overrightarrow{f_3}] \cdot [\mathcal{O}]$. Since $\overrightarrow{f_3} \sim \mathcal{O}$, we need to reverse a signed circuit or cocircuit containing f_3, and then reverse f_3 again. $-\overrightarrow{f_1} + \overrightarrow{f_3} + \overrightarrow{f_4}$ is a signed cocircuit containing f_3. Hence $[(+,-,+,-)]$ is the circuit-cocircuit equivalence class of $[\overrightarrow{f_3}] \cdot [\mathcal{O}]$.

As such, any bijection between $\mathbf{G}(M)$ and $\mathbf{B}(M)$ yields a simply transitive group action of $S(M)$ on $\mathbf{B}(M)$ via composing with the canonical action.

2.3 Fourientations

The notion of fourientations was introduced and systematically studied by Backman and Hopkins [3]. The first author applied this notion in [9] to study the connection between the BBY bijections and Lawrence polytopes. We find the language of fourientations also helpful in the proof of our main theorem.

Definition 2.10. Given a set E, a *fourientation* $\mathcal{F}$ is a map from E to the set $\{\emptyset, -, +, \pm\}$. We denote the set of fourientations on the ground set of a matroid M by $\mathbf{F}(M)$.

As with orientations, for $x \in E$, we write $\mathcal{F}\langle x \rangle$ for the output of the map at x. Intuitively, each element of the ground set can be oriented in either direction, bi-oriented, or unoriented.

For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(M)$, we write $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{F}_1 \cap \mathcal{F}_2$ for the fourientations obtained by taking pointwise union or intersection (treating $-,+,\pm$ as $\{-\},\{+\},\{-,+\}$). Furthermore, we define $-\mathcal{F}_1$ to be $\mathcal{F}_1$ with $-$ and $+$ swapped, and $\mathcal{F}_1^c$ to be $-\mathcal{F}_1$ with $\emptyset$ and $\pm$ swapped.

2.4 Triangulating Signatures and the Backman-Baker-Yuen Bijection

In [2], Backman, Baker, and the fourth author defined a family of explicit bijections between $\mathbf{G}(M)$ and $\mathbf{B}(M)$. These maps were generalized in [4] and [9] to the context we use in this paper. Below, we give their constructions in the language of fourientations.

Definition 2.11. A *circuit signature* $\sigma \subset \overrightarrow{\mathbf{C}}(M)$ is a collection of signed circuits of M such that for each circuit $C \in \mathbf{C}(M)$, exactly one of the two signed circuits supported on C is contained in σ. We write $\sigma(C)$ for the circuit supported on C that is contained in σ.

Define a *cocircuit signature* $\sigma^* \subset \overrightarrow{\mathbf{C}^*}(M)$ analogously. For a cocircuit C^*, we write $\sigma^*(C^*)$ for the signed cocircuit supported on C^* that is contained in σ^*.

Figure 2: The circuit signature of Example 2.12 (left panels) and the cocircuit signature of Example 2.12 (right panels).

By a *circuit-cocircuit signature* we mean pair consisting of a circuit signature and a cocircuit signature.

Example 2.12. For the graph of Figure 1, the signed circuits $\overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_3}$, $\overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_4}$ and $-\overrightarrow{f_3} + \overrightarrow{f_4}$ form a circuit signature. The signed cocircuits $-\overrightarrow{f_1} + \overrightarrow{f_3} + \overrightarrow{f_4}$, $-\overrightarrow{f_1} - \overrightarrow{f_2}$, and $\overrightarrow{f_2} + \overrightarrow{f_3} + \overrightarrow{f_4}$ form a cocircuit signature. See also Figure 2.

Definition 2.13. Fix a circuit-cocircuit signature (σ, σ^*) and a basis B. For each $e \notin B$, let C_e be the unique circuit contained in $B \cup \{e\}$ (known as the *fundamental circuit* of e with respect to B). For each $e \in B$, let C_e^* be the unique cocircuit contained in $(E \setminus B) \cup \{e\}$ (known as the *fundamental cocircuit* of e with respect to B).

We denote by $\mathcal{F}(B, \sigma)$ the fourientation where all $e \in B$ are bi-oriented and all the $e \in E \setminus B$ are oriented according to $\sigma(C_e)$. Similarly, we denote by $\mathcal{F}(B, \sigma^*)$ the fourientation where all $e \in E \setminus B$ are bi-oriented and all $e \in B$ are oriented according to $\sigma^*(C_e^*)$.

Example 2.14. For the graph of Figure 1, take the spanning tree T consisting of edges f_1 and f_3. Take the circuit-cocircuit signature (σ, σ^*) of Example 2.12. Then $\mathcal{F}(T, \sigma) = (\pm, -, \pm, +)$ and $\mathcal{F}(T, \sigma^*) = (-, \pm, +, \pm)$ respectively for edges f_1, f_2, f_3 and f_4.

The BBY bijection will depend on a circuit-cocircuit signature, but in order to obtain a bijection, we need some "niceness" for the signatures, notably, the following.

Definition 2.15. [9] A circuit signature σ (resp. cocircuit signature σ^*) is called *triangulating* if for any distinct $B_1, B_2 \in \mathbf{B}(M)$, the fourientation $\mathcal{F}(B_1, \sigma) \cap -\mathcal{F}(B_2, \sigma)$ is not compatible with any $\overrightarrow{C} \in \overrightarrow{\mathbf{C}}(M)$ (resp. $\mathcal{F}(B_1, \sigma^*) \cap -\mathcal{F}(B_2, \sigma^*)$ is not compatible with any $\overrightarrow{C}^* \in \overrightarrow{\mathbf{C}^*}(M)$). A circuit-cocircuit signature (σ, σ^*) is *triangulating* if σ and σ^* are both triangulating.

Here a simple 1-chain $\overrightarrow{P}$ (for example $\overrightarrow{C}$ or $\overrightarrow{C}^*$) is *compatible* with a fourientation $\mathcal{F}$ if for all $f \in P$, either $\mathcal{F}\langle f \rangle = \pm$ or the sign of $\overrightarrow{P}\langle f \rangle$ matches the sign of $\mathcal{F}\langle f \rangle$.

Definition 2.16. Let M be a regular matroid and (σ, σ^*) be a circuit-cocircuit signature. An orientation $\mathcal{O}$ is *σ-compatible* (resp. *σ^*-compatible*) if every signed circuit (resp. cocircuit) compatible with $\mathcal{O}$ is in σ (resp. σ^*). An orientation is *(σ, σ^*)-compatible* if it is both σ-compatible and σ^*-compatible.

Fix a triangulating signature (σ, σ^*). By [9, Proposition 1.21(1)] there is a unique (σ, σ^*)-compatible orientation in each equivalence class in $\mathbf{G}(M)$. Hence the following notion is well-defined.

Definition 2.17. Given an orientation $\mathcal{O}$, let $\mathcal{O}^\circ$ be the (unique) (σ, σ^*)-compatible orientation in the same reversal class as $\mathcal{O}$. Likewise, given $[\mathcal{O}] \in \mathbf{G}(M)$, let $[\mathcal{O}]^\circ = \mathcal{O}^\circ$. Furthermore, let $\mathbf{O}^\circ(M)$ be the set of all (σ, σ^*)-compatible orientations.

Note that $\mathcal{O}^\circ$ and $\mathbf{O}^\circ(M)$ both depend on the choice of circuit-cocircuit signature. We omit a reference to this signature in the notation for readability.

We can also directly define the canonical action on the set $\mathbf{O}^\circ(M)$, namely, for $g \in S(M)$ and $\mathcal{O} \in \mathbf{O}^\circ(M)$, we define $g \cdot \mathcal{O} := (g \cdot [\mathcal{O}])^\circ$.

Example 2.18. Let us return to Example 2.9, and take the signature of Example 2.12. It can be checked that this is triangulating. With this, $[\overrightarrow{f_1}] \cdot (-,-,+,+) = (+,-,+,+)$ and $[\overrightarrow{f_3}] \cdot (-,-,+,+) = (+,-,-,+)$.

Now we are in the position to introduce the BBY bijection.

Definition 2.19 (BBY bijection)**.** Fix a regular matroid M and a pair (σ, σ^*) of triangulating signatures. The map $\beta_{(M,\sigma,\sigma^*)} : \mathbf{B}(M) \to \mathbf{O}(M)$ is given by $B \mapsto \mathcal{F}(B, \sigma) \cap \mathcal{F}(B^*, \sigma^*)$.

Theorem 2.20. *[4, 9] For a regular matroid M and a pair (σ, σ^*) of triangulating signatures. The BBY map $\beta_{(M,\sigma,\sigma^*)}$ is a bijection between $\mathbf{B}(M)$ and $\mathbf{O}^\circ(M)$. In particular, this map induces a bijection between $\mathbf{B}(M)$ and $\mathbf{G}(M)$.*

The bijection $\beta_{(M,\sigma,\sigma^*)}$ together with the canonical action in Section 2.2 induces a simply transitive group action of $S(M)$ on $\mathbf{B}(M)$ that we call the *BBY action*.

Example 2.21. Take the graphic matroid M of Figure 1 with the circuit-cocircuit signature (σ, σ^*) of Example 2.12. Let $T = \{f_1, f_3\}$. Then $\beta_{(M,\sigma,\sigma^*)}(T) = (-,-,+,+)$.

Let us compute the BBY action of $\overrightarrow{f_1}$ on T. We have $[\overrightarrow{f_1}] \cdot (-,-,+,+) = (+,-,+,+)$ by Example 2.18. One can check that for $T' = \{f_2, f_3\}$ we have $\beta_{M,\sigma,\sigma^*}(T') = (+,-,+,+)$. Hence $[\overrightarrow{f_1}] \cdot \beta_{(M,\sigma,\sigma^*)}(T) = \beta_{(M,\sigma,\sigma^*)}(T')$.

2.5 The Main Theorem

Before stating the main theorem, we remark that any triangulating circuit-cocircuit signature (σ, σ^*), and any $e \in E$ that is not a loop or coloop naturally yields triangulating circuit-cocircuit signatures $(\sigma \setminus e, \sigma^* \setminus e)$ and $(\sigma / e, \sigma^* / e)$ on $M \setminus e$ and M/e respectively. The following theorem says that the BBY algorithm is consistent.

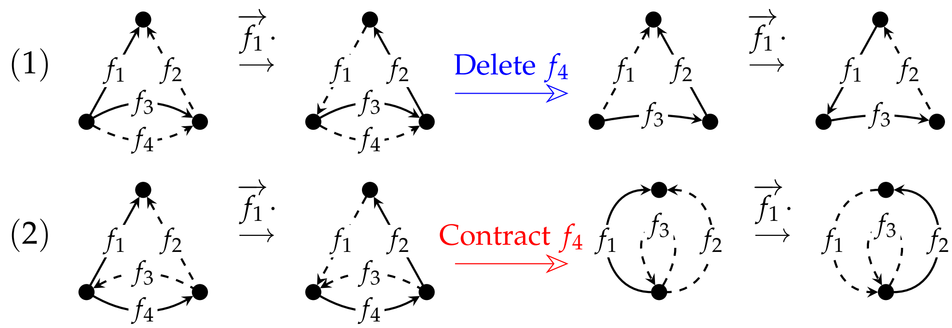


Figure 3: Above are illustrations for the first two parts of Theorem 2.22. See Example 2.23 for details.

Theorem 2.22. *Let M be a regular matroid and (σ, σ^*) be a triangulating circuit-cocircuit signature. Suppose that $\overrightarrow{f}$ is an arc and $B_1, B_2 \in \mathbf{B}(M)$ such that*

$$[\overrightarrow{f}] \cdot \beta_{(M,\sigma,\sigma^*)}(B_1) = \beta_{(M,\sigma,\sigma^*)}(B_2).$$

1. *For any $e \in (B_1^c \cap B_2^c) \setminus f$, we have*

$$[\overrightarrow{f}] \cdot \beta_{(M\setminus e,\sigma\setminus e,\sigma^*\setminus e)}(B_1) = \beta_{(M\setminus e,\sigma\setminus e,\sigma^*\setminus e)}(B_2).$$

2. *For any $e \in (B_1 \cap B_2) \setminus f$, we have*

$$[\overrightarrow{f}] \cdot \beta_{(M/e,\sigma/e,\sigma^*/e)}(B_1 \setminus e) = \beta_{(M/e,\sigma/e,\sigma^*/e)}(B_2 \setminus e).$$

3. *If e and f are in different connected components of M, then $e \in B_1 \iff e \in B_2$.*

Theorem 2.22 is a generalization of [10, Theorem 4.6] from plane graphs to regular matroids.

Example 2.23. Take the graphic matroid M from Figure 1 with the cycle-cocycle signature (σ, σ^*) from Example 2.12. The first row of Figure 3 demonstrates Theorem 2.22(1) and the second row demonstrates Theorem 2.22(2). The depicted orientations are the circuit-cocircuit minimal orientations assigned to the spanning trees by the BBY bijection.

Let us explain the first row. The upper left panel shows action of $\overrightarrow{f_1}$ on the basis $\{f_1, f_3\}$: The action produces $\{f_2, f_3\}$ as explained in Example 2.18. We have $\sigma \setminus f_4 = \{\overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_3}\}$, and $\sigma^* \setminus f_4 = \{-\overrightarrow{f_1} + \overrightarrow{f_3}, -\overrightarrow{f_1} - \overrightarrow{f_2}, \overrightarrow{f_2} + \overrightarrow{f_3}\}$. One can check that indeed $[\overrightarrow{f_1}] \cdot \beta_{(M\setminus f_4,\sigma\setminus f_4,\sigma^*\setminus f_4)}(\{f_1, f_3\}) = \beta_{(M\setminus f_4,\sigma\setminus f_4,\sigma^*\setminus f_4)}(\{f_2, f_3\})$.

3 Some Proof Ingredients

We now introduce one of the main tools that we used to prove our main result, Theorem 2.22. The purpose of this theorem is to localize the changes in the BBY action that we need to analyze. We include the statement here for it may be of independent interest.

Theorem 3.1. *Let $\overrightarrow{f}$ be an arc and $\mathcal{O}_1, \mathcal{O}_2 \in \mathbf{O}^\circ(M)$ such that $\overrightarrow{f} \cdot \mathcal{O}_1 = \mathcal{O}_2$. Then $\mathcal{O}_1$ can be transformed to $\mathcal{O}_2$ by the following (at most) three step process.*

1. *Reverse at most one signed circuit or cocircuit containing f that is compatible with $\mathcal{O}_1$.*
2. *Reverse $\overrightarrow{f}$.*
3. *Reverse at most one signed circuit or cocircuit containing f that is compatible with the new orientation.*

Furthermore, the following conditions hold.

a. *A reversal occurs during step 1 (respectively, step 3) if and only if $\overrightarrow{f} \sim \mathcal{O}_1$ (respectively, $-\overrightarrow{f} \sim \mathcal{O}_2$).*

b. *If reversals occur during both step 1 and step 3, one of these is a circuit reversal while the other is a cocircuit reversal.*

Theorem 2.22(3) follows immediately from Theorem 3.1. For the rest, by duality, it suffices to focus on Theorem 2.22(1). The deletion of an edge e does not affect the cocircuit reversals that occur in Theorem 3.1 in an essential way. The case that the edge e appears in the circuits reversed in Theorem 3.1 is the main obstacle. However, we prove that this cannot happen using fourientations.

Here is a short illustration of how the fourientations help with the proof. Under the assumption of Theorem 2.22, denote

$$\begin{aligned} \mathcal{O}_1 &= \mathcal{F}(B_1,\sigma) \cap \mathcal{F}(B_1,\sigma^*), & \mathcal{O}_2 &= \mathcal{F}(B_2,\sigma) \cap \mathcal{F}(B_2,\sigma^*), \\ \mathcal{F} &= \mathcal{F}(B_1,\sigma) \cap -\mathcal{F}(B_2,\sigma), \quad \text{and} & \mathcal{F}^* &= \mathcal{F}(B_1,\sigma^*) \cap -\mathcal{F}(B_2,\sigma^*). \end{aligned}$$

Theorem 3.1 describes the difference between $\mathcal{O}_1$ and $\mathcal{O}_2$. For the edges where $\mathcal{O}_1$ and $\mathcal{O}_2$ coincide, the following lemma transfers the information to fourientations.

Lemma 3.2. *[9, Lemma 2.8] For any $x \in E$, if $\mathcal{O}_1\langle x\rangle = \mathcal{O}_2\langle x\rangle$, then $\mathcal{F}\langle x\rangle = \mathcal{F}^*\langle x\rangle$.*

We have a similar lemma when $\mathcal{O}_1$ and $\mathcal{O}_2$ differ, which is more technical and omitted here. We also know that $\mathcal{F}$ (resp. $\mathcal{F}^*$) is not compatible with any $\overrightarrow{C} \in \overrightarrow{\mathbf{C}}(M)$ (resp. $\overrightarrow{C}^* \in \overrightarrow{\mathbf{C}^*}(M)$) from Definition 2.15. Combining all this information on the two fourientations, we are able to prove the desired result.

We show the power of the fourientation language by giving a short proof [9, Remark 2.10] of the following result in [2], which was first proven using a geometric argument.

Corollary 3.3. $\beta_{(M,\sigma,\sigma^*)}$ *is injective.*

Proof. If $\mathcal{O}_1 = \mathcal{O}_2$ comes from two distinct bases via $\beta_{(M,\sigma,\sigma^*)}$, then Lemma 3.2 and the triangulating assumption of (σ, σ^*) imply that $\mathcal{F} = \mathcal{F}^*$ is not compatible with any signed circuit/cocircuit, which contradicts the *3-painting axiom* [6, Theorem 3.4.4] in oriented matroid theory. See [9, Lemma 2.3] for a fourientation version of the 3-painting axiom. ∎

4 Special Instances of Consistency

4.1 Acyclic signatures

The notion of acyclic signatures was introduced in [20, 2]. A circuit signature σ is *acyclic* if the only set of nonnegative λ_C values satisfying $\sum_{\overrightarrow{C} \in \sigma} \lambda_C \overrightarrow{C} = \mathbf{0}$ is where every λ_C is zero. Acyclic cocircuit signatures are defined analogously.

The seemingly technical definition arrives naturally in the context of polyhedral geometry. By [9, Lemma 3.4], acyclic signatures are triangulating, and it can be proven that the property of being acyclic is preserved under deletion or contraction of signatures. Hence we have the following corollary, which was Conjecture 6.11 of [10].

Corollary 4.1. *The BBY actions with respect to acyclic signatures are consistent.*

4.2 The Planar Case

For a plane graph, circuits oriented counterclockwise form a triangulating circuit signature [2]. Also, for any graph, the signed cocircuits "oriented away" from a fixed vertex v form a triangulating cocircuit signature. Notice that the circuit and cocircuit signature given in Example 2.12 fall into the above cases.

Combining [5, Theorem 7.1] and [2, Example 1.1.3], the rotor-routing torsor action of a plane graph is equal to the BBY action with respect to this circuit-cocircuit signature. Moreover, it is apparent (and can be proven rigorously) that an embedding-preserving deletion (respectively, contraction) of a plane graph induces the deletion (respectively, contraction) of the circuit-cocircuit signature. As a final corollary of all these, we have:

Corollary 4.2. *[10, Theorem 4.6] The rotor-routing torsor algorithm is consistent.*

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Asymptotics of bivariate algebraico-logarithmic generating functions

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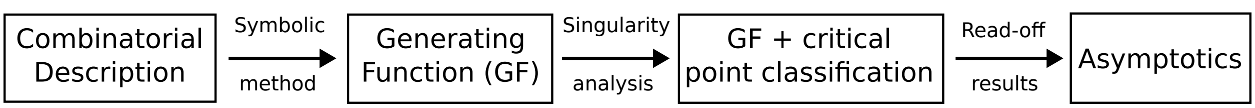
Abstract. We derive asymptotic formulae for the coefficients of bivariate generating functions with algebraic and logarithmic factors. Logarithms appear when encoding cycles of combinatorial objects, and also implicitly when objects can be broken into indecomposable parts. Asymptotics are quickly computable and can verify combinatorial properties of sequences and assist in randomly generating objects. While multiple approaches for algebraic asymptotics have recently emerged, we find that the contour manipulation approach can be extended to these D-finite generating functions.

Keywords: algebraic singularities, logarithms, bivariate, asymptotics, ACSV

A goal in analytic combinatorics in several variables (ACSV) is to derive asymptotic estimates for multivariate arrays that encode combinatorial information. Asymptotic formulae are useful for computing highly accurate estimates of sequences, determining what large structures look like, and randomly generating objects. In contrast, even when exact formulae can be found, they may be be cumbersome to evaluate or interpret [17]. The schema for generating asymptotic estimates follows.



The read-off results depend on the form of the generating function (GF). Here, we broaden the read-off results to bivariate algebraico-logarithmic GFs, for several reasons:

1. Algebraico-logarithmic GFs appear widely, including problems involving cycles, Pólya enumeration, Pólya urns [12], necklaces [11], and more. Logarithms also appear in problems where a combinatorial family of objects can be described implicitly as components within other objects [7, Section III.7.3].

2. Our main theorem, Theorem 1, involves D-finite GFs, while most ACSV results have focused on rational or algebraic GFs.

3. Differing approaches for computing algebraic asymptotics have recently emerged [9, 10, 1]. Although the direct contour manipulations of [9] are technical, here we show that this approach is readily applied to algebraico-logarithmic GFs.

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1 Univariate asymptotics

The *symbolic method* is now standard [7], and it converts common combinatorial operations on sets into algebraic operations on GFs. For example, if the GF $F(z)$ encodes the sequence counting objects of size n from some set $\mathcal{F}$ as n varies, then the GF $1/(1-F(z))$ encodes the numbers of ordered sequences of objects of any length from $\mathcal{F}$ whose total size is n. Other common operations include taking powersets, multisets, or ordered tuples from a set, or highlighting a component within a combinatorial object. Of particular importance here is the *cycle construction* for ordinary or exponential GFs, which involves sums of logarithms or a single logarithm, respectively. Once we have a GF, we aim for asymptotics of the form

$$[z^n]F(z) \sim Cn^r(\log^s n)\rho^n \tag{1.1}$$

as $n \to \infty$ where C, r, s, and ρ do not depend on n. The location of a GF's closest singularities to the origin determines ρ, and the behavior of the GF near these singularities determines C, r, and s. Example 1 below illustrates the efficiency of computing univariate asymptotics.

Example 1 (Logarithms of Catalan numbers)**.** Logarithms of the Catalan number GF were considered in Knuth's 2014 Christmas Tree lecture [14] and the American Mathematical Monthly [15]. They have since been studied [4] and can encode cycles of Dyck paths and families of labelled paths [13]. In particular, consider

$$D^{(m)}(z) = \left[\log\left(\frac{1-\sqrt{1-4z}}{2z}\right)\right]^m.$$

The singularity at $z = 0$ is removable. Otherwise, $D^{(m)}(z)$ has algebraic singularities determined by the zero set within the square root, $\{z : 1-4z = 0\}$, and when the input to the log is 0, so $\{z : (1-\sqrt{1-4z})/(2z) = 0\}$. The input to the logarithm is never zero (since the point $z = 0$ is a removable singularity), so the only singularity of $D^{(m)}(z)$ is at $z = 1/4$. Expanding near $z = 1/4$ reveals

$$\begin{aligned} D^{(m)}(z) &= \left(\log 2 + \sqrt{1-4z} - \frac{7}{2}(1-4z) + O(1-4z)^{3/2}\right)^m \\ &= \log^m 2 - \binom{m}{1}[\log^{m-1} 2]\sqrt{1-4z} + \text{higher order terms in } (1-4z). \end{aligned}$$

In this expansion, the next term with an algebraic singularity is $O(1-4z)^{3/2}$, so the transfer theorem from Flajolet and Odlyzko [6] immediately yields

$$[z^n]D(z) = \frac{m\log^{m-1} 2}{2\sqrt{\pi}} \cdot n^{-3/2} \cdot 4^n + O(4^n n^{-5/2}).$$

---

2 Multivariate asymptotics background

Multivariate GFs encode arrays of numbers, useful for tracking combinatorial parameters. Alternatively, multivariate GFs can assist univariate analyses, including lattice walk enumeration [3]. Let $F(x, y)$ be the bivariate GF encoding the array $a_{r,s}$, so that $F(x, y) = \sum a_{r,s} x^r y^s$. For a fixed *direction* $\hat{\mathbf{r}} := (r_1, r_2) \in \mathbb{R}^2_{>0}$, we search for an asymptotic expression for $[x^{r_1 n} y^{r_2 n}] F(x, y)$ as $n \to \infty$. As in the univariate case, asymptotics are often in the form $C \cdot n^r (\log^s n) \cdot \rho^n$ for appropriate choices of constants r, s, and ρ. However, the idea of *closest* singularity to the origin needs to be refined in multiple variables.

ACSV connects geometry and combinatorics, as there is a diverse set of possible geometries of GF singularities. Our focus here is the simplest but most common scenario, when a GF has a smooth minimal critical point. Other cases for rational GFs are covered in [16]. Let $\mathcal{V} := \{(x, y) : H(x, y) = 0\}$ be a singular variety for a GF, defined by when some analytic function $H(x, y)$ is zero. For rational GFs, H is the denominator, while for algebraico-logarithmic GFs, H may be the input of a square root, a logarithm, or the product of several such inputs. Smooth critical points for the direction $\hat{\mathbf{r}} = (r_1, r_2)$ satisfy

$$H = 0, \quad r_2 x H_x = r_1 y H_y,$$

where H_x and H_y represent the partial derivatives of H with respect to x and y [16]. By design, these equations yield points minimizing the exponential growth rate in Equation (6.1) below. Additionally, a smooth critical point must be a location where $\mathcal{V}$ is locally a smooth manifold. This can be checked by verifying that H, H_x and H_y never simultaneously vanish, since the implicit function theorem guarantees a local smooth parameterization near any point where at least one partial is nonzero. When H is a polynomial, all conditions for smooth critical points are also defined by polynomial equations, which means that identifying critical points can be done efficiently with Gröbner bases.

Theorem 1 requires that the critical point (p, q) is *minimal*, meaning there are no singularities coordinate-wise smaller in $\mathcal{V}$. Minimality is simpler to check when a GF has only non-negative coefficients (the *combinatorial* case). Minimality is crucial here to allow an explicit Cauchy integral contour manipulation that reaches the critical point (p, q).

3 Generating function classifications and logarithms

Multivariate rational GFs cover many combinatorial scenarios: not only do they count arrays enumerating the output of discrete finite automata, but they are also useful when more complicated sequences can be expressed as the diagonal of a rational GF. Nonetheless, there are combinatorial situations where a sequence cannot be encoded as a diagonal of the terms of a rational GF, such as when the asymptotics of a sequence are not of the form $Cn^{-s}\rho^{-n}$ for $s \in \mathbb{Z}/2$. However, the dictionary of asymptotic results is incomplete for GFs beyond the rational domain.

A GF $F(\mathbf{z})$ can be classified according to what kind of equation F satisfies. Rational GFs satisfy linear equations with coefficients in $\mathbf{z}$, while algebraic GFs satisfy polynomial equations. Even more broadly, D-finite GFs satisfy linear PDEs with coefficients in $\mathbf{z}$.

Several distinct approaches recently advanced asymptotic formulae for algebraic GFs. In [9], a change of variables and a direct contour manipulation of the Cauchy integral formula lead to results for bivariate algebraic GFs. This technical process is currently limited to two dimensions, but could be extended to more dimensions with some additional overhead. Another possible approach [10] embedded the coefficients of an algebraic GF into a rational GF in more variables. Accessing the singular variety of an algebraic GF directly from its minimal polynomial sometimes gives faster and cleaner results [1]. Finally, [2, 8, 5] give probabilistic interpretations of the coefficients of algebraic GFs.

It is less clear how to approach D-finite GFs. Here, we modify the contour approach from [9] to attack a concrete class of bivariate D-finite GFs that include logarithms. In contrast, it is not obvious how the other approaches could be adapted to this setting.

4 Result

Our focus is bivariate GFs of the form $F(x,y) = H(x,y)^{-\alpha}[\log H(x,y)]^{\beta}$, where $H(x,y)$ is analytic near the origin with only non-negative power series coefficients, and where $\beta \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbb{R}$ is not in $\mathbb{Z}_{\leq 0}$. This form is motivated by the results in [7].

Theorem 1. *Let $H(x,y)$ be an analytic function near the origin whose power series expansion at $(0,0)$ has non-negative coefficients. Define $\mathcal{V} = \{(x,y) : H(x,y) = 0\}$. Assume that there is a single smooth strictly minimal critical point of $\mathcal{V}$ at (p,q) within the domain of analyticity of H where p and q are real and positive. Let $\lambda = \frac{r+O(1)}{s}$ as $r,s \to \infty$ with r and s integers. Define the following quantities:*

$$\chi_1 = \frac{H_y(p,q)}{H_x(p,q)} = \frac{p}{\lambda q},$$

$$\chi_2 = \frac{1}{2H_x}\left(\chi_1^2 H_{xx} - 2\chi_1 H_{xy} + H_{yy}\right)\Big|_{(x,y)=(p,q)},$$

$$M = -\frac{2\chi_2}{p} - \frac{\chi_1^2}{p^2} - \frac{1}{\lambda q^2}.$$

Assume that $H_x(p,q)$ and M are nonzero. Fix $\alpha \in \mathbb{R}$ where $\alpha \notin \mathbb{Z}_{\leq 0}$ and $\beta \in \mathbb{Z}_{\geq 0}$. Then, the following expression holds as $r,s \to \infty$:

$$[x^r y^s]H(x,y)^{-\alpha}\log^{\beta}(H(x,y)) \sim (-1)^{\beta}\frac{(-pH_x(p,q))^{-\alpha} r^{\alpha-1}}{\Gamma(\alpha)\sqrt{-2\pi q^2 M r}} p^{-r} q^{-s} \log^{\beta} r \left[1 + \sum_{j\geq 1}\frac{\mathcal{E}_j}{\log^j r}\right],$$

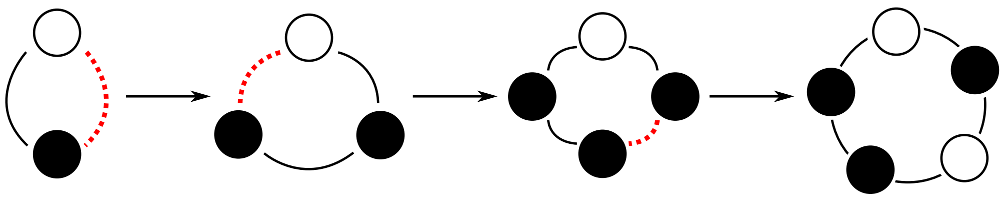

Figure 1: Start with one white bead and one black bead. Then, pick any edge in the necklace. If both neighboring beads are black, insert a white bead on this edge. Otherwise, insert a black bead. The necklace illustrated above has multiple constructions but contributes once to the x^5y^2 term in the bivariate GF in Example 2.

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where $\mathcal{E}_j = \sum_{k=0}^{j} \binom{\beta}{j}\binom{j}{k} \log^k \left(\frac{-1}{pH_x(p,q)}\right) \Gamma(\alpha)\ \frac{d^{j-k}}{dt^{j-k}} \frac{1}{\Gamma(t)}\Big|_{t=\alpha}$.

The theorem applies when the singularity nearest the origin is determined by a zero in the logarithm. In other cases, the singularity may be algebraic, in which case existing algebraic results apply (see Example 4). Also, when the closest singularity is instead due to a pole of H, rewriting $\log(H) = -\log(1/H)$ allows us to apply Theorem 1.

Theorem 1 could be extended to the case where $\beta \notin \mathbb{Z}_{\geq 0}$, although this adds additional complexity because the logarithmic term in the GF then contributes an additional branch cut. When $\beta \in \mathbb{Z}_{\geq 0}$, the series in Theorem 1 is finite, but in general it is infinite. Also, the theorem statement is still true when $\alpha = 0$ (or even $\alpha \in \mathbb{Z}$) by defining $(1/\Gamma(\alpha))$ and $d^j/dt^j(1/\Gamma(t))\big|_{t=\alpha}$ by their limits at $\alpha = 0$, as described in the univariate case in [7]. The analogous asymptotic expansion for $\alpha = 0$ can then be computed, still with descending powers of $\log(r)$ and leading term given by

$$[x^r y^s] \log^\beta H(x,y) \sim (-1)^\beta \frac{\beta r^{-3/2} p^{-r} q^{-s}}{\sqrt{-2\pi q^2 M}} \log^{\beta-1} r.$$

5 Examples

The examples below and more are analyzed in the `SageMath` worksheet here:

`https://cocalc.com/Tristan-Larson/FPSAC-algebraico-logarithmic/Examples`

Example 2 (Necklaces)**.** As in [11], consider necklaces with black and white beads where no two white beads are adjacent. These are analyzed in [11] via a univariate GF, and they can be constructed by a "necklace process" shown in Figure 1 that relates to network communication models. Let φ be Euler's totient function. We consider the bivariate GF

$$N(x,y) = \sum_{k\geq 1} \frac{\varphi(k)}{k} \log\left(\frac{1-x^k}{1-x^k-y^kx^{2k}}\right),$$

in which the coefficient $[x^r y^s]N(x,y)$ counts the number of necklaces with r total beads and s white beads. The GF can be derived by viewing necklaces as cycles of {white beads followed by a positive number of black beads}.

Consider seeking asymptotics in the direction $(\ell, 1)$ with $\ell > 2$. Combinatorially, $\ell > 2$ corresponds to necklaces having more than twice as many total beads as white beads. For each k, define $H_k = 1 - x^k - y^k x^{2k}$, which contributes k^2 critical points satisfying

$$x^k = \frac{\ell - 2}{\ell - 1}, \; y^k = \frac{\ell - 1}{(\ell - 2)^2}.$$

Define (p_k, q_k) to be the positive real solution. We verify that there are no nonsmooth critical points by checking for each k that $[H_k = 0, \frac{\partial}{\partial x} H_k = 0, \frac{\partial}{\partial y} H_k = 0]$ has no solutions. Furthermore, the exponential growth rate near any critical point (p_k, q_k) is given by $|p_k^\ell q_k|^{-1}$, which can easily be verified as maximized when $k = 1$. Thus, we need only to consider the contributions from the critical point (p_1, q_1) determined by the first term in the sum, $\log\left(\frac{1-x}{1-x-yx^2}\right)$. For this, we find that

$$\chi_1 = \frac{(\ell-2)^3}{\ell(\ell-1)^2}, \; \chi_2 = -\frac{(2\ell-1)(\ell-2)^5}{\ell^3(\ell-1)^3}, \text{ and } M = -\frac{(\ell-2)^5}{\ell^3(\ell-1)}.$$

Then the asymptotic enumeration formula of Theorem 1 gives

$$[x^{\ell n} y^n]N(x,y) \sim \frac{n^{-3/2}\ell^{5/2}}{\sqrt{2\pi}} \frac{(\ell-1)^{(2\ell n - 2n + 3)/2}}{(\ell-2)^{(2\ell n - 4n + 9)/2}}.$$

To illustrate accuracy, consider $\ell = 3$. The approximation becomes $[x^{pn} y^n]N(x,y) \sim 18n^{-3/2}4^n\sqrt{3/\pi}$, yielding (for instance) the estimate $[x^{225}y^{75}]N(x,y) \approx 6.199 \times 10^{41}$. The actual value is $[x^{225}y^{75}]N(x,y) = 6.188\ldots \times 10^{41}$, with an error of only 0.167%.

Example 3 (Cyclical Interlaced Permutations)**.** Let $\mathcal{C}$ be the set of circular arrangements of the bicolored set $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{m}\}$, with $m + n \geq 1$, as illustrated in Figure 2. For fixed m and n, there are $(n+m)!/(n+m)$ arrangements. So, if x tracks the number of black elements and y tracks the number of the red (barred) elements, $\mathcal{C}$ has the GF,

$$C(x,y) = \log\left(\frac{1}{1-x-y}\right) = \sum_{n+m\geq 1} \frac{1}{n+m}\binom{n+m}{n} x^n y^m.$$

Note that this GF is the logarithm of the GF in [16, Examples 2.2, 8.13, 9.10]. The labelled objects here lead to an exponential GF with a single logarithm, in contrast to the unlabelled objects in Example 2 that yield an ordinary GF with a sum of logarithms.

We will now compute the asymptotics in the direction $(1, \ell)$, where $\ell > 0$. There is a unique minimal smooth critical point at $(1/(1+\ell), \ell/(1+\ell))$. For the quantities defined in Theorem 1, we have $\chi_1 = 1, \chi_2 = 0$, and $M = -(1+\ell)^3/\ell$, yielding

$$[x^r y^{\ell r}]C(x,y) \sim \frac{r^{-3/2}\ell^{-r\ell}(1+\ell)^{(1+\ell)r}}{\sqrt{2\pi\ell(1+\ell)}}.$$

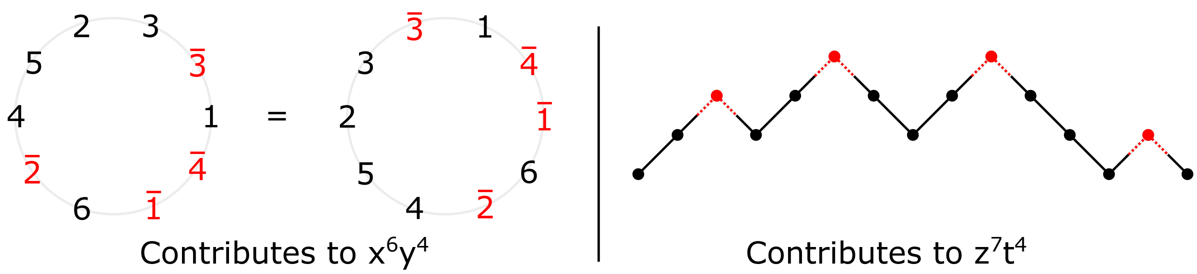

Figure 2: On the left, a cyclical interlaced permutation is drawn as enumerated in Example 3. Because there are 6 black numbers and 4 red numbers, this contributes to x^6y^4. Rotating the numbers gives the same configuration. On the right, a Dyck path with 14 steps and 4 peaks is drawn, which contributes to z^7t^4 in the GF from Example 4.

Example 4 (Logarithm of Narayana numbers)**.** The Narayana numbers refine Example 1: let $a_{n,s}$ be the number of Dyck paths with length n and number of peaks s, as in Figure 2. This example illustrates how algebraic singularities may still determine asymptotics for a non-algebraic GF. Let $N(z,t) := \sum a_{n,s} z^n t^s$ be the GF for the Narayana numbers, and let $P(z,t)$ count the Dyck paths that never return to the x-axis except at their start and end. Then, from the symbolic method, N and P satisfy the relations,

$$N(z,t) = \frac{1}{1-P(z,t)}, \quad P(z,t) = tz + z(N(z,t)-1).$$

$$N(z,t) = \frac{1+z-tz-\sqrt{(1+z-tz)^2-4z}}{2z}.$$

Consider the growth rate of the coefficients $[z^n t^s] \log^r N(z,t)$ in the direction $(\ell,1)$ for any $\ell > 1$. To determine the singularities of $\log^r N(z,t)$, note that N has a removable singularity at $z=0$, with limit 1. Thus, $\log^r N(z,t)$ has singularities from the logarithm determined by $N(z,t)=0$ (with $z \neq 0$) and algebraic singularities determined by the zero set of $H(z,t) := (1+z-tz)^2-4z$. A simple analysis determines that $N(z,t) \neq 0$ for any values of z and t. Thus, we find a single smooth critical point at $(p,q) := ([1-1/\ell]^2, 1/[\ell-1]^2)$, and there are no nonsmooth critical points.

To use the results in [9], we must also ensure that the critical point is minimal. This is slightly more difficult here, but because $N(z,t)$ is combinatorial, there must be a minimal singularity with positive real coordinates by Pringsheim's Theorem. To verify minimality, consider points of the form $(z,t) = (vp, wq)$ for real paramters $0 \le v, w \le 1$, and search for values of v and w where $H(vp, wq) = 0$. Because H is quadratic, we can solve for v in terms of w and ℓ, and then verify that for all $\ell > 1$ and all $v, w \in [0,1]$, $dw/dv < 0$. Ultimately, this implies that there are no solutions where v and w are both less than 1, so that (p,q) indeed must be minimal.

Near the critical point (p,q) we can expand $\log^r N(z,t)$ as in Example 1 to obtain

$$\log^r N(z,t) = \log^r\left(\frac{1+p-pq}{2p}\right) - r\log^{r-1}\left(\frac{1+p-pq}{2p}\right)\frac{\sqrt{H}}{1+p-pq} + \cdots,$$

from which we conclude that the $\sqrt{H}$ term determines the dominant asymptotics. Applying Corollary 2 of [9] yields our final result,

$$[z^{\ell n}t^n]\log^r N(z,t) \sim \frac{r}{2\pi}\log^{r-1}\left(\frac{\ell}{\ell-1}\right)\cdot n^{-2}\cdot(\ell-1)^{-2n(\ell-1)-1}\,\ell^{2\ell n-1}.$$

6 Proof sketch

We prove Theorem 1 by using the Cauchy integral formula,

$$[x^r y^s]H(x,y)^{-\alpha}\log^{\beta}(H(x,y)) = \frac{-1}{4\pi^2}\iint_T H(x,y)^{-\alpha}\log^{\beta}(H(x,y))x^{-r-1}y^{-s-1}dxdy, \quad (6.1)$$

where T is a torus centered at $(0,0)$ that is small enough that it does not enclose any singularities of $H(x,y)^{-\alpha}\log^{\beta}(H(x,y))$.

6.1 Step 1: Change of variables

A key idea in [9] is to use this change of variables, with χ_1 and χ_2 as in Theorem 1:

$$u = x + \chi_1(y-q) + \chi_2(y-q)^2, \quad v = y.$$

Call $H(x,y) := \tilde{H}(u,v)$. Expanding $\tilde{H}(u,v) = \sum_{m,n\geq 0} d_{m,n}(u-p)^m(v-q)^n$, we find $d_{0,0} = d_{0,1} = d_{0,2} = 0$. For functions of the form $F(u,v) = [\log\tilde{H}(u,v)]^{\beta}(\tilde{H}(u,v))^{-\alpha}$, it turns out that having these three terms equal to zero is enough to approximate F near (p,q) with the product of a function in u and a function in v.

6.2 Step 2: Choose a convenient contour

In order to justify that F can be written as a product, we first decide how to deform the torus T in Equation (6.1). We focus on the details of the contour when v is near the critical point q, since the contour away from the critical point does not contribute to the asymptotics. We choose approximately a product contour, with a Hankel contour in the u variable contour and a circle of radius q in the v variable.

The u variable contour will wrap around a point that shifts slightly depending on the v variable: more precisely, since (p,q) is a smooth critical point, the zero set $\mathcal{V} := \{(u,v) : H(u,v) = 0\}$ can be parameterized with a smooth function $G(v)$ such that

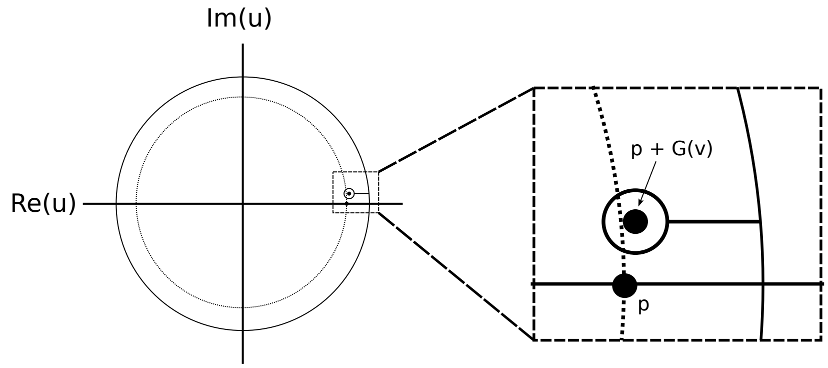

Figure 3: The torus T in the Cauchy integral is deformed so that near the critical point, it is approximately a product contour. For v close to q, the contour is exactly a circle. For u close to p, the contour expands beyond the critical point p by using a Hankel-like wrapping around the zero set of H, which is parameterized in terms of v by $p + G(v)$.

$H(p + G(v), v) = 0$ locally near $v = q$. Thus, we center the u contour at the point $p + G(v)$. See Figure 3 for a diagram of the u contour near the point (p, q). Because we assume (p, q) is a unique minimal critical point of H, for v values away from q, we can expand the contour to circles with radii larger than $|p|$ and $|q|$ making this portion of the contour negligible. The transition between regimes is described in greater detail in [9].

6.3 Step 3: Approximate the integrand with a product integral

After the change of variables to (u, v) coordinates, we can estimate the resulting Cauchy integrand with a product of a function in u and a function in v.

Lemma 1. *Let $\mathcal{C}_r$ be the portion of the contour defined in Section 6.2 where v is close to q. Then,*

$$\iint_T (\tilde{H}(u,v)^{-\alpha} \log^\beta(\tilde{H}(u,v)))(u - \chi_1(v-q) - \chi_2(v-2)^2)^{-r-1} v^{-s-1} du dv$$
$$\sim \iint_{\mathcal{C}_r} \Bigg([H_x(p,q)(u-p)]^{-\alpha} \log^\beta(H_x(p,q)(u-p)) u^{-r-1} v^{-s-1} \times$$
$$\left[1 - \frac{\chi_1(v-q) + \chi_2(v-q)^2}{p}\right]^{-r-1} \Bigg) du dv.$$

The full proof of this lemma is technical, generalizing a similar proof in [9]. Near (p, q), tedious computations reveal $\tilde{H}(u, v)$ may be estimated by truncating its power series. Away from (p, q), the contributions to the integral are exponentially smaller than the parts near (p, q), and hence they may be ignored. The addition of the logarithm adds some new technicalities to the proof. To begin, we define correction factors K, L, and N:

$$K = \left(\frac{1 - \frac{\chi_1(v-q)+\chi_2(v-q)^2}{u}}{1 - \frac{\chi_1(v-q)+\chi_2(v-q)^2}{p}} \right)^{-r-1}, \quad L = \left(\frac{\tilde{H}(u,v)}{C(u-p)} \right)^{-\alpha}, \quad N = \left(\frac{\log \tilde{H}(u,v)}{\log[C(u-p)]} \right)^{\beta},$$

with $C := H_x(p,q)$. The point of these factors is that the integrands of the left and right sides in Lemma 1 are equal up to $K \cdot L \cdot N$. Thus, in a neighborhood near (p,q), the goal is to show that $K, L, N = 1 + o(1)$ uniformly. Lemmas 4 and 5 of [9] state the result for K and L, so it remains to show the equivalent result for N.

Lemma 2. *When u and v are sufficiently close to (p,q), the following holds uniformly as $r, s \to \infty$ with $\lambda = \frac{r+O(1)}{s}$:*

$$N(u,v) = 1 + o(1).$$

Proof. Again with $C := H_x(p,q)$, we write

$$\log \tilde{H}(u,v) = \log[C(u-p)] + \log \frac{\tilde{H}(u,v)}{C(u-p)} = \log[C(u-p)] - \frac{1}{\alpha} \log L(u,v),$$

$$N(u,v) = \left[1 - \frac{\frac{1}{\alpha} \log L(u,v)}{\log C(u-p)} \right]^{\beta}.$$

From Lemma 5 of [9], $L(u,v) = 1 + o(1)$ in this region. Thus, $\log L(u,v) = o(1)$ as $r \to \infty$. Additionally, $|\log C(u-p)|$ is bounded away from zero as u is close to p, implying that $N(u,v) = [1 + o(1)]^{\beta} = 1 + o(1)$ as desired. □

6.4 Step 4: Evaluate the product integral

We can now split Equation (6.1) into two univariate integrals. The v integral is a standard Fourier-Laplace type integral that is identical to the case without logarithms.

Lemma 3 (Lemma 9 of [9])**.** *The following holds uniformly as $r, s \to \infty$ with $\lambda = \frac{r+O(1)}{s}$:*

$$\int_V v^{-s-1} \left[1 - \frac{\chi_1(v-q) + \chi_2(v-q)^2}{p} \right]^{-r-1} dv = iq^{-s} \sqrt{\frac{2\pi}{-q^2 M r}} + o\left(q^{-s} r^{-\frac{1}{2}} \right).$$

Thus, the final step in proving Theorem 1 is to evaluate the u integral.

Lemma 4. *Define $\mathcal{E}_j = \sum_{k=0}^{j} \binom{\beta}{j} \binom{j}{k} \log \left(\frac{-1}{pH_x(p,q)} \right) \Gamma(\alpha) \left. \frac{d^{j-k}}{dt^{j-k}} \frac{1}{\Gamma(t)} \right|_{t=\alpha}$. Then, as $r \to \infty$,*

$$\frac{1}{2\pi i} \int_U (H_x(p,q)(u-p))^{-\alpha} \left[\log^{\beta}(H_x(p,q)(u-p)) \right] u^{-r-1} du$$
$$= (-1)^{\beta} \frac{(-pH_x(p,q))^{-\alpha} r^{\alpha-1}}{\Gamma(\alpha)} p^{-r} \left[\log^{\beta} r \right] \left[1 + \sum_{j \geq 1} \frac{\mathcal{E}_j}{\log^j r} \right],$$

where U is a small circle near 0 that does not enclose any other singularities of the integrand.

Proof. We begin with some factoring:

$$\frac{1}{2\pi i}\int_U (H_x(p,q)(u-p))^{-\alpha}\left[\log^\beta(H_x(p,q)(u-p))\right]u^{-r-1}du$$
$$= (-1)^\beta \frac{(-pH_x(p,q))^{-\alpha}}{2\pi i}\int_U (1-u/p)^{-\alpha}\left[\log\frac{1}{1-u/p}+L\right]^\beta u^{-r-1}du,$$

where $L = \log\frac{-1}{pH_x(p,q)}$. Substitute $u = pz$ and expand $\left[\log\frac{1}{1-z}+L\right]^\beta$ as a series:

$$(-1)^\beta \frac{(-pH_x(p,q))^{-\alpha}}{2\pi i}p^{-r}\sum_{k\geq 0}\binom{\beta}{k}L^k\int_U (1-z)^{-\alpha}\log^{\beta-k}\frac{1}{1-z}z^{-r-1}dz.$$

By [6, Theorem 3A], we have for each k that

$$\frac{1}{2\pi i}\int_U (1-z)^{-\alpha}\log^{\beta-k}\frac{1}{1-z}z^{-r-1}dz \sim \frac{r^{\alpha-1}}{\Gamma(\alpha)}\log^{\beta-k}r\left[1+\sum_{j\geq 1}\frac{c_j^{(k)}}{\log^j r}\right],$$

where $c_j^{(k)} := \binom{\beta-k}{j}\Gamma(\alpha)\left.\frac{d^j}{dt^j}\frac{1}{\Gamma(t)}\right|_{t=\alpha}$. So

$$(-1)^\beta \frac{(-pH_x(p,q))^{-\alpha}}{2\pi i}p^{-r}\sum_{k\geq 0}\binom{\beta}{k}L^k\int_U (1-z)^{-\alpha}\log^{\beta-k}\frac{1}{1-z}z^{-r-1}dz$$
$$\sim (-1)^\beta \frac{(-pH_x(p,q))^{-\alpha}r^{\alpha-1}}{\Gamma(\alpha)}p^{-r}\sum_{k\geq 0}\binom{\beta}{k}L^k\log^{\beta-k}r\left[1+\sum_{j\geq 1}\frac{c_j^{(k)}}{\log^j r}\right].$$

Letting $e_j^{(k)} = \binom{\beta}{k}L^k c_j^{(k)}$, we can rewrite the double sum as

$$\sum_{k\geq 0}\binom{\beta}{k}L^k\log^{\beta-k}r\left[1+\sum_{j\geq 1}\frac{c_j^{(k)}}{\log^j r}\right] = \log^\beta r\sum_{k\geq 0}\binom{\beta}{k}L^k\left[\frac{1}{\log^k r}+\sum_{j\geq k+1}\frac{c_{j-k}^{(k)}}{\log^j r}\right]$$
$$= \log^\beta r\left[1+\sum_{j\geq 1}\sum_{k=0}^{j}\frac{e_{j-k}^{(k)}}{\log^j r}\right] = \log^\beta r\left[1+\sum_{j\geq 1}\frac{\mathcal{E}_j}{\log^j r}\right],$$

with $\mathcal{E}_j$ defined above. With this, we have the result as desired. □

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Two-Row Set-Valued Tableaux: Catalan^{+k} Combinatorics

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Abstract. Set-valued standard Young tableaux are a generalization of standard Young tableaux due to Buch (2002) with applications in algebraic geometry. The enumeration of set-valued SYT is significantly more complicated than in the ordinary case, although product formulas are known in certain special cases. In this work we study the case of two-rowed set-valued SYT with a fixed number of entries. These tableaux are a new combinatorial model for the Catalan, Narayana, and Kreweras numbers, and can be shown to be in correspondence with both 321-avoiding permutations and a certain class of bicolored Motzkin paths. We also introduce a generalization of the set-valued comajor index studied by Hopkins, Lazar, and Linusson (2023), and use this statistic to find seemingly new q-analogs of the Catalan and Narayana numbers.

Keywords: Catalan numbers, set-valued tableaux, pattern-avoidance

1 Introduction

1.1 Set-Valued Tableaux

Let $\lambda \vdash n$. A *set-valued Young tableau* of shape λ is a filling S of the cells of the Ferrers diagram of λ with nonempty sets of positive integers. They were introduced by Buch [2] to study the K-theory of the Grassmannian, and have since appeared in both algebro-geometric and combinatorial contexts (see, inter alia, [1, 4, 5, 6, 8, 11]).

A set-valued Young tableau is *standard* if:

1. The sets in the cells of λ form a set partition of $[n+k]$ for some $k \geq 0$, and
2. If u is (weakly) northwest of v in λ then $\max S(u) < \min S(v)$.

We write $\mathrm{SYT}^{+k}(\lambda)$ for the set of set-valued standard Young tableaux of λ with entries in $[n+k]$. Intuitively, a set-valued SYT S can be thought of an integer filling of λ (filling each cell u with $\min S(u)$) along with k *extra elements*. The combinatorics of these objects

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is much more intricate than in the ordinary case; there is no known analog of the hook-length formula for counting set-valued SYT in general, although Anderson, Chen, and Tarasca [1] proved a determinantal formula for counting them.

For the purposes of enumerating the elements of $\mathrm{SYT}^{+k}(\lambda)$, it is sometimes useful to view a set-valued tableaux S from a different perspective.

Proposition 1. *A standard set-valued Young tableau of shape λ is equivalent to the following data:*

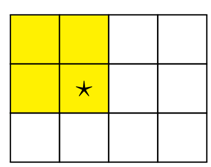
1. *A standard Young tableau S^* of shape λ,*
2. *A weak chain $\lambda^\bullet$ of subshapes $\emptyset = \lambda_0 \subsetneq \lambda_1 \subseteq \cdots \subseteq \lambda_k \subseteq \lambda_{k+1} = \lambda$,*
3. *A choice of a corner cell u_i of λ_i for each $1 \leq i \leq k$.*

In lieu of a proof, consider the following illustrative example.

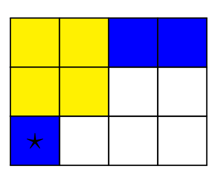
Example 2. *Consider the following set-valued SYT $T \in \mathrm{SYT}^{+4}(3 \times 4)$:*

1	2	7	8
3	4, 5	11	13
6, 9, 10	12	14, 15	16

There are cells with extra entries at matrix coordinates $(2,2)$, $(3,1)$, and $(3,3)$. Among these, the cell at $(2,2)$ has the smallest extra entry: 5. We define λ_1 to be the subshape of 3×4 for which the entries of T are between 1 and 5:

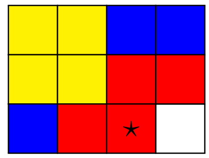


The starred cell at matrix position $(2,2)$ is u_1. The next extra entry is 9, at matrix position $(3,1)$. We define λ_2 to be the subshape for which the entries of T are between 1 and 9:



Then since 9 belongs to the starred cell $(3,1)$, we define that cell to be u_2. The next extra entry is 10, which is in the same cell as the extra entry 9. Then $\lambda_3 = \lambda_2$ and $u_3 = u_2$.

The last extra entry is 15, at matrix position $(3,3)$. We have that λ_4 is the subshape $(4,4,3)$ consisting of the cells of T whose entries are between 1 and 15.



Since 15 *belongs to the cell at position* $(3,3)$, *we define* u_4 *to be that corner cell. Finally,* λ_5 *is the entire shape.*

We obtain T^* *from* T *by removing the* 4 *extra entries from* T *and decrementing the remaining entries of the cells in* $\lambda_i \setminus \lambda_{i-1}$ *by* $i-1$ *for each* i*:*

1	2	6	7
3	4	8	10
5	9	11	12

The entries in the yellow cells are decremented by 0*; those in the blue cells are decremented by* 1*; those in the red cells are decremented by* 3 *(notice there are no cells in* $\lambda_3 \setminus \lambda_2$*); and the entry of the bottom right square is decremented by* 4*.*

This construction allows us to define a version of the *comajor index* for set-valued tableaux. Let $S \in \mathrm{SYT}^{+k}(\lambda)$, and decompose S into ℓ chunks $T_1, \ldots, T_\ell$ and k additional elements $x_1, \ldots, x_k$ as in Example 2. A *(natural) descent* of T_i is an entry j of T_i such that $j+1$ is also an entry of T_i and is in a higher row[1].

We write $\mathrm{D}(T_i)$ for the descent set of T_i, and we define the *set-valued descents* of S to be

$$\mathrm{D}^{+k}(S) := \bigsqcup D(T_i) \sqcup \{x_1, \ldots, x_k\}.$$

The *set-valued comajor index* of S is then defined as

$$\mathrm{comaj}^{+k}(S) := \sum_{x \in \mathrm{D}^{+k}(S)} (n+k-x).$$

Example 3. *Continuing from Example 2:* $D^{+4}(S) = \{5,6,9,10,12,15\}$, *so* $\mathrm{comaj}^{+4}(S) = 38$*.*

The $k=1$ version set-valued comajor index was recently used by Hopkins, Lazar, and Linusson [7] to find a product formula for $\displaystyle\sum_{S \in \mathrm{SYT}^{+1}(a \times b)} q^{\mathrm{comaj}^{+1}(S)}$ analogous to Stanley's hook-content formula. Our generalized version is motivated by the probabilistic reasoning used in [7] — when one attempts to extend their arguments to general SYT^{+k}, the comaj^{+k} statistic emerges quite naturally and yields extensions of some of the results of [7] to the general case (to appear in forthcoming work).

[1] This is different from the usual definition of a descent in a Young tableau; our definition instead comes from the theory of P-partitions.

1.2 Main Results

The present work considers set-valued SYT from a different perspective. Rather than fixing the shape and the number of extra elements of a set-valued tableau S, we instead fix the number of rows and total number of elements.

This change of perspective has proven to be fruitful; if we restrict our attention to the case of two-row tableaux and fix the total number of elements while letting the number of columns vary, we obtain several new results:

- For fixed n and i, exact counts of $\bigsqcup_{2b-i+k=n} \mathrm{SYT}^{+k}(b, b-i)$ for all $0 \leq i \leq b$. For $i = 0$, that is, rows of equal length, it is the Catalan number (Equation (2.1)) and for general i it is a ballot number plus a binomial coefficient (Theorem 13).
- New models for the Catalan (Proposition 7), Narayana, and Kreweras (Proposition 8) numbers (proved via a bijection with 321-avoiding permutations).
- A new summation formula for the 321-avoiding permutations by the number of peaks (Corollary 10).
- Exact counts of several families of lattice paths arising from these tableaux (Proposition 12 and Theorem 13).
- Seemingly-new families of q-Catalan and q-Narayana numbers (Section 5.1).

2 Bijection to 321-avoiding Permutations

In this section we will use a bijection to 321-avoiding permutations of length $n-1$ prove that for any $n \geq 2$

$$\sum_{2b+k=n} \#\mathrm{SYT}^{+k}(2 \times b) = \mathrm{Cat}(n-1). \tag{2.1}$$

A permutation $\pi = \pi_1 \dots \pi_n$ is called *321-avoiding* if it does not have three elements $\pi_i > \pi_j > \pi_k$ for $1 \leq i < j < k \leq n$. Another well-known way to describe 321 avoiding permutations is as follows. Recall that a *right-to-left minimum* in a permutation π is an element π_i such that $\pi_i < \pi_j$ for all $j > i$. The right-to-left minima of any permutation form an increasing sequence (when read from the left). The condition that a permutation is 321-avoiding is equivalent to asking that the elements that are not right-to-left minima also form an increasing sequence. This characterization dates back to the early 1900s; see [10, Vol. I, Section V, Chapter III]. [2] Visualising the permutation with a permutation matrix, the right-to-left minima will be on or below the main diagonal and the other elements above the diagonal. Forming a lattice path around the elements on or above

[2] The text considers 123-avoiding permutations, which are the reverses of 321-avoiding permutations.

$$\begin{array}{c}3\\5\\1\\2\\7\\8\\4\\10\\11\\6\\9\end{array}\begin{pmatrix}0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\end{pmatrix}$$

Figure 1: The permutation matrix of π from Example 4, along with its associated lattice path.

the diagonal gives a direct bijection to south-east lattice paths above the diagonal, which is one of the many standard representations of Catalan objects. Alternatively, one can draw a lattice path below the right-to left minima and then rotate the drawing by a half turn. We also need to define an *inner valley*[3] in a permutation $\pi \in S_n$ as an element $\pi_j, 1 < j < n$ such that $\pi_{j-1} > \pi_j < \pi_{j+1}$.

Example 4. *The permutation* $\pi = 3\ 5\ \overline{1}\ \overline{2}\ 7\ 8\ \overline{4}\ 10\ 11\ \overline{6}\ \overline{9}$ *is* 321*-avoiding. We overline the right-to-left-minima.*

For a fixed n we now define a map

$$\alpha : \bigsqcup_{2b+k=n} \mathrm{SYT}^{+k}(2\times b) \mapsto \{321\text{-avoiding permutations of } [n-1]\}.$$

Let $T \in \mathrm{SYT}^{+k}(2\times b)$, with $k+2b=n$.

(1) Remove the largest element n from T, so it contains the numbers from 1 to $n-1$.

(2) The permutation $\alpha(T)$ starts with all except the largest element in the top left box, followed by the entries of the box directly below it, and then the largest element of the top left box. The permutation continues with the elements in the second box in the top row except the largest, then all elements in the box below it, then the largest element in the second box in the top row. We continue in this way, placing the elements of the ith box from the left in the bottom row immediately before the largest element of the ith box in the top row.

[3]An inner valley differs from an ordinary valley in that neither the first nor the last position can be an inner valley.

Example 5.

1, 2	3, 4, 6	7	10
5, 8	9	11, 12	13, 14

$\stackrel{\alpha}{\mapsto} \pi = \overline{1}\ 5\ 8\ \overline{2}\ \overline{3}\ \overline{4}\ 9\ \overline{6}\ 11\ 12\ \overline{7}\ 13\ \overline{10}$

The resulting permutation $\alpha(T)$ will by construction have the numbers in the top row as its right-to-left minima. The elements in the bottom row (except n, which has been deleted) will form another increasing sequence. The permutation formed is thus 321-avoiding. Note that the largest elements in the top boxes in columns $1, \dots, b-1$ will be inner valleys in the permutation and there are no other inner valleys.

The inverse of α is reasonably simple; however, the full description requires checking several cases so we omit most of the details here. Intuitively, given a 321-avoiding permutation π, the right-to-left minima are inserted into the top row (with the first run of right-to-left minima needing special handling), while the ith run of elements that are *not* right-to-left minima is inserted into the ith box of the bottom row.

Example 6. *We reuse the permutation from Example 4 to illustrate α^{-1}.*

$\pi = \ 3\ 5\ \overline{1}\ \overline{2}\ 7\ 8\ \overline{4}\ 10\ 11\ \overline{6}\ \overline{9} \stackrel{\alpha^{-1}}{\mapsto}$

1	2, 4	6	9
3, 5	7, 8	10, 11	12

We summarize some basic properties of α.

Proposition 7. *For all $n \geq 2$:*

- *The map α is a bijection from $\bigsqcup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b)$ to the set of 321-avoiding permutations of $[n-1]$.*
- *The elements in the top row of T form the sequence of right-to-left minima in $\alpha(T)$.*
- *If T has b columns, then $\alpha(T)$ will have $b-1$ inner valleys.*

An *inner peak* in a permutation $\pi \in S_n$ is an element $\pi_j, 1 < j < n$ such that $\pi_{j-1} < \pi_j > \pi_{j+1}$. For the set of 321-avoiding permutations the involution formed by rotating the permutation matrix a half turn shows that inner peaks and inner valleys are equidistributed for 321-avoiding permutations.[4]

[4]We thank FindStat [12], which helped us find that the refinement into columns was equidistributed with number of inner peaks. This equidistribution was a key insight into finding the bijection α.

Recall from the theory of Catalan numbers that the number of Dyck paths of length $2n$ is counted by the Catalan number $\mathrm{Cat}(n)$, also the number of such paths with m peaks is enumerated by the Narayana number $N_{n,m} = \frac{1}{m}\binom{n}{m-1}\binom{n-1}{m-1}$. There is even one further refinement. Let c_i be the number of upsteps in the Dyck paths directly before peak number i in the path, which gives a partition $\mathbf{c} = (c_1, \ldots, c_m)$ of n, that is $\sum_i c_i = n$. Further let μ_j be the number of c_i that equals j. Thus μ (or sometimes written $[1^{\mu_1}2^{\mu_2} \ldots n^{\mu_n}]$) is the **type** of the composition $\mathbf{c}$ and of the Dyck path. The number of Dyck paths with m peaks and of type μ is known to be the Kreweras number $\mathrm{Krew}(n, m, \mu) = \frac{n(n-1)\ldots(n-m+1)}{\prod_j \mu_j!}$[9]. In the bijection α, a tableau with m elements in the top row will be mapped to a 321-avoiding permutation with m right-to-left minima. As discussed above, we can draw a lattice path under these in the permutation matrix and by rotating half a turn obtain a bijection to Dyck paths with m peaks. The distance between two consecutive elements in the top row is mapped to the number of upsteps c_i of the Dyck path. This proves the following proposition.

Proposition 8. *For any $b, k \geq 1$ we have the following refinements:*

1. *The number of tableaux in $\cup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b)$ with m elements in the top row is the Narayana number $N_{n-1,m} = \frac{1}{m}\binom{n-1}{m-1}\binom{n-2}{m-1}$*

2. *The number of tableaux in $\cup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b)$ with elements $a_1, \ldots, a_m$ in the top row is the Kreweras number $\mathrm{Krew}(n, m, \mu)$, where μ is the type of $(c_1, \ldots, c_m)$ with $c_i = a_{i+1} - a_i$ and $a_{m+1} := n$.*

The bijection α also implies the following.

Proposition 9. *For $n \geq 3$*

1. $|\cup_{2b+k+1=n} \mathrm{SYT}^{+k}(b+1, b)| = \mathrm{Cat}(n) - \mathrm{Cat}(n-1) = \frac{3}{n+1}\binom{2n-2}{n}$.

2. $|\cup_{2b+k=n} \mathrm{SYT}^{+k}(b+1, b)/(1)| = \mathrm{Cat}(n) - 2\,\mathrm{Cat}(n-1) + \mathrm{Cat}(n-2)$.

3 Enumeration According to Peaks

In [1, Corollary 5.4], Anderson, Chen, and Tarasca give a formula for the Euler characteristic of a certain *Brill–Noether space*, which they had earlier shown to be equal to the (signed) count of a certain class of set-valued tableaux. Specializing to the two row case and translating into our notation, their formula becomes:

$$\#\{\mathrm{SYT}^{+k}(2 \times b)\} = \frac{1}{k!} \sum_{c=0}^{\lfloor \frac{k}{2} \rfloor} f^{(k-c,c)} f^{(b+k-c,b+c)} (b+k-c-1)_{(k-c)} (b+c-2)_{(c)}, \qquad (3.1)$$

where f^λ is the number of SYT of shape λ, and $(x)_{(a)}$ is the falling factorial $x(x-1)\cdots(x-a+1)$.

For our purposes, it is convenient to use the hook length formula to rewrite Equation 3.1 purely in terms of factorials and binomial coefficients:

$$\begin{aligned}
&\#\{\mathrm{SYT}^{+k}(2\times b)\} && (3.2)\\
&\quad = \frac{1}{k!}\sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} \frac{k!(k-2c+1)}{c!(k-c+1)!}\frac{(2b+k)!(k-2c+1)}{(b+c)!(b+k-c+1)!}\frac{(b+k-c-1)!}{(b-1)!}\frac{(b+c-2)!}{(b-2)!}\\
&\quad = \sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} \frac{(k-2c+1)^2}{(k-c+1)(b+k-c+1)}\binom{b+c-2}{c}\binom{b+k-c-1}{b-1}\binom{2b+k}{b+c}. && (3.3)
\end{aligned}$$

In light of our bijection between $\mathrm{SYT}^{+k}(2\times b)$ and the set of 321-avoiding permutations of $n=2b+k$ with exactly k peaks, we immediately have the following result:

Corollary 10.

$$\#\{\mathfrak{S}^{321}_{2b+k}\} = \sum_{c=0}^{\lfloor \frac{k}{2}\rfloor} \frac{(k-2c+1)^2}{(k-c+1)(b+k-c+1)}\binom{b+c-2}{c}\binom{b+k-c-1}{b-1}\binom{2b+k}{b+c}.$$

This sort of closed form expression is a somewhat pleasant surprise; in [3, Theorem 3], the authors give the following generating function formula for the sequence $a^{\mathrm{pk}}_{n,k}(321)$ of 321-avoiding permutations of $[n]$ with k peaks:

$$\sum_{n\geq 0}\sum_{k\geq 0} a^{\mathrm{pk}}_{n,k}(321)q^k z^n = 1 + z\left(-\frac{-1+\sqrt{-4z^2q+4z^2-4z+1}}{2z(zq-z+1)}\right)^2, \tag{3.4}$$

It is not at all obvious how one would obtain Equation (3.3) from Equation (3.4) (or vice-versa) using only elementary techniques.

4 Motzkinlike and Ballotlike Paths

In addition to 321-avoiding permutations, we can interpret the $\mathrm{SYT}^{+k}(2\times b)$ in terms of a certain class of bicolored Motzkin paths.

We recall that a *Motzkin path* of length n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$ consisting of *up steps* $U=(1,1)$, *down steps* $D=(1,-1)$, and *horizontal steps* $H=(1,0)$ in some order, with the property that the path never goes below the x-axis.

We will color the horizontal steps of the Motzkin paths with u (for upstairs or umber) and d (downstairs or denim). We will consider the following two restrictions on the coloring:

(1) umber horizontal steps do not occur at height zero;

(2) denim horizontal steps do not occur before the first down step.

We use $\mathrm{Motz}(n)$ to denote the set of bicolored Motzkin paths of length n, and $\mathrm{Motz}^1(n)$, $\mathrm{Motz}^2(n)$, and $\mathrm{Motz}^{1,2}(n)$ to denote the set of paths which satisfies the restrictions (1), (2), and both (1) and (2) respectively.

Extending the well-known bijection between two-rowed rectangular SYTs and Dyck paths, we have the following bijection.

Proposition 11. *There is a bijection β between $\mathrm{SYT}^k(2 \times b)$ and $\mathrm{Motz}^{1,2}(2b+k)$ with k horizontal steps. A tableau S maps to the path Γ for which:*

- *up steps of Γ occur at the minimal entries of boxes in the first row of S;*
- *down-steps of Γ occur at the minimal entries in the second row;*
- *we color a horizontal step of Γ umber if its index is a (non-minimal) entry of a box in the first row of S, and denim if its index is a (non-minimal) entry in the second row.*

The proof of Proposition 11 is reasonably straightforward and is omitted in this abstract.

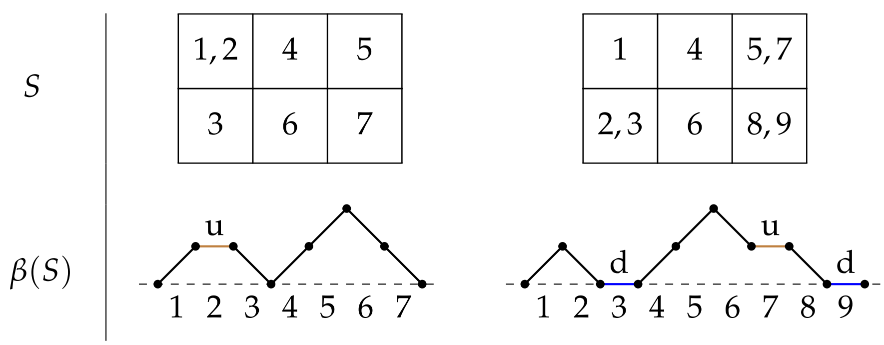


Figure 2: Examples of the bijection β between two-rowed rectangular set-valued SYTs and birestricted bicolored Motzkin paths.

The first two equalities in the following proposition are well-known but the other two seem to be new. We construct a bijection that, when concatenated with β in Proposition 11, gives us a second bijective proof of (2.1). The details are omitted in this extended abstract.

Proposition 12. *The Catalan numbers enumerate all four possible restriction on Motzkin paths:* $|\mathrm{Motz}(n)| = \mathrm{Cat}(n+1)$, $|\mathrm{Motz}^1(n)| = \mathrm{Cat}(n)$, $|\mathrm{Motz}^2(n)| = \mathrm{Cat}(n)$, *and* $|\mathrm{Motz}^{1,2}(n)| = \mathrm{Cat}(n-1)$, *for* $n \geq 2$.

4.1 Ballotlike paths

We can also consider a larger class of paths which we call *ballotlike*. A ballotlike path P is a lattice path in the 1st quadrant starting at $(0,0)$ and ending at (n,i) which uses the steps $U = (1,1)$, $D = (1,-1)$, $u = (1,0)^{\text{umber}}$ and $d = (1,0)^{\text{denim}}$, subject to the same conditions on u and d steps from the definition of $\mathrm{Motz}^{1,2}(n)$. We write $\mathrm{Bal}^*(n,i)$ for the set of ballotlike paths ending at (n,i). The enumeration turns out to be the sum of a classical ballot number and a binomial coefficient.

Theorem 13. *For any (n,i) with $0 \le i \le n$, we have*

$$\#\,\mathrm{Bal}^*(n,i) = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

Moreover, if we take the obvious extension of the bijection between $\mathrm{Motz}^{1,2}(n)$ and set-valued SYT of shape $2 \times b$, we have for any (n,i) with $0 \le i \le n$,

$$\# \bigsqcup_{2b+k-i=n} \mathrm{SYT}^{+k}(b, b-i) = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

Example 14. *When $n = 4$ and $i = 2$ we have 6 set-valued SYT.*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1,2,3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1,2 & 3,4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2,3,4 \\ \hline \end{array}$$

5 Future Work

5.1 q-Catalan and q-Narayana

Given the numerology for $\mathrm{SYT}^{+k}(2 \times b)$, it is natural to consider the following q-analogs:

$$\widetilde{\mathrm{Cat}}_n(q) := \sum_{2b+k=n+1} \left(\sum_{S \in \mathrm{SYT}^{+k}(2\times b)} q^{\mathrm{comaj}^{+k}(S)} \right)$$

$$\widetilde{N}_{n,m}(q) := \sum_{2b+k=n+1} \left(\sum_{\substack{S \in \mathrm{SYT}^{+k}(2\times b) \\ m \text{ elts in top row}}} q^{\mathrm{comaj}^{+k}(S)} \right).$$

Using the bijection in Proposition 11 and a double recursion we can compute the polynomials $\widetilde{\mathrm{Cat}}_q$ and $\widetilde{N}_{n,m}(q)$ for small values of n, m (details omitted in this extended abstract). They do not seem to match any statistic we have found in the literature.

Question 15. *Are there better formulas for $\widetilde{\mathrm{Cat}}_n(q)$ and $\widetilde{N}_{n,m}(q)$?*

n	$\widetilde{\mathrm{Cat}}_n(q)$
1	1
2	$q+1$
3	q^3+2q^2+q+1
4	$q^6+2q^5+3q^4+3q^3+2q^2+2q+1$
5	$q^{10}+2q^9+3q^8+7q^7+6q^6+5q^5+6q^4+7q^3+3q^2+q+1$

Figure 3: The first six $\widetilde{\mathrm{Cat}}_n(q)$ polynomials.

n/m	**1**	**2**	**3**	**4**
1	1			
2	1	q		
3	q	$2q^2+1$	q^3	
4	q^3	$2q^4+q^3+q^2+q+1$	$2q^5+q^4+q^3+q^2+q$	q^6

Figure 4: $\widetilde{N_{n,m}}(q)$ for $2 \leq n \leq 5$.

5.2 Expected Number of Columns

The results of [7] draw heavily upon the language of probability theory. In particular, the authors consider several families of probability distributions on subshapes of the $a \times b$ rectangular partition, and compute the expected value of the number of corners of the subshapes with respect to these distributions. In this spirit, we consider the number of columns of a randomly-selected $S \in \bigsqcup_{2b+k=n} \mathrm{SYT}^{+k}(2 \times b)$ (equivalently, the number of inner peaks of a randomly-selected 321-avoiding permutation).

Conjecture 16. *If we sample such an S uniformly at random, we have for $n \geq 3$ that*

$$\mathbb{E}\left(\#\text{ of columns of } S\right) = \left(\binom{n}{2} + n - 3\right) \Big/ (2n-3).$$

Question 17. *Is there a nice formula for the q-version? Specifically, is there a better formula for*

$$\mathbb{E}_q\left(\#\text{ of columns of } S\right) = \sum_{2b+k=n+1} b \cdot \left(\sum_{S \in \mathrm{SYT}^{+k}(2\times b)} q^{\mathrm{comaj}^{+k}(S)}\right) \Big/ \widetilde{\mathrm{Cat}}_n(q)?$$

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Skein relations for punctured surfaces

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Abstract. We use a combinatorial expansion formula for cluster algebras of surface type via order ideals of posets to give explicit skein relations for elements of a cluster algebra arising from a punctured surfaces. An immediate corollary of this is that the bangles and bracelets of Musiker, Schiffler, and Williams, which are known to provide a basis in the unpunctured case, form a spanning set in the punctured case.

Keywords: skein relation, triangulated surfaces, cluster algebras

1 Introduction

Subsequent to the original introduction of cluster algebras by Fomin and Zelevinsky in 2002 [5], a significant amount of effort has been devoted to studying cluster algebras of *surface type*, as defined in [3, 4]. Such cluster algebras are particularly appealing objects of study because they admit constructions of a variety of combinatorial objects - including snake graphs, T-paths, and posets - that can be used to prove important structural results about positivity or the existence of bases. In this extended abstract, we use a cluster expansion formula from [11, 13] which expresses elements of a cluster algebra as generating functions of order ideals of certain posets. We use this expansion formula to prove *skein relations*, i.e. relations used to resolve intersections or incompatibilities of arcs. Topologically, a skein relation takes a pair of intersecting arcs or an arc with self-intersection and replaces this configuration with two sets of arcs which avoid the intersection in two different ways. This method gives a generalization and new perspective to *snake graph calculus*, as defined in [2]. Skein relations for unpunctured surfaces were given in [10, 1]. Skein relations on punctured surfaces in the coefficient-free case were discussed in [7] and specific forms of skein relations in the principal coefficient case (so called "tidy exchange relations") were given in [13]. Here, we give explicit formulae and show all skein relations on (potentially punctured) surfaces contain a term that is not divisible by any coefficient variable y_i. Consequently, we observe that the *bangles* and *bracelets* defined in [9] form spanning sets and are linearly independent.

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2 Background

Cluster algebras are a type of recursively generated commutative ring with distinguished generators, called *cluster variables*, that appear in fixed-size subsets $\mathbf{x} = (x_1, \ldots, x_n)$ called *clusters*. Each cluster $\mathbf{x}$ has an associated set of *coefficients* $\mathbf{y} = (y_1, \ldots, y_n)$. Clusters can be obtained from each other via an involutive process called *mutation*. A single mutation μ_k uniquely exchanges a cluster variable $x_k \in \mathbf{x}$ for some $x'_k \notin \mathbf{x}$. The relation between x_k and x'_k is referred to as an *exchange relation*. Given a cluster $\mathbf{x}$, it is always possible to mutate at every $x_i \in \mathbf{x}$. A single cluster is sufficient to generate the entire cluster algebra.

Two of the most celebrated properties of cluster algebras are the *Laurent phenomenon* and *positivity*, which together state that every cluster algebra element can be written as a Laurent polynomial with positive integer coefficients in terms of any choice of cluster.

Triangulated surfaces provide a well-known geometric model for *ordinary cluster algebras of surface type* [3, 4]. Let S be a surface with (potentially empty) boundary and a non-empty set of *marked points* M, where there is at least one marked point on each boundary component. Marked points in the interior of S are referred to as *punctures*. Every such marked surface (S, M) has an associated cluster algebra $\mathcal{A}_S$. Clusters of $\mathcal{A}_S$ correspond to distinct triangulations of (S, M), with individual cluster variables corresponding to individual arcs (i.e., curves with endpoints in M and no self-intersections). Coefficients correspond to *laminations* [4], i.e. additional collections of curves on (S, M) that meet certain conditions. Following the restrictions in [8], we do not allow (S, M) to be a closed surface with exactly two punctures, a monogon with less than two punctures, an unpunctured bigon or triangle, or a sphere with less than four punctures.

In the surface model, mutation at x_k is represented by *flipping* the corresponding arc γ in a triangulation T - that is, by replacing γ with a different arc γ', which corresponds to x'_k, such that $T - \{\gamma\} \cup \{\gamma'\}$ is still a valid triangulation. To provide complete geometric models for cluster algebras from punctured surfaces [3] introduced the more general notion of *tagged arcs*. A *tagged arc* is an arc whose ends have been tagged either *plain* or *notched* such that: the arc does not cut out a once-punctured monogon, any end on ∂S is tagged plain, and both ends of a loop have the same tagging.

If η is a tagged arc with endpoints p and q, we write η^0 to denote the underlying plain arc. If we wish to emphasize the notching of η, we will write $\eta^{(p)}$ when η has a single notched end at p and $\eta^{(pq)}$ when η is notched at both endpoints. Two tagged arcs α and β are *compatible* if and only if the following properties hold: the isotopy classes of α^0 and β^0 contain non-intersecting representatives; if $\alpha^0 = \beta^0$ then at least one end of α has the same tagging as the corresponding end of β; and if $\alpha^0 \neq \beta^0$ have a shared endpoint, then α and β must have the same tagging at that endpoint. A *tagged triangulation* is a maximal collection of pairwise compatible tagged arcs. We will work with clusters associated to triangulations with only plain arcs.

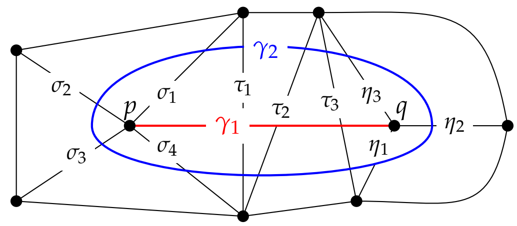


Figure 1: An example of an arc γ_1 and closed curve γ_2 on a triangulated surface.

3 Cluster expansion formula

3.1 The poset for an arc

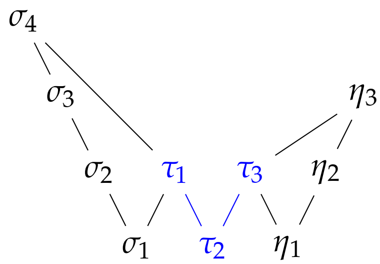
Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of a surface (S, M). For any arc γ on (S, M), we construct a corresponding poset P_γ, following [11, 13]. We note that the posets P_γ will be exactly the poset of join-irreducibles in the lattice of perfect matchings of the snake graph $\mathcal{G}_\gamma$, as in [8, 14].

First, suppose that γ is an arc with both endpoints tagged plain. Fix an orientation for γ and let $\tau_{i_1}, \ldots \tau_{i_d}$ be the list of arcs of T crossed by γ, in the order determined by our choice of orientation of γ. We will place a poset structure on $[d]$ in the following way. Any two consecutive arcs crossed by γ, τ_{i_j} and $\tau_{i_{j+1}}$, border a triangle Δ_j that γ passes through between these crossings. Let s_j denote the shared endpoint of τ_{i_j} and $\tau_{i_{j+1}}$ which is a vertex of Δ_j. If s_j lies to the right of γ (with respect to the orientation placed on γ), then we set $j \gtrdot j+1$. Otherwise, we set $j \lessdot j+1$. The resulting poset is sometimes referred to as a *fence poset* since its Hasse diagram is a path graph. The process is the same if γ is a *generalized arc*, so that it has self-intersections.

Next, suppose that $\gamma^{(p)}$ is notched at its starting point $s(\gamma) = p$. Begin by drawing the fence poset for γ^0. Suppose the first triangle γ passes through is Δ_0. Necessarily, Δ_0 is bordered by τ_{i_1} and two spokes at p. Label these spokes σ_1, σ_m where σ_1 is the clockwise neighbor of τ_{i_1}. Label the remaining spokes in clockwise order. Then, we include elements $1^s, \ldots, m^s$ in the poset, and set $m^s \lessdot (m-1)^s \lessdot \cdots \lessdot 1^s$, $1^s \gtrdot 1$ and $m^s \lessdot 1$. If we have an arc which is instead notched at its terminal point, we repeat this process with elements $1^t, \ldots, m^t$, and we combine these processes for an arc tagged at both endpoints. We call the resulting posets *loop fence posets* as they correspond to the loop graphs given by Wilson in [14]. We say that the elements $1^s, \ldots, m^s$ are in a *loop*. If we wish to refer to a loop fence poset P with the loop portion removed, we will denote this P^0, so that $P_{\gamma^0} = P^0_\gamma$.

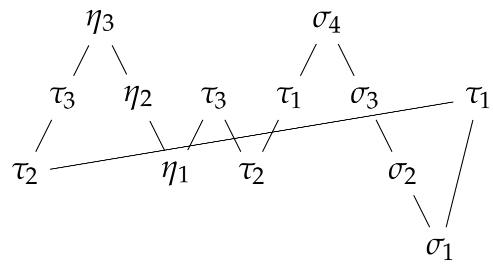
Finally, suppose that γ is a closed curve. Choose an point a of γ which is not a point of intersection between γ and T. Treat γ like an arc with $s(\gamma) = t(\gamma) = a$, choose an orientation of γ, and form the fence poset on $[d]$ associated to this arc. It must be that τ_{i_1} and τ_{i_d} share an endpoint which is an endpoint of the triangle containing a. If this

$P_{\gamma_1^{(pq)}}$



$$\mathbf{a}_{\gamma_1^{(pq)}} = \mathbf{e}_{\sigma_1} + \mathbf{e}_{\tau_2} + \mathbf{e}_{\eta_1}$$
$$\mathbf{b}_{\gamma_1^{(pq)}} = \mathbf{e}_{\tau_1} + \mathbf{e}_{\tau_3}$$
$$\mathbf{r}_{\gamma_1^{(pq)}} = 0$$

P_{γ_2}



$$\mathbf{a}_{\gamma_2} = \mathbf{e}_{\eta_1} + \mathbf{e}_{\tau_2} + \mathbf{e}_{\sigma_1}$$
$$\mathbf{b}_{\gamma_2} = \mathbf{e}_{\eta_3} + \mathbf{e}_{\tau_3} + \mathbf{e}_{\sigma_4}$$
$$\mathbf{r}_{\gamma_2} = 0$$

Table 1: The loop fence poset $P_{\gamma_1^{(pq)}}$ and circular fence poset P_{γ_2} for the arcs from Figure 1. Note that the fence poset $P_{\gamma_1} = P^0_{\gamma_1^{(pq)}}$ for the plain arc γ_1 appears as a subposet of $P_{\gamma_1^{(pq)}}$, indicated in blue, and has $\mathbf{a}_{\gamma_1} = \mathbf{e}_{\tau_2}$, $\mathbf{b}_{\gamma_1} = 0$, and $\mathbf{r}_{\gamma_1} = \mathbf{e}_{\sigma_1} + \mathbf{e}_{\eta_1}$.

endpoint is to the right of γ with the chosen orientation, we set $d \gtrdot 1$; otherwise we set $d \lessdot 1$. These posets are called *circular fence posets* since the underlying graph of such a Hasse diagram is a cycle. To improve readability, we will often refer to all of these types of posets as fence posets unless the specific type is relevant, in which case we use the specific term. See Table 1 for several examples; note here and for the remainder of the paper, we label the poset elements with the arcs they correspond to and we conflate these two notions when context is clear.

3.2 Minimal Terms

Let $\mathbf{a}_\gamma = (a_1, \ldots, a_n)$ where a_j is the number of times there is a minimal element $\tau_{i_k} \in P_\gamma$ such that $\tau_{i_k} = \tau_j$. Let $\mathbf{b}_\gamma = (b_1, \ldots, b_n)$ where b_j is the number of times there is an element $\tau_{i_k} \in P_\gamma$ which covers at least two elements and is not in a loop such that $\tau_{i_k} = \tau_j$. Note that one or both of the elements which τ_{i_k} covers can be in a loop.

Suppose γ is an plain arc and there exists $\tau_i, \tau_j \in T$ such that τ_i follows τ_1 in clockwise order in Δ_0, the first triangle γ passes through, and similarly τ_j follows τ_d in clockwise order in Δ_d, the last triangle γ passes through. Then we set $\mathbf{r}_\gamma = \mathbf{e}_i + \mathbf{e}_j$ where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^n$. If γ is instead notched at an endpoint or the clockwise neighbor of τ_1 or τ_d is on the boundary of (S, M), then we omit its contribution.

Given any arc or closed curve γ, we define $\mathbf{g}_\gamma := -\mathbf{a}_\gamma + \mathbf{b}_\gamma + \mathbf{r}_\gamma$. We remark that this notation is inspired by the notation for the g-vector of a string module, as in [12].

Geiß, Labardini-Fragoso, and Schröer studied these $\mathbf{g}$-vectors for plain arcs and closed curves in [6]. In particular, using Proposition 10.14 and Remark 11.1, they showed that $\mathbf{x}^{\mathbf{g}_\gamma}$ is the unique term in x_γ^T which is not divisible by any variable y_i. We show the same statement for a notched arc γ.

Lemma 1. *Let T be a triangulation of a surface without self-folded triangles. The monomial $\mathbf{x}^{\mathbf{g}_\gamma}$ is the unique term in the expansion of x_γ^T which is not divisible by any variable y_i.*

Given an arc $\tau_i \in T$, let $x_{CCW}(\tau_i) = x_{\tau_j} x_{\tau_k}$ if there are two arcs $\tau_j, \tau_k \in T$ that are counterclockwise neighbors of τ_i within the two triangles that it borders. If one or both of those neighbors is a boundary arc, then we ignore its contribution. The monomial $x_{CW}(\tau_i)$ is defined analogously using the clockwise neighbors of τ_i. We set $\widehat{y}_{\tau_i} := (x_{CCW}(\tau_i)/x_{CW}(\tau_i))\, y_{\tau_i}$. Let $J(P)$ denote the poset of lower order ideals of a poset P_γ. Each $I \in J(P)$ has an associated *weight* $w(I) = \prod_{j \in I} \widehat{y}_{\tau_{i_j}}$.

Proposition 1. *Let γ be an arc or closed curve on a marked surface (S, M) with triangulation T such that $\gamma \notin T$. Then, the associated element x_γ of the cluster algebra $\mathcal{A}(S, M)$ written with respect to the cluster corresponding to T can be expressed by*

$$x_\gamma^T = \mathbf{x}^{\mathbf{g}_\gamma} \sum_{I \in J(P_\gamma)} w(I).$$

Proof. If γ is not an arc such that $\gamma^0 \in T$, then this follows from combining Proposition 3.2 in [11] with Lemma 1. If $\gamma \neq \gamma^0$ and $\gamma^0 \in T$, we prove this expansion formula by using the algebraic identities that relate a singly-notched arc to plain arc and Theorem 12.9 in [8], which relates a doubly-notched arc to plain and singly-notched arcs. □

Example 1. Applying Proposition 1 to the arc $\gamma_1^{(pq)}$ from Table 1 produces

$$x_{\gamma_1^{(pq)}} = \frac{x_{\tau_1} x_{\tau_3}}{x_{\sigma_4} x_{\tau_2} x_{\eta_3}} \left[\frac{x_{\sigma_3} y_{\sigma_4} y_{\tau_2}}{x_{\sigma_1} x_{\tau_1}^2 x_{\tau_3}} + \frac{x_{\eta_1} y_{\eta_3}}{x_{\tau_1} x_{\tau_3}^2 x_{\eta_2}} + \frac{x_{\sigma_3} x_{\eta_1} y_{\sigma_4} y_{\eta_3}}{x_{\tau_1} x_{\sigma_1} x_{\tau_3} x_{\eta_2}} + \frac{y_{\eta_2} y_{\eta_3}}{x_{\tau_3}} + \frac{x_{\sigma_2} x_{\sigma_3} y_{\sigma_3}}{x_{\sigma_1} x_{\sigma_4}} + \cdots \right]$$

where we have explicitly shown only the terms arising from order ideals of size two.

4 Skein Relations

Let γ_1 and γ_2 be two curves with a point of incompatibility s; by this, we mean that either γ_1 and γ_2 intersect, or $\gamma_1^0 \neq \gamma_2^0$ share an endpoint and have opposite taggings at the endpoint. In some cases, γ_1 and γ_2 cross the same set of arcs before or after passing through s; if s is an intersection point, as we vary the representatives of γ_1 and γ_2 in their isotopy classes, the point s can lie on any of these arcs. We call such a configuration a *crossing overlap*. When the set-up is understood, we refer to this set of commonly crossed arcs as R.

If two arcs cross and this point of intersection is near the endpoint of one arc, then when we form some of the arcs in the resolution, these will *pivot* at this endpoint. For example, in the left diagram in Table 2, the left arc C^- pivots across σ_2 in counterclockwise direction and the right arc C^- pivots across σ_6 in clockwise direction. Some of these pivots will also affect the y-monomial in the resolution. For a pair of crossing curves, we define the *sweep set*, denoted Sw, to be the set of arcs that an arc in the resolution pivots past in in clockwise (resp counterclockwise) direction at a plain (resp notched) endpoint. Now suppose we instead have two arcs with incompatible taggings at a puncture p. Suppose $\gamma_1^{(p)}$ is tagged at p and γ_2 is not. Then, we define the sweep set to be the set of arcs from T which lie counterclockwise of γ_1 and clockwise of γ_2. See Table 2 for examples.

Given two arcs with an incompatibility and associated sets $R \cup Sw$, one can show that one of the sets of arcs in the resolution at the incompatibility will not cross any of the arcs in $R \cup Sw$. We will label the sets of arcs (called *multicurves*) in the resolution as C^+ and C^- where C^- is the set which does not cross any arcs in R and Sw.

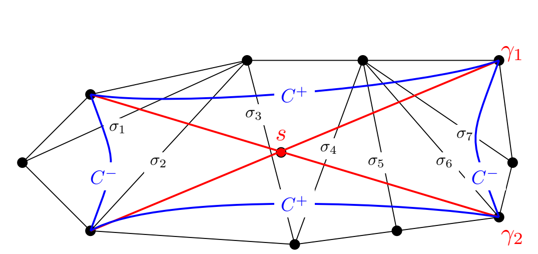
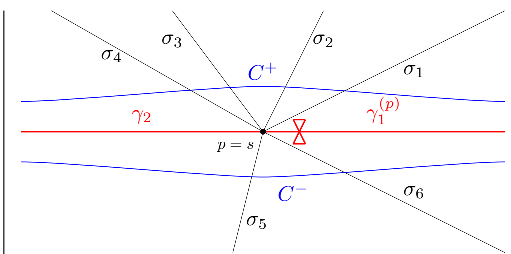
	
$R = \{\sigma_3, \sigma_4, \sigma_5\}$ $Sw = \{\sigma_6\}$	$R = \varnothing$ $Sw = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$

Table 2: Examples of R and Sw for a transverse crossing (left) and an incompatibility at a puncture (right).

Theorem 1. *1. Let $\{\gamma_1, \gamma_2\}$ be a multicurve of arcs or closed curves which are incompatible. Choose one point of incompatibility and let C^+ and C^- be the resolution at this intersection. Then, $x_{\gamma_1}x_{\gamma_2} = x_{C^+} + Y_R Y_{Sw} x_{C^-}$.*

2. Let γ_1 be an arc or closed curve which is incompatible with itself. Choose one point of incompatibility and let C^+ and C^- be the resolution at this intersection. Then, $x_{\gamma_1} = x_{C^+} + Y_R Y_{Sw} x_{C^-}$.

We prove Theorem 1 in cases. In section 4.1, we will explain our proof method which can be used for all cases. In Sections 4.2 and 4.3, respectively, we will explicitly prove this

Theorem for a pair of arcs with incompatible taggings at a puncture and a pair of arcs with a transverse crossing. The relations that we explicitly discuss in these sections are helpful in unifying some of the cases outlined in [9]. For example, the relation discussed in 4.2 can be used to handle cases 6-11 from [9]. For the sake of brevity, we only include explicit proofs for these two examples.

4.1 General approach

Let γ_1 and γ_2 be two curves with a point of incompatibility and resolutions C^+ and C^-. Set $P_i := P_{\gamma_i}$ and $\mathbf{g}_i := \mathbf{g}_{\gamma_i}$. In light of Proposition 1, we can write $x_{\gamma_1}x_{\gamma_2}$ as

$$\mathbf{x}^{\mathbf{g}_1+\mathbf{g}_2} \sum_{(I_1,I_2)\in J(P_1)\times J(P_2)} w(I_1)w(I_2).$$

We set $w(I_1, I_2)$ to be the product of the weights of the components $w(I_1)w(I_2)$. If $C^+ = \{\gamma_3, \gamma_4\}$, then we set $J(C^+) = J(P_3) \times J(P_4)$; otherwise, C^+ is a singleton $\{\gamma_3\}$ and we set $J(C^+) = J(P_3)$. We define $J(C^-)$ similarly. Our method of proof centers on finding a partition of $J(\mathcal{P}_1) \times J(\mathcal{P}_2) = A \sqcup B$ such that $(\emptyset,\emptyset) \in A$, and bijections Φ_A between A and $J(C^+)$ and Φ_B between B and $J(C^-)$. Moreover, we require that the bijection between A and $J(C^+)$ is weight-preserving, so that $w(I_1)w(I_2) = w(\Phi_A(I_1, I_2))$ and that the bijection between B and $J(C^-)$ is weight preserving up to a unique monomial, so that for some monomial Z in x and y variables, $w(I_1)w(I_2) = Zw(\Phi_B(I_1, I_2))$. Let $Z = XY$ be the decomposition of Z into x and y variables. The final step of each proof is to show that $\mathbf{g}_1 + \mathbf{g}_2$ is equal to the sum of the $\mathbf{g}$-vectors for the posets in C^+ (denoted $\mathbf{g}_{C^+}$) and $\mathbf{g}_1 + \mathbf{g}_2 + \deg(X)$ is equal to the the sum of the $\mathbf{g}$-vectors for the posets in C^- (denoted $\mathbf{g}_{C^-}$) where $\deg(X) = (\deg_{x_{\tau_1}}(X), \ldots, \deg_{x_{\tau_n}}(X))$. Then, we can rewrite $x_{\gamma_1}x_{\gamma_2}$ as

$$\begin{aligned} &= \mathbf{x}^{\mathbf{g}_1+\mathbf{g}_2} \sum_{(I_1,I_2)\in A} w(\Phi_A(I_1, I_2)) + \mathbf{x}^{\mathbf{g}_1+\mathbf{g}_2} \sum_{(I_1,I_2)\in B} Zw(\Phi_B(I_1, I_2)) \\ &= \mathbf{x}^{\mathbf{g}_{C^+}} \sum_{\boldsymbol{I}\in J(C^+)} w(\boldsymbol{I}) + \mathbf{x}^{\mathbf{g}_{C^-}} Y \sum_{\boldsymbol{I}\in J(C^-)} w(\boldsymbol{I}) = x_{C^+} + Yx_{C^-}, \end{aligned}$$

where x_{C^+} is the product of x variables associated to the arcs in C^+ In each example, Z will be a product of $\hat{y}$-variables that corresponds to the preimage of a tuple of emptysets in $J(C^-)$. For part (2) of Theorem 1, we have similar statements with just one poset I_1. When resolving a self-intersection, it is possible for one arc to have a contractible kink, in which case we remove the kink and multiply the associated expression by -1; in this case, the bijections are adjusted to account for the difference in sign.

4.2 Incompatibility at punctures

Consider two arcs, $\gamma_1^{(p)}$ and γ_2 which are incompatible at a puncture p as on the right hand side of Table 2. Recall from Section 2 that this means $\gamma_1^{(p)}$ and γ_2 have opposite taggings at p and $\gamma_1^0 \neq \gamma_2^0$. Orient $\gamma_1^{(p)}$ and γ_2 to both begin at p. Let the spokes at p from T be $\sigma_1, \ldots, \sigma_m$, labeled in counterclockwise order such that the first triangle that $\gamma_1^{(p)}$ passes through is bounded by σ_1 and σ_m. If $\gamma_2 \notin T$, let $1 \leq k \leq m$ be such that the first triangle γ_2 passes through is bounded by σ_k and σ_{k+1}, where we interpret σ_{m+1} as σ_1. If $\gamma_2 \in T$, then we let k be such that $\gamma_2 = \sigma_k$.

Draw a small circle h that encompasses p and does not cross any arcs of T except the spokes at p. We define $\gamma_1^{-1} \circ_{CCW} \gamma_2$ as the arc which results from following γ_1 from $t(\gamma_1)$ with reverse orientation until its intersection with h, following h counterclockwise until its intersection with γ_2, and then following γ_2 until $t(\gamma_2)$. We define $\gamma_1^{-1} \circ_{CW} \gamma_2$ similarly. Set $\gamma_3 := \gamma_1^{-1} \circ_{CCW} \gamma_2$ and $\gamma_4 := \gamma_1^{-1} \circ_{CW} \gamma_2$ and note that γ_3 crosses $\sigma_1, \ldots, \sigma_k$ and γ_4 crosses $\sigma_{k+1}, \ldots, \sigma_m$. On the right hand side of Table 2, $k = 4$, γ_3 is the arc denoted C^+ and γ_4 is the arc denoted C^-.

When $k = m$ and $\gamma_2 \notin T$, so that the first triangles $\gamma_1^{(p)}$ and γ_2 pass through are the same, then we have two additional cases based on whether $\gamma_1^{(p)}$ is clockwise or counterclockwise of γ_2 at p. Since these cases produce different sets Sw, we differentiate them. We refer to the case where $\gamma_1^{(p)}$ lies clockwise from γ_2 as the $k = 0$ case.

It is only in the $k = 0$ and $k = m$ cases when $\gamma_2 \notin T$ that we will have a crossing overlap. If $\tau_{i_1}, \ldots, \tau_{i_{d_1}}$ and $\tau_{j_1}, \ldots, \tau_{j_{d_2}}$ are the ordered sequences of arcs from T crossed by γ_1 and γ_2 respectively, and $w \geq 1$ is the largest number such that $\tau_{i_r} = \tau_{j_r}$ for all $1 \leq r \leq w$, then $R = \{\tau_{i_1}, \ldots, \tau_{i_w}\}$, regarded as a multiset. When $\gamma_2 \in T$, then there is no possible case for $k = 0$, and $R = \emptyset$ in the $k = m$ case.

Proposition 2. *Let $\gamma_1^{(p)}$ and γ_2 be arcs which are incompatible at a puncture p. For k and R as defined above, set*

$$Y_R = \prod_{\tau \in R} y_\tau \quad \text{and} \quad Y_{Sw} = \prod_{\sigma_i \in Sw} y_{\sigma_i} = \prod_{i=1}^{k} y_{\sigma_i}.$$

Then, we have $x_{\gamma_1^{(p)}} x_{\gamma_2} = C^+ + C^-$ where C^+ and C^- are defined as follows:

	C^+	C^-
$k \neq 0, m$	x_{γ_3}	$Y_{Sw} x_{\gamma_4}$
$k = 0$	x_{γ_4}	$Y_R x_{\gamma_3}$
$k = m$	x_{γ_3}	$Y_{Sw} Y_R x_{\gamma_4}$

Proof. We detail the $k \neq 0, m$ and $\gamma_2 \notin T$ case; the special cases follow from various modifications to these overarching ideas. The posets $P_{\gamma_1^{(p)}}, P_{\gamma_3}$ and P_{γ_4} are provided in Table 3; we suppress the poset P_{γ_2} as its structure is not important for the proof.

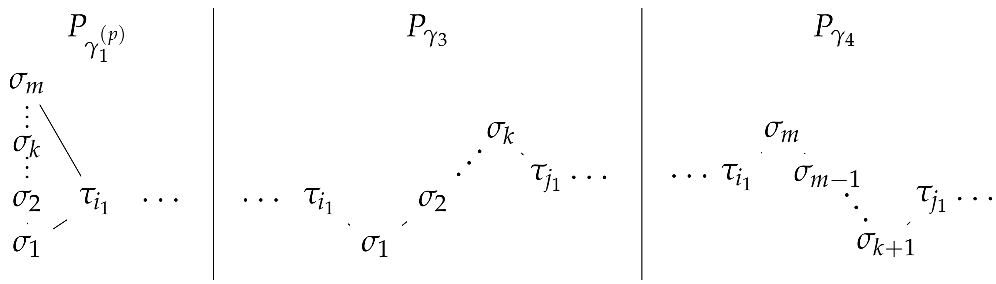


Table 3: Posets for a resolution of an incompatibility for the $k \neq 0, m$ cases. Recall that τ_{i_1} is the first arc crossed by γ_1 and γ_2 is the first arc crossed by γ_2.

Let $A_1 \subseteq J(P_{\gamma_1^{(p)}}) \times J(P_{\gamma_2})$ consist of all pairs (I_1, I_2) such that $\sigma_k \notin I_1$ and let A_2 consist of all pairs such that $\sigma_k \in I_1, \sigma_{k+1} \notin I_1$, and $\tau_{j_1} \in I_2$. Let B be the complement of $A_1 \sqcup A_2$; in other words, B consists of pairs (I_1, I_2) such that $\tau_{j_1} \in I_2$ only if $\sigma_{k+1} \in I_1$.

It is clear that A_1 is in bijection with $\{I_3 \in J(P_3) : \sigma_k \notin I_3\}$ and A_2 is in bijection with $\{I_3 \in J(P_3) : \sigma_k \in I_3\}$, where this bijection sends each element to its image in P_{γ_3}. Similarly, we have a bijection $B \cong P_{\gamma_4}$ which sends $(I_1, I_2) \in B$ to $(I_1 \backslash \langle \sigma_k \rangle) \cup I_2$. The description of B ensures that this set is an order ideal so that this map is well-defined.

We now compare the g-vectors. Let $\delta_{\tau_{i_1} > \tau_{i_2}} = 1$ if τ_{i_2} exists and $\tau_{i_1} > \tau_{i_2}$. We have that $\mathbf{g}_{\gamma_1^{(p)}} = -\mathbf{e}_{\sigma_1} + \delta_{\tau_{i_1} > \tau_{i_2}} \mathbf{e}_{\tau_{i_1}} + \mathbf{g}_1'$ where $\mathbf{g}_1'$ involves contributions from τ_{i_ℓ} for $\ell > 1$. For simplicity, suppose $\tau_{j_1} < \tau_{j_2}$. Then, $\mathbf{g}_{\gamma_2} = \mathbf{e}_{\sigma_k} - \mathbf{e}_{\tau_{j_1}} + \mathbf{g}_2'$ for similarly defined $\mathbf{g}_2'$. We see immediately that $\mathbf{g}_{\gamma_3} = -\mathbf{e}_{\sigma_1} + \mathbf{e}_{\sigma_k} + \delta_{\tau_{i_1} > \tau_{i_2}} \mathbf{e}_{\tau_{i_1}} + \mathbf{g}_1' - \mathbf{e}_{\tau_{j_1}} + \mathbf{g}_2' = \mathbf{g}_1 + \mathbf{g}_2$. Now, we compute $\mathbf{g}_{\gamma_4} = -\mathbf{e}_{\sigma_{k+1}} + \mathbf{e}_{\sigma_m} - (1 - \delta_{\tau_{i_1} > \tau_{i_2}}) \mathbf{e}_{\tau_{i_1}} + \mathbf{g}_1' + \mathbf{g}_2'$, so that $\mathbf{g}_{\gamma_4} - (\mathbf{g}_{\gamma_1^{(p)}} + \mathbf{g}_{\gamma_2}) = \mathbf{e}_{\sigma_m} + \mathbf{e}_{\sigma_1} + \mathbf{e}_{\tau_{j_1}} - \mathbf{e}_{\sigma_k} - \mathbf{e}_{\sigma_{k+1}} - \mathbf{e}_{\tau_{i_1}}$. Let $\sigma_{[i]}$ denote the third arc in the triangle formed by σ_i and σ_{i+1}. Then, from the definition, we have $\hat{y}_{\sigma_i} = y_{\sigma_i} \frac{x_{\sigma_{i-1}} x_{\sigma_{[i]}}}{x_{\sigma_{i+1}} x_{\sigma_{[i-1]}}}$. One can see that $\hat{y}_{\sigma_1} \cdots \hat{y}_{\sigma_k} = (y_{\sigma_1} \cdots y_{\sigma_k}) \frac{x_{\sigma_m} x_{\sigma_1} x_{\sigma_{[k]}}}{x_{\sigma_k} x_{\sigma_{k+1}} x_{\sigma_{[0]}}}$, and the claim follows after noting that $\sigma_{[k]} = \tau_{j_1}$ and $\sigma_{[0]} = \sigma_{[m]} = \tau_{i_1}$. One can repeat similar calculations if $\tau_{j_1} > \tau_{j_2}$.

□

4.3 Transverse Crossings

Here, we consider two arcs, γ_1 and γ_2 that have a point of intersection. For brevity, here we will assume these arcs have a crossing overlap, so that $R \neq \emptyset$. If not, we have two more cases based on the fact that the point of intersection must occur in the first or last triangle of one or both of the arcs.

We orient γ_1 and γ_2 so that they pass through the arcs in R in the same direction. With our fixed point of intersection s, let $\gamma_1 \circ \gamma_2$ denote the arc given by following γ_1 along its orientation until s and then following γ_2. Let $\gamma_3 = \gamma_1 \circ \gamma_2, \gamma_4 = \gamma_2 \circ \gamma_1, \gamma_5 = \gamma_1 \circ \gamma_2^{-1}$, and $\gamma_6 = \gamma_2^{-1} \circ \gamma_1$, where -1 denotes using the reverse orientation. Note that γ_3 and γ_4 both pass through R, though they do not have a crossing overlap here, while γ_5 and γ_6 avoid the intersections with the arcs in R. Therefore, $C^+ = \{\gamma_3, \gamma_4\}$ and $C^- = \{\gamma_5, \gamma_6\}$.

Proposition 3. *Let γ_1 and γ_2 be two arcs which intersect in a crossing overlap R. Let the resolution be $\{\gamma_3, \gamma_4\} \cup \{\gamma_5, \gamma_6\}$. Then,*

$$x_{\gamma_1} x_{\gamma_2} = x_{\gamma_3} x_{\gamma_4} + Y_R Y_{Sw} x_{\gamma_5} x_{\gamma_6}$$

where $Y_R = \prod_{\tau \in R} y_\tau$ and $Y_{Sw} = \prod_{\tau \in Sw} y_\tau$.

In the proof, we will use a poset-theoretic version of a tool from [1]. Let posets P_1 and P_2 have a crossing overlap in a region R. Index the elements in $P_1 \cap R$ as $P_1(1), \ldots, P_1(m)$ such that $P_1(i)$ only has cover relations with $P_1(i-1)$ and $P_1(i+1)$, when these exist, and index the elements in $P_2 \cap R$ analogously such that $P_1(i)$ and $P_2(i)$ are equivalent for each i. Given $I_1 \in J(P_1)$ and $I_2 \in J(P_2)$, let the *switching position* be the smallest value j such that $P_1(j) \in I_1$ if and only if $P_2(j) \in I_2$. One can show that a switching position exists unless $R \subseteq I_1$ and $R \cap I_2 = \varnothing$ or vice versa.

Proof. We say that a subset R of a poset P is on *top* if there is no $j \in P \backslash R$ such that j is larger than an element in R and define a subset being on *bottom* similarly. One can show that, when γ_1 and γ_2 have a crossing overlap, up to relabeling, R_1 is on top and R_2 is on bottom. In the following, suppose that γ_1 crosses arcs $\alpha_1, \ldots, \alpha_{d_1}$ in T and γ_2 crosses $\eta_1, \ldots, \eta_{d_2}$. We assume that these arcs have a crossing overlap in regions $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$. Let $1 \leq s \leq t \leq d_1$ and $1 \leq s' \leq t' \leq d_2$ be such that $R_1 = \{\alpha_s, \ldots, \alpha_t\}$ and $R_2 = \{\eta_{s'}, \ldots, \eta_{t'}\}$.

We focus on one case which includes a nonempty set Sw as an illustrative proof. We will omit discussion of **g**-vectors as the previous proof already illustrated all relevant ideas. Suppose $s' = 1$ and $s(\gamma_2)$ is notched. It must be that $s > 1$ in order for γ_1 and γ_2 to have an intersection. Necessarily, the arc α_{s-1} is a spoke incident to the puncture $s(\gamma_2)$. Index this set of spokes as $\sigma_1, \ldots, \sigma_m$ in counterclockwise order such that $\alpha_{s-1} = \sigma_1$. Suppose that γ_1 crosses $\sigma_1, \ldots, \sigma_k$ and let β be the arc which γ_1 crosses right before crossing σ_k, if it exists. We will assume $t < d_1$ and $t' < d_2$; we can repeat these arguments two times if we also have one of these cases. Table 4 provides the posets P_1, P_2, P_5, and P_6. If β does not exist, then P_5 is the chain between σ_{k+1} and σ_{k-1}, with order as in the Table. The poset P_3 is obtained by taking P_1 and replacing $R_1 > \alpha_{t+1}$ with $R_3 < \eta_{t'+1}$ and P_4 is obtained dually from P_2.

We set A to be the union of pairs (I_1, I_2) such that one of the following holds: (1) there is a switching position between R_1 and R_2, (2) $R_1 \subseteq I_1$ and $R_2 \cap I_2 = \varnothing$, (3) $R_2 \subseteq I_2, R_1 \cap I_1 = \varnothing, \alpha_{t+1} \in I_1$ and $\eta_{t'+1} \notin I_2$, or (4) $R_2 \subseteq I_2, R_1 \cap I_1 = \varnothing, \alpha_{t+1} \notin I_1$, $\sigma_k \in I_2$ only if $\beta \in I_1$ and $\sigma_{k+1} \notin I_2$ if the highest element σ_m is in I_1. If β does not exist,

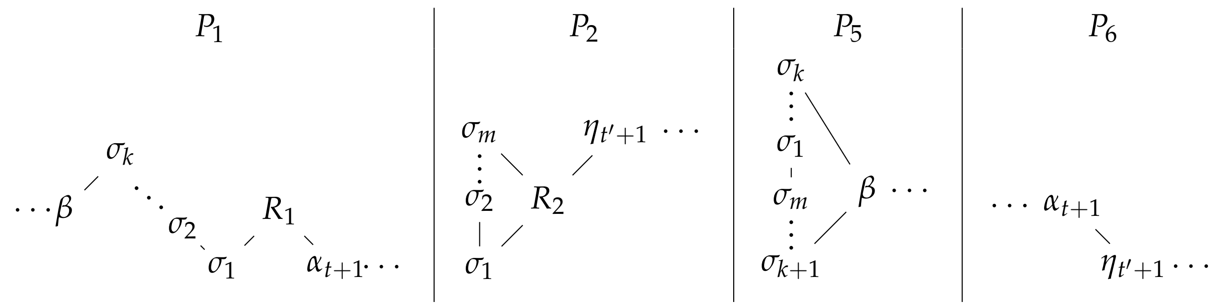


Table 4: Some of the posets for a resolution of a transverse crossing between γ_1 and γ_2

the condition involving β is removed. We define Φ_A as follows. If (I_1, I_2) has a switching position, which is j in R_1 and j' in R_2, then we set $\Phi_A(I_1, I_2) = (I_3, I_4)$ where I_3 is the result of taking all elements of I_1 up to α_j and all elements of I_2 after $\eta_{j'}$ and I_4 is the result of taking all elements of I_2 up to $\eta_{j'}$ and all elements of I_1 after α_j. Since $\alpha_j \in I_1$ if and only if $\eta_{j'} \in I_2$, these form order ideals. If a pair (I_1, I_2) is from item (2) we send R_1 to R_3, if from item (3) we send R_2 to R_4, and if from item (4) we send R_2 to R_3. Some of the elements σ_i do not have one clear image in $P_3 \times P_4$, so care is taken in these latter items to send them to appropriate places so that the resulting sets are still order ideals.

We let B be the complement of A in $J(P_1) \times J(P_2)$; explicitly, B is the set of tuples such that $R_1 \cap I_1 = \emptyset$, $R_2 \cup \langle \sigma_k \rangle \subseteq I_2$, $\alpha_{t+1} \in I_1$ only if $\eta_{t'+1} \in I_2$, and $\beta \in I_1$ only if $\sigma_{k+1} \in I_2$. Our definition of B implies that the restrictions of $I_1 \sqcup (I_2 \backslash (R_2 \cup \langle \sigma_k \rangle))$ to P_5 and P_6 are order ideals. This defines our bijection Φ_B. $\square$

5 Implications

In [9], given a surface (S, M), Musiker, Schiffler, and Williams define two sets of arcs, bangles $\mathcal{C}^\circ$ and bracelets $\mathcal{C}$, and show that the set of elements of $\mathcal{A}_S$ arising from each ($\mathcal{B}^\circ$ and $\mathcal{B}$ respectively) forms a basis of $\mathcal{A}_S$. They leave as a question whether these sets could also give the basis of $\mathcal{A}_S$ when (S, M) has punctures; the lack of skein relations in the punctured setting is a large reason why they did not extend their basis to this case.

Our skein relations show that a product $x_{\gamma_1} x_{\gamma_2}$ of incompatible arcs can be written in terms of $\mathcal{B}^\circ$ and of $\mathcal{B}$, which shows that these sets are still spanning in the punctured case. Moreover, because our relations are always of the form $x_{\gamma_1} x_{\gamma_2} = x_{C^+} + Y x_{C^-}$, we know that Lemma 6.3 from [9] remains true. As explained in Section 8.5 of the same article, this will show that these sets are also linearly-independent.

Lemma 2. *Let γ_1 and γ_2 be multicurves with at least one point of incompatibility on (S, M). Then the expansion*

$$x_{\gamma_1} x_{\gamma_2} = \sum_i Y_i M_i,$$

where $M_i \in \mathcal{B}^\circ$ and the Y_i represent monomials in the coefficient variables, has a unique index j such that $Y_j = 1$.

As future work, it remains for us to verify that $\mathcal{B}$ and $\mathcal{B}^\circ$ are still subsets of $\mathcal{A}_S$. Although we expect this to be true, it is non-trivial to prove and will, as a consequence, complete the proof that $\mathcal{B}$ and $\mathcal{B}^0$ remain bases in the punctured setting.

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Charmed roots and the Kroweras complement

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Abstract. Although both noncrossing partitions and nonnesting partitions are uniformly enumerated for Weyl groups, the exact relationship between these two sets of combinatorial objects remains frustratingly mysterious. In this abstract, we give a precise combinatorial answer in the case of the symmetric group using a new definition of *charmed roots*.

Résumé. Les partitions non-croisées et les partitions non-emboîtées soient uniformément énumérées pour les groupes de Weyl, la relation exacte entre ces deux ensembles d'objets combinatoires reste frustrante. Dans cet abstrait, nous donnons une réponse combinatoire précise dans le cas du groupe symétrique en utilisant une nouvelle définition de *racines charmées*.

Keywords: Catalan combinatorics, noncrossing, nonnesting, Kreweras complement

1 Introduction

1.1 Noncrossing and nonnesting partitions

Let $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group acting in its reflection representation on a complex vector space V of dimension r with reflections T [10, 13]. Our results will mostly concern the symmetric group $W = S_n$, where the set of reflections is the set of all transpositions (i, j). The ring of W-invariants $\mathbb{C}[V]^W$ is a polynomial ring generated by invariants of degrees $d_1 \leq d_2 \leq \cdots \leq d_r$. The *Coxeter number* of a well-generated W (that is, W is generated by r reflections) is $h = d_r$ and the *W-Catalan number* is

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$$\mathrm{Cat}(W) := \prod_{i=1}^{r} \frac{h+d_i}{d_i}. \tag{1.1}$$

The *c-noncrossing partition lattice* $\mathrm{NC}(W,c)$ is the interval $[e,c]_T$ in the absolute order. For S_n, the absolute length of a permutation $w \in S_n$ is n minus the number of cycles of w; the cycles of $w \in \mathrm{NC}(W,c)$ are the blocks of a noncrossing partition. Until very recently, the number of noncrossing partitions had only been computed case-by-case; a uniform proof was found in the case of real W in [8]. For W a well-generated finite complex reflection group with Coxeter element c, $|\mathrm{NC}(W,c)| = \mathrm{Cat}(W)$.

Let now W be a Weyl group (a crystallographic real reflection group), with positive roots Φ^+. In this abstract, we will systematically replace positive roots by their corresponding reflections. The positive root poset is the partial order on Φ^+ defined by $\alpha \leq \beta$ iff $\beta - \alpha$ is a nonnegative sum of positive roots; for S_n, Φ^+ is the partial order defined on the transpositions (i,j) with covering relations $(i+1,j) \lessdot (i,j) \lessdot (i,j+1)$ (see Figure 1 for an example). The *nonnesting partitions* $\mathrm{NN}(W)$ are the order ideals in the positive root poset [16, Remark 2]. There are uniform proofs that $|\mathrm{NN}(W)| = \mathrm{Cat}(W)$.

Despite the fact that they are both counted by $\mathrm{Cat}(W)$, there are at least two Incongruities between $\mathrm{NC}(W,c)$ and $\mathrm{NN}(W)$:

1. $\mathrm{NC}(W,c)$ is defined for well-generated complex reflection groups, while $\mathrm{NN}(W)$ is only defined for Weyl groups;

2. the definition of $\mathrm{NC}(W,c)$ requires the choice of a Coxeter element, while $\mathrm{NN}(W)$ has no such dependence;

The exact relationship between noncrossing and nonnesting partitions remains frustratingly mysterious, and finding a uniform "natural" bijection is perhaps the biggest open question in Coxeter–Catalan combinatorics. To our taste, there are two approaches to this problem: the first approach is based on the case-by-case combinatorial models available in the classical types A, B, D [9, 19, 6, 11, 18, 3]; the second approach was pioneered in [2] based on observations in [14, 4], and uses a mysterious coincidence of two cyclic actions to induce a bijection. Our main theorem refines both of these approaches in the special case of the symmetric group S_n.

1.2 Cyclic actions

For W a well-generated complex reflection group and c a Coxeter element, the *c-Kreweras complement* on the noncrossing partition lattice is the anti-automorphism of $\mathrm{NC}(W,c)$ defined by $\mathrm{Krew}_c(\pi) = \pi^{-1}c$ [1, Section 4.2], [12]. Since $\mathrm{Krew}_c^2(\pi) = c^{-1}\pi c$ and c has order h, Krew_c has order h if $-1 \in W$, and $2h$ otherwise.

For W a Weyl group, *rowmotion* on nonnesting partitions is the map $\mathrm{Row}(p) = \min_{\Phi^+}\{\alpha \mid \alpha \not\leq \beta \text{ for any } \beta \in p\}$. Panyushev conjectured that the order of Row on $\mathrm{NN}(W)$ was h if $-1 \in W$ and $2h$ otherwise [14], and Bessis and Reiner refined Panyushev's conjecture by observing that Row had the same orbit structure on $\mathrm{NN}(W)$ as Krew on $\mathrm{NC}(W,c)$ [4]. This was proven by Armstrong, Stump, and Thomas [2].

But Incongruity (2) remains—while the definitions of $\mathrm{NC}(W,c)$ and Krew_c depend on the choice of a Coxeter element, the set $\mathrm{NN}(W)$ and its action Row do not depend on any such choice. We address this lack of dependence on the Coxeter element c by modifying the definition of Row [20, 21]. It is well-known that rowmotion can be written as a sequence of local moves as follows [5, 17]. A *toggle* $\mathrm{tog}_\alpha(p)$ of a nonnesting partition p at a positive root α either adds α to p (when $\alpha \notin p$) or removes α from p (when $\alpha \in p$), provided that the result is again a nonnesting partition. For nonnesting partitions, Row can be computed by toggling each root of the root poset in order of height (or by *row*). It is natural to modify the order of these toggles.

For c a standard Coxeter element and c a particular choice of reduced word for c, the *c-sorting word* for the long element $w_\circ$ is the leftmost reduced word in simple reflections for $w_\circ$ in c^∞. Write $\mathsf{w}_\circ(\mathsf{c}) = [r_1, r_2, \ldots, r_N]$, with each $r_i \in S$ and define the inversion sequence $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c})) = [t_1, t_2, \ldots, t_N]$, where $t_a := (r_1 r_2 \cdots r_{a-1}) r_a (r_1 r_2 \cdots r_{a-1})^{-1}$. Then $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c}))$ totally orders the reflections of W.

We can now address Incongruity (2) by defining a modification of rowmotion to accomodate a Coxeter element c:

$$\begin{aligned} \mathrm{Krow}_c : \mathrm{NN}(W) &\to \mathrm{NN}(W) \\ p &\mapsto \left(\mathrm{tog}_{t_N} \circ \cdots \circ \mathrm{tog}_{t_2} \circ \mathrm{tog}_{t_1}\right)(p). \end{aligned} \tag{1.2}$$

We call this map the *c-Kroweras complement*.

1.3 Main Theorem

Our main theorem uses the c-Kreweras and c-Kroweras complements to relate noncrossing and nonnesting partitions. Recall that the *support* of a noncrossing partition $\pi \in \mathrm{NC}(W,c)$ is the set $\mathrm{Supp}(\pi)$ of simple reflections required to write a reduced word in simple reflections for π; similarly, the *support* of a nonnesting partition $p \in \mathrm{NN}(W)$ is the set $\mathrm{Supp}(p)$ of simple roots that lie in p (as an order ideal of Φ^+).

Theorem 1. *Let S_n be the symmetric group, and fix a standard Coxeter element $c \in S_n$. Then there is a unique bijection $\mathrm{Charm}_c\colon \mathrm{NC}(S_n,c) \to \mathrm{NN}(S_n)$ satisfying $\mathrm{Charm}_c \circ \mathrm{Krew}_c = \mathrm{Krow}_c \circ \mathrm{Charm}_c$ and $\mathrm{Supp} = \mathrm{Supp} \circ \mathrm{Charm}_c$.*

In particular, for any standard Coxeter element $c \in S_n$, the order of Krow_c on $\mathrm{NN}(W)$ is $2h$. The statement of the main theorem in Armstrong-Stump-Thomas [2] can be obtained from the statement of our Theorem 1 by replacing Krow_c by Row, and replacing the symmetric group by any finite Weyl group; the main difference from the result in [2] is that we resolve Incongruity (2), constructing truly different bijections between noncrossing and nonnesting partitions for each Coxeter element.

2 Coxeter elements and charmed roots

Recall that a *(standard) Coxeter element* c is a product of the simple reflections in any order. To ease notation, we reserve the symbols r_i to refer to a fixed ordering $\mathsf{c} := [r_1, r_2, \ldots, r_{n-1}]$ of S and define $c := r_1 r_2 \cdots r_{n-1}$. We also reserve the symbol $s_k = r_1$ for the first simple reflection in the chosen reduced word, and write $c' = r_2 \cdots r_{n-1} r_1$.

It is easy to show that the cycle notation of any Coxeter element in the symmetric group has a particularly simple form: $c \in S_n$ consists of a single cycle with an initial increasing subsequence starting at 1 and ending at n, followed by a decreasing sequence of the remaining unused entries. Let c be a Coxeter element with cycle notation $(w_1, w_2, \ldots, w_m, w_{m+1} \ldots, w_n)$, where $1 = w_1 < w_2 < \cdots < w_m = n$ and $n = w_m > w_{m+1} > \cdots > w_n > w_1 = 1$. Write

$$L_c := \{w_2, \ldots, w_{m-1}\} \text{ and } R_c := \{w_{m+1}, \ldots, w_n\}.$$

Definition 1. *For $1 < i < j < n$, we say that a root (i,j) is c-charmed if $i \in L_c$ and $j \in R_c$ or if $i \in R_c$ and $j \in L_c$ and c-ordinary otherwise. We write $\heartsuit_c$ for the set of c-charmed roots.*

In figures, we depict c-charmed roots with a $\heartsuit$ and ordinary roots by a circle. The root poset of type A_8 is illustrated in Figure 1.

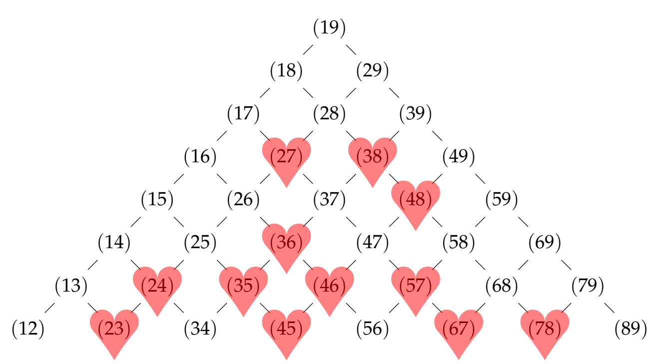

Figure 1: The Hasse diagram of the positive root poset Φ^+ of type A_8. For $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$, the c-charmed roots from Definition 1 are marked using hearts.

Example 1. *Consider the Coxeter element* $c = s_2s_1s_3s_6s_5s_4s_8s_7$ *in* S_9*. The element* c *has cycle notation* $(1,3,4,7,9,8,6,5,2)$*, so that*

$$L_c = \{3,4,7\} \text{ and } R_c = \{8,6,5,2\}.$$

We visualize the cycle notation of c by drawing it as points labeled $w_1, w_2, \ldots, w_n$ counter-clockwise around a circle. We visualize a root (i,j) by connecting the vertices labeled i and j by a line segment. For any $a,b,c,d \in [n]$, we say that (a,b) *crosses* (c,d) if and only if (a,b) and (c,d) are crossing in their interior (note that (a,b) does not cross itself). For $1 < i < j < n$, it is easy to check that a root (i,j) is c-charmed if and only if (i,j) crosses $(i-1,j+1)$. Figure 2 illustrates this visualization.

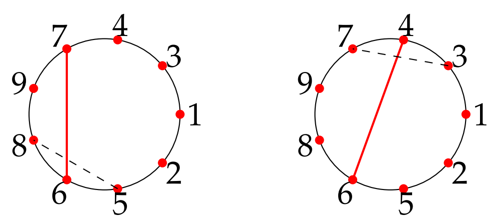

Figure 2: The visualization of the cycle notation $(1,3,4,7,9,8,6,5,2)$ of $c = s_2s_1s_3s_6s_5s_4s_8s_7$. *Left:* the initial c-charmed simple root $(6,7)$ (in red) intersecting the root $(5,8) = (6-1,7+1)$ (dashed). *Right:* the c-charmed root $(4,6)$ (in red) intersecting the root $(3,7) = (4-1,6+1)$ (dashed).

3 Charmed bijections

In this section, we define a general family of *charmed bijections* between *balanced pairs* of subsets and nonnesting partitions. Our charmed bijections depend on a choice of decoration of the roots in $\Phi^+ = \Phi^+(A_{n-1})$, and use certain *intimate* families of lattice paths as intermediate objects. Specializing to the c-charmed roots coming from a Coxeter element c, we obtain our *c-charmed bijections* between $\mathrm{NC}(S_n, c)$ and $\mathrm{NN}(S_n)$.

3.1 Balanced pairs and noncrossing partitions

Definition 2. *Say that a pair of sets* (O, I) *with* $O, I \subseteq [n]$ *is* balanced *if* $|O| = |I|$ *and* $|O \cap [k]| \geq |I \cap [k]|$ *for all* $1 \leq k \leq n$*. Write* $\mathrm{Bal}(n)$ *for all balanced pairs of subsets of* $[n]$*.*

We first show that balanced pairs are naturally in bijection with c-noncrossing partitions. Let $\pi \in \mathrm{NC}(S_n, c)$. We define $O(\pi)$ to be the set of integers i for which there exists an $j > i$ in the same block as i, and we define the set $I(\pi)$ to be the set of integers j such that there exists an $i < j$ in the same block as j. It is immediate from the definition that $(O, I) \in \mathrm{Bal}(n)$.

Proposition 1. *The map*[1] $\pi \mapsto (O(\pi), I(\pi))$ *is a bijection between* $\mathrm{NC}(S_n, c)$ *and* $\mathrm{Bal}(n)$.

Proof. We construct its inverse. For a given pair $(O, I) \in \mathrm{Bal}(n)$, we can construct a $\pi \in \mathrm{NC}(S_n, c)$ with $O(\pi) = O$ and $I(\pi) = I$ as follows. The *closed singletons* of π are the integers that are neither in O nor I. We place each integer in $O \setminus I$ in its own block and call these blocks *open*. Then we add iteratively the integers in I to the open blocks, starting with the smallest integer, such that the intermediate partition is always noncrossing. This is achieved by adding an integer x to the first open block we visit when walking from x towards n via 1 in the cycle notation of c. If an integer in $I \setminus O$ is added to a block we call this block *closed* and thereafter do not add any integers to it. By construction we have $O(\pi) = O$ and $I(\pi) = I$. $\square$

The bijection of Proposition 1 is illustrated in Figure 3.

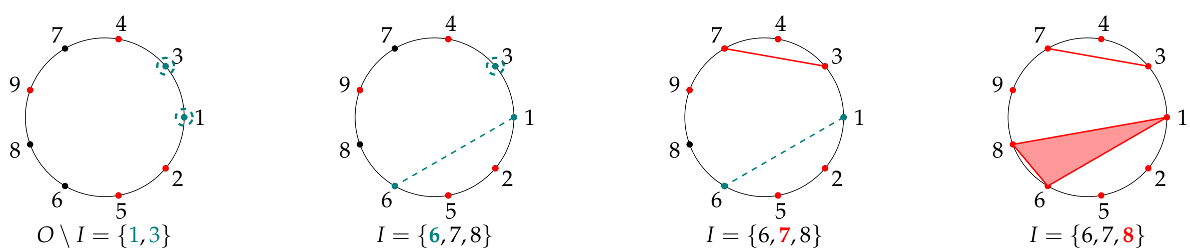

Figure 3: The construction of a noncrossing partition in $\mathrm{NC}(S_9, c)$ for $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 = (1\,3\,4\,7\,9\,8\,6\,5\,2)$ starting with the outgoing set $O = \{1, 3, 6\}$ and incoming set $I = \{6, 7, 8\}$. We depict open blocks in dashed teal, and closed blocks in solid red; we circle open and closed singletons using the same color code.

3.2 Intimate families

We draw the root poset for S_n by placing the root (i, j) in the plane with coordinates $((i + j - 1)/2, (j - i)/2)$ and—since the label (i, j) is implied by the position—we may omit the labels on the roots. For $1 \leq i \leq n$, we draw n additional points labeled by i at coordinates $(i - 1/2, 0)$ and call these extra points *integral vertices*. For $\heartsuit \subseteq \Phi^+$, we call a root (i, j) *charmed* if $(i, j) \in \heartsuit$ and *ordinary* otherwise. We depict charmed roots using

[1] To be precise, both O and I depend on c, however since c will always be clear from the context we omit it in the notation.

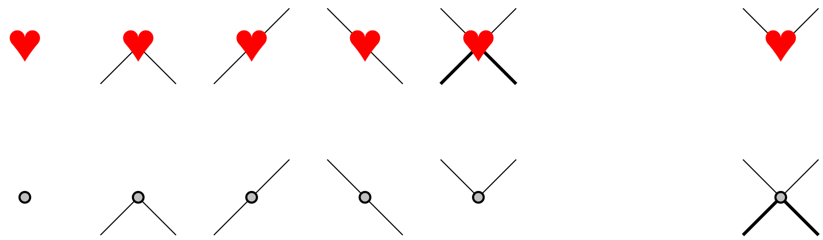

Figure 4: *Left:* the five allowed local configurations for charmed families. *Right:* the two local configurations are forbidden for charmed families.

hearts ♥ and ordinary roots using circles—an example of c-charmed roots is illustrated in Figure 1.

A *path* is a lattice path with step set $\{(1/2, 1/2), (1/2, -1/2)\}$ that starts and ends at an integral vertex and stays strictly above the x-axis. We call $(1/2, 1/2)$-steps *up*, and $(1/2, -1/2)$-steps *down*; a *peak* (resp. *valley*) of a path is a root contained in an up step to its left (resp. right) and a down step to its right (resp. left). Two paths are *kissing* if they do not cross or share edges—they may *meet* at a vertex, where they are said to *kiss*. (Note, though, that kisses are not required for two paths to be kissing.) A family of paths is *kissing* if they are pairwise kissing. A path *feints* at a root (i, j) if (i, j) is a valley, but the path does not kiss any path at (i, j). These definitions are illustrated on the right of Figure 4: the top configuration is a feint at a charmed root, while the bottom one is a kiss at an ordinary root.

For $\heartsuit \subseteq \Phi^+$, a family $\mathcal{L}$ of kissing paths is called *♥-charmed* if:

- paths only kiss at charmed roots and
- paths only feint at ordinary roots.

In other words, a family of paths is charmed if it avoids the two local configurations shown on the right of Figure 4.

Definition 3. *A family $\mathcal{L}$ of ♥-charmed kissing paths is called* ♥-intimate *if:*

- *every ordinary root either lies above all paths in $\mathcal{L}$ or is contained in some path in $\mathcal{L}$ and*
- *no path contains a root above a charmed peak of a path in $\mathcal{L}$, unless that charmed peak is the location of a kiss.*

3.3 Balanced pairs and intimate families

We now relate balanced pairs and intimate families of paths. For a family $\mathcal{L}$ of paths, we call an integral vertex on the x-axis *outgoing* if it is incident to an up step and *incoming* if it is incident to a down step (an integral vertex can be both outgoing and incoming). Denote by $\mathrm{Out}(\mathcal{L})$ (resp. $\mathrm{In}(\mathcal{L})$) the set of labels of outgoing (resp. incoming) vertices of $\mathcal{L}$. It is clear that $(\mathrm{Out}(\mathcal{L}), \mathrm{In}(\mathcal{L}))$ is balanced.

Lemma 1. *Let* $\heartsuit \subseteq \Phi^+$ *and let* $(O, I) \in \mathrm{Bal}(n)$*. Then there is a unique* $\heartsuit$*-inimate family* $\mathcal{L}_{(O,I)}$ *with* $\mathrm{Out}(\mathcal{L}_{(O,I)}) = O$ *and* $\mathrm{In}(\mathcal{L}_{(O,I)}) = I$*.*

Proof. We first construct a well-formed word of parentheses from the subsets O and I. For i from 1 to n, write:

- a $(_i$ parenthesis if $i \in O \setminus I$,
- a $)_i$ parenthesis if $i \in I \setminus O$, and
- $)_i(_i$ parentheses if $i \in O \cap I$.

We now construct an $\heartsuit$-intimate family $\mathcal{L}$ recursively, starting with the empty family of paths. At each step, we pick neighbouring parentheses of the form $(_i)_j$, delete them, and add the path P that starts at i and ends at j that takes a down step whenever possible without violating the condition that the family is charmed, and an up step otherwise. Then $\mathcal{L} \cup \{P\}$ is intimate:

- If there were an ordinary root below $\mathcal{L} \cup \{P\}$ that wasn't part of a path, then that root would lie between p and $\mathcal{L}$, since $\mathcal{L}$ was intimate. But then P took an up step instead of a possible down step, contradicting the definition of P.
- If a previously constructed path P' in $\mathcal{L}$ started at an integral vertex after i, ended before j, and had a charmed peak which is not the location of a kiss, then our new path P will kiss P' at that charmed peak.

The order of choosing two neighbouring parentheses is irrelevant. The family produced is unique, since if at any point a path uses a step different from those prescribed by the algorithm above, then the resulting family of paths will be non-intimate. This non-intimacy will persist, regardless of how the family is extended. □

An example of the algorithm used in the proof of Lemma 1 is given in Figure 5.

Let $\mathcal{L}$ be an $\heartsuit$-intimate family of paths. We define the order ideal $J(\mathcal{L})$ of $\mathcal{L}$ to be the set of all roots (i, j) which lie on or below a path in $\mathcal{L}$. It is clear that $J(\mathcal{L})$ is an order ideal and hence is in $\mathrm{NN}(S_n)$.

Lemma 2. *Let* $\heartsuit \subseteq \Phi^+$ *and* $J \in \mathrm{NN}(S_n)$*. Then there exists a unique* $\heartsuit$*-intimate family* $\mathcal{L}_J$ *with* $J(\mathcal{L}_J) = J$*.*

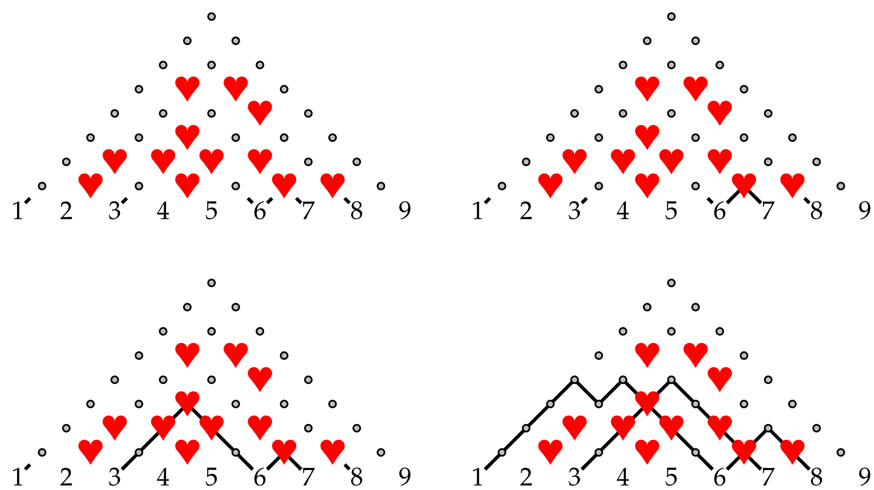

Figure 5: The construction of an intimate family of paths with outgoing set $\{1,3,6\}$ and incoming set $\{6,7,8\}$. The corresponding word of parentheses is $(_1(_3)_6(_6)_7)_8$.

Proof. We construct an ♥-intimate family $\mathcal{L}$ recursively, starting with the empty family of paths. At each step, we add a maximal path p to $\mathcal{L}$ such that all roots contained in p lie in J. We then replace J by the order ideal generated by all ordinary roots in J not contained in a path of $\mathcal{L}$, all charmed feints of paths of $\mathcal{L}$, and roots in J not lying below a path of $\mathcal{L}$. The recursion stops when J is empty. It is clear that the resulting family $\mathcal{L}$ is the unique ♥-intimate family of paths with order ideal J. □

An example of the algorithm used in the proof of Lemma 2 is given in Figure 6.

3.4 Charmed bijections between balanced pairs and nonnesting partitions

As a direct consequence of Lemmas 1 and 2, we obtain the following family of bijections between balanced pairs and nonnesting partitions.

Proposition 2. *Fix a collection of charmed roots* $♥ \subseteq \Phi^+$. *Then the map* $J_♥ \colon \mathrm{Bal}(n) \to \mathrm{NN}(S_n)$ *defined by* $J_♥(O,I) = J(\mathcal{L}_{(O,I)})$ *is a bijection.*

Charmed roots along the upper boundary of Φ^+ do not affect the bijection of Proposition 2. On the other hand, each of the $2^{\binom{n-2}{2}}$ charming choices for the the roots (i,j) with $1 < i < j < n$ gives rise to a distinct bijection between $\mathrm{NN}(S_n)$ and $\mathrm{Bal}(n)$.

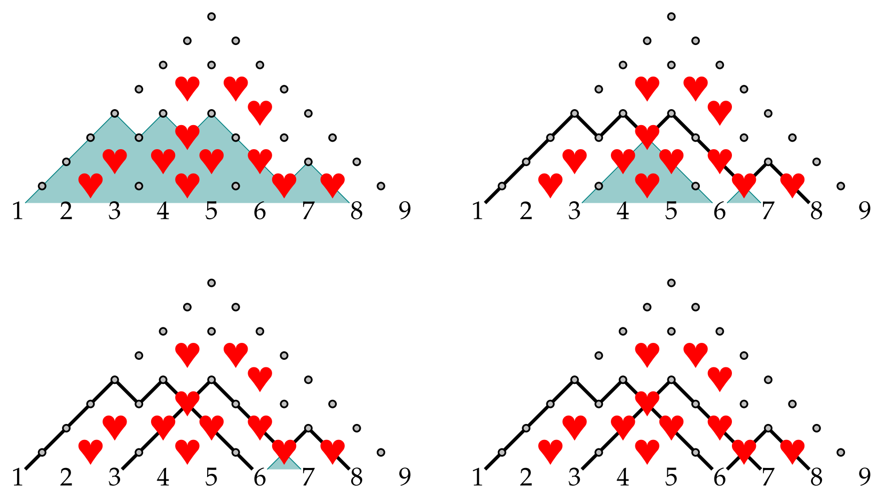

Figure 6: The construction of the intimate family of paths with order ideal J, where J contains the roots in the grey-shaded region of the top left picture. At each step, the order ideal under consideration consists of the roots contained in the teal-shaded region.

4 Charmed bijections between noncrossing and nonnesting partitions

Let $c \in S_n$ be a Coxeter element. Using Proposition 1 and Proposition 2, we now construct a bijection between c-noncrossing partitions and nonnesting partitions *that depends on the choice of Coxeter element c*, resolving Incongruity (2).

Definition 4. *The c-charmed bijection between c-noncrossing partitions and nonnesting partitions is given by*

$$\begin{aligned} \mathrm{Charm}_c \colon \mathrm{NC}(S_n, c) &\to \mathrm{NN}(S_n) \\ \pi &\mapsto J_{\heartsuit_c}(O(\pi), I(\pi)), \end{aligned}$$

where the set $\heartsuit_c$ of c-charmed roots is defined in Definition 1 and the map $J_{\heartsuit_c}$ is defined in Proposition 2.

Theorem 2. *For all Coxeter elements c, the bijection Charm_c is the unique support-preserving bijection between $\mathrm{NC}(S_n, c)$ and $\mathrm{NN}(S_n)$ satisfying $\mathrm{Krow}_c \circ \mathrm{Charm}_c = \mathrm{Charm}_c \circ \mathrm{Krew}_c$.*

For reasons of space, we only sketch the idea of the proof and refer the reader to the full version of this extended abstract for the details [7]. We say that a simple reflection s is *initial* in c if $\ell_S(sc) \leq \ell_S(c)$. If $s = s_k = (k, k+1)$ is initial in c, then $c' = scs$ is also a Coxeter element of S_n, and we will denote this by writing $c \xrightarrow{k} c'$. Theorem 2 (and hence Theorem 1) are proven using *Cambrian induction*—that is, we show that the theorem holds for a particular Coxeter element c_1 (the *base case*), and then we show that if $c \xrightarrow{k} c'$ and the theorem holds for c, then the theorem also holds for c' (the *inductive step*). Since all Coxeter elements in S_n are conjugate by a sequence of conjugations by initial simple reflections [15, Lemma 1.7], the theorem holds for all Coxeter elements.

As a consequence of our proof, we obtain a simple description for reading the blocks of the noncrossing partition from the corresponding intimate family.

Corollary 1. *Let $\mathcal{L}$ be an $\heartsuit_c$-intimate family and $\pi \in \mathrm{NC}(S_n, c)$ the corresponding noncrossing partition with $(\mathrm{Out}(\mathcal{L}), \mathrm{In}(\mathcal{L})) = (O(\pi), I(\pi))$. The blocks of π consist of the integers which are connected by paths in $\mathcal{L}$ after reinterpreting each kiss between a pair of paths in $\mathcal{L}$ at a charmed root as a crossing.*

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Fragmenting any Parallelepiped into a Signed Tiling

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Abstract. It is broadly known that any parallelepiped tiles space by translating copies of itself along its edges. In earlier work relating to higher-dimensional sandpile groups, the second author discovered a novel construction which fragments the parallelpiped into a collection of smaller tiles. These tiles fill space with the same symmetry as the larger parallelepiped. Their volumes are equal to the components of the multi-row Laplace determinant expansion, so this construction only works when all these signs are non-negative (or non-positive).

In this work, we extend the construction to work for all parallelepipeds, without requiring the non-negative condition. This naturally gives tiles with negative volume, which we understand to mean canceling out tiles with positive volume. In fact, with this cancellation, we prove that every point in space is contained in exactly one more tile with positive volume than tile with negative volume. This is a natural definition for a signed tiling.

Our main technique is to show that the net number of signed tiles doesn't change as a point moves through space. This is a relatively indirect proof method, and the underlying structure of these tilings remains mysterious.

Keywords: periodic tiling, signed tiling, parallelepiped, determinant expansion

1 Introduction

To motivate our work, we begin with an illustrative two-dimensional example of our main construction. Consider the matrices

$$K = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \qquad S_{\{1\}}(K) = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \qquad \text{and} \qquad S_{\{2\}}(K) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

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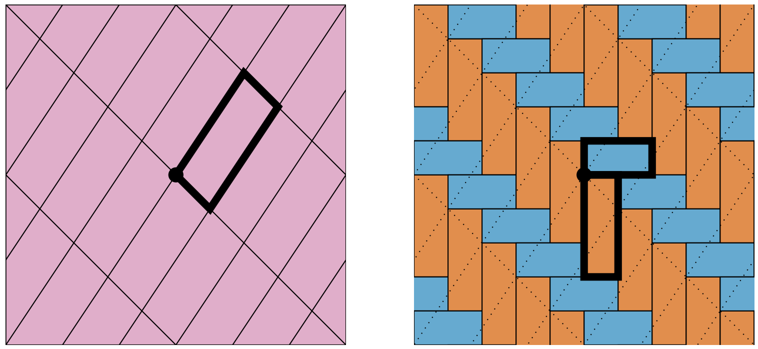


Figure 1: On the left is the tiling given by translations of the parallelepiped $\prod(K)$. On the right is the tiling given by translations of the fundamental parallelepipeds of the *fragment matrices* $S_{\{1\}}(K)$ and $S_{\{2\}}(K)$.

The matrices $S_{\{1\}}(K)$ and $S_{\{2\}}(K)$ are called the *fragment matrices* of K. They are obtained by negating the second row and then zeroing out a diagonal. Directly from the Laplace expansion for determinants, we can see that

$$-\det(K) = \det(S_{\{1\}}(K)) + \det(S_{\{2\}}(K)).$$

Given a matrix N, let $\Pi(N)$ be the (half-open) *fundamental parallelepiped* of N (see Definition 2.2 for details). It is broadly known that for any nonsingular matrix N, copies of $\Pi(N)$ can be used to form a periodic tiling of space. For example, the tiling on the left of Figure 1 is formed by copies of the parallelepiped $\Pi(K)$ that are translated by the integer linear combinations of columns of K (see Lemma 2.4).

Curiously, there also exists a tiling on the same lattice that is formed by the parallelepipeds $\Pi(S_{\{1\}}(K))$, and $\Pi(S_{\{2\}}(K))$. In particular, the tiling on the right of Figure 1 is formed by $\Pi(S_{\{1\}}(K))$ and $\Pi(S_{\{2\}}(K))$, along with their translates by all of the integer combinations of columns of K.

This tiling is a two dimensional example of a construction which was introduced by the second author to define *matrix-tree multijections* [4, 5]. This construction can be applied to any invertible $(r+k) \times (r+k)$ matrix M, and produces a collection of $\binom{r+k}{r}$ *fragment matrices* of M. When the determinants of the fragment matrices are all non-negative (or all non-positive), translating them by integer linear combinations of the columns of M produces a periodic tiling of $\mathbb{R}^{r+k}$.

In this paper, we prove that the elegant tiling structure of the fragment matrices is still present even without the restriction on M that all the fragment matrices have non-negative determinant. In particular, while the translates do not always form a traditional tiling with no overlap or gaps, they always produce a *signed tiling*.

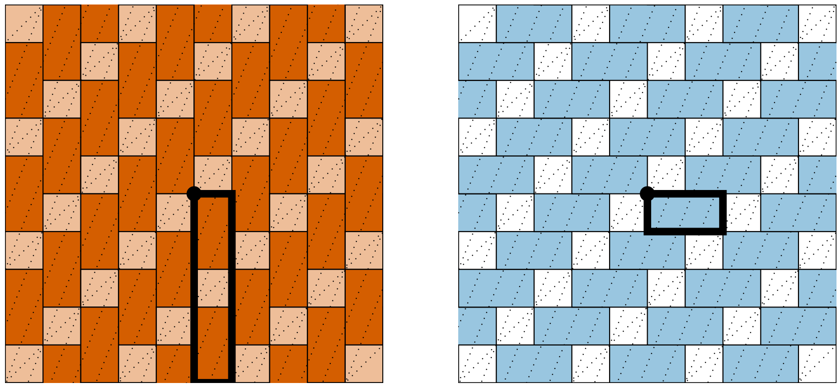


Figure 2: On the left is the tiling obtained by translating the fundamental parallelepipeds of $S_{\{1\}}(L)$ by integer combinations of columns of L. The darker regions indicate where two parallelepipeds overlap, while the lighter region is the portion covered by a single parallelepiped. On the right is the tiling obtained by translating the fundamental parallelepipeds of $S_{\{2\}}(L)$ by integer combinations of columns of L. This time, there are no overlaps, but the white region is formed by gaps between parallelepipeds. By Theorem 2.9, the shaded region on the right precisely corresponds to the darker region on the left.

To illustrate this signed version of the tiling, we give another 2-dimensional example. This time, the determinants of the fragment matrices have opposite signs.

Let

$$L = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}, \qquad S_{\{1\}}(L) = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}, \qquad \text{and} \qquad S_{\{2\}}(L) = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$$

As in the previous example, the fragment matrices $S_{\{1\}}(L)$ and $S_{\{2\}}(L)$ are formed by negating the second row and zeroing a diagonal. Next, we consider translates of the fragment matrices by integer linear combinations of the columns of L. In this case, the tiles no longer perfectly fill space, and instead overlap, see Figure 2.

In our previous example, the determinants of $S_{\{1\}}(K)$ and $S_{\{2\}}(K)$ were both negative. In this example, $S_{\{1\}}(L)$ is negative, but $S_{\{2\}}(L)$ is positive. Moreover, the positively signed tiles overlap. Nevertheless, an elegant tiling structure can still be found.

Consider the two partial tilings given in Figure 2. Every point in the plane is covered by either one translate of $\Pi(S_{\{1\}}(L))$ or two translates of $\Pi(S_{\{1\}}(L))$ and one translate of $\Pi(S_{\{2\}}(L))$. This means that if we define translates of $\Pi(S_{\{1\}}(L))$ to be *positive* tiles and translates of $\Pi(S_{\{2\}}(L))$ to be *negative* tiles, then for any point $\mathbf{p} \in \mathbb{R}^2$, the signed total of all tiles containing $\mathbf{p}$ is always 1.

This surprising alignment of positive and negative tiles works in general. Reiterating the previous setting, we let M be an invertible $(r + k) \times (r + k)$ matrix. We break this matrix into two parts, the first r rows and the last k rows. The two tiles from the two dimensional case become $\binom{r+k}{r}$ many tiles, indexed by which r columns are preserved in the top r rows (see Definition 2.5 for details).

We generalize the cancellation observed in the example with L by introducing a function f. This function counts the number of positively signed tiles at a point, minus the number of negatively signed tiles at that point. Our main result is the following.

Theorem 2.9. *The function $f : \mathbb{R}^{r+k} \to \mathbb{Z}$, defined by*

$$f(\mathbf{p}) := \left(\sum_{T \in \mathbf{T}^+(M)} \mathbb{1}_T(\mathbf{p}) \right) - \left(\sum_{T \in \mathbf{T}^-(M)} \mathbb{1}_T(\mathbf{p}) \right),$$

is constant with value $(-1)^k \operatorname{sgn}(\det(M))$.

In Section 2, we describe the general construction and introduce the notation necessary to understand the statement of Theorem 2.9. In Section 3, we give a high level description of the general proof argument. In Section 4, we give an example of a four dimensional signed tiling, which we visualize in 2 dimensions. Finally, in Section 5, we consider future extensions and pose questions we think will be interesting to explore. For more details, see our full paper on ArXiv [3].

2 Signed Tiling Construction

Fix positive integers r and k as well as an $(r+k) \times (r+k)$ matrix M with real entries. Additionally, fix a generic direction vector $\mathbf{w} \in \mathbb{R}^{r+k}$. More precisely, $\mathbf{w}$ can be anything but a set of measure 0 that depends on N.

Remark 2.1. Even more precisely, $\mathbf{w}$ is sufficiently generic for our purposes if it is not spanned by any collection of $r+k-1$ column vectors of any of the $(r+k) \times (r+k)$ matrices we will be working with. Specifically, these matrices are M along with $S_\sigma(M)$ for $\sigma \in \binom{[r+k]}{r}$ (See Definition 2.5).

In this paper, we work extensively with parallelepipeds. The vector $\mathbf{w}$ gives a consistent way to define *half-open parallelepipeds*.

Definition 2.2. Let N be an $(r+k) \times (r+k)$ matrix with real entries. Define $\Pi(N)$ to be the set of $\mathbf{p} \in \mathbb{R}^{r+k}$ such that for all sufficiently small $\epsilon > 0$, the point $\mathbf{p} + \epsilon\mathbf{w}$ is in

$$\sum_{i \in [r+k]} \{x_i N_i : 0 \leq x_i \leq 1\}.$$

The set $\Pi(N)$ is called the *(half-open) parallelepiped of N (with respect to* $\mathbf{w}$*)*.

Although Definition 2.2 depends on $\mathbf{w}$, we omit it in our notation for conciseness.

Remark 2.3. The genericity conditions for $\mathbf{w}$ (which are discussed in Remark 2.1) are precisely the conditions necessary to ensure the following condition. For all matrices N that we will be working with and all points $\mathbf{p} \in \mathbb{R}^{r+k}$, there exists some $\epsilon > 0$ such that the segment from $\mathbf{p}$ to $\mathbf{p} + \epsilon\mathbf{w}$ does not intersect the boundary of the fundamental parallelepiped of N (except possibly at $\mathbf{p}$).

Before we get to our first lemma, let us quickly clarify some confusing notation. The term *disjoint union* can be used in two different ways in mathematics, so we will denote these with two different symbols. For sets A and B, we write $A \bigsqcup B$ for the set $A \cup B$ with the added restriction that $A \cap B = \emptyset$. We use the notation $A \biguplus B$ to indicate the other kind of disjoint union, where A and B are considered as separate objects.

We now present a simple observation about translating parallelepipeds, which will be the foundation of our construction.

Lemma 2.4. *For any choice of M, we have*

$$\mathbb{R}^{r+k} = \bigsqcup_{\mathbf{z} \in \mathbb{Z}^{r+k}} \left(\Pi(M) + M\mathbf{z}\right).$$

This lemma follows from the fact that the unit cube tiles space, and the displacement between cubes in this tiling is all $\mathbb{Z}$-valued vectors. The lemma describes this same tiling, after applying M as a linear transformation. Our main construction is of a more complicated tiling under the same translation lattice, which is formed by *fragmenting M*.

Definition 2.5. Let $\sigma \in \binom{[r+k]}{r}$, i.e., $\sigma \subset [r+k]$ with $|\sigma| = r$. The *σ-fragment matrix* of M, written $S_\sigma(M)$, is the matrix obtained from M by the following 3 step process:

1. For each $i \notin \sigma$, replace the *first* r entries of column i with 0.

2. For each $i \in \sigma$, replace the *last* k entries of column i with 0.

3. Negate all of the entries in the last k rows.

Example 2.6. Let $r = k = 2$. Any $(r+k) \times (r+k)$ matrix M has 6 associated fragment matrices corresponding to the subsets of $\binom{[4]}{2}$. For example, if

$$M = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \end{bmatrix} \text{ and } \sigma = \{1,4\}, \text{ then } S_\sigma(M) = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}.$$

To form a signed tiling, we parameterize tiles formed by translating the fundamental parallelepiped of fragment matrices by integer combinations of the columns of M.

Definition 2.7. For any $\mathbf{z} \in \mathbb{Z}^{r+k}$ and $\sigma \in \binom{[r+k]}{r}$, the *tile* parameterized by the pair $(\mathbf{z}, \sigma)$ is defined as

$$\mathcal{T}(\mathbf{z}, \sigma) := \Pi(S_\sigma(M)) + M\mathbf{z}.$$

Note that since $\Pi(S_\sigma(M))$ depends on $\mathbf{w}$, the tile $\mathcal{T}(\mathbf{z}, \sigma)$ will depend on $\mathbf{w}$ as well. Nevertheless, the precise choice of $\mathbf{w}$ is not important for our results as long as it remains fixed (and sufficiently generic, see Remark 2.1). Also, note that we usually think of a tile $\mathcal{T}(\mathbf{z}, \sigma)$ as a polytope made up of a collection of points, not the points themselves. With this perspective in mind, we introduce the following definition.

Definition 2.8. Consider the sets of tiles

$$\mathbf{T}^+(M) := \biguplus_{\mathbf{z} \in \mathbb{Z}^{r+k}} \left(\biguplus_{\sigma \in \binom{[r+k]}{r},\ \det(S_\sigma(M)) > 0} \mathcal{T}(\mathbf{z}, \sigma) \right),$$

$$\text{and } \mathbf{T}^-(M) := \biguplus_{\mathbf{z} \in \mathbb{Z}^{r+k}} \left(\biguplus_{\sigma \in \binom{[r+k]}{r},\ \det(S_\sigma(M)) < 0} \mathcal{T}(\mathbf{z}, \sigma) \right).$$

The set $\mathbf{T}^+(M)$ is called the set of *positive tiles*, while $\mathbf{T}^-(M)$ is called the set of *negative tiles*. We also write $\mathbf{T}(M) := \mathbf{T}^+(M) \uplus \mathbf{T}^-(M)$. Note that we don't include the tiles where $\det(S_\sigma(M)) = 0$, but in this case, $S_\sigma(M)$ is not invertible, and $\Pi(S_\sigma(M))$ is empty.

Definition 2.8 allows us to cleanly state our main result. Note that we write $\mathbb{1}_T$ for the *indicator function* of a tile T.

Theorem 2.9. *The function $f : \mathbb{R}^{r+k} \to \mathbb{Z}$, defined by*

$$f(\mathbf{p}) := \left(\sum_{T \in \mathbf{T}^+(M)} \mathbb{1}_T(\mathbf{p}) \right) - \left(\sum_{T \in \mathbf{T}^-(M)} \mathbb{1}_T(\mathbf{p}) \right),$$

is constant with value $(-1)^k \operatorname{sgn}(\det(M))$.

When one of $\mathbf{T}^+(M)$ or $\mathbf{T}^-(M)$ is empty, Theorem 2.9 specializes to a result about more traditional tilings. For this result, we will treat each $\mathcal{T}(\mathbf{z}, \sigma)$ as a collection of points in $\mathbb{R}^{r+k}$. We state only the version where $\mathbf{T}^-(M)$ is empty, but the same statement holds if "non-negative" is replaced with "non-positive".

Corollary 2.10. *[5, Corollary 9.2.8] If the sign of $\det(S_\sigma(M))$ is non-negative for each $\sigma \in \binom{[r+k]}{r}$, then*

$$\mathbb{R}^{r+k} = \bigsqcup_{\mathbf{z} \in \mathbb{Z}} \left(\bigsqcup_{\sigma \in \binom{[r+k]}{r}} \mathcal{T}(\mathbf{z}, \sigma) \right).$$

Remark 2.11. The conditions required on M for Corollary 2.10 to apply are discussed in [5, Section 6.7]. The original proof of the corollary relies on these properties, so we needed different methods to prove the more general Theorem 2.9. A special case of Corollary 2.10 was used in [4] to define a family of *multijections* between the *sandpile group* and *cellular spanning forests* for a large class of cell complexes. This generalizes a construction of Backman Baker and Yuen which used *zonotopal tilings* to answer questions about *chip-firing* on *regular matroids*[1].

3 An Outline of the Proof

Our proof of Theorem 2.9 is structured in the following way.

1. First, we show that the average value of f is $(-1)^k \operatorname{sgn}(\det(M))$.
2. Next, we group the facets of the tiles into collections that lie in the same hyperplane.
3. After this, we imagine a particle crossing a point contained in one of these collections of facets. We show that when doing so, it crosses exactly two facets. Furthermore, in one crossing it *enters* a *positive* tile or *exits* a *negative* tile, while in the other crossing, it *exits* a *positive* tile or *enters* a *negative* tile.
4. From these observations, we conclude that f is constant. Theorem 2.9 then follows from our first observation.

To find the average value of f, we use the multiple row version of Laplace's determinant expansion formula as well as some basic calculus techniques. One important observation is the following chain of equalities, which holds for any $\sigma \in \binom{[r+k]}{r}$.

$$\sum_{\mathbf{z}\in\mathbb{Z}^{r+k}} \int_{\Pi(M)} \mathbb{1}_{\mathcal{T}(\mathbf{z},\sigma)}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{r+k}} \mathbb{1}_{\mathcal{T}(\mathbf{0},\sigma)}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{r+k}} \mathbb{1}_{\Pi(S_\sigma(M))}(\mathbf{x})\mathrm{d}\mathbf{x} = |\det(S_\sigma(M))|.$$

The longest and most technical part of our proof is the facet grouping result. This argument required careful bookkeeping and several applications of Cramer's rule.

4 Lower Dimensional Slices

While Theorem 2.9 gives a signed tiling of $\mathbb{R}^{r+k}$, it is also possible to visualize the tiling in $\mathbb{R}^k$ or $\mathbb{R}^r$ by fixing the first r or last k entries respectively. We conclude with an example of a 2-dimensional slice of a 4-dimensional signed tiling.

Example 4.1. For the matrix M from Example 2.6, the set $\mathbf{T}(M)$ consists of 6 families of 4-dimensional parallelepipeds, where each family contains infinitely many translations of a single fragment.

By taking the determinant of each fragment, we find that

$$\mathbf{T}^{+}(M) = \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \left(\bigsqcup_{\sigma\in\{(1,2),(1,3),(1,4),(2,3),(2,4)\}} \mathcal{T}(\mathbf{z},\sigma) \right), \text{ and}$$
$$\mathbf{T}^{-}(M) = \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \mathcal{T}(\mathbf{z},\{3,4\}).$$

Confirming that Theorem 2.9 holds for this example is not a completely straightforward task, even with the help of a computer. Nevertheless, regardless of the choice of $\mathbf{w}$, one can show that each $\mathbf{p} \in \mathbb{R}^4$ is contained in

- one tile in $\mathbf{T}^{+}(M)$ and no tiles in $\mathbf{T}^{-}(M)$,
- two tiles in $\mathbf{T}^{+}(M)$ and one tile in $\mathbf{T}^{-}(M)$, or
- three tiles in $\mathbf{T}^{+}(M)$ and two tiles in $\mathbf{T}^{-}(M)$.

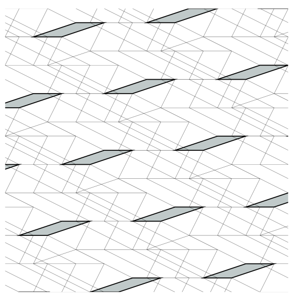
In each case, the value of $f(\mathbf{p})$ is 1, which is also the sign of $\det(M)$.

It is possible to visualize this tiling by taking a 2-dimensional slice which fixes the last 2 coordinates in $\mathbb{R}^4$. Each of the six families of tiles are given in Figure 3. In Figure 4, we combine the positive tiles and the negative tiles. Notice that if the negative tiles are "subtracted" from the positive tiles, the region formed by the difference covers the plane. This demonstrates Theorem 2.9.

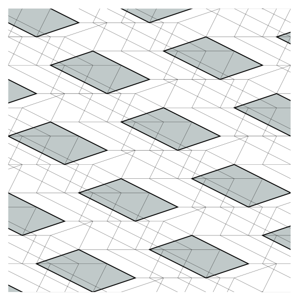
5 Open Problems

The main motivation for this project was an attempt to gain a deeper understanding of a curious phenomenon (in particular Corollary 2.10). While we were successful at generalizing this statement to Theorem 2.9, this new result is just as surprising. We expect that a deeper exploration of this problem will lead more surprises in the future, and we have several specific directions in mind the explore.

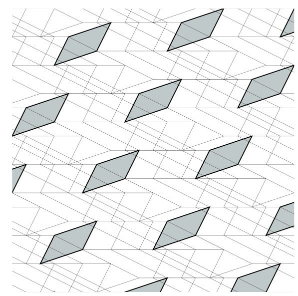
Our initial approach when attempting to prove Theorem 2.9 was to consider an arbitrary point in $\mathbb{R}^{r+k}$ (or $\Pi(M)$) and compute which tiles contain this point. A direct proof of this form would give additional insight about the tiling, since it would allow us to calculate the number and type of tiles containing a given point. However, this method was more challenging than we expected, and we ended up relying on an indirect method by focusing on the facets and proving that f is constant.



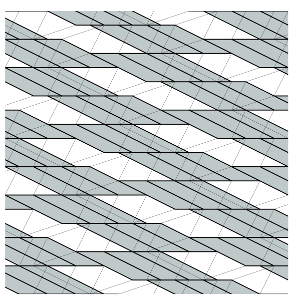
$\det(S_{\{1,2\}}(M)) = 2$



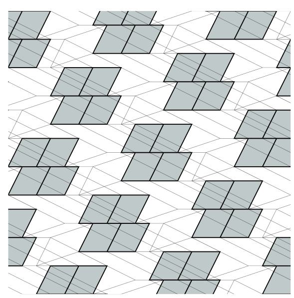
$\det(S_{\{1,3\}}(M)) = 10$



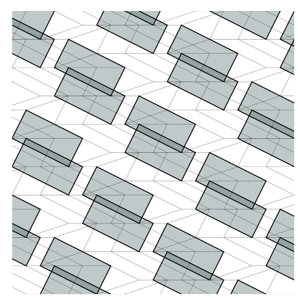
$\det(S_{\{1,4\}}(M)) = 5$



$\det(S_{\{2,3\}}(M)) = 24$



$\det(S_{\{2,4\}}(M)) = 16$



$\det(S_{\{3,4\}}(M)) = -20$

Figure 3: Here we show the contributions of each of the six classes of tiles in Example 4.1 to a 2-dimensional slice of the tiling. Notice that the proportion of the plane that is covered by a specific class of tiles (with multiplicity for any overlapping tiles) is proportional to the magnitude of the determinant of the corresponding fragment matrix.

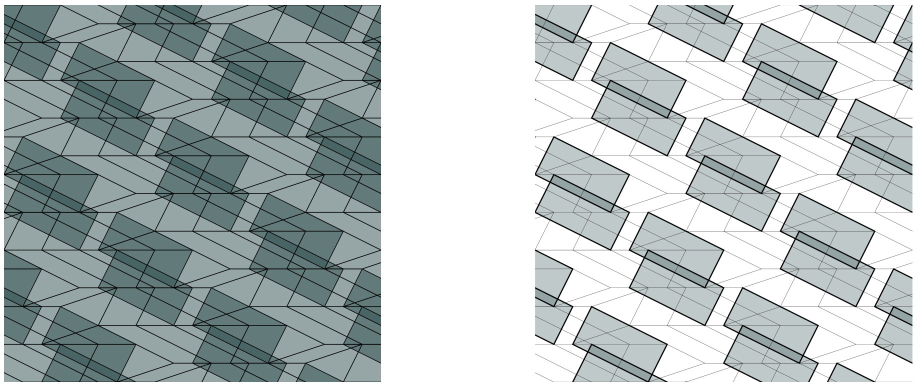


Figure 4: This image on the left is formed by overlapping the 5 positive tiles in Figure 3 while the image on the right is given by the single negative tile. By Theorem 2.9, each point is covered by exactly one more positive tile than negative tile.

Open 5.1. *What is the best algorithm to determine which tiles contain a given point? Can such an algorithm be used to give a more direct proof of Theorem 2.9?*

Another promising method to prove Theorem 2.9 is to use Fourier analysis, applying similar methods to those used in [2] (see also [6]). Perhaps these ideas could lead to a more elegant proof once the background is established.

Open 5.2. *Is there a proof for Theorem 2.9 using Fourier analysis?*

In addition to an alternate proof of the main theorem, we would also be interested in generalizing this result. As written, our construction relies on a choice of coordinates. While it should be possible to translate the statement into coordinate-free language, this is not a trivial task. Nevertheless, such a generalization would likely provide additional insight into the underlying phenomenon behind our construction.

Open 5.3. *Is there a coordinate-free analogue to Theorem 2.9 or Corollary 2.10?*

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Ehrhart polynomials, Hecke series, and affine buildings

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Abstract. Given a lattice polytope P and a prime p, we define a function from the set of primitive symplectic p-adic lattices to the rationals that extracts the ℓth coefficient of the Ehrhart polynomial of P relative to the given lattice. Inspired by work of Gunnells and Rodriguez Villegas in type A, we show that these functions are eigenfunctions of a suitably defined action of the spherical symplectic Hecke algebra. Although they depend significantly on the polytope P, their eigenvalues are independent of P and expressed as polynomials in p. We define local zeta functions that enumerate the values of these Hecke eigenfunctions on the vertices of the affine Bruhat–Tits buildings associated with p-adic symplectic groups. We compute these zeta functions by enumerating p-adic lattices by their elementary divisors and, simultaneously, one Hermite parameter. We report on a general functional equation satisfied by these local zeta functions, confirming a conjecture of Vankov.

Keywords: Ehrhart polynomials, Hecke series, affine buildings, Satake isomorphism, symplectic lattices

1 Introduction

Let P be a fixed full-dimensional lattice polytope in $\mathbb{R}^n$, i.e. the convex hull of finitely many points $V(P)$ in $\Lambda_0 = \mathbb{Z}^n$. Given a lattice Λ such that $\Lambda_0 \subseteq \Lambda \subseteq \mathbb{Q}^n$, we denote the *Ehrhart polynomial of P with respect to* Λ by

$$E^{\Lambda}(P) = \sum_{\ell=0}^{n} c_{\ell}^{\Lambda}(P) T^n \in \mathbb{Q}[T]. \tag{1.1}$$

It is of interest to describe the variation of the coefficients $c_{\ell}^{\Lambda}(P)$ with Λ as compared to $c_{\ell}(P) = c_{\ell}^{\Lambda_0}(P)$; write $E(P)$ for $E^{\Lambda_0}(P)$. For $g \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$ we define

$$g \cdot P = \mathrm{conv}\{g \cdot v \mid v \in V(P)\},$$

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which is again a lattice polytope. We write Λ_g for the lattice generated by the rows of $g \in \mathrm{GL}_n(\mathbb{Q})$. Thus, for every $g \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$, we have

$$E(g \cdot P) = E^{\Lambda_{g^{-1}}}(P). \tag{1.2}$$

We note that $\Lambda_g \subseteq \mathbb{Z}^n \subseteq \Lambda_{g^{-1}} \subseteq \mathbb{Q}^n$ for $g \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$ with $|\det(g)| > 1$.

Gunnells and Rodriguez Villegas [3] consider how the coefficients of $E^{\Lambda}(P)$ from Equation (1.1) relate to $E(P)$ for lattices Λ such that $\Lambda_0 \subseteq \Lambda \subseteq p^{-1}\Lambda_0 \subseteq \mathbb{Q}^n$. In Section 2.1 we revisit these results from our perspective. In addition, we consider a symplectic analogue of the work of Gunnells and Rodriguez Villegas.

1.1 Zeta functions of Ehrhart coefficients

For a prime p, we write $\mathbb{Z}_p$ for the ring of p-adic integers and $\mathbb{Q}_p$ for its field of fractions. Below we define, for each $n \in \mathbb{N} = \{1, 2, \dots\}$ and $\ell \in [2n]_0 = \{0, \dots, 2n\}$, local zeta functions which we call *Ehrhart–Hecke zeta functions*. These functions are Dirichlet series in a complex variable s encoding the ratio of ℓth coefficients of the Ehrhart polynomial of P, as the lattice Λ varies among symplectic lattices in $\mathbb{Q}_p^{2n}$.

Recall the group scheme GSp_{2n} of symplectic similitudes. For a ring K its K-rational points are, with $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$,

$$\mathrm{GSp}_{2n}(K) = \left\{ A \in \mathrm{GL}_{2n}(K) \;\middle|\; AJA^{\mathrm{t}} = \mu(A)J, \text{ for some } \mu(A) \in K^{\times} \right\}.$$

We set $G_n = \mathrm{GSp}_{2n}(\mathbb{Q}_p)$, $\Gamma_n = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$, and $G_n^+ = \mathrm{GSp}_{2n}(\mathbb{Q}_p) \cap \mathrm{Mat}_{2n}(\mathbb{Z}_p)$. The set G_n^+/Γ_n is in bijection with the set of special vertices of the affine building associated with the group $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$, which is of type $\widetilde{C}_n$.

We define the (*local*) *Ehrhart–Hecke zeta function* (of type C) as

$$\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s) = \sum_{g \in G_n^+/\Gamma_n} \frac{c_\ell^{\Lambda_{g^{-1}}}(P)}{c_\ell(P)} |\Lambda_{g^{-1}} : \mathbb{Z}_p^n|^{-s}.$$

Informally speaking, the zeta function $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)$ hence encodes the average ℓth coefficient of the Ehrhart polynomial of P across certain symplectic lattices.

1.2 Symplectic Hecke series

The zeta functions of Section 1.1 are closely connected to formal power series over the Hecke algebra associated with the pair (G_n^+, Γ_n). To explain this connection, we establish additional notation. For $m \in \mathbb{N}$ we define

$$D_n^{\mathsf{C}}(m) = \{A \in G_n^+ \mid AJA^{\mathrm{t}} = mJ\}.$$

Let $\mathcal{H}_p^{\mathsf{C}} = \mathcal{H}^{\mathsf{C}}(G_n^+, \Gamma_n)$ be the spherical Hecke algebra. The Hecke operator $T_n^{\mathsf{C}}(m)$ is

$$T_n^{\mathsf{C}}(m) = \sum_{g \in \Gamma_n \backslash D_n^{\mathsf{C}}(m)/\Gamma_n} \Gamma_n g \Gamma_n.$$

The (formal) symplectic Hecke series is defined as

$$\sum_{\alpha \geqslant 0} T_n^{\mathsf{C}}(p^\alpha) X^\alpha \in \mathcal{H}_p^{\mathsf{C}}[\![X]\!]. \tag{1.3}$$

Shimura's conjecture [6] that the series in (1.3) is a rational function in X was proved by Andrianov [1]. Explicit formulae, however, seem only to be known for $n \leqslant 4$; see [9].

We consider the image of the Hecke series in (1.3) under the Satake isomorphism $\Omega : \mathcal{H}_p^{\mathsf{C}} \to \mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ mapping onto the ring of invariants of W, the Weyl group of G_n. For variables $\boldsymbol{x} = (x_0, \ldots, x_n)$, we define the (*local*) *Satake generating function* as

$$R_{n,p}(\boldsymbol{x}, X) = \sum_{\alpha \geqslant 0} \Omega(T_n^{\mathsf{C}}(p^\alpha)) X^\alpha \in \mathbb{C}[\mathbf{x}^{\pm 1}][\![X]\!].$$

and the (*local*) *primitive local Satake generating function* as

$$R_{n,p}^{\mathrm{pr}}(\boldsymbol{x}, X) = (1 - x_0 X)\,(1 - x_0 x_1 \cdots x_n X)\, R_{n,p}(\boldsymbol{x}, X). \tag{1.4}$$

We write $V(\mathscr{X}_n)$ for the set of vertices of $\mathscr{X}_n$, the affine building $\mathscr{X}_n$ of type $\widetilde{\mathsf{A}}_{n-1}$ associated with the group $\mathrm{GL}_n(\mathbb{Q}_p)$, viz. homothety classes of full lattices in $\mathrm{GL}_n(\mathbb{Q}_p)$. In [2, Section 3.3] Andrianov shows, in essence, that $R_{n,p}^{\mathrm{pr}}$ can be interpreted as a sum over $V(\mathscr{X})$; see Theorem 1.1 below.

For a lattice $\Lambda \leqslant \mathbb{Z}_p^n$, set $\boldsymbol{\nu}(\Lambda) = (\nu_1 \leqslant \cdots \leqslant \nu_n) \in \mathbb{N}_0^n$ if $\mathbb{Z}_p^n/\Lambda \cong \mathbb{Z}/p^{\nu_1} \oplus \cdots \oplus \mathbb{Z}/p^{\nu_n}$. Setting $\nu_0 = 0$, we define

$$\boldsymbol{\mu}(\Lambda) = (\mu_1, \ldots, \mu_n) = (\nu_n - \nu_{n-1},\ \ldots,\ \nu_1 - \nu_0).$$

Having chosen a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p^n$ we associate to each lattice $\Lambda \leqslant \mathbb{Z}_p^n$ a unique matrix

$$M_\Lambda = \begin{pmatrix} p^{\delta_1} & m_{12} & \cdots & m_{1n} \\ & p^{\delta_2} & \cdots & m_{2n} \\ & & \ddots & \vdots \\ & & & p^{\delta_n} \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{Z}_p), \tag{1.5}$$

whose rows generate Λ and with $0 \leqslant v_p(m_{ij}) \leqslant \delta_j$ for all $1 \leqslant i < j \leqslant n$. The matrix M_Λ in (1.5) is said to be in Hermite normal form. We set $\delta(\Lambda) = (\delta_1, \ldots, \delta_n)$. Clearly each homothety class $[\Lambda]$ contains a unique representative $\Lambda_{\mathrm{m}} \leqslant \mathbb{Z}_p^n$ such that $p^{-1}\Lambda_{\mathrm{m}} \nleqslant \mathbb{Z}_p^n$.

Theorem 1.1 (Andrianov)**.** *Let $n \in \mathbb{N}$, $\boldsymbol{a} = (1, 2, \ldots, n) \in \mathbb{N}^n$, $\boldsymbol{d} = (n, n-1, \ldots, 1) \in \mathbb{N}^n$, and let $\langle , \rangle$ be the usual dot product. Then*

$$R_{n,p}^{\mathrm{pr}}(\boldsymbol{x}, X) = \sum_{[\Lambda] \in V(\mathscr{X}_n)} p^{\langle \boldsymbol{d}, \boldsymbol{\nu}(\Lambda_{\mathrm{m}}) \rangle - \langle \boldsymbol{a}, \boldsymbol{\delta}(\Lambda_{\mathrm{m}}) \rangle} x_1^{\delta_1(\Lambda_{\mathrm{m}})} \cdots x_n^{\delta_n(\Lambda_{\mathrm{m}})} (x_0 X)^{\nu_n(\Lambda_{\mathrm{m}})}.$$

1.3 The Hermite–Smith generating function

We define a generating function enumerating finite-index sublattices of $\mathbb{Z}_p^n$ simultaneously by their Hermite and Smith normal forms. For $n \in \mathbb{N}$, let $\boldsymbol{X} = (X_1, \dots, X_n)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ be variables. The *Hermite–Smith generating function* is

$$\mathrm{HS}_{n,p}(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{\Lambda \leqslant \mathbb{Z}_p^n} \boldsymbol{X}^{\boldsymbol{\mu}(\Lambda)} \boldsymbol{Y}^{\delta(\Lambda)} = \sum_{\Lambda \leqslant \mathbb{Z}_p^n} \prod_{i=1}^{n} X_i^{\mu_i(\Lambda)} Y_i^{\delta_i(\Lambda)} \in \mathbb{Z}[\![\boldsymbol{X}, \boldsymbol{Y}]\!]. \tag{1.6}$$

Clearly, if $\Lambda \leqslant \mathbb{Z}_p^n$ has finite index, then so does $p^m \Lambda$ for all $m \in \mathbb{N}_0$. This allows us to extract a "homothety factor" from the sum defining $\mathrm{HS}_{n,p}(\boldsymbol{X}, \boldsymbol{Y})$. The *primitive Hermite–Smith generating function* is

$$\mathrm{HS}^{\mathrm{pr}}_{n,p}(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{[\Lambda] \in V(\mathscr{X}_n)} \boldsymbol{X}^{\boldsymbol{\mu}(\Lambda_{\mathrm{m}})} \boldsymbol{Y}^{\delta(\Lambda_{\mathrm{m}})} = (1 - X_n Y_1 \cdots Y_n) \, \mathrm{HS}_{n,p}(\boldsymbol{X}, \boldsymbol{Y}). \tag{1.7}$$

With this generating function we may obtain the primitive local Satake generating function of Section 1.2, as follows. We define a ring homomorphism

$$\begin{aligned} \Phi : \mathbb{Q}[\![X_1, X_2, \dots, Y_1, Y_2, \dots]\!] &\longrightarrow \mathbb{Q}[\![x_0, x_1, \dots, X]\!] \\ X_i &\longmapsto p^{\binom{i+1}{2}} x_0 X, \\ Y_i &\longmapsto p^{-i} x_i \end{aligned} \tag{1.8}$$

for all $i \in \mathbb{N}_0$. By design of Φ and Theorem 1.1 we have $\Phi(\mathrm{HS}^{\mathrm{pr}}_{n,p}) = R^{\mathrm{pr}}_{n,p}$.

Example 1.2. For $n = 2$, the Hermite–Smith generating function is

$$\begin{aligned} \mathrm{HS}_{2,p}(\boldsymbol{X}, \boldsymbol{Y}) &= \frac{1 - X_1^2 Y_1 Y_2}{(1 - X_1 Y_1)(1 - p X_1 Y_2)(1 - X_2 Y_1 Y_2)}, \\ R_{2,p}(\boldsymbol{x}, X) &= \frac{1 - p^{-1} x_0^2 x_1 x_2 X^2}{(1 - x_0 X)(1 - x_0 x_1 X)(1 - x_0 x_2 X)(1 - x_0 x_1 x_2 X)}. \end{aligned}$$

2 Main results

Interpreting the ℓ-th coefficients of the Ehrhart polynomial of the polytope P as a function on a set of (homothety classes of) p-adic lattices invites the definition of an action of the spherical Hecke algebra $\mathcal{H}_p^{\mathsf{C}}$. The latter is generated by a set of $n+1$ generators $T_n^{\mathsf{C}}(p, 0)$, $T_n^{\mathsf{C}}(p^2, 1), \dots, T_n^{\mathsf{C}}(p^2, n)$. It suffices to explain how these generators act. For $k \in [n]$, define diagonal matrices in G_n^+ as follows:

$$D_0 = \mathrm{diag}(\underbrace{1, \dots, 1}_{n}, \underbrace{p, \dots, p}_{n}), \qquad D_k = \mathrm{diag}(\underbrace{1, \dots, 1}_{n-k}, \underbrace{p, \dots, p}_{k}, \underbrace{p^2, \dots, p^2}_{n-k}, \underbrace{p, \dots, p}_{k}).$$

Set $\mathscr{D}^{\mathsf{C}}_{n,k} = \Gamma_n D_k \Gamma_n/\Gamma_n$. The set $\mathscr{D}^{\mathsf{C}}_{n,k}$ can be interpreted as the set of symplectic lattices with symplectic elementary divisors equal to those of D_k. We define

$$T^{\mathsf{C}}_n(p,0)E(P) = \sum_{g\in\mathscr{D}^{\mathsf{C}}_{n,0}} E(g\cdot P), \qquad T^{\mathsf{C}}_n(p^2,k)E(P) = \sum_{g\in\mathscr{D}^{\mathsf{C}}_{n,k}} E(g\cdot P).$$

For $\ell \geqslant \mathbb{N}_0$, we define functions

$$\mathscr{E}_{n,p,\ell,P} : G^+_n/\Gamma_n \to \mathbb{C}, \quad \Gamma_n g \mapsto c_\ell(E^{\Lambda_{g^{-1}}}(P)).$$

Lastly, for all $T \in \mathcal{H}^{\mathsf{C}}_p$ set

$$T\mathscr{E}_{n,p,\ell,P}(\Gamma_n g) = c_\ell(TE^{\Lambda_{g^{-1}}}(P)).$$

Recall that P is full-dimensional; for $k \in [n]$, and $\ell \in [2n]_0$, we define

$$\nu^{\mathsf{C}}_{n,0,\ell}(p) = \frac{c_\ell(T^{\mathsf{C}}_n(p,0)E(P))}{c_\ell(E(P))}, \qquad \nu^{\mathsf{C}}_{n,k,\ell}(p) = \frac{c_\ell(T^{\mathsf{C}}_n(p^2,k)E(P))}{c_\ell(E(P))}.$$

The notation suggests that the value $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ is independent of the polytope P, which is justified by Theorem A. General properties of the Ehrhart polynomial imply that

$$\nu^{\mathsf{C}}_{n,n,\ell}(p) = p^\ell, \qquad \nu^{\mathsf{C}}_{n,k,0}(p) = \#\mathscr{D}^{\mathsf{C}}_{n,k}.$$

Every $\mathbb{Q}$-linear homomorphism $\lambda : \mathcal{H}^{\mathsf{C}}_p \to \mathbb{C}$ is uniquely determined by parameters $(a_0,\dots,a_n) \in \mathbb{C}^{n+1}$ such that if $\psi : \mathbb{C}[x_0^{\pm 1},\dots,x_n^{\pm 1}] \to \mathbb{C}$ is given by $x_i = a_i$ then $\lambda = \psi\circ\Omega$; see [2, Proposition 3.3.36].

Theorem A. *The functions $\mathscr{E}_{n,p,\ell,P}$ are Hecke eigenfunctions under the action defined above; specifically, for all $k \in [n]$, we have*

$$T^{\mathsf{C}}_n(p,0)\mathscr{E}_{n,p,\ell,P} = \nu^{\mathsf{C}}_{n,0,\ell}(p)\mathscr{E}_{n,p,\ell,P}, \qquad T^{\mathsf{C}}_n(p^2,k)\mathscr{E}_{n,p,\ell,P} = \nu^{\mathsf{C}}_{n,k,\ell}(p)\mathscr{E}_{n,p,\ell,P},$$

where the $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ are polynomials in p with integer coefficients which are independent of P. Moreover, the parameters associated to $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ are $(p^\ell, p, p^2, \dots, p^{n-1}, p^{n-\ell})$.

Table 1 lists the values of $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ for small values of n and k.

Theorem A enables us to relate $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)$ to $R_{n,p}(\boldsymbol{x}, X)$. Let $\psi_{n,\ell}$ be the ring homomorphism from $\mathbb{C}[\![x_0, x_1, \dots, X]\!] \to \mathbb{C}[t]$ given by

$$X \mapsto t^n \qquad x_0 \mapsto p^\ell, \qquad x_n \mapsto p^{n-\ell}, \qquad x_i \mapsto p^i.$$

Corollary B. *For $n \in \mathbb{N}$ and $\ell \in [2n]_0$ we have, writing $t = p^{-s}$,*

$$(\psi_{n,\ell}\circ\Phi)(\mathrm{HS}^{\mathrm{pr}}_{n,p}(\boldsymbol{X},\boldsymbol{Y})) = \psi_{n,\ell}(R^{\mathrm{pr}}_{n,p}) = \mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)\left(1-p^{\ell-s}\right)\left(1-p^{\binom{n+1}{2}-s}\right).$$

ℓ	$\nu^{\mathsf{C}}_{2,0,\ell}(p)$	$\nu^{\mathsf{C}}_{2,1,\ell}(p)$
4	$p^5+p^4+p^3+p^2$	$p^8+p^7+p^6+p^5$
3	$p^4+p^3+p^3+p^2$	$2p^6+p^5+2p^4-p^3$
2	$p^3+p^3+p^2+p^2$	$p^5+3p^4+p^3-p^2$
1	$p^3+p^2+p^2+p^1$	$2p^4+p^3+2p^2-p$
0	p^3+p^2+p+1	$p^4+p^3+p^2+p$

Table 1: The polynomials $\nu^{\mathsf{C}}_{2,k,\ell}(p)$ for $k \in \{0,1\}$ and $\ell \in [4]_0$.

Thanks to Corollary B, we can work with $\mathrm{HS}_{n,p}$ to prove that $R_{n,p}$ and $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}$ satisfy a self-reciprocity property, which proves the conjecture in [9, Remark 4].

Theorem C. *Let $n \in \mathbb{N}$. Then $\mathrm{HS}_{n,p}(\boldsymbol{X},\boldsymbol{Y})$ is a rational function in $\boldsymbol{X}$ and $\boldsymbol{Y}$. Furthermore, for $\boldsymbol{X}^{-1} = (X_1^{-1},\dots,X_n^{-1})$ and $\boldsymbol{Y}^{-1} = (Y_1^{-1},\dots,Y_n^{-1})$, we have*

$$\mathrm{HS}_{n,p}(\boldsymbol{X}^{-1},\boldsymbol{Y}^{-1})\Big|_{p\to p^{-1}} = (-1)^n p^{\binom{n}{2}} X_n Y_1\cdots Y_n \cdot \mathrm{HS}_{n,p}(\boldsymbol{X},\boldsymbol{Y}).$$

We prove Theorem C by writing $\mathrm{HS}_{n,p}$ as a p-adic integral and applying results of [10], where the operation of inverting p is also explained.

Corollary D. *For $n \in \mathbb{N}$ and $\ell \in [2n]_0$, we have*

$$\begin{aligned}\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{n+1}p^{n^2+\ell-2ns}\cdot\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s),\\ R_{n,p}(\mathbf{x},X)\Big|_{p\to p^{-1}} &= (-1)^{n+1}p^{\binom{n}{2}}x_0^2x_1\dots x_nX^2\cdot R_{n,p}(\mathbf{x},X).\end{aligned}$$

In the next theorem, we determine a formula for the specialization of $\mathrm{HS}^{\mathrm{pr}}_{n,p}$ which yields $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}$ by Corollary B. To this end we define

$$\overline{\mathrm{HS}}_{n,p}(\boldsymbol{X},Y) = \mathrm{HS}^{\mathrm{pr}}_{n,p}(\boldsymbol{X},1,\dots,1,Y).$$

We prove that $\overline{\mathrm{HS}}_{n,p}$ is a rational function in the $n+1$ variables $\boldsymbol{X}$ and Y and, in addition, the prime p. In order to describe the formula, we define additional notation. For $I = \{i_1 < \cdots < i_\ell\} \subseteq [n-1]$, with $i_{\ell+1} = n$, $k \in [\ell+1]$, and a variable Z, we set

$$\begin{aligned}I^{(k)} &= \{i_j \mid j<k\} \cup \{i_j - 1 \mid j \geqslant k\}\\ \mathscr{G}_{n,I,k}(Z,\boldsymbol{X},Y) &= \left(\prod_{j=1}^{k-1}\frac{Z^{i_j(n-i_j-1)}X_{i_j}}{1-Z^{i_j(n-i_j-1)}X_{i_j}}\right)\left(\prod_{j=k}^{\ell}\frac{Z^{i_j(n-i_j)}X_{i_j}Y}{1-Z^{i_j(n-i_j)}X_{i_j}Y}\right).\end{aligned}$$

Theorem E. *Let $n \in \mathbb{N}$. For $I = \{i_1 < \cdots < i_\ell\}_< \subseteq [n-1]$, set*

$$W_{n,I}(Z, \boldsymbol{X}, Y) = \sum_{k=1}^{\ell+1} Z^{-(n-i_k)} \binom{n-1}{I^{(k)}}_{Z^{-1}} \mathscr{G}_{n,I,k}(Z, \boldsymbol{X}, Y) + \sum_{k=1}^{\ell} \frac{(1-Z^{-i_j})\mathscr{G}_{n,I,k}(Z, \boldsymbol{X}, Y)}{1 - Z^{i_j(n-i_j-1)}X_{i_j}} \left(\sum_{m=k+1}^{\ell+1} Z^{-(n-i_m)} \right) \binom{n-1}{I^{(k+1)}}_{Z^{-1}}.$$

Then

$$\overline{\mathrm{HS}}_{n,p}(\boldsymbol{X}, Y) = \sum_{I \subseteq [n-1]} W_{n,I}(p, \boldsymbol{X}, Y) \in \mathbb{Z}(p, \boldsymbol{X}, Y).$$

Via the various substitutions given above, Theorem E yields explicit formulae for the functions $R_{n,p}$ and, specifically,

$$Z^{\mathsf{C}}_{n,\ell,p}(s) = (1-p^{\ell-s})^{-1}(1-p^{\binom{n+1}{2}-s})^{-1} \sum_{I \subseteq [n-1]} W_{n,I}\left(p, \left(p^{\binom{i+1}{2}+\ell-ns}\right)_{i=1}^{n}, p^{-\ell}\right).$$

In the next theorem we show that the primitive local Satake generating function can be viewed as a "p-analogue" of the fine Hilbert series of a Stanley–Reisner ring. Let V be a finite set. If $\Delta \subseteq 2^V$ is a simplicial complex on V, then the Stanley–Reisner ring of Δ over a ring K is

$$K[\Delta] = K[X_v \mid v \in V]/(\prod_{v \in \sigma} X_v \mid \sigma \in 2^V \setminus \Delta).$$

Theorem F. *For all $n \in \mathbb{N}$, let Δ_n be the n-simplex with vertices $[n]$ and $\Delta = \mathrm{sd}(\partial\Delta_n)$, the barycentric subdivision of boundary of Δ_n, with vertices given by the nonempty subsets of $[n]$. Let $\boldsymbol{y} = (y_I : \varnothing \neq I \subseteq [n])$ and $\varphi : \mathbb{Z}[\![\boldsymbol{y}]\!] \to \mathbb{Z}[\![\boldsymbol{x}, X]\!]$ via $y_I \mapsto x_0 X \prod_{i \in I} x_i$. Then*

$$R^{\mathrm{pr}}_{n,p}(\boldsymbol{x}, X)\big|_{p \to 1} = \varphi(\mathrm{Hilb}(\mathbb{Z}[\Delta]; \boldsymbol{y})) = \sum_{\sigma \in \Delta} \prod_{J \in \sigma} \frac{\varphi(y_J)}{1 - \varphi(y_J)}.$$

With Theorem F, we come full circle and relate the local Satake generating function $R_{n,p}$ to the Ehrhart series of the n-cube.

Corollary 2.1. *For all $n \in \mathbb{N}$, let P be the n-cube. Then*

$$R_{n,p}(\mathbf{1}, X)\big|_{p \to 1} = \mathrm{Ehr}_P(X) = \frac{\mathrm{E}_n(X)}{(1-X)^{n+1}},$$

where $\mathrm{E}_n(X) = \sum_{\sigma \in S_n} X^{\mathrm{des}(\sigma)}$ is the Eulerian polynomial.

Proof. It follows from Theorem F that

$$(1-X)^2\, R_{n,p}(\mathbf{1}, X)\big|_{p \to 1} = \sum_{\sigma \in \Delta} \prod_{J \in \sigma} \frac{X}{1-X}, \tag{2.1}$$

where Δ is the barycentric subdivision of the boundary of the n-simplex. From [5, Theore. 9.1] and Equation (2.1) it follows that

$$R_{n,p}(\mathbf{1}, X)\big|_{p\to 1} = \frac{\mathrm{E}_n(X)}{(1-X)^{n+1}} = \sum_{k\geqslant 0}(k+1)^n X^k = \mathrm{Ehr}_P(X). \qquad \square$$

2.1 The type–A story

Our work was inspired by Gunnells and Rodriguez Villegas. In [3] they considered type-A versions of some of the questions outlined above. We paraphrase parts of [3] from the perspective of our work in type C. For a prime p we define the (*local*) *Ehrhart–Hecke zeta function* (of type A) as

$$\mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = \sum_{\substack{\mathbb{Z}_p^n \leqslant \Lambda \leqslant \mathbb{Q}_p^n \\ |\Lambda : \mathbb{Z}_p^n| < \infty}} \frac{c_\ell^\Lambda(P)}{c_\ell(P)} |\Lambda : \mathbb{Z}_p^n|^{-s}. \tag{2.2}$$

Let $\Gamma_n^{\mathsf{A}} = \mathrm{GL}_n(\mathbb{Z})$ and $G_n^{\mathsf{A}} = \mathrm{Mat}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$. For $m \in \mathbb{N}$, let

$$D_n^{\mathsf{A}}(m) = \{g \in G_n^{\mathsf{A}} \mid |\det(g)| = m\},$$

so $D_n^{\mathsf{A}}(m)$ is a finite union of double cosets relative to Γ_n^{A}. We define

$$T_n^{\mathsf{A}}(m) = \sum_{g \in \Gamma_n^{\mathsf{A}} \backslash D_n^{\mathsf{A}}(m) / \Gamma_n^{\mathsf{A}}} \Gamma_n^{\mathsf{A}} g \Gamma_n^{\mathsf{A}},$$

where the sum runs over a set of representatives of the double cosets, which is an element of the Hecke algebra determined by $(\Gamma_n^{\mathsf{A}}, G_n^{\mathsf{A}})$. Moreover, if $\gcd(m, m') = 1$, then

$$T_n^{\mathsf{A}}(m) T_n^{\mathsf{A}}(m') = T_n^{\mathsf{A}}(mm').$$

For $k \in [n]_0$ define $\pi_k(p) = \mathrm{diag}(1, \dots, 1, \overbrace{p, \dots, p}^{k})$ and $T_n^{\mathsf{A}}(p,k) = \Gamma_n^{\mathsf{A}} \pi_k(p) \Gamma_n^{\mathsf{A}}$, which decomposes into a finite (disjoint) union of right cosets relative to Γ_n^{A}.

Gunnells and Rodriguez Villegas [3] considered the following action of the Hecke algebra on the Ehrhart polynomial $E(P) = E^{\Lambda_0}(P)$ of P:

$$T_n^{\mathsf{A}}(p,k) E(P) = \sum_{g \in \Gamma_n^{\mathsf{A}} \pi_k(p) \Gamma_n^{\mathsf{A}} / \Gamma_n^{\mathsf{A}}} E(g \cdot P), \tag{2.3}$$

where the sum runs over a set of right coset representatives. The action in (2.3) is independent of the chosen representatives since Γ_n^{A} comprises bijections of $\mathbb{Z}^n$. Our definition in (2.3) differs from [3] only cosmetically via (1.2).

Denote by $\mathrm{Gr}(\ell, n, p)$ the set of ℓ-dimensional subspaces in $\mathbb{F}_p^n$. For $n \in \mathbb{N}$, $\ell, k \in [n]_0$, and $U \in \mathrm{Gr}(\ell, n, p)$, define

$$\nu^{\mathsf{A}}_{n,k,\ell}(p) = \sum_{W \in \mathrm{Gr}(k,n,p)} \#(U \cap W).$$

Let $\psi^{\mathsf{A}}_{n,\ell} : \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{Q}$ be given by $x_n \mapsto p^\ell$ and $x_i \mapsto p^i$ for all $i \in [n-1]$. Let further ω denote the Satake isomorphism from the p-primary part of the Hecke algebra associated with $(\Gamma^{\mathsf{A}}_n, G^{\mathsf{A}}_n)$, written $\mathcal{H}^{\mathsf{A}}_p$, to the symmetric subring of $\mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Let $s_{n,k}(x_1, \dots, x_n)$ be the (homogeneous) elementary symmetric polynomial of degree k, and set $s_{n,-1} = 0$.

Theorem 2.2 ([3]). *For $n \in \mathbb{N}$, $k, \ell \in [n]_0$, and a prime p, we have*

$$\nu^{\mathsf{A}}_{n,k,\ell}(p) = p^k \binom{n-1}{k}_p + p^\ell \binom{n-1}{k-1}_p = \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p,k))).$$

Moreover,

$$\mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = (1 - p^{\ell - s})^{-1} \prod_{k=1}^{n-1} (1 - p^{k-s})^{-1}.$$

Proof. First we prove the claims concerning $\nu^{\mathsf{A}}_{n,k,\ell}(p)$. Therefore,

$$\begin{aligned}
\nu^{\mathsf{A}}_{n,k,\ell}(p) &= \binom{n}{k}_p - \binom{n-1}{k-1}_p + p^\ell \binom{n-1}{k-1}_p && \text{([3, Lem. 3.3])} \\
&= p^k \binom{n-1}{k}_p + p^\ell \binom{n-1}{k-1}_p && \text{(Pascal identity)} \\
&= p^k s_{n-1,k}(1, p, \dots, p^{n-2}) + p^\ell s_{n-1,k-1}(1, p, \dots, p^{n-2}) && \text{([4, Ex. I.2.3])} \\
&= p^{-\binom{k}{2}} \psi^{\mathsf{A}}_{n,\ell}(s_{n,k}) \\
&= \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p,k))). && \text{([2, Lem. 3.2.21])}
\end{aligned}$$

We now tend to the last claim. Tamagawa [7] established the identity

$$\sum_{m \geqslant 0} T^{\mathsf{A}}_n(p^m) X^m = \left(\sum_{k=0}^{n} (-1)^k p^{\binom{k}{2}} T^{\mathsf{A}}_n(p,k) X^k \right)^{-1} \in \mathcal{H}^{\mathsf{A}}_p[\![X]\!]. \tag{2.4}$$

Applying $\psi^{\mathsf{A}}_{n,\ell} \circ \omega$ to (2.4) and setting $X = p^{-s}$, we have

$$\sum_{m \geqslant 0} \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p^m))) p^{-ms} = \left(\sum_{k=0}^{n} \psi^{\mathsf{A}}_{n,\ell}(s_{n,k}) (-p)^{-ks} \right)^{-1} = (1 - p^{\ell-s})^{-1} \prod_{k=1}^{n-1} (1 - p^{k-s})^{-1}.$$

Since $\nu^{\mathsf{A}}_{n,k,\ell}(p)$ is an eigenvalue for $T_n(p,k)$, it follows that

$$\mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = \sum_{m \geqslant 0} \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p^m))) p^{-ms}. \qquad \square$$

Corollary 2.3. *Let $\zeta(s)$ be the Riemann zeta function. For $n \in \mathbb{N}$ and $\ell \in [n]_0$, we have*

$$\prod_{\text{prime } p} \mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = \zeta(s-\ell)\prod_{k=1}^{n-1}\zeta(s-k).$$

3 Examples

3.1 Hecke eigenfunctions

We give some explicit examples, showing in Figure 3.1 that the eigenfunctions of Theorem A depend significantly on the polytope. We do this by displaying a graph whose vertices correspond to homothety classes of lattices. We evaluate the functions $\mathscr{E}_{n,p\ell,P}$ on Λ_{m} for each homothety class $[\Lambda]$.

3.2 Local Ehrhart–Hecke zeta functions

For $n \in [3]$ and $\ell \in [2n]_0$, we record the rational functions $W_{n,\ell}(X,Y) \in \mathbb{Q}(X,Y)$ where, for all primes, $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s) = W_{n,\ell}(p, p^{-ns})$. We computed these with SageMath [8].

$$\begin{aligned}
W_{1,\ell}(X,Y) &= \frac{1}{(1-XY)(1-X^{\ell}Y)} \\
W_{2,\ell}(X,Y) &= \frac{1-X^{2+\ell}Y^2}{(1-X^2Y)(1-X^3Y)(1-X^{\ell}Y)(1-X^{\ell+1}Y)} \\
W_{3,\ell}(X,Y) &= \frac{1+(X^{1+\ell}+X^4)Y - A_{\ell}(X)Y^2 + (X^{6+2\ell}+X^{9+\ell})Y^3 + X^{10+2\ell}Y^4}{(1-X^3Y)(1-X^5Y)(1-X^6Y)(1-X^{\ell}Y)(1-X^{2+\ell}Y)(1-X^{3+\ell}Y)} \\
W_{4,\ell}(X,Y) &= \frac{N_{4,\ell}(X,Y)}{D_{4,\ell}(X,Y)},
\end{aligned}$$

where $A_{\ell}(X) = X^{7+\ell} + 2X^{6+\ell} + 2X^{4+\ell} + X^{3+\ell}$,

$$\begin{aligned}
N_{4,\ell}(X,Y) = 1 &+ (X^5+X^6+X^7+X^8+X^{1+\ell}+X^{2+\ell}+X^{3+\ell}+X^{4+\ell})Y + (X^{13} \\
&- X^{4+\ell} - 2X^{5+\ell} - 2X^{6+\ell} - 2X^{7+\ell} - 2X^{8+\ell} - 2X^{9+\ell} - 3X^{10+\ell} \\
&- 2X^{11+\ell} - 2X^{12+\ell} - 2X^{13+\ell} - X^{14+\ell} + X^{5+2\ell})Y^2 + (X^{14+\ell} \\
&- X^{18+\ell} + X^{10+2\ell} - X^{14+2\ell})Y^3 - (X^{23+\ell} - X^{14+2\ell} - 2X^{15+2\ell} \\
&- 2X^{16+2\ell} - 2X^{17+2\ell} - 3X^{18+2\ell} - 2X^{19+2\ell} - 2X^{20+2\ell} - 2X^{21+2\ell} \\
&- 2X^{22+2\ell} - 2X^{23+2\ell} - X^{24+2\ell} + X^{15+3\ell})Y^4 - (X^{24+2\ell} + X^{25+2\ell} \\
&+ X^{26+2\ell} + X^{27+2\ell} + X^{20+3\ell} + X^{21+3\ell} + X^{22+3\ell} + X^{23+3\ell})Y^5 \\
&- X^{28+3\ell}Y^6,
\end{aligned}$$

$$D_{4,\ell}(X,Y) = (1-X^4Y)(1-X^7Y)(1-X^9Y)(1-X^{10}Y) \\ \times (1-X^{\ell}Y)(1-X^{3+\ell}Y)(1-X^{5+\ell}Y)(1-X^{6+\ell}Y).$$

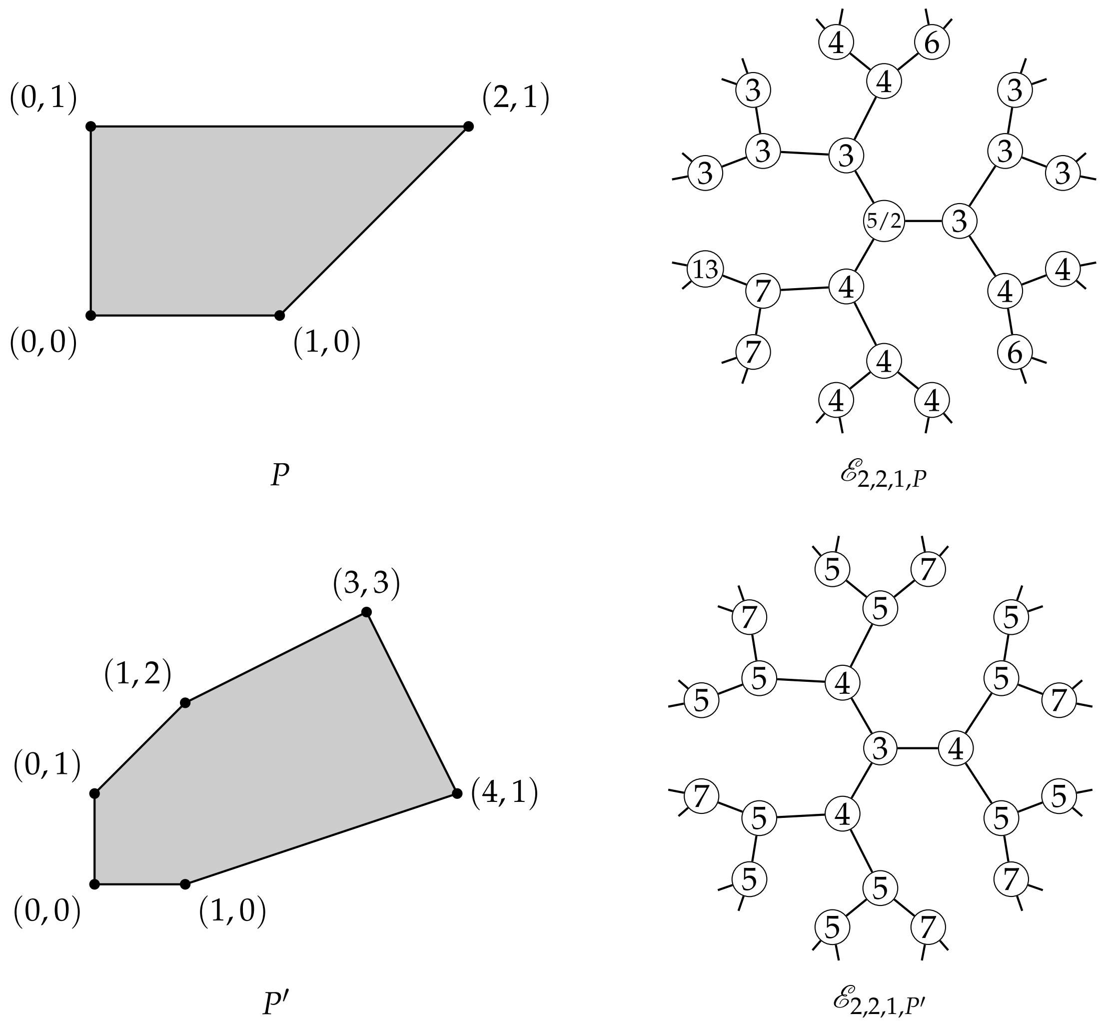

Figure 3.1: Polytopes and some values of $\mathscr{E}_{2,2,1,P}$ displayed on lattices in the affine building of type $\widetilde{\mathsf{A}}_1$ associated with the group $\mathrm{GSp}_2(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)$. The center vertex corresponds to the homothety class of the identity, and the values are the linear coefficients of the Ehrhart polynomials with respect to the corresponding lattices.

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Cluster algebras and tilings for the $m = 4$ amplituhedron

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Abstract. The *amplituhedron* $\mathcal{A}^Z_{n,k,m}$ is the image of the positive Grassmannian $\mathrm{Gr}^{\geq 0}_{k,n}$ under the map $\tilde{Z} : \mathrm{Gr}^{\geq 0}_{k,n} \to \mathrm{Gr}_{k,k+m}$ induced by a positive linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. It was originally introduced in physics in order to give a geometric interpretation of scattering amplitudes. More specifically, one can compute scattering amplitudes in $\mathcal{N} = 4$ SYM by 'tiling' the $m = 4$ amplituhedron $\mathcal{A}^Z_{n,k,4}$ — that is, decomposing $\mathcal{A}^Z_{n,k,4}$ into 'tiles' (closures of images of $4k$-dimensional cells of $\mathrm{Gr}^{\geq 0}_{k,n}$ on which $\tilde{Z}$ is injective). In this article we deepen both our understanding of tiles and tilings of the $m = 4$ amplituhedron and the connection with cluster algebras. Firstly, we prove the *cluster adjacency conjecture* for BCFW tiles of $\mathcal{A}^Z_{n,k,4}$, which says that facets of tiles are cut out by collections of compatible cluster variables for $\mathrm{Gr}_{4,n}$. Secondly, we describe each BCFW tile as the semialgebraic set in $\mathrm{Gr}_{k,k+4}$ where certain cluster variables have particular signs. Finally, we prove the *BCFW tiling conjecture*, which says that any way of iterating the BCFW recurrence gives rise to a tiling of the amplituhedron $\mathcal{A}^Z_{n,k,4}$. Along the way, we introduce a method to construct seeds for $\mathrm{Gr}_{4,n}$ comprised of high-degree cluster variables, which may be of independent interest in the study of cluster algebras.

Keywords: positroid, amplituhedron, cluster algebras, tile, tiling, BCFW

1 Introduction

The (tree) *amplituhedron* $\mathcal{A}^Z_{n,k,m}$ is the image of the positive Grassmannian $\mathrm{Gr}^{\geq 0}_{k,n}$ under the *amplituhedron map* $\tilde{Z} : \mathrm{Gr}^{\geq 0}_{k,n} \to \mathrm{Gr}_{k,k+m}$. It was introduced by Arkani-Hamed and Trnka [4] in order to give a geometric interpretation of *scattering amplitudes* in $\mathcal{N} = 4$ super Yang Mills theory (SYM): in particular, one can compute $\mathcal{N} = 4$ SYM scattering amplitudes

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by 'tiling' the $m = 4$ amplituhedron $\mathcal{A}^Z_{n,k,4}$ — that is, by decomposing the amplituhedron into smaller 'tiles' — and summing the 'volumes' of the tiles. While the case $m = 4$ is most important for physics, the amplituhedron is defined for any positive n, k, m with $k + m \leq n$, and has a very rich geometric and combinatorial structure. It generalizes cyclic polytopes (when $k = 1$), cyclic hyperplane arrangements [19] (when $m = 1$), and the positive Grassmannian (when $k = n - m$), and it is connected to the hypersimplex and the positive tropical Grassmanian [23, 26] (when $m = 2$). The amplituhedron is also an example of a *Grassmann polytope* ('Grasstope') and conjectured to be a *positive geometry* [1, 21]. The followings are two of the guiding problems about the amplituhedron.

The first is the *cluster adjacency conjecture*, which says that facets of tiles are cut out by collections of compatible cluster variables. This was motivated by physics where cluster algebras were shown to describe singularities of scattering amplitudes in $\mathcal{N} = 4$ SYM [16]. In particular, [7, 8] conjectured that the terms in tree-level amplitudes coming from the BCFW recursions are rational functions whose poles correspond to compatible cluster variables of the Grassmannian $\mathrm{Gr}_{4,n}$, see also [25]. The cluster adjacency conjecture, formulated for the $m = 2$ and $m = 4$ amplituhedron in [22] and [17], was proved for all tiles of the $m = 2$ amplituhedron in [26].

The second is the *BCFW tiling conjecture*, which says that any way of iterating the BCFW recurrence gives rise to a collection of cells whose images tile the $m = 4$ amplituhedron $\mathcal{A}^Z_{n,k,4}$. This arose alongside the definition of the amplituhedron [4] in order to give a geometric interpretation of the recurrence Britto–Cachazo–Feng–Witten [6] introduced to compute scattering amplitudes. BCFW-like tilings of the $m = 1$ and $m = 2$ amplituhedron were proved in [19] and [5], building on [3] and [20]. Finally, extending the work of [20], it was proved in [10] that the 'standard' way of performing the BCFW recursion gives a tiling for the $m = 4$ amplituhedron.

Main results. In this paper we build on [26] and [10] to give a very complete picture of the $m = 4$ amplituhedron. We show that arbitrary *BCFW cells* give tiles (Theorem 3.5) and that they satisfy the cluster adjacency conjecture (Theorem 3.15). We strengthen the connection with cluster algebras by associating to each BCFW tile a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$ (Definition 3.11), which we use to describe the tile as a semialgebraic set in $\mathrm{Gr}_{k,k+4}$ (Theorem 3.13). For 'standard' BCFW tiles, one can also give a non-recursive description of these cluster variables and the underlying quiver, and define an associated cluster algebra [9, Sections 8, 9]. Finally, we use these results to prove the BCFW tiling conjecture for the $m = 4$ amplituhedron (Theorem 3.17).

Further motivation. From the point of view of cluster algebras, the study of tiles for the amplituhedron $\mathcal{A}_{n,k,m}$ is useful because it is closely related to the cluster structure on the Grassmannian $\mathrm{Gr}_{m,n}$, as was shown for $m = 2$ in [26] and as this paper demonstrates for $m = 4$. In particular, for $m = 4$, the *BCFW product* (Definition 3.2) used to recursively build tiles (Definition 3.3) has a cluster quasi-homomorphism counterpart called *product promotion* (Definition 3.6), that can be used to recursively construct cluster variables and seeds in $\mathrm{Gr}_{4,n}$ (Theorem 3.7).

In the closely related field of *total positivity*, one prototypical problem is to give an efficient characterization of the 'positive part' of a space as the subset where a certain minimal collection of functions take on positive values [13] ('positivity test'). For example, for any cluster $\mathbf{x}$ for $\mathrm{Gr}_{k,n}$ [28], the *positive Grassmannian* $\mathrm{Gr}_{k,n}^{>0}$ can be described as the region in $\mathrm{Gr}_{k,n}$ where all the cluster variables of $\mathbf{x}$ are positive.

We think of Theorem 3.13 as a 'positivity test' for membership in a BCFW tile of the amplituhedron. See [26, Theorem 6.8] for an analogous result for $m = 2$, and [9, Conjecture 7.17] for some conjectures for general m.

From the point of view of discrete geometry, it is interesting to study tiles and more generally Grasstopes because one can think of them as a generalization of polytopes in the Grassmannian. In particular, the positivity tests for the positive Grassmannian and BCFW tiles can be thought of as analogues of the hyperplane description of polytopes. Finally, it would be interesting to show that tiles are positive geometries.

2 Background

2.1 The (positive) Grassmannian

The *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{F})$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{F}^n$. Let $[n]$ denote $\{1,\dots,n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. We can represent a point $V \in \mathrm{Gr}_{k,n}(\mathbb{F})$ as the row-span of a full-rank $k \times n$ matrix C with entries in $\mathbb{F}$. Then for $I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k}$, we let $\langle I \rangle_V = \langle i_1\, i_2\, \dots\, i_k \rangle_V$ be the $k \times k$ minor of C using the columns I. The $\langle I \rangle_V$ are called the *Plücker coordinates* of V, and are independent of the choice of matrix representative C (up to common rescaling). The *Plücker embedding* $V \mapsto \{\langle I \rangle_V\}_{I \in \binom{[n]}{k}}$ embeds $\mathrm{Gr}_{k,n}(\mathbb{F})$ into projective space[1] . If C has columns $v_1,\dots,v_n$, we may also identify $\langle i_1\, i_2\, \dots\, i_k \rangle$ with $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, hence e.g. $\langle i_1\, i_2\, \dots\, i_k \rangle = -\langle i_2\, i_1\, \dots\, i_k \rangle$. In this paper we will often be working with the *real* Grassmannian $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{R})$. We will also denote by $\mathrm{Gr}_{k,N}$ the Grassmannians of k-planes in a vector space with basis indexed by a set $N \subset [n]$.

Definition 2.1 (Positive Grassmannian)**.** [24, 27] We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if (up to a global change of sign) $\langle I \rangle_V \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is *totally positive* if $\langle I \rangle_V > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$, respectively. $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the *totally nonnegative Grassmannian*, or sometimes just the *positive Grassmannian*.

[1] We will sometimes abuse notation and identify C with its row-span; we will also drop the subscript V on Plücker coordinates when it does not cause confusion.

If we partition $\mathrm{Gr}_{k,n}^{\geq 0}$ into strata based on which Plücker coordinates are strictly positive and which are 0, we obtain a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ into *positroid cells* [27]. Each positroid cell S gives rise to a matroid $\mathcal{M}$, whose bases are precisely the k-element subsets I such that the Plücker coordinate $\langle I \rangle$ does not vanish on S; $\mathcal{M}$ is called a *positroid*.

There are many ways to index positroid cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ [27], such as *plabic graphs*:

Definition 2.2. Let G be a *plabic graph*, i.e. a planar bipartite graph[2] embedded in a disk, with black vertices $1, 2, \ldots, n$ on the boundary of the disk. An *almost perfect matching* M of G is a collection of edges which covers each internal vertex of G exactly once. The *boundary* of M, denoted ∂M, is the set of boundary vertices covered by M. The positroid associated to G is the collection $\mathcal{M} = \mathcal{M}(G) := \{\partial M : M \text{ an almost perfect matching of } G\}$.

Both $\mathrm{Gr}_{k,n}$ and $\mathrm{Gr}_{k,n}^{\geq 0}$ admit the following set of operations, which will be useful to us.

Definition 2.3 (Operations on the Grassmannian)**.** We define the following maps on $\mathrm{Mat}_{k,n}$, which descends to maps on $\mathrm{Gr}_{k,n}$ and $\mathrm{Gr}_{k,n}^{\geq 0}$, which we denote in the same way:

- (cyclic shift) We define the *cyclic shift* as the map $\mathrm{cyc} : \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,n}$ which sends $v_1 \mapsto (-1)^{k-1} v_n$ and $v_i \mapsto v_{i-1}, 2 \leq i \leq n$, and in terms of Plückers: $\langle I \rangle \mapsto \langle I - 1 \rangle$.
- (reflection) We define *reflection* as the map $\mathrm{refl} : \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,n}$ which sends $v_i \mapsto v_{n+1-i}$ and rescales a row by $(-1)^{\binom{k}{2}}$, and in terms of Plückers: $\langle I \rangle \mapsto \langle n + 1 - I \rangle$.
- (zero column) We define the map $\mathrm{pre}_i : \mathrm{Mat}_{k,[n]\setminus\{i\}} \to \mathrm{Mat}_{k,n}$ which adds a zero column at i, and in terms of Plückers: $\langle I \rangle \mapsto \langle I \rangle$.

Here, $I - 1$ is obtained from $I \in \binom{[n]}{k}$ by subtracting 1 (mod n) from each element of I and $n + 1 - I$ is obtained from I by subtracting each element of I from $n + 1$.

2.2 The amplituhedron

Building on [2, 18], Arkani-Hamed and Trnka [4] introduced the *(tree) amplituhedron*, which they defined as the image of the positive Grassmannian under a positive linear map. Let $\mathrm{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 2.4 (Amplituhedron)**.** Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$, where $k + m \leq n$. The *amplituhedron map* $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $\mathrm{Gr}_{k,n}^{\geq 0}$, and CZ is a $k \times (k + m)$ matrix representing an element of $\mathrm{Gr}_{k,k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}^{Z} \subset \mathrm{Gr}_{k,k+m}$ is the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$.

In this article we will be concerned with the case where $m = 4$.

[2] We will always assume that plabic graphs are *reduced* [27, Definition 12.5].

Definition 2.5 (Tiles)**.** Fix k, n, m with $k + m \leq n$ and choose $Z \in \mathrm{Mat}^{>0}_{n,k+m}$. Given a positroid cell S of $\mathrm{Gr}^{\geq 0}_{k,n}$, we let $Z^\circ_S := \tilde{Z}(S)$ and $Z_S := \overline{\tilde{Z}(S)} = \tilde{Z}(\overline{S})$. We call Z_S and Z°_S a *tile* and an *open tile* for $\mathcal{A}^Z_{n,k,m}$ if $\dim(S) = km$ and $\tilde{Z}$ is injective on S.

Definition 2.6 (Tilings)**.** A *tiling* of $\mathcal{A}^Z_{n,k,m}$ is a collection $\{Z_S \mid S \in \mathcal{C}\}$ of tiles, such that their union equals $\mathcal{A}^Z_{n,k,m}$ and the open tiles $Z^\circ_S, Z^\circ_{S'}$ are pairwise disjoint.

There is a natural notion of *facet* of a tile, generalizing the notion of facet of a polytope.

Definition 2.7 (Facet of a cell and a tile)**.** Given two positroid cells S' and S, we say that S' is a *facet* of S if $S' \subset \partial S$ and S' has codimension 1 in $\overline{S}$. If S' is a facet of S and Z_S is a tile of $\mathcal{A}^Z_{n,k,m}$, we say that $Z_{S'}$ is a *facet* of Z_S if $Z_{S'} \subset \partial Z_S$ and has codimension 1 in Z_S.

Definition 2.8 (Twistor coordinates)**.** Fix $Z \in \mathrm{Mat}^{>0}_{n,k+m}$ with rows $Z_1, \ldots, Z_n \in \mathbb{R}^{k+m}$. Given $Y \in \mathrm{Gr}_{k,k+m}$ with rows $y_1, \ldots, y_k$, and $\{i_1, \ldots, i_m\} \subset [n]$, we define the *twistor coordinate* $\langle\langle i_1 i_2 \cdots i_m \rangle\rangle$ to be the determinant of the matrix with rows $y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m}$.

Note that the twistor coordinates are defined only up to a common scalar multiple. An element of $\mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates [19]. Moreover, $\mathrm{Gr}_{k,k+m}$ can be embedded into $\mathrm{Gr}_{m,n}$ so that the twistor coordinate $\langle\langle i_1 \ldots i_m \rangle\rangle$ is the pullback of the Plücker coordinate $\langle i_1, \ldots, i_m \rangle$ in $\mathrm{Gr}_{m,n}$.

Definition 2.9. We refer to a homogeneous polynomial in twistor coordinates as a *functionary*. For $S \subseteq \mathrm{Gr}^{\geq 0}_{k,n}$, we say a functionary F has a definite sign $s \in \{\pm 1\}$ (or vanishes) on Z°_S if for all $Z \in \mathrm{Mat}^{>0}_{n,k+4}$ and for all $Y \in Z^\circ_S$, $F(Y)$ has sign s (or 0, respectively).

Functionaries will be crucial to describe tiles of the amplituhedron, to prove the main theorems about cluster adjacency and BCFW tilings, and to connect with cluster algebras.

2.3 Cluster Algebras

Cluster algebras were introduced by Fomin and Zelevinsky in [14], motivated by the study of total positivity; see [12] for an introduction. We give a quick definition of cluster algebras from quivers. All cluster algebras here will be of *geometric type*.

A *quiver* Q is an oriented graph given by a finite set of vertices. For a quiver without oriented cycles of length 1 and 2, one can define a *quiver mutation* $\mu_k(Q)$ at each vertex k of Q. This operation, described in [14], is an involution: $\mu^2_k(Q) = Q$.

Definition 2.10. Choose $s \geq r$ positive integers. Let $\mathcal{F}$ be an *ambient field* of rational functions in r independent variables over $\mathbb{C}(x_{r+1}, \ldots, x_s)$. A *labeled seed* in $\mathcal{F}$ is a pair $(\mathbf{x}, Q)$, where $\mathbf{x} = (x_1, \ldots, x_s)$ forms a free generating set for $\mathcal{F}$ and Q is a quiver with vertices $1, 2, \ldots, r$ called *mutable*, and vertices $r + 1, \ldots, s$ called *frozen*.

We call $\mathbf{x}$ a *cluster* and its elements $\{x_1, \ldots, x_s\}$ *cluster variables*. The variables $x_1, \ldots, x_r$ are called *mutable*, and the variables $c = \{x_{r+1}, \ldots, x_s\}$ are called *frozen*.

$$
\begin{tikzcd}
\langle 4\,5\,6\,7\rangle \arrow[dr] & & & & \\
& \langle 3\,5\,6\,7\rangle \arrow[r] \arrow[d] & \langle 3\,4\,6\,7\rangle \arrow[r] \arrow[d] & \langle 3\,4\,5\,7\rangle \arrow[r] \arrow[d] & \langle 3\,4\,5\,6\rangle \arrow[dl] \\
& \langle 2\,5\,6\,7\rangle \arrow[r] \arrow[d] & \langle 2\,3\,6\,7\rangle \arrow[r] \arrow[d] \arrow[ul] & \langle 2\,3\,4\,7\rangle \arrow[r] \arrow[d] \arrow[ul] & \langle 2\,3\,4\,5\rangle \arrow[ul] \\
& \langle 1\,5\,6\,7\rangle & \langle 1\,2\,6\,7\rangle \arrow[ul] & \langle 1\,2\,3\,7\rangle \arrow[ul] & \langle 1\,2\,3\,4\rangle \arrow[ul]
\end{tikzcd}
$$

Figure 1: The rectangle seed $\Sigma_{4,7}$. Mutable variables are in the colored box.

Definition 2.11 (*Seed mutations*)**.** Let $(\mathbf{x}, Q)$ be a labeled seed in $\mathcal{F}$, and let $k \in \{1, \ldots, r\}$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, Q)$ into the labeled seed $\mu_k(\mathbf{x}, Q) = (\mathbf{x}', \mu_k(Q))$, where the cluster $\mathbf{x}' = (x'_1, \ldots, x'_s)$ is defined as follows: $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_k\, x_k = \prod_{i:\, i \to k} x_i + \prod_{i:\, i \leftarrow k} x_i\,. \tag{2.1}$$

Where $i \to k$ (or $i \leftarrow k$) denotes an edge oriented from vertex i to k (or k to i). Note that one omits arrows between two frozen vertices as they do not affect seed mutation.

Definition 2.12. Let $\mathbb{T}_r$ be an *r-regular tree* whose edges are labeled by $1, \ldots, r$, so that edges emanating from each vertex receive different labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seed assigned to the endpoint of an edge k emanated from t is obtained by mutating Σ_t in direction k.

Definition 2.13 (*Cluster algebra*)**.** Given a cluster pattern, we denote as $\mathcal{X}$ the union of all mutable variables of all the seeds in the pattern. Let $\mathbb{C}[c^{\pm 1}]$ be the *ground ring* consisting of Laurent polynomials in the frozen variables. The *cluster algebra* $\mathcal{A}$ associated with a given pattern is the $\mathbb{C}[c^{\pm 1}]$-subalgebra of the ambient field $\mathcal{F}$ generated by all mutable variables, with coefficients which are Laurent polynomials in the frozen variables: $\mathcal{A} = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, Q)$, where $(\mathbf{x}, Q)$ is any seed in the underlying cluster pattern. We say that $\mathcal{A}$ has *rank* r because each cluster contains r mutable variables. Cluster variables that belong to a common cluster are said to be *compatible*.

The Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$ has a cluster structure [28], defined starting from particularly nice seeds called *rectangles seed* $\Sigma_{k,n}$, see Figure 1 and the exposition of [11].

Theorem 2.14 ([28])**.** *Let $\mathrm{Gr}^{\circ}_{k,n}$ be the open subset of the Grassmannian where the frozen variables don't vanish. Then the coordinate ring $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}]$ of the affine cone over $\mathrm{Gr}^{\circ}_{k,n}$ is the cluster algebra $\mathcal{A}(\Sigma_{k,n})$.*

Moreover, the operations on the Grassmannian $\mathrm{cyc}, \mathrm{refl}, \mathrm{pre}$ in Definition 2.3 induce maps on $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,N}]$ which are compatible with the cluster structure of Theorem 2.14:

Proposition 2.15. *The maps* $\mathrm{cyc}, \mathrm{refl} : \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}] \to \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}], \mathrm{pre}_i : \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,[n]\setminus\{i\}}] \to \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{k,n}]$ *take cluster variables to cluster variables and preserve compatibility and exchange relations.*

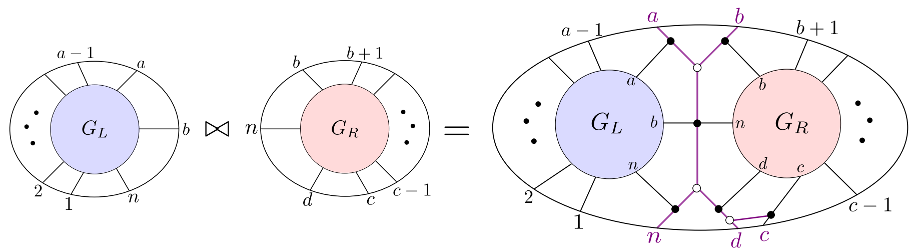

Figure 2: The BCFW product $S_L \bowtie S_R$ of S_L and S_R in terms of their plabic graphs.

3 Results

Our first main result is proving that a class of cells called *BCFW cells* give tiles for $\mathcal{A}^Z_{n,k,4}$. We will build BCFW cells recursively using the *BCFW product*. Let us first introduce some notation we will use throughout this section.

Notation 3.1. Choose integers $1 \le a < b < c < d < n$ with a, b and c, d, n consecutive. Let[3] $N_L = \{n, 1, 2, \dots, a, b\}, N_R = \{b, \dots, c, d, n\}$ and $D = (a, b, c, d, n)$. Also fix $k \le n$ and two nonnegative integers $k_L \le |N_L|$ and $k_R \le |N_R|$ such that $k_L + k_R + 1 = k$. Note that, for any set of indices $N \subset [n]$, our results hold with N instead of $[n]$, by replacing 1 and n in the definition with the smallest and largest elements of N, respectively.

Definition 3.2 (BCFW product)**.** Let $S_L \subseteq \mathrm{Gr}_{k_L,N_L}, S_R \subseteq \mathrm{Gr}^{\ge 0}_{k_R,N_R}$ as in Notation 3.1 and G_L, G_R be the respective plabic graphs. The *BCFW product* of S_L and S_R is the positroid cell $S_L \bowtie S_R \subseteq \mathrm{Gr}_{k,n}$ corresponding the plabic graph in the right-hand side of Figure 2. When it is not clear from the context, we will say $\bowtie$ is performed 'with indices D'.

We now introduce the family of *BCFW cells* to be the set of positroid cells which is closed under the operations in Definitions 2.3 and 3.2:

Definition 3.3 (BCFW cells)**.** The set of *BCFW cells* is defined recursively. For $k = 0$, let the trivial cell $\mathrm{Gr}^{>0}_{0,n}$ be a BCFW cell. If S is a BCFW cell, so is the cell obtained by applying cyc, refl, pre to S. If S_L, S_R are BCFW cells, so is their BCFW product $S_L \bowtie S_R$.

Example 3.4. For $k = 1$, the BCFW cells in $\mathrm{Gr}^{\ge 0}_{1,n}$ are as in Figure 3 (left). They have Plücker coordinates $\langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle, \langle e\rangle > 0$ and all others zero. In Figure 3 (right), $S_{ex} \subset \mathrm{Gr}^{\ge 0}_{2,7}$ is obtained as $S_L \bowtie S_R$, with S_L, S_R BCFW cells in $\mathrm{Gr}^{\ge 0}_{1,N_L}, \mathrm{Gr}^{\ge 0}_{0,N_R}$ respectively, with $N_L = \{7, 1, 2, 3, 4\}, N_R = \{4, 5, 6, 7\}$ and $D = (3, 4, 5, 6, 7)$.

Theorem 3.5 (BCFW tiles)**.** *The amplituhedron map is injective on each BCFW cell. That is, the closure $Z_S := \overline{\tilde{Z}(S)}$ of the image of a BCFW cell S is a* tile*, which we refer to as a* BCFW tile.

[3]Note that we will overload the notation and let n index an element of a vector space basis for different vector spaces; however, in what follows, the meaning should be clear from context.

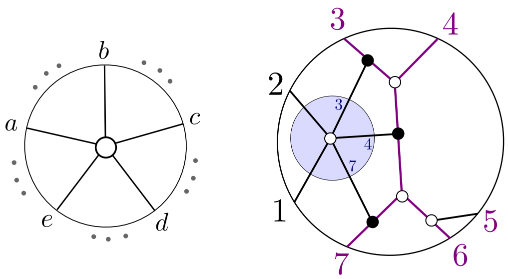

Figure 3: Plabic graphs of: a BCFW cell in $\mathrm{Gr}_{1,n}^{\geq 0}$ (left); a BCFW cell $S_{ex} \subset \mathrm{Gr}_{2,7}^{\geq 0}$ (right).

A key ingredient to prove Theorem 3.5 is inverting the amplituhedron map on BCFW tiles [9, Theorem 7.7] by using *product promotion* – an operation which interacts nicely both with the cluster structure on the Grassmannian and with the BCFW product.

Definition 3.6. Using Notation 3.1, *product promotion* is the homomorphism

$$\Psi_D = \Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_L}) \times \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_R}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n}),$$

induced by the following substitution:

$$\text{on } \widehat{\mathrm{Gr}}_{4,N_L}\colon b \mapsto \frac{(ba)\cap(cdn)}{\langle a\,c\,d\,n\rangle}, \quad \text{on } \widehat{\mathrm{Gr}}_{4,N_R}\colon n \mapsto \frac{(ba)\cap(cdn)}{\langle a\,b\,c\,d\rangle}, \; d \mapsto \frac{(dc)\cap(abn)}{\langle a\,b\,c\,n\rangle}.$$

The vector $(ij)\cap(rsq) := v_i\langle j\,r\,s\,q\rangle - v_j\langle i\,r\,s\,q\rangle = -v_r\langle i\,j\,s\,q\rangle + v_s\langle i\,j\,r\,q\rangle - v_q\langle i\,j\,r\,s\rangle$ is in the intersection of the 2-plane and the 3-plane spanned by v_i, v_j and v_r, v_s, v_q, respectively.

We show[4] that Ψ is in fact a *quasi-homomorphism* (see [15]) from the cluster algebra[5] $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$ to a sub-cluster algebra of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,n}]$. See [15, Definition 3.1, Proposition 3.2] for the precise definition of a quasi-homomorphism.

Theorem 3.7. *Product promotion Ψ is a quasi-homomorphism of cluster algebras. In particular, Ψ maps a cluster variable (respectively, cluster) of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$, to a cluster variable (respectively, sub-cluster) of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,n}]$, up to multiplication by Laurent monomials in $\mathcal{T}' := \{\langle a\,b\,c\,n\rangle, \langle a\,b\,c\,d\rangle, \langle b\,c\,d\,n\rangle, \langle a\,c\,d\,n\rangle\}$.*

Remark 3.8. Definition 3.6 and Theorem 3.7 extend also to the degenerate cases, e.g. for $a = 1$ (*upper promotion*), where $\Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_R}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n})$, see [9, Section 4.3].

Definition 3.9. Let x be a cluster variable of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}]$ or $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$. We define the *rescaled product promotion* $\overline{\Psi}(x)$ of x to be the cluster variable of $\mathrm{Gr}_{4,n}$ obtained from $\Psi(x)$ by removing[6] the Laurent monomial in $\mathcal{T}'$ (c.f. Theorem 3.7).

[4] We will sometime omit the dependence on the indices $D = \{a, b, c, d, n\}$ in Ψ (and $\overline{\Psi}$) for brevity.

[5] $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$ is a cluster algebra where each seed is the disjoint union of a seed of each factor.

[6] If $x = \langle bcdn\rangle$, then $\overline{\Psi}(x) = \Psi(x) = x$.

The fact that product promotion is a cluster quasi-homomorphism may be of independent interest in the study of the cluster structure on $\mathrm{Gr}_{4,n}$. Much of the work thus far on the cluster structure of the Grassmannian has focused on cluster variables which are polynomials in Plücker coordinates with low degree; by constrast, the cluster variables we obtain can have arbitrarily high degree in Plücker coordinates (e.g. see the *chain polynomials* in [9, Theorem 8.3]). We introduce the following notation:

$$\langle a\,b\,c\,|\,d\,e\,|\,f\,g\,h\rangle := \langle a,b,c,(d\,e)\cap(f\,g\,h)\rangle = \langle a\,b\,c\,d\rangle\,\langle e\,f\,g\,h\rangle - \langle a\,b\,c\,e\rangle\,\langle d\,f\,g\,h\rangle. \tag{3.1}$$

Example 3.10. For N_L and N_R as in Example 3.4, the only Plücker which changes is: $\Psi(\langle 1\,2\,4\,7\rangle) = \langle 1\,2\,7|3\,4|5\,6\,7\rangle/\langle 3\,4\,6\,7\rangle$, and $\overline{\Psi}(\langle 1\,2\,4\,7\rangle) = \langle 1\,2\,7|3\,4|5\,6\,7\rangle$ which is a quadratic cluster variable in $\mathrm{Gr}_{4,7}$, e.g. obtained by mutating $\langle 2367\rangle$ in $\Sigma_{4,7}$ of Figure 1.

Using rescaled product promotion and the operations in Proposition 2.15, we associate to each BCFW tile Z_S a collection of compatible cluster variables $\mathbf{x}(S)$ for $\mathrm{Gr}_{4,n}$.

Definition 3.11 (Cluster variables for BCFW tiles)**.** Let $S \subset \mathrm{Gr}_{k,n}^{\geq 0}$ be a BCFW cell. We define the set of *coordinate cluster variables* $\mathbf{x}(S)$ for S recursively as follows:

- If $S = S_L \bowtie S_R$ with indices $D_k = (a_k, b_k, c_k, d_k, n_k)$, then
$$\mathbf{x}(S) = \overline{\Psi}_{D_k}(\mathbf{x}(S_L) \cup \mathbf{x}(S_R)) \cup \{\langle I\rangle, I \in \binom{D_k}{4}\}, \tag{3.2}$$
- If $S = \begin{cases} \mathrm{pre}_i(S') \\ \mathrm{cyc}(S') \\ \mathrm{refl}(S') \end{cases}$ then $\mathbf{x}(S) = \begin{cases} \mathbf{x}(S') \\ \mathrm{cyc}^{-1}(\mathbf{x}(S')) \\ \mathrm{refl}(\mathbf{x}(S')) \end{cases}$,

and for the base case $k = 0$, we set $\mathbf{x}(S) = \emptyset$. Here, $\mathrm{cyc}^{-1} = \mathrm{cyc}^{n-1}$.

For a BCFW cell S, $\mathbf{x}(S)$ depends on the sequence of operations in Definition 3.3 used to build S, but we will drop this dependence for brevity.

Note that $\mathbf{x}(S)$ is a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$ [9, Lemma 7.6].

Example 3.12. From Example 3.4, $S_{ex} = S_L \bowtie S_R$ and $\mathbf{x}(S_L) = \{\langle I\rangle, I \in \binom{D_L}{4}\}, \mathbf{x}(S_R) = \emptyset$, where $D_L = \{1,2,3,4,7\}$. Then by Example 3.10 the coordinate cluster variables $\mathbf{x}(S_{ex})$ are: $\overline{\Psi}(\mathbf{x}(S_L)) = \mathbf{x}(S_L) \setminus \{\langle 1247\rangle\} \cup \{\langle 127|34|567\rangle\}$ together with $\{\langle I\rangle, I \in \binom{D}{4}\}$.

Given a cluster variable x in $\mathrm{Gr}_{4,n}$, we will denote as $x(Y)$ the functionary on $\mathrm{Gr}_{k,k+4}$ (cf. Definition 2.9) obtained by identifying Plücker coordinates $\langle I\rangle$ in $\mathrm{Gr}_{4,n}$ with twistor coordinates $\langle\langle I\rangle\rangle$ in $\mathrm{Gr}_{k,k+4}$ (cf. Definition 2.8). Then interpreting each cluster variable as a functionary, we describe each BCFW tile as the semialgebraic subset of $\mathrm{Gr}_{k,k+4}$ where the coordinate cluster variables take on particular signs.

Theorem 3.13 (Sign description of BCFW tiles)**.** *Let Z_S be a BCFW tile. For each element x of $\mathbf{x}(S)$, the functionary $x(Y)$ has a definite sign s_x on Z_S° and*

$$Z_S^\circ = \{Y \in \mathrm{Gr}_{k,k+4} : s_x\, x(Y) > 0 \text{ for all } x \in \mathbf{x}(S)\}.$$

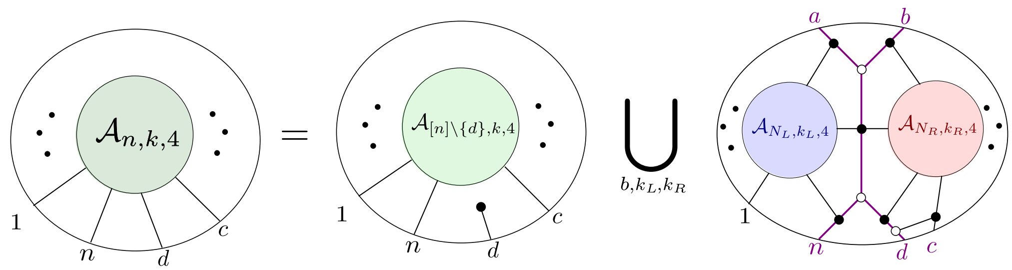

Figure 4: BCFW tiling for $\mathcal{A}_{n,k,4}$. On the right: the first term is obtained by tiling $\mathcal{A}_{[n]\setminus\{d\},k,4}$ (from $\mathcal{T}_{pre}$); the second term is the union over b, k_L, k_R as in Definition 3.16 of the collections of tiles obtained by tiling $\mathcal{A}_{N_L,k_L,4}$ and $\mathcal{A}_{N_R,k_R,4}$ (from $\mathcal{T}_{k_L,k_R,b,n}$).

Example 3.14. The open tile $Z^\circ_{ex} := \tilde{Z}(S_{ex})$, with S_{ex} from Example 3.4, is the semialgebraic set in $\mathrm{Gr}_{2,6}$ where the functionaries $x(Y)$, with $x \in \mathbf{x}(S_{ex})$ of Example 3.12 are positive, except when $x \in \{\langle 3567\rangle, \langle 3457\rangle, \langle 2347\rangle, \langle 3567\rangle\}$, for which $x(Y)$ are negative.

One can study facets of tiles (see Definition 2.7) by describing associated functionaries which vanish on them. Given a functionary $F(\langle\langle I\rangle\rangle)$ on $\mathrm{Gr}_{k,k+4}$, we can obtain a polynomial $F(\langle I\rangle)$ in the Plücker coordinates in $\mathrm{Gr}_{4,n}$. Then the coordinate cluster variables in Definition 3.11 are a key tool in the proof of cluster adjacency conjecture for BCFW tiles:

Theorem 3.15 (Cluster adjacency for BCFW tiles)**.** *Let Z_S be a BCFW tile of $\mathcal{A}^Z_{n,k,4}$. Each facet $Z_{S'}$ of Z_S lies on a hypersurface cut out by a functionary $F_{S'}(\langle\langle I\rangle\rangle)$ such that $F_{S'}(\langle I\rangle)$ is in $\mathbf{x}(S)$. Thus $\{F_{S'}(\langle I\rangle) : Z_{S'}$ a facet of $Z_S\}$ is a collection of compatible cluster variables of* $\mathrm{Gr}_{4,n}$.

Finally, we show how to use BCFW tiles to tile $\mathcal{A}^Z_{n,k,4}$ (Definition 2.6). Theorems 3.5 and 3.13 are important ingredients to prove our last main result Theorem 3.17. We use Notation 3.1, fix $n \geq k+4$, and define $b_{min} := 2$ if $k_L = 0$ and otherwise $b_{min} := k_L + 3$.

Definition 3.16 (BCFW collections)**.** We say that a collection $\mathcal{T}$ of $4k$-dimensional BCFW cells in $\mathrm{Gr}^{\geq 0}_{k,n}$ is a *BCFW collection of cells* for $\mathcal{A}_{n,k,4}$ if it has the following recursive form:

- If $k = 0$ or $k = n-4$, $\mathcal{T}$ is the single BCFW cell $\mathrm{Gr}^{>0}_{0,n}$ or $\mathrm{Gr}^{>0}_{n-4,n}$, respectively.
- If $\mathcal{T} = \{S\}$ is a BCFW collection of cells, so is $\{\mathrm{refl}\, S\}_{S\in\mathcal{T}}$ and $\{\mathrm{cyc}\, S\}_{S\in\mathcal{T}}$.
- Otherwise $\mathcal{T} = \mathcal{T}_{pre} \bigsqcup_{b,k_L,k_R} \mathcal{T}_{k_L,k_R,b,n}$, where
  - b ranges from b_{min} to $n-3-k_R$, and k_L, k_R as in Notation 3.1;
  - $\mathcal{T}_{pre} = \{\mathrm{pre}_d(S)\}_{S\in\mathcal{C}}$, where $\mathcal{C}$ is a BCFW collection of cells for $\mathcal{A}_{[n]\setminus\{d\},k,4}$;
  - $\mathcal{T}_{k_L,k_R,b,n} = \{S_L \bowtie S_R\}_{(S_L,S_R)\in\mathcal{C}_L\times\mathcal{C}_R}$ where $\mathcal{C}_L$ and $\mathcal{C}_R$ are BCFW collections of cells for $\mathcal{A}_{N_L,k_L,4}$ and $\mathcal{A}_{N_R,k_R,4}$.

Theorem 3.17 (BCFW tilings)**.** *Every BCFW collection of cells $\mathcal{T} = \{S\}$ as in Definition 3.16 gives rise to a tiling $\{Z_S\}_{S\in\mathcal{T}}$ of the amplituhedron $\mathcal{A}^Z_{n,k,4}$, which we refer to as a* BCFW tiling.

See Figure 4 for an illustration. This generalizes the main result of [10], which proved the same result for the *standard* BCFW cells, and proves the main conjecture of [4].

Non-BCFW tiles are also expected to satisfy cluster adjacency, have a sign description in terms of cluster variables, and appear in tilings of $\mathcal{A}^Z_{n,k,4}$, see [9, Section 12.2].

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Quasisymmetric expansion of Hall-Littlewood symmetric functions

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Abstract. In our previous works we introduced a q-deformation of the generating functions for enriched P-partitions. We call the evaluation of this generating functions on labelled chains, the q-fundamental quasisymmetric functions. These functions interpolate between Gessel's fundamental ($q = 0$) and Stembridge's peak ($q = 1$) functions, the natural quasisymmetric expansions of Schur and Schur's Q-symmetric functions. In this paper, we show that our q-fundamental functions provide a quasisymmetric expansion of Hall-Littlewood S-symmetric functions with parameter $t = -q$.

Résumé. Dans nos travaux précédents, nous avons introduit une q-déformation des fonctions génératrices pour les P-partitions enrichies. Nous nommons l'évaluation de ces fonctions génératrices sur les chaînes étiquetées, les fonctions quasisymétriques q-fondamentales. Ces fonctions interpolent entre les fonctions fondamentales de Gessel ($q = 0$) et les fonctions de pics de Stembridge ($q = 1$) qui sont les expansions quasisymétriques naturelles des fonctions symétriques de Schur et Q Schur. Dans cet article, nous montrons que nos fonctions q-fondamentales fournissent une expansion quasisymétrique des fonctions symétriques Hall-Littlewood S avec paramètre $t = -q$.

Keywords: Hall-Littlewood, quasisymmetric functions, enriched P-partitions

1 Introduction

We define the q-fundamental quasisymmetric functions as the q-deformed generating functions for enriched P-partitions on labelled chains [5, 6]. These functions naturally interpolate between I. Gessel's fundamental ([1], $q = 0$) and J. Stembridge's peak ([12], $q = 1$) quasisymmetric functions and exhibit most of the nice properties of these two classical families. In particular, when q is not a complex root of unity they span the ring of quasisymmetric functions (QSym). When q is a root of unity, a subfamily of our q-fundamentals is the basis of the algebra of extended peaks [6], a proper subalgebra of QSym that coincides with Stembridge's algebra of peaks when $q = 1$. Fundamental and peak functions indexed by standard Young tableaux of shape λ are respectively the

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quasisymmetric expansions of Schur and Schur's Q-symmetric functions indexed by λ. Finding the analogous families of symmetric functions for general q appears as a natural question. We find out that q-fundamental functions provide a similar quasisymmetric expansion of the family $(S_\lambda(X;t))_\lambda$, the Hall-Littlewood S-symmetric functions with parameter $t = -q$. After recalling the required definitions, we state and prove our main result. Finally, we look at some important consequences regarding the quasisymmetric extension of the classical homorphism between Λ, the algebra of symmetric functions and the subalgebra of Λ spanned by Hall-Littlewood functions as well as some Cauchy like formulas for the $S_\lambda(X;t)$'s.

1.1 Integer partitions, Young tableaux and permutation statistics

Let $\mathbb{P}$ be the set of positive integers and $\mathbb{P}^{\pm}$ be the set of positive and negative integers ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \ldots$. We embed $\mathbb{P}$ into $\mathbb{P}^{\pm}$ and let $-\mathbb{P} \subseteq \mathbb{P}^{\pm}$ be the set of all $-n$ for $n \in \mathbb{P}$. For $n \in \mathbb{P}$ write $[n] = \{1, \ldots, n\}$ and $\mathfrak{S}_n$ the symmetric group on $[n]$. A *partition* λ of an integer n, denoted $\lambda \vdash n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of $\ell(\lambda) = p$ parts sorted in decreasing order such that $|\lambda| = \sum_i \lambda_i = n$. We denote the one part partition (n) simply n. A partition λ is represented as a Young diagram of $n = |\lambda|$ boxes arranged in $\ell(\lambda)$ left justified rows so that the i-th row from the top contains λ_i boxes. Given a second partition μ with $\ell(\mu) \leq \ell(\lambda)$ such that $\mu_i \leq \lambda_i$, $(i \leq \ell(\mu))$ delete the μ_i leftmost boxes of the i-th row to get the diagram of *shape* λ/μ. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a *semistandard Young tableau*. If the entries are consecutive and strictly increasing along the rows, we call it a *standard Young tableau* and we denote $SYT(\lambda/\mu)$ (resp. $SSYT(\lambda/\mu)$) the set of standard (resp. semistandard) Young tableaux of shape λ/μ. A *marked semistandard Young tableau* is a Young diagram filled with integers in $\mathbb{P}^{\pm}$ such that the entries are increasing along rows and columns and such that each row contains at most once each negative integer and that each column contains at most once each positive integer. Denote $SSYT^{\pm}(\lambda/\mu)$ the

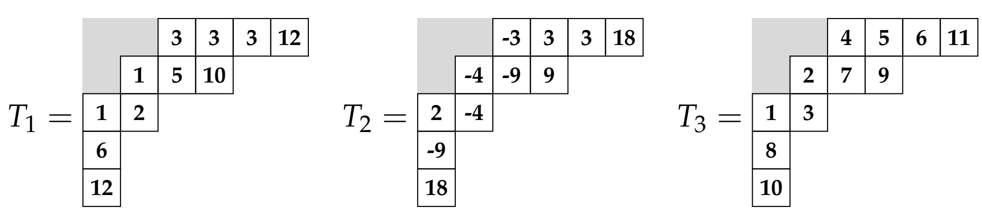

Figure 1: A semistandard, marked semistandard and standard tableau of shape $(6,4,2,1,1)/(2,1)$. The descent set of T_3 is $\{2,6,7,9\}$ while its peak set is $\{2,6,9\}$. T_2 has $neg(T_2) = 5$ negative entries.

set of marked semistandard Young tableaux of shape λ/μ. Define the *descent set of a standard Young tableau* T as $Des(T) = \{1 \leq i \leq n-1 \mid i$ is in a strictly higher row than $i+1\}$ and it *peak set* as $Peak(T) = \{2 \leq i \leq n-1 \mid i \in Des(T)$ and $i-1 \notin Des(T)\}$. Finally, denote the number of negative entries of a marked tableau T as $neg(T)$.

Example 1. *Figure 1 depicts a semistandard, a marked semistandard and a standard Young tableau with their shape and descent and peak set.*

Similarly, the descent set and peak set of a permutation π in $\mathfrak{S}_n$ are the subsets of $[n-1]$ defined as $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ and $\mathrm{Peak}(\pi) = \{2 \leq i \leq n-1 | \pi(i-1) < \pi(i) > \pi(i+1)\}$. Finally the *Robinson-Schensted (RS) correspondence* ([9, 10]) is a bijection between permutations π in $\mathfrak{S}_n$ and ordered pairs of standard Young tableaux (P, Q) of the same shape $\lambda \vdash n$. This bijection is descent preserving in the sense that $Des(\pi) = Des(Q)$, and $Des(\pi^{-1}) = Des(P)$.

1.2 Hall-Littlewood symmetric functions

Consider the set of indeterminates $X = \{x_1, x_2, x_3, \ldots\}$. Let Λ denote the ring of symmetric functions over $\mathbb{C}$. We use notations consistent with [7]. Namely, for $\lambda \vdash n$, denote $m_\lambda(X)$, $h_\lambda(X)$, $e_\lambda(X)$, $p_\lambda(X)$ and $s_\lambda(X)$ the *monomial*, *complete homogeneous*, *elementary*, *power sum* and *Schur* symmetric functions over X indexed by λ. Fix a parameter $t \in \mathbb{C}$ and define $q_n(X;t) \in \Lambda$ as $q_0(X;t) = 1$ and for any positive integer n as:

$$q_n(X;t) = (1-t)\sum_i x_i^n \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}. \tag{1.1}$$

The generating function for the q_n is given by

$$\sum_{n \geq 0} q_n(X;t)u^n = \prod_i \frac{1 - x_i tu}{1 - x_i u}. \tag{1.2}$$

The family $(q_n(X;t))_n$ generates a subalgebra of Λ that we denote Λ_t. In particular, Λ_t is a proper subalgebra of Λ when t is a root of unity.

Definition 1 (Hall-Littlewood S-symmetric functions)**.** *Let λ/μ be a skew shape, define the Hall-Littlewood S-symmetric functions indexed by λ/μ as*

$$S_{\lambda/\mu}(X;t) = det\left(q_{\lambda_i - \mu_j - i + j}(X;t)\right)_{i,j} \tag{1.3}$$

As a direct consequence of Definition 1, setting $t = 0$ yields $S_{\lambda/\mu}(X;0) = s_{\lambda/\mu}(X)$. When $t = -1$, $S_{\lambda/\mu}(X;-1)$ is a variant of *Schur's Q-function* indexed by λ/μ. We end this section with the definition of a classical ring homomorphism.

Definition 2 (Ring homomorphism)**.** *Define a ring homomorphism θ_t: $\Lambda \longrightarrow \Lambda_t$ by setting for any non-negative integer n,*

$$\theta_t(h_n)(X) = q_n(X;t).$$

In particular, one has $\theta_t(e_n)(X) = q_n(X;t)$, $\theta_t(p_n)(X) = (1-t^n)p_n(X)$ and, as a consequence of Definition 1,

$$\theta_t(s_{\lambda/\mu})(X) = S_{\lambda/\mu}(X;t).$$

1.3 Enriched P-partitions and q-deformed generating functions

We recall the main definitions regarding weighted posets, enriched P-partitions and their q-deformed generating functions. See [1, 4, 5, 11, 12] for more details.

Definition 3 (Labelled weighted poset, [4])**.** *A* labelled weighted poset *is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a* labelled poset, *i.e., an arbitrary partial order $<_P$ on the set $[n]$ and $\epsilon : [n] \longrightarrow \mathbb{P}$ is a map (called the* weight function*). If $\epsilon(i) = 1$ for all $i \in [n]$, we may simply omit it.*

Each node of a labelled weighted poset is marked with its label and weight (Figure 2).

$$\begin{tikzcd}
 & 2,\ \epsilon(2)=5 & & \\
3,\ \epsilon(3)=2 \arrow[ur] & & 1,\ \epsilon(1)=1 \arrow[ul] \arrow[r] & 4,\ \epsilon(4)=2 \\
 & 5,\ \epsilon(5)=2 \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Figure 2: A 5-vertex labelled weighted poset. Arrows show the covering relations.

Definition 4 (Enriched P-partition, [12])**.** *Given a labelled weighted poset $P = ([n], <_P, \epsilon)$, an* enriched P-partition *is a map $f : [n] \longrightarrow \mathbb{P}^{\pm}$ that satisfies the two following conditions:*

(i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.

(ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.

We let $\mathcal{L}_{\mathbb{P}^{\pm}}(P)$ denote the set of enriched P-partitions.

Definition 5 (q-Deformed generating function, [5])**.** *Consider the ring $\mathbb{C}[[X]]$ of formal power series on X and let $q \in \mathbb{C}$ be an additional parameter. Given a labelled weighted poset $([n], <_P, \epsilon)$, define its generating function $\Gamma^{(q)}([n], <_P, \epsilon) \in \mathbb{C}[[X]]$ as*

$$\Gamma^{(q)}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P, \epsilon)} \prod_{1 \le i \le n} q^{[f(i)<0]} x_{|f(i)|}^{\epsilon(i)},$$

where $[f(i) < 0] = 1$ if $f(i) < 0$ and 0 otherwise.

Finally, let $X^{\pm} = \{x_{-1}, x_1, x_{-2}, x_2, \ldots\}$. In the sequel we denote ϖ the homomorphism $\varpi : \mathbb{C}[[X^{\pm}]] \longrightarrow \mathbb{C}[[X]]$ defined by setting $\varpi(x_i) = q^{[i<0]} x_{|i|}$ for $x_i \in X^{\pm}$.

1.4 q-fundamental quasisymmetric functions

We recall results from [5] and [6].

Definition 6 (q-Fundamental quasisymmetric functions)**.** *Given a permutation $\pi = \pi_1 \ldots \pi_n$ of $\mathfrak{S}_n$, we let $P_\pi = ([n], <_\pi, 1^n)$ be the labelled weighted poset on the set $[n]$, where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$ and where all the weights are equal to 1 (see Figure 3). Define the q-*fundamental quasisymmetric function

$$L_\pi^{(q)} = \Gamma^{(q)}([n], <_\pi, 1^n).$$

$$\pi_1 \longrightarrow \pi_2 \longrightarrow \cdots\cdots\cdots \longrightarrow \pi_n$$

Figure 3: The labelled weighted poset P_π.

The q-fundamental quasisymmetric functions belong to the subalgebra of $\mathbb{C}\,[[X]]$ called the ring of *quasisymmetric functions (*QSym*)*, i.e. for any strictly increasing sequence of indices $i_1 < i_2 < \cdots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$. The specialisations of $L_\pi^{(q)}$ to $L_\pi = L_\pi^{(0)}$ and $K_\pi = L_\pi^{(1)}$ are respectively the Gessel's fundamental [1] and Stembridge's peak [12] quasisymmetric functions indexed by permutation π. We have the following explicit expression.

$$L_\pi^{(q)} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} q^{|\{j \in \mathrm{Des}(\pi) | i_j = i_{j+1}\}|} (q+1)^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1} x_{i_2} \ldots x_{i_n}. \tag{1.4}$$

Furthermore q-fundamental quasisymmetric functions admit a closed-form product and coproduct.

Proposition 1. *Let $q \in \mathbb{C}$, let π and σ be two permutations in $\mathfrak{S}_n$ and $\mathfrak{S}_m$. The product of $L_\pi^{(q)}$ and $L_\sigma^{(q)}$ is given by*

$$L_\pi^{(q)} L_\sigma^{(q)} = \sum_{\tau \in \pi \sqcup\!\sqcup \overline{\sigma}} L_\tau^{(q)}, \tag{1.5}$$

where $\overline{\sigma} = n + \sigma_1\, n + \sigma_2 \ldots n + \sigma_m$. Moreover, the coproduct $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ of the Hopf algebra QSym *(see [3, §5.1]) acts on the q-fundamental quasisymmetric functions as follows:*

$$\Delta(L_\pi^{(q)}) = \sum_{i=0}^{n} L^{(q)}_{\mathrm{std}(\pi_1\pi_2\ldots\pi_i)} \otimes L^{(q)}_{\mathrm{std}(\pi_{i+1}\pi_{i+2}\ldots\pi_n)}.$$

Here, if γ is a sequence of non-repeating integers, $\mathrm{std}(\gamma)$ *is the permutation whose values are in the same relative order as the entries of γ.*

According to Equation (1.4), $L_\pi^{(q)}$ depends only on the descent set of π. We reindex our q-fundamentals by an integer n and a subset of $[n-1]$. We recall two significant results.

Proposition 2 ([5]). *$(L_{n,I}^{(q)})_{n\geq 0, I\subseteq[n-1]}$ is a basis of* QSym *if and only if $q \in \mathbb{C}$ is not a root of unity.*

Proposition 3 ([6]). *Let $p \in \mathbb{P}$ and $\rho_p \in \mathbb{C}$ such that $-\rho_p$ is a primitive $p+1$-th root of unity. For a subset I of $[n-1]$, write $I \subseteq_p [n-1]$ if $I \cup \{0\}$ does not contain more than p consecutive elements. Then $(L_{n,I}^{(\rho_p)})_{n\geq 0, I\subseteq_p[n-1]}$ is a basis of a proper subalgebra $\mathcal{P}^p$ of* QSym.

For general $q \in \mathbb{C}$ denote $\mathcal{P}^{(q)}$ the subalgebra of QSym spanned by the $(L_{n,I}^{(q)})_{n,I}$. If q is not a root of unity then $\mathcal{P}^{(q)} = \mathrm{QSym}$. If $q = \rho_p$ for some $p \in \mathbb{P}$ then $\mathcal{P}^{(q)} = \mathcal{P}^p$.

2 Relating Hall-Littlewood and q-fundamentals functions

The ring of symmetric functions Λ is a subalgebra of QSym and any symmetric function may be expanded in quasisymmetric bases. The relation between Schur functions (i.e Hall-Littlewood S-functions with parameter $t = 0$) and fundamental quasisymmetric functions is of particular interest. Let λ/μ be a skew shape, Gessel shows in [1]

$$S_{\lambda/\mu}(X;0) = s_{\lambda/\mu}(X) = \sum_{T\in SYT(\lambda/\mu)} L_{Des(T)}^{(0)}(X). \tag{2.1}$$

On the other hand, Stembridge shows in [12] that

$$S_{\lambda/\mu}(X;-1) = \sum_{T\in SYT(\lambda/\mu)} L_{Des(T)}^{(1)}(X). \tag{2.2}$$

As a result, understanding how these relations generalise for general q seems to be a very legitimate question. We state our result and some significant consequences.

2.1 Computing the q-deformed generating functions on skew diagrams

Let $n \in \mathbb{P}$ and λ and μ be two partitions such that λ/μ is a skew shape and $|\lambda| - |\mu| = n$. Label the skew Young diagram of shape λ/μ with the successive integers of $[n]$ from left to right and bottom to top. Define the partial order $<_{\lambda/\mu}$ on $[n]$ as $i <_{\lambda/\mu} j$ if and only if i lies northwest of j and denote the labelled poset $P_{\lambda/\mu} = ([n], <_{\lambda/\mu})$. As a direct consequence the set of enriched $P_{\lambda/\mu}$-partitions are precisely the marked semistandard Young tableaux of shape λ/μ, i.e. $\mathcal{L}_{\mathbb{P}^\pm}(P_{\lambda/\mu}) = SSYT^\pm(\lambda/\mu)$.

$$
\begin{tikzcd}
 & & 8 \arrow[r] \arrow[d] & 9 \arrow[r] \arrow[d] & 10 \arrow[r] & 11 \\
 & 5 \arrow[r] \arrow[d] & 6 \arrow[r] & 7 & & \\
3 \arrow[r] \arrow[d] & 4 & & & & \\
2 \arrow[d] & & & & & \\
1 & & & & &
\end{tikzcd}
$$

Figure 4: The labelled weighted poset $P_{(6,4,2,1,1)/(2,1)}$.

Theorem 1. *Let $n \in \mathbb{P}$ and λ/μ be a skew shape such that $|\lambda| - |\mu| = n$. The q-deformed generating function of $P_{\lambda/\mu}$ is exactly the Hall-Littlewood S-symmetric function with parameter $t = -q$.*

$$S_{\lambda/\mu}(X; -q) = \Gamma^{(q)}([n], <_{\lambda/\mu}). \tag{2.3}$$

The proof is postponed to Section 3. As a consequence to Theorem 1, we give an explicit quasisymmetric expansion of the Hall-Littlewood S-symmetric functions that is a natural generalisation of Equations (2.1) and (2.2).

Theorem 2. *Let λ/μ be a skew shape. The Hall-Littlewood S-symmetric function with parameter $t = -q$ is related to q-fundamental quasisymmetric functions through*

$$S_{\lambda/\mu}(X; -q) = \sum_{T \in SYT(\lambda/\mu)} L^{(q)}_{Des(T)}(X). \tag{2.4}$$

Proof. Let $n = |\lambda| - |\mu|$. Given a marked semistandard Young tableau $T \in SSYT^{\pm}(\lambda/\mu)$, define its standardisation as the standard tableau $T_0 \in SYT(\lambda/\mu)$ obtained by relabelling the boxes of T with the integers in $[n]$ such that:

- The entries of T and T_0 are in the same relative order
- Identical negative entries of T are relabelled from top to bottom
- Identical positive entries of T are relabelled from left to right.

Denote $T^{st} = T_0$. For instance, in Figure 1, $T_1^{st} = T_2^{st} = T_3$. Further denote $X^{|T|} = \prod_{i \in \mathbb{P}^{\pm}} x_{|i|}^{t_i}$ where t_i is the number of entries equal to i in T. Finally, use Theorem 1 to get

$$
\begin{aligned}
S_{\lambda/\mu}(X; -q) = \Gamma^{(q)}(P_{\lambda/\mu}) &= \sum_{T \in SSYT^{\pm}(\lambda/\mu)} q^{neg(T)} X^{|T|} \\
&= \sum_{T_0 \in SYT(\lambda/\mu)} \left(\sum_{T \in SSYT^{\pm}(\lambda/\mu),\, T^{st} = T_0} q^{neg(T)} X^{|T|} \right)
\end{aligned}
$$

End the proof by noticing that the part between parentheses is exactly $L^{(q)}_{Des(T)}(X)$. $\square$

Recall the ring homomorphism θ_t of Definition 2. The following result is a consequence of Theorem 2.

Theorem 3. *There is a ring homomorphism Θ_q: QSym $\longrightarrow \mathcal{P}^{(q)}$ such that for any positive integer n and any subset $I \subseteq [n-1]$, $\Theta_q\left(L^{(0)}_{n,I}\right) = L^{(q)}_{n,I}$. Then the restriction of Θ_q to Λ is exactly θ_{-q} and the ring map diagram of Figure 5 is commutative.*

$$\begin{tikzcd} \mathrm{QSym} \arrow[r, "\Theta_q"] & \mathcal{P}^{(q)} \\ \Lambda \arrow[u] \arrow[r, "\theta_{-q}"] & \Lambda_{-q} \arrow[u] \end{tikzcd}$$

Figure 5: Map diagram relating QSym, $\mathcal{P}^{(q)}$, Λ and Λ_{-q}. Vertical maps are inclusion.

Proof. The existence and proper definition of Θ_q is a consequence of Equation (1.5). To end the proof, it suffices to show that for any n non-negative integer, $\Theta_q(h_n)(X) = q_n(X;-q)$. Indeed, one has

$$\begin{aligned}\Theta_q(h_n)(X) = \Theta_q(s_n)(X) &= \Theta_q(L^{(0)}_{n,\emptyset})(X) \\ &= L^{(q)}_{n,\emptyset}(X) = S_n(X;-q) \\ &= q_n(X;-q)\end{aligned}$$

This is the desired result. $\square$

Remark 1. *Applying the morphism Θ_q to both the left-hand and right-hand sides of Equation (2.1) gives an alternative proof that $\theta_t\left(s_{\lambda/\mu}\right)(X) = S_{\lambda/\mu}(X;t)$. Indeed*

$$\Theta_q\left(s_{\lambda/\mu}\right)(X) = \sum_{T \in SYT(\lambda/\mu)} \Theta_q\left(L^{(0)}_{Des(T)}\right)(X) = \sum_{T \in SYT(\lambda/\mu)} L^{(q)}_{Des(T)}(X) = S_{\lambda/\mu}(X;-q)$$

2.2 Cauchy like formula for Hall-Littlewood symmetric functions

We use Theorem 2 to provide an alternative proof of a classical Cauchy like formula for Hall-Littlewood S-symmetric functions. Denote $Y = \{y_1, y_2, \ldots,\}$ an additional alphabet of commutating indeterminate independent of and commuting with X and denote the product alphabet $XY = \{x_i y_j\}_{i,j}$. We first show the following proposition.

Proposition 4. *Let $\pi \in \mathfrak{S}_n$ be a permutation. Extend the definition of $\Gamma^{(q)}$ to the alphabet XY by considering P_π-partitions $(f,g) : i \mapsto (f(i), g(i))$ with value in $\mathbb{P} \times \mathbb{P}^{\pm}$ that we equip with the lexicographic order. Assume also that for $(i,j) \in \mathbb{P} \times \mathbb{P}^{\pm}$, (i,j) is negative if and only if j is negative. We have*

$$\Gamma^{(q)}(P_\pi)(XY) = \sum_{(f,g)\in\mathcal{L}_{\mathbb{P}\times\mathbb{P}^{\pm}}([n],<_\pi)} \prod_{1\le i\le n} q^{[g(i)<0]} x_{f(i)} y_{|g(i)|}$$

The q-fundamental indexed by π on the product of indeterminate XY satisfies

$$L_\pi^{(q)}(XY) = \Gamma^{(q)}(P_\pi)(XY) = \sum_{\sigma\circ\tau=\pi} L_\sigma^{(0)}(X) L_\tau^{(q)}(Y). \tag{2.5}$$

Proof. The proof is similar to the one in [8, thm 6.11] and not detailed here. □

In [7, III. 4. Eq. (4.7)], Macdonald provides a Cauchy like formula for Hall-Littlewood symmetric functions.

$$q_n(XY;t) = \sum_{\lambda\vdash n} s_\lambda(X) S_\lambda(Y;t). \tag{2.6}$$

Proposition 5. *Equation (2.6) is a direct consequence of Proposition 4 and Theorem 2.*

Proof. Fix $q \in \mathbb{C}$ and use Proposition 4 to write

$$q_n(XY;-q) = L_{id_n}^{(q)}(XY) = \sum_{\sigma\in\mathfrak{S}_n} L_{\sigma^{-1}}^{(0)}(X) L_\sigma^{(q)}(Y),$$

where $id_n \in \mathfrak{S}_n$ is the identity permutation. The RS correspondence allows to reindex the sum over standard Young tableaux.

$$\begin{aligned} q_n(XY;-q) &= \sum_{\lambda\vdash n} \sum_{T,U\in SYT(\lambda)} L_{Des(T)}^{(0)}(X) L_{Des(U)}^{(q)}(Y) \\ &= \sum_{\lambda\vdash n} \left(\sum_{T\in SYT(\lambda)} L_{Des(T)}^{(0)}(X) \right) \left(\sum_{U\in SYT(\lambda)} L_{Des(U)}^{(q)}(Y) \right) \end{aligned}$$

Applying Theorem 2 yields Equation (2.6). □

3 Proof of Theorem 1

Let $\prec$ be a total order on $\mathbb{P}^{\pm}$. Define the binary relation R as follows. For any two elements $i, j \in \mathbb{P}^{\pm}$, set

$$(i\ R\ j) \iff (i \preccurlyeq j \text{ but not } i = j \in -\mathbb{P}).$$

We define a formal power series on the alphabet $X^{\pm} = \{x_{-1}, x_1, x_{-2}, x_2, \dots\}$.

Definition 7. *For each non-negative integer n, define the formal power series*

$$H_n(X^{\pm}) = \sum_{\substack{(i_1,i_2,\ldots,i_n)\in(\mathbb{P}^{\pm})^n; \\ i_1 \ R \ i_2 \ R \ \cdots \ R \ i_n}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

Moreover, set $H_n = 0$ for all $n < 0$.

Define an alternative version of the generating function for enriched P-partitions $\Gamma^{\pm}([n], <_P) \in \mathbb{C}[[X^{\pm}]]$ as

$$\Gamma^{\pm}([n], <_P) = \sum_{f\in\mathcal{L}_{\mathbb{P}^{\pm}}([n],<_P)} \prod_{1\le i\le n} x_{f(i)}.$$

Recall the homomorphism $\varpi : \mathbb{C}[[X^{\pm}]] \longrightarrow [[X]]$, such that $\varpi(x_i) = q^{[i<0]}x_{|i|}$ for $x_i \in X^{\pm}$. Clearly

$$\varpi(\Gamma^{\pm}([n], <_P)) = \Gamma^{(q)}([n], <_P).$$

Proposition 6. *Let λ and μ be two partitions such that λ/μ is a skew shape. We have*

$$\Gamma^{\pm}([n], <_{\lambda/\mu}) = \det\left(H_{\lambda_i-\mu_j-i+j}\right)_{i,j\in[k]} \tag{3.1}$$

Proof. We want to apply [2, §7]. To this end, we introduce a new relation. Let $\overline{R}$ be the complement of the binary relation R. (Thus, $\overline{R}$ is the binary relation on $\mathbb{P}^{\pm}$ defined by $\left(i\ \overline{R}\ j\right) \iff (\text{not } i\ R\ j)$.) It is easy to see that both relations R and $\overline{R}$ are transitive. Hence, the relation R is semitransitive (meaning that if $a, b, c, d \in \mathbb{P}^{\pm}$ satisfy $a\ R\ b\ R\ c$, then $a\ R\ d$ or $d\ R\ c$). Therefore, [2, Theorem 11] yields that the power series $s^R_{\lambda/\mu}$ (defined in [2, §7]) counts R-tableaux of shape λ/μ. But the R-tableaux of shape λ/μ are precisely the enriched $P_{\lambda/\mu}$-partitions. $\square$

In order to prove Theorem 1 from Equation (3.1), we need to show that for any non-negative integer n, $\varpi(H_n(X^{\pm})) = q_n(X; -q)$. We proceed in three steps. First we have the following proposition.

Proposition 7. *Let $n \in \mathbb{N}$. Then,*

$$H_n = \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \left(\prod_{u\in U} x_u\right)\left(\prod_{v\in V} x_v\right)$$

(where the product over $v \in V$ takes each element with its multiplicity). In particular, H_n does not depend on the order $\prec$.

Secondly, we express $q_n(X; -q)$ in terms of elementary and complete homogeneous symmetric functions.

Lemma 1. *Let n be a non-negative integer and $q \in \mathbb{C}$.*

$$q_n(X;-q) = \sum_{k=0}^{n} q^k e_k h_{n-k}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}
\sum_n q_n(X;-q)u^n &= \prod_{i\geq 1} \frac{1+qx_iu}{1-x_iu} = \underbrace{\left(\prod_{i\geq 1}(1+qx_iu)\right)}_{=\sum_n q^n e_n t^n} \underbrace{\left(\prod_{i\geq 1}\frac{1}{1-x_iu}\right)}_{=\sum_n h_n u^n} \\
&= \left(\sum_n q^n e_n u^n\right)\left(\sum_n h_n u^n\right) = \sum_n \left(\sum_{k=0}^{n} q^k e_k h_{n-k}\right) u^n.
\end{aligned}$$

Extracting coefficients in u^n on both sides yields the desired result. $\square$

Finally, use Proposition 7 and Lemma 1 to relate H_n and q_n.

Proposition 8. *Let $n \in \mathbb{Z}$. Then,*

$$\varpi\left(H_n(X^{\pm})\right) = q_n(X;-q)$$

Proof. From Proposition 7, we know that

$$H_n = \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \left(\prod_{u\in U} x_u\right)\left(\prod_{v\in V} x_v\right).$$

Applying the map ϖ to both sides of this equality, we obtain

$$\begin{aligned}
\varpi(H_n) &= \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \left(\prod_{u\in U} \underbrace{\varpi(x_u)}_{\substack{=qx_{-u} \\ (\text{since } u\in -\mathbb{P})}}\right)\left(\prod_{v\in V} \underbrace{\varpi(x_v)}_{\substack{=x_v \\ (\text{since } v\in \mathbb{P})}}\right) \\
&\qquad (\text{since } \varpi \text{ is a continuous } \mathbf{k}\text{-algebra homomorphism}) \\
&= \sum_{k=0}^{n} \sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \underbrace{\left(\prod_{u\in U} (qx_{-u})\right)}_{\substack{=q^k(\prod_{u\in U} x_{-u}) \\ (\text{since } |U|=k)}} \left(\prod_{v\in V} x_v\right) \\
&= \sum_{k=0}^{n} q^k \underbrace{\left(\sum_{\substack{U \text{ is a size-}k \\ \text{subset of } -\mathbb{P}}} \prod_{u\in U} x_{-u}\right)}_{=\underbrace{\sum_{\substack{U \text{ is a size-}k \\ \text{subset of } \mathbb{P}}} \prod_{u\in U} x_u}_{=e_k}} \underbrace{\left(\sum_{\substack{V \text{ is a size-}(n-k) \\ \text{multisubset of } \mathbb{P}}} \prod_{v\in V} x_v\right)}_{=h_{n-k}}
\end{aligned}$$

As a result $\varpi(H_n) = \sum_{k=0}^{n} q^k e_k h_{n-k} = q_n(X; -q)$ by Equation (3.2). $\square$

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Geometry of C-Matrices for Mutation-Infinite Quivers

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Abstract. The set of forks is a class of quivers introduced by M. Warkentin, where every connected mutation-infinite quiver is mutation equivalent to infinitely many forks. Let Q be a fork with n vertices, and $\boldsymbol{w}$ be a fork-preserving mutation sequence. We show that every c-vector of Q obtained from $\boldsymbol{w}$ is a solution to a quadratic equation of the form

$$\sum_{i=1}^{n} x_i^2 + \sum_{1\le i<j\le n} \pm q_{ij}x_ix_j = 1,$$

where q_{ij} is the number of arrows between the vertices i and j in Q. From the proof of this result, when Q is a rank 3 mutation-cyclic quiver, every c-vector of Q is a solution to a quadratic equation of the same form.

Keywords: quivers, c-vectors, forks, quadratic equations

1 Introduction

The mutation of a quiver Q was discovered by S. Fomin and A. Zelevinsky in their seminal paper [12] where they introduced cluster algebras. It also appeared in the context of Seiberg duality [10]. The c-vectors (and C-matrices) of Q were defined through mutations in further developments of the theory of cluster algebras [13], and together with their companions, g-vectors (and G-matrices), played fundamental roles in the study of cluster algebras (for instance, see [7, 14, 19, 20, 22]). When Q is acyclic, positive c-vectors are actually real Schur roots, that is, the dimension vectors of indecomposable rigid modules over Q [5, 15, 25]. Moreover, they appear as the denominator vectors of non-initial cluster variables of the cluster algebra associated to Q [4].

Due to the multifaceted appearance of c-vectors in important constructions, there have been various results related to the description of c-vectors (or real Schur roots)

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of an acyclic quiver [1, 15, 16, 23, 24, 25]. In [18], K.-H. Lee and K. Lee conjectured a correspondence between real Schur roots of an acyclic quiver and non-self-crossing curves on a Riemann surface and proposed a new combinatorial/geometric description. The conjecture is now proven by A. Felikson and P. Tumarkin [9] for acyclic quivers with multiple edges between every pair of vertices. Recently, S. D. Nguyen [21] proved the conjecture for an arbitrary acyclic (valued) quiver.

For a given (not necessarily acyclic) quiver Q, the set of quivers that are mutation equivalent to Q is called the mutation equivalence class of Q and denoted by $\mathrm{Mut}(Q)$. The quiver Q is said to be *mutation-infinite* if $|\mathrm{Mut}(Q)|$ is not finite, and *mutation-finite* if $|\mathrm{Mut}(Q)| < \infty$. The mutation-finite quivers are completely classified, and relatively well studied. On the other hand, mutation-infinite quivers still await further investigations.

A reader-friendly version of our main theorem may be stated as follows.

Theorem 1.1. *Let n be any positive integer. Let P be a mutation-infinite connected quiver with n vertices. Then there exist an infinite number of pairs of a quiver $Q \in Mut(P)$ and $k \in \{1, ..., n\}$ such that every c-vector of Q obtained from any mutation sequence not starting with k is a solution to a quadratic equation of the form*

$$\sum_{i=1}^{n} x_i^2 + \sum_{1 \le i < j \le n} \pm q_{ij} x_i x_j = 1, \tag{1.2}$$

where q_{ij} is the number of arrows between the vertices i and j in Q. There does not seem to be a simple way of determining the exact signs of the $x_i x_j$ terms.

To state a more precise theorem, we need to recall the definition of forks. An *abundant quiver* is a quiver such that there are two or more arrows between every pair of vertices.

Definition 1.3. [26, Definition 2.1] A *fork* is an abundant quiver F, where F is not acyclic and where there exists a vertex r, called the point of return, such that

- For all $i \in F^-(r)$ and $j \in F^+(r)$ we have $f_{ji} > f_{ir}$ and $f_{ji} > f_{rj}$, where $F^-(r)$ is the set of vertices with arrows pointing towards r and $F^+(r)$ is the set of vertices with arrows coming from r.
- The full subquivers induced by $F^-(r)$ and $F^+(r)$ are acyclic.

An example of a fork is given by

$$\begin{tikzcd} & r \arrow[dr, "4"] & \\ i \arrow[ur, "3"] & & j \arrow[ll, "5"] \end{tikzcd},$$

where r is the point of return.

It is known that "most" quivers in $\mathrm{Mut}(Q)$ of any connected mutation-infinite quiver Q are forks, as Theorem 1.4 and Proposition 1.5 imply.

Theorem 1.4. [26, Theorem 3.2] *A connected quiver is mutation-infinite if and only if it is mutation-equivalent to a fork.*

Proposition 1.5. [26, Proposition 5.2] *Let G be the exchange graph of a connected mutation-infinite quiver. A simple random walk on G will almost surely leave the fork-less part and never come back.*

A *fork-preserving* mutation sequence is a reduced sequence of mutations that starts with a fork and does not mutate at its point of return. A more precise version of our main theorem is as follows.

Theorem 1.6. *Let Q be a fork, and let $\boldsymbol{w}$ be a fork-preserving mutation sequence. Every c-vector of Q obtained from $\boldsymbol{w}$ is a solution to a quadratic equation of the form* (1.2)*.*

A quiver Q is called *mutation-acyclic* if it is mutation-equivalent to an acyclic quiver, else it is called *mutation-cyclic*. Notably, we have discovered a counterexample to Theorem 1.6 for truly arbitrary mutation-sequences $\boldsymbol{w}$ in the case of quivers on four vertices (to appear in the full version of this abstract [8]), but the proof of the theorem provides a stronger corollary in the three vertex case. Ahmet Seven informed us that he had independently discovered this result.

Corollary 1.7. *Let Q be a mutation-cyclic quiver with 3 vertices. Then every c-vector of Q is a solution to a quadratic equation of the form* (1.2) *with $n = 3$.*

As a byproduct of our proof, we also obtain the following theorem, which is closely related to a result of Fomin and Neville [11, Lemma 6.14].

Theorem 1.8. *Let $\boldsymbol{w}$ be a fork-preserving mutation sequence. The sign-vector (see Definition 2.3) of $C^{\boldsymbol{w}}$ depends only on the signs of entries of initial exchange matrix B. In other words, the sign-vector is independent of the number of arrows between vertices of the initial quiver Q.*

Corollary 1.9. *Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, the corresponding n-tuple of reflections $(r_1^{\boldsymbol{w}}, r_2^{\boldsymbol{w}}, \dots, r_n^{\boldsymbol{w}})$ (see Definition 2.6) depends only on the signs of entries of the initial exchange matrix B.*

From this, we are able to prove that the product of reflections is equal to a Coxeter element. More precisely, we have the following.

Theorem 1.10. *Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, we have*

$$r^{\boldsymbol{w}}_{\lambda(1)} \dots r^{\boldsymbol{w}}_{\lambda(n)} = r_{\rho(1)} \dots r_{\rho(n)}$$

for some permutations $\lambda, \rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on $\{1, \dots, n\}$ and $r_1, \dots, r_n$ are the initial reflections, where λ is determined by $\boldsymbol{w}$ and ρ is fixed by the first mutation of $\boldsymbol{w}$.

Corollary 1.11. *Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, there exist pairwise non-crossing and non-self-crossing admissible curves $\eta_i^{\boldsymbol{w}}$ (see Definition 2.10) such that $r_i^{\boldsymbol{w}} = \nu(\eta_i^{\boldsymbol{w}})$ for every $i \in \{1, ..., n\}$.*

The above results are explored more thoroughly in our forthcoming paper [8], and they all rely heavily on our use of l-vectors and generalized intersection matrices.

2 Preliminaries

2.1 C-matrices

Let n be a positive integer. If $B = [b_{ij}]$ is an $n \times n$ skew-symmetric matrix, then B is in correspondence with a quiver Q on n vertices: if $b_{ij} > 0$ and $i \neq j$, then Q has b_{ij} arrows from vertex i to vertex j. The statements of some theorems have been formulated in terms of Q; however, we prefer to work with B since the description of c-vectors is more clear in this setting. Also, for a nonzero vector $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, we write $c > 0$ if all c_i are non-negative, and $c < 0$ if all c_i are non-positive.

Assume that $M = [m_{ij}]$ is an $n \times 2n$ matrix with integer entries. Let $\mathcal{I} := \{1, 2, \ldots, n\}$ be the set of indices. For $\boldsymbol{w} = [i_1, i_2, \ldots, i_\ell]$, $i_j \in \mathcal{I}$, we define the matrix $M^{\boldsymbol{w}} = [m_{ij}^{\boldsymbol{w}}]$ inductively: the initial matrix is M for $\boldsymbol{w} = [\,]$, and assuming we have $M^{\boldsymbol{w}}$, define the matrix $M^{\boldsymbol{w}[k]} = [m_{ij}^{\boldsymbol{w}[k]}]$ for $k \in \mathcal{I}$ with $\boldsymbol{w}[k] := [i_1, i_2, \ldots, i_\ell, k]$ by

$$m_{ij}^{\boldsymbol{w}[k]} = \begin{cases} -m_{ij}^{\boldsymbol{w}} & \text{if } i = k \text{ or } j = k, \\ m_{ij}^{\boldsymbol{w}} + \operatorname{sgn}(m_{ik}^{\boldsymbol{w}}) \max(m_{ik}^{\boldsymbol{w}} m_{kj}^{\boldsymbol{w}}, 0) & \text{otherwise,} \end{cases} \tag{2.1}$$

where $\operatorname{sgn}(a) \in \{1, 0, -1\}$ is the signature of a. The matrix $M^{\boldsymbol{w}[k]}$ is called the *mutation of $M^{\boldsymbol{w}}$* at index (or label) k, $\boldsymbol{w}$ and $\boldsymbol{w}[k]$ are called *mutation sequences*, and n is the *rank*.

Let B be a $n \times n$ skew-symmetric matrix. Consider the $n \times 2n$ matrix $\begin{bmatrix} B & I \end{bmatrix}$ and a mutation sequence $\boldsymbol{w} = [i_1, \ldots, i_\ell]$. After the mutations at the indices $i_1, \ldots, i_\ell$ consecutively, we obtain $\begin{bmatrix} B^{\boldsymbol{w}} & C^{\boldsymbol{w}} \end{bmatrix}$. Write their entries as

$$B^{\boldsymbol{w}} = \left[b_{ij}^{\boldsymbol{w}} \right], \qquad C^{\boldsymbol{w}} = \left[c_{ij}^{\boldsymbol{w}} \right] = \begin{bmatrix} c_1^{\boldsymbol{w}} \\ \vdots \\ c_n^{\boldsymbol{w}} \end{bmatrix}, \tag{2.2}$$

where $c_i^{\boldsymbol{w}}$ are the row vectors.

Definition 2.3. The matrix $C^{\boldsymbol{w}}$ is called a C-matrix of B for any w [1]. The row vectors $c_i^{\boldsymbol{w}}$ are called *c-vectors* of B for any i and $\boldsymbol{w}$. Each non-zero entry of $c_i^{\boldsymbol{w}}$ will share the same sign [6], allowing us to define the *sign-vector* of $C^{\boldsymbol{w}}$, where the i-th entry is 1 if $c_i^{\boldsymbol{w}} > 0$ and -1 if $c_i^{\boldsymbol{w}} < 0$.

[1] This is slightly different from the original definition by Fomin and Zelevinsky

2.2 Reflections and L-matrices

In order to prove Theorem 1.6, we needed to study the L-matrices arising from reflections and a particular generalized intersection matrix associated to our exchange matrix.

Definition 2.4. A *generalized intersection matrix* (GIM) is a square matrix $A = [a_{ij}]$ with integral entries such that (1) for diagonal entries, $a_{ii} = 2$; (2) $a_{ij} > 0$ if and only if $a_{ji} > 0$; (3) $a_{ij} < 0$ if and only if $a_{ji} < 0$.

Let $\mathcal{A}$ be the (unital) $\mathbb{Z}$-algebra generated by s_i, e_i, $i = 1, 2, \ldots, n$, subject to the following relations:

$$s_i^2 = 1, \quad \sum_{i=1}^{n} e_i = 1, \quad s_i e_i = -e_i, \quad e_i s_j = \begin{cases} s_i + e_i - 1 & \text{if } i = j, \\ e_i & \text{if } i \neq j, \end{cases} \quad e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\mathcal{W}$ be the subgroup of the units of $\mathcal{A}$ generated by s_i, $i = 1, \ldots, n$. Note that $\mathcal{W}$ is (isomorphic to) the universal Coxeter group. An element $r \in \mathcal{W}$ is called a reflection if $r^2 = 1$. Let $\mathfrak{R} \subset \mathcal{W}$ be the set of reflections.

From now on, let $A = [a_{ij}]$ be an $n \times n$ symmetric GIM. Let $\Gamma = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$ be the lattice generated by the formal symbols $\alpha_1, ..., \alpha_n$. Define a representation $\pi : \mathcal{A} \to \mathrm{End}(\Gamma)$ by

$$\pi(s_i)(\alpha_j) = \alpha_j - a_{ji}\alpha_i \quad \text{and} \quad \pi(e_i)(\alpha_j) = \delta_{ij}\alpha_i, \quad \text{for } i, j \in \{1, ..., n\}.$$

We suppress π when we write the action of an element of $\mathcal{A}$ on Γ.

Given a skew-symmetric matrix B, for each linear ordering $\prec$ on $\{1, ..., n\}$, we define the associated GIM $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} b_{ij} & \text{if } i \prec j, \\ 2 & \text{if } i = j, \\ -b_{ij} & \text{if } i \succ j. \end{cases} \tag{2.5}$$

An ordering $\prec$ provides a certain way for us to regard the skew-symmetric matrix B as acyclic even when it is not.

Definition 2.6. When $\boldsymbol{w} = [\,]$, we let $r_i = s_i \in \mathfrak{R}$ for each $i \in \{1, ..., n\}$. For each mutation sequence $\boldsymbol{w}$ and each $i \in \{1, ..., n\}$, define $r_i^{\boldsymbol{w}} \in \mathfrak{R}$ inductively as follows:

$$r_i^{\boldsymbol{w}[k]} = \begin{cases} r_k^{\boldsymbol{w}} r_i^{\boldsymbol{w}} r_k^{\boldsymbol{w}} & \text{if } b_{ik}^{\boldsymbol{w}} c_k^{\boldsymbol{w}} > 0, \\ r_i^{\boldsymbol{w}} & \text{otherwise.} \end{cases} \tag{2.7}$$

Clearly, each $r_i^{\boldsymbol{w}}$ is written in the form

$$r_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}} s_i (g_i^{\boldsymbol{w}})^{-1}, \quad g_i^{\boldsymbol{w}} \in \mathcal{W}, \quad i \in \{1, ..., n\}.$$

Definition 2.8. Let sgn $= \{1, -1\}$ be the group of order 2, and consider the natural group action sgn $\times \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, where we identify Γ with $\mathbb{Z}^n$. Choose an ordering $\prec$ on $\{1, ..., n\}$ to fix a GIM A, and define

$$l_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}}(\alpha_i) \in \mathbb{Z}^n / \operatorname{sgn}, \qquad i \in \{1, ..., n\},$$

where we set $\alpha_1 = (1, 0, ..., 0), ..., \ \alpha_n = (0, ..., 0, 1)$. Then the *L-matrix* $L^{\boldsymbol{w}}$ associated to A is defined to be the $n \times n$ matrix whose i^{th} row is $l_i^{\boldsymbol{w}}$ for $i \in \{1, ..., n\}$, i.e., $L^{\boldsymbol{w}} = \begin{bmatrix} l_1^{\boldsymbol{w}} \\ \vdots \\ l_n^{\boldsymbol{w}} \end{bmatrix}$, and the vectors $l_i^{\boldsymbol{w}}$ are called the *l-vectors of* A. Note that the L-matrix and l-vectors associated to a GIM A implicitly depend on the representation π which is suppressed from the notation.

With the above machinery, we show the following, which further implies Theorem 1.6.

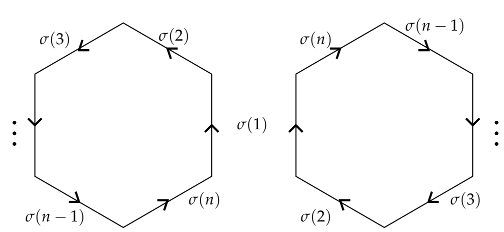
Theorem 2.9. *Let Q be a fork with n vertices, and let $\boldsymbol{w}$ be a fork-preserving mutation sequence. For each $i \in \{1, ..., n\}$, there exists a diagonal matrix $D_i^{\boldsymbol{w}}$ such that $(D_i^{\boldsymbol{w}})^2 = 1$ and $l_i^{\boldsymbol{w}} = c_i^{\boldsymbol{w}} D_i^{\boldsymbol{w}}$. In other words, the entries of l-vectors are equal to the entries of c-vectors up to sign.*

2.3 Geometry of reflections

Here we review the definition of admissible curves [18, 17].

Let $\mathcal{Q}$ be a fork with n vertices labeled by $I := \{1, ..., n\}$ and point of return r. Let σ be the linear ordering given by $r \prec a_{n-1} \prec a_{n-2} \prec \cdots \prec a_1$, where $a_1, a_2, \ldots, a_{n-1}$ are the vertices of $Q \setminus \{r\}$ and $a_i \prec a_j$ if and only if there is an arrow from j to i.

We define a labeled Riemann surface Σ_σ[2] as follows. Let G_1 and G_2 be two identical copies of a regular n-gon. Label the edges of each of the two n-gons by $T_{\sigma(1)}, \ldots, T_{\sigma(n)}$ counter-clockwise. Fix the orientation of every edge of G_1 (resp. G_2) to be counter-clockwise (resp. clockwise) as in the following picture.



[2]The punctured discs appeared in Bessis' work [3]. For better visualization, here we prefer to use an alternative description using compact Riemann surfaces with one or two marked points.

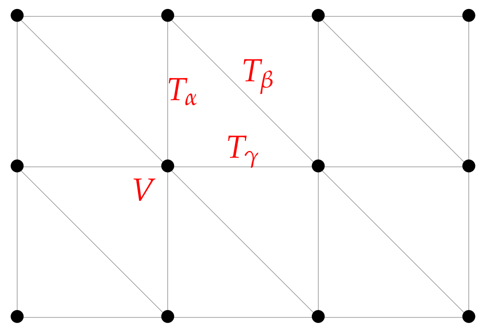


Figure 1: This picture illustrates a portion of the universal cover Σ_σ, and the three arcs T_α, T_β, and T_γ.

Let Σ_σ be the (compact) Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ obtained by gluing together the two n-gons with all the edges of the same label identified according to their orientations. The edges of the n-gons become N different curves in Σ_σ. If n is odd, all the vertices of the two n-gons are identified to become one point in Σ_σ and the curves obtained from the edges are loops. If n is even, two distinct vertices are shared by all curves. Let $\mathcal{T}$ be the set of all curves, i.e., $\mathcal{T} = T_1 \cup \cdots \cup T_n \subset \Sigma_\sigma$, and V be the set of the vertex (or vertices) on $\mathcal{T}$.

For simplicity, here we give a precise definition of an admissible curve for rank 3 quivers only, but it is straightforward to generalize to quivers of higher rank. For our geometric model on rank 3 quivers, we consider the (triangulated) torus with one marked point along with admissible curves (see Definition 2.10). The key point here is that there is a map from the set of admissible curves to $\mathfrak{R}$.

For each $\sigma \in \mathfrak{S}_3$, let Σ_σ be the closed Riemann surface of genus 1 with a single marked point V, and let $\widetilde{\Sigma_\sigma}$ be the universal cover of Σ_σ, which can be regarded as $\mathbf{R}^2$. Let $\alpha = \sigma(1)$, $\beta = \sigma(2)$, and $\gamma = \sigma(3)$. Fix three arcs T_α, T_β, and T_γ on Σ_σ and the projection $p : \widetilde{\Sigma_\sigma} \longrightarrow \Sigma_\sigma$ such that $p^{-1}(T_\alpha) = \mathbf{Z} \times \mathbf{R} \subset \mathbf{R}^2$, $p^{-1}(T_\beta) = \{(x,y) : x+y \in \mathbf{Z}\} \subset \mathbf{R}^2$, $p^{-1}(T_\gamma) = \mathbf{R} \times \mathbf{Z} \subset \mathbf{R}^2$, and $p^{-1}(V) = \mathbf{Z}^2 \subset \mathbf{R}^2$. Hence T_α is the vertical line segment, T_β is the diagonal, and T_γ is the horizontal line segment. Let $T = T_1 \cup T_2 \cup T_3$. See Figure 1.

Definition 2.10. An *admissible curve* is a pair consisting of a continuous function $\eta : [0,1] \longrightarrow \Sigma_\sigma$ and a sequence $\{i_\ell\}_{\ell=1}^k$ of entries with in $i_\ell \in \{1,2,3\}$ such that

1) $\eta(x) = V$ if and only if $x \in \{0,1\}$;

2) if $\eta(x) \in T \setminus \{V\}$ then $\eta([x-\epsilon, x+\epsilon])$ meets T transversally for sufficiently small $\epsilon > 0$;

3) $\eta(x_\ell) \in T_{i_\ell}$ and $\ell \in \{1,...,k\}$, where

$$\{x_1 < \cdots < x_k\} = \{x \in (0,1) : \eta(x) \in T\}$$

4) $v(\eta) \in \mathfrak{R}$, where $v(\eta) := r_{i_1} \cdots r_{i_k} \in \mathcal{W}$.

Example 2.11. In Example 3.5, when $\boldsymbol{w} = [1,2,3]$, the admissible curve η_2^w has

$$v(\eta_2^w) = r_2 r_1 r_3 r_1 r_2 r_1 r_3 r_1 r_2.$$

Note that η_2^w crosses T_2, T_1, T_3, T_1, T_2, T_1, T_3, T_1, T_2 in this order.

3 Examples

In this section, we will consider the following two quivers to demonstrate our theorems:

$$P = \begin{tikzcd} & 2 \arrow[dr, "3"] & \\ 1 \arrow[ur, "3"] & & 3 \arrow[ll, "6"] \end{tikzcd} \quad \text{and} \quad Q = \begin{tikzcd} & 2 \arrow[dr, "4"] & \\ 1 \arrow[ur, "3"] & & 3 \arrow[ll, "5"] \end{tikzcd}$$

Both quivers are mutation-cyclic [2]. Also, P and Q are forks and are mutation-equivalent to only forks. In this section, we will consider the c-vectors of both P and Q under three mutation sequences, namely, $\boldsymbol{w} = [1]$, $\boldsymbol{w} = [1,2]$, and $\boldsymbol{w} = [1,2,3]$.

Example 3.1. An example of Theorem 1.8 is given in the table below:

Mutation Sequence	$[B^w \mid C^w]$-matrix for P	$[B^w \mid C^w]$-matrix for Q
$\boldsymbol{w} = [1]$	$\left[\begin{array}{ccc\|ccc} 0 & -3 & 6 & -1 & 0 & 0 \\ 3 & 0 & -15 & 0 & 1 & 0 \\ -6 & 15 & 0 & 6 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{ccc\|ccc} 0 & -3 & 5 & -1 & 0 & 0 \\ 3 & 0 & -11 & 0 & 1 & 0 \\ -5 & 11 & 0 & 5 & 0 & 1 \end{array}\right]$
$\boldsymbol{w} = [1,2]$	$\left[\begin{array}{ccc\|ccc} 0 & 3 & -39 & -1 & 0 & 0 \\ -3 & 0 & 15 & 0 & -1 & 0 \\ 39 & -15 & 0 & 6 & 15 & 1 \end{array}\right]$	$\left[\begin{array}{ccc\|ccc} 0 & 3 & -28 & -1 & 0 & 0 \\ -3 & 0 & 11 & 0 & -1 & 0 \\ 28 & -11 & 0 & 5 & 11 & 1 \end{array}\right]$
$\boldsymbol{w} = [1,2,3]$	$\left[\begin{array}{ccc\|ccc} 0 & -582 & 39 & -1 & 0 & 0 \\ 582 & 0 & -15 & 90 & 224 & 15 \\ -39 & 15 & 0 & -6 & -15 & -1 \end{array}\right]$	$\left[\begin{array}{ccc\|ccc} 0 & -305 & 28 & -1 & 0 & 0 \\ 305 & 0 & -11 & 55 & 120 & 11 \\ -28 & 11 & 0 & -5 & -11 & -1 \end{array}\right]$

For each quiver, the sign vector of the C-matrix for $\boldsymbol{w} = [1]$, $\boldsymbol{w} = [1,2]$, and $\boldsymbol{w} = [1,2,3]$ is $(-1,1,1)$, $(-1,-1,1)$, and $(-1,1,-1)$).

Example 3.2. The quadratic equation for the quiver P is given by

$$x^2 + y^2 + z^2 - 3xy - 6xz + 3yz = 1.$$

and the quadratic equation for Q is given by

$$x^2 + y^2 + z^2 - 3xy - 5xz + 4yz = 1.$$

It is easy to verify that the c-vectors

$$(x, y, z) = (90, 224, 15) \text{ and } (x, y, z) = (-6, -15, -1)$$

both satisfy the quadratic equation for P and that the c-vectors

$$(x, y, z) = (55, 120, 11) \text{ and } (x, y, z) = (-5, -11, -1)$$

both satisfy the quadratic equation for Q.

Example 3.3. In this example, we demonstrate Corollary 1.9. If we mutate the reflections for both of P and Q with $\boldsymbol{w} = [1]$, then we arrive at

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2, \quad r_3^{\boldsymbol{w}} = r_1 r_3 r_1.$$

If we mutate both of them with $\boldsymbol{w} = [1, 2]$, then we arrive at

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2, \quad r_3^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2.$$

If we mutate both of them with $\boldsymbol{w} = [1, 2, 3]$, then we arrive at

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2 r_1 r_3 r_1 r_2, \quad r_3^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2.$$

We can see that both of these are fork-preserving mutation sequences with the same initial orientation for the B matrix.

Example 3.4. In this example, we demonstrate Theorem 1.10. If we take the three mutated reflections from Example 3.3 for $\boldsymbol{w} = [1]$, then

$$r_1^{\boldsymbol{w}} r_3^{\boldsymbol{w}} r_2^{\boldsymbol{w}} = r_3 r_1 r_2.$$

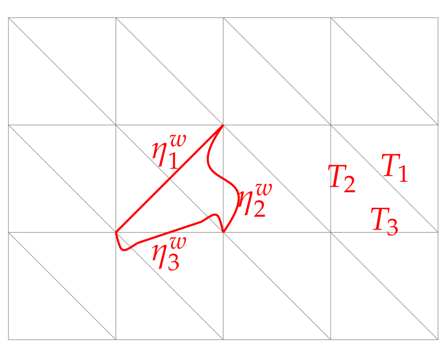
For $\boldsymbol{w} = [1, 2]$, we have

$$r_1^{\boldsymbol{w}} r_2^{\boldsymbol{w}} r_3^{\boldsymbol{w}} = r_3 r_1 r_2.$$

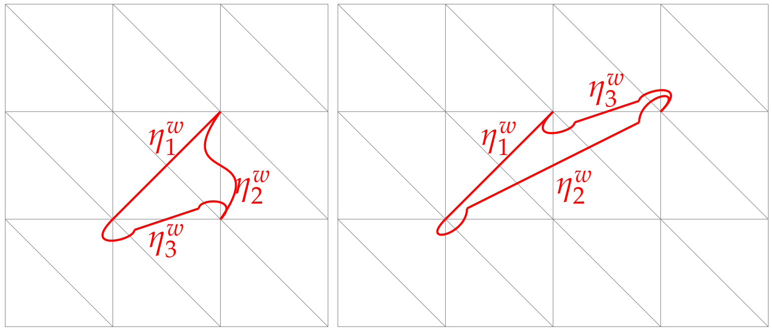
Finally, for $\boldsymbol{w} = [1, 2, 3]$, we have

$$r_1^{\boldsymbol{w}} r_3^{\boldsymbol{w}} r_2^{\boldsymbol{w}} = r_3 r_1 r_2.$$

Example 3.5. In this example, we demonstrate Corollary 1.11. If we take the three mutated reflections from Example 3.3 for $\boldsymbol{w} = [1]$, then we get the following admissible curves:



For $\boldsymbol{w} = [1,2]$ and $\boldsymbol{w} = [1,2,3]$, we get the following admissible curves respectively:



Note that these curves are pairwise non-crossing as well as non-self-crossing. Also, using the labeling of the 3 arcs from the picture for the first set of non-crossing curves, we can recover the sequence of reflections from the curves in each picture and confirm the correspondence.

Example 3.6. To demonstrate how to calculate l-vectors, we consider $l_2^{\boldsymbol{w}}$ for the quiver Q with $\boldsymbol{w} = [1,2,3]$ and linear ordering $2 \prec 1 \prec 3$. First, construct the GIM

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -3 & 2 & 4 \\ -5 & 4 & 2 \end{bmatrix}.$$

Then consider the following matrices in $M_{3\times 3}(\mathbb{Z})$.

$$S_1 = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 1 \end{bmatrix}, \qquad S_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}.$$

Using the sequence of reflections from Example 3.3 and the definition of l-vectors, we know that

$$\begin{aligned} l_2^{\boldsymbol{w}} &= s_2 s_1 s_3 s_1(\alpha_2) \\ &= (S_2^T S_1^T S_3^T S_1^T(\alpha_2^T))^T \\ &= \alpha_2 S_1 S_3 S_1 S_2 \\ &= (\alpha_2) S_1 S_3 S_1 S_2 \\ &= (3\alpha_1 + \alpha_2) S_3 S_1 S_2 \\ &= (3\alpha_1 + \alpha_2 + 11\alpha_3) S_1 S_2 \\ &= (55\alpha_1 + \alpha_2 + 11\alpha_3) S_2 \\ &= 55\alpha_1 + 120\alpha_2 + 11\alpha_3. \end{aligned}$$

These calculations can then be used to demonstrate Theorem 2.9. Compare the table below with the one given in Example 3.1.

Mutation Sequence	L-matrix for P	L-matrix for Q
$w = [1]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$
$w = [1, 2]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 15 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$
$w = [1, 2, 3]$	$\begin{bmatrix} 1 & 0 & 0 \\ 90 & 224 & 15 \\ 6 & 15 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 55 & 120 & 11 \\ 5 & 11 & 1 \end{bmatrix}$

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Higher Specht Polynomials and Tableaux Bijections for Hessenberg Varieties

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Abstract. The cohomology rings of regular semisimple Hessenberg varieties are only completely understood in some cases. One such case is when the Hessenberg function is $h = (h(1), n, \ldots, n)$, and is described by Abe, Horiguchi, and Masuda in 2017. We define an alternative basis for the cohomology ring in this case, which is a higher Specht basis. We give combinatorial bijections between the monomials in this basis and sets of P-tableaux, motivated by the work of Gasharov in 2008 and Shareshian and Wachs in 2016. This bijection illustrates the connection between the symmetric group action on these cohomology rings and the Schur expansion of chromatic symmetric functions. We further use the inversion formula for P-tableaux to give a new combinatorial proof of the known Poincaré polynomial for these Hessenberg varieties.

Keywords: Hessenberg varieties, P-tableaux, higher Specht bases

1 Introduction

In this extended abstract, we exhibit new connections between the combinatorics of Hessenberg varieties, P-tableaux, and chromatic symmetric functions, and illustrate their use in proving geometric results using combinatorial tools. In particular, when S is a regular semisimple matrix and $h = (h(1), n, \ldots, n)$, we construct a higher Specht basis for the cohomology ring $H^*(\mathrm{Hess}(S, h))$, display combinatorial bijections between these higher Specht basis elements and sets of tableaux, and use a new combinatorial method to find the Poincaré polynomial for $\mathrm{Hess}(S, h)$. Full proofs of the results in this paper are forthcoming in [13].

Hessenberg varieties, initially defined and studied in [11, 12], are linear subvarieties of the full flag variety $Fl(\mathbb{C}^n)$. They are connected to chromatic symmetric and quasisymmetric functions due to the action of the symmetric group S_n on their cohomology rings defined by Tymozcko in [18]. The geometry of Hessenberg varieties has been – and continues to be – extensively studied, including in [1, 9, 7, 8, 10, 17].

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1.1 Background on Hessenberg Varieties and Specht Modules

Given a proper coloring $\kappa : V \to \mathbb{N}$ of a finite simple graph $G = (V, E)$ on a totally-ordered vertex set V, define an **ascent** to be an edge $\{v, w\}$ such that $v < w$ and $\kappa(v) < \kappa(w)$. Define $\operatorname{asc}(\kappa)$ to be the total number of ascents in κ. In this paper, we consider graphs with $V = [n] = \{1, 2, \ldots, n\}$. In [14], Shareshian and Wachs defined the chromatic quasisymmetric function, a graded analogue of Stanley's chromatic symmetric function [15], using the ascent statistic:

Definition 1.1. *Given a finite simple graph $G = (V, E)$, the* ***chromatic quasisymmetric function*** *for G is*

$$X_G(\mathbf{x}; q) = \sum_{\kappa: V \to \mathbb{N}} \left(\prod_{i \in V} x_{\kappa(i)} \right) q^{\operatorname{asc}(\kappa)}$$

where the sum ranges over all proper colorings κ of the vertices of G.

Recall that the **(full) flag variety** of $\mathbb{C}^n$ is the variety $Fl(\mathbb{C}^n)$ whose points are flags $F_\bullet = F_0 \subset F_1 \subset \cdots \subset F_n$ such that $\dim(F_i) = i$. We will define Hessenberg varieties to be a subvariety of the flag variety, given a matrix $X : \mathbb{C}^n \to \mathbb{C}^n$ and a function h. First, a **Hessenberg function** is a function $h : [n] \to [n]$ such that for all i, we have $i \leq h(i)$, and $h(i) \leq h(i+1)$. We usually denote this as a vector $h = (h(1), h(2), \ldots, h(n))$.

Definition 1.2. *Given a matrix $X : \mathbb{C}^n \to \mathbb{C}^n$ and a Hessenberg function $h : [n] \to [n]$, define the* ***Hessenberg variety*** *to be:*

$$\operatorname{Hess}(X, h) = \left\{ F_\bullet \in Fl(\mathbb{C}^n) \mid X(F_i) \subseteq F_{h(i)} \text{ for all } 1 \leq i \leq n \right\} \tag{1.1}$$

In [18], Tymoczko defined an action of the symmetric group on the cohomology ring $H^*(\operatorname{Hess}(S, h))$ when S is a regular semisimple matrix, allowing us to study the structure of this ring as an S_n-module. For each Hessenberg function, we can also construct a poset P_h on $[n]$, using h to determine which elements are comparable: We say that $i <_{P_h} j$ if and only if $h(i) < j$. Let G_h be the incomparability graph of P_h, which has an edge $\{i, j\}$ whenever i and j are incomparable in P_h. Notably, posets formed by this construction are $(3+1)$- and $(2+2)$-avoiding, so their incomparability graphs are relevant to the following conjecture of Stanley and Stembridge on chromatic symmetric functions.

Conjecture 1.3 ([16] Conjecture 5.1)**.** *If P is a $(3+1)$-free poset, then $X_{\operatorname{inc}(P)}(\mathbf{x})$ is e-positive, that is, it can be written with positive coefficients when expanded in the elementary basis of symmetric functions.*

In [5], Guay-Paquet proved that it suffices to prove the conjecture for posets which are $(3+1)$- and $(2+2)$-avoiding. Shareshian and Wachs conjectured that there is a connection between chromatic quasisymmetric functions and the graded cohomology ring

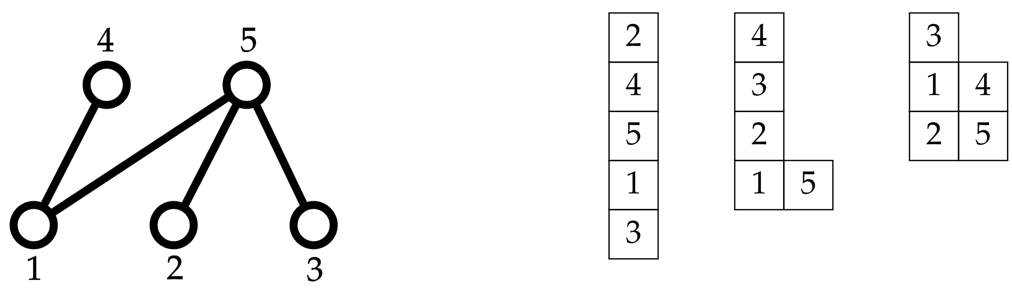

Figure 1: On the left, the poset P_h for $h = (3,4,4,5,5)$. On the right, three P_h-tableaux (in French notation). The leftmost tableau has 4 inversions given by the pairs $(1,3), (2,3), (2,4)$, and $(4,5)$, but does not have the inversion $(2,5)$ since $2 <_{P_h} 5$.

of regular semisimple Hessenberg varieties, using the ascent formula for the incomparability graph G_h. This connection, stated below, was proven by Brosnan and Chow in [2] and separately by Guay-Paquet in [6].

Proposition 1.4 ([2, 6]). *Let S be a regular semisimple $n \times n$ matrix and $h : [n] \to [n]$ be a Hessenberg function. Let G_h be the incomparability graph for P_h. Then*

$$\omega X_{G_h}(\mathbf{x}; q) = \sum_{k=0}^{|E|} \mathrm{Frob}(H^{2k}(\mathrm{Hess}(S,h)))q^k$$

Above, ω is the standard involution on symmetric functions which sends the Schur function s_λ to $s_{\lambda'}$, where λ' is the transpose of λ, and Frob is the Frobenius characteristic map which sends the irreducible S_n-module V_λ to the Schur function s_λ.

Definition 1.5 ([3]). *Let P be a poset and λ be a partition of n. A* ***P-tableau of shape λ*** *is a filling of the Young diagram of λ with entries from P such that:*

- *Each entry in P is used at most once.*
- *Adjacent entries in rows are P-increasing from left to right.*
- *Adjacent entries in columns are P-nondecreasing from bottom to top.*

We say a **P-inversion** in a P-tableau is a pair of entries (i, j) such that $i < j$ as integers, i is in a higher row than j, and i and j are incomparable in P. Define $\mathrm{inv}(T)$ to be the number of P-inversions in T. In [3], Gasharov used P-tableaux to show that the chromatic symmetric functions of incomparability graphs of $(3+1)$-free posets are Schur-positive. Using this inversion statistic on P-tableaux, Shareshian and Wachs extended this result to the chromatic quasisymmetric case, as stated below.

Proposition 1.6 ([14] Theorem 6.3)**.** *Let G be the incomparability graph of a $(3+1)$-free poset P, and let $PT(\lambda)$ be the set of P-tableaux of shape λ. Then we have:*

$$X_G(\mathbf{x};q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\mathrm{inv}(T)} \right) s_\lambda$$

Combining the results of Propositions 1.4 and 1.6, we can connect the graded cohomology of $\mathrm{Hess}(S,h)$ with P-tableaux in the following way. If S is a regular semisimple matrix, and h is a Hessenberg function with poset P_h and incomparability graph G_h, then

$$\sum_{k=0}^{|E|} \mathrm{Frob}(H^{2k}(\mathrm{Hess}(S,h)))q^k = \omega X_{G_h}(\mathbf{x};q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\mathrm{inv}_h(T)} \right) s_{\lambda'} \tag{1.2}$$

where λ' is the transpose partition of λ, and inv_h is the inversion statistic for P_h.

The formula above gives us a nice way of understanding the decomposition of the S_n-module $H^*(\mathrm{Hess}(S,h))$ into irreducible modules. Irreducible S_n-modules are isomorphic to the Specht modules, which have a basis indexed by standard tableaux:

Definition 1.7. *Given a standard tableau T of shape λ, define the* ***Specht polynomial*** *to be*

$$F_T = \prod_{C \in \lambda} \left(\prod_{i<j \in C} (x_j - x_i) \right)$$

where the first product is over all columns in the Young diagram, and the second product is over all pairs of entries $i < j$ in the column C. If $SYT(\lambda)$ is the set of all standard tableaux of shape λ, then the ***Specht module*** *V_λ is the subspace of $\mathbb{Q}[x_1, \ldots, x_n]$ generated by $\{F_T\}_{T \in SYT(\lambda)}$.*

An immediate consequence of this definition is that the dimension of the Specht module V_λ is the number of standard tableaux of shape λ, which we denote $\#SYT(\lambda)$. We define a higher Specht basis for an S_n-module as follows.

Definition 1.8 ([4], Definition 1.5)**.** *If R is an S_n-module which decomposes into irreducible S_n-modules as*

$$R = \bigoplus_{\lambda} c_\lambda V_\lambda\,,$$

then a ***higher Specht basis of*** *R is a set of elements $\mathcal{B}$ with a decomposition $\mathcal{B} = \bigcup_\lambda \bigcup_{i=1}^{c_\lambda} \mathcal{B}_{i,\lambda}$ such that the elements of $\mathcal{B}_{i,\lambda}$ are a basis of the i-th copy of V_λ in the decomposition of R.*

Hence, higher Specht bases of S_n-modules are a natural way to understand the action of S_n, and allow us to more easily identify the decomposition into irreducible modules.

2 Higher Specht basis for the cohomology ring

In this section, let S be a regular semisimple $n \times n$ matrix, and $h = (h(1), n, \ldots, n)$ be a Hessenberg function. In [9] (Theorem 4.3), Abe, Horiguchi, and Masuda give a presentation of the cohomology ring $H^*(\mathrm{Hess}(S,h))$ as a quotient of a polynomial ring in $2n$ variables. Further, in Remark 4.5, they describe a set of basis elements for this ring. We name the two sets of different types of elements below:

$$B_1 = \left\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{not containing the factor } \prod_{\ell=1}^{h(1)} x_\ell \right\} \tag{2.1}$$

$$B_2 = \left\{ x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k \quad \text{not containing the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \right\} \tag{2.2}$$

over all $0 \leq i_j \leq n - j$ in B_1, and over all $0 \leq \ell_j \leq n - 1 - j$, and $1 \leq k \leq n - 1$ in B_2.

The symmetric group S_n acts on the above monomials by fixing the set of x_i and permuting the set of y_i in the natural way. This group action gives a representation of S_n, which decomposes into the direct sum of trivial representations (corresponding to the Specht module $V_{(n)}$) and standard representations (corresponding to the Specht module $V_{(n-1,1)}$).

We define B_3 to be the following set of monomials:

$$B_3 = \left\{ x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_{k+1} - y_1) \quad \text{not containing the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \right\} \tag{2.3}$$

over all $0 \leq \ell_j \leq n - 1 - j$ and $1 \leq k \leq n - 1$.

Notice that there are natural projections from B_1 to the Specht module $V_{(n)}$ and from B_3 to the Specht module $V_{(n-1,1)}$ given by forgetting the x_i variables.

Theorem 2.1 ([13]). *The set $B_1 \cup B_3$ forms a higher Specht basis of $H^*(\mathrm{Hess}(S,h))$.*

The proof (see [13] for full details) uses the fact that $B_1 \cup B_2$ forms a $\mathbb{Z}-$basis of $H^*(\mathrm{Hess}(S,h))$, and constructs the transition matrix from $B_1 \cup B_2$ to $B_1 \cup B_3$ using the relations given in [9] to express the new elements in terms of the old basis. Then, we prove that this transition matrix is invertible, which requires the following lemma.

Lemma 2.2 ([13]). *If $f(x_1, \ldots, x_n)$ is a homogeneous polynomial in the ring $H^*(\mathrm{Hess}(S,h))$, then f can be expressed solely in terms of basis elements from B_1.*

Knowing that $B_1 \cup B_3$ forms a higher Specht basis of $H^*(\mathrm{Hess}(S,h))$ allows us to obtain a more direct proof of the following fact, by counting the number of monomials of each type in B_1 and B_3.

Corollary 2.3. *The dot action of S_n on $H^*(\mathrm{Hess}(S,h))$ decomposes into $h(1)(n-1)!$ copies of the trivial representation, and $(n - h(1))(n-2)!$ copies of the standard representation.*

3 Bijections between basis elements and P-tableaux

As seen in Section 1.1, there is a bijection between the set of standard tableaux of shape λ with basis elements of the Specht module V_λ, given by the construction of the basis elements. Further, from Equation 1.2, we have an explicit connection between the number of basis elements of $H^*(\mathrm{Hess}(S,h))$ of each degree and the set of P_h-tableaux with each number of inversions. In particular, there should be bijections between the higher Specht basis elements and the sets of P_h-tableaux with shape corresponding to the Specht polynomials in the basis.

3.1 Regular Nilpotent Hessenberg Varieties

In the case of regular nilpotent Hessenberg varieties, a polynomial presentation of the cohomology ring is known for any Hessenberg function h.

Proposition 3.1 ([7], Corollary 7.3)**.** *Let N be a regular nilpotent matrix, and let $h : [n] \to [n]$ be a Hessenberg function. Then the following set of monomials form an additive basis for $H^*(Hess(N,h))$:*

$$\mathcal{N}_h := \left\{ x_1^{i_1} \cdots x_n^{i_n} \mid 0 \le i_k \le h(k) - k \text{ for } 1 \le k < n \right\}$$

In [1], Abe et al. show that the cohomology rings of regular nilpotent Hessenberg varieties are isomorphic to the fixed points of the cohomology rings for regular semisimple Hessenberg varieties. In particular, these are the pieces of $H^*(\mathrm{Hess}(S,h))$ which decompose into trivial S_n-modules, corresponding to the Specht module $V_{(n)}$.

Define $PT(h,\lambda)$ to be the set of P_h-tableaux of shape λ. Taking the transpose partition for $\lambda = (n)$ (because of Equation 1.2), we form a map φ between $\mathcal{N}_h$ and $PT(h,(1^n))$ for any Hessenberg function h.

Definition 3.2. *Let $x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{N}_h$.*

- *Begin with a P_h-tableau T of a single box whose entry is n.*
- *For each $k = n-1, \ldots 1$:*
  - *If $i_k = 0$, insert k into a new box at the bottom of T, so that k occurs directly below some $\ell > k$.*
  - *If $i_k > 0$, then since $i_k \le h(k) - k$, we have $k < h(k)$. List the entries $k+1, \ldots, h(k)$, which already exist in T, in order from the lowest to highest row position in T. Insert k in a new box directly above the i_k-th lowest entry of this list.*

Define $\varphi(x_1^{i_1} \cdots x_n^{i_n}) \in PT(h,(1^n))$ to be the resulting tableau from this process.

Example 3.3. *Let $h = (2,3,5,5,5)$, and consider the monomial $x_1x_3x_4 \in \mathcal{N}_h$. We construct $\varphi(x_1x_3x_4)$ as follows.*

$$\begin{array}{|c|}\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 5\\\hline\end{array}\qquad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 5\\\hline 2\\\hline\end{array}\qquad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 5\\\hline 1\\\hline 2\\\hline\end{array}$$

We start with a single box containing a 5. Then, to insert 4 with $i_4 = 1$ P_h-inversion, we insert the 4 above the 5. To insert 3 with $i_3 = 1$ P_h-inversion, we insert the 3 above the 5 but below the 4. Notice that at each step, the number of elements in P_h greater than k that are incomparable to k is $h(k) - k$, which is also the largest possible power i_k.

In [13], we prove that φ is a bijection, which is weight preserving in the following way: If m is a monomial of degree d, then $\varphi(m)$ has d P_h-inversions.

Theorem 3.4 ([13])**.** *The map φ is a well-defined, weight-preserving bijection.*

3.2 Regular Semisimple Hessenberg Varieties

Now we turn our attention to regular semisimple Hessenberg varieties. In this section, S is a regular semisimple matrix and $h = (h(1), n, \ldots, n)$. Recall the partial set of basis elements B_1 defined in Equation 2.1, and recall that S_n fixes these monomials, since they contain no y_i variable. These correspond to basis elements of the trivial Specht module $V_{(n)}$. Again, we take the transpose partition, and define a map ψ between B_1 and $PT(h, (1^n))$ as follows.

Definition 3.5. *Let $x_1^{i_1} \cdots x_n^{i_n} \in B_1$.*

- *Begin with a P_h-tableau T of a single box whose entry is n.*
- *For each $k = n-1, \ldots, 1$, insert k into T above exactly i_k of the existing entries.*
- *Let k' be the smallest index in $1, \ldots, h(1)$ such that $i_{k'} = 0$, which exists by the definition of B_1. By this construction, after inserting n through 1, k' will be on the bottom of T.*
  - *If $k' = 1$, then define $\psi(x_1^{i_1} \cdots x_n^{i_n})$ to be T.*
  - *If $1 < k' \le h(1)$, then slide the entry k' up until it is directly below the 1, and define $\psi(x_1^{i_1} \cdots x_n^{i_n})$ to be T after this slide.*

Example 3.6. *Let $h = (3,5,5,5,5)$, and consider the monomial $x_1^2x_3x_4 \in B_1$. We construct $\varphi(x_1^2x_3x_4)$ as follows.*

$$\begin{array}{c}\boxed{5}\end{array} \qquad \begin{array}{c}\boxed{4}\\ \boxed{5}\end{array} \qquad \begin{array}{c}\boxed{4}\\ \boxed{3}\\ \boxed{5}\end{array} \qquad \begin{array}{c}\boxed{4}\\ \boxed{3}\\ \boxed{5}\\ \boxed{2}\end{array} \qquad \begin{array}{c}\boxed{4}\\ \boxed{3}\\ \boxed{1}\\ \boxed{5}\\ \boxed{2}\end{array} \qquad \begin{array}{c}\boxed{4}\\ \boxed{3}\\ \boxed{1}\\ \boxed{2}\\ \boxed{5}\end{array}$$

We start with a single box containing a 5. Then we insert the 4 above one existing entry, the 3 above one existing entry, the 2 above no existing entries, and the 1 above two existing entries. Since $1 <_{P_h} 5$ are comparable, the resulting tableau is not a P_h-tableau, so we shift the 2 (which is incomparable to the 1) to be directly below the 1. In the second-to-last tableau, the number of P_N-inversions with each k as the smaller entry is exactly i_k, so reading these inversions returns the monomial $x_1^2x_3x_4$.

In [13], we find the inverse map of ψ to prove the following theorem.

Theorem 3.7 ([13])**.** *The map ψ is a well-defined bijection.*

We now construct a map for the set of basis elements B_3 defined in Equation 2.3. Define $PSPT(h, \lambda)$ to be the set of pairs (S, T) where S is a standard tableau and T is a P_h-tableau, both of shape λ. Since the monomials in B_3 correspond to the Specht polynomials in the Specht module $V_{(n-1,1)}$, we construct the map τ to the set $PSPT(h,(2,1^{n-2}))$, with the x_i variables corresponding to the P_h-tableau and the y_i variables corresponding to the standard tableau.

Definition 3.8. *Let $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) \in B_3$.*

- *Define S to be the unique standard tableau of shape $(2,1^{n-2})$ with entries 1 and k in the bottom row.*
- *Let j be the largest entry among $\{h(1)+1,\dots,n\}$ such that the exponent ℓ_{n-j+1} on x_j is zero, which exists by the definition of B_3.*
- *Initialize a tableau T with a single row of two boxes, containing a 1 and j.*
- *For each $i = 2,\dots,n$, other than j, insert i into the left column so that it is under exactly $(i-2) - \ell_{n-i+1}$ of the current entries in the left column.*

Define $\tau(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)) \in PSPT(h,(2,1^{n-2}))$ to be the pair (S,T) resulting from this construction.

Example 3.9. *Let $h = (3,5,5,5,5)$, and consider the monomial $x_5^2 x_3(y_3 - y_1) \in B_3$. Note that $j = 4$ is the largest index where x_j has an exponent of zero. We construct $\tau(x_5^2 x_3(y_2 - y_1))$ as follows.*

$$S = \begin{array}{|c|c} \hline 5 & \multicolumn{1}{c}{} \\ \hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 3 & \multicolumn{1}{c}{} \\ \hline 5 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array} = T$$

S is defined to be the unique standard Young tableaux of shape $(2, 1^{n-2})$ with a 1 and 3 in the bottom row. Then, to form T, we start with a single row containing a 1 and a 4. We then insert a 2 in the left column underneath $(2-2) - 0 = 0$ entries, a 3 in the left column underneath $(3-2) - 1 = 0$ entries, and a 5 in the left column underneath $(5-2) - 2 = 1$ entry.

In [13], we again find the inverse map to prove the following theorem.

Theorem 3.10 ([13])**.** *The map τ is a well-defined bijection.*

This map is almost weight-reversing, since the exponents on the x_i terms correspond to inversions that are missing in the tableaux, since we insert i so that it forms $(i-2) - \ell_{n-i+1})$ inversions. In future work, we hope to use bijections like these to extrapolate potential bases for $H^*(\mathrm{Hess}(S, h))$ in other cases.

4 Poincaré polynomials of Hessenberg varieties

Given a graded vector space V over a field k, if $V = \bigoplus_{i \in \mathbb{N}} V_i$ with each subspace V_i consisting of vectors of degree i being finite dimensional, then the **Poincaré polynomial** of V is

$$\mathrm{Poin}(V, q) = \sum_{i \in \mathbb{N}} \dim_k(V_i) q^i$$

Then, for an algebraic variety X with graded cohomology ring $H^*(X)$, we define the Poincaré polynomial of X to be $\mathrm{Poin}(X, q) := \mathrm{Poin}(H^*(X), q)$. From Equation 1.2, we can write the Poincaré polynomial of a regular semisimple Hessenberg variety in the following way:

$$\mathrm{Poin}(\mathrm{Hess}(S, h), q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{PT}(h, \lambda)} q^{\mathrm{inv}_h(T)} \right) \#\mathrm{SYT}(\lambda) \tag{4.1}$$

since the dimension of the irreducible Specht module V_λ is the number of standard tableaux of shape λ. We use this formula to provide an alternate proof of the formula of the Poincaré polynomial for $\mathrm{Hess}(S, h)$ when $h = (h(1), n, \ldots, n)$, originally calculated by Abe, Horiguchi, and Masuda in [9]. Recall that the q-analogue of n is $(n)_q = (1 + q + \cdots + q^{n-1})$, and the q-analogue of $n!$ is $(n)_q! = (n)_q(n-1)_q \cdots (1)_q$. We present the full proof here, as it illustrates the new combinatorial method using P_h-tableaux.

Theorem 4.1 ([9], Lemma 3.2)**.** *If $h = (h(1), n, \ldots, n)$, then the Poincaré polynomial of $\mathrm{Hess}(S, h)$ is given by*

$$\begin{aligned}\mathrm{Poin}(\mathrm{Hess}(S,h),q) &= \frac{1-q^{h(1)}}{1-q}\prod_{j=1}^{n-1}\frac{1-q^j}{1-q} + (n-1)q^{h(1)-1}\frac{1-q^{n-h(1)}}{1-q}\prod_{j=1}^{n-2}\frac{1-q^j}{1-q}\\ &= h(1)_q(n-1)_q! + (n-1)q^{h(1)-1}(n-h(1))_q(n-2)_q!\end{aligned}$$

Proof. From above, we know that

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = \sum_{\lambda\vdash n}\left(\sum_{T\in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)}\right)\#\mathrm{SYT}(\lambda).$$

Let $h = (h(1), n, \ldots, n)$. All chains in P_h have length two and include the element 1. Since distinct rows in a P_h tableaux need to contain entries from distinct chains in P_h, the only shapes λ with a nonzero number of P_h-tableaux are $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$. Further, we have that $\#\mathrm{SYT}(\lambda) = 1$ and $\#\mathrm{SYT}(\mu) = n - 1$.

For $\lambda = (1^n)$, we need to count the P_h-inversions in the P_h tableaux of this shape. Since the element 1 is incomparable to 2 through $h(1)$, it can form between 0 and $h(1) - 1$ inversions as the smaller entry. For each $i = 2, \ldots, n$, the entry i can form up to $n - i$ inversions as the smaller entry. Hence, we get that

$$\sum_{T\in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} = (1 + q + \cdots + q^{h(1)-1})(1 + q + \cdots + q^{n-2})! = h(1)_q(n-1)_q!.$$

For $\mu = (2, 1^{n-2})$, the bottom row of any P_h-tableaux of shape μ must be filled with entries from a chain in P_h, so it contains a 1 and an i for some $i = h(1) + 1, \ldots, n$. Then, since $i > 1$, it is incomparable with all other $j \neq 1$, so the entry i in the bottom row forms inversions as the larger entry with the entries $2, \ldots, i - 1$, of which there are $i - 2$. So this entry contributes between $h(1) - 1$ and $n - 2$ inversions to the P_h-tableaux as the larger entry. Then, for the column entries of $j = 2, \ldots, n$ and $j \neq i$, if $j < i$, then j forms an inversion with i where j is the smaller entry (which was already counted), and can form an inversion as the smaller entry with the other $n - j - 1$ entries larger than j. If $j > i$, then j does not form an inversion with i, and can form an inversion as the smaller entry with any of the $n - j$ entries larger than j. In each case, there is a unique placement for j giving each set of inversions. Hence we have

$$\sum_{T\in \mathrm{PT}(h,\mu)} q^{\mathrm{inv}_h(T)} = (q^{h(1)-1} + \cdots + q^{n-1})(1 + \cdots + q^{n-3})! = q^{h(1)-1}(n-h(1))_q(n-2)_q!$$

Therefore, for $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$, we have the Poincaré polynomial of $\mathrm{Hess}(S, h)$ as follows:

$$\begin{aligned}\mathrm{Poin}(\mathrm{Hess}(S,h),q) &= \sum_{T \in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} + (n-1) \sum_{T \in \mathrm{PT}(h,\mu)} q^{\mathrm{inv}_h(T)} \\ &= h(1)_q (n-1)_q! + (n-1) q^{h(1)-1} (n-h(1))_q (n-2)_q!\end{aligned}$$

This completes the proof. □

These methods provide a new, combinatorial means of finding Poincaré polynomials of regular semisimple Hessenberg varieties, which may be useful in further understanding the basis decomposition of their cohomology rings.

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Charge formulas for Macdonald polynomials at $t = 0$ from multiline queues and diagrams

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Abstract. Multiline queues are combinatorial objects coming from probability theory that give formulas for the q-Whittaker specialization $P_\lambda(X; q, 0)$ of the Macdonald polynomials. We define a charge statistic and an RSK-esque procedure on multiline queues that naturally recovers the Schur expansion of $P_\lambda(X; q, 0)$. We extend these results to *generalized multiline queues*, which are in bijection with binary matrices, and obtain a new family of formulas for $P_\lambda(X; q, 0)$ in terms of these objects. Multiline diagrams are the plethystic analogs of multiline queues that were recently found to give a formula for the modified Hall–Littlewood polynomials $\widetilde{H}_\lambda(X; q, 0)$. We obtain formulas for the latter through a cocharge statistic and an RSK-esque procedure on multiline diagrams.

Keywords: multiline queues, multiline diagrams, Macdonald polynomials, q-Whittaker, Hall–Littlewood, crystal operators, RSK, charge, cocharge.

1 Introduction

Macdonald polynomials $P_\lambda(X; q, t)$ [10] are symmetric functions in the variables $X = x_1, x_2, \dots$ with coefficients in $\mathbb{Q}(q, t)$. They are indexed by partitions, and characterized as the unique basis satisfying certain triangularity and orthogonality axioms. They contain as specializations the *q-Whittaker polynomials* $P_\lambda(X; q, 0)$, the Hall–Littlewood polynomials $P_\lambda(X; 0, t)$, the Schur functions $s_\lambda = P_\lambda(X; 0, 0)$, and are connected to many other important families of symmetric functions. The *modified Macdonald polynomials* $\widetilde{H}_\lambda(X; q, t)$ were introduced by Garsia and Haiman [6] as a combinatorial version of $P_\lambda(X; q, t)$. They are obtained through plethysm from a scaled form J_λ of P_λ as $\widetilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda[X/(1 - t^{-1}); q, t^{-1}]$ (see [7] for details). The *modified Hall–Littlewood polynomial* is the specialization $\widetilde{H}_\lambda(X; q, 0)$ (which is equal to $\widetilde{H}_{\lambda'}(X; 0, q)$).

Expanding the q-Whittaker and the modified Hall–Littlewood polynomials in the Schur basis yields Kotska–Foulkes and modified Kotska–Foulkes coefficients:

$$P_\lambda(X; q, 0) = \sum_{\mu \le \lambda} K_{\lambda\mu}(q, 0) s_\mu(X) \quad \text{and} \quad \widetilde{H}_\lambda(X; q, 0) = \sum_{\mu \le \lambda} \widetilde{K}_{\lambda\mu}(q, 0) s_\mu(X). \tag{1.1}$$

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At $q = 0$, Lascoux and Schützenberger [8] gave a charge formula for $K_{\lambda\mu}(0,t)$ as a sum over semistandard tableaux of shape λ and content μ. The relation to the Kotska–Foulkes coefficients is given by the following set of formulas (see Definition 2.3):

$$K_{\lambda\mu}(0,t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(T)}, \quad K_{\lambda\mu}(q,0) = K_{\lambda'\mu'}(0,q),$$
$$\widetilde{K}_{\lambda\mu}(0,t) = \widetilde{K}_{\lambda\mu'}(t,0) = t^{n(\mu)}K_{\lambda\mu}(0,1/t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cocharge}(T)}.$$

In this abstract, we study $P_\lambda(X;q,0)$ and $\widetilde{H}_\lambda(X;q,0)$ through multiline queues and multiline diagrams. Our constructions recover classical results, and provide variations and simplifications of formulas for these polynomials. This abstract is based on [11].

In Section 3, we introduce a weight-preserving RSK-esque procedure on multiline queues from which several classical results immediately follow, including (1.1) and the Cauchy identities. In Section 4, we obtain a new formula for the modified Hall–Littlewood polynomials via a cocharge statistic on multiline diagrams.

Theorem 1.1. *Let λ be a partition. The modified Hall–Littlewood polynomial is given by*

$$\widetilde{H}_\lambda(x_1,\ldots,x_n;q,0) = \sum_{D \in \mathrm{MLD}(\lambda,n)} q^{\widetilde{\mathrm{maj}}(D)} x^D = \sum_{D \in \mathrm{MLD}(\lambda,n)} q^{\mathrm{cocharge}(\widetilde{\mathrm{cw}}(D))} x^D. \tag{1.2}$$

We also get new formulas for $K_{\lambda\mu}(q,0)$ and $\widetilde{K}_{\lambda\mu}(q,0)$, bypassing the charge statistic.

Theorem 1.2. *For a partition ν, let $B_\nu \in \mathrm{MLQ}_0(\nu,\ell(\nu))$ be the multiline queue with all balls left-justified and let $\widetilde{B}_\nu \in \mathrm{MLD}_0(\nu,\ell(\nu))$ be the diagonal multiline diagram of type ν. Then*

$$K_{\lambda\mu}(q,0) = \sum_{\substack{M \in \mathrm{MLQ}(\mu',\lambda') \\ \rho_N(M)=B_{\lambda'}}} q^{\mathrm{maj}(M)} \quad \textit{and} \quad \widetilde{K}_{\lambda\mu}(q,0) = \sum_{\substack{D \in \mathrm{MLD}(\mu',\lambda) \\ \widetilde{\rho}_N(D)=\widetilde{B}_\lambda}} q^{\widetilde{\mathrm{maj}}(D)}.$$

Finally, in Section 5 we extend our results to generalized multiline queues, obtaining a new family of formulas, indexed by compositions, for the q-Whittaker polynomials.

Theorem 1.3. *Let λ be a partition, n an integer, and let α be a composition with $\alpha^+ = \lambda'$. Then*

$$P_\lambda(x_1,\ldots,x_n;q,0) = \sum_{M \in \mathrm{GMLQ}(\alpha,n)} q^{\mathrm{maj}_G(M)} x^M.$$

2 Preliminaries

Definition 2.1. *The* charge *of a permutation $\sigma \in \mathfrak{S}_n$ is* $\mathrm{charge}(\sigma) = \sum_{i \notin Des(\sigma)} (n-i)$, *where* $\mathrm{Des}(\sigma) = \{i : \sigma^{-1}(i) > \sigma^{-1}(i-1)\}$. *The* cocharge *is* $\mathrm{cocharge}(\sigma) = \binom{n}{2} - \mathrm{charge}(\sigma)$.

This definition generalizes to words with partition content by splitting the word into *charge subwords*. Let w be a word with content $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. Extract the first subword $w^{(1)}$ by scanning w from left to right and finding the first occurrence of its largest letter $k := \mu_1'$, then $k-1, \ldots, 2, 1$, looping back around the word whenever needed. This subword $w^{(1)}$ is then extracted from w, and the remaining charge subwords are obtained recursively from the remaining letters, which now have content $(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_k - 1)$. For each i, $w^{(i)}$ can be thought of as permutations in $\mathfrak{S}_{\mu_i'}$.

Definition 2.2. *For a word w with partition content $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, its* charge *is given by* $\mathrm{charge}(w) = \mathrm{charge}(w^{(1)}) + \mathrm{charge}(w^{(2)}) + \ldots + \mathrm{charge}(w^{(k)})$.

Definition 2.3. *For a semistandard Young tableau T (in French notation), define its* row reading word, *denoted by* $\mathrm{rrw}(T)$, *to be the word obtained by recording the entries of the rows of T from top to bottom and from left to right within each row. If T has partition content, the* charge *of T is given by* $\mathrm{charge}(T) = \mathrm{charge}(\mathrm{rrw}(T))$.

We will make use of two types of related operations acting on words, described below. See [11] for details on how $\mathrm{charge}(w)$ can be restated in terms of these operations.

Definition 2.4 (Classical and cylindrical matching operators)**.** *Let n be a positive integer and let w be a word in the alphabet $\{1, \ldots, n\}$. For $1 \leq i < n$, define $\pi_i(w)$ to be a word in open and closed parentheses $\{(,)\}$ that is obtained by reading w from left to right and recording a "(" for each $i+1$ and a ")" for each i. The signature rule (see, e.g. [3]) is the procedure of iteratively matching pairs of open and closed parentheses whenever they are adjacent or whenever there are only matched parentheses in between. Then $\pi_i(w)$ contains the data of which instances of i and $i+1$ in w are matched or unmatched following the signature rule applied to $\pi_i(w)$.*

Let $\pi_i^c(w)$ represent the word $\pi_i(w)$ on a circle, so that open and closed parentheses may match by wrapping around the word. Then the cylindrically unmatched *$i+1$'s and i's in w correspond respectively to the (cylindrically) unmatched open and closed parentheses in $\pi_i^c(w)$, according to the signature rule executed on a circle. The* wrapping *$i+1$'s and i's in w correspond respectively to the cylindrically matched open and closed parentheses in $\pi_i^c(w)$ that are unmatched in $\pi_i(w)$.*

Example 2.5. *For $w = 312214342131232$, the unmatched parentheses are show in red in* $\pi_1(w) = $) (( ) ( )) ((*and* $\pi_1^c(w) = $) (( ) ( )) ((. *This corresponds to the unmatched* 1*'s and* 2*'s indicated by* ˆ *and the cylindrically unmatched* 2 *underlined:* $w = 3\hat{1}2214342131\underline{\hat{2}}3\hat{2}$.

3 Multiline queues and charge

The multiline queues we study are the $t = 0$ specialization of the multiline queues and their statistics defined by Corteel, Williams, and the first author in [4], and are in correspondence with the classical multiline queues introduced by Ferrari and Martin [5].

Definition 3.1. *Fix a partition λ, an integer $n \geq \ell(\lambda)$, and set $L := \lambda_1$. A* multiline queue *of* shape (λ, n) *is an arrangement of balls on an array with L rows numbered 1 through L from bottom to top and n columns numbered 1 through n from left to right, such that row j contains λ'_j balls. Denote the set of multiline queues of shape (λ, n) by $\mathrm{MLQ}(\lambda, n)$.*

A multiline queue can be viewed as a binary matrix by corresponding balls to 1's and vacancies to 0's. We represent a multiline queue as a tuple $M = (B_1, \ldots, B_L)$ of L subsets of $\{1, \ldots, n\}$ where $B_j = (b_1, \ldots, b_{\lambda'_j})$ is the set of labels of columns containing balls in row j of M. A site (r, j) of M refers to the cell in column j of row r of M; we say the site is empty if $j \notin B_r$, and contains a ball otherwise.

Definition 3.2. *The* column word *of a multiline queue M, denoted $\mathrm{cw}(M)$, is obtained by recording the row number of each ball by scanning the columns of M from* left to right *and from* top to bottom *within each column. For M in Example 3.14, $\mathrm{cw}(M) = 421|3|41|521|32$.*

Definition 3.3. *Let $n > 0$ and $S, T \subseteq [n]$, where we shall consider (S, T) as rows 1 and 2 of a multiline queue. Then $\pi(S, T) = \pi_1(\mathrm{cw}(S, T))$. Unmatched open parenthesis are referred to as* unmatched above *elements, and unmatched closed parenthesis are* unmatched below.

Example 3.4. *Let $S = \{2, 3, 5\}$ and $T = \{1, 4, 5, 6\}$ corresponding to rows 1 and 2 from B in Example 3.18. Then $\pi(S, T) = (\,)\,)\,(\,(\,)\,($ where the unmatched parentheses are in red, corresponding to $4, 6 \in T$ unmatched above and $3 \in S$ unmatched below.*

The Ferrari–Martin pairing process is an algorithm that deterministically assigns a label to each ball in a multiline queue M to obtain a labelled multiline queue $L(M)$.

Definition 3.5 (Ferrari–Martin algorithm)**.** *Let $M = (B_1, \ldots, B_L)$ be a multiline queue of shape (λ, n). Define the* labelled multiline queue $L(M)$ *by replicating M and sequentially labelling the balls, as follows. For each row r for $r = L, L-1, \ldots, 2$, each unlabelled ball in B_r is labelled r. Next, for $\ell = L, L-1, \ldots, r$, let $\mathrm{cw}(M)^{(r,\ell)}$ be the restriction of $\mathrm{cw}(M)$ to the balls labelled ℓ in B_r and the unlabelled balls in B_{r-1}. The balls in row $i-1$ that are cylindrically matched in $\pi^c_{r-1}(\mathrm{cw}(M)^{(r,\ell)})$ acquire the label ℓ. To complete the process, all unpaired balls in row 1 are labelled "1". Such a labelling is shown in Example 3.14.*

Definition 3.6. *Let $M \in \mathrm{MLQ}(\lambda, n)$ with labelling $L(M)$, and let $m_{r,\ell}$ be the number of wrapping balls labelled ℓ when cylindrically matched from row r to row $r-1$ in $L(M)$. Then*

$$\mathrm{maj}(M) = \sum_{2 \leq r \leq L} \sum_{r \leq \ell \leq L} m_{r,\ell}\,(\ell - r + 1).$$

When we restrict to multiline queues with major index equal to zero we obtain a set of objects that is in bijection with semistandard tableaux [11].

Definition 3.7. *If M satisfies* $\mathrm{maj}(M) = 0$*, we call it* non-wrapping. *We will denote the set of non-wrapping multiline queues of shape (λ, n) by $\mathrm{MLQ}_0(\lambda, n)$.*

We have an expression for Schur functions in terms of multiline queues [4].

$$s_\lambda(x_1, \dots, x_n) = \sum_{M \in \mathrm{MLQ}_0(\lambda,n)} x^M. \tag{3.1}$$

Theorem 3.8. *Let M be a multiline queue. Then* $\mathrm{maj}(M) = \mathrm{charge}(\mathrm{cw}(M))$.

Notably, the theorem above eliminates the need for the Ferrari–Martin algorithm to determine $\mathrm{maj}(M)$. Thus we obtain the following formula for $P_\lambda(X; q, 0)$.

Theorem 3.9. *Let λ be a partition. The q-Whittaker polynomial is given by*

$$P_\lambda(x_1, \dots, x_n; q, 0) = \sum_{M \in \mathrm{MLQ}(\lambda,n)} q^{\mathrm{maj}(M)} x^M = \sum_{M \in \mathrm{MLQ}(\lambda,n)} q^{\mathrm{charge}(\mathrm{cw}(M))} x^M \tag{3.2}$$

where the first equality is due to [4].

3.1 Collapsing on multiline queues via row operators

Let $\mathcal{M}_{(2)}$ be the set of binary matrices with finite support, and let $\mathcal{M}_{(2)}(L, n)$ be the set of such matrices with size $L \times n$. For $B \in \mathcal{M}_{(2)}(L, n)$ and every $1 \leq j \leq L$, let $B_j \subseteq [n]$ be the set of column labels of the balls (1's) of row j of B.

Definition 3.10. *Let $B \in \mathcal{M}_{(2)}$. The* dropping operator *e_i acts on B by dropping the ball corresponding to the leftmost unmatched above element in $\pi(B_i, B_{i+1})$ from B_{i+1} to B_i. Define $e_i^\star(B)$ to drop all balls that are unmatched above from B_{i+1} to B_i. By definition, $e_i(e_i^\star) = e_i^\star$.*

For $M \in \mathrm{MLQ}(\lambda, n)$, the operators e_i act on M as the classical crystal operators E_i (the standard lowering crystal operators in type A on words; see [3]) act on $\mathrm{cw}(M)$, so that $\mathrm{cw}(e_i(M)) = E_i(\mathrm{cw}(M))$. Moreover, the operators $e_i^\star$, which maximally apply e_i, satisfy the braid relations (i) $e_i^\star e_{i+1}^\star e_i^\star = e_{i+1}^\star e_i^\star e_{i+1}^\star$, and (ii) $e_i^\star e_j^\star = e_j^\star e_i^\star$ whenever $|i - j| \geq 2$. Applying the operators $e_i^\star$ from bottom to top defines a procedure that we call collapsing.

Definition 3.11. *For a pair of integers a and b with $a \leq b$, let $[a, b]$ be the interval of integers. Define $e_{[a,b]}^\star := e_a^\star e_{a+1}^\star \cdots e_b^\star$, where we use multiplicative notation for composition.*

Definition 3.12 (Collapsing). *Let $L, n > 0$. Define the* collapsing *map on $\mathcal{M}_{(2)}(L, n)$ as*

$$\rho \ : \ \mathcal{M}_{(2)}(L, n) \longrightarrow \bigcup_\mu \mathrm{MLQ}_0(\mu, n) \times \mathrm{SSYT}(\mu') \tag{3.3}$$

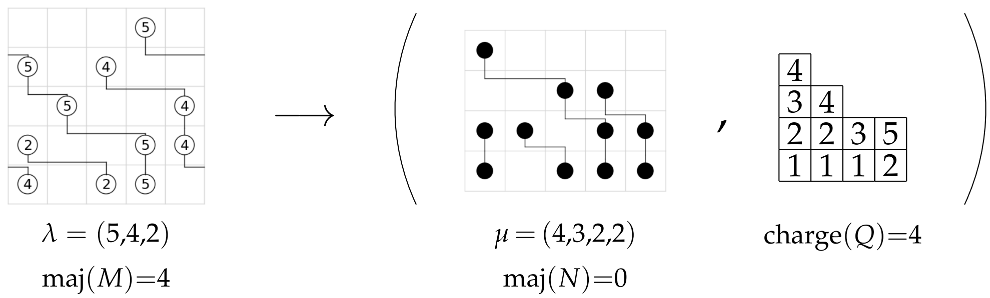
$$B \longmapsto (\rho_N(B), \rho_Q(B)) \tag{3.4}$$

where $\rho_N(B)$ is given by $\rho_N(B) = e_{[1,L-1]}^\star e_{[1,L-2]}^\star \cdots e_{[1,2]}^\star e_{[1,1]}^\star(B)$, and $\rho_Q(B)$ is the semistandard tableaux whose entries i record the difference in row content between $e_{[1,i]}^\star e_{[1,i-1]}^\star \cdots e_{[1,1]}^\star(B)$ and $e_{[1,i-1]}^\star e_{[1,i-2]}^\star \cdots e_{[1,1]}^\star(B)$ for $2 \leq i \leq L-1$, and between $e_{[1,1]}^\star(B)$ and B for $i = 1$.

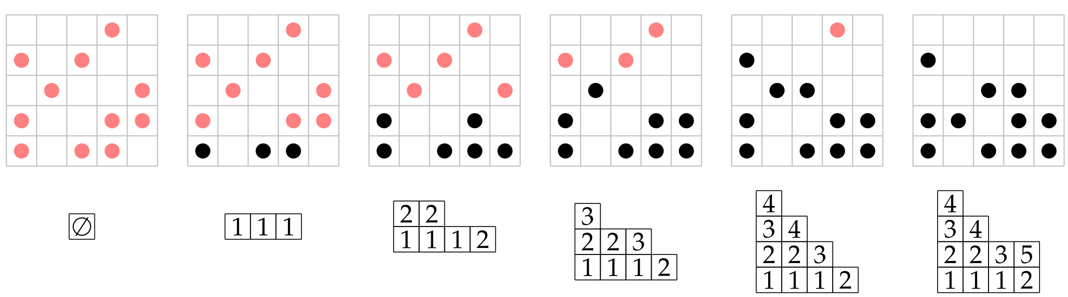
Restricting the previous map to the set of multiline queues $\mathrm{MLQ}(\lambda, n)$ yields a bijection to pairs of a non-wrapping multiline queue on n columns and a semistandard tableau of the conjugate shape with content λ'. By taking the preimage of $\{B_{\lambda'}\} \times \mathrm{SSYT}(\lambda, \mu)$ under this map we obtain Theorem 1.2 using the following result.

Theorem 3.13. *Let $M \in \mathrm{MLQ}(\lambda, n)$ be a multiline queue. Then* $\mathrm{maj}(M) = \mathrm{charge}(\rho_Q(M))$.

Example 3.14. *We show the collapsing $\rho(M) = (N, Q)$ of a multiline queue $M \in \mathrm{MLQ}(\lambda, 5)$ with $\lambda = (5, 4, 2)$.*



The step-by-step collapsing of the rows from bottom to top is shown, where the black balls are collapsed particles and the red/shaded balls are the remaining rows of the starting multiline queue, along with the recording tableaux corresponding to each step.



3.2 Multiline queue RSK

In [9], commuting crystal operators on rows and columns of integer matrices are introduced to recover some classical tableaux operations such as the RSK correspondence and jeu de taquin. These operators correspond to bi-directional collapsing in the setting of multiline queues (and in Section 4.2, multiline diagrams).

Definition 3.15. *For a matrix $B \in \mathcal{M}_{(2)}$, define $\mathrm{rot}(B)$ to be the rotation of B by 90° counterclockwise. We use the same notation to describe the rotation of a multiline queue M by identifying it with its associated binary matrix. We define $e_i^{\downarrow} = e_i$ from Definition 3.10 and $e_i^{\leftarrow} = \mathrm{rot}^{-1} \circ e_i \circ \mathrm{rot}$ as the operator that drops unmatched balls to the left. We also define $\rho^{\downarrow}(B) := \rho_N(B)$, and $\rho^{\leftarrow}(B) := \mathrm{rot}^{-1}(\rho_N(\mathrm{rot}(B)))$. See Example 3.18.*

Theorem 3.16. *Let $P(L, n)$ be the set of partitions λ with $\ell(\lambda) \le n$ and $\ell(\lambda') \le L$. The map*

$$\mathrm{mRSK} : \mathcal{M}_{(2)}(L, n) \longrightarrow \bigcup_{\lambda \in P(L,n)} \mathrm{MLQ}_0(\lambda, n) \times \mathrm{MLQ}_0(\lambda', L)$$

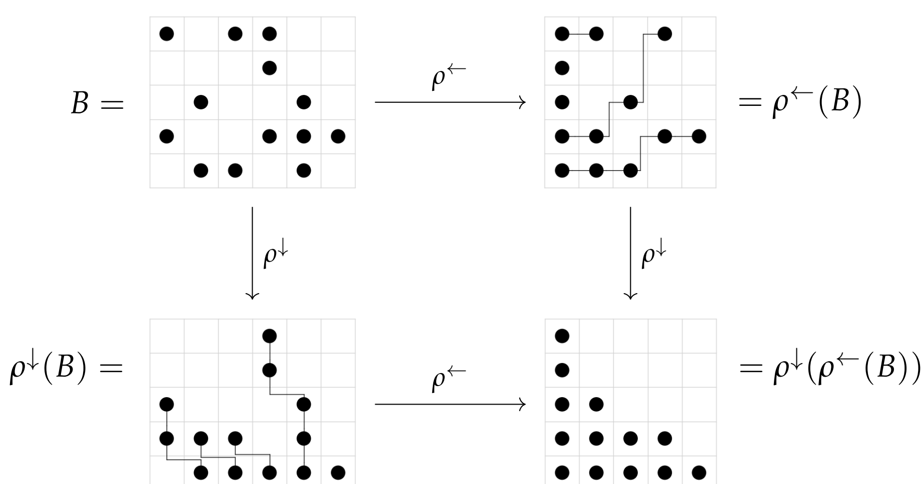
given by $\mathrm{mRSK}(B) = (\rho^{\downarrow}(B), \rho^{\leftarrow}(B))$ *is a bijection.*

The following fact can be obtained from [9, Lemma 1.3.7].

Lemma 3.17. *Let $B \in \mathcal{M}_{(2)}$. Then $e_i^{\downarrow}(e_j^{\leftarrow}(B)) = e_j^{\leftarrow}(e_i^{\downarrow}(B))$ for all i and j. Moreover, if B is a multiline queue,* $\mathrm{maj}(e_i^{\leftarrow}(B)) = \mathrm{maj}(B)$.

The previous lemma implies that $\rho^{\downarrow}(\rho^{\leftarrow}(B)) = \rho^{\leftarrow}(\rho^{\downarrow}(B))$. Since the major index is preserved while collapsing to the left when $M \in \mathrm{MLQ}(\lambda)$, examining the construction of the recording tableau $\rho_Q(\rho^{\leftarrow}(M))$ leads to a simple proof of Theorem 3.13. Furthermore, Theorem 3.16 gives a bijective proof of the dual Cauchy identity in view of Equation (3.1): $\sum_\lambda s_\lambda(X) s_{\lambda'}(Y) = \prod_{i,j}(1 + x_i y_j)$.

Example 3.18. *For the matrix $B \in \mathcal{M}_{(2)}(5, 6)$ in the upper left, we show $\rho^{\leftarrow}(B)$ in the upper right, $\rho^{\downarrow}(B)$ in the bottom left, and the double collapsing in the bottom right.*



4 Multiline diagrams and cocharge

A *multiline diagram* is a configuration of balls on a rectangular grid with no restriction on the number of balls occupying each cell, and such that the number of balls in each row is weakly decreasing from bottom to top. Multiline diagrams have appeared in the context of a family of statistical mechanics processes called the *totally asymmetric zero range process* (see [2]). They are also in bijection with inversion-free Haglund–Haiman–Loehr tableaux [7] and in (weight preserving) bijection with queue-inversion-free tableaux [2], which give formulas for the modified Hall–Littlewood polynomials. Thus, as a reference to the plethystic correspondence between the q-Whittaker polynomials $P_\lambda(X;q,0)$ and the modified Hall–Littlewood polynomials $\widetilde{H}_\lambda(X;q,0)$, we think of multiline diagrams as the *plethystic analog* of multiline queues.

Definition 4.1. *Let λ be a partition and $n > 0$. A* multiline diagram *of shape (λ, n) is a configuration of particles on a $\lambda_1 \times n$ grid, such that each site can contain any number of particles, and row j contains λ'_j particles (labelled from bottom to top). Denote the set of multiline diagrams of type (λ, n) by* $\mathrm{MLD}(\lambda, n)$.

We represent a multiline diagram by the tuple $D = (D_1, \ldots, D_{\lambda_1})$, where each D_i is a *multiset* of $[n]$ of size λ'_i

Definition 4.2. *For a word $w = w_1 \ldots w_n$, define $\mathrm{rev}(w) = w_n \ldots w_1$. Define the multiline diagram column reading word as $\widetilde{\mathrm{cw}}(D) := \mathrm{rev}(\mathrm{cw}(D))$, where $\mathrm{cw}(D)$ is given by the multiline queue reading order. See Example 4.11 for reference.*

Definition 4.3. *Let $n > 0$ and let S, T be multisets in $[n]$; we shall consider (S, T) as rows 1 and 2 of a multiline diagram. Then $\widetilde{\pi}(S, T) = \pi(\widetilde{\mathrm{cw}}(S, T))$.*

Example 4.4. *Let $S = \{2, 3, 3, 3\}$ and $T = \{1, 3, 4, 4\}$, corresponding to the second and third rows of Example 4.11. Then $\widetilde{\pi}(S, T) = (\,(\,)\,)\,)\,(\,)\,($ where the unmatched parentheses are in red, corresponding to $1 \in T$ unmatched above and $3 \in S$ unmatched below.*

There is a pairing process on multiline diagrams, where particles are paired *strictly to the left*, that is analogous to the Ferrari–Martin algorithm and produces a major index statistic. See Example 4.11.

Definition 4.5 (Major index for multiline diagrams)**.** *The* major index *of a multiline diagram D, denoted by $\widetilde{\mathrm{maj}}(D)$, is determined by the non-wrapping pairings. Let $m_{r,\ell}(D)$ be the number of balls labelled ℓ that wrap when matched from row r to row $r-1$. Then*

$$\widetilde{\mathrm{maj}}(D) = \sum_{r,\ell} (\lambda'_r - m_{r,\ell})(r - \ell + 1).$$

The following lemma implies one of our main results, Theorem 1.1.

Lemma 4.6. *Let $D = (D_1, \ldots, D_L)$ be a multiline diagram. Then* $\widetilde{\mathrm{maj}}(D) = \mathrm{cocharge}(\widetilde{\mathrm{cw}}(D))$.

The lemma follows from the same argument as the proof of Theorem 3.8 with an appropriate modification to the parentheses matching algorithm.

4.1 Collapsing on multiline diagrams via row operators

Definition 4.7. *A multiline diagram $D \in \mathrm{MLD}(\lambda, n)$ is called* non-wrapping *if there are no wrapping pairings between any pair of rows. Denote the set of non-wrapping multiline diagrams by $\mathrm{MLD}_0(\lambda, n)$. Note that these multiline diagrams satisfy $\widetilde{\mathrm{maj}}(D) = n(\lambda')$.*

For each non-wrapping multiline diagram $D \in \mathrm{MLD}_0(\lambda, n)$, there is a unique semistandard tableau in $\mathrm{SSYT}(\lambda', n)$ whose row contents match those of D, implying that

$$s_\lambda(x_1, \ldots, x_n) = \sum_{D \in \mathrm{MLD}_0(\lambda', n)} x^D. \tag{4.1}$$

Definition 4.8. *Let $D = (D_1, \ldots, D_L)$ be a multiline diagram. The* dropping operator $\widetilde{e}_i$ *acts on D by moving the rightmost element unmatched above in $\widetilde{\pi}(D_i, D_{i+1})$ from D_{i+1} to D_i. The operator $\widetilde{e}_i^{\star}$ is defined as the operator that maximally applies $\widetilde{e}_i$, as an analog to Definition 3.10.*

Let $\mathcal{M}$ denote the set of nonnegative matrices with finite support, and let $\mathcal{M}(L, n)$ be the subset of such matrices of on L rows and n columns.

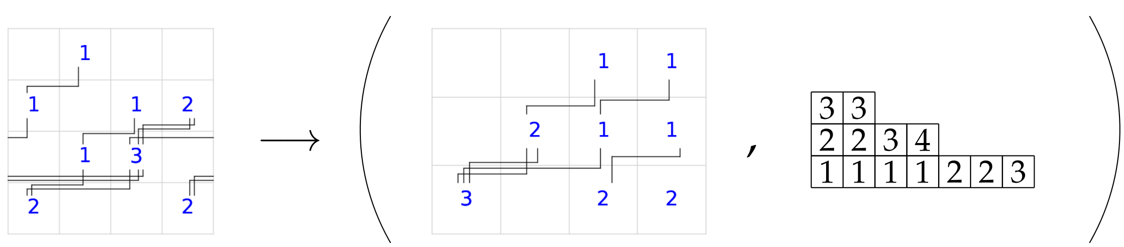
Definition 4.9 (Collapsing)**.** *Let L, n be positive integers. In analogy to Definition 3.12, define* collapsing *for nonnegative integer matrices $\mathcal{M}(L, n)$ by $\widetilde{\rho}(D) = (\widetilde{\rho}_N(D), \widetilde{\rho}_Q(D))$:*

$$\widetilde{\rho} \;:\; \mathcal{M}(L, n) \longrightarrow \bigcup_{\mu} \mathrm{MLD}_0(\mu, n) \times \mathrm{SSYT}(\mu'). \tag{4.2}$$

Restricting the collapsing map to the set of multiline diagrams $\mathrm{MLD}(\lambda, n)$, $\widetilde{\rho}$ yields a bijection to pairs of non-wrapping multiline diagrams and semistandard tableau of the conjugate shape with content λ. By taking the preimage of $\{\widetilde{B}_\lambda\} \times \mathrm{SSYT}(\lambda, \mu)$, we obtain the formula for $\widetilde{K}_{\lambda\mu}(q, 0)$ from Theorem 1.2 using the following theorem.

Theorem 4.10. *Let $D \in \mathrm{MLD}(\lambda, n)$ be a multiline diagram. Then $\widetilde{\mathrm{maj}}(D) = \mathrm{cocharge}(\widetilde{\rho}_Q(D))$.*

Example 4.11. *The collapsing of $D = (\{1, 1, 4, 4\}, \{2, 3, 3, 3\}, \{1, 3, 4, 4\}, \{2\}) \in \mathrm{MLD}(\lambda, 4)$ with $\lambda = (4, 3, 3, 3)$ is shown, with integers representing the number of particles at each site.*



$\widetilde{\mathrm{maj}}(D)$=9 $\quad$ $\mu \;=\; (3, 3, 2, 2, 1, 1, 1)$ $\quad$ $\mathrm{cocharge}(\widetilde{\rho}_Q(D)) = 9$

$\widetilde{\mathrm{cw}}(D)$=1133.2223.24.113

4.2 Multiline diagram RSK

Recall that multiline diagram pairing is done strictly to the left. With the appropriate modification on the construction of the parenthesis word $\widetilde{\pi}(S,T)$ we can set the pairing direction to be strictly to the right to get an equivalent set of objects.

Definition 4.12. *Let $P \in \{L,R\}$ (left or right) be a direction of pairing. The set of multiline diagrams of shape (λ,n) with pairing direction P is denoted by $\mathrm{MLD}_P(\lambda,n)$. Similarly, the set of non-wrapping multiline diagrams with pairing in direction P is denoted by $\mathrm{MLD}_{0,P}(\lambda,n)$.*

Definition 4.13. *For a pairing direction $P \in \{L,R\}$, let $\tilde{e}^{\downarrow}_{P,i}$ be the operator acting on matrices $D \in \mathcal{M}$ that drops the ball that is furthest in the opposite direction of P among balls in row $i+1$ that are unmatched above. Extending the definition of the rotation operators* rot *on matrices, we similarly define leftward operators $\tilde{e}^{\leftarrow}_{P,i} = \mathrm{rot}^{-1} \circ \tilde{e}^{\downarrow}_{P,i} \circ \mathrm{rot}$.*

The interplay between the pairing and collapsing directions plays an important role when defining the RSK analog for multiline diagrams. In particular, opposite pairing directions are required for the following crucial lemma to hold.

Lemma 4.14. *Let $D \in \mathcal{M}$. Then $\tilde{e}^{\downarrow}_{L,i}(\tilde{e}^{\leftarrow}_{R,j}(D)) = \tilde{e}^{\leftarrow}_{R,j}(\tilde{e}^{\downarrow}_{L,i}(D))$ for all i and j.*

From the definition of $\tilde{e}^{\downarrow}_{P,i}$ and in analogy to the presented collapsing procedures, collapsing downwards and leftwards with pairing direction left and right can be defined from these operators and from the rotation operator.

Theorem 4.15. *Let L,n be positive integers, and let $\mathcal{M}(L,n)$ represent the set of $L \times n$ nonnegative integer matrices. The following map, given by $\mathrm{dRSK}(B) = (\widetilde{\rho}^{\downarrow}_{L}(B), \widetilde{\rho}^{\leftarrow}_{R}(B))$, is a bijection:*

$$\mathrm{dRSK} \ : \ \mathcal{M}(L,n) \longrightarrow \bigcup_{\lambda \,:\, \lambda_1 \leq \min\{L,n\}} \mathrm{MLD}_{0,L}(\lambda,n) \times \mathrm{MLD}_{0,R}(\lambda,L).$$

This theorem, together with Equation (4.1), gives a bijective proof, using multiline diagrams, of the Cauchy identity $\sum_\lambda s_\lambda(X)s_\lambda(Y) = \prod_{i,j}(1-x_iy_j)^{-1}$. Moreover, by Theorem 4.15 and the fact that multiline queues and diagrams are in bijection [11], we have the following formulations of the (q,t)-Kostka polynomials at $t=0$ in terms of multiline queues and multiline diagrams.

Corollary 4.16. *The (modified) Kostka polynomial at $t=0$ is given by*

$$K_{\lambda\mu}(q,0) = \sum_{M \in \mathrm{MLQ}_0(\lambda,\mu)} q^{\mathrm{maj}(\mathrm{rot}(M))} \quad \textit{and} \quad \widetilde{K}_{\lambda\mu}(q,0) = \sum_{D \in \mathrm{MLD}_0(\lambda',\mu)} q^{\widetilde{\mathrm{maj}}(\mathrm{rot}(D))}.$$

5 Generalized multiline queues

A generalized multiline queue is a multiline queue in which we relax the condition that the number of balls in each row must be weakly decreasing from bottom to top.

Denote by α^+ the partition obtained by rearranging the parts of the composition α.

Definition 5.1. *Let λ be a partition, α a composition such that $\alpha^+ = \lambda'$, and $n > \ell(\lambda)$ a positive integer. A* generalized multiline queue *of type (α, n) is a tuple of subsets $(B_1, \dots, B_L)$ such that $B_j \subseteq [n]$ and $|B_j| = \alpha_j$ for $1 \leq j \leq L$. Denote the set of generalized multiline queues corresponding to a composition α by $\mathrm{GMLQ}(\alpha, n)$. Then $\mathrm{MLQ}(\lambda, n) = \mathrm{GMLQ}(\lambda', n)$.*

In generalized multiline queues we consider the vacancies to be "anti-particles". There is a pairing algorithm that generalizes the Ferrari–Martin procedure by sequentially assigning labels to both the particles and the anti-particles in M, by pairing sites between adjacent rows from top to bottom such that *particles are paired weakly to the right*, while *anti-particles are paired weakly to the left*, and propagating the labels upon pairing. This is done in a certain priority order: see [1, Section 2] for the details of the procedure. When applied to a (regular) multiline queue, the labelling of the particles coincides with that in Definition 3.5.

Definition 5.2. *Let $M \in \mathrm{GMLQ}(\alpha, n)$ with an associated labelling. For $1 \leq r, \ell \leq L$, let $m_{r,\ell}$ (*resp. *$a_{r,\ell}$) be the number of particles (*resp. *anti-particles) of type ℓ that wrap when pairing to the right (*resp. *left) from row r to row $r-1$, as shown in Example 5.5. Define*

$$\mathrm{maj}_G(M) = \sum_{1 \leq r, \ell \leq L} m_{r,\ell}(\ell - r + 1) - a_{r,\ell}(\ell - r + 1).$$

When $M \in \mathrm{MLQ}(\lambda, n)$, every anti-particle at row r is labelled $r-1$, so $\mathrm{maj}_G(M) = \mathrm{maj}(M)$.

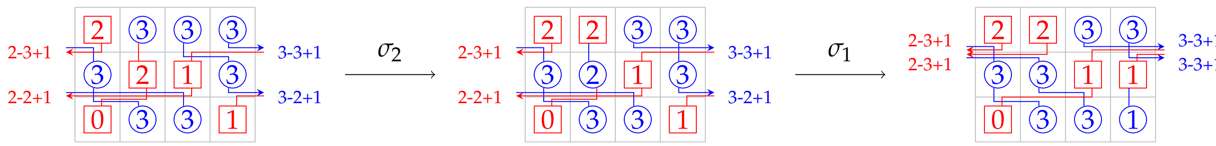
In [1], a row-swapping involution acting on GMLQ is defined to show that certain statistics and distributions are preserved between the set $\mathrm{GMLQ}(\alpha)$ and the set $\mathrm{MLQ}(\lambda)$, where $\alpha^+ = \lambda$. We generalize the result of [1] by showing that the distribution of the maj_G statistic is also preserved, thus recovering Theorem 1.3, which is a formula for $P_\lambda(x_1, \dots, x_n; q, 0)$ as a sum over $\mathrm{GMLQ}(\alpha, n)$ where $\alpha^+ = \lambda'$.

Definition 5.3. *For $(B_1, \dots, B_L) \in \mathrm{GMLQ}(\alpha, n)$ and $1 \leq i \leq L-1$, define the involution σ_i by* exchanging *cylindrically unmatched particles in $\pi_i^c(\mathrm{cw}(B_i, B_{i+1}))$ between B_i and B_{i+1}.*

Proposition 5.4. *Let α be a composition with $\alpha^+ = \lambda'$, $L := \ell(\alpha)$, $M \in \mathrm{GMLQ}(\alpha)$, and let $1 \leq i \leq L-1$. Then $\rho^{\downarrow}(M) = \rho^{\downarrow}(\sigma_i(M))$ and $\mathrm{maj}_G(M) = \mathrm{maj}_G(\sigma_i(M))$.*

Since the σ_i's satisfy the Moore–Coxeter relations and $\mathrm{maj}_G(M) = \mathrm{maj}(M)$ when $M \in \mathrm{MLQ}(\lambda)$, we obtain Theorem 1.3.

Example 5.5. *We show the labelled anti-particles (squares) and particles (circles) corresponding to* $M = (\{2,3\},\{1,4\},\{2,3,4\}) \in \mathrm{GMLQ}((2,2,3),4)$*,* $\sigma_2(M) = (\{2,3\},\{1,2,4\},\{3,4\}) \in \mathrm{GMLQ}((2,3,2),4)$ *and* $\sigma_1(\sigma_2(M)) = (\{2,3,4\},\{1,2\},\{3,4\}) \in \mathrm{GMLQ}((3,2,2),4)$*. We show the positive and negative contributions to* maj_G *for each, totalling* $\mathrm{maj}_G = 2$ *in each case.*



If M is a multiline queue, $\mathrm{maj}(M)$ can be computed directly from $\mathrm{charge}(\mathrm{cw}(M))$, bypassing the Ferrari–Martin procedure. There is a natural question of whether one could compute charge directly from a GMLQ without the anti-particles, and without the operators σ_i. This would allow us to define charge on generalized MLDs to get analogous results for $\widetilde{H}_\lambda(X;q,0)$.

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Slit-slide-sew bijections for constellations and quasiconstellations

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Abstract. We extend so-called slit-slide-sew bijections to constellations and quasiconstellations. We present an involution on the set of hypermaps given with an orientation, one distinguished corner, and one distinguished edge leading away from the corner while oriented in the given orientation. This involution reverts the orientation, exchanges the distinguished corner with the distinguished edge in some sense, slightly modifying the degrees of the incident faces in passing, while keeping all the other faces intact.

The involution specializes into a bijection interpreting combinatorial identities and allows to recover the counting formula for constellations or quasiconstellations with a given face degree distribution.

Keywords: bijection, plane map, hypermap, constellation, map enumeration.

1 Introduction

In the present work, we pursue the investigation of so-called *slit-slide-sew* bijections, introduced in [1] on forests and plane quadrangulations, and then further developed in [2, 3] on plane bipartite and quasibipartite maps. Here, we focus on a generalization of the latter, called *constellations* and *quasiconstellations*.

Hypermaps. Recall that a *plane map* is an embedding of a finite connected graph (possibly with multiple edges and loops) into the sphere, considered up to orientation-preserving homeomorphisms. Now fix an integer $p \geq 2$. A (plane) *p-hypermap* is a plane map whose faces are shaded either *dark* or *light* in such a way that

- adjacent faces do not have the same shade (one is dark, the other light);
- each dark face has degree p.

These actually generalize maps, which correspond to 2-hypermaps. In the terminology of hypermaps, light faces generalize faces and might be called *hyperfaces*, whereas dark faces generalize edges and are called *hyperedges*. We do not use this terminology here.

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A (plane) *p-constellation* is a p-hypermap such that the degrees of its light faces are all multiples of p. In a *p-hypermap*, a light face whose degree is not a multiple of p will be called a *flawed face*. A p-constellation is thus a p-hypermap without flawed faces. A *quasi-p-constellation* is a p-hypermap with exactly two flawed faces. Note that, in a p-hypermap, the sum of the degrees of the light faces is necessarily a multiple of p, since it is equal to the sum of the degrees of the dark faces, which are all p. As a result, a p-hypermap cannot have a single flawed face and, in a quasi-p-constellation, the two flawed faces have, modulo p, degrees $+k$ and $-k$ for some $0 < k < p$.

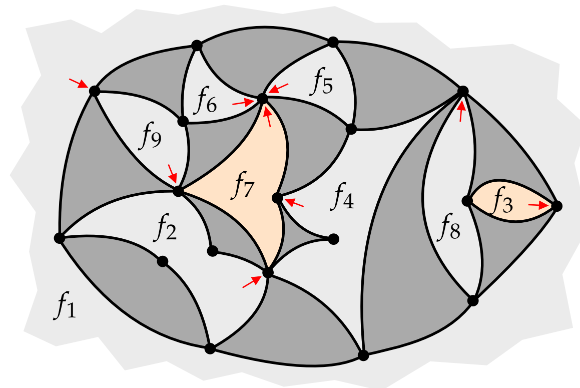


Figure 1: A quasi-3-constellation of type $(9, 6, 2, 6, 3, 3, 4, 3, 3)$. The two flawed faces, highlighted in orange, are f_3 and f_7. Every light face has a marked corner, always represented by a red arrowhead.

Enumeration. For an r-tuple $\boldsymbol{a} = (a_1, \ldots, a_r)$ of positive integers, let us denote by $C(\boldsymbol{a})$ the number of p-hypermaps with exactly r light faces, numbered $f_1, \ldots, f_r$ and of respective degrees $a_1, \ldots, a_r$, each bearing a marked corner[1]. The r-tuple $\boldsymbol{a}$ will be called the *type* of such p-hypermaps. See Figure 1. By elementary considerations and Euler's characteristic formula, the integers

$$E(\boldsymbol{a}) := \sum_{i=1}^{r} a_i\,, \qquad D(\boldsymbol{a}) := \frac{E(\boldsymbol{a})}{p}\,, \qquad \text{and} \qquad V(\boldsymbol{a}) := E(\boldsymbol{a}) - D(\boldsymbol{a}) - r + 2$$

are respectively the numbers of edges, dark faces, and vertices of p-hypermaps of type $\boldsymbol{a}$. Generalizing Tutte's so-called *formula of slicings* [6], it has been computed that, when at most two a_i's are not in $p\mathbb{N}$, that is, for p-constellations [4] or quasi-p-constellations [5], it holds that

$$C(\boldsymbol{a}) = c_{\boldsymbol{a}}\,\frac{\big(E(\boldsymbol{a}) - D(\boldsymbol{a}) - 1\big)!}{V(\boldsymbol{a})\,!}\prod_{i=1}^{r}\alpha(a_i), \qquad \text{where } \alpha(x) := \frac{x!}{\lfloor x/p\rfloor!\,\big(x - \lfloor x/p\rfloor - 1\big)!}$$
$$\text{and } c_{\boldsymbol{a}} = \begin{cases} 1 & \text{if } p \text{ divides every } a_i \\ p-1 & \text{otherwise} \end{cases}. \tag{1.1}$$

[1] Recall that a *corner* is an angular sector delimited by two consecutive edges around a vertex.

Combinatorial identities. In the present work, we give a bijective interpretation for the following combinatorial identity, which transfers one degree from one face to another.

Proposition 1 (Transferring one degree from f_1 to f_2)**.** *Let $\boldsymbol{a} = (a_1, \ldots, a_r)$ be an r-tuple of positive integers such that $a_1 \geq 2$, and with coordinates equal modulo p to*

(i) either $(k, -k, 0, \ldots, 0)$ for some $k \in \{0, \ldots, p-1\}$,

(ii) or $(1, 0, \ldots, 0, -1, 0, \ldots, 0)$, with the -1 in any position from 3 to r.

Let also $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_r) := (a_1 - 1, a_2 + 1, a_3, \ldots, a_r)$. Then the following identity holds:

$$\left(a_1 - \lceil a_1/p \rceil\right) \left(a_2 + 1\right) \mathrm{C}(\boldsymbol{a}) = \left(\tilde{a}_1 + 1\right) \left(\tilde{a}_2 - \lceil \tilde{a}_2/p \rceil\right) \mathrm{C}(\tilde{\boldsymbol{a}}) . \tag{1.2}$$

To obtain (1.2) from (1.1), one might first observe that, for any $x \in \mathbb{N}$,

$$\frac{\alpha(x)}{\alpha(x-1)} = d_x \, \frac{x}{x - \lceil x/p \rceil} \qquad \text{where} \quad d_x = \begin{cases} p-1 & \text{if } p \mid x \\ 1 & \text{if } p \nmid x \end{cases},$$

and then that, in both cases (*i*) and (*ii*), $c_{\boldsymbol{a}} d_{a_1} = c_{\tilde{\boldsymbol{a}}} d_{\tilde{a}_2} = p - 1$.

We furthermore treat the case of a degree 1-face, which may easily be obtained as above.

Proposition 2 (Transferring the degree of a degree 1-face f_1 to f_2)**.** *Let $\boldsymbol{a} = (1, a_2, \ldots, a_r)$ and $\tilde{\boldsymbol{a}} = (\tilde{a}_2, \ldots, \tilde{a}_r) := (a_2 + 1, a_3, \ldots, a_r)$ be respectively an r-tuple and an $r-1$-tuple of positive integers, both having at most two coordinates not lying in $p\mathbb{N}$. Then the following identity holds:*

$$\left(a_2 + 1\right) \mathrm{C}(\boldsymbol{a}) = V(\tilde{\boldsymbol{a}}) \left(\tilde{a}_2 - \lceil \tilde{a}_2/p \rceil\right) \mathrm{C}(\tilde{\boldsymbol{a}}) . \tag{1.3}$$

It is easy to see that the number of p-constellations with exactly one light face of degree pn is equal to the known number of p-ary trees with n nodes. Using this as initial condition, Propositions 1 and 2 provide yet another proof of (1.1).

Methodology. In order to bijectively interpret (1.2) and (1.3), the idea is to distinguish elements, such as edges, vertices, faces, corners, etc., in such a way that each side of an equation of interest counts maps given with such distinguished elements. Remark that we will always use the word "distinguished" to designate these extra elements, keeping the word "marked" only for the marked corners, which we see as inherent to the hypermaps into consideration.

Once both sides of the considered equation are properly interpreted as cardinalities of sets of maps with distinguished elements, we bijectively go from one set to the other as follows. Using the distinguished elements, we construct a directed path in the map, called *sliding path*. We then slit the map along this sliding path and sew back together the sides of the slit after sliding by one unit, in the sense that the left side of the i-th edge is sewn back on the right side of the $i \pm 1$-th edge (the ± 1 being the same for all edges and determined by some rule). This mildly modifies the map along the path but does not affect its faces, except the two that are around the extremities of the sliding path. In the process, new distinguished elements naturally appear in the resulting map; these allow us to recover the sliding path in order to slide back.

Organization of the paper. The remainder of the document is structured in the following manner. We start by giving in Section 2 the definitions and conventions we use, as well as a combinatorial interpretation of the prefactor $\left(a - \lceil a/p \rceil\right)$ appearing in the identities (1.2) and (1.3). We then present in Section 3 our bijective interpretation of these identities through a more general involution on the set of maps given with an orientation, a distinguished corner, and a distinguished edge satisfying an extra constraint.

2 Preliminaries

Distinguishing a corner. Following previous works on slit-slide-sew bijections, we use the convention, depicted in Figure 2, that the marked corner of a face creates two possible corners to distinguish. One might think of the marked corner as a dangling half-edge, with one corner on each side. As a result, a face of degree a bearing its marked corner has $a + 1$ possible corners to distinguish.

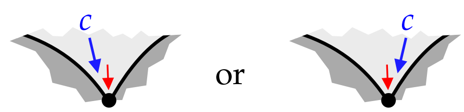


Figure 2: Distinguishing a corner around the marked corner.

Edge orientation. As is customary, we will orient the edges of the hypermaps we consider, in such a way that light faces always lie to the same side of the oriented edges (and thus dark faces always lie to the same other side). These orientations will be called the *light-left orientation* when the light faces[2] all lie to the left, and the *light-right orientation* when the light faces all lie to the right. In other words, in the light-right orientation, the edges are oriented clockwise around light faces and counterclockwise around dark faces. See Figure 3. We will need to use both orientations in the present paper. We will always clearly mention which orientation we use whenever it matters. Without specific mention, both orientations can be used. **Once one of the two possible orientations is fixed**, we will use the following conventions.

[2]Recall that the light faces are the main objects of focus.

Given an edge e, we will respectively denote by e^- and e^+ the origin and end of the edge e, oriented as convened. The corner *preceding* e is defined as the corner c_e delimited by e and the edge that precedes e in the contour of the incident light face, in the convened orientation. Similarly, we denote by c^+ the vertex incident to a corner c.

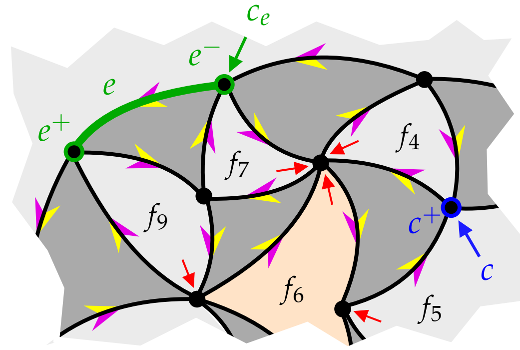


Figure 3: Edge orientation and definitions. Here, the light-right orientation is depicted.

Paths. A *path* from a vertex v to a vertex v' is a finite sequence $\wp = (e_1, e_2, \ldots, e_k)$ of edges such that $e_1^- = v$, for $1 \le i \le k-1$, $e_i^+ = e_{i+1}^-$, and $e_k^+ = v'$. Its *length* is the integer k, which we denote by $[\wp] := k$. A path is called *simple* if the vertices it visits are all distinct.

Beware that a path is only made of edges oriented in the convened orientation. In other words, edges cannot be used "backward." In particular, this means that all the faces lying to the left of a path are of the same shade (either all light or all dark), whereas all the faces lying to its right are of the other shade. The side of the path where the faces are all light will be called its *light side*, whereas the other side will be called its *dark side*.

Directed metric and geodesics. We will use the directed metric associated with the convened orientation: given two vertices v, v' in a p-hypermap, we denote by $\vec{\mathrm{d}}(v,v')$ the smallest k for which there exists a path from v to v' of length k. (We put an arrow on top in the notation to keep in mind that this is only a directed metric.) A *geodesic* from v to v' is such a path.

There are generally several geodesics from a given vertex v to a target vertex v'. Among all of these, one will be of particular interest in this work: the *lightest geodesic*, constructed as follows. It is only well defined from a starting edge or corner e_0 such that $e_0^+ = v$. (The starting element e_0 does not belong to the path.) Then, provided e_0, e_1, …, e_j have already been constructed and the path is not complete (that is, $e_j^+ \neq v'$), we set the subsequent edge e_{j+1} as the one, among the edges e such that $e^- = e_j^+$ and $\vec{\mathrm{d}}(e^+,v') = \vec{\mathrm{d}}(e_j^+,v') - 1$, that comes first while turning around e_j^+ in the direction incoming edge, light face. See Figure 4. In other words, the lightest geodesic is the leftmost geodesic if the convened orientation is the light-left orientation and the rightmost geodesic if the convened orientation is the light-right orientation.

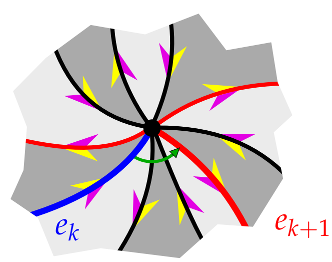


Figure 4: Definition of the lightest geodesic. The edges going closer to v' are in red.

Edge types. Given a fixed vertex v in a p-hypermap, we may differentiate three types of edges: an edge e is said to be

- *leaving* v if $\vec{\mathsf{d}}(v,e^+) = \vec{\mathsf{d}}(v,e^-) + 1$;
- *approaching* v if $\vec{\mathsf{d}}(v,e^+) = \vec{\mathsf{d}}(v,e^-) + 1 - p$;
- *irregular with respect to* v if $\vec{\mathsf{d}}(v,e^+) - \vec{\mathsf{d}}(v,e^-) \not\equiv 1 \bmod p$.

Observe that $1 - p \le \vec{\mathsf{d}}(v,e^+) - \vec{\mathsf{d}}(v,e^-) \le 1$ since there is always a path of length 1, namely the path consisting of the single edge e, as well as a path from e^+ to e^- of length $p-1$, made of all the other edges incident to the dark face incident to e. As a result, if e is irregular with respect to v, then it holds that $\vec{\mathsf{d}}(v,e^+) - \vec{\mathsf{d}}(v,e^-) \in \{2-p, 3-p, \ldots, 0\}$.

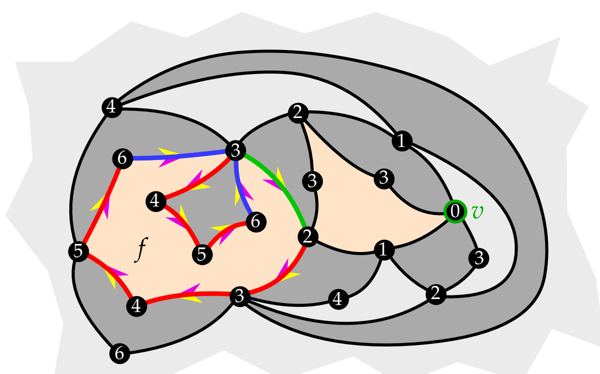


Figure 5: The different types of edges incident to a flawed face.The distances to v are written in the vertices. Around f, the $\big(10 - \lceil 10/4 \rceil\big) = 7$ red edges are leaving v; the $\lfloor 10/4 \rfloor = 2$ blue edges are approaching v; the green edge is irregular with respect to v.

The following proposition gives the number of each type among edges incident to a given face in a p-constellation or a given flawed face in a quasi-p-constellation; this provides an interpretation to the prefactor $\big(a - \lceil a/p \rceil\big)$ appearing in (1.2) and (1.3). We refer the reader to the extended version of this paper for a proof.

Proposition 3. *We consider a vertex v and a light face f of degree a in a p-hypermap.*

(1) If the p-hypermap is a p-constellation then, among the a edges incident to f, $a - a/p$ are leaving v; a/p are approaching v; none are irregular with respect to v.

(2) If the p-hypermap is a quasi-p-constellation and f a flawed face then, among the a edges incident to f, $\big(a - \lceil a/p \rceil\big)$ are leaving v; $\lfloor a/p \rfloor$ are approaching v; one is irregular with respect to v.

3 Bijective interpretation

3.1 Slit slide sew

Let us first describe the operation at the heart of our construction. See Figure 6. Assume that, on some p-hypermap $\mathbf{m}$, we have a simple path $\wp = (e_1, e_2, \dots, e_k)$ linking some corner c in some light face f to a different corner c' in some light face f' (which may possibly be equal to f), that is, such that $e_1^- = c^+$ and $e_k^+ = c'^+$. We may then follow $\wp$, **entering from the corner** c and **exiting through the corner** c'. This creates a simple path on the sphere, starting inside the face f and finishing inside f'. We may slit the sphere along this path, thus doubling the sides of the path. In the hypermap $\mathbf{m}$, this doubles the path $\wp$, making up two copies, one incident to light faces and the slit, and one incident to dark faces and the slit. We denote by $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$ the former and by $\boldsymbol{d} = (d_1, \dots, d_k)$ the latter.

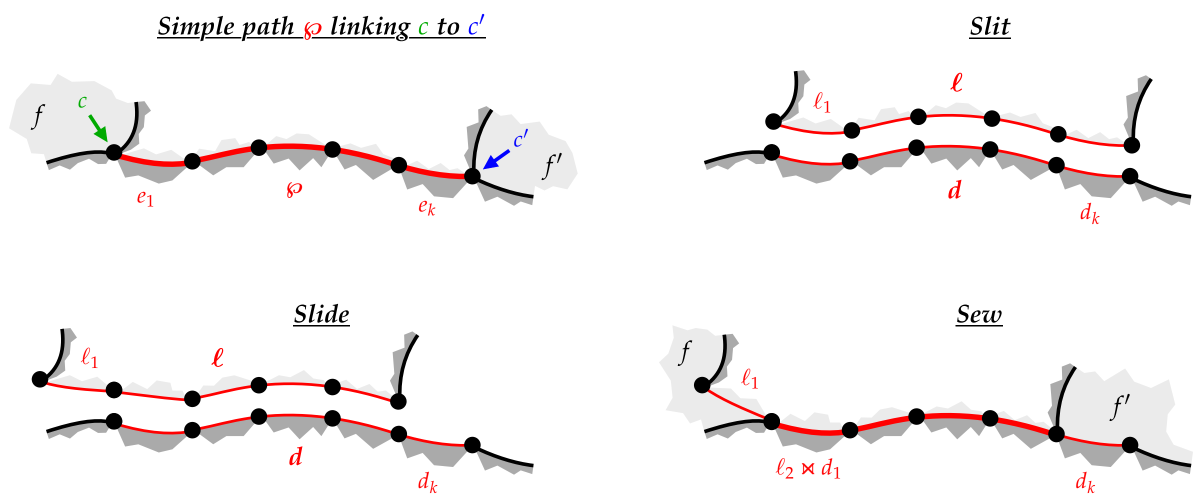


Figure 6: The slit-slide-sew operation on a p-hypermap.

Note that the data of $\wp$ is not sufficient to properly define this operation; one needs to know from which corner to enter $\wp$ in order to decide if an edge incident to e_1^- becomes incident whether to ℓ_1^- or to d_1^-. Similarly, one needs to know through which corner to exit $\wp$.

We then sew back $\boldsymbol{\ell}$ onto $\boldsymbol{d}$ but only after sliding by one unit, in the sense that we match ℓ_{i+1} with d_i, for every $1 \le i \le k-1$. For further reference, we denote by $\ell_{i+1} \bowtie d_i$ the resulting edge. Observe that, except from f and f', the faces are not altered by the process. Observe also that ℓ_1 and d_k are not matched with anything:

- d_k is still incident to the original dark face and is now also incident to f';
- ℓ_1 is still incident to the original light face and is now also incident to f.

Consequently, the result is no longer a p-hypermap since ℓ_1 is incident to light faces from both sides. However, in the case where ℓ_1 is actually a dangling edge (an edge with one extremity of degree 1), removing it provides a p-hypermap. This happens if and only if c is the corner preceding e_1 ; this will always be the case in the present work.

3.2 Face of degree two or more

We now present the bijective interpretation for the identity (1.2) of Proposition 1.

Involution. We define a mapping Φ on the set $\mathcal{H}$ of quadruples $(\mathcal{o}, \mathbf{m}, c, e)$, where

- $\mathcal{o}$ is an orientation (either light-left or light-right);
- $\mathbf{m}$ is a p-hypermap;
- c is a distinguished corner of some light face;
- e is a distinguished edge leaving c^+ in the orientation $\mathcal{o}$.

We break down the process into the following steps. See Figure 7.

1. Reorientation
From now on, we convene to use the reverse orientation, which we denote by $\tilde{\mathcal{o}}$.

2. Sliding path
We consider the corner c_e preceding e and the lightest geodesic γ from c_e to c^+.

3. Slitting, sliding, sewing
We slit, slide, sew along γ from c_e to c as described in the previous section: along γ, the light side of an edge is now matched with the dark side of the previous edge.

4. Output
The unmatched light side of the first edge of γ yields a dangling edge; we remove it and denote the resulting corner by $\tilde{c}$. We denote the edge corresponding to the unmatched dark side of the final edge of γ by $\tilde{e}$. We let $\tilde{\mathbf{m}}$ be the resulting map. Finally, the output of the construction is the quadruple $\Phi(\mathcal{o}, \mathbf{m}, c, e) := (\tilde{\mathcal{o}}, \tilde{\mathbf{m}}, \tilde{c}, \tilde{e})$.

Theorem 4. *The mapping $\Phi \colon \mathcal{H} \to \mathcal{H}$ is an involution.*

We refer the reader to the extended version of this work for the proof of Theorem 4.

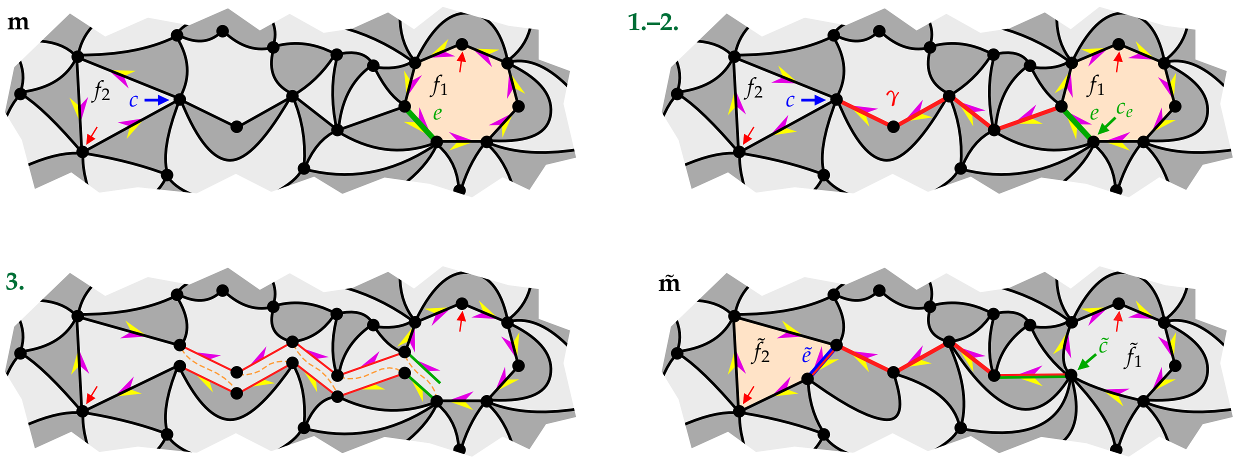


Figure 7: The involution $\Phi\colon \mathcal{H} \to \mathcal{H}$. Only the orientation around the faces of interest and along γ are depicted. **Top left.** The input. **Top right.** We changed the orientation and defined the sliding path γ. **Bottom left.** We slit along the path. The dashed lines indicate to sew back after sliding. **Bottom right.** The output.

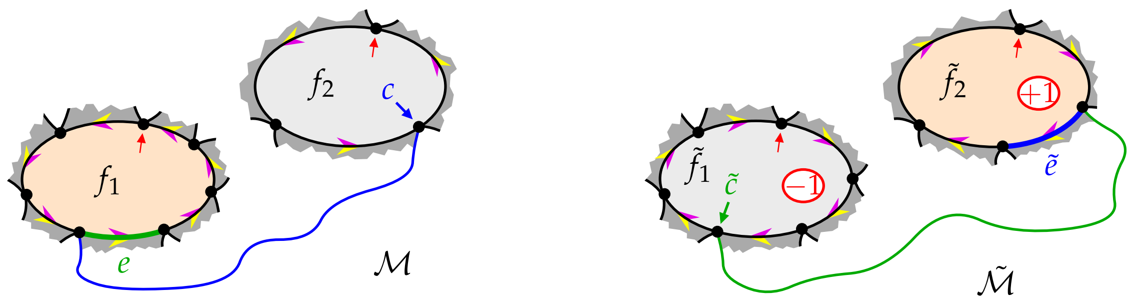
Specialization. We now see how Φ specializes into a bijection interpreting (1.2). We let

$$\boldsymbol{a} = (a_1, \ldots, a_r) \qquad \text{and} \qquad \tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_r) := (a_1 - 1, a_2 + 1, a_3, \ldots, a_r)$$

be as in the statement of Proposition 1. Note that this means that p-hypermaps of type $\boldsymbol{a}$ are either p-constellations or quasi-p-constellations whose **first** face is flawed. Similarly, p-hypermaps of type $\tilde{\boldsymbol{a}}$ are either p-constellations or quasi-p-constellations whose **second** face is flawed.

We fix an orientation $\mathcal{o}$ and define the following sets, whose cardinalities are respectively the left-hand side and the right-hand side of (1.2), by Proposition 3 (recall also the convention at the begining of Section 2 for distinguishing corners).

- We let $\mathcal{M}$ be the set of p-hypermaps of type $\boldsymbol{a}$ carrying
  - one distinguished corner c in the **second** face,
  - one distinguished edge e incident to the **first** face and leaving c^+, for the **orientation** $\mathcal{o}$.
- We let $\tilde{\mathcal{M}}$ be the set of p-hypermaps of type $\tilde{\boldsymbol{a}}$ carrying
  - one distinguished corner $\tilde{c}$ in the **first** face,
  - one distinguished edge $\tilde{e}$ incident to the **second** face and leaving $\tilde{c}^+$, for $\tilde{\mathcal{o}}$.



Here, $p = 3$*, we are in the case (ii) of Proposition 1, and* $\circ =$ light-left.

The pictograph above summarizes the definitions of $\mathcal{M}$ and $\tilde{\mathcal{M}}$. The red ± 1 on the right shows the increase or decrease of the degree of the face in $\tilde{\mathcal{M}}$ in comparison with the one of the corresponding face in $\mathcal{M}$. In order to avoid confusion, we denote the first and second faces of maps in $\mathcal{M}$ by f_1 and f_2 as before, and use $\tilde{f}_1$ and $\tilde{f}_2$ instead, for maps in $\tilde{\mathcal{M}}$. The paths symbolize the fact that the edges are leaving the corners.

Remark 1. Note that the convention on the orientation of edges is not the same in the definitions of the sets $\mathcal{M}$ and $\tilde{\mathcal{M}}$. This clearly bears no effects from an enumeration point of view but is of crucial importance for our bijections.

Corollary 5. *The mapping* Φ *specializes into a bijection from* $\{(\circ, \mathbf{m}, c, e) : (\mathbf{m}, c, e) \in \mathcal{M}\}$ *onto* $\{(\tilde{\circ}, \tilde{\mathbf{m}}, \tilde{c}, \tilde{e}) : (\tilde{\mathbf{m}}, \tilde{c}, \tilde{e}) \in \tilde{\mathcal{M}}\}$*, thus providing a bijection between* $\mathcal{M}$ *and* $\tilde{\mathcal{M}}$*.*

3.3 Face of degree one

We proceed to the bijective interpretation for the identity (1.3) of Proposition 2, which works in a similar fashion as before.

Setting. Let $\boldsymbol{a} = (1, a_2, \dots, a_r)$ and $\tilde{\boldsymbol{a}} = (\tilde{a}_2, \dots, \tilde{a}_r) := (a_2 + 1, a_3, \dots, a_r)$ be tuples of positive integers, both with at most two coordinates not lying in $p\mathbb{N}$. In order not to be confused by the index shift in $\tilde{\boldsymbol{a}}_2$, we denote the faces of p-hypermaps of type $\tilde{\boldsymbol{a}}$ by $\tilde{f}_2, \dots, \tilde{f}_r$. In particular, p-hypermaps of type $\tilde{\boldsymbol{a}}$ are either p-constellations, or are quasi-p-constellations whose face $\tilde{f}_2$ (the one with degree $\tilde{a}_2$) is flawed. We fix an orientation $\circ$ and define the following sets, whose cardinalities are the sides of (1.3), again by Proposition 3 for the right-hand side.

- We let $\mathcal{N}$ be the set of p-hypermaps of type $\boldsymbol{a}$ carrying
  - one distinguished corner c in the face f_2.
- We let $\tilde{\mathcal{N}}$ be the set of p-hypermaps of type $\tilde{\boldsymbol{a}}$ carrying
  - one distinguished vertex $\tilde{v}$,
  - one distinguished edge $\tilde{e}$ incident to $\tilde{f}_2$ and leaving $\tilde{v}$ for $\tilde{\circ}$.

We put f_1 on the pictograph since we think of it as the "missing" distinguished element for $\mathcal{N}$. Note that we do not need to specify an orientation for maps in $\mathcal{N}$; we will however use the orientation o for these maps in due time. The bijections between $\mathcal{N}$ and $\tilde{\mathcal{N}}$ can be thought of as degenerate versions of the one of the previous section. Here, we do not have an involution; we need to describe both mappings. We break them down into similar steps as above. See Figure 8.

Suppressing a face. We consider $(\mathbf{m}, c) \in \mathcal{N}$.

1. From this point on, we use the reverse orientation $\tilde{o}$.
2. We consider the lightest geodesic γ from the unique corner of f_1 to c^+.
3. We denote by d_0 the unique edge incident to f_1. We slit, slide, sew along γ from the unique corner of f_1 to c as described in Section 3.1, while furthermore matching the unmatched light side of the first edge with d_0.
4. We set $\Psi_-(\mathbf{m}, c) := (\tilde{\mathbf{m}}, \tilde{v}, \tilde{e})$, where $\tilde{\mathbf{m}}$ is the resulting map, $\tilde{e}$ is the edge corresponding to the unmatched dark side of the final edge of γ, and $\tilde{v}$ is the origin of γ.

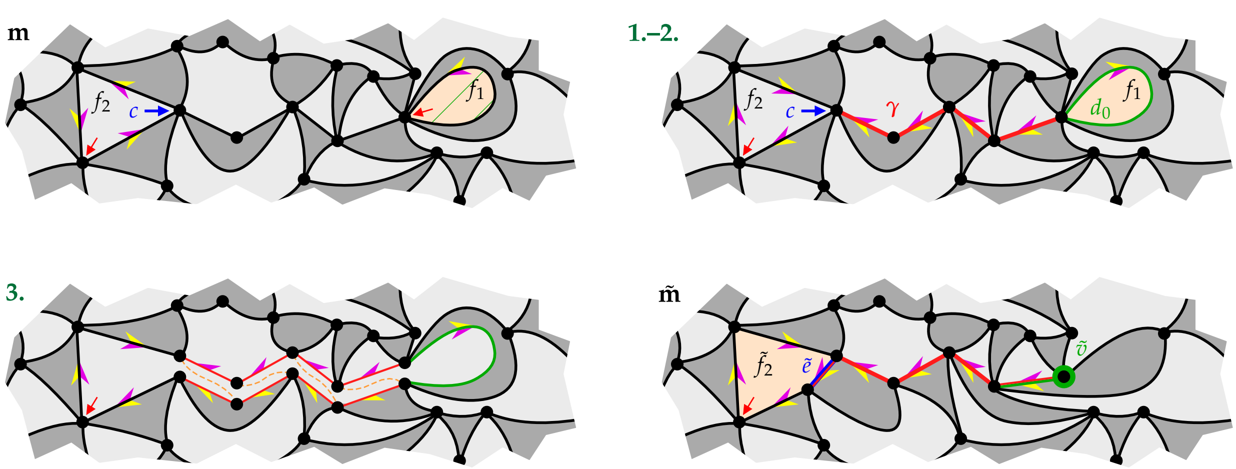


Figure 8: The bijection in the case of a degree 1-face, from $\mathcal{N}$ to $\tilde{\mathcal{N}}$.

Adding a face. We consider $(\tilde{\mathbf{m}}, \tilde{v}, \tilde{e}) \in \tilde{\mathcal{N}}$.

1. From this point on, we use the orientation $\mathcal{O}$.
2. We consider the lightest geodesic $\tilde{\gamma}$ from the corner $c_{\tilde{e}}$ preceding $\tilde{e}$ to $\tilde{v}$.
3. We slit $\tilde{\mathbf{m}}$ along $\tilde{\gamma}$, entering from $c_{\tilde{e}}$ and stopping at $\tilde{v}$, **without disconnecting the map** at $\tilde{v}$, slide by one unit, and sew back as before. Now the unmatched dark side of the final edge creates a loop enclosing an extra face, which we denote by f_1 and mark at its unique corner.
4. We replace $\tilde{\ell}_1$ with a corner c, let $\mathbf{m}$ be the resulting map, and set $\Psi_+(\tilde{\mathbf{m}}, \tilde{v}, \tilde{e}) := (\mathbf{m}, c)$.

Theorem 6. *The mappings $\Psi_- \colon \mathcal{N} \to \tilde{\mathcal{N}}$ and $\Psi_+ \colon \tilde{\mathcal{N}} \to \mathcal{N}$ are well defined and inverse bijections.*

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Crystal Chute Moves on Pipe Dreams

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Abstract. Schubert polynomials represent a basis for the cohomology of the complete flag variety and thus play a central role in geometry and combinatorics. In this context, Schubert polynomials are generating functions over various combinatorial objects, such as rc-graphs or reduced pipe dreams. By restricting Bergeron and Billey's chute moves on rc-graphs, we define a Demazure crystal structure on the monomials of a Schubert polynomial. As a consequence, we provide a method for decomposing Schubert polynomials as sums of key polynomials, complementing related work of Assaf and Schilling via reduced factorizations with cutoff, as well as Lenart's coplactic operators on biwords.

Keywords: Schubert polynomial, pipe dream, rc-graph, chute move, Demazure crystal

1 Introduction

Schubert polynomials are fundamental objects which lie at the intersection of geometry, representation theory, and algebraic combinatorics. By a classical theorem of Borel, the cohomology of the manifold of complete flags in $\mathbb{C}^n$ with integer coefficients is canonically isomorphic to the quotient of $\mathbb{Z}[x_1, \ldots, x_n]$ by the ideal generated by the symmetric polynomials without constant term [5]. The geometry of the flag variety is best captured by the cohomology classes of the Schubert varieties, which correspond to Schubert polynomials under Borel's isomorphism, generalizing the role of the Schur polynomials in the cohomology of the Grassmannian. In addition to encoding geometric information about the flag variety, individual Schubert polynomials also exhibit rich combinatorial and representation theoretic structures, as developed in [16, 14, 21, 17, 1] and explored further in the present work.

1.1 Schubert and key polynomials

Given any permutation $w \in S_n$, the *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$ can be calculated recursively using a sequence of divided difference operators, by the original

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definition of Lascoux and Schützenberger [15], inspired by the work of Demazure [7] and Bernstein–Gel′fand–Gel′fand [3]. Based on a conjecture of Stanley, the first combinatorial formula for Schubert polynomials was given by Billey, Jockusch, and Stanley using the language of *rc-graphs* [4], with an alternate proof by Fomin and Stanley [9]. An equivalent combinatorial description for Schubert polynomials was later provided by Fomin and Kirillov [10], rebranded by Knutson and Miller as *reduced pipe dreams* [13], following the conventions of Bergeron and Billey [2]. Besides being attractive ways to visually represent Schubert polynomials, pipe dreams generalize to flag manifolds the role of the semistandard Young tableaux for Grassmannians, while admitting generalizations to other cohomological contexts.

Many combinatorial models for Schubert polynomials also involve a family of operators, which permute the individual monomials. To highlight several examples most closely related to this work, Bergeron and Billey define *chute and ladder moves* on rc-graphs [2], the inspiration for which they attribute to Kohnert's thesis [14]. Miller provides a mitosis algorithm which lists reduced pipe dreams recursively by induction on the weak order on S_n [19]. Lenart develops operations on biwords which correspond to the coplactic operators on tableaux [17]. Morse and Schilling define a family of operators on *reduced factorizations* in [20], which restricts to an action on Schubert polynomials via the semi-standard key tableaux of Assaf and Schilling [1].

All of the operators mentioned above encode useful combinatorics about Schubert polynomials; however, some of them additionally carry representation-theoretic information. The most natural approach to track the representation theory is often through Kashiwara's *crystals* [11], which are graphical models for the irreducible representations of a complex semisimple Lie algebra. Lenart summarizes many results in [17] using the language of crystal operators rooted in a *pairing process* on rc-graphs, though the details are carried out via jeu de tacquin on biwords, most naturally associated with the combinatorics of semistandard Young tableaux. More explicitly, Assaf and Schilling prove in [1, Theorem 5.11] that the set of all reduced factorizations for $w \in S_n$ satisfying an additional *cutoff criterion* decomposes as a union of Demazure crystals.

The decomposition of a combinatorial model for Schubert polynomials into a union of Demazure crystals thus also yields a description of how $\mathfrak{S}_w$ is expressed as a sum of key polynomials κ_a, as in [1, Corollary 5.12]. Tableaux versions of such formulas include the original of Lascoux and Schützenberger [16], a related result of Reiner and Shimozono [21] on factorized row-frank words, and so on. The main goal of this paper is to provide such a decomposition for Schubert polynomials as sums of key polynomials, expressed in terms of reduced pipe dreams.

1.2 Main results

Inspired by the chute moves of [2] on rc-graphs, we develop a crystal structure on the monomials of a Schubert polynomial, giving a method for decomposing Schubert polynomials as sums of key polynomials, complementing the closely related works [1, 17]. Our *crystal chute moves* on reduced pipe dreams are either raising or lowering operators, denoted e_i and f_i, respectively. If the raising operator $e_i(D)$ applied to a reduced pipe dream D for the given permutation $w \in S_n$ equals zero for all $1 \leq i < n$, then we say $D \in RP(w)$ is a *highest weight pipe dream*. We direct the reader to Section 2 for precise definitions of all relevant terminology.

The highest weight pipe dreams naturally index the key polynomials in the decomposition below, as they are in bijection with a pair consisting of a partition λ_D having n parts and a permutation $\pi_D \in S_n$, such that $\mathfrak{a}_D = \pi_D(\lambda_D)$ for a unique composition a_D.

Theorem 1. *Given any $w \in S_n$, the Schubert polynomial may be expressed as*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{\substack{D \in RP(w) \\ e_i(D)=0,\ \forall 1 \leq i < n}} \kappa_{a_D}(x_1, \ldots, x_n),$$

where the composition $a_D = \mathrm{wt}(\widetilde{D})$ for a diagram $\widetilde{D}$ constructed from the highest weight pipe dream D; see Algorithm 1 for details.

Figure 1 on the next page shows how $\mathfrak{S}_{[21543]}$ decomposes as the sum of three key polynomials, indexed by the three pipe dreams with no incoming lowering edges, having weights $\lambda_D \in \{(2,1,1,0), (2,2,0,0), (3,1,0,0)\}$ recording the number of crosses in each row, with respective truncating permutations $\pi_D \in \{s_2s_1s_3, s_2, s_3s_2\}$ read from the edges.

2 A Crystal Structure on Pipe Dreams

In this section, we review the combinatorics of Schubert polynomials in the language of reduced pipe dreams. We then define crystal chute moves by restricting the chute moves of [2] on rc-graphs via a pairing process.

2.1 Schubert polynomials and pipe dreams

Reduced pipe dreams index the monomials of Schubert polynomials, as we review in Theorem 2. Fix an $n \in \mathbb{N}$ and consider the $n \times n$ grid, indexed such that the box in row i from the top and column j from the left is labeled by (i, j), as for matrix entries. A *pipe dream* is a diagram D obtained by covering each box on the grid with one of two square tiles: a cross $+$ or an elbow $\rotatebox{0}{$\smile$}$. Further, crosses are only permitted in boxes (i, j) such

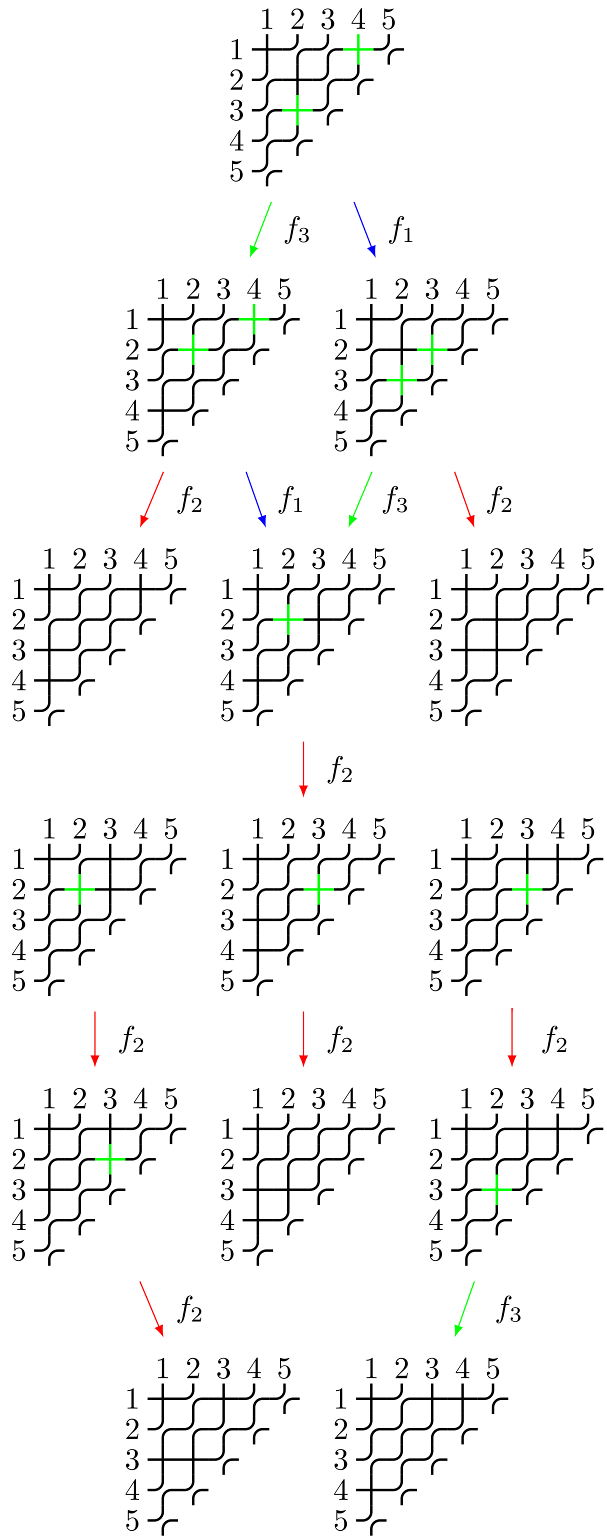

Figure 1: The Demazure crystal structure on reduced pipe dreams for $w = [21543]$. Crosses that will be moved by the lowering operators f_i are in shown green.

that $i+j \leq n$, so we will typically only draw the portion of D which lies on or above the main anti-diagonal.

By connecting the crosses and elbows on each tile in the unique possible way, as shown in Figure 1, we can view the resulting diagram as a network of *pipes* moving north and east, with water flowing in from the left of the grid and out at the top. The water in each pipe enters and exits from a unique pair of row and column indices, so that each pipe dream corresponds to a permutation on the set $[n] = \{1, \ldots, n\}$ as follows. The one-line notation for a permutation $w \in S_n$ records the action of w on $[n]$ in the form $w = [w_1 \cdots w_n]$, where we write $w_i = w(i)$ for brevity. A diagram D is a pipe dream *for the permutation* $w = [w_1 \cdots w_n]$ if the pipe entering row i exits from column w_i for all $i \in [n]$. For example, each of the diagrams in Figure 2 below is a pipe dream for the same permutation $w = [21543] \in S_5$.

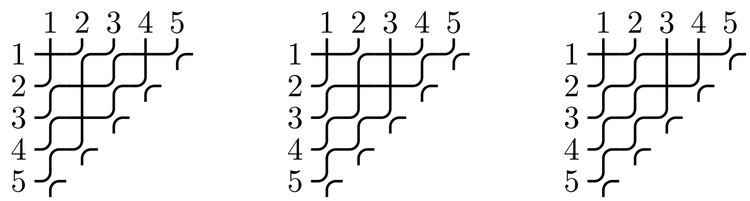

Figure 2: Several reduced pipe dreams for $w = [21543] \in S_5$.

A pipe dream is *reduced* if each pair of pipes crosses at most once, as in Figure 2. Denote by $RP(w)$ the set of all reduced pipe dreams for a given permutation w. We denote by D_+ the set of all boxes of D which are covered by a cross; note that D_+ uniquely determines D. Provided that the pipe dream is reduced, [13, Lemma 1.4.5] says that the number of crosses in $D \in RP(w)$ equals the length of the permutation, or the number of its inversions, given by $|D_+| = \ell(w) = \#\{i < j \mid w_i > w_j\}$.

The *weight of a pipe dream* $D \in RP(w)$, denoted by $\mathrm{wt}(D)$, is the weak composition of $\ell(w)$ whose i^{th} coordinate equals the number of crosses in row i of D. For example, the three weight vectors corresponding to the pipe dreams from Figure 2 below are $(2,1,1,0), (2,2,0,0)$, and $(3,1,0,0)$ recorded from left to right, all of which happen to be partitions in this example.

Schubert polynomials are generating functions over reduced pipe dreams, as illustrated by the following result, originally proved by Billey, Jockusch and Stanley [4], later reproved by Fomin and Stanley [9], and recorded here in the language of pipe dreams.

Theorem 2 (Corollary 2.1.3 [13])**.** *Let $w \in S_n$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{D \in RP(w)} \mathbf{x}^{\mathrm{wt}(D)}. \tag{2.1}$$

We use $\mathbf{x}$ to denote a monomial in the variables $x_1, \ldots, x_n$. Given any vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}^n_{\geq 0}$, the notation $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \cdots x_n^{v_n}$ is used throughout.

2.2 Crystal chute moves

In this section, we describe a family of operators on the set $RP(w)$ of reduced pipe dreams for a given permutation, which we show in our main theorem produces a Demazure crystal structure on the monomials of $\mathfrak{S}_w$.

Definition 1. *Given a reduced pipe dream D for a permutation in S_n, fix a row index $i \in [n]$. Denote the rightmost cross in row i by c. (Since crosses only occur in boxes (i,j) such that $i+j \leq n$, then D has no crosses in row n.) We define a* pairing process on row $1 \leq i < n$ *of D as follows:*

1. *Look for an unpaired cross c_+ in row $i+1$ such that c_+ lies weakly to the right of c in D. If there are multiple such c_+, choose the leftmost c_+.*
   - *(a) If such c_+ exists, we say that c and c_+ are* paired*.*
   - *(b) If no such c_+ exists, we say that c is* unpaired*.*
2. *Denote by c' the cross in row i which is both closest to c and lies to the left of c.*
   - *(a) If such c' exists, we reset $c := c'$ and start again from step (1).*
   - *(b) If no such c' exists, the pairing process on row i is complete.*

We illustrate the pairing process on the righthand pipe dream from Figure 2 below.

Example 1. *Fix $i = 1$ and identify $c = (1,4)$ as the rightmost cross in row 1. Since there are no crosses in row 2 which lie weakly right of c, then c_+ does not exist and c is unpaired in step (1b).*

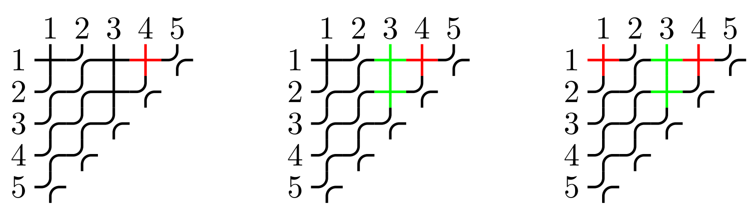

Figure 3: The pairing process applied to row 1 of a reduced pipe dream. We color paired crosses green and unpaired crosses red.

In step (2), we identify $c' = (1,3)$ as the cross in row 1 closest to and left of the original $c = (1,4)$. We thus return to step (1) applied to $c = (1,3)$. We identify $c_+ = (2,3)$ as a cross in row 2 which is weakly right of $c = (1,3)$, and so these crosses get paired in step (1a).

The only remaining cross $c' = (1,1)$ is unpaired since all crosses in row 2 are now paired. The pairing process is complete, having analyzed all crosses in row 1.

After running the pairing process on row i of $D \in RP(w)$, we define an operator f_i on D which produces another element of $RP(w)$ whenever it is nonzero.

Definition 2. *Let $D \in RP(w)$ for $w \in S_n$. Fix an $1 \leq i < n$ and run the pairing process on row i of D. If all crosses in row i are paired, then set $f_i(D) = 0$. Otherwise, denote by $(i, j) \in D_+$ the leftmost unpaired cross in row i.*

If $(i, k) \in D_+$ for all $1 \leq k \leq j$, then set $f_i(D) = 0$. Otherwise, define $m \in \mathbb{N}$ such that:

(a) $(i, j - m), (i + 1, j - m) \notin D_+$ and

(b) $(i, j - k), (i + 1, j - k) \in D_+$ for all $1 \leq k < m$.

Define a new diagram $f_i(D)$ by

$$f_i(D)_+ = \{D_+ \backslash (i, j)\} \cup \{(i + 1, j - m)\}.$$

The family of operators f_i for $1 \leq i < n$ are called (lowering) crystal chute moves.

In words, the crystal chute move f_i exchanges the leftmost unpaired cross at (i, j) and the elbow at $(i + 1, j - m)$, where m is chosen such that the rectangle strictly between this pair of tiles is filled by crosses.

We now illustrate how to apply the crystal chute moves on a reduced pipe dream.

Example 2. *Consider the sequence shown in Figure 4, in which we instead begin with the lefthand pipe dream D for $w = [21543]$ from Figure 2. If we run the pairing process on row 1, the leftmost unpaired cross is $(1, 4) \in D_+$. Properties (a) and (b) hold for $m = 1$, and the corresponding rectangle of crosses between $(1, 4) \in D_+$ and the elbow at $(2, 3)$ is empty in this case. To apply f_1, the red cross in $(1, 4)$ moves to the blue elbow in position $(2, 3)$, resulting in the middle diagram in Figure 4.*

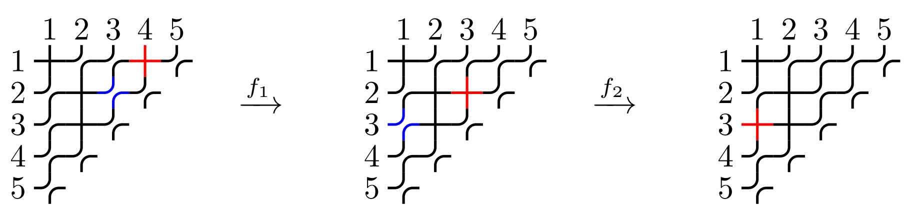

Figure 4: Applying a sequence of crystal chute moves to a reduced pipe dream.

Running the pairing process next on row 2 of the middle pipe dream, $(2, 3)$ is the leftmost unpaired cross, and $m = 2$, corresponding to the tile of paired crosses in rows 2 and 3 which are preserved under applying f_2. Here instead, the red cross at $(2, 3)$ jumps over this rectangle of crosses to the blue elbow in position $(3, 1)$, resulting in the third diagram in Figure 4.

We now define a second family of operators e_i to be precisely the inverse of the crystal chute moves from Definition 2.

Definition 3. *Let $D \in RP(w)$ for $w \in S_n$. Fix an $1 \leq i < n$ and run the pairing process on row i of D. If all crosses in row $i+1$ are paired, then set $e_i(D) = 0$. Otherwise, denote by $(i+1, \ell) \in D_+$ the rightmost unpaired cross in row $i+1$.*

Let $n > \ell$ be minimal such that $(i+1, n) \notin D_+$. Define a new diagram $e_i(D)$ by

$$e_i(D)_+ = \{D_+ \backslash (i+1, \ell)\} \cup \{(i, n)\}.$$

The family of operators e_i for $1 \leq i < n$ are called (raising) crystal chute moves*.*

We now have well-defined raising and lowering operators on reduced pipe dreams.

Proposition 1. *The raising crystal chute move $e_i : RP(w) \to RP(w) \cup \{0\}$ is well-defined for all $1 \leq i < n$, satisfying $\mathrm{wt}(e_i(D)) = \mathrm{wt}(D) + \alpha_i$ for any $D \in RP(w)$. Moreover, the raising and lowering crystal chute moves are mutually inverse.*

The pipe dreams D on which $e_i(D) = 0$ for all $1 \leq i < n$ play a distinguished role in the statement of Theorem 3 below, so we highlight them here.

Definition 4. *If $e_i(D) = 0$ for all $1 \leq i < n$, then D is a* highest weight pipe dream*.*

2.3 Demazure crystals and the main theorem

We refer the reader to [6] for more background on crystals. Given a partition λ with n parts, the type A_{n-1} crystal of highest weight λ is denoted by $B(\lambda)$, and the character of the crystal $B(\lambda)$ is the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$.

Demazure crystals are subsets of $B(\lambda)$ truncated by a permutation which restricts the set of raising and lowering operators. More precisely, for any subset $X \subseteq B(\lambda)$ and any index $1 \leq i < n$, we define $\mathfrak{D}_i$ in terms of lowering operators as

$$\mathfrak{D}_i(X) = \{b \in B(\lambda) \mid b \in f_i^k(X) \text{ for some } k \geq 0\}.$$

Now given any $\pi \in S_n$, write $\pi = s_{i_1} \cdots s_{i_p}$ as a product of simple transpositions $s_i = (i, i+1)$ where the expression for π is reduced, meaning that $p = \ell(\pi)$ is minimal. If u_λ denotes the highest weight element of $B(\lambda)$, the *Demazure crystal* associated to the pair (λ, π) is defined by

$$B_\pi(\lambda) = \mathfrak{D}_{i_1} \cdots \mathfrak{D}_{i_p}(u_\lambda).$$

The character of the Demazure crystal $B_\pi(\lambda)$ generalizes the Demazure characters of [8], as conjectured by Littelmann [18] and proved by Kashiwara [12]. Moreover, the character of the Demazure crystal $B_\pi(\lambda)$ is the key polynomial $\kappa_a(x_1, \ldots, x_n)$ indexed by the composition a such that $a = \pi(\lambda)$.

Our main theorem says that the set of reduced pipe dreams for a permutation admits a Demazure crystal structure determined by the crystal chute moves from Section 2.2.

Theorem 3. *Given any $w \in S_n$, the operators e_i and f_i for $1 \leq i < n$ define a type A_{n-1} Demazure crystal structure on $RP(w)$. That is,*

$$RP(w) \cong \bigcup_{\substack{D \in RP(w) \\ e_i(D)=0,\ \forall 1 \leq i < n}} B_{\pi_D}(\mathrm{wt}(D)),$$

where the truncating permutation π_D is the shortest permutation such that $\mathrm{wt}(\widetilde{D}) = \pi_D(\mathrm{wt}(D))$, for a diagram $\widetilde{D}$ constructed algorithmically from the highest weight pipe dream D; see Theorem 4.

Theorem 3 is the pipe dream analog of [1, Theorem 5.11], phrased there in terms of reduced factorizations for w meeting a cutoff condition. Refer to Figure 1 in the introduction to see how $RP([21543])$ decomposes into the union of three Demazure crystals.

3 Permutation Indexing the Demazure Crystal

This section explains the algorithm for identifying the truncating permutation from a highest weight pipe dream, equivalently the composition defining the corresponding key polynomial. We begin by describing how to obtain a new diagram $\widetilde{D}$ from any highest weight pipe dream D.

Algorithm 1. *Let D be a highest weight pipe dream.*

1. *For each cross in row i, shift it to the right by $i - 1$.*
2. *For each row, beginning in the lowest row, move the leftmost cross down to the row such that its row and column index match. Fix these crosses.*
3. *Set $\ell = 2$.*
   (a) *Beginning at the bottom row containing unfixed crosses, consider the leftmost unfixed cross. Move that cross down to the lowest possible row, remaining in its current column, such that:*
      i. *The cross may not move through other crosses;*
      ii. *The cross is the ℓ^{th} cross from the left in its new row; and*
      iii. *The cross does not have any previously fixed crosses to its right in the new row.*
   (b) *Fix this moved cross.*
   (c) *Repeat steps (a) and (b) untill all rows with unfixed crosses have been considered.*
4. *Increment ℓ by 1, and repeat step (3).*

Once all crosses are fixed, the algorithm terminates. Denote the resulting diagram by $\widetilde{D}$.

We illustrate Algorithm 1 on an example.

Example 3. *Consider the permutation $w = [4726315] \in S_7$. One of its highest weight pipe dreams D is depicted in Figure 5. The result after applying steps (1) and (2) of Algorithm 1 to D is in Figure 6, with fixed crosses marked in red.*

	1	2	3	4	5	6	7
1	+	+	+	·	+	+	·
2	+	+	+	·	·	·	
3	+	·	+	+	·		
4	+	·	·	·			
5	+	·	·				
6	·	·					
7	·						

Figure 5: A highest weight pipe dream D for $w = [4726315] \in S_7$.

	1	2	3	4	5	6	7
1	+	+	+		+	+	
2		+	+	+			
3			+		+	+	
4				+			
5					+		
6							
7							

Figure 6: The result of applying steps (1) and (2) of Algorithm 1 to D.

We now move to the iterative step (3). Set $\ell = 2$. We begin on the lowest row with an unfixed cross, that being row 3. We move the leftmost unfixed cross in this row, that being the cross at $(3,5)$, down in its column to a position that meets criteria (i) through (iii). We first observe that there is a cross at $(5,5)$, meaning that we are unable to move our cross to row 5 or any row below it. Our only option is to move this cross to row 4. Observe that a cross at $(4,5)$ would be the second cross in its row. Thus, we move the cross at $(3,5)$ to $(4,5)$ and fix it there.

The two crosses at $(2,3)$ and $(1,2)$ cannot move lower without violating (i). These two crosses are thus also fixed, completing the round of moves for $\ell = 2$. At the end of this round, we obtain the diagram shown in Figure 7. We then increment ℓ to 3, and repeat the process. We omit the details, but the final result $\widetilde{D}$ is shown in Figure 8.

Finally, the truncating permutation π_D is obtained from the diagram $\widetilde{D}$ as follows.

Theorem 4. *Let $D \in RP(w)$ be a highest weight pipe dream for $w \in S_n$. Then $\pi_D \in S_n$ from Theorem 3 is the unique shortest permutation such that $\mathrm{wt}(\widetilde{D}) = \pi_D(\mathrm{wt}(D))$. In addition, the composition $a_D = \mathrm{wt}(\widetilde{D})$ from Theorem 1 indexes the key polynomial corresponding to $(\pi_D, \mathrm{wt}(D))$.*

We conclude by extracting the truncating permutation π_D and the composition a_D from Example 3 via Theorem 4.

	1	2	3	4	5	6	7
1	+	+	+		+	+	
2		+	+	+			
3			+			+	
4				+	+		
5					+		
6							
7							

Figure 7: The diagram after completing the first iteration of step (3).

	1	2	3	4	5	6	7
1	+	+	+				
2		+	+	+	+	+	
3			+				
4				+	+	+	
5					+		
6							
7							

Figure 8: The diagram $\widetilde{D}$ after completing Algorithm 1.

Example 4. *For the highest weight pipe dream D in Example 3, we have* $\mathrm{wt}(D) = (5,3,3,1,1,0)$. *After applying Algorithm 1, we obtained the diagram $\widetilde{D}$ in Figure 8 such that* $a_D = \mathrm{wt}(\widetilde{D}) = (3,5,1,3,1,0)$. *The shortest permutation π_D such that $a_D = \pi_D(\mathrm{wt}(D))$ equals $\pi_D = s_1s_3$, since* $(3,5,1,3,1,0) = s_1s_3(5,3,3,1,1,0)$.

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A Regular Unimodular Triangulation of the Matroid Base Polytope

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Abstract. We produce the first regular unimodular triangulation of an arbitrary matroid base polytope. We then extend our triangulation to integral generalized permutahedra. Prior to this work it was unknown whether each matroid base polytope admitted a unimodular cover.

Keywords: matroid, polytope, triangulation

1 Introduction

Despite considerable interest, very little is known about triangulations of matroid base polytopes. There are a few motivations for wanting to have nice triangulations of matroid base polytopes. The first motivation comes from White's conjecture whose weakest version states that the toric ideal of a matroid base polytope is quadratically generated [34][26]. Herzog and Hibi asked whether the toric ideal of every matroid base polytope has a quadratic Gröbner basis [20]. It follows by a result of Sturmfels [33] combined with an observation of Ohsugi and Hibi [27] that the existence of a quadratic Gröbner basis is equivalent to the existence of a quadratic triangulation, i.e. a regular unimodular flag triangulation. The existence of a quadratic triangulation is known for base sortable matroids, e.g. positroids [31, 33, 6, 24, 25]. For transversal matroids, a result of Conca [9] establishes that the toric ring is Koszul, which is stronger than quadratic generation of the toric ideal but weaker than a quadratic triangulation.

The second motivation comes from Ehrhart theory. A formula for the volume of a matroid base polytope was calculated by Ardila–Doker–Benedetti [1], but no formula is currently known which is cancellation free, i.e. involves no subtraction. If a polytope P admits a unimodular triangulation $\mathcal{T}$, then the volume of P is equal to the number of maximal simplices in $\mathcal{T}$. The volume of a polytope occurs as the leading coefficient of the Ehrhart polynomial. Several researchers have investigated Ehrhart polynomials

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for matroid base polytopes [8][21] [16] largely motivated by the conjecture of De Loera–Haws–Köppe [11] that matroid base polytopes are Ehrhart positive—this conjecture was recently disproven by Ferroni [13], but various other questions about these polynomials remain open. The volume of a polytope P is also given by the evaluation of the h^*-polynomial at 1. Another conjecture by De Loera–Haws–Köppe, which remains open, is that the h^*-vectors of matroid base polytopes are unimodal [11]. Ferroni further conjectures that the h^*-polynomial of a matroid polytope (more generally an integral generalized permutahedron) is real-rooted [16, 14]. It has been conjectured that if a polytope P has the integer decomposition property (is IDP), then P has a unimodal h^*-vector [30], and it is known that every matroid base polytope is IDP [20]. We note that the property of admitting a unimodular triangulation is strictly stronger than the property of being IDP [7]. We refer the reader to [15] for a comprehensive survey of results in this area. It is known that the h^*-vector of a polytope is equal to the h-vector of any unimodular triangulation of the polytope [32][5], thus one might hope that such a triangulation could shed some light on this conjecture.

A natural question which sits in between these various results and conjectures is whether each matroid base polytope admits a (not necessarily flag) regular unimodular triangulation. That the matroid base polytope admits a (not necessarily regular) unimodular triangulation was conjectured by Haws in their 2009 thesis [19]. In this paper we give an affirmative answer to this question by providing a regular unimodular triangulation of an arbitrary matroid base polytope. We then apply this result to produce a regular unimodular triangulation of an arbitrary integral generalized permutahedron, and explain how this gives a regular unimodular triangulation of the matroid independence polytope. We emphasize that prior to this work it was unknown whether every matroid base polytope admitted a unimodular cover (this was also conjectured by Haws [19]) let alone a unimodular triangulation. Our construction produces many different triangulations, but at the time of writing we do not know if any of them are flag. We invite other researchers to try their hand at applying our triangulation to the topics above. See Remark 3.6.

2 Preliminaries

We recommend the following texts for an introduction to matroids [28], polytope theory [35], and triangulations [10][18]. Let $[n]$ denote the set of integers $\{1, \ldots, n\}$. Given $S \subseteq [n]$ we will employ the notation $x_S := \sum_{i \in S} x_i$. We identify $\{0,1\}^n$ with the collection of all subsets of $[n]$. We denote the standard basis vectors for $\mathbb{R}^n$ by e_i for $1 \leq i \leq n$.

Definition 2.1. A *matroid* is a pair $M = (E, \mathcal{B})$ where E is a finite set called the *ground set*, and $\mathcal{B}$ is a nonempty collection of subsets of E called the *bases* which satisfy the following basis exchange condition:

- For any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists some $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

A set $I \subseteq E$ is *independent* if there exists some basis $B \in \mathcal{B}$ such that $I \subseteq B$. The collection of independent sets is denoted $\mathcal{I}$. The *rank* of a set $S \subseteq E$, written $r(S)$, is the maximum cardinality of an independent set contained in S.

Matroid independence polytopes and the matroid base polytopes were introduced by Edmonds [12].

Definition 2.2. Given a matroid M on ground set $E = [n]$, the *matroid base polytope* P_M is the convex hull of the indicator vectors for the bases of M, and the *matroid independence polytope* $P_{\mathcal{I}}$ is the convex hull of the indicator vectors of the independent sets. More explicitly, given $S \subseteq E$, we define the indicator vector $\chi_S \in \mathbb{R}^n$ by

$$\chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

Thus $P_M = \text{conv}\{\chi_B : B \in \mathcal{B}\}$ and $P_{\mathcal{I}} = \text{conv}\{\chi_I : I \in \mathcal{I}\}$.

The matroid base polytope is the distinguished face of the matroid independence polytope where the sum of the coordinates is maximized. The matroid independence polytope will be discussed at the end of this article (see Corollary 3.4).

Gelfand–Goresky–MacPherson–Serganova uncovered a connection between matroid base polytopes and the geometry of the Grassmannian [17]. They showed that torus orbit closure of a linear space L in the Grassmannian is a normal toric variety whose weight polytope is the matroid base polytope $P_{M(L)}$, where $M(L)$ is the matroid determined by L. See Katz [22] for an overview of this story. By standard toric theory, our regular unimodular triangulation of P_M gives a projective Crepant resolution of the toric variety associated to the cone over a matroid base polytope.

Matroid bases polytopes allow for a polytopal characterization of matroids.

Theorem 2.3. *[12][17] A polytope P is a matroid base polytope for some matroid M if and only if P is a 0-1 polytope whose edge directions are of the form $e_i - e_j$.*

Polymatroids are a generalization of matroids described by monotonic submodular fuctions taking values in the nonnegative reals. Their base polytopes are equivalent by translation to the generalized permutahedra of Postnikov [29]. See [2] for a careful treatment of the following definition.

Definition 2.4. A *generalized permutahedron* $P \subseteq \mathbb{R}^n$ is a polytope defined by any one of the following equivalent conditions:

1. The edge directions for P are all of the form $e_i - e_j$,

2. The normal fan of P is a coarsening of the braid arrangement,

3. P is defined by inequalities $x_S \leq f(S)$ where $f : \{0,1\}^n \to \mathbb{R}$ is a submodular function, together with a single equation $x_{[n]} = f([n])$.

An *integral generalized permutahedron* P is a generalized permutahedron whose vertices have integer coordinates. The following is well-known, and follows from the unimodularity of the set of primitive ray generators of each chamber in the braid arrangement.

Lemma 2.5. *Let P be a generalized permutahedron determined by a submodular function f as in condition (3) of Definition 2.4. If f is an integer-valued function then P is an integral generalized permutahedron. Moreover, if P is an integral generalized permutahedron then f may be chosen to be integer-valued.*

In our proof, we will use condition (2) from Definition 2.4 as this allows us to describe the affine span of a face of a matroid base polytope.

Lemma 2.6. *Let P be an integral generalized permutahedron and* $\mathrm{aff}(P)$ *its affine span. Then*

$$\mathrm{aff}(P) = \bigcap_{i=1}^{j} \{x_{S_i} = b_i\}$$

for some flag of subsets $\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_j = [n]$ and some $b_i \in \mathbb{Z}$.

We note that when P is a matroid base polytope, the b_i in the lemma above is equal to the rank of the set S_i viewed as a subset of the ground set of the matroid.

Definition 2.7. A *subdivision* of a polytope P is a collection of polytopes $\mathcal{S} = \{P_1, \ldots, P_k\}$ such that

1. $\bigcup_{i=1}^k P_i = P$

2. for each $P_i \in \mathcal{S}$ and F a face of P_i, there exists some j such that $F = P_j$

3. for any i and j with $1 \leq i, j \leq k$, the intersection $P_i \cap P_j$ is a face of both P_i and P_j.

A maximal polytope in $\mathcal{S}$ is a *cell* of $\mathcal{S}$.

Definition 2.8. A *triangulation* of a polytope P is a subdivision $\mathcal{T} = \{T_1, \ldots, T_k\}$ of P such that each polytope T_i is a simplex.

Definition 2.9. Let $P \subset \mathbb{R}^n$ be a polytope and S a finite subset of P containing the vertices of P. Given a function $f : S \to \mathbb{R}$, the subdivision *induced* by f is the subdivision of P formed by projecting the lower faces of the polytope

$$\mathrm{conv}\{(x, f(x)) : x \in S\} \subset \mathbb{R}^{n+1}.$$

A subdivision is *regular* if it is induced by some function f.

Given a set $S \subseteq \mathbb{R}^n$, let $\mathrm{aff}(S)$ denote the affine span of S. Let $\mathrm{lin}(S)$ denote the linear subspace of $\mathbb{R}^n$ with the same dimension and parallel to $\mathrm{aff}(S)$.

Definition 2.10. A lattice simplex T is *unimodular* if it has normalized volume 1. Equivalently, if T has vertices $v_0, \ldots, v_n \in \mathbb{Z}^n$, then T is unimodular whenever a maximal linearly independent set of edge vectors $\{v_i - v_j\}$ form a lattice basis for $\mathrm{lin}(T) \cap \mathbb{Z}^n$.

Definition 2.11. The *resonance arrangement* $\mathcal{A}_n$ is the hyperplane arrangement in $\mathbb{R}^n$ consisting of all hyperplanes $H_S = \{x \in \mathbb{R}^n : x_S = 0\}$ where $\emptyset \subsetneq S \subseteq [n]$.

For an introduction to the resonance arrangement (also called the *all subsets arrangement*) we refer the reader to [23]. A *flat* of a hyperplane arrangement $\mathcal{H}$ is an intersection of hyperplanes in $\mathcal{H}$.

Definition 2.12. We say that an affine functional $\ell : \mathbb{R}^n \to \mathbb{R}$ is *generic* if it is non constant on each positive dimensional flat of the resonance arrangement.

We note that a generic point p on the n-th moment curve

$$C_n = \{(t, t^2, \ldots, t^n) : t \in \mathbb{R}\}$$

produces a generic linear functional $x \mapsto \langle x, p \rangle$.

3 A deletion-contraction triangulation

In this section we establish the main result of this paper.

Theorem 3.1. *Every matroid base polytope has a regular unimodular triangulation.*

Before providing a proof, we briefly give some context for our construction. Two fundamental operations on a matroid are the deletion and contraction of an element, and many important constructions in matroid theory proceed by an inductive appeal to these operations. If e is a loop or coloop, then the matroid base polytope P_M is translation equivalent to $P_{M/e}$ and $P_{M \setminus e}$. If e is neither a loop nor a coloop then P_M is the convex hull of $P_{M/e}$ and $P_{M \setminus e}$. In this way, our recursive construction fits into the paradigm of deletion-contraction.

Let $M = (E, \mathcal{B})$ be a matroid with ground set $E = [n]$, and $P_M \subset \mathbb{R}^n$ its matroid base polytope. We will use $\mathrm{verti}(P_M)$ to denote the vertices of P_M. We show P_M has a unimodular triangulation by induction on n. If $n = 1$, then P_M is a point and we are done.

Assume $n \geq 2$. Let P_0 and P_1 be the polytopes in $\mathbb{R}^{n-1}$ such that $P_0 \times \{0\} = P_M \cap \{x_1 = 0\}$ and $P_1 \times \{1\} = P_M \cap \{x_1 = 1\}$. Note that P_0 or P_1 may be empty, which occurs if 1 is a loop or coloop. If P_0 is nonempty then it is the matroid base polytope of $M \setminus 1$, and if P_1 is nonempty then it is the matroid base polytope of $M/1$.

By the inductive hypothesis, P_0 and P_1 have regular unimodular triangulations. (We assume an empty polytope has a regular unimodular triangulation induced by a function with empty domain.) Let $f_0 : \mathrm{verti}(P_0) \to \mathbb{R}$ and $f_1 : \mathrm{verti}(P_1) \to \mathbb{R}$ be functions which induce these triangulations. Let $\ell_0, \ell_1 : \mathbb{R}^{n-1} \to \mathbb{R}$ be affine functionals such that $\ell_0 - \ell_1$ is generic. Let $\epsilon > 0$ be sufficiently small, and define $f : \mathrm{verti}(P_M) \to \mathbb{R}$ to be the function

$$f(x) = \begin{cases} \ell_0(x_2, \dots, x_n) + \epsilon f_0(x_2, \dots, x_n) & \text{if } x_1 = 0 \\ \ell_1(x_2, \dots, x_n) + \epsilon f_1(x_2, \dots, x_n) & \text{if } x_1 = 1. \end{cases}$$

In our full paper [4], we prove that f induces a unimodular triangulation of P_M.

The following theorem is more explicit version of Theorem 3.1.

Theorem 3.2. *Let $P \in \mathbb{R}^n$ be a matroid base polytope. For each string $s \in \bigsqcup_{k=1}^{n-1} \{0,1\}^k$, let $\ell_s : \mathbb{R}^{n-|s|} \to \mathbb{R}$ be an affine functional, where $|s|$ is the length of s. Assume that $\ell_{s'0} - \ell_{s'1}$ is generic for all strings s'. Then for $1 \gg \epsilon_1 \gg \epsilon_2 \gg \cdots \gg \epsilon_{n-1} > 0$, the function $f : \mathrm{verti}(P) \to \mathbb{R}$ defined by*

$$f(x) = \sum_{k=1}^{n-1} \epsilon_k \ell_{x_1 \dots x_k}(x_{k+1}, \dots, x_n)$$

induces a regular unimodular triangulation on P_M.

Proof. This is obtained by unwinding the induction in the proof of Theorem 3.1. □

We now explain how to extend our triangulation to all integral generalized permutahedra.

Corollary 3.3. *Every integral generalized permutahedron has a regular unimodular triangulation.*

Proof. Let $P \in \mathbb{R}^n$ be an integral generalized permutahedron. By translating P if necessary, we may assume without loss of generality that there is some positive integer R such that $P \subset \{x : 0 \leq x_k \leq R \text{ for all } 1 \leq k \leq n\}$. It is known that dicing P by the hyperplanes $\{x_k = c\}$ where c and k are integers with $1 \leq k \leq n$ and $0 \leq c \leq R$ gives a regular integral subdivision $\mathcal{X}$ of P, and every cell of the subdivision is a translation of a matroid base polytope[1]. Let $g : P \cap \mathbb{Z}^n \to \mathbb{R}$ be a function which induces $\mathcal{X}$.

For each $s \in \bigsqcup_{k=1}^{n-1} \{0, \dots, R\}^k$, choose an affine functional $\ell_s : \mathbb{R}^{n-|s|} \to \mathbb{R}$ so that $\ell_{s'i} - \ell_{s'(i+1)}$ is generic for all strings s' and integers i. For $1 \gg \epsilon_1 \gg \epsilon_2 \gg \cdots \gg \epsilon_{n-1} > 0$, define the function $f : P \cap \mathbb{Z}^n \to \mathbb{R}$ by

$$f(x) = g(x) + \sum_{k=1}^{n-1} \epsilon_k \ell_{x_1 \dots x_k}(x_{k+1}, \dots, x_n).$$

[1] This can be verified by appealing to the submodularity description of generalized permutahedra, Lemma 2.5, and Theorem 2.3.

Then f induces a subdivision of P which refines $\mathcal{X}$. Moreover, by Theorem 3.2, the restriction of f to each cell of $\mathcal{X}$ induces a unimodular triangulation. □

Corollary 3.4. *Every matroid independence polytope has a regular unimodular triangulation.*

Proof. Each matroid independence polytope $P_{\mathcal{I}}$ is unimodularly equivalent to an integral generalized permutahedron: given a point $v = (v_1, \dots v_n) \in P_{\mathcal{I}}$, let $\psi(v) = (v_0, v_1, \dots v_n) \in \mathbb{R}^{n+1}$, where $v_0 = r(E) - \sum_{i=1}^{n} v_i$. The map ψ is unimodular and its image is an integral generalized permutahedron[2]. We can apply our triangulation to $\psi(P_{\mathcal{I}})$ and then map this triangulation back to $P_{\mathcal{I}}$ to obtain a regular unimodular triangulation of the latter. □

Example 3.5. We provide an example of our triangulation for the cycle matroid of the complete graph K_4. Let $\text{verti}(K_4) = \{v_0, v_1, v_2, v_3\}$. To simplify notation we denote the edges of K_4 by integers:

$$v_0v_1 = 0,\ v_1v_2 = 1,\ v_0v_1 = 2,\ v_1v_3 = 3,\ v_0v_3 = 4,\ v_2v_3 = 5.$$

The bases are in the following order:

0. {0 1 3}
1. {1 2 3}
2. {1 3 4}
3. {0 1 4}
4. {0 1 5}
5. {1 2 5}
6. {1 4 5}
7. {1 2 4}
8. {0 2 4}
9. {2 3 4}
10. {0 2 3}
11. {0 2 5}
12. {2 3 5}
13. {0 3 5}
14. {3 4 5}
15. {0 4 5}

We take the height function described in Theorem 3.2 as follows: if s is a string ending is 0, the function ℓ_s is 0. If a string ends in 1, and the string has length k, the function $\ell_s = (-3^{n-k-1}, -3^{n-k-2}, \dots, 1)$. The cells of the associated triangulation are

{3 7 8 9 12 14}
{3 5 7 8 12 14}
{3 5 6 7 8 14}
{3 5 8 11 12 14}
{3 5 6 8 11 14}
{3 4 6 11 14 15}
{3 4 5 6 11 14}
{3 4 5 11 12 14}
{3 6 8 11 14 15}
{0 3 8 11 14 15}
{0 3 4 5 11 12}
{0 3 4 5 12 14}
{0 3 4 11 12 14}
{0 3 4 5 6 14}
{0 3 4 11 14 15}
{0 4 11 12 13 14}
{0 4 11 13 14 15}
{0 3 5 8 11 12}

[2] It is implicit in [3] that the independence polytope is unimodularly equivalent to a generalized permutahedron.

{0 3 8 11 12 14}
{0 3 5 6 7 14}
{0 10 11 12 13 14}
{0 8 10 11 14 15}
{0 8 10 11 12 14}
{0 8 9 10 12 14}
{0 10 11 13 14 15}
{0 3 5 7 12 14}
{0 3 5 7 8 12}
{0 2 3 6 7 14}
{0 2 3 7 9 14}
{0 1 7 9 10 12}
{0 1 5 7 10 12}
{0 5 8 10 11 12}
{0 7 8 9 10 12}
{0 5 7 8 10 12}
{0 1 2 6 7 14}
{0 1 2 7 9 14}
{0 1 7 9 12 14}
{0 1 5 7 12 14}
{0 1 5 6 7 14}
{0 3 7 9 12 14}
{0 3 8 9 12 14}
{0 3 7 8 9 12}.

Remark 3.6. The authors, Matt Larson, and Sam Payne attempted to apply the construction of this article to produce quadratic triangulations of graphic matroid base polytopes, i.e. spanning tree polytopes. We convinced ourselves that it not possible to do so using only ℓ_s above which are exponential. We welcome others to attempt to apply our triangulation to White's conjecture.

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Analogues of two classical pipedream results on bumpless pipedreams

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Abstract. Schubert polynomials are distinguished representatives of Schubert cycles in the cohomology of the flag variety. Pipedreams (PD) and bumpless pipedreams (BPD) are two combinatorial models of Schubert polynomials. There are many classical results on PDs. For instance, Fomin and Stanley represented each PD as an element in the nil-Coexter algebra. Lenart and Sottile converted each PD into certain chains in the Bruhat order. This paper establishes the BPD analogues of both viewpoints. Our results lead to a bijection between PDs and BPDs via Lenart's growth diagram.

1 Introduction

Fix $n \in \mathbb{Z}_{\geqslant 0}$. For a permutation $w \in S_n$, Lascoux and Schützenberger [12] recursively define the *Schubert polynomial* $\mathfrak{S}_w$. The base case is $\mathfrak{S}_{w_0} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ where w_0 is the permutation with one-line notation $[n, n-1, \cdots, 1]$. To compute $\mathfrak{S}_w$ for other $w \in S_n$, we need the *divided difference operator* $\partial_i(f) := \frac{f - f(\cdots, x_{i+1}, x_i, \cdots)}{x_i - x_{i+1}}$. Let $s_i \in S_n$ denote the transposition that swaps i and $i+1$. Then for any $w \in S_n$ and $i \in [n-1]$:

$$\partial_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } w(i) > w(i+1), \\ 0 & \text{if } w(i) < w(i+1). \end{cases}$$

The Schubert polynomials represent Schubert cycles in flag varieties and have been extensively investigated. Schubert polynomials have two distinct combinatorial formulas involving "pipes": pipedreams (PD) [1, 3] and bumpless pipedreams (BPD) [11]. Both are fillings of grids with certain tiles. When we refer to cells of a grid, we use the matrix coordinates: row 1 is the topmost row and column 1 is the leftmost column. A *pipedream* is a filling of a staircase grid: The grid has a cell in row i column j for each $i + j \leqslant n + 1$. The rightmost cell in each row is . The rest of the cells can be (crossing) or (bump), but two pipes cannot cross more than once. A *bumpless pipedream (BPD)* is a consistent filling of an $n \times n$ grid with six types of cells: , , , , and (blank). Pipes enter from each cell on the bottom and exit on the right edge. In addition, two

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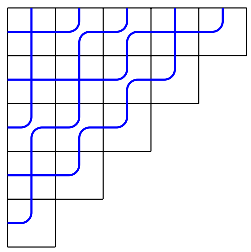
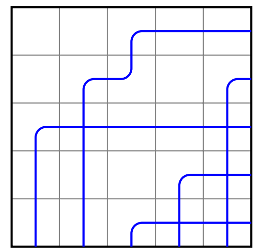
pipes cannot cross more than once. The permutation associated to each PD (resp. BPD) can be read off as follows: Label the pipes $1, 2, \ldots, n$ along the top (resp. bottom) edge, follow the pipes, and read the labels from top to bottom on the left (resp. right) edge.

Example 1.1. When $n = 5$, we present a PD and a BPD associated with $[2, 5, 1, 4, 3]$:

Let $\mathsf{PD}(w)$ (resp. $\mathsf{BPD}(w)$) be the set of all PDs (resp. BPDs) associated with $w \in S_n$. For $P \in \mathsf{PD}(w)$ (resp. $P \in \mathsf{BPD}(w)$), the weight of P, denoted as $\mathsf{wt}(P)$, is a sequence of $n-1$ integers where the i^{th} entry is the number of $\boxplus$ (resp. $\square$) on row i. For instance, the PD and BPD in Example 1.1 both have weight $(2, 2, 0, 1)$. If $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$ is a sequence of $n-1$ non-negative integers, we use x^α to denote the monomial $x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}$.

Theorem 1.2. *[1, 3, 11] For $w \in S_n$, $\mathfrak{S}_w = \sum_{P \in \mathsf{PD}(w)} x^{\mathsf{wt}(P)} = \sum_{D \in \mathsf{BPD}(w)} x^{\mathsf{wt}(D)}$.*

There is a recent surge of research connecting BPDs with PDs and finding BPD analogue of classical PD apparatus [7, 10, 8, 17]. This paper establishes the BPD analogue of two classical stories on PDs:

- The nil-Coexter algebra $\mathcal{N}_n$ is generated by $u_1, \cdots, u_{n-1}$. Fomin and Stanley [6] defined the following elements in $\mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{N}_n$:

  $$A_i(x_i) := (1 + x_i u_{n-1})(1 + x_i u_{n-2}) \cdots (1 + x_i u_i) \text{ and } \mathfrak{S}^{\mathsf{PD}} := A_1(x_1) \cdots A_{n-1}(x_{n-1}).$$

  Combinatorially, after expanding $\mathfrak{S}^{\mathsf{PD}}$, each term $x^\alpha u_{i_1} \cdots u_{i_k}$ naturally corresponds to a $P \in \mathsf{PD}(w)$ with $\alpha = \mathsf{wt}(P)$ and $i_1 \cdots i_k$ is a reduced word of w. Algebraically, Fomin and Stanley proved $\mathfrak{S}^{\mathsf{PD}} = \sum_{w \in S_n} \mathfrak{S}_w u_{i_1} \cdots u_{i_l}$ where $i_1 \cdots i_l$ is any reduced word of w. Consequently, they obtain an operator theoretic proof of the PD fomula.

- The Bruhat order is a partial order on S_n. Lenart and Sottile [14] defined a bijection from $\mathsf{PD}(w)$ to chains $(w_1, w_2, \cdots, w_n)$ in the Bruhat order where $w_1 = w$, $w_n = w_0$ and there is an increasing i-chain from w_i to w_{i+1} for $i \in [n-1]$ (See Section 2.2).

Since the introduction of BPDs, finding a BPD analogue of the Fomin-Stanley construction has been an open problem. Instead of the nil-Coexter algebra, we consider the Fomin-Kirillov algebra $\mathcal{E}_n$ [4]. It is generated by $d_{i,j}$ for $1 \leqslant i < j \leqslant n$ and has a right action on $\mathbb{Q}[S_n]$ denoted as $\odot$. Define the following elements in $\mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{E}_n$:

$$R_i(x_i) := (x_i + d_{1,i+1} + \cdots + d_{i,i+1})(x_i + d_{1,i+2} + \cdots + d_{i,i+2}) \cdots (x_i + d_{1,n} + \cdots + d_{i,n}), \text{ and}$$

$$\mathfrak{S}^{\mathsf{BPD}} := w_0 \odot (R_1(x_1)R_2(x_2)\cdots R_{n-1}(x_{n-1})).$$

Combinatorially, after expanding $\mathfrak{S}^{\mathsf{BPD}}$, we show each term $x^\alpha w$ naturally corresponds to a $D \in \mathsf{BPD}(w)$ with $\alpha = \mathsf{wt}(D)$. Algebraically, we establish Theorem 4.3, obtaining an operator theoretic proof of the BPD formula.

Theorem 4.3. *We have* $\mathfrak{S}^{\mathsf{BPD}} = \sum_{w\in S_n} \mathfrak{S}_w w$.

A crucial tool to understand $\mathfrak{S}^{\mathsf{BPD}}$ is a novel encoding algorithm Φ that encodes each element of $\mathsf{BPD}(w)$ as partial fillings of a staircase grid which we call *flagged tableaux*. We denote the image of $\mathsf{BPD}(w)$ under Φ as $\mathsf{FT}(w)$. Each $T \in \mathsf{FT}(w)$ corresponds to a chain in the Bruhat order denoted as $\mathsf{chain}(T) = (w_n, \cdots, w_1)$. Then we establish Theorem 3.9, obtaining a BPD analogue of Lenart and Sottile's work.

Theorem 3.9. *The map* $\mathsf{chain}(\cdot)$ *is a bijection from* $\mathsf{FT}(w)$ *to chains* $(w_n, \cdots, w_1)$ *in the Bruhat order where* $w_n = w$, $w_1 = w_0$ *and there is an increasing i-chain from* w_{i+1} *to* w_i. *Consequently,* $\mathsf{chain} \circ \Phi$ *is a bijection from* $\mathsf{BPD}(w)$ *to such chains.*

In other words, PDs and BPDs can both be viewed as certain chains in the Bruhat order, exhibiting a duality. Finally, we use Lenart's growth diagram [13] to obtain a bijection between these chains, obtaining a bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. We conjecture this bijection agrees with the existing bijection of Gao and Huang [7]. This conjecture has been verified on S_7.

Organization: In §2, we cover some necessary background. In §3, we define the encoding map $\Phi : \mathsf{BPD}(w) \to \mathsf{FT}(w)$ and establish Theorem 3.9. In §4, we construct our BPD analogue of the Fomin-Stanley construction. In §5, we use Lenart's growth diagram to build a bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. In §6, we describe one conjecture that extends the chain formulas of $\mathfrak{S}_w$ to double Schubert polynomials.

2 Background

2.1 Fomin-Stanley construction

A *reduced word* of $w \in S_n$ is a word $i_1 i_2 \cdots i_l$ such that $w = s_{i_1} \cdots s_{i_l}$ and l is minimized. One can read off a reduced word of w from every $P \in \mathsf{PD}(w)$ as follows: Go through its crossings from top to bottom and right to left in each row. For a crossing in row r column c, read off $r + c - 1$. For instance, the PD in Example 1.1 gives 41324 which is a reduced word of $[2, 5, 1, 4, 3]$.

The *nil-Coexter algebra* $\mathcal{N}_n$ is generated by $u_1, \cdots, u_{n-1}$ satisfying:

$$\begin{cases} u_i^2 & = 0, \\ u_i u_j & = u_j u_i \text{ if } |i-j| \geqslant 2\,, \\ u_i u_{i+1} u_i & = u_{i+1} u_i u_{i+1} \text{ if } i \in [n-2]. \end{cases}$$

Consider $a = u_{i_1} \cdots u_{i_l} \in \mathcal{N}_n$, we have $a \neq 0$ if and only if $i_1 \cdots i_l$ is a reduced word of some $w \in S_n$. In this case, $a = u_{j_1} \cdots u_{j_{l'}}$ if and only if $j_1 \cdots j_{l'}$ is a reduced word for the same w. Fomin and Stanley [6] defined the following elements in $\mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{N}_n$:

$$\begin{aligned} A_i(x_i) &:= (1 + x_i u_{n-1})(1 + x_i u_{n-2}) \cdots (1 + x_i u_i) \text{ for } i \in [n-1], \text{ and} \\ \mathfrak{S}^{\mathsf{PD}} &:= A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1}). \end{aligned}$$

Combinatorially, $\mathfrak{S}^{\mathsf{PD}} = \sum_P x^{\mathsf{wt}(P)} u_{i_1} \cdots u_{i_l}$ where the sum runs over all PD and $i_1 \cdots i_l$ is the reduced word read off from the PD. Algebraically, Fomin and Stanley showed that

$$\mathfrak{S}^{\mathsf{PD}} = \sum_{w \in S_n} \mathfrak{S}_w u_{i_1} \cdots u_{i_l}, \tag{2.1}$$

where $i_1 \cdots i_l$ is an arbitrary reduced word of w. This formula would imply the PD formula in Theorem 1.2. Fomin and Stanley proved (2.1) by showing $\partial_i(\mathfrak{S}^{\mathsf{PD}}) = \mathfrak{S}^{\mathsf{PD}} u_i$ for any $i \in [n-1]$. This equation then reduces to $\partial_i(R_i(x_i) R_{i+1}(x_{i+1})) = R_i(x_i) R_{i+1}(x_{i+1}) u_i$. In §4, we present the BPD analogue of (2.1) and establish our equation in a similar way.

2.2 Bruhat order

For $1 \leqslant i < j \leqslant n$, we use $t_{i,j}$ to denote the permutation that swaps i and j. For $w \in S_n$, let $\ell(w) := |\{(i,j) : i < j, w(i) > w(j)|$. Let $\leqslant$ be the *Bruhat order* on S_n, where the cover relation is given by $u \lessdot w$ if $w = u t_{i,j}$ and $\ell(w) = \ell(u) + 1$. We say $C = (w_1, w_2, \cdots, w_d)$ is a *Bruhat chain* from w_1 to w_d if $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_d$. The length of C is $d-1$. The *weight* of C, denoted as $\mathsf{wt}(C)$, is a sequence of length $d-1$ where the i^{th} entry is $\ell(w_{i+1}) - \ell(w_i)$. The chain is *saturated* if $w_1 \lessdot w_2 \lessdot \cdots \lessdot w_d$. We may represent a saturated chain as

$$w_1 \xrightarrow{t_{a_1,b_1}} w_2 \xrightarrow{t_{a_2,b_2}} \cdots \xrightarrow{t_{a_{d-1},b_{d-1}}} w_d,$$

where $a_i < b_i$ and $w_{i+1} = w_i t_{a_i,b_i}$.

Take $k \in [n-1]$. We use $\leqslant_k$ to denote the *k-Bruhat order* on S_n. Its cover relation is given by $u \lessdot_k w$ if $u \lessdot w$ and $w = u t_{i,j}$ for some $i \leqslant k < j$. Similarly, we can define k-Bruhat chains and saturated k-Bruhat chains. For simplicity, we say "*k-chains*" in place of "k-Bruhat chains". The k-Bruhat order can be used to describe the Monk's rule [15]:

$\mathfrak{S}_w(x_1 + \cdots + x_k) = \sum_{w \lessdot_k u} \mathfrak{S}_u$ for any $w \in S_n$ and $k \in [n-1]$ such that $w(j) = n$ for some $j > k$. Sottile generalized the Monk's rule by considering multiplying $\mathfrak{S}_w$ with

$$h_d(x_1, \cdots, x_k) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant k} x_{i_1} \cdots x_{i_d},$$

where $k \in [n-1]$ and $d \in \mathbb{Z}_{>0}$. Say a saturated k-chain $w_1 \xrightarrow{t_{a_1,b_1}} w_2 \xrightarrow{t_{a_2,b_2}} \cdots \xrightarrow{t_{a_{d-1},b_{d-1}}} w_d$ is *increasing* if $w_1(a_1) < w_2(a_2) < \cdots < w_{d-1}(a_{d-1})$. In other words, the smaller number swapped is increasing. It is not hard to show for any $u, w \in S_n$ and $k \in [n-1]$, there is at most one increasing k-chain from u to w.

Theorem 2.1. *[16] Take $u \in S_n$ and $d \in \mathbb{Z}_{\geqslant 0}$. For any $k \in [n-1]$ such that $n, n-1, \cdots, n-d+1$ are among $w(k+1), \cdots, w(n)$, then*

$$\mathfrak{S}_u \times h_d(x_1, \cdots, x_k) = \sum_w \mathfrak{S}_w.$$

The sum is over all w such that there is an increasing k-chain from u to w with length d.

Lenart and Sottile [14] view PDs as certain Bruhat chains. We introduce the following definition to describe their chains in a more general way.

Definition 2.2. We say a Bruhat chain $C = (w_1, w_2, \cdots, w_l, w_{l+1})$ is *compatible* with a sequence $(k_1, \cdots, k_l)$ if there exists an increasing k_i-chain from w_i to w_{i+1} for each $i \in [l]$.

Lenart and Sottile [14] described a bijection from $\mathsf{PD}(w)$ to chains from w to w_0 compatible with $(1, 2, \cdots, n-1)$: Take $P \in \mathsf{PD}(w)$. For $i \in [n]$, let P_i be the pipedream obtained from P by changing all bumps above row i into crossings. Let w_i be the permutation associated with P_i. Then $(w_1, \cdots, w_n)$ is the resulting chain. In addition, if we change bumps in row i of P_i into crossings from left to right, permutations of the intermediate pipedreams will form the increasing i-chain from w_i to w_{i+1}.

Example 2.3. Let P be the pipedream in Example 1.1. Then its corresponding chain is $([2,5,1,4,3], [5,3,1,4,2], [5,4,1,3,2], [5,4,3,2,1], [5,4,3,2,1])$. The increasing 1-chain from $[2,5,1,4,3]$ to $[5,3,1,4,2]$ is given by: $[2,5,1,4,3] \xrightarrow{t_{1,5}} [3,5,1,4,2] \xrightarrow{t_{1,2}} [5,3,1,4,2]$.

If a pipedream P is sent to the chain C, then $\mathsf{wt}(C) = (n-1, \cdots, 1) - \mathsf{wt}(P)$ where the subtraction is entry-wise. Thus, this bijection recovers a result of Bergeron and Sottile:

Corollary 2.4. *[2] For $w \in S_n$, $\mathfrak{S}_w = \sum_C x^{(n-1,\cdots,1)-\mathsf{wt}(C)}$, where the sum is over all chains from w to w_0 compatible with $(1, 2, \cdots, n-1)$.*

We end this section by extending Corollary 2.4 using the following observation:

Proposition 2.5. *Pick $u, w \in S_n$, $k_1, k_2 \in [n-1]$ and $d_1, d_2 \in \mathbb{Z}_{\geqslant 0}$. The number of chains from u to w compatible with (k_1, k_2) and has weight (d_1, d_2) matches the number of chains from u to w compatible with (k_2, k_1) and has weight (d_2, d_1).*

Proof. By Theorem 2.1, the number of chains (u, v, w) compatible with (k_1, k_2) and has weight (d_1, d_2) is the coefficient of $\mathfrak{S}_w$ in $\mathfrak{S}_u \times h_{d_1}(x_1, \cdots, x_{k_1}) \times h_{d_2}(x_1, \cdots, x_{k_2})$. The proof is finished by the commutativity of polynomial multiplication. □

Since we have two sets with the same size, it would be natural to ask:

Problem 2.6. Find an explicit bijection between the two set of chains in Proposition 2.5.

In §5, we show Lenart's growth diagram [13] solves Problem 2.6 in a special case.

Combining Corollary 2.4 and Proposition 2.5, we deduce:

Corollary 2.7. *Take $w \in S_n$ and $\gamma \in S_{n-1}$. If $(d_1, \cdots, d_{n-1})$ is a sequence of numbers, let $\gamma^{-1}(d_1, \cdots, d_{n-1}) := (d_{\gamma^{-1}(1)}, \cdots, d_{\gamma^{-1}(n-1)})$. We also view γ as a sequence of numbers. Then $\mathfrak{S}_w = \sum_C x^{(n-1,\cdots,1)-\gamma^{-1}(\mathrm{wt}(C))}$, summing over all chains from w to w_0 compatible with γ.*

This corollary implies that we have a combinatorial formula of $\mathfrak{S}_w$ involving Bruhat chains for each choice of $\gamma \in S_{n-1}$. Under Lenart and Sottile's bijection, the PD formula is identified with the Bruhat chain formula when $\gamma = [1, 2, \cdots, n-1]$. In §3, we identify the BPD formula with the Bruhat chain formula when $\gamma = [n-1, n-2, \cdots, 1]$.

3 Encoding BPDs as flagged tableaux and chains

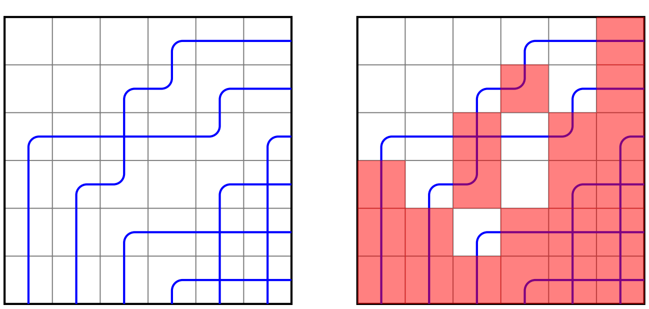
We first encode each BPD as the following combinatorial object.

Definition 3.1. A *flagged tableau* is a staircase grid with a cell in row i column j if $i + j \leqslant n$. Moreover, each cell in row i is empty or filled with a number in $[i]$.

We define an encoding map Φ from $\mathsf{BPD}(w)$ to the set of flagged tableaux.

Definition 3.2. Take $D \in \mathsf{BPD}(w)$ for some $w \in S_n$. For $i \in [n]$, there are $(i-1)$ pipes exiting from the top from row i of D, so there are $(i-1)$, and . We mark these cells, and then mark the rightmost unmarked cell in row i. There will be $n - i$ unmarked cells. To fill the cell in row i column j of $\Phi(D)$, we look at the j^{th} leftmost unmarked cell in row i of D. If it is a blank, we leave the cell in $\Phi(D)$ unfilled. Otherwise, it contains a pipe that ends in row p for some $p \leqslant i$. We fill the cell in $\Phi(D)$ by p.

Example 3.3. Assume $n = 6$. Take $D \in \mathsf{BPD}([2, 1, 6, 5, 3, 4])$ as depicted on the left. Then we perform the encoding algorithm and mark certain cells red. Finally, we obtain $\Phi(D)$.



$\Phi(D) =$

			1	1
		1	2	
2	2	2		
1				
5				

To precisely describe the image of $\mathsf{BPD}(w)$ under Φ, we need the following definition.

Definition 3.4. The *reading word* of a flagged tableau T, denoted as $\mathsf{word}(T)$, is a sequence of pairs obtained as follows. Go through entries of T from top to bottom, and right to left in each row. When we see the number i in column c, we write the pair $(i, n+1-c)$.

By the definition of flagged tableaux, for each pair in the reading word, the first entry is smaller than the second.

Example 3.5. In Example 3.3, $\mathsf{word}(\Phi(D)) = (1,2)(1,3)(2,3)(1,4)(2,4)(2,5)(2,6)(1,6)(5,6)$.

Let T be a flagged tableau with reading word $(a_1, b_1), \cdots, (a_d, b_d)$. For $i \in [d]$, we let $w_i = w_0 t_{a_1,b_1} \cdots t_{a_i,b_i}$. Then we say T is *associated* with the permutation w_d if

$$w_d \xrightarrow{t_{a_d,b_d}} w_{d-1} \xrightarrow{t_{a_{d-1},b_{d-1}}} \cdots \xrightarrow{t_{a_2,b_2}} w_1 \xrightarrow{t_{a_1,b_1}} w_0$$

is a saturated Bruhat chain. Let $\mathsf{FT}(w)$ consist of all flagged tableaux associated with w.

Example 3.6. In Example 3.3, $\Phi(D)$ is associated with $[2,1,6,5,3,4]$ because:

$$\begin{aligned}&[2,1,6,5,3,4] \xrightarrow{t_{5,6}} [2,1,6,5,4,3] \xrightarrow{t_{1,6}} [3,1,6,5,4,2] \xrightarrow{t_{2,6}} [3,2,6,5,4,1] \xrightarrow{t_{2,5}} [3,4,6,5,2,1]\\ &\xrightarrow{t_{2,4}} [3,5,6,4,2,1] \xrightarrow{t_{1,4}} [4,5,6,3,2,1] \xrightarrow{t_{2,3}} [4,6,5,3,2,1] \xrightarrow{t_{1,3}} [5,6,4,3,2,1] \xrightarrow{t_{1,2}} [6,5,4,3,2,1]\end{aligned}$$

is a saturated Bruhat chain from $[2,1,6,5,3,4]$ to w_0. Notice that $D \in \mathsf{BPD}([2,1,6,5,3,4])$.

For a flagged tableau T, define the *weight* of T, denoted as $\mathsf{wt}(T)$, to be a sequence of $n-1$ numbers whose i^{th} entry is the number of blanks in row i. Then we have:

Proposition 3.7. *For $w \in S_n$, Φ is a weight-preserving bijection from $\mathsf{BPD}(w)$ to $\mathsf{FT}(w)$.*

We may turn T into a chain compatible with $(n-1, \cdots, 2, 1)$ as follows. Suppose T has reading word $(a_1, b_1), \cdots, (a_d, b_d)$ and set $w_i = w_0 t_{a_1,b_1} \cdots t_{a_i,b_i}$ for $i \in [d]$. Let m_i be the number of non-empty cells above row $i+1$ of T for $i = 0, 1. \cdots, n-1$. Clearly, $(w_{m_i}, w_{m_i - 1}, \cdots, w_{m_{i-1}})$ is an i-chain. Moreover, we can check it is an increasing i-chain. Then define $\mathsf{chain}(T) := (w_{m_{n-1}}, \cdots, w_{m_1}, w_0)$, which is compatible with $(n-1, \cdots, 2, 1)$.

Example 3.8. Let T be the $\Phi(D)$ in Example 3.3. Then $\mathsf{chain}(T)$ is

$$([2,1,6,5,3,4],[2,1,6,5,4,3],[3,1,6,5,4,2],[3,5,6,4,2,1],[4,6,5,3,2,1],[6,5,4,3,2,1]).$$

Theorem 3.9. *The map $\mathsf{chain}(\cdot)$ is a bijection from $\mathsf{FT}(w)$ to Bruhat chains from w to w_0 compatible with $(n-1, \cdots, 2, 1)$. Consequently, $\mathsf{chain} \circ \Phi$ is a bijection from $\mathsf{BPD}(w)$ to such chains.*

The bijection $\mathsf{chain} \circ \Phi$ is an analogue of Lenart and Sottile's bijection [14] on $\mathsf{PD}(w)$. Notice that for $D \in \mathsf{BPD}(w)$, if $\mathsf{wt}(D) = (\alpha_1, \cdots, \alpha_{n-1})$ then

$$\mathsf{wt}(\mathsf{chain}(\Phi(D))) = (1 - \alpha_{n-1}, \cdots, n-2-\alpha_2, n-1-\alpha_1).$$

Thus, we have identified the BPD formula of $\mathfrak{S}_w$ with the Bruhat chain formula in Corollary 2.7 with $\gamma = [n-1, \cdots, 2, 1]$.

4 Analogue of Fomin-Stanley construction on BPDs

We now construct $\mathfrak{S}^{\mathsf{BPD}}$, our analogue of $\mathfrak{S}^{\mathsf{PD}}$, as a generating function of the flagged tableaux, or equivalently BPDs. Instead of the nil-Coexter algebra $\mathcal{N}_n$, our construction uses the *Fomin-Kirillov algebra* [5] $\mathcal{E}_n$, generated by $\{d_{i,j} : 1 \leqslant i < j \leqslant n\}$ satisfying:

$$\begin{cases} d_{i,j}^2 & = 0 \text{ if } i < j \text{ ,} \\ d_{i,j}d_{j,k} & = d_{i,k}d_{i,j} + d_{j,k}d_{i,k} \text{ if } i < j < k \text{ ,} \\ d_{j,k}d_{i,j} & = d_{i,j}d_{i,k} + d_{i,k}d_{j,k} \text{ if } i < j < k \text{ ,} \\ d_{i,j}d_{k,l} & = d_{k,l}d_{i,j} \quad \text{if } i < j,\, k < l \text{ and } i,j,k,l \text{ distinct.} \end{cases}$$

Fomin and Kirillov described an action of $\mathcal{E}_n$ on $\mathbb{Q}[S_n]$. In this paper, we adopt a slightly different convention and consider a right action of $\mathcal{E}_n$ on $\mathbb{Q}[S_n]$. For $w \in S_n$,

$$w \odot d_{i,j} := \begin{cases} wt_{i,j} & \text{if } wt_{i,j} \lessdot w \\ 0 & \text{otherwise.} \end{cases}$$

Define $A := \mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{E}_n$. It acts on $\mathbb{Q}[x_1, \cdots, x_{n-1}][S_n]$ from the right: $(fw) \odot (g \otimes e) = (fg)(w \odot e)$ for any $f, g \in \mathbb{Q}[x_1, \cdots, x_{n-1}]$, $w \in S_n$ and $e \in \mathcal{E}_n$. We may identify $\mathcal{E}_n$ and $\mathbb{Q}[x_1, \cdots, x_{n-1}]$ as subalgebras of A.

Definition 4.1. Take $i \in [n-1]$. For $i < j$, define $B_{i,j} \in \mathcal{E}_n$ as $B_{i,j} := d_{1,j} + \cdots + d_{i,j}$. Define $R_i(x_i) \in A$ as $R_i(x_i) := (x_i + B_{i,i+1})(x_i + B_{i,i+2}) \cdots (x_i + B_{i,n})$. Finally, define $\mathfrak{S}^{\mathsf{BPD}} \in \mathbb{Q}[x_1, \cdots x_{n-1}][S_n]$ as $\mathfrak{S}^{\mathsf{BPD}} := w_0 \odot (R_1(x_1)R_2(x_2) \cdots R_{n-1}(x_{n-1}))$.

We show $\mathfrak{S}^{\mathsf{BPD}}$ is a generating function of flagged tableaux, or equivalently all BPDs:

Proposition 4.2. *We have*

$$\mathfrak{S}^{\mathsf{BPD}} = \sum_{w \in S_n} \sum_{T \in \mathsf{FT}(w)} x^{\mathsf{wt}(T)} w = \sum_{w \in S_n} \sum_{D \in \mathsf{BPD}(w)} x^{\mathsf{wt}(D)} w.$$

Proof. If we expand $R_i(x_i)$, each term corresponds to one way of filling row i of a flagged tableau. The expression $(x_i + B_{i,j})$ in $R_i(x_i)$ corresponds to ways of filling the cell at row i and column $n+1-j$: x_i means to leave the box empty and $d_{p,j}$ means to fill it with p. If we expand $R_1(x_1) \cdots R_{n-1}(x_{n-1})$, for each term $x^\alpha d_{a_1,b_1} \cdots d_{a_k,b_k}$, there is a flagged tableau T with $\mathsf{wt}(T) = x^\alpha$ and $\mathsf{word}(T) = (a_1, b_1) \cdots (a_k, b_k)$. Let $w = w_0 \odot d_{a_1,b_1} \cdots d_{a_k,b_k}$. If $w = 0$, we know T is not associated with any permutation. Otherwise, $T \in \mathsf{FT}(w)$. Thus, we have the first equation. The second equation follows from Proposition 3.7. □

Now we establish the BPD analogue of (2.1).

Theorem 4.3. *We have* $\mathfrak{S}^{\mathsf{BPD}} = \sum_{w \in S_n} \mathfrak{S}_w w$.

Our proof is similar to the arguments of Fomin and Stanley. Consider a right action of $\mathcal{N}_n$ on S_n with $w \odot u_i = wt_{i,i+1}$ if $w(i) < w(i+1)$ and $w \odot u_i = 0$ otherwise. We may extend this action to $\mathbb{Q}[x_1, \cdots, x_{n-1}][S_n]$ by setting $f \odot u_i = f$ for all $f \in \mathbb{Q}[x_1, \cdots, x_{n-1}]$. Similar to Fomin and Stanley's approach, Theorem 4.3 reduces to:

Proposition 4.4. *For each $i \in [n-1]$, $\partial_i(\mathfrak{S}) = \mathfrak{S} \odot u_i$.*

Proof Sketch. The left hand side is just $w_0 \odot R_1(x_1) \cdots \partial_i(R_i(x_i)R_{i+1}(x_{i+1})) \cdots R_{n-1}(x_{n-1})$. We turn the right hand side into $w_0 \odot R_1(x_1) \cdots R_i(x_i)\, u_{i,i+1}\, R_{i+1}(x_{i+1}) \cdots R_{n-1}(x_{n-1})$. Then we show $w_0 \odot R_1(x_1) \cdots R_{i-1}(x_{i-1})$ is in the span of terms $x^\alpha w$ where x^α is a monomial involving $x_1, \cdots, x_{i-1}$ and $w \in S_n$ satisfies $w(i+1) > \cdots > w(n)$. We just need

$$x^\alpha w \odot \partial_i((R_i(x_i)R_{i+1}(x_{i+1})) = x^\alpha w \odot R_i(x_i)\, u_{i,i+1}\, R_{i+1}(x_{i+1}) \text{ for such } x^\alpha w.$$

We then establish this equation via a complicated but routine computation. □

Fomin and Kirillov [4] defined the *Dunkl element* $\theta_i := -\sum_{j<i} d_{j,i} + \sum_{j>i} d_{i,j} \in \mathcal{E}_n$ for $i \in [n]$. They showed the Dunkl elements $\theta_1, \cdots, \theta_n$ commute with each other. We end this subsection by providing an alternative way to write $\mathfrak{S}^{\mathsf{BPD}}$ using Dunkl elements.

Proposition 4.5. *We have $\mathfrak{S}^{\mathsf{BPD}} = w_0 \odot \prod_{1 \leqslant i < j \leqslant n}(x_i - \theta_j)$. Notice that terms multiplied on the right hand side commute with each other, so the $\prod$ notation makes sense.*

Remark 4.6. Sergey Fomin kindly informed the author that $w_0 \odot \prod_{1 \leqslant i < j \leqslant n}(x_i - \theta_j)$ seems related to the following variation of Cauchy identity of Schubert polynomials:

$$\prod_{1 \leqslant i < j \leqslant n} (x_i - y_j) = \sum_{w \in S_n} \mathfrak{S}_w(x_1, \cdots, x_{n-1})\mathfrak{S}_{ww_0}(-y_n, \cdots, -y_2). \tag{4.1}$$

Indeed, by the Monk's rule, (4.1) is equivalent to $w_0 \odot \prod_{1 \leqslant i < j \leqslant n}(x_i - \theta_j) = \sum_{w \in S_n} \mathfrak{S}_w w$. In other words, Theorem 4.3 and Proposition 4.5 form an alternative proof of (4.1).

5 Bijection between pipedreams and bumpless pipedreams

In this section, we present a weight preserving bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. By [14] and Theorem 3.9, we just need a weight reversing bijection between chains from w to w_0 compatible with $(1, \cdots, n-1)$ and those compatible with $(n-1, \cdots, 1)$.

This task can be done by Lenart's growth diagram [13], which can be viewed as the following algorithm. Given $k_1, k_2 \in [n-1]$ and chains C_1, C_2, where C_1 (resp. C_2) is a saturated k_1-chain from u to v (resp. k_2-chain from v to w), the algorithm outputs a saturated k_2-chain from u to v' and a saturated k_1-chain from v' to w. Moreover, the k_1-chain (resp. k_2-chain) in the output has the same length as C_1 (resp. C_2).

Assume $C_1 = (u_1, \cdots, u_{d_1})$ and $C_2 = (w_1, \cdots, w_{d_2})$ where $u_{d_1} = w_1$. We first draw:

$$u_1 \xrightarrow{k_1} u_2 \xrightarrow{k_1} \cdots \xrightarrow{k_1} u_{d_1-1} \xrightarrow{k_1} w_1 \xrightarrow{k_2} w_2 \xrightarrow{k_2} \cdots \xrightarrow{k_2} w_{d_2}.$$

We start from this labeled chain and apply a local move: Find a part of the chain that looks like $a \xrightarrow{k_1} b \xrightarrow{k_2} c$. We must have $a \lessdot_{k_1} b \lessdot_{k_2} c$. There exists a unique $b' \in S_n$ such that $b' \neq b$ and $a \lessdot b' \lessdot c$. If $a \lessdot_{k_2} b' \lessdot_{k_1} c$, we replace this part of the chain by $a \xrightarrow{k_2} b' \xrightarrow{k_1} c$. Otherwise, we must have $a \lessdot_{k_2} b' \lessdot_{k_1} c$ and we replace this part by $a \xrightarrow{k_2} b \xrightarrow{k_1} c$. We keep applying this local move until the labeled chain looks like:

$$u'_1 \xrightarrow{k_2} u'_2 \xrightarrow{k_2} \cdots \xrightarrow{k_2} u'_{d_2-1} \xrightarrow{k_2} w'_1 \xrightarrow{k_1} w'_2 \xrightarrow{k_1} \cdots \xrightarrow{k_1} w'_{d_1}.$$

Then we output the k_2-chain $(u'_1, \cdots, u'_{d_2-1}, w'_1)$ and the k_1-chain $(w'_1, \cdots, w'_{d_1})$.

Example 5.1. Say the inputs are: $k_1 = 2$, $k_2 = 3$, $C_1 = ([2,1,4,3],[2,4,1,3],[3,4,1,2])$, and $C_2 = ([3,4,1,2],[3,4,2,1])$. We start from the following labeled chain and apply local moves:

$$[2,1,4,3] \xrightarrow{2} [2,4,1,3] \xrightarrow{2} [3,4,1,2] \xrightarrow{3} [3,4,2,1].$$

$$[2,1,4,3] \xrightarrow{2} [2,4,1,3] \xrightarrow{3} [2,4,3,1] \xrightarrow{2} [3,4,2,1],$$

$$[2,1,4,3] \xrightarrow{3} [2,3,4,1] \xrightarrow{2} [2,4,3,1] \xrightarrow{2} [3,4,2,1].$$

Therefore, the outputs are $([2,1,4,3],[2,3,4,1])$ and $([2,3,4,1],[2,4,3,1],[3,4,2,1])$.

We may use Lenart's growth diagram to define a map $\mathsf{growth}_{k_1,k_2}$.

Definition 5.2. Take a chain (u,v,w) that is compatible with (k_1,k_2). Let C_1 (resp. C_2) be the increasing k_1-chain (resp. k_2-chain) from u to v (resp. v to w). Input C_1, C_2, k_1, k_2 to Lenart's growth diagram, obtaining a k_2-chain from u to v' and a k_1-chain from v' to w. Then define $\mathsf{growth}_{k_1,k_2}(u,v,w)$ as (u,v',w).

The map $\mathsf{growth}_{k_1,k_2}$ does not solve Problem 2.6. When (u,v,w) is compatible with (k_1,k_2), $\mathsf{growth}_{k_1,k_2}(u,v,w)$ might not be compatible with (k_2,k_1): By Example 5.1, we have

$$\mathsf{growth}_{2,3}([2,1,4,3],[3,4,1,2],[3,4,2,1]) = ([2,1,4,3],[2,3,4,1],[3,4,2,1]),$$

which is not compatible with $(3,2)$, but $([2,1,4,3],[3,4,1,2],[3,4,2,1])$ is compatible with $(2,3)$. Nevertheless, $\mathsf{growth}_{k_1,k_2}$ solves Problem 2.6 in the following special case.

Lemma 5.3. *Take $1 \leqslant k_2 < k_1 \leqslant n-1$ and $u, w \in S_n$ such that $w(k_1+1) > w(k_1+2) > \cdots > w(n)$ and $w(j) = n+1-j$ for each $j \in [k_2]$. Then $\mathsf{growth}_{k_1,k_2}$ is a weight reversing bijection from chains (u,v,w) compatible with (k_1,k_2) to chains (u,v',w) compatible with (k_2,k_1).*

Now we use $\mathsf{growth}_{k_1,k_2}$ to derive a map Growth. This map is defined on a chain $C = (w_n, \cdots, w_1)$ from w to w_0 compatible with $(n-1, \cdots, 2, 1)$. It first applies $\mathsf{growth}_{2,1}$, $\mathsf{growth}_{3,1}$, $\cdots$, $\mathsf{growth}_{n-1,1}$ to get a chain compatible with $(1, n-1, \cdots, 2)$. Then it applies $\mathsf{growth}_{3,2}, \cdots, \mathsf{growth}_{n-1,2}$ to get a chain compatible with $(1, 2, n-1, \cdots, 3)$. Eventually, it produces a chain compatible with $(1, 2, \cdots, n-1)$ defined as $\mathsf{Growth}(C)$. We can check when we apply each $\mathsf{growth}_{k_1,k_2}$, the condition in Lemma 5.3 is satisfied.

Proposition 5.4. *For $w \in S_n$, the map* Growth *is a weight-reversing bijection from* $\{$*chains from w to w_0 compatible with* $(n-1, \cdots, 1)\}$ *to* $\{$*chains from w to w_0 compatible with* $(1, \cdots, n-1)\}$.

By [14] and Theorem 3.9, Growth leads to a weight preserving bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$, which we conjecture agrees with the bijection of Gao-Huang [7].

Example 5.5. Consider the chain $([2,1,4,3], [2,3,4,1], [2,4,3,1], [4,3,2,1])$ which is compatible with $(3,2,1)$ and has weight $(1,1,2)$. We apply $\mathsf{growth}_{2,1}$ and then $\mathsf{growth}_{3,1}$ to get $([2,1,4,3], [4,1,3,2], [4,2,3,1], [4,3,2,1])$ which is compatible with $(1,3,2)$ and has weight $(2,1,1)$. Finally, use $\mathsf{growth}_{3,2}$ to get $([2,1,4,3], [4,1,3,2], [4,3,1,2], [4,3,2,1])$ which is compatible with $(1,2,3)$ and has weight $(2,1,1)$.

6 Extending Corollary 2.7 to double Schubert polynomials

The *double Schubert polynomial* $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ is in $x_1, \cdots, x_{n-1}$ and $y_1, \cdots, y_{n-1}$. It recovers $\mathfrak{S}_w$ after setting each y_i to 0 and can be computed using PDs and BPDs: For $P \in \mathsf{PD}(w)$ (resp. $\mathsf{BPD}(w)$), let $\mathsf{WT}(P)$ be the product over $\boxplus$ (resp. $\square$) in P, where the tile in row i column j gives $(x_i - y_j)$. By [9, 17], $\mathfrak{S}_w(\mathbf{x}, \mathbf{y}) = \sum_{P \in \mathsf{PD}(w)} \mathsf{WT}(P) = \sum_{P \in \mathsf{BPD}(w)} \mathsf{WT}(P)$.

Take $\gamma \in S_{n-1}$ and let $C = (w_1, \cdots, w_n)$ be a chain compatible with γ. Define $\mathsf{WT}_\gamma(C)$ as $\prod_{i=1}^{n-1} \prod_t (x_{\gamma_i} - y_{w_i(t)})$, where t runs over all $t > \gamma_i$ such that $w_i(t) = w_{i+1}(t)$. After setting all y_i to 0, $\mathsf{WT}_\gamma(C)$ recovers $x^{(n-1,\cdots,1) - \gamma^{-1}(\mathsf{wt}(C))}$. The following conjecture extends Corollary 2.7 and has been checked for all $w \in S_n$ for $n \leqslant 8$ and all $\gamma \in S_{n-1}$:

Conjecture 6.1. *For $\gamma \in S_{n-1}$, we have* $\mathfrak{S}_w\,(\boldsymbol{x}, \boldsymbol{y}) = \sum_{C:\textit{chain from } w \textit{ to } w_0 \textit{ compatible with } \gamma} \mathsf{WT}_\gamma(C)$.

This conjecture agrees with the PD and BPD formula when $\gamma = [1, \cdots, n-1]$ and $\gamma = [n-1, \cdots, 1]$ respectively via the bijections in [14] and Theorem 3.9.

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Asymptotic log-concavity of dominant lower Bruhat intervals via Brunn–Minkowski inequality

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Abstract. Björner and Ekedahl [Ann. of Math. (2), 170.2(2009), pp. 799-817] pioneered the study of length-counting sequences associated with parabolic lower Bruhat intervals in crystallographic Coxeter groups. In this extended abstract, we study the asymptotic behavior of these sequences in affine Weyl groups. Let W be an affine Weyl group with corresponding Weyl group W_f, and ${}^fW$ be the set of minimal representatives for the right cosets $W_f \backslash W$. Let t_λ be the translation by a dominant coroot lattice element λ and ${}^fb_i^{t_\lambda}$ be the number of elements of length i below t_λ in the Bruhat order on ${}^fW$, which is the $2i$-dimensional Betti number of a Schubert variety in a certain affine Grassmannian. We show that the sequence $\{{}^fb_i^{t_\lambda}\}_i$ is "asymptotically log-concave" in the following sense: the "shape" of the k-fold dilated sequence $\{{}^fb_i^{t_{k\lambda}}\}_i$, as k tends to infinity, converges to a continuous function obtained from a certain polytope P^λ; by the Brunn–Minkowski inequality, this function is log-concave.

Keywords: asymptotic log-concavity, affine Weyl group, dominant Bruhat intervals, dominant lattice formula, Brunn–Minkowski inequality

1 Background

Studying classes of Schubert varieties in the cohomology ring of the generalized flag variety leads to important results in enumerative geometry (the classical "Schubert calculus"), while the study of their intersection cohomology plays a fundamental role in representation theory (the "Kazhdan–Lusztig theory"). Following Björner and Ekedahl [1], we are interested in the behavior of the Betti numbers of Schubert varieties.

More precisely, consider a complex Kac–Moody group G with Borel subgroup B and maximal torus T. The corresponding Weyl group W has the structure of a crystallographic Coxeter system (W, S), where S is the generating set, and we denote by $\ell \colon W \to \mathbb{N}$ the length function. For any $J \subset S$, there is a parabolic subgroup $W_J := \langle s \in J \rangle$ of W and a corresponding subgroup $P_J := BW_JB$ of G.

The quotient $P_J \backslash G$ is a projective (ind-)variety called the *generalized (partial) flag variety*. We have the well-known Bruhat decomposition $P_J \backslash G = \bigsqcup_{w \in {}^JW} P_J \backslash P_J wB$, where

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${}^JW$ is the set of minimal representatives for the right cosets $W_J\backslash W$. The component $C_w := P_J\backslash P_JwB$ is called the *Schubert cell* associated with $w \in {}^JW$. Topologically, C_w is an $\ell(w)$-dimensional affine space $\mathbb{A}^{\ell(w)}$. Its closure $X_w := \overline{C_w}$ is called the *Schubert variety* associated with w. There is a partial order $\leq$ on ${}^JW$ called the *Bruhat–Chevalley order* defined by $v \leq w$ if $C_v \subseteq X_w$. Furthermore, we have the decomposition

$$X_w = \bigsqcup\nolimits_{v\in {}^JW, v\leq w} P_J\backslash P_JvB. \tag{1.1}$$

Question 1. *How many complex i-dimensional cells occur in the decomposition* (1.1) *of X_w?*

Let us denote this number by ${}^Jb_i^w$. Equation (1.1) gives the equality

$${}^Jb_i^w = \mathrm{Card}\{v \in {}^JW \mid v \leq w \text{ and } \ell(v) = i\}, \tag{1.2}$$

which also equals the $2i$-dimensional Betti number of X_w (the odd dimensional Betti numbers of X_w are 0).

Question 1 is difficult to answer in general. If X_w is smooth, the Poincaré duality implies that ${}^Jb_i^w = {}^Jb_{\ell(w)-i}^w$. While the hard Lefschetz theorem implies that the sequence $\{{}^Jb_i^w\}_i$ is *unimodal*, that is, it goes up and then goes down. But X_w is singular in general, hence Poincaré duality and hard Lefschetz theorem usually fail. By means of deep results in Hodge theory, Björner and Ekedahl [1] showed that the sequence $\{{}^Jb_i^w\}_i$ satisfies the following two sets of inequalities

$${}^Jb_i^w \leq {}^Jb_{\ell(w)-i}^w \text{ for } i \leq \frac{\ell(w)}{2}, \quad \text{and} \quad {}^Jb_0^w \leq {}^Jb_1^w \leq \cdots \leq {}^Jb^w_{\left\lceil\frac{\ell(w)}{2}\right\rceil-1} \leq {}^Jb^w_{\left\lceil\frac{\ell(w)}{2}\right\rceil}. \tag{1.3}$$

The first set of inequalities is rephrased as the sequence being *top-heavy*, while the second is the fact that the sequence is weakly increasing in the "lower half part".

Some variants of Question 1 have been studied. By Equation (1.2), one can formulate an analog of Question 1 for general Coxeter groups. Using Soergel bimodules and the Hodge theory established by Elias and Williamson in [11], it is proven that the inequalities in (1.3) hold for a general Coxeter group W in the non-parabolic case (that is, $J = \emptyset$, see [15]). For the parabolic case, we believe that a proof of these inequalities should follow from the Hodge theory of singular Soergel bimodules [18]. On the other hand, in the context of Schubert varieties of hyperplane arrangements, Huh and Wang [14], and Braden et al. [3] proved Dowling and Wilson's "Top-Heavy conjecture" for matroids.

Despite these great achievements, the unimodality of $\{{}^Jb_i^w\}_i$ for the "upper half part" remains an interesting open problem. To the best of our knowledge, there is no partial result yet. However, conjectures related to this problem have been made. Before we get into these, let us recall that a sequence $a_0, a_1, \ldots, a_n$ of positive real numbers is said to be *log-concave* if

$$a_{i-1}a_{i+1} \leq a_i^2 \text{ for all } 0 < i < n.$$

This notion is stronger than unimodality: a log-concave sequence is always unimodal. Regarding log-concavity of Bruhat intervals, Brenti conjectured the following:

Conjecture 2 ([4, Conjecture 2.11]). *Let W be a (finite) Weyl group, and $u, v \in W$. The sequence $\{b_i^{[u,v]}\}_i$ is log-concave, where $b_i^{[u,v]} := \mathrm{Card}\,\{w \in W \mid u \leq w \leq v \text{ and } \ell(w) = i\}$.*

It is known that the parabolic analog of Conjecture 2 does not hold. For example, the Betti numbers of the Schubert variety $X_{(8,8,4,4)}$ inside the Grassmannian $\mathrm{Gr}(4, 12)$ gives a non-unimodal sequence. See [20] for details.

2 Our results

Let $W = \mathbb{Z}\Phi^\vee \rtimes W_f$ be an affine Weyl group with finite Weyl group W_f and root system Φ of rank r. Let $(E, (-|-))$ be the r-dimensional Euclidean space where Φ lives in. Let ${}^fW$ be the set of minimal representatives for the right cosets $W_f \backslash W$. Denote by C_+ the dominant Weyl chamber. Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ be a dominant coroot lattice element, and $t_\lambda \in W$ be the translation by λ. Let ${}^f[e, t_\lambda] := \{w \in {}^fW \mid w \leq t_\lambda\}$ be the dominant lower Bruhat interval. For $0 \leq i \leq \ell(t_\lambda)$, we define

$$ {}^f b_i^{t_\lambda} := \mathrm{Card}\big\{w \in {}^f[e, t_\lambda] \bigm| \ell(w) = i\big\}. $$

This is the $2i$-dimensional Betti number of a (spherical) Schubert variety in the affine Grassmannian $\mathcal{G}r := G(F)/G(\mathcal{O})$, where $F = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$, and G is the semisimple and simply connected complex algebraic group with root system Φ. We prove that the sequence $\{{}^f b_i^{t_\lambda}\}_i$ is *asymptotically log-concave* in the following sense:

- The "shape" of the length-counting sequences of the dilated intervals ${}^f[e, t_{k\lambda}]$ converges to a continuous function when k tends to infinity (Theorem 3).
- This continuous function is log-concave (Corollary 7).

2.1 Asymptotic convergence

Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ be a fixed dominant lattice element. We define the convex polytope

$$ P^\lambda := \mathrm{Conv}\{w\lambda \mid w \in W_f\} \cap \overline{C_+} \subset E, $$

where $\mathrm{Conv}\{-\}$ is the convex hull of a set of points. Let $\mathrm{ht}\colon P^\lambda \to \mathbb{R}$ be the *height function* $\mathrm{ht}(x) := (2\rho|x)$, where ρ is the half sum of positive roots. In particular, $\mathrm{ht}(\lambda) = \ell(t_\lambda)$. We denote by Vol_r the Lebesgue measure on E and by $\mathrm{ht}_*\mathrm{Vol}_r$ the corresponding push-forward measure on $\mathbb{R}$. That is, for any Borel set $U \subseteq \mathbb{R}$,

$$ (\mathrm{ht}_*\mathrm{Vol}_r)(U) := \mathrm{Vol}_r(\mathrm{ht}^{-1} U) = \mathrm{Vol}_r\big(\big\{x \in P^\lambda \bigm| (2\rho|x) \in U\big\}\big). $$

We also denote by Vol_{r-1} the Lebesgue measure on affine hyperplanes of E. Then, the density function of $\mathrm{ht}_*\mathrm{Vol}_r$ is

$$g(z) = \|2\rho\|^{-1}\,\mathrm{Vol}_{r-1}(\mathrm{ht}^{-1}(z)).$$

Let δ_x denote the Dirac measure (that is, point mass) at the point $x \in \mathbb{R}$. For any positive integer k, we define a discrete measure $\mathfrak{m}_k$ supported on $[0, \ell(t_\lambda)]$ by

$$\mathfrak{m}_k := k^{-r} \sum\nolimits_{0 \le i \le k\ell(t_\lambda)} {}^f b_i^{t_{k\lambda}} \delta_{\frac{i}{k}}. \tag{2.1}$$

Intuitively, we distribute the sequence $\{{}^f b_i^{t_{k\lambda}}\}_i$ evenly on the interval $[0, \ell(t_\lambda)]$. We also define a step function $S_k \colon [0, \ell(t_\lambda)] \to \mathbb{R}$ as follows. For any $x \in [0, \ell(t_\lambda)]$, there exists a unique $i \in \{0, 1, \dots, k\ell(t_\lambda)\}$ such that $x \in [\frac{i}{k}, \frac{i+1}{k})$. We define

$$S_k(x) := k^{-(r-1)} \cdot {}^f b_i^{t_{k\lambda}}.$$

The function S_k records the numbers $\{{}^f b_i^{t_{k\lambda}}\}_i$ and behaves like the "density function" of $\mathfrak{m}_k$. The following is our main theorem.

Theorem 3. *Let* $\mathrm{Vol}_r(A_+)$ *be the volume of the fundamental alcove* A_+*.*

(1) (Weak convergence of measures) *The sequence of measures* $\{\mathfrak{m}_k\}_k$*, as k tends to infinity, converges weakly to* $\frac{1}{\mathrm{Vol}_r(A_+)}\mathrm{ht}_*\mathrm{Vol}_r$*.*

(2) (Uniform convergence of functions) *The sequence of functions* $\{S_k\}_k$*, as k tends to infinity, converges uniformly to the function* $\frac{1}{\mathrm{Vol}_r(A_+)}g$*.*

See Section 3 for an explicit example.

Remark 4. *If* λ *is strongly dominant, that is,* $\lambda \in C_+$*, then* P^λ *is combinatorially equivalent to a hypercube (see [5]).*

2.2 The dominant lattice formula

We define the *Poincaré polynomial* $\pi^{t_\lambda}(q)$ of the sequence $\{{}^f b_i^{t_\lambda}\}_i$ by

$$\pi^{t_\lambda}(q) := \sum\nolimits_{0 \le i \le \ell(t_\lambda)} {}^f b_i^{t_\lambda} q^i = \sum\nolimits_{w \in {}^f[e, t_\lambda]} q^{\ell(w)}.$$

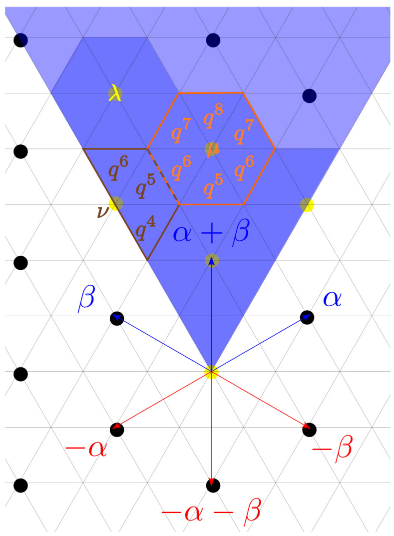
Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots of Φ, and $\{s_1, \dots, s_r\}$ be the set of corresponding simple reflections. For any $\mu \in \mathbb{Z}\Phi^\vee$, we denote by W_μ the standard parabolic subgroup of W_f generated by $\{s_i \mid 1 \le i \le r, (\mu|\alpha_i) = 0\}$ and by ${}^\mu W_f$ the set of minimal representatives for the right cosets $W_\mu \backslash W_f$. We also define *the Poincaré polynomial* ${}^\mu\pi_f(q)$ of the set ${}^\mu W_f$ by ${}^\mu\pi_f(q) := \sum_{w \in {}^\mu W_f} q^{\ell(w)}$.

The following theorem is one of our most important results, and plays an important role in the proof of Theorem 3.

Theorem 5 (Dominant lattice formula)**.** *Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ as before. Then*

$$\pi^{t_\lambda}(q) = \sum\nolimits_{\mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee} q^{(2\rho|\mu)} \cdot {}^{\mu}\pi_f(q^{-1}). \tag{2.2}$$

This formula serves as a bridge between the discrete nature of $\{{}^f b_i^{t_\lambda}\}_i$ and the continuous nature of the geometry of P^λ. See Figure 1 for an illustration.



Description of two of the summands of the dominant lattice formula when W is of affine type A_2 and $\lambda = 2\alpha + 3\beta$, where $\alpha := \alpha_1^\vee$ and $\beta := \alpha_2^\vee$. The yellow points are the lattice points inside P^λ. The alcoves of the interval ${}^f[e, t_\lambda]$ are colored with dark blue. There are 6 dominant alcoves arranged around the strongly dominant lattice point $\mu := 2\alpha + 2\beta$, and 3 around $\nu := \alpha + 2\beta$ which is on the wall. The summand corresponding to μ in the formula is given by $q^8 \cdot {}^{\mu}\pi_f(q^{-1}) = q^5 + 2q^6 + 2q^7 + q^8$. The terms of this polynomial are colored orange and placed inside their corresponding alcoves in the picture. The summand corresponding to ν is given by $q^6 \cdot {}^{\nu}\pi_f(q^{-1}) = q^4 + q^5 + q^6$, whose terms are colored with brown.

Figure 1: Illustration for the dominant lattice formula.

2.3 Log-concavity and a conjecture on unimodality

The following theorem taken from [17, p. 270] can be deduced from the classical Brunn–Minkowski inequality.

Theorem 6 (Brunn–Minkowski, see [17, p. 270])**.** *Let L_1 be a real vector space and let $M \subset L_1$ be a convex body. Let $p\colon L_1 \to L_2$ be a linear transformation. Then*

$$x \mapsto \big(\mathrm{Vol}\big(p^{-1}(x) \cap M\big)\big)^{1/(\dim M - \dim p(M))}$$

is a concave function on $p(M)$.

Applying the above theorem to the map $\mathrm{ht}\colon P^\lambda \to \mathbb{R}$ and taking logarithm (which is a concave function), we have immediately the following corollary.

Corollary 7. *The density function g of the measure $\mathrm{ht}_*\mathrm{Vol}_r$ is log-concave, that is, $\log g$ is a concave function.*

Remark 8. *The sequence $\{{}^f b_i^{t_\lambda}\}_i$ is not necessarily log-concave. For example, from the step function in Figure 2a, we observe that the sequence contains the consecutive terms $(4, 4, 5)$.*

We propose the following conjecture:

Conjecture 9. *The sequence* $\{{}^f b_i^{t_\lambda}\}_i$ *is unimodal.*

This conjecture has been tested for different choices of λ in affine Weyl groups of rank ≤ 4 (and also type $\widetilde{A_5}$) with the help of `SageMath`.

3 An example of Theorem 3

Let W be the affine Weyl group associated with the root system Φ of type C_3 and simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$. Then, $r = 3$. Following [2, Plate III], we write $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, and $\alpha_3 = 2\epsilon_3$. Let

$$\lambda = 3\alpha_1^\vee + 6\alpha_2^\vee + 7\alpha_3^\vee.$$

We have that $\mathrm{ht}(\lambda) = 32$. For convenience, we define $(a, b, c)_\Phi := a\alpha_1^\vee + b\alpha_2^\vee + c\alpha_3^\vee$. The polytope P^λ is the convex polyhedron with six vertices given by

$$\{(0,0,0)_\Phi, (3,3,3)_\Phi, (3,5,7)_\Phi, (3,6,6)_\Phi, (7/3, 14/3, 7)_\Phi, (3,6,7)_\Phi\},$$

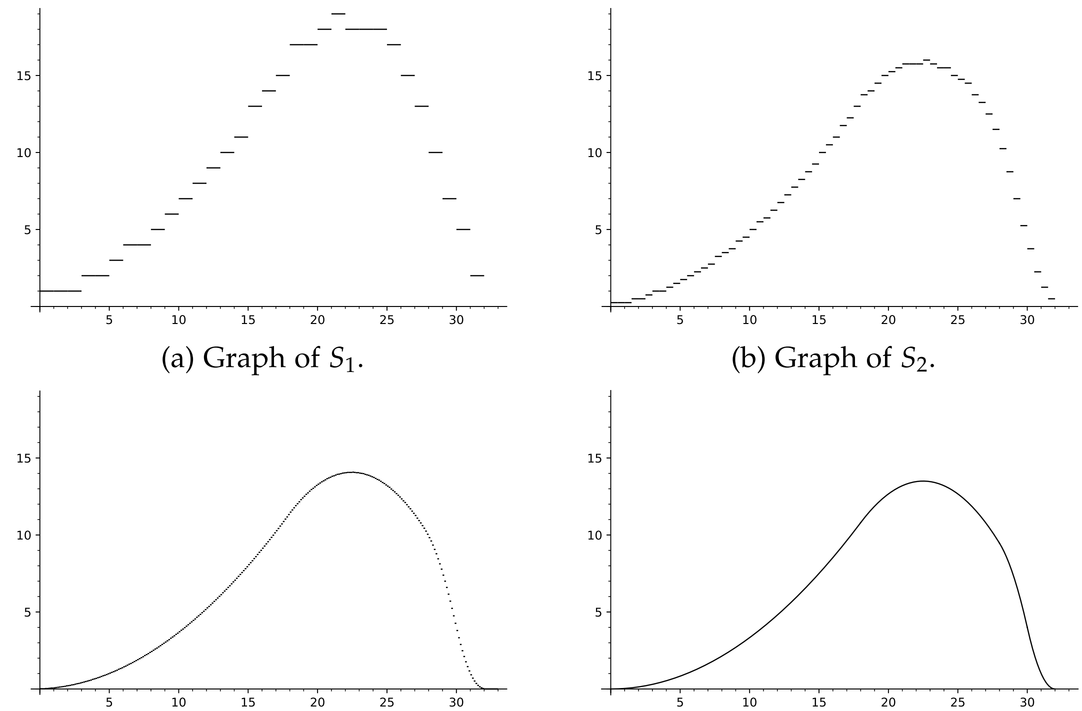
which is an example of a non-lattice polytope. Since $\rho = (3,5,3)_\Phi$, we get $||\rho|| = \sqrt{14}$. From [8, Equation 2.4], or by direct computations, we have that $\mathrm{Vol}_3(A_+) = 1/48$. In view of Theorem 3, the only missing ingredient to compute the limit function is to determine the area function $\mathrm{Vol}_2(\mathrm{ht}^{-1}(x))$. From the theory of convex polytopes, this function is a piece-wise quadratic polynomial and its exact form can be obtained by Lagrange interpolation. We omit the details and just give a graph of the function $g/\,\mathrm{Vol}_3(A_+)$ in Figure 2d.

We can use Theorem 3 to give quick estimates of the terms in the sequence $\{{}^f b_i^{t_{k\lambda}}\}_i$ when k is big enough. For instance, when $k = 8$, the value of ${}^f b_{196}^{t_{8\lambda}}$ is virtually impossible to obtain in a computer directly from definitions. Let us pick $x = 24.5 (= 196/8)$. From our theorem we have

$$S_8(24.5) = {}^f b_{196}^{t_{8\lambda}}/8^2 \sim 48g(24.5) = 389/30$$

giving ${}^f b_{196}^{t_{8\lambda}} \sim 829.86$.

On the other hand, Theorem 5 gives the exact values of the terms in the sequence $\{{}^f b_i^{t_{k\lambda}}\}_i$. We can compute the value of the function S_8 (which takes a considerable time to get in a computer.) In particular, we have ${}^f b_{196}^{t_{8\lambda}} = 863$. Our quick estimate of 829.86 from before was off by 3.84%. In various examples, we observed that the error of the estimation decreases roughly linearly with the growth of k. See Figure 2 for the graphs of the step functions S_1, S_2, and S_8.



(a) Graph of S_1. (b) Graph of S_2.

(c) Graph of S_8. (d) Graph of $g/\operatorname{Vol}_r(A_+)$.

Figure 2: In the affine Weyl group W of affine type C_3, we consider $\lambda = 3\alpha_1^\vee + 6\alpha_2^\vee + 7\alpha_3^\vee$. These pictures illustrate how the sequence of step functions $S_k\colon [0, \ell(t_\lambda)] \to \mathbb{R}$ converges uniformly to the continuous function $g/\operatorname{Vol}_r(A_+)$.

4 Connections with asymptotic representation theory

The formulation of Theorem 3(1) borrows ideas from the construction of the now-called Duistermaat–Heckman measure [13] and Okounkov's work [16] on the asymptotic log-concavity for multiplicities of representations. Let G be a compact connected Lie group and λ be a dominant weight. In [13], Heckman constructed a discrete measure

$$\frac{\sum_\mu \dim V(k\lambda)_\mu \delta_{\frac{\mu}{k}}}{\sum_\mu \dim V(k\lambda)_\mu}$$

supported on the weight polytope $\operatorname{Conv}\{w\lambda \mid w \in W_f\}$, where $\dim V(k\lambda)_\mu$ is the weight multiplicity of the irreducible representation of G with highest weight $k\lambda$. He proved that this sequence of discrete measures, as k tends to infinity, converges weakly to the push-forward of the Liouville measure of the coadjoint orbit of λ under the moment map. The density function of the limit measure is a piecewise polynomial function [9] and Graham proved that it is log-concave via Hodge–Riemann inequalities [12]. Later, Okounkov [16] introduced the now-called Newton–Okounkov bodies to prove, in a similar weak limit sense, that for any reductive group G and any representation V of G, the multiplicities of irreducible G-modules in the homogeneous coordinate ring of a G-stable irreducible subvariety of $\mathbb{P}(V)$ are log-concave.

It is not hard to see the similarity between our construction (2.1) and the one of Heckman, and it is indeed similar to the one of Okounkov. However, our proof technique is quite different from theirs. Moreover, it is not obvious that our original cell-counting problem has such a critical relation to the geometry of a convex polytope. Theorem 3(1) is the analog of theirs, while a result like Theorem 3(2) is novel in this kind of setting.

5 Relation with Ehrhart's theory

For an r-dimensional lattice polytope P (that is, all vertices of P are points of a given lattice L), the *Ehrhart polynomial* [10] is a polynomial in k that counts the number of lattice points in the k-fold dilation kP of P. The leading coefficient is equal to the r-dimensional volume $\mathrm{Vol}_r(P)$ of P, divided by the volume $d(L)$ of the fundamental region of the lattice L. This implies that

$$\mathrm{Vol}_r(P) = \lim_{k\to\infty} \frac{d(L)\cdot \mathrm{Card}\{\text{lattice points in } kP\}}{k^r}. \tag{5.1}$$

If X is the toric variety corresponding to the normal fan of P, then P defines an ample line bundle on X. The Ehrhart polynomial of P coincides with the Hilbert polynomial of this line bundle, and the asymptotic result (5.1) is a consequence of the asymptotic Riemann-Roch theorem [19, Tag 0BJ8].

The problem in our work is analogous to the one in Ehrhart's theory, while we count alcoves in each length rather than all lattice points in the polytope P^λ. When the polytope P^λ is not a lattice polytope, it has no Ehrhart polynomial. We want to raise the following question related to Theorem 3(2):

Question 10. *Is ${}^f b_{ki}^{t_{k\lambda}}$ a quasi-polynomial in k of degree $(r-1)$ for k sufficiently large, with*

$$\frac{\mathrm{Vol}_{r-1}(\mathrm{ht}^{-1}(i))}{\mathrm{Vol}_r(A_+)\cdot \|2\rho\|}$$

as the leading coefficient?

6 Main ideas in the proofs of Theorem 3 and Theorem 5

For complete proofs, see [6].

6.1 Theorem 5

Our cell-counting problem can be translated into "counting alcoves" thanks to the natural bijection between the affine Weyl group W and the set of alcoves. In particular, $w \in {}^fW$ if and only if the corresponding alcove A_w is dominant, that is, it is contained in C_+. On the other hand, we have the following well-known result.

Lemma 11. *Suppose $\lambda, \mu \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$. The following are equivalent:*

(1) $t_\mu \leq t_\lambda$ *in the Bruhat–Chevalley order.*

(2) $\mu \in \mathrm{Conv}\{w\lambda \mid w \in W_f\}$.

These facts motivate the definition of the polytope $P^\lambda = \mathrm{Conv}\{w\lambda \mid w \in W_f\} \cap \overline{C_+}$. They also lead to a description of the interval ${}^f[e, t_\lambda]$ in terms of the lattice points in P^λ:

$$ {}^f[e, t_\lambda] = \{t_\mu w \in W \mid \mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee, w \in {}^\mu W_f\}. $$

Then, the dominant lattice formula (Theorem 5) follows from comparing the lengths of the elements on both sides of this equality.

6.2 Theorem 3

The following is the main geometric intuition in our proof of Theorem 3: a dominant Bruhat interval can be realized as a bounded region—a union of finitely many alcoves—inside C_+; after dilating the lattice element λ of the interval ${}^f[e, t_\lambda]$ and re-scaling the region back, the alcoves in the region get smaller and smaller, and the region approaches P^λ. This is illustrated in Figure 3. Other works relating Euclidean geometry and Bruhat intervals in affine Weyl groups can be found in [8, 7].

The following corollary of Theorem 5 is crucial in the proof of Theorem 3:

Corollary 12. *We define*

$$ \pi_f(q) := \textstyle\sum_{w \in W_f} q^{\ell(w)}, \quad \pi^\lambda(q) = \sum_{\mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee} q^{(2\rho|\mu)}, \quad \pi^\lambda_+(q) = \sum_{\mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee \cap C_+} q^{(2\rho|\mu)}. $$

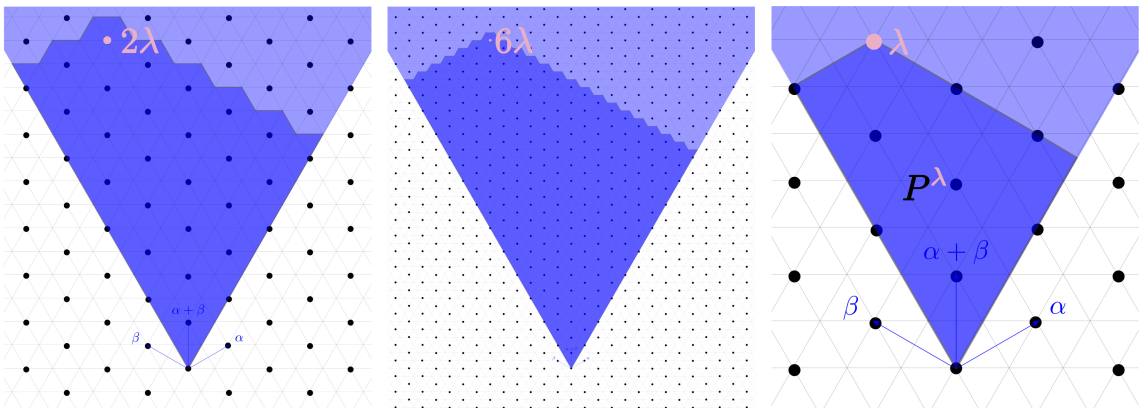
Then we have

$$ \pi^\lambda_+(q) \cdot \pi_f(q^{-1}) \leq \pi^{t_\lambda}(q) \leq \pi^\lambda(q) \cdot \pi_f(q^{-1}), \tag{6.1} $$

where the inequalities between these Laurent polynomials mean to be coefficient-wise.

Considering the coefficients in the inequality (6.1), we are able to approximate ${}^f b_i^{t_{k\lambda}}$ using the numbers

$$ \mathrm{Card}\left(P^\lambda \cap \frac{1}{k}\mathbb{Z}\Phi^\vee \cap \mathrm{ht}^{-1}(y) \right), \tag{6.2} $$



(a) The interval ${}^f[e, t_{2\lambda}]$. (b) The interval ${}^f[e, t_{6\lambda}]$. (c) The polytope P^λ.

Figure 3: Behavior of the intervals ${}^f[e, t_{k\lambda}]$ when W is of affine type A_2 and $\lambda = 3\alpha + 4\beta$, where $\alpha := \alpha_1^\vee$ and $\beta := \alpha_2^\vee$. In each picture, the set of small triangles corresponds to the set of alcoves. The coroot lattice is indicated by black bullets and the dominant Weyl chamber is colored blue. In the first two pictures, the alcoves corresponding to the elements in the intervals are filled with darker blue. So is the polytope P^λ in the third picture.

where y runs over a particular set of numbers near i/k. For these y, it turns out that $(\frac{1}{k}\mathbb{Z}\Phi^\vee) \cap \mathrm{ht}^{-1}(y)$ is a lattice of rank $r-1$ in the affine hyperplane $\mathrm{ht}^{-1}(y)$.

We construct a Riemann sum using the numbers from (6.2). As k tends to infinity, this sum converges to a quantity related to the volume of a part of P^λ. From basic results about weak convergence, this leads to a proof of Theorem 3(1).

The proof of Theorem 3(2) is more technical than the proof of Theorem 3(1). First of all, it suffices to prove that $S_k(x)$ converges uniformly for $x \in [0, \ell(t_\lambda)]$ of the form i/k, because of the definition of S_k and the continuity of g. For this, we need to estimate the value of the step function S_k at those $x = i/k$, which is $k^{1-r} \cdot {}^f b_i^{t_{k\lambda}}$. As before, we switch this estimation to the estimation of the numbers in (6.2). Let y be as before.

We choose a fundamental domain B_k of the lattice $(\frac{1}{k}\mathbb{Z}\Phi^\vee) \cap \mathrm{ht}^{-1}(0)$ containing the origin point of $\frac{1}{k}\mathbb{Z}\Phi^\vee$. If we join all the translations of B_k by points in $P^\lambda \cap (\frac{1}{k}\mathbb{Z}\Phi^\vee) \cap \mathrm{ht}^{-1}(y)$, we obtain the region

$$\mathcal{R} := \bigsqcup \left\{ l + B_k \,\middle|\, l \in P^\lambda \cap \frac{1}{k}\mathbb{Z}\Phi^\vee \cap \mathrm{ht}^{-1}(y) \right\}$$

in the hyperplane $\mathrm{ht}^{-1}(y)$.

Because we can compute the volume of B_k directly from Φ, estimating the value of (6.2) is equivalent to estimating the value of $\mathrm{Vol}_{r-1}(\mathcal{R})$. The proof of the convergence is then achieved by comparing $\mathrm{Vol}_{r-1}(\mathcal{R})$ with $\mathrm{Vol}_{r-1}(P^\lambda \cap \mathrm{ht}^{-1}(x))$. This, as well as the uniformity, requires the use of some "Euclidean geometries" to carefully estimate the volume of some open neighborhood of the boundary of $P^\lambda \cap \mathrm{ht}^{-1}(x)$ (see Figure 4 for an example of such a neighborhood). When k is large enough, for any x and y, the boundary of $\mathcal{R}$ is contained in such a neighborhood. Because the volume of such a neighborhood can be sufficiently small, this implies that $\mathrm{Vol}_{r-1}(\mathcal{R})$ is sufficiently close to $\mathrm{Vol}_{r-1}(P^\lambda \cap \mathrm{ht}^{-1}(x))$. This leads to the proof of the uniform convergence.

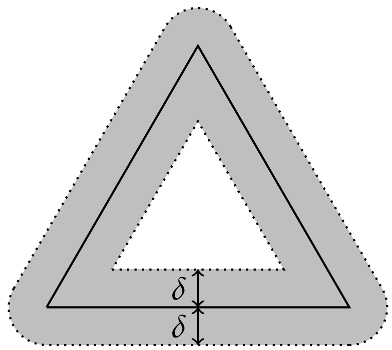


Figure 4: A triangle T and the neighbourhood $\mathcal{N}(\partial T, \delta)$.

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Combinatorics of Boundary Algebras

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Abstract. Boundary algebras are an important tool in the categorification, by Jensen–King–Su and by Pressland, of cluster structures on positroid varieties, defined by Scott and by Galashin–Lam. Each connected positroid has a corresponding boundary algebra. We give a combinatorial way to recover a positroid from its boundary algebra. We then describe the set of algebras which arise as the boundary algebra of some positroid. Finally, we give the first complete description of the minimal relations in the boundary algebra. We expect this description to be helpful in extending results known for Grassmannian boundary algebras to more general settings.

Keywords: Categorification, Positroids, Cluster Algebras

1 Introduction

An *open positroid variety*, defined by Knutson–Lam–Speyer [8], is the variety of points in the Grassmannian realizing a given positroid. They broaden the positroid stratification of the nonnegative Grassmannian [11] to the full Grassmannian. As conjectured in [10, 9] and proven in [14, 15, 6], the coordinate ring of an open positroid variety has the structure of a *cluster algebra*. Such a cluster structure is a combinatorially rich algebraic structure that in particular interacts well with nonnegativity [5]. For instance, the positivity of the cluster variables in a single cluster is enough to guarantee the positivity of all the other, possibly infinitely many, cluster variables.

Boundary algebras appear in the context of *categorification* of the cluster structure on an open positroid variety. Categorification is a process by which structures from other areas of math are realized using category theory, often through module categories. In 2006, Scott [14] showed that the Grassmannian has a cluster structure, which was categorified by Jensen–King–Su [7] as the category of *Gorenstein-projective modules* over the *circle algebra*. In 2016, Baur–King–Marsh [4] connected this with dimer models by realizing the circle algebra as a completed boundary algebra of a Grassmannian dimer model.

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Pressland [12] extended this setup in 2022 by showing that the cluster structure defined by Galashin–Lam on an arbitrary connected positroid variety [6] is categorified by the Gorenstein-projective category of the completed boundary algebra of an appropriate dimer model.

This categorification has proven useful in understanding the Galashin–Lam cluster structure on positroid varieties. Pressland [13] used it to prove a conjecture of Muller–Speyer that two a priori different cluster are closely related [10]. In particular, he shows that the source-labelled and target-labelled cluster structures on a positroid variety quasi-coincide. Much work is being done on the cluster structures on Grassmannians, including studying the Gorenstein-projective modules over the circle algebra corresponding to rank 2 and 3 cluster variables [2, 3]. This work is difficult to extend to all positroid varieties because there is no generators-and-relations description in the literature for boundary algebras of general positroid varieties. Forthcoming work of the first author and Khrystyna Serhiyenko contains a combinatorial construction of the boundary algebra, but only describes the relations up to an operation called *cancellative closure*. We build on this construction and give a combinatorial description of the boundary algebra of a connected positroid variety, including a minimal set of relations. We isolate combinatorial data which determines the boundary algebra, and call it a **boundary chart**. We characterize boundary charts of connected positroids and provide an explicit bijection between realizable boundary charts and connected positroids. This gives us a new cryptomorphism of connected positroids.

Our new description of boundary algebras gives additional tools for studying the Gorenstein-projective modules over these algebras. We expect our results to be useful in generalizing the work mentioned above from the Grassmannian setting to general positroid varieties.

2 Background

2.1 Positroids

A *positroid* is a special type of realizable matroid which reflects the combinatorial structure of the *totally nonnegative Grassmannian*. See [11] for background on positroids. In this section, we introduce *perfect orientations* and *decorated permutations*, which are two of many equivalent descriptions of positroids.

Definition 2.1. *A **plabic graph** (planar bi-colored graph) is an undirected planar graph embedded in a disc with n vertices on the boundary, labelled b_i for $i \in [n]$ in clockwise order. Plabic graphs may have additional vertices in the interior of the disc which are each assigned one of two colors, either white ($\circ$) or black ($\bullet$). Boundary vertices must be incident to exactly one edge. We consider plabic graphs modulo homotopy.*

There are moves and reductions that can be applied to plabic graphs which preserve the key combinatorial properties we are interested in. Using these, we may, and will, assume that our plabic graphs are bipartite (for this purpose, we ignore boundary vertices) and *reduced* [11, Def. 12.5]. An example of such a graph is illustrated in Figure 1.

We may define positroids in terms of *perfect orientations* of plabic graphs. These are orientations $\mathcal{O}$ of the edges of the plabic graph such that each white internal vertex is incident to exactly one incoming edge and each black internal vertex is incident to exactly one outgoing edge. The source set of $\mathcal{O}$ is then $\{i \mid b_i \text{ is a source in } \mathcal{O}\}$. Fix a plabic graph G on n boundary vertices. All perfect orientations have source sets of the same size k. We say such a plabic graph is of *type* (k, n). The set consisting of the source sets of all perfect orientations forms a positroid $\mathcal{P}(G)$ of rank k on $[n]$ [11].

Move equivalent plabic graphs give the same positroid. One way to distinguish move equivalence classes of plabic graphs is by using their *trip permutations*. These are permutations π defined as follow: For each $i \in [n]$, construct a trip which starts from b_i and follow the edges of the plabic graph according to the "rules of the road": At each white vertex turn right, and at each black vertex turn left. This trip will end at some boundary vertex b_j and we define $\pi(i) = j$. We obtain a fixed point $\pi(i) = i$ if and only if b_i is connected by a single edge to a leaf. At fixed points, we additionally keep track of the color of this leaf. With this additional data, we refer to π as a *decorated permutation*. Decorated permutations of $[n]$ are in bijection with positroids on $[n]$. We denote by $\mathcal{P}_\pi$ the positroid corresponding to a decorated permutation π. In this abstract, we will be primarily concerned with connected positroids, in which case the decorated permutations are *stabilized-interval-free permutations* [1]. These have no fixed points and so, in particular, are undecorated permutations.

Consider $\mathrm{Gr}_{k,n}$, the Grassmannian of k-planes in $\mathbb{C}^n$, embedded in $\mathbb{CP}^{\binom{n}{k}-1}$ by *Plücker coordinates* Δ_I, where I is a k-element subset of $[n]$. To define *open positroid varieties*, we will need to label faces F of a plabic graph G of type (k, n) by the set of $i \in [n]$ such that F lies to the left of the trip terminating at i. One can show that each such label will have size k and, if $\mathcal{P}(G) = \mathcal{P}(G')$, then the boundary faces of G and G' will have the same labels [10].

Definition 2.2. *Fix a positroid $\mathcal{P}$ of rank k on $[n]$. Let G be a reduced plabic graph such that $\mathcal{P} = \mathcal{P}(G)$. Label the faces of G as above. The* ***open positroid variety*** $\Pi^\circ_{\mathcal{P}}$ *is the subset of* $\mathrm{Gr}_{k,n}$ *where $\Delta_I = 0$ for all $I \notin \mathcal{P}$ and $\Delta_I \neq 0$ for all I which label a boundary face of G.*

Finally, we construct the *quiver* of a plabic graph.

Definition 2.3. *A* ***quiver*** $Q = (Q_0, Q_1)$ *is a directed graph with vertices Q_0 and arrows Q_1, with no loops or oriented 2-cycles. Some vertices $F_0 \subset Q_0$ may be marked as frozen.*

For a bipartite plabic graph G, define the quiver $Q(G)$ as follows: Place a vertex at each internal face of the plabic graph. Faces incident to the boundary contain frozen

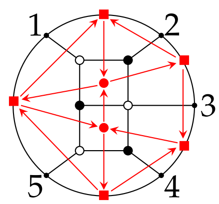

Figure 1: A plabic graph on 5 boundary vertices, with its quiver indicated in red, where frozen vertices are squares and unfrozen vertices are circles. The decorated permutation, obtained by following the rules of the road, is $\pi = 45123$.

vertices. For each edge e of G which is incident to a white vertex, add an arrow α to the quiver between the faces on either side of e such that the white endpoint of e is to the right of α. The only other edges of G are those which connect boundary vertices to black vertices. In this case, add an arrow α such that the black vertex is to the left of α. The quiver of a plabic graph is illustrated in Figure 1.

Galashin and Lam have shown that the coordinate ring of an open positroid variety $\Pi^\circ_{\mathcal{P}(G)}$ is isomorphic to the cluster algebra with quiver $Q(G)$ [6].

2.2 Boundary Algebras

We now introduce the *boundary algebra*, which is important for the categorification of the cluster structure on an open positroid variety introduced by Pressland [12].

Definition 2.4. *For a quiver Q, the **path algebra** $\mathbb{C}Q$ is spanned by finite paths in Q, including empty paths at each vertex. The product in the algebra between paths p and q is the concatenation of p and q, if p ends at the start of q, and 0 otherwise.*

Fix a stabilized-interval-free permutation π. Let G be any plabic graph with trip permutation π, and let $Q = Q(G)$. Each internal face of Q is bounded by an oriented cycle. Each edge d not between two frozen boundary vertices is incident to two faces and thus part of oriented cycles c_d^+ and c_d^- bounding those faces. Say c_d^+ factors as dp_d^+ and c_d^- factors as dp_d^-. Let e_i be the empty path at the boundary vertex v_i of $Q(G)$, and let $e = \sum_{i=1}^n e_i$.

Definition 2.5. *The **dimer algebra** A_Q is $\mathbb{C}Q$ modulo the relations $p_d^+ = p_d^-$ for all edges d of Q which are not between two frozen vertices.*

Definition 2.6. *The **boundary algebra** of the positroid $\mathcal{P}_\pi$ is $B_\pi = eA_{Q(G)}e$ for any G such that $\mathcal{P} = \mathcal{P}(G)$.*

It is not obvious, but if $\mathcal{P}(G) = \mathcal{P}(G')$, then $eA_{Q(G)}e = eA_{Q(G')}e$, so this is well defined. Multiplication by e on both sides discards each path which neither originates from nor terminates at a boundary vertex. Thus, B_π can be thought of as the algebra of paths between boundary vertices, modulo the relations $p_d^+ = p_d^-$.

We define the following elements of B_π: For each $i \in [n]$, let x_i be a minimal path from v_i to v_{i+1} and let y_i be a minimal path from v_{i+1} to v_i, where all indices are taken modulo n. Let $x = \sum_{i=1}^n x_i$ and $y = \sum_{i=1}^n y_i$. Let $t = xy$; then t is central in B_π. Let p be a path in B_π between two nonadjacent vertices. Define $\tau(p)$ and $\eta(p)$ by the condition that p is directed from $v_{\tau(p)}$ to $v_{\eta(p)}$. Suppose p does not factor as a product of other paths. We then say p is a *nonadjacent arrow* and define $\text{reach}_p = \eta(p) - \tau(p)$ taken modulo n so that $\text{reach}_p \in [n]$. One can show that p satisfies $pt^{X_p} = x_{\tau(p)}x_{\tau(p)+1}\cdots x_{\eta(p)-1}$ for a suitable positive integer X_p. We represent the boundary algebra by putting the vertices v_i in clockwise order around a circle and drawing in the nonadjacent arrows. For example, the second subfigure of Figure 2 shows the representation of a boundary algebra. with a nonadjacent arrow from v_4 to v_1 in black. In this figure, the arrows x_i and y_i are shown in gray.

In forthcoming work, the first author and Khrystyna Serhiyenko prove the following two results showing how to calculate the boundary algebra B_π directly from a stabilized-interval-free permutation π. In order to state these results, we must represent the permutation π as a directed graph on vertices w_i with directed edges from w_i to $w_{\pi(i)}$ for $i \in [n]$. We will draw the permutation graph such that w_i lies between vertices v_i and v_{i+1}. We refer to edges of the permutation graph as *strands*. Figure 2 shows three examples of permutation graphs, in red.

Definition 2.7. *For $i \in [n]$, we define the* ***i-shifted linear order*** $<_i$ *on $[n]$ by $i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1$. We say $(a_1 \leq \cdots \leq a_m) \in [n]^m$ is a* ***cyclic ordering*** *if there exists some $i \in [n]$ such that $a_1 \leq_i a_2 \leq_i \cdots \leq_i a_m$. We will allow ourselves to replace some or all of the inequalities with strict inequalities if consecutive terms are not allowed to be equal.*

Definition 2.8. *Define $[i,j] = \{l \mid (i \leq l \leq j) \text{ is a clockwise ordering}\}$ to be the (closed) clockwise interval between i and j. We similarly define the clockwise intervals $(v_i, v_j]$, $[v_i, v_j)$, and (v_i, v_j) by excluding one or both of the endpoints.*

Definition 2.9. *Let v_j be a boundary vertex of Q. Consider a strand α from r to $\pi(r)$ in the permutation graph of π. We say that v_j is* ***to the right of*** *α if $j \in [\pi(r), r)$ and otherwise v_j is* ***to the left of*** *α. Let p be an arrow between nonadjacent vertices v_i and v_j. We say that α is* ***left-supporting*** *to p if $(\pi(r) \leq r \leq \eta(p) \leq \tau(p))$ is a cyclic ordering. We say that the strand α is* ***right-supporting*** *to p if $(r \leq \pi(r) \leq \tau(p) \leq \eta(p))$ is a cyclic ordering. In either case, we say that this strand is* ***supporting*** *to p.*

Informally, a strand left (resp. right) supports an arrow p if it points in the opposite direction of p and lies to its left (resp. right).

Theorem 2.10. *Fix a connected positroid $\mathcal{P}$ with permutation π. Fix distinct nonadjacent boundary vertices v_i and v_j. The arrow p from v_i to v_j is a nonadjacet arrow of B_π if and only if*

1. *for all $l \in (j, i)$, there is a right-supporting strand α of p with v_l to its left, and*

2. *for all $l \in (i, j)$, there is a left-supporting strand α of p with v_l to its right.*

*In this case, the relations $pt^{X_p} = x_i x_{i+1} \cdots x_{j-1}$ and $pt^{Y_p} = y_{i-1} y_{i-2} \cdots y_j$ hold in B_π, where X_p is the number of left-supporting strands and Y_p is the number of right-supporting strands of p. We call X_p the **left relation number** of p, and we call Y_p the **right relation number** of p.*

Definition 2.11. *An ideal I of a path algebra $\mathbb{C}Q$ (generated by commutation relations) is **cancellative** if, for any $a, p, q, b \in I$ with $\eta(a) = \tau(p) = \tau(q)$ and $\eta(p) = \eta(q) = \tau(b)$, we have $apb - aqb \in I \iff p - q \in I$. The **cancellative closure** of an ideal I, denoted $\operatorname{CancClos}(I)$, is the smallest cancellative ideal containing I.*

Theorem 2.12. *Let Q_π° be the quiver on v_i, for $i \in [n]$, whose arrows are x_i, y_i and the nonadjacent arrows of B_π. Let I_π° be the cancellative closure of the ideal I generated by the relations given in Theorem 2.10, the relations $xy - yx$ and $x^k - y^{n-k}$:*

$$I_\pi^\circ = \operatorname{CancClos}\left(\left\langle xy - yx, x^k - y^{n-k}, \sum_{p \text{ nonadjacent arrow}} pt^{X_p} - x_{\tau(p)} x_{\tau(p)+1} \cdots x_{\eta(p)-1} \right\rangle\right).$$

Then $B_\pi \equiv \mathbb{C}Q_\pi^\circ / I_\pi^\circ$.

Together, Theorem 2.10 and Theorem 2.12 give a way to calculate the boundary algebra of a positroid from its decorated permutation. However, this is obfuscated by the necessity of taking a cancellative closure. We address this in Section 3.3.

Example 2.13. *Let π be the permutation depicted in the middle of Figure 2. We see that there is an arrow p from v_1 to v_4. Since p has one left-supporting strand (from 6 to 4) and one right-supporting strand (from 1 to 3), it is labelled with $X_p : Y_p = 1 : 1$. By Theorem 2.12, the boundary algebra is $B_\pi \cong \mathbb{C}Q_\pi^\circ / I_\pi^\circ$, where the arrows of Q_π° are p along with the greyed-out arrows (representing x_i and y_i) and the ideal I_π° is generated up to cancellative closure by*

$$\{xy - yx,\ x^k - y^{n-k},\ pt - x_4 x_5 x_6\}.$$

3 Combinatorial construction of the boundary algebra

3.1 From Boundary Algebras to Permutations

Our work involves a new combinatorial object, which we call a *boundary chart*.

Definition 3.1. *A **boundary chart** consists of data $C = (k, n, S, X)$ as follows, where k and n are integers satisfying $1 \le k \le n$ and S is a set of arrows on vertices v_i, for $i \in [n]$, such that arrows are not between v_i and $v_{i\pm1}$, with indices taken cyclically, and there is at most one arrow from v_i to v_j for any $i, j \in [n]$. Finally, $X \in \mathbb{Z}_{>0}^{|S|}$ gives a positive integer for each arrow.*

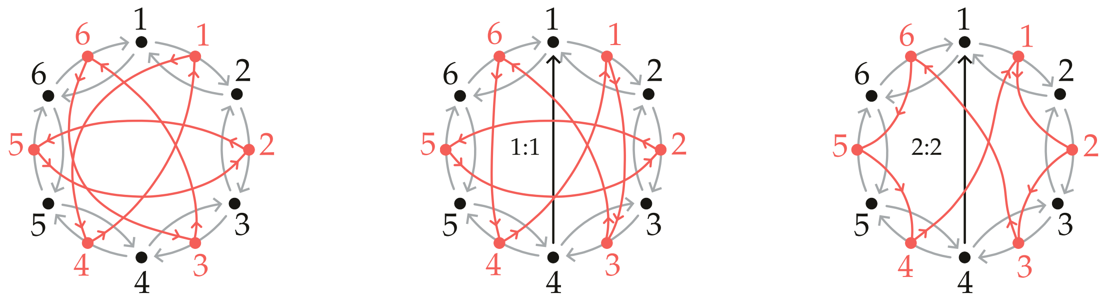

Figure 2: Three boundary charts on six vertices with $k = 3$ and their corresponding decorated permutations depicted in red. The arrows x_i and y_i are greyed out. For clarity, we use a red i in place of w_i and a black i in place of v_i.

Given a positroid, the boundary chart is precisely the data determined in Theorem 2.10. Here, we start with a connected positroid of rank k on $[n]$ with permutation π, and the set S consists of the nonadjacent arrows of B_π. The numbers X_p are as in the theorem. Observe that it is implicit in Theorem 2.10 that there is at most one arrow between any two vertices. By Theorem 2.12, this information fully determines B_π.

Definition 3.2. *A boundary chart is* ***realizable*** *if it is obtained from a positroid via this process.*

We introduce the following auxiliary piece of data for a boundary chart: Let $Y \in \mathbb{Z}_{>0}^{|S|}$ be a vector of positive integers, indexed by the arrows in S, such that $Y_p = X_p + k - \text{reach}_p$ for each $p \in S$. Using $xy = yx$ and $x^k = y^{n-k}$ from Theorem 2.12, one can show that the Y_p here coincide with the Y_p in Theorem 2.10.

We represent the data in a boundary chart by placing the vertices v_i around a circle, in clockwise order. We draw in the arrows and mark each arrow $p \in S$ with the pair of integers $(X_p : Y_p)$. We refer to these as the left and right relation numbers of p, respectively. Note that knowing X_p and Y_p suffices to recover k when the set S of nonadjacent arrows is nonempty. Three examples are illustrated in black in Figure 2 (the red and grey parts are not in the boundary chart). For the realizable boundary chart obtained from $\mathcal{P} = \mathcal{P}_\pi$, this coincides with the representation of B_π described in section Section 2.2. We will need the following terminology:

Definition 3.3. *Let p and q be two arrows in S.*

- *If $(\tau(p) < \tau(q) < \eta(p) < \eta(q))$ is a cyclic ordering, we say p and q* ***cross****.*
- *If $(\tau(p) \leq \tau(q) < \eta(q) \leq \eta(p))$ is a cyclic ordering, then p lies to the right of q. We say that p and q are* ***parallel****, with p* ***right-parallel*** *of q. Define* ***left-parallel*** *similarly.*
- *If $(\eta(p) \leq \tau(q) < \eta(q) \leq \tau(p))$ is a cyclic ordering, then p lies to the right of q. We say p and q are* ***antiparallel****, with p* ***right-antiparallel*** *of q. Define* ***left-antiparallel*** *similarly.*

Visually, crossing arrows are arrows that intersect on their interiors, like the arrow from 9 to 4 and the arrow from 7 to 2 in Figure 3. Parallel arrows are, roughly, arrows that go in the "same direction", like the arrow from 9 to 4 and the arrow from 10 to 1 in Figure 3, while antiparallel arrows are arrows which neither cross nor are parallel.

Given a realizable boundary chart $C = (k, n, S, X)$, we describe how to recover the permutation of its associated boundary algebra. We will construct the permutation as a permutation graph on vertices w_i such that w_i lies between vertices v_i and v_{i+1}. We refer to the vertices w_i as strand vertices. First, we must discuss the ideas of visibility and of influence in a boundary chart. We do this inductively. We will give an example of these definitions in Example 3.8. We start with a reformulation of the right and left relation numbers which will make the following construction easier to state.

Definition 3.4. *Let C be a boundary chart with left and right relation numbers X and Y, respectively. For $p \in S$, let $L(p)$ (resp. $R(p)$) be the set of arrows left-parallel (resp. right-parallel) to p. Then the **adjusted left relation numbers** are defined inductively by $X'_p = X_p - \sum_{q \in L(p)} X'_q$ and the **adjusted right relation numbers** are defined inductively by $Y'_p = Y_p - \sum_{q \in R(p)} Y'_q$.*

Definition 3.5. *Let α be an arrow of a realizable boundary chart $C = (k, n, S, X)$. Inductively define the following:*

1. *The **right-head-visible** strand vertices to α are those strand vertices to the right of α which are not right-head-influenced by a right-parallel arrow of α or left-head-influenced by a right-antiparallel arrow of α (this condition is vacuous if there are no arrows to the right of α). The **right-head-influenced** strand vertices of α are the Y'_α right-head-visible strand vertices most immediately clockwise of $\eta(\alpha)$.*

2. *The **left-head-visible** strand vertices of α are those strand vertices to the left of α which are not left-head-influenced by a left-parallel arrow of α or right-head-influenced by a left-antiparallel arrow of α. The **left-head-influenced** strand vertices of α are the X'_α left-head-visible strand vertices most immediately counterclockwise of $\eta(\alpha)$.*

*We will use the phrase **head-influenced** to mean either left- or right-head-influenced. We define **left-tail visible** and **right-tail-visible** strand vertices as above, swapping "head" with "tail," "$\eta(\alpha)$" with "$\tau(\alpha)$," and "clockwise" with "counter-clockwise."*

Construction 3.6. *Let $C = (k, n, S, X)$ be a realizable boundary chart. Let α be an arrow of C. Let $w_{\sigma_1}, \ldots, w_{\sigma_{X'(\alpha)}}$ be the right-head-influenced strand vertices of α, ordered clockwise so that w_{σ_1} is most immediately clockwise of $\eta(\alpha)$. Let $w_{\sigma'_1}, \ldots, w_{\sigma'_{X'(\alpha)}}$ be the right-tail-influenced strand vertices of α, ordered clockwise so that $w_{\sigma_{X'(\alpha)}}$ is most immediately counter-clockwise of $\tau(\alpha)$. Then define $\phi_\alpha(\sigma_j) = \sigma'_j$ for $j \in X'(\alpha)$. Symmetrically define ϕ_α on the left-head-influenced vertices of α. We define a function (indeed, we will see, a permutation) π on $[n]$ by*

$$\pi(j) = \begin{cases} \phi_\alpha(j) & w_j \text{ is head-influenced by an arrow } \alpha \in C \\ j - k & \text{otherwise.} \end{cases}$$

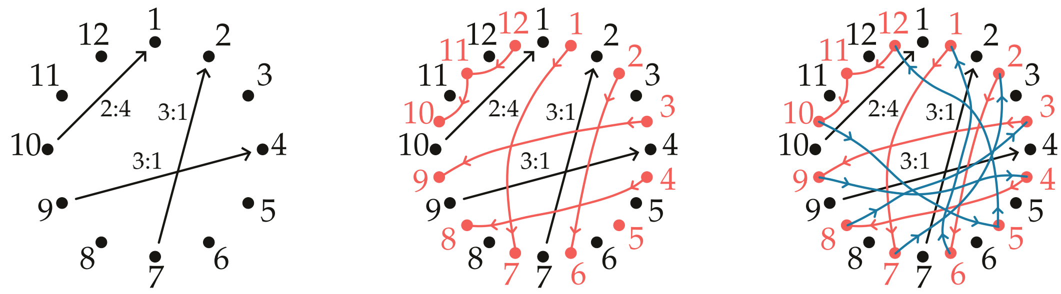

Figure 3: Application of Construction 3.6 to a boundary chart with 12 vertices and $k = 5$. For clarity, we use a red i in place of w_i and a black i in place of v_i.

It is not immediately obvious that π is well-defined; for example, some strand vertex w_j may be head-influenced by two arrows α and β. In fact, whenever this happens, $\phi_\alpha(j) = \phi_\beta(j)$.

Theorem 3.7. *The map π is a well-defined stable-interval-free permutation, and C is the boundary chart of B_π.*

This process is most easily understood visually, in an example. We denote nonadjacent arrows in the boundary algebra from v_i to v_j by $p_{i\to j}$.

Example 3.8. *We look at Figure 3. The first subfigure shows a boundary chart on 12 vertices. We have $Y'_{p_{7\to 2}} = 1$, hence the only right-head-influenced strand vertex of $p_{7\to 2}$ is w_2 and its only right-tail-influenced strand vertex is w_6 and we see that $\pi(2) = 6$. Similarly, the right-influence of $p_{9\to 4}$ induces $\pi(4) = 8$. We have $Y'_{p_{10\to 1}} = Y_{p_{10\to 1}} - Y_{p_{7\to 2}} - Y_{p_{9\to 4}} = 2$, so the right-head-influence of $p_{10\to 1}$ is $\{w_1, w_3\}$ (skipping over w_2, which is right-head-influenced by the right-parallel arrow $p_{7\to 2}$) and the right-tail-influence of $p_{10\to 1}$ is $\{w_7, w_9\}$ (skipping over w_8, which is right-tail-influenced by $p_{9\to 4}$). Then we see $\pi(1) = 7$ and $\pi(3) = 9$. The middle of Figure 3 shows in red all strands induced by the influence of an arrow; the right completes the permutation graph by adding in blue the remaining strands from w_j to w_{j-k}.*

3.2 Realizable Boundary Charts

We next classify realizable boundary charts.

Theorem 3.9. *Let $C = (k, n, S, X)$ be a boundary chart with left and right relation numbers X and Y, respectively, and with adjusted left and right relation numbers X' and Y', respectively. For $p \in S$, let $R_{\nparallel}(p)$ and $L_{\nparallel}(p)$ denote the sets of arrows right and left-antiparallel to p, respectively. Then C is realizable if and only if the following hold.*

1. *For all $p \in S$, $X_p + \sum_{q \in L_{\nparallel}(p)} Y'_q < \mathrm{reach}_p$ and $Y_p + \sum_{q \in R_{\nparallel}(p)} X'_q < n - \mathrm{reach}_p$.*

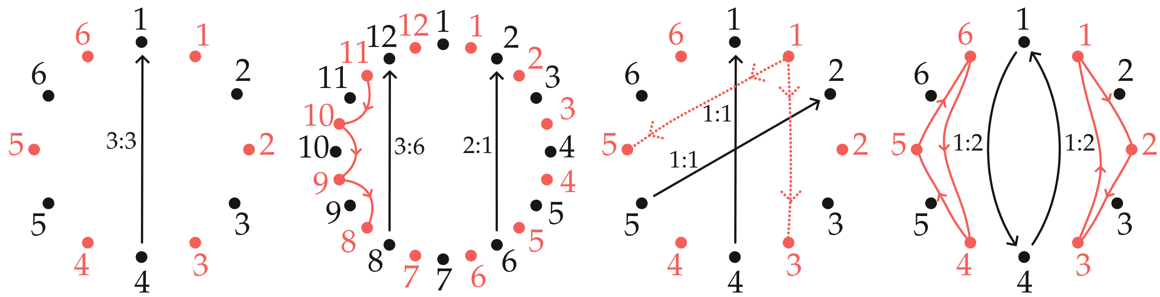

Figure 4: Four *nonrealizable* boundary charts violating the conditions of Theorem 3.9. For clarity, we use a red i in place of w_i and a black i in place of v_i.

2. *For all $p \in S$, $X'_p \geq 0$ and $Y'_p \geq 0$.*

   (a) *If $X'_p = 0$ (resp. $Y'_p = 0$), there must be crossing arrows q and r, both left (rep. right) parallel to p, such that $\tau(p) = \tau(q)$ and $\eta(p) = \eta(r)$.*

3. *Let $p, q \in S$ be crossing such that $(v_i = \eta(p) < v_j = \eta(q) < \tau(p) < \tau(q))$ is a cyclic ordering. Let A_p denote the set of arrows r right-parallel to p such that $\eta(r) \in [\eta(p), \eta(q))$. Let A_q denote the set of arrows r left-parallel to q such that $\eta(r) \in (\eta(p), \eta(q)]$. Then $Y'_p + X'_q + \sum_{r \in A_p} Y'_r + \sum_{r \in A_q} X'_r \leq j - i$, where the right side is taken modulo n so that it lies in $[n]$.*

4. *If $p, q \in S$ form an oriented digon, then $X_p + X_q + Y_p + Y_q < n$.*

One may attempt to apply Construction 3.6 to general boundary algebras. However, the conditions of Theorem 3.9 are necessary in order for the result to be a well-defined stable-interval-free permutation. For example, condition 1 ensures that there are enough strand vertices to the right of any $p \in S$ to count out Y'_p right-influenced strand vertices. In the first subfigure of Figure 4, there are not enough red vertices for $p_{4 \to 1}$ to have three left-supporting strands. The second subfigure in Figure 4 violates condition 2, as $X'(p_{6 \to 2}) = -1$. The left-supporting strands of $p_{8 \to 12}$ also left-support $p_{6 \to 2}$, so $p_{6 \to 2}$ has too many left-supporting strands. Condition 3 ensures that, if p and q are arrows of C which both influence w_j, then $\phi_p(j) = \phi_q(j)$. See the third subfigure of Figure 4, where the two crossing arrows are pulling the strand starting at w_1 in different directions. Condition 4 ensures the permutation constructed in Construction 3.6 is stable-interval-free; see the fourth subfigure of Figure 4, where the permutation fixes $[1, 3]$.

The sufficiency of these conditions is more surprising. We prove sufficiency by showing that Construction 3.6 and the map of Theorem 2.10 compose to the identity on the boundary charts satisfying the conditions of Theorem 3.9. Hence, we may view the combinatorial conditions of Theorem 3.9 as a new cryptomorphism for connected positroids.

3.3 Minimal relations

Recall the presentation of the boundary algebra $B_\pi \cong \mathbb{C}Q_\pi^\circ / I_\pi^\circ$ given in Theorem 2.12, which has the drawback of the ideal I_π° being defined using a cancellative closure. In this section, we give a description of the minimal relations of the ideal I_π° using the information of the permutation π and the boundary chart $C = (k, n, S, X)$.

Definition 3.10. *Given an arrow $p \in S$ of C, let R_p (respectively T_p) be the vertex w_i of the permutation graph most immediately clockwise of $\eta(p)$ (resp. $\tau(p)$) which is the start (resp. end) of a strand which crosses p (i.e., which starts to the right of p and ends to the left, or vice versa).*

- *Two (necessarily parallel) arrows p and q are* ***stitch-equivalent*** *if $R_p = R_q$ and $T_p = T_q$.*
- *A strand is* ***relation-defining*** *if it does not travel from R_p to T_p for any arrow p.*

Example 3.11. *In Figure 3, the arrows $p_{10\to 1}$ and $p_{9\to 4}$ are stitch-equivalent to each other with $T_{10\to 1} = T_{9\to 4} = 12$ and $R_{10\to 1} = R_{9\to 4} = 5$, but not to the arrow from 7 to 2, with $T_{7\to 2} = 8$ and $R_{7\to 2} = 3$. Every strand is relation-defining with the exception of the strand from 5 to 12.*

Definition 3.12. *For v_a and v_b boundary vertices of B_π, define the* ***aggressive clockwise path*** $\mathbf{ACL}(v_a, v_b)$ ***from*** *v_a* ***to*** *v_b by starting at v_a and repeatedly taking the non-y_j arrow which ends most immediately counter-clockwise of v_b. Similarly define the* ***aggressive counter-clockwise path*** $\mathbf{ACC}(v_a, v_b)$. *When these paths are equivalent, we say that the* ***aggressive relation from*** *v_a* ***to*** *v_b is $[\mathrm{ACL}(v_a, v_b)] - [\mathrm{ACC}(v_a, v_b)]$.*

Theorem 3.13. *The following relations of $\mathbb{C}Q_\pi^\circ$, along with the relation $x_i y_i = y_{i-1} x_{i-1}$ for each $i \in [n]$, form a minimal generating set for I_π°:*

1. *For every relation-defining strand from w_a to w_b, take the aggressive relation from v_b to v_a.*
2. *Let $\{p_1, \ldots, p_m\}$ be a stitch-equivalence class, ordered left to right, with $T_{p_i} = T$ and $R_{p_i} = R$ for all $i \in [m]$. Define $v_{a_{m+1}} := T$, $v_{b_0} := R$, and $v_{a_i} := \tau(p_i)$, $v_{b_i} := \eta(p_i)$ for $i \in [m]$. Then, take the aggressive relation from v_{a_i} to $v_{b_{i-1}}$ for each $i \in [m+1]$.*

Example 3.14. *Consider the boundary chart of Figure 3. The strand from w_1 to w_7 is relation-defining and yields the relation $p_{7\to 2} y_1 = x_7 x_8 x_9 p_{10\to 1}$. All of the strands $w_j \mapsto w_{j-k}$, drawn in blue, give relations composed only of x's and y's. For example, the strand from w_5 to w_{12} gives $x_{12} x_1 \cdots x_4 = y_{11} y_{10} \cdots y_5$. There are two stitch-equivalence classes $\{p_{10\to 1}, p_{9\to 4}\}$ and $\{p_{7\to 2}\}$. The former gives $\{y_{11} y_{10} p_{10\to 1} = x_{12},\ p_{10\to 1} x_1 x_2 x_3 = y_9 p_{9\to 4},\ p_{9\to 4} x_4 = y_8 y_7 y_6\}$ and the latter gives $\{p_{7\to 2} x_2 = y_6 y_5 y_4 y_3,\ x_8 x_9 p_{10\to 1} x_1 = y_7 p_{7\to 2}\}$. These five relations and those given by relation-defining strands make up all minimal relations of I_π°.*

Note that Theorem 3.13 uses both the boundary chart and the permutation obtained from it by Construction 3.6. It would be hard to rephrase the theorem in terms of one or the other. This indicates that boundary charts and stabilized-interval-free permutations, while both cryptomorphisms of connected positroids, highlight different information.

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The commutant of divided difference operators, Klyachko's genus, and the comaj statistic

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Abstract. In [5, 12, 13] are studied certain operators on polynomials and power series that commute with all divided difference operators ∂_i. We introduce a second set of "martial" operators $♂_i$ that generate the full commutant, and show how a Hopf-algebraic approach naturally reproduces the operators ξ^ν from [12]. We then pause to study Klyachko's homomorphism $H^*(Fl(n)) \to H^*$(the permutahedral toric variety), and extract the part of it relevant to Schubert calculus, the "affine-linear genus". This genus is then re-obtained using Leibniz combinations of the $♂_i$. We use Nadeau-Tewari's q-analogue of Klyachko's genus to study the equidistribution of ℓ and comaj on $\binom{[n]}{k}$, generalizing known results on S_n.

Keywords: divided difference operators, Schubert calculus, comaj statistic

1 The martial operators $♂_\pi$

1.1 The ring of Schubert symbols

Given a Dynkin diagram D with Weyl group $W(D)$, define the **ring of Schubert symbols** $H(D)$ as the cohomology ring of the associated (possibly infinite-dimensional) flag variety, with the usual Schubert basis $\{\mathcal{S}_w \colon w \in W(D)\}$. The Dynkin diagrams that will interest us are primarily the semi-infinite $A_{\mathbb{Z}_+}$ and the biinfinite $A_{\mathbb{Z}}$. In these type A cases $W(D)$ is the group of finite permutations of $\mathbb{Z}_+$ or of $\mathbb{Z}$. An important difference between the two is that $H(A_{\mathbb{Z}_+})$ is generated by $\{\mathcal{S}_{r_i} \colon i \in \mathbb{Z}_+\}$, where r_i is a simple transposition, so the multiplication is entirely determined by Monk's rule, whereas $H(A_{\mathbb{Z}})$ requires additional generators $\{\mathcal{S}_{r_1 r_2 \cdots r_k}\}$ and determining its multiplication involves also the flag Pieri rule. With all that in mind we largely abandon the geometry and work with these rings symbolically.

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For each vertex α of D hence generator $r_\alpha \in W(D)$, we have an operator $\partial_\alpha \circlearrowright H(D)$, pronounced "partial α":

$$\partial_\alpha \mathcal{S}_\pi := \begin{cases} \mathcal{S}_{\pi r_\alpha} & \text{if } \pi r_\alpha < \pi \\ 0 & \text{if } \pi r_\alpha > \pi \end{cases}$$

from which we can well-define ∂_π for any $\pi \in W(D)$ using products.

Theorem 1 (Lascoux-Schützenberger). *There is an isomorphism $H(A_{\mathbb{Z}_+}) \to \mathbb{Z}[x_1, x_2, \ldots]$ taking the Schubert symbol $\mathcal{S}_\pi$ to its "Schubert polynomial" $S_\pi(x_1, x_2, \ldots)$. On the target ring, ∂_α acts by Newton's divided difference operation.*

Call a ring homomorphism from $H(D)$ to some other ring a **genus**,[1] making the above isomorphism the **Lascoux-Schützenberger genus**.

It was observed in [5, 13] that the operator $\nabla := \sum_i \frac{d}{dx_i}$ on the target ring has two nice properties: it commutes with each ∂_i, and its application to any Schubert polynomial is a positive combination of Schubert polynomials. Our goal in this section is to characterize operations of the first type, with an eye toward the second. To study this commutant it will be handy to work with algebra actions.

1.2 Two commuting actions of the nil Hecke algebra

Let $Nil(D)$ denote formal (potentially infinite) linear combinations of elements $\{d_\pi \colon \pi \in W(D)\}$, with a multiplication defined by $d_\pi d_\rho := \begin{cases} d_{\pi\rho} & \text{if } \ell(\pi\rho) = \ell(\pi) + \ell(\rho) \\ 0 & \text{if } \ell(\pi\rho) < \ell(\pi) + \ell(\rho). \end{cases}$ This multiplication extends to infinite sums in a well-defined way, insofar as any $w \in W(D)$ has only finitely many length-additive factorizations. Slightly abusing[2] terminology, we call this $Nil(D)$ the **nil Hecke algebra**. The association $d_\pi \mapsto \partial_\pi$ gives an action of the opposite algebra $Nil(D)^{op}$ on $H(D)$; the infinitude of the sums in $Nil(D)$ is again not problematic, because $H(D)$'s elements are finite sums of Schubert symbols.

Define $♂_\alpha$ (pronounced "martial α") by

$$♂_\alpha \mathcal{S}_\pi := \begin{cases} \mathcal{S}_{r_\alpha \pi} & \text{if } r_\alpha \pi < \pi \\ 0 & \text{if } r_\alpha \pi > \pi \end{cases}$$

We can well-define $♂_{\prod Q} := \prod_{q \in Q} ♂_q$ for each reduced word Q.

[1] This terminology is stolen from the study of various cobordism rings of a point, e.g. the "Hirzebruch genus" and "Witten genus" are ring homomorphisms to $\mathbb{Z}$.

[2] One ordinarily considers only finite linear combinations, but we have need of certain infinite ones, and this simplifies the statement of Theorem 2.

Theorem 2. *The map $d_\pi \mapsto \male_\pi$ defines an action of $Nil(D)$ on $H(D)$ (unspoiled by the potential infinitude), commuting with the $Nil(D)^{op}$-action by the operators ∂_α. Conversely, each operator on $H(D)$ that commutes with all operators ∂_α arises as the action of a unique element of $Nil(D)$.*

In short, $Nil(D)$ and $Nil(D)^{op}$ are one another's commutants in their actions on $H(D)$.

This then characterizes the operators that commute with all ∂_i; we don't know any significance of the resulting algebra again being $Nil(D)$.

Proof. For $h \in H(D)$, let $\int h$ denote the coefficient of $\mathcal{S}_e$ in h. Each $h = \sum_\pi h_\pi \mathcal{S}_\pi$ is determined by the values

$$\int (\partial_\rho h) = \int \sum_\pi h_\pi \partial_\rho \mathcal{S}_\pi = \sum_\pi h_\pi \int \partial_\rho \mathcal{S}_\pi = \sum_\pi h_\pi \delta_{\pi,\rho} = h_\rho$$

We'll make use of the easy fact $\int \male_\pi \, \partial_\rho \, \mathcal{S}_\sigma = \begin{cases} 1 & \text{if } \sigma = \pi^{-1}\rho \text{ and } \ell(\sigma) = \ell(\pi) + \ell(\rho) \\ 0 & \text{otherwise.} \end{cases}$

Now let C be an operator on $H(D)$ commuting with all ∂_π. For each $\pi \in W(D)$, let $c_\pi := \int C(\mathcal{S}_{\pi^{-1}})$. We then confirm $C = \sum_\pi c_\pi \male_\pi$ using the determination above:

$$\begin{aligned}
\int \partial_\rho C(\mathcal{S}_\sigma) &= \int C(\partial_\rho \mathcal{S}_\sigma) = \int C\left(\mathcal{S}_{\sigma\rho^{-1}} \left[\ell(\sigma\rho^{-1}) = \ell(\sigma) - \ell(\rho)\right]\right) \\
&= \left[\ell(\sigma\rho^{-1}) = \ell(\sigma) - \ell(\rho)\right] \; c_{\rho\sigma^{-1}} \\
\int \partial_\rho \left(\sum_\pi c_\pi \male_\pi\right)(\mathcal{S}_\sigma) &= \sum_\pi c_\pi \int \male_\pi \, \partial_\rho \, \mathcal{S}_\sigma \;=\; \sum_\pi c_\pi \left[\sigma = \pi^{-1}\rho\right] \left[\ell(\sigma) = \ell(\pi) + \ell(\rho)\right] \\
&= c_{\rho\sigma^{-1}} \left[\ell(\rho\sigma^{-1}) = \ell(\sigma) - \ell(\rho)\right]
\end{aligned}$$

Here $[P] = 1$ if P is true, $[P] = 0$ if P is false, for a statement P. □

Example. The action of $\nabla := \sum_i \frac{d}{dx_i}$ on polynomials, pulled back to an action on $H(A_{\mathbb{Z}_+})$, is given by the operator $\sum_{n \in \mathbb{N}_+} n \male_n$. What is particularly special about ∇ is that it is a differential (i.e. satisfies the Leibniz rule), and is of degree -1.

Theorem 3. *Let $\sum_\alpha c_\alpha \male_\alpha \in Nil(D)$ be an operator of degree -1. If it is a differential (and D is simply-laced, for convenience) then each c_α is $\frac{1}{2}\sum_\beta c_\beta$ where the βs are α's neighbors in D.*

In particular if D is of finite type ADE, the only system of coefficients (c_α) is zero. If $D = A_{\mathbb{Z}_+}$, the only options are multiples of $c_i \equiv i$. If $D = A_{\mathbb{Z}}$, the space of such systems is two-dimensional, spanned by $c_i \equiv i$ and $c_i \equiv 1$.

Hence the ∇ discovered in [5] in the $A_{\mathbb{Z}_+}$ case was the only such operator available. In [12] it is explained that $\xi = \sum_{i \in \mathbb{Z}} \male_i$ is special to the back-stable situation of $A_{\mathbb{Z}}$; here we see that it is the only new option. (The result [12, Theorem 6] is very similar.)

Proof sketch. The proof amounts to applying $\sum_\alpha c_\alpha \male_\alpha$ to $(\mathcal{S}_{r_\alpha})^2 = \sum_\beta \mathcal{S}_{r_\beta r_\alpha}$ (computed using the Chevalley-Monk rule). □

2 Not-quite-Hopf algebras and Nenashev operators

2.1 The dual algebras

Define a pairing $Nil(D) \otimes_{\mathbb{Z}} H(D) \to \mathbb{Z}$ by

$$p \otimes s \mapsto \text{coefficient of } \mathcal{S}_e \text{ in } p(s)$$

and from there a map $Nil(D) \to H(D)^* := Hom_{\mathbb{Z}}(H(D), \mathbb{Z})$.

The following is well-known to the experts, if not usually expressed exactly this way (see e.g. [2], [9, §7.2]).

Theorem 4. *This map $Nil(D) \to H(D)^*$ is an isomorphism. Unfortunately the induced comultiplication $H(D) \to H(D) \otimes H(D)$ is not a ring homomorphism (example below), so the two are not thereby dual Hopf algebras. (There is an alternative statement explored in [10].)*

There is an analogue of Theorem 1 for $H(A_{\mathbb{Z}})$, taking each $\mathcal{S}_\pi$ to its **back-stable Schubert function** BS_π invented by the third author (and independently by Buch and by Lee), which were studied in [9, 12]. Define a **back-stable function** $p \in \mathbb{Z}[[\ldots, x_{-1}, x_0, x_1, x_2, \ldots]]$ to be a power series

- of finite degree, such that
- p depends only on the variables $\{x_k, k < N\}$ for some $N \gg 0$, and
- for some $M \ll 0$, p is symmetric in the variables $\{x_i, i \leq M\}$.

One way (as appears in [12]) to think of the ring of back-stable functions is as the image of the injection

$$\begin{array}{rcl} Symm \otimes_{\mathbb{Z}} \mathbb{Z}[\ldots, x_{-1}, x_0, x_1, \ldots] & \to & \mathbb{Z}[[\ldots, x_0, \ldots]] / \langle \text{elementary symmetric functions} \rangle \\ p \otimes q & \mapsto & p(\ldots, x_{-2}, x_{-1}, x_0)\, q \end{array}$$

For $\pi \in W(A_{\mathbb{Z}})$ considered as a finite permutation of $\mathbb{Z}$, and $shift_N(i) := i + N$, observe for $N \gg 0$ that $\pi[N] := shift_N(i) \circ \pi \circ shift_{-N}(i)$ is a permutation of $\mathbb{Z}$ that leaves $-\mathbb{N}$ in place, and thus has a well-defined Schubert polynomial. Define the **back-stable Schubert function**

$$BS_\pi := \lim_{N \to \infty} S_{\pi[N]}(x_{1-N}, x_{2-N}, \ldots)$$

where the limit is computed coefficient-wise (note that any single coefficient settles down to a constant value for all large enough N).

Theorem 5. *[9, Theorem 3.5] The back-stable Schubert functions lie in, and are a $\mathbb{Z}$-basis of, the ring of back-stable functions.*

In this coördinatization we can compute the comultiplication on $H(A_{\mathbb{Z}})$ and bound its failure to be a ring homomorphism. Transposing the multiplication from §1.2 of $d_\pi d_\rho$, we obtain $\Delta(BS_\sigma) = \sum\{BS_\pi \otimes BS_\rho \colon \sigma = \pi\rho,\ \ell(\sigma) = \ell(\pi) + \ell(\rho)\}$. Then, alas,

$$\begin{aligned} \Delta(BS_{r_2}^2) = \Delta(BS_{r_1 r_2} + BS_{r_3 r_2}) &= (BS_{r_1 r_2}\otimes 1) + (BS_{r_1}\otimes BS_{r_2}) + (1\otimes BS_{r_1 r_2}) \\ &\quad + (BS_{r_3 r_2}\otimes 1) + (BS_{r_3}\otimes BS_{r_2}) + (1\otimes BS_{r_3 r_2}) \\ \neq \Delta(BS_{r_2})\Delta(BS_{r_2}) = (BS_{r_2}\otimes 1 + 1\otimes BS_{r_2})^2 &= (BS_{r_1 r_2}\otimes 1) + (BS_{r_3 r_2}\otimes 1) + (BS_{r_2}\otimes BS_{r_2}) \\ &\quad + (BS_{r_2}\otimes BS_{r_2}) + (1\otimes BS_{r_1 r_2}) + (1\otimes BS_{r_3 r_2}) \end{aligned}$$

Luckily $\Delta(BS_\pi BS_{\rho[N]}) = \Delta(BS_{\pi\circ(\rho[N])}) = \Delta(BS_\pi)\Delta(BS_{\rho[N]})$ for $N \gg 0$. Call this property **"separated Hopfness"**, to be used below.

2.2 The Fomin-Greene–Nenashev operators ξ^ν

With these identifications, and the self-duality of the Hopf algebra *Symm* of symmetric functions, we can interpret some results of Nenashev [12]:

$$\begin{array}{ccccccc} H(A_{\mathbb{Z}}) & \xrightarrow{\sim} & \{\text{back-stable functions}\} & \xleftarrow{\sim} & Symm \otimes_{\mathbb{Z}} \mathbb{Z}[\ldots, x_{-1}, x_0, x_1, \ldots] & \twoheadrightarrow & Symm \\ Nil(A_{\mathbb{Z}}) & & & \longleftarrow & & & Symm \end{array}$$

The map $\twoheadrightarrow$ is the **Stanley genus**: it takes $\mathcal{S}_\pi$ to its **Stanley symmetric function** $St_\pi = \sum_\lambda a_\pi^\lambda\, Schur_\lambda$. The lower map, its transpose, takes $Schur_\lambda$ to $\sum_\pi a_\pi^\lambda\, d_\pi$. If we let this operator act on $H(A_{\mathbb{Z}})$ under the $d_\pi \mapsto \partial_\pi$ action, we get the **Fomin-Greene–Nenashev operator** $\xi^\lambda := \sum_\pi a_\pi^\lambda\, \partial_\pi$ [3, 12]. (See also the j_λ operators in the "Peterson subalgebra" defined in [9, §9.3], which are a double version of the ξ^λ.)

Let $\mathfrak{m}$ denote the kernel of the map $H(A_{\mathbb{Z}}) \twoheadrightarrow Symm$. Using the separated Hopfness and the fact that $BS_{\pi[N]} - BS_\pi \in \mathfrak{m}$, one shows that each $\Delta(pq) - \Delta(p)\Delta(q)$ (which serves as a measure of non-Hopfness) lies in $\mathfrak{m} \otimes H(A_{\mathbb{Z}}) + H(A_{\mathbb{Z}}) \otimes \mathfrak{m}$. Hence the map $H(A_{\mathbb{Z}}) \to Symm$ factors through a map of Hopf algebras. Dually, the transpose map is a Hopf map to a Hopf sub-bialgebra of $Nil(A_{\mathbb{Z}})$. In particular this Hopf map explains Nenashev's formulæ [12, §4.4]

$$\xi^\lambda \xi^\mu = \sum_\nu c_{\lambda\mu}^\nu\, \xi^\nu \qquad\qquad \xi^\nu(pq) = \sum_{\lambda,\mu} c_{\lambda\mu}^\nu\, \xi^\lambda(p)\, \xi^\mu(q)$$

2.3 Interlude (not used elsewhere): topological origin of the $\{BS_\pi\}$

The stability property underlying Lascoux-Schützenberger's definition of Schubert polynomials is the fact that each $S_\pi \in H^*(Fl(n))$ is the pullback $\iota_{n+1}^*(S_{\pi\oplus 1})$ along a map $\iota_{n+1} : Fl(n) \hookrightarrow Fl(n+1)$ taking $(E_\bullet)$ to $(F_\bullet \colon F_{i\leq n} = E_i \oplus 0,\ F_{n+1} = E_n \oplus \mathbb{C})$. Chaining these together, one builds an element of the inverse limit of the cohomology rings, a ring $\mathbb{Z}[[x_1, x_2, \ldots]] / \langle \text{elementary symmetric functions } e_i \rangle$. It was then Lascoux-Schützenberger's pleasant surprise that these "inverse limit Schubert classes" lie in (and exactly span) the image of the injective ring homomorphism $\mathbb{Z}[x_1, x_2, \ldots]$ into this algebra.

This admits of a parallel story, based on a different map $\iota_{1+2n+1} : Fl(2n) \hookrightarrow Fl(2n+2)$ taking $(E_\bullet)$ to $(F_\bullet\colon F_{i\in[1,2n+1]} = \mathbb{C} \oplus E_{i-1} \oplus 0,\ F_{2n+2} = \mathbb{C} \oplus \mathbb{C}^{2n} \oplus \mathbb{C})$. Now, in order to achieve a coherent labeling (as n varies) we index the classes in $H^*(Fl(2n))$ using permutations of $[1-n, n]$ rather than of $[1, 2n]$. Once again the inverse limit is a power series ring modulo elementary symmetrics, but it is *no longer true* that the inverse limit Schubert classes are representable by polynomials; rather, they can be represented by back-stable functions. (And again, they form a basis thereof.)

One advantage of ι_{1+2n+1} is that it is equivariant w.r.t. the *duality* endomorphism of $Fl(2n)$, which takes $(E_\bullet)$ to $(E_\bullet^\perp)$, defined w.r.t. the symplectic form pairing coördinates i and $1-i$, for $i \in [1, n]$. On the level of classes, this takes $BS_\pi \mapsto BS_{w_0 \pi w_0}$ where $w_0(i) := 1-i$. On the level of back-stable functions, it takes $x_i \mapsto -x_{1-i}$, $e_i(x_{\leq 0}) \mapsto e_i(x_{\leq 0})$.

Since this duality respects Schubert classes and the alphabet (x_i), it takes Monk's rules to Monk's rules. In particular it turns the transition formula (a specific Monk's rule)

$$BS_\pi = x_i BS_{\pi'} + \sum_{\text{certain } \pi''} BS_{\pi''} \qquad \text{into} \qquad BS_\rho = -x_j BS_{\rho'} + \sum_{\text{certain } \rho''} BS_{\rho''}$$

which implies (unstably) the cotransition formula $x_j \mathcal{S}_{\rho'} = -\cancel{\mathcal{S}_\rho} + \sum_{\text{certain } \rho''} \mathcal{S}_{\rho''}$ of [8].

3 Relation to Klyachko's genus

3.1 Klyachko's ideal and its prime factors

Let $T \leq GL_n(\mathbb{C})$ denote the group of diagonal matrices, and $TV_{perm} \subseteq Fl(n)$ be the **permutahedral toric variety** obtained as the closure of a generic T-orbit on the flag manifold $Fl(n)$. This subvariety arises as a Hessenberg variety (see e.g. [1]) and is of key importance in [6, 11].

The inclusion $\iota\colon TV_{perm} \hookrightarrow Fl(n)$ induces a map backwards on cohomology, which is neither injective nor surjective. Klyachko [7] presented its image $im(\iota^*)$ (with rational coefficients), and a formula for ι^* evaluated on Schubert symbols:

$$\begin{array}{rcl} H^*(Fl(n);\mathbb{Q}) & \to & im(\iota^*) \cong \mathbb{Q}[k_0, \ldots, k_n] \Big/ \left\langle \begin{array}{c} k_i(-k_{i-1} + 2k_i - k_{i+1}) = 0,\ \ 1 \leq i \leq n-1 \\ k_0 = k_n = 0 \end{array} \right\rangle \\ \mathcal{S}_\pi & \mapsto & \dfrac{1}{\ell(\pi)!} \displaystyle\sum_{Q \in RW(\pi)} \prod_{q \in Q} k_q \quad \text{where } RW(\pi) \text{ is the set of reduced words} \end{array}$$

Taking forward- and back-stable limits, while leaving behind geometry, we get the **Klyachko genus** $H(A_{\mathbb{Z}}) \to \mathbb{Q}[\ldots,k_{-1},k_0,k_1,\ldots] \Big/ \langle\, k_i(-k_{i-1}+2k_i-k_{i+1}) = 0 \;\forall i \in \mathbb{Z}\,\rangle$ whose map on Schubert symbols is given by the same formula. We use this to recover a result of Nenashev, foreshadowing some results in §5:

Theorem 6. *[12, Proposition 3 and discussion after] Let $RW(\pi)$ denote the set of reduced words for π. There must* **exist** *(but the proof doesn't find one) a "rectification" map*

$$\{\textit{shuffles of any word in } RW(\pi) \textit{ with any word in } RW(\rho)\} \to \coprod_\sigma RW(\sigma)$$

whose fiber over any reduced word for σ has size $c^\sigma_{\pi\rho}$, the coefficient from $\mathcal{S}_\pi\mathcal{S}_\rho = \sum_\sigma c^\sigma_{\pi\rho}\,\mathcal{S}_\sigma$.

Proof. Apply the Klyachko genus to that last equation, then set all $k_i = 1$, obtaining

$$\frac{1}{\ell(\pi)!}\sum_{P\in RW(\pi)}\prod_P 1 \quad \frac{1}{\ell(\rho)!}\sum_{R\in RW(\rho)}\prod_R 1 = \sum_\sigma c^\sigma_{\pi\rho}\,\frac{1}{\ell(\sigma)!}\sum_{S\in RW(\sigma)}\prod_S 1$$

Since $c^\sigma_{\pi\rho} = 0$ unless $\ell(\sigma) = \ell(\pi)+\ell(\rho)$, we can restrict to those σ. Multiplying through:

$$\#RW(\pi)\;\#RW(\rho)\binom{\ell(\pi)+\ell(\rho)}{\ell(\pi)} = \sum_\sigma c^\sigma_{\pi\rho}\,\#RW(\sigma)$$

Let $C^\sigma_{\pi\rho}$ be a set with cardinality $c^\sigma_{\pi\rho}$ (and wouldn't you like to know one?). Then we can interpret the above as

$$\#\{\text{shuffles of any word in } RW(\pi) \text{ with any word in } RW(\rho)\} \;=\; \#\coprod_\sigma (C^\sigma_{\pi\rho}\times RW(\sigma))$$

Hence there exists a bijection; compose it with the projection to $\coprod_\sigma RW(\sigma)$. □

We can further simplify the target of this genus by modding out by each of the minimal prime ideals that contain the Klyachko ideal. We get ahold of these using the Nullstellensatz,[3] i.e. by looking at the components of the solution set to Klyachko's equations.

Proposition 1. *Consider $\mathbb{Z}$-ary tuples $(k_i)_{i\in\mathbb{Z}}$ of complex numbers satisfying the Klyachko equalities. This set is the (nondisjoint) union of the following countable set of 2-planes:*

- *For $a,b\in\mathbb{C}$, let $k_m = am+b$.*
- *For $i\le j$ each in $\mathbb{Z}$, and $x,y\in\mathbb{C}$ a pair of "slopes", let* $k_m = \begin{cases} x(m-i) & \text{if } k\le i \\ 0 & \text{if } k\in[i,j] \\ y(m-j) & \text{if } k\ge j. \end{cases}$

[3]This isn't quite fair, in that we are working in infinite dimensions, but we won't worry about it. All we're really trying to do here is choose, for each i, which factor of $k_i(-k_{i-1}+2k_i-k_{i+1})$ to mod out.

After completing this work, we learned of a very similar calculation in [11, §3.4], so we omit the proof of proposition 1 (obtainable as a sort of $q \to 1$ limit of theirs).

Each component defines a quotient of the Klyachko ring, namely

$$\mathbb{Q}[\ldots, k_{-1}, k_0, k_1, \ldots] \Big/ \langle -k_{m-1} + 2k_m - k_{m+1} = 0 \ \forall m \in \mathbb{Z} \rangle$$

$$\forall i \leq j, \qquad \mathbb{Q}[\ldots, k_{-1}, k_0, k_1, \ldots] \Big/ \left\langle \begin{array}{cc} k_m = 0 & \forall m \in [i, j] \\ -k_{m-1} + 2k_m - k_{m+1} = 0 & \forall m \notin [i, j] \end{array} \right\rangle$$

Call the map of $H(A_{\mathbb{Z}})$ to the first quotient the **affine-linear genus**.

There is a slight subtlety in that the Klyachko ideal is not radical, and as such, the map from the Klyachko ring to the direct sum of these quotients is not injective. We will return to this minor matter below.

3.2 Dropping the other genera

The other components (besides the one giving the affine-linear genus) are useless, in the following senses. Say $k_m = 0$ for some m; then there are three situations.

1. *Some reduced word for a permutation π uses the letter m.* Then *all* reduced words do, with the effect that $\mathcal{S}_\pi \mapsto 0$ in the quotient ring.

2. *Each reduced word for π uses some letters $> m$ and some $< m$.* Then $\pi = \pi_{<m}\pi_{>m}$ where each uses only letters $< m$, $> m$ respectively. In this case $\mathcal{S}_\pi = \mathcal{S}_{\pi_{<m}}\mathcal{S}_{\pi_{>m}}$.

3. *Each reduced word for π only uses letters on one side of m.* At this point there is nothing to be gained by setting $k_m = 0$; we could work with just the affine-linear genus.

Our principal interest in genera is to study **Schubert calculus**, the structure constants $c_{\pi\rho}^{\sigma}$ of the multiplication of Schubert symbols. That is hard to do if the symbols map to zero (situation #1), silly to do directly if the symbols are are themselves products (situation #2), and in situation #3 might as well be done using the affine-linear genus. As such, at this point we cast aside the Klyachko genus in favor of the affine-linear genus γ:

$$\gamma : \ H(A_{\mathbb{Z}}) \ \to \ \mathbb{Q}[a, b], \qquad\qquad \mathcal{S}_\pi \mapsto \frac{1}{\ell(\pi)!} \sum_{P \in RW(\pi)} \prod_{i \in P} (ai + b)$$

The assiduous reader might be guessing now that the information lost when passing from the Klyachko ideal to its radical is similarly negligible for Schubert calculus purposes. And indeed: if we factor the Klyachko ideal as an intersection of primary instead of prime components, we run into the ideals

$$\forall i \leq j, \qquad \mathbb{Q}[\ldots, k_{-1}, k_0, k_1, \ldots] \Big/ \left\langle \begin{array}{cc} k_m^2 = 0 & \forall m \in [i+1, j-1] \\ k_i = k_j = 0 & \\ -k_{m-1} + 2k_m - k_{m+1} = 0 & \forall m \notin [i, j] \end{array} \right\rangle$$

These would let us study π, ρ, σ whose reduced words use only the letters in the range $[i+1, j-1]$, and each at most once. This is an extremely limited case.

4 The affine-linear genus γ from the martial derivations

Recall the derivations

$$\nabla = \sum_m m \male_m \qquad\qquad \xi = \sum_m \male_m$$

Being derivations, they exponentiate to automorphisms of $\mathbb{Q} \otimes_{\mathbb{Z}} H(A_{\mathbb{Z}})$ (where the $\mathbb{Q}$ is necessitated by the denominators in the exponential series).

Theorem 7. *The following triangle commutes:*

$$\begin{tikzcd} & H(A_{\mathbb{Z}}) \arrow[dl, "e^{a\nabla+b\xi}"'] \arrow[dr, "\gamma"] & \\ \mathbb{Q}[a,b] \otimes_{\mathbb{Z}} H(A_{\mathbb{Z}}) \arrow[rr, two heads] & & \mathbb{Q}[a,b] \\ \mathcal{S}_\pi \arrow[rr, mapsto] & & \delta_{\pi,e} \end{tikzcd}$$

Proof. The proof is not conceptual; we compute both sides and compare. Indeed, we find the statement intriguing exactly because we know of no geometric reason the two maps should be related.

$$e^{a\nabla+b\xi} \cdot \mathcal{S}_\pi \;=\; \sum_n \frac{1}{n!}(a\nabla + b\xi)^n \cdot \mathcal{S}_\pi \;\mapsto\; \frac{(a\nabla + b\xi)^{\ell(\pi)} \cdot \mathcal{S}_\pi}{\ell(\pi)!} \;=\; \frac{\left(\sum_i (ai+b)\male_i\right)^{\ell(\pi)} \cdot \mathcal{S}_\pi}{\ell(\pi)!}$$

Expanding $\left(\sum_i (ai+b)\male_i\right)^{\ell(\pi)}$, the nonvanishing terms correspond to reduced words of length $\ell(\pi)$, and only those that multiply to π^{-1} survive application to $\mathcal{S}_\pi$. □

In particular the proof of Theorem 6 essentially amounts to applying $\exp(\xi)$. (Oddly, the original proof in [12] is closer to an application of $\exp(\nabla)$.)

There is a fascinating q-**Klyachko genus** introduced in [11, §3.4]:

$$\begin{aligned} \gamma_q :\ H(A_{\mathbb{Z}}) \ &\to\ \mathbb{Q}(q)[\alpha, \beta] \\ \mathcal{S}_\pi \ &\mapsto\ \frac{1}{\ell(\pi)^{q}_{\bullet}} \sum_{Q:\ \prod Q = \pi} q^{\mathrm{comaj}(Q)} \prod_{i \in Q} \left(\alpha q^i + \beta\right) \end{aligned}$$

Here $m^{q}_{\bullet}$ is the q-torial $\prod_{j=1}^{m} [j]_q$, and $\mathrm{comaj}(Q)$ is the sum of the positions of the ascents. We looked for a long time for a q-analogue of Theorem 7, to no avail: it would provide an automorphism of $H(A_{\mathbb{Z}})(q)[\alpha, \beta]$ whose $\ell = 0$ part is the q-Klyachko genus.

5 Rectification and the q-statistic

We pursue a q-analogue of (Nenashev's) Theorem 6. Applying Nadeau-Tewari's q-Klyachko genus to $\mathcal{S}_\pi \mathcal{S}_\rho = \sum_\sigma c_{\pi\rho}^\sigma \mathcal{S}_\sigma$ we get

$$\frac{1}{\ell(\pi)^q_\bullet} \sum_{P \in RW(\pi)} q^{\mathrm{comaj}(P)} \prod_{i \in P} (\alpha q^i + \beta) \quad \frac{1}{\ell(\rho)^q_\bullet} \sum_{R \in RW(\rho)} q^{\mathrm{comaj}(Q)} \prod_{i \in R} (\alpha q^i + \beta)$$
$$= \sum_\sigma c_{\pi\rho}^\sigma \frac{1}{\ell(\sigma)^q_\bullet} \sum_{S \in RW(\sigma)} q^{\mathrm{comaj}(S)} \prod_{i \in S} (\alpha q^i + \beta)$$

Multiplying through, we get

$$\binom{\ell(\pi)+\ell(\rho)}{\ell(\pi)}_q \sum_{\substack{P \in RW(\pi) \\ R \in RW(\rho)}} q^{\substack{\mathrm{comaj}(P) \\ +\mathrm{comaj}(R)}} \prod_{i \in P \coprod R} (\alpha q^i + \beta) \quad = \quad \sum_\sigma c_{\pi\rho}^\sigma \sum_{S \in RW(\sigma)} q^{\mathrm{comaj}(S)} \prod_{i \in S} (\alpha q^i + \beta)$$

Let's interpret both sides at $\alpha = \beta = q = 1$, again using a mystery set $C_{\pi\rho}^\sigma$ with cardinality $c_{\pi\rho}^\sigma$. Define a **barred word** for π as a reduced word in which some letters are overlined, e.g. $12\overline{1}$ for (13). Then the left side of the above equation counts pairs (P, R) of barred words, shuffled together, where the barring indicates "use the αq^i term" rather than the β term. Meanwhile, the right side counts pairs (τ, S) where S is a barred word for some σ, and τ is in $C_{\pi\rho}^\sigma$.

Theorem 8. *Define the q-statistic of a barred word as the sum of the locations of the ascents, plus the sum of the barred letters.*

Define the q-statistic of a shuffle ш *of a pair (P, R) of barred words as the sum of the two q-statistics, plus the number of inversions in the shuffle (letters in R leftward of letters in P).*

Then there **exists** *(but the proof doesn't find one) a "rectification" map*

$$\{\textit{shuffles of pairs } (P, R) \textit{ of barred words for } \pi, \rho\} \quad \to \quad \coprod_\sigma \{\textit{barred words for } \sigma\}$$

preserving the number of bars and the q-statistic, whose fiber over each word for σ is of size $c_{\pi\rho}^\sigma$.

We note that the affine-linear genus doesn't let one produce such a combinatorial result, insofar as the factors $ai + b$ can involve $i < 0$ (in the back-stable setting of $A_\mathbb{Z}$).

Example. These examples get large very quickly, so we restrict to the fully barred case. Let $\pi = \rho = 12463578$, chosen to give a $c_{\pi\rho}^\sigma > 1$ (and chosen stably enough that the terms in the product don't move $-\mathbb{N}$). Each of π and ρ have two reduced words (354 and 534, comajs 1 and 2, each of total 12), and there are $\binom{6}{3}$ ways to shuffle, for a total of $2 \cdot 2 \cdot \binom{6}{3} = 80$; the resulting q-statistics range from $26 = 1 + 12 + 1 + 12 + 0$

to $37 = 2 + 12 + 2 + 12 + 3 \cdot 3$. There are 7 terms $\mathcal{S}_\sigma$ in the product $\mathcal{S}_\pi \mathcal{S}_\rho$ (one with coefficient 2) with various numbers of reduced words.

q-statistic:	26	27	28	29	30	31	32	33	34	35	36	37	
	1	3	5	8	11	12	12	11	8	5	3	1	total = 80
σ													
23561478	1	1	2	1	2	1	1						
14562378		1		1	1	1		1					
13572468			1	2	2	3	3	2	2	1			
13572468			1	2	2	3	3	2	2	1			again
23471568		1	1	2	2	2	1	1					
13482567					1	1	2	2	2	1	1		
12673458					1		1	1	1		1		
12583467						1	1	2	1	2	1	1	

In the line with "total = 80", we count the number of fully barred shuffles with given q-statistic. In each of the lower lines, we put σ on the left, and on the right we list the number of fully barred words for it with given q-statistic. Then the theorem asserts that each number atop a column is the sum of the numbers below. There is a silly rotational near-symmetry tracing to the fact that π and ρ are Grassmannian permutations for self-conjugate partitions.

6 Equidistribution of inversion number vs. comaj on $\binom{[n]}{m}$

Let $J \subseteq \mathbb{Z}$ be a set of n numbers, no two adjacent. Then the product $\prod_{j \in J} r_j$ is well-defined i.e. is independent of the order; indeed, the reduced words for $\prod_{j \in J} r_j$ are in correspondence with permutations of J. The same holds when multiplying subsets of J.

Fix $K \subseteq J$ and let $\rho = \prod_K r_k$, $\pi = \prod_{J \setminus K} r_j$. Then $\mathcal{S}_\pi \mathcal{S}_\rho = \mathcal{S}_{\pi\rho}$, and Theorem 8 (again in the fully barred case) predicts a bijection

$$\{\text{insertions of reduced words } R \text{ for } \rho \text{ into reduced words } P \text{ for } \pi\} \quad \to \quad RW(\pi\rho)$$

such that $\left[\text{comaj}(P) + \text{comaj}(R) + \text{the inversion number of the shuffle}\right]$ matches comaj of the resulting word ш. Note that the obvious map (just insert R where the shuffle suggests) does *not* correspond these two statistics!

If we break J not into two subsets, but all the way down into individual letters, this recovers the equidistribution on S_n of the statistics ℓ and comaj (or maj); see e.g. [14, Proposition 1.4.6].

This hints at a stronger result: that for any two strings P, R such that PR has no repeats, on the set $\{\text{shuffles ш}\}$ the distributions of the statistic $\text{comaj}(P) + \text{comaj}(R) + \ell(\text{ш})$ and the statistic $\text{comaj}(\text{ш})$ match. (Theorem 8 only guarantees this *after summing* over all $P \in RW(\pi), R \in RW(\rho)$.) And indeed, this stronger claim holds [4].

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Jack Derangements

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Abstract.

For each integer partition $\lambda \vdash n$ we give a simple combinatorial expression for the sum of the Jack character θ_α^λ over the integer partitions of n with no singleton parts. For $\alpha = 1, 2$ this gives closed forms for the eigenvalues of the permutation and perfect matching derangement graphs, resolving an open question in algebraic graph theory. A byproduct of the latter is a simple combinatorial formula for the immanants of the matrix $J - I$ where J is the all-ones matrix, which might be of independent interest. Our proofs center around a Jack analogue of a hook product related to Cayley's Ω–process in classical invariant theory, which we call *the principal lower hook product*.

Keywords: Symmetric Functions, Jack Polynomials, Derangements, Algebraic Graph Theory, Young Tableaux, Umbral Calculus.

1 Introduction

Let $\lambda \vdash n$ be an integer partition and consider the power sum expansion of the *Jack polynomials*, i.e., $J_\lambda = \sum_{\mu \vdash n} \theta_\alpha^\lambda(\mu) p_\mu$ [32]. The θ_α^λ's are often called the *Jack characters* because they are a deformation of a normalization of the *irreducible characters* χ^λ of *the symmetric group* S_n. In particular, the Jack polynomials at $\alpha = 1, 2$ recover the integral forms of the *Schur* and *Zonal polynomials* respectively. These specializations have been widely studied in algebraic combinatorics due to their connections with S_n and the set $\mathcal{M}_{2n}$ of *perfect matchings of the complete graph* K_{2n}, but for arbitrary $\alpha \in \mathbb{R}$ many open questions remain [2, 32, 22]. This state of affairs has led to an investigation of the Jack characters since they provide *dual* information about Jack polynomials that may shed light on these open questions; however, the dual path towards understanding Jack polynomials is paved with its own conjectures [10, 16, 17]. We make some progress in this vein by taking *sums* of $\theta_\alpha^\lambda(\mu)$'s rather than single $\theta_\alpha^\lambda(\mu)$'s.

Let $\mathrm{fp}(\mu)$ be the number of singleton parts of μ. Define the λ-*Jack derangement sum*

$$\eta_\alpha^\lambda := \sum_{\substack{\mu \vdash n \\ \mathrm{fp}(\mu)=0}} \theta_\alpha^\lambda(\mu)$$

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to be the sum of the Jack character θ_α^λ over the *derangements*, i.e., partitions $\mu \vdash n$ with no singleton parts. To motivate this definition, recall that if $\lambda \vdash n$ is the cycle type of a permutation $\pi \in S_n$, then π is a *derangement* if and only if $\mathrm{fp}(\lambda) = 0$. Let $D_n \subseteq S_n$ be the set of derangements of S_n. One can show that η_1^λ is a scaled character sum over D_n, i.e.,

$$\eta_1^\lambda = \sum_{\substack{\mu \vdash n \\ \mathrm{fp}(\mu)=0}} \theta_1^\lambda(\mu) = \sum_{\substack{\mu \vdash n \\ \mathrm{fp}(\mu)=0}} \frac{|C_\mu|}{\chi^\lambda(1)}\chi^\lambda(\mu) = \frac{1}{\chi^\lambda(1)} \sum_{\pi \in D_n} \chi^\lambda(\pi)$$

where $C_\mu \subseteq S_n$ is the conjugacy class corresponding to $\mu \vdash n$. For $\alpha = 2$, an analogous result holds for the so-called *perfect matching derangements* of $\mathcal{M}_{2n}$ (see [18], for example). We are unaware of combinatorial models for $\alpha \neq 1, 2$, but it is natural to view η_α^λ as the α-analogue of the character sum over derangements, which is our main focus.

While little is known about the Jack derangement sums for arbitrary $\alpha \in \mathbb{R}$, the $\alpha = 1, 2$ cases have received special attention in algebraic graph theory because they are in fact the eigenvalues of the so-called *derangement graphs*. The set $\{\eta_1^\lambda\}_{\lambda \vdash n}$ is the spectrum of the *permutation derangement graph*: $\Gamma_{n,1} := (S_n, E)$ where $\pi\sigma \in E \Leftrightarrow \sigma\pi^{-1} \in D_n$, i.e., the normal Cayley graph of S_n generated by D_n. See [7, Ch. 14] or [29] for more details on the permutation derangement graph. The set $\{\eta_2^\lambda\}_{\lambda \vdash n}$ is the spectrum of the *perfect matching derangement graph*: $\Gamma_{n,2} := (\mathcal{M}_{2n}, E)$ where $mm' \in E \Leftrightarrow m \cap m' = \emptyset$. For more details on the perfect matching derangement graph, see [7, Ch. 15] or [18].

These graphs made their debut in *Erdős–Ko–Rado combinatorics*, a branch of extremal combinatorics that studies how large families of combinatorial objects can be subject to the restriction that any two of its members intersect. By design, the *independent sets* (sets of vertices that are pairwise non-adjacent) of $\Gamma_{n,\alpha}$ are in one-to-one correspondence with the so-called *intersecting families* of permutations and perfect matchings for $\alpha = 1, 2$, and the spectra of these graphs have been used to give tight upper bounds and characterizations of the largest intersecting families of S_n and $\mathcal{M}_{2n}$. We refer the reader to [7] for a comprehensive account of algebraic techniques in Erdős–Ko–Rado combinatorics.

The derangement graphs are interesting in their own right since they are natural analogues of the celebrated *Kneser graph*, a cornerstone of algebraic graph theory [9]. Because the algebraic combinatorics of permutations and perfect matchings are more baroque than that of subsets, the eigenvalues of the derangement graphs have proven to be far more challenging to understand. We briefly overview the results in this area.

The first non-trivial recursion for the eigenvalues of the permutation derangement graph was derived by Renteln [29] using determinantal formulas for the *shifted Schur functions* [26], which he used to calculate the minimum eigenvalue of the permutation derangement graph. Using different techniques, Ellis [5] later computed the minimum eigenvalue of the permutation derangement graph. Deng and Zhang [4] determined the second largest eigenvalue. In [13], Ku and Wales investigated some interesting properties of the eigenvalues of the permutation derangement graph. In particular, they proved

The Alternating Sign Theorem, namely, that sgn $\eta_1^\lambda = (-1)^{|\lambda|-\lambda_1}$ for all λ, and they offered a conjecture on the magnitudes of the eigenvalues known as the *Ku–Wales Conjecture*. In [14], Ku and Wong proved this conjecture by deriving another recursive formula using shifted Schur functions that led to a simpler proof of the Alternating Sign Theorem.

It was soon noticed that the algebraic properties of the perfect matching derangement graph parallel those of the permutation derangement graph. The minimum eigenvalue of the perfect matching derangement graph was computed by Godsil and Meagher [8] and later by Lindzey [19, 20]. An analogue of the Alternating Sign Theorem was conjectured in [18, 7] which was recently proven by both Renteln [30] and Koh et al [12]. In an earlier effort to prove this conjecture, Ku and Wong [15] give recursive formulas for η_2^λ and a few closed forms for select shapes. In [31], Srinivasan gives more computationally efficient formulas for the eigenvalues of the perfect matching derangement graph. Godsil and Meagher ask whether an analogue of the Ku–Wales conjecture holds for the perfect matching derangement graph [7, pg. 316]. The latter has remained open since the eigenvalues of the perfect matching derangement graph have defied nice recursive expressions akin to permutation derangement graph. This is because the aforementioned determinantal formulas for shifted Schur functions do not exist for shifted Zonal polynomials or shifted Jack polynomials.

The main shortcoming of the known eigenvalue formulas for the derangement graphs is that they cannot be evaluated efficiently, i.e., they lack “good formulas". Indeed, finding closed forms was deemed a difficult open problem [7, pg. 316], perhaps due to the formal hardness of evaluating the irreducible characters of the symmetric group [28, 11, 27]. Our results show that good formulas do in fact exist.

To state our main results we need a few definitions. Let $h_*^\lambda(i,j) := \alpha a_\lambda(i,j) + l_\lambda(i,j) + 1$ be the *lower hook length* of the cell $(i,j) \in \lambda$ where $a_\lambda(i,j)$ and $l_\lambda(i,j)$ denote *arm length* and *leg length* respectively. We define $H_*^1(\lambda) := h_*^\lambda(1,1)h_*^\lambda(1,2)\cdots h_*^\lambda(1,\lambda_1)$ to be the *principal lower hook product* of the integer partition λ. For $\alpha = 1$, the lower hook length is just the usual notion of hook length, in which case we call $H_*^1(\lambda)$ the *principal hook product*. Note that the principal hook product for $\lambda = (n)$ is simply $n!$.

It turns out that the principal hook product for arbitrary λ arises naturally in classical invariant theory, namely, in the evaluation of a differential operator known as *Cayley's* Ω*–process* (see [3]). Independently, Filmus and Lindzey [6] observe a similar phenomenon in their study of harmonic polynomials on perfect matchings, wherein they show that the principal lower hook product appears in the evaluation of a family of differential operators acting polynomial spaces associated with perfect matchings. From the results of [6], we show in Section 3 that the principal hook product $H_*^1(\lambda)$ counts an interesting class of colored permutations $\mathcal{S}_\lambda$, defined as follows.

For each $i \in [n] := \{1, 2, \ldots, n\}$, we assign a list of colors $L(i) \subseteq [m]$ for some $m \in \mathbb{N}$. We define a *colored permutation* (c, σ) to be an assignment of colors $c = c_1, c_2, \ldots, c_n$ such that $c_i \in L(i)$ and a permutation $\sigma \in \mathrm{Sym}([n])$ such that $\sigma(i) = j \Rightarrow c_i = c_j$, i.e., each cycle of the permutation is monochromatic. Any partition λ defines a color list on each element i of the symbol set $[\lambda_1]$ by setting $L(i) := [\lambda'_i]$ where λ' denotes the *transpose* or *conjugate* partition of λ. We define $\mathcal{S}_\lambda$ to be the set of all such colored permutations, formally, $\mathcal{S}_\lambda := \{(c \in [\lambda'_1] \times \cdots \times [\lambda'_{\lambda_1}], \sigma \in S_{\lambda_1}) : \sigma(i) = j \Rightarrow c_i = c_j \text{ for all } i \in [\lambda_1]\}$. We say that a colored permutation $(c, \sigma) \in \mathcal{S}_\lambda$ is a *derangement* if $\sigma(i) = i \Rightarrow c_i \neq 1$ for all $1 \leq i \leq \lambda_1$. In other words, these are the colored permutations that have no colored cycles in common with $(1, \ldots, 1, ()) \in \mathcal{S}_\lambda$. Let $\mathcal{D}^\lambda$ be the set of derangements of $\mathcal{S}_\lambda$, and let $\mathcal{D}^\lambda_k$ be the set of derangements of $\mathcal{S}_\lambda$ with exactly k disjoint cycles. We define $D^\lambda := |\mathcal{D}^\lambda|$ and $d^\lambda_k := |\mathcal{D}^\lambda_k|$, so that $D^\lambda = d^\lambda_1 + d^\lambda_2 + \cdots + d^\lambda_{\lambda_1}$. For any $\alpha \in \mathbb{R}$, let $D^\lambda_\alpha := \sum_{k=1}^{\lambda_1} d^\lambda_k \alpha^{\lambda_1 - k}$ be the *λ-Jack derangement number*. Our first main result is Theorem 1, that the Jack derangement sums equal the Jack derangement numbers (up to sign).

Theorem 1. *For all $\alpha \in \mathbb{R}$, we have $\eta^\lambda_\alpha = (-1)^{|\lambda| - \lambda_1} D^\lambda_\alpha$*

Theorem 1 gives cleaner and more general proofs of all the previous results on the derangement graphs.

Corollary 1 (Alternating Sign Theorem)**.** *For all $\alpha \geq 0$, we have* $\operatorname{sgn} \eta^\lambda_\alpha = (-1)^{|\lambda| - \lambda_1}$.

Corollary 2 (Ku–Wales Theorem)**.** *For all $\mu, \lambda \vdash n$ such that $\mu_1 = \lambda_1$ and $\alpha \geq 0$, we have $\mu \trianglelefteq \lambda \Rightarrow |\eta^\mu_\alpha| \leq |\eta^\lambda_\alpha|$.*

Setting $\alpha = 2$ in Corollary 2 answers Godsil and Meagher's question on the Ku–Wales conjecture for the perfect matching derangement graph [7, pg. 316].

Corollary 3. *For all $\alpha \geq 1$ and $n \geq 6$, we have $(n) = \arg\max_{\lambda \vdash n} \eta^\lambda_\alpha$, $(n-1, 1) = \arg\min_{\lambda \vdash n} \eta^\lambda_\alpha$, and $(n-1, 1) = \arg\max_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} |\eta^\lambda_\alpha|$.*

Our second main result is a closed-form expression for the eigenvalues of $\Gamma_{n,1}$ and $\Gamma_{n,2}$.

This work can be seen as a companion paper to [21], where less explicit but more general formulas for a variety of different "disjointness" and derangement graphs are given.

2 Shifted Jack Polynomials

We overview standard terminology associated with Jack polynomials. For any cell $(i, j) \in \lambda$, the *leg length* $l_\lambda(i, j)$ of (i, j) is the number of cells below (i, j) in the same column of λ, and the *arm length* $a_\lambda(i, j)$ of (i, j) is the number of cells to the right of (i, j) in the same row of λ, i.e., $a_\lambda(i, j) = |\{(i, k) \in \lambda : k > j\}|$ and $l_\lambda(i, j) = |\{(k, j) \in \lambda : k > i\}|$. Note that arm length and leg length remain well-defined even when λ is replaced by a set of cells that does not form an integer partition. Let $h^\lambda_*(i, j) := \alpha a_\lambda(i, j) + l_\lambda(i, j) + 1$ and $h^*_\lambda(i, j) := \alpha(a_\lambda(i, j) + 1) + l_\lambda(i, j)$ be the *lower hook length* and *upper hook length* of

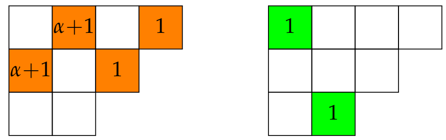


Figure 1: Let $\mu = (4,3,2) \vdash 9$. The colored cells $S = \{(2,1),(1,2),(2,3),(1,4)\}$ on the left is a 4-transversal of μ with α-weight $w_\alpha(S) = (\alpha+1)^2$. The colored cells $S' = \{(1,1),(3,2)\}$ on the right is a 2-transversal of μ with α-weight $w_\alpha(S') = 1$. Each colored cell is labeled with its lower hook length with respect to S and S'.

$(i,j) \in \lambda$, respectively. Let $H_*^\lambda = \prod_{(i,j)\in\lambda} h_*^\lambda(i,j)$ and $H_\lambda^* = \prod_{(i,j)\in\lambda} h_\lambda^*(i,j)$ be the *lower hook product* and *upper hook product* of λ, respectively. Note that the lower and upper hook product remain well-defined even when λ is replaced by a set of cells that does not form an integer partition.

Theorem 2 is a simple but opaque expression for η_α^λ in terms of the (integral form) shifted Jack polynomials $J_\lambda^\star(x;\alpha)$ (see [25], for example). These expressions are already known for η_1^λ and η_2^λ in terms of the determinantal formula for the shifted Schur polynomials [29] and more recently for the shifted Zonal polynomials [30]. Theorem 2 is simply the Jack analogue of these results.

Theorem 2. *For all λ and $\alpha \in \mathbb{R}$, we have $\eta_\alpha^\lambda = \sum_{k=0}^{|\lambda|}(-1)^{|\lambda|-k} J_k^\star(\lambda)/k!$.*

3 Tableau Transversals and Principal Hook Products

We now leverage some combinatorial results of [1, 6] to give a more tractable combinatorial formulation of Theorem 2, which we use to prove Theorem 1 for $\alpha = 1, 2$.

A *k-transversal* T of λ is a set of k cells of T which forms a partial transversal of the columns of λ, that is, no two cells of T lie in the same column of λ. Define the *α-weight* of a k-transversal T to be the lower hook product of T, i.e., $w_\alpha(T) = H_*^T$, with the convention that $w_\alpha(\varnothing) = 1$ (see Figure 1 for examples). Let $\mathcal{T}_\lambda^k$ be the collection of k-transversals of λ.

In [1, Theorem 5.12], Alexandersson and Féray show that $J_k^\star(\lambda)/k! = \sum_{T\in\mathcal{T}_\lambda^k} w_\alpha(T)$. Independently, Filmus and Lindzey [6] prove the following identity: $J_{\lambda_1}^\star(\lambda)/\lambda_1! = \sum_{T\in\mathcal{T}_\lambda^{\lambda_1}} w_\alpha(T) = H_*^1(\lambda)$. For $\alpha = 1$, we note that this identity can be deduced from Naruse's hook-length formula for standard skew-tableaux [23]. We write $\mu \preceq_k \lambda$ if μ is a subshape λ obtained by removing k columns of λ. There are $\binom{\lambda_1}{k}$ such subshapes, and we let the sigma notation $\sum_{\mu\preceq_k\lambda}$ denote the sum over all $\binom{\lambda_1}{k}$ subshapes μ of λ obtained by removing k columns.

Theorem 3. *For any shape λ and $\alpha \in \mathbb{R}$, we have $\eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1} \sum_{k=0}^{\lambda_1} (-1)^k \sum_{\mu \preceq_k \lambda} H_*^1(\mu)$.*

Theorem 3 and Theorem 4 can now already be used to give an elementary combinatorial proof of Theorem 1 for $\alpha = 1, 2$ via the principle of inclusion-exclusion. This is because λ-colored permutations $\mathcal{S}_\lambda$ (see Section 1) and *λ-colored perfect matchings* $\mathcal{M}_\lambda$ (see full version) are *bona fide* combinatorial objects, and their sizes are counted by the principal hook product $H_*^1(\lambda)$.

Theorem 4. [6] *For any shape λ, we have $|\mathcal{S}_\lambda|, |\mathcal{M}_\lambda| = H_*^1(\lambda)$ for $\alpha = 1, 2$, respectively.*

In Section 5 we generalize this proof of Theorem 1 to all $\alpha \in \mathbb{R}$, but along the way we collect several results concerning principal lower hook products, perhaps of independent interest, that allow us to give more explicit expressions of Theorem 1. Specializing these expressions to $\alpha = 1, 2$ yields closed-form expressions for the eigenvalues of derangement graphs, our second main result.

4 Minors of the Principal Hook Product

In this section we prove a few technical lemmas concerning the principal hook product that are needed for closed-form expressions of Theorem 1. Let λ^{-i} be the shape obtained by removing the ith column of λ. Let $\lambda^{-i_1-i_2-\cdots-i_k}$ be the shape obtained by removing (distinct) columns $i_1, i_2, \ldots, i_k$ of λ. It is useful to think of the $H_*^1(\lambda^{-i})$'s as the *first minors* of λ, and the $H_*^1(\lambda^{-i_1-\cdots-i_k})$'s as *k-minors* of λ. The ordering of the i_j's is immaterial, i.e., $\lambda^{-i_1-i_2-\cdots-i_k} = \lambda^{-i_{\sigma(1)}-i_{\sigma(2)}-\cdots-i_{\sigma(k)}}$ for all $\sigma \in S_k$. Let $\lambda^{\underline{k}}$ be the shape obtained by removing the last k columns of λ. We adopt the shorthand $h_j := h_*^\lambda(1, j)$ henceforth. Lemma 1 gives a Laplace-like expansion that relates the principal lower hook product to its first minors.

Lemma 1 (Laplace Expansion)**.** *We have $\sum_{i=1}^{\lambda_1} H_*^1(\lambda^{-i}) = \frac{1}{\alpha}\left(H_*^1(\lambda) + (\alpha - h_{\lambda_1}) H_*^1(\lambda^{\underline{1}})\right)$, equivalently, $H_*^1(\lambda) = \sum_{i=1}^{\lambda_1-1} \alpha H_*^1(\lambda^{-i}) + h_{\lambda_1} H_*^1(\lambda^{-\lambda_1})$.*

For $\alpha \geq 1$, we are now in a position to give a short proof of both the Alternating Sign Theorem and a useful upper bound on the magnitudes of the Jack derangement sums.

Proposition 1. *For all $\alpha \geq 1$, we have* $\operatorname{sgn} \eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1}$. *Moreover, $|\eta_\alpha^\lambda| \leq H_*^1(\lambda)$.*

For any λ and integer $0 \leq j \leq \lambda_1 - 1$, let $f_\lambda^*(j) := \prod_{i=0}^{j}((j+1)\alpha - h_{\lambda_1 - i})$, and define $f_\lambda^*(j) := 1$ for all negative integers j. Lemma 2 is a generalization of Lemma 1 that we will be needed in order to give a more explicit version of [1, Theorem 5.12].

Lemma 2. *For all shapes λ and $0 \leq j \leq \lambda_1 - 1$, we have $\sum_{i=1}^{\lambda_1} f_{\lambda^{-i}}^*(j-1)\ H_*^1((\lambda^{-i})^{\underline{j}}) = \frac{1}{\alpha}\left(f_\lambda^*(j-1)\ H_*^1(\lambda^{\underline{j}}) + f_\lambda^*(j)\ H_*^1(\lambda^{\underline{j+1}})\right)$.*

Theorem 5 is a more explicit form for [1, Theorem 5.12], perhaps of independent interest.

Theorem 5. *For all* $\alpha \in \mathbb{R}$, *we have* $\frac{J^\star_{\lambda_1-k}(\lambda)}{(\lambda_1-k)!} = \sum_{\mu \preceq_k \lambda} H^1_*(\mu) = \frac{1}{\alpha^k}\sum_{j=0}^k (-1)^j \frac{\prod_{i=1}^{\lambda_1}(h_i - j\alpha)}{(k-j)!j!}$, *equivalently,* $\frac{H^*_k}{(\lambda_1-k)!}J^\star_{\lambda_1-k}(\lambda) = \sum_{j=0}^k (-1)^j \binom{k}{j}\prod_{i=1}^{\lambda_1}(h_i - j\alpha)$.

Those familiar with the umbral calculus or the calculus of finite differences may recognize the right-hand side of the second equation in Theorem 5 as essentially the kth-order forward difference Δ^k of the univariate degree-λ_1 polynomial $\mathbf{H}^1_*(\lambda, x) := \prod_{i=1}^{\lambda_1}(h_i - x\alpha)$ in x at the origin, i.e., $H^*_{(k)}J^\star_{\lambda_1-k}(\lambda)/(\lambda_1-k)! = (-1)^k\Delta^k[\mathbf{H}^1_*(\lambda, x)](0)$ where we define $\Delta^k[f](x) := \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x+i)$ for any function $f(x)$. Forward differences of this kind are connected to polynomial interpolation in the falling factorial basis $x^{\underline{k}} := x(x-1)(x-2)\cdots(x-k+1)$, in particular, the *Newton (interpolation) polynomial* $N(x)$ of a set of points $S = \{(x_i, p(x_i))\}_{i=0}^d$:

$$N(x) := [p(x_0)]x^{\underline{0}} + [p(x_0), p(x_1)]x^{\underline{1}} + \cdots + [p(x_0), p(x_1), \ldots, p(x_d)]x^{\underline{d}}$$

where $[p(x_0), \ldots, p(x_j)]$ is the notation for the so-called *jth divided difference*. Note that if $p(x)$ is a degree-d polynomial and $|S| > d+1$, then $[p(x_0), \ldots, p(x_j)] = 0$ for all $j > d$.

Finally, we recall the fact that if $x_i = i$ for all $0 \le i \le d$, then $[p(x_0), p(x_1), \ldots, p(x_j)] = \Delta^j[p](0)/j!$, and the Newton interpolation polynomial is of the form

$$N(x) = \frac{p(0)}{0!}x^{\underline{0}} + \frac{\Delta^1[p](0)}{1!}x^{\underline{1}} + \cdots + \frac{\Delta^d[p](0)}{d!}x^{\underline{d}}. \tag{4.1}$$

See Stanley [33, Ch. 1.9] for a more in-depth discussion of the calculus of finite differences and its connections to combinatorics. In the next section, we show that each Jack derangement number is the sum of the coefficients of a Newton polynomial (Theorem 6).

5 Proof of Theorem 1

Building off the results of the previous sections, we sketch a proof of Theorem 1 in this section. For all $j > 0$, define $H^1_*(\lambda, j) := \prod_{i=1}^{\lambda_1}(h_i - j\alpha)$ to be the *j-shifted principal lower hook product*. It will be convenient to think of the shifted principal lower hook product as a univariate polynomial in x, i.e., $\mathbf{H}^1_*(\lambda, x) := \prod_{i=1}^{\lambda_1}(h_i - x\alpha)$. We let $d^{(\alpha)}_{n,k}$ denote the α-generalization of the *rencontres numbers*, that is, $d^{(\alpha)}_{n,k} := \frac{\alpha^n n!}{\alpha^k k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{\alpha^i i!}$.

Theorem 6. *For all* λ, $\alpha \in \mathbb{R}$, *and* $n \ge \lambda_1$, *we have* $\eta^\lambda_\alpha = (-1)^{|\lambda|-\lambda_1}\frac{1}{\alpha^n n!}\sum_{j=0}^n d^{(\alpha)}_{n,j} H^1_*(\lambda, j)$.

Theorem 6 allows us to connect the Jack derangement sums to the Poisson distribution. For all $\alpha \in \mathbb{R}$, a simple induction shows that $\sum_{j=0}^n d^{(\alpha)}_{n,j}/\alpha^n n! = 1$, and moreover, that

$\lim_{n\to\infty} d_{n,k}^{(\alpha)}/\alpha^n n! = e^{-1/\alpha}/\alpha^k k!$. For $\alpha > 0$, the limiting distribution is the Poisson distribution with expected value $1/\alpha$. After taking limits, for all $\alpha \in \mathbb{R}$, we have

$$\eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1} e^{-1/\alpha} \sum_{x=0}^{\infty} \frac{H_*^1(\lambda, x)}{\alpha^x x!}. \tag{5.1}$$

For $\alpha > 0$, we may interpret the Jack derangement sum as some type of "generalized factorial moment" of the Poisson distribution (up to sign), i.e., $\eta_\alpha^\lambda = (-1)^{|\lambda|-\lambda_1}\mathbb{E}[\mathbf{H}_*^1(\lambda, x)]$. A combinatorial interpretation of these moments will follow as a corollary of Theorem 1. Recall that the factorial moments of the Poisson distribution have a remarkably simple form, namely, for all $\alpha \in \mathbb{R}$, we have $\lim_{x\to\infty} x^{\underline{k}_\alpha}/\alpha^x x! = e^{1/\alpha}$ where $x^{\underline{k}_\alpha} := \alpha^k x^{\underline{k}}$. In light of Equation (5.1), the foregoing suggests that we should express the polynomial $\mathbf{H}_*^1(\lambda, x)$ in the *α-falling factorial basis* $\{x^{\underline{k}_\alpha}\}$, which we determine below for λ such that $\lambda_1 = 1, 2, 3$. Let λ' denote the transpose of λ. If $\lambda_1 = 1$, then we have $\mathbf{H}_*^1(\lambda, x) = -x^{\underline{1}_\alpha} + \lambda_1' x^{\underline{0}_\alpha}$. If $\lambda_1 = 2$, then we have $\mathbf{H}_*^1(\lambda, x) = x^{\underline{2}_\alpha} - (\lambda_2' + \lambda_1')x^{\underline{1}_\alpha} + \lambda_2'(\alpha + \lambda_1')x^{\underline{0}_\alpha}$. If $\lambda_1 = 3$, then we may write $\mathbf{H}_*^1(\lambda, x)$ as

$$-x^{\underline{3}_\alpha} + (\lambda_3' + \lambda_2' + \lambda_1')x^{\underline{2}_\alpha} - ((\alpha + \lambda_1')\lambda_3' + (\alpha + \lambda_1')\lambda_2' + (\alpha + \lambda_2')\lambda_3')x^{\underline{1}_\alpha} + \lambda_3'(\alpha + \lambda_2')(2\alpha + \lambda_1').$$

Indeed, the following proposition shows that each coefficient of $\mathbf{H}_*^1(\lambda, x)$ expressed in the α-falling factorial basis is a polynomial $c_k^\lambda(\alpha)$ that admits a combinatorial interpretation.

Proposition 2. *Let $\hat{\lambda}$ be the partition obtained by removing the first column of λ, and let $\#\mathrm{cyc}(\sigma)$ denote the number of cycles of a permutation σ. For all shapes λ and $\alpha \in \mathbb{R}$, we have $\mathbf{H}_*^1(\lambda, x) = \sum_{k=0}^{\lambda_1} c_k^\lambda(\alpha) x^{\underline{k}_\alpha}$ where $c_k^\lambda(\alpha) = (\alpha(\lambda_1 - 1 - k) + \lambda_1')c_k^{\hat{\lambda}}(\alpha) - c_{k-1}^{\hat{\lambda}}(\alpha)$, $c_k^\lambda(\alpha) := 0$ if $k > \lambda_1$, $c_{-1}^\lambda(\alpha) := 0$. Moreover, we have*

$$(-1)^k[\alpha^{\lambda_1-k-j}]c_k^\lambda(\alpha) = \sum_{\substack{I\subseteq[\lambda_1]\\|I|=k}} |\{(c,\sigma) \in \mathcal{S}_\lambda : \#\mathrm{cyc}(\sigma) = k + j \text{ and } c_i = 1, \sigma(i) = i\ \forall i \in I\}|.$$

Upon expressing Equation (5.1) in the α-falling factorial basis via the Proposition 2, the proof of Theorem 1 becomes straightforward (see the full version for more details).

6 Eigenvalues of the Permutation Derangement Graph

The known recursive expressions for the eigenvalues of the permutation derangement graph originate from [34, Ex. 7.63a], where Stanley considers the sum $d_\lambda := \sum_{\pi\in D_n} \chi^\lambda(\pi)$ and shows it can be written in terms of the complete homogeneous symmetric functions:

$$\sum_{\lambda\vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n} (-1)^{n-k} n^{\underline{k}} h_1^{n-k} h_{n-k}.$$

For hook shapes, both Stanley [34, Ex. 7.63b] and Okazaki [24, Corollary 1.3] prove that

$$d_{(j,1^{n-j})} = (-1)^{n-j}\binom{n}{j}|D_j| + (-1)^{n-1}\binom{n-1}{j} = (-1)^{n-j}\binom{n-1}{j}((n-j)|D_{j-1}| + |D_j|).$$

Recalling that $\eta_1^\lambda = d_\lambda / f^\lambda$ where $f^\lambda := \chi^\lambda(1)$ is the number of standard Young tableaux of shape λ, the following generalizes Stanley and Okazaki's results to all partitions λ.

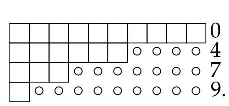
Corollary 4. $d_\lambda = (-1)^{|\lambda|-\lambda_1} f^\lambda D^\lambda$.

This suggests a natural combinatorial interpretation of $|d_\lambda|$ in terms of standard Young tableaux t of shape λ and colored derangements $(c,\sigma) \in \mathcal{D}^\lambda$. Indeed, the set $\mathcal{D}^\lambda$ is in bijection with permutations σ' defined on λ_1 cells of a fixed Young diagram t of shape λ that satisfy the following criteria: if $\sigma'(i) = j$, then the cells containing i and j belong to the same row of t; no two cells involved in the permutation σ' lie in the same column of t; and if $\sigma'(i) = i$, then the cell containing i does not belong to the first row of t. We obtain the desired count by letting t vary over all standard Young tableaux of shape λ. For $\lambda = (1^n)$ this gives a notably different proof of the well-known identity $d_{1^n} = \sum_{\pi \in D_n} \mathrm{sgn}(\pi) = \sum_{\pi \in D_n} (-1)^{\mathrm{inv}(\pi)} = (-1)^{n-1}(n-1)$, i.e., that the number of odd derangements versus even derangements differ by $\pm(n-1)$. More generally, for any integer partition $\lambda \vdash n$, we define the *immanant* of a $n \times n$ matrix A to be $\mathrm{Imm}_\lambda(A) := \sum_{\pi \in S_n} \chi^\lambda(\pi) A_{i,\pi(i)}$. If we consider the adjacency matrix of the complete graph $K_n = J_n - I_n$ where J_n is the $n \times n$ all-ones matrix, then we see that the immanants of the complete graph admit an elegant combinatorial interpretation:

$$\mathrm{Imm}_\lambda(K_n) = \sum_{\pi \in S_n} \chi^\lambda(\pi) \prod_{i=1}^{n} (K_n)_{i,\pi(i)} = \sum_{\pi \in D_n} \chi^\lambda(\pi) = d_\lambda.$$

Recall that Theorem 1 gives an expression for the Jack derangement numbers as a polynomial in α with non-negative coefficients $D_\alpha^\lambda = d_1^\lambda \alpha^{\lambda_1 - 1} + d_2^\lambda \alpha^{\lambda_1 - 2} + \cdots + d_{\lambda_1}^\lambda$ where d_k^λ is the number of colored permutations of $\mathcal{D}^\lambda$ that have precisely k disjoint cycles. One issue with this formula is that the d_k^λ's are hard to compute for general shapes λ, as they are at least as difficult as the associated Stirling numbers of the first kind. Theorem 6 offers a more concrete but less combinatorial form, which for arbitrary α seems to be as good as it gets; however, for $\alpha = 1,2$, we show that Theorem 6 can be massaged into an explicit combinatorial closed form in terms of what we call *extended hook products*. Before we begin, we require a few more tableau-theoretic definitions.

Let $\lambda^c := (\lambda_1 - \lambda_1, \lambda_1 - \lambda_2, \cdots, \lambda_1 - \lambda_{\ell(\lambda)})$ be the *complement* of λ. In other words, the complement of λ is the subset of cells of the shape $(\lambda_1)^{\ell(\lambda)}$ that do not lie in λ. For $\lambda = (10,6,3,1)$, the complement $\lambda^c = (0,4,7,9)$ is the set of dots below:



Let $\mathrm{rev}(\lambda^c)$ be the partition obtained by reversing the order of the rows of λ^c. We also let $\mathrm{rev} : \lambda^c \to \mathrm{rev}(\lambda^c)$ denote the natural bijection defined on their cells, e.g.,

$$\mathrm{rev}\left(\begin{array}{cccccccccc} & & & & & & u & t & s & r \\ & & & q & p & o & n & m & l & k \\ j & i & h & g & f & e & d & c & b & a \end{array}\right) = \begin{array}{cccccccccc} a & b & c & d & e & f & g & h & i & j \\ k & l & m & n & o & p & q & & & \\ r & s & t & u & & & & & & \end{array} .$$

For any cell $\square \in \lambda^c$, we define its *upper hook length* to be $h^*_{\lambda^c}(\square) = h^*_{\mathrm{rev}(\lambda^c)}(\mathrm{rev}(\square))$, and similarly for lower hook lengths. For example, we have the following upper hook lengths for $\alpha = 1$ and $\mu = (10, 6, 3, 1)$:

13	11	10	8	7	6	4	3	2	1
8	6	5	3	2	1	1	2	3	4
4	2	1	1	2	3	5	6	7	8
1	1	2	4	5	6	8	9	10	11

Let $H^*_i(\lambda)$ be the *ith principal upper hook product*, i.e., the product of the upper hook lengths along the ith row of λ. We define the *extended ith principal upper hook product* to be $H^+_i(\lambda) := H^*_i(\lambda)H^*_i(\lambda^c)$. Continuing the example above, we see that $H^+_3(\mu) = 4 \cdot 2 \cdot 1 \cdot 8!/4 = 80640$. Note that $H^*_1(\lambda) = H^+_1(\lambda)$ for all λ since $(\lambda^c)_1 = 0$.

Let $d_{n,k}$ be the *kth rencontres number*, i.e., the number of permutations of S_n with precisely k fixed points. Let $p_{n,k} = d_{n,k}/n!$ be the probability of drawing a permutation (uniformly at random) from S_n with precisely k fixed points. The *Frobenius coordinates* of λ are given by $\lambda = (a_1, \ldots, a_d \mid b_1, \ldots, b_d)$ where $a_i := \lambda_i - i$ is the number of boxes to the right of the diagonal in row i, and $b_i := \lambda'_i - i$ is the number of boxes below the diagonal in column i. By default, we define $a_{d+1} := -1$. We are finally in a position to state our second main result, namely, good closed forms for the eigenvalues of $\Gamma_{n,1}$.

Theorem 7 (Eigenvalues of $\Gamma_{n,1}$)**.** *For all $\lambda = (\lambda_1, \ldots, \lambda_\ell) = (a_1, \ldots, a_d \mid b_1, \ldots, b_d) \vdash n$, we have $\eta^\lambda_1 = (-1)^n \sum_{i \le \lambda_i + 1} (-1)^{\lambda_i} p_{\lambda_1, a_1 - a_i}\, H^+_i(\lambda)$.*

Explicit closed-form expressions for the eigenvalues of the perfect matching derangement graph $\Gamma_{n,2}$ can be derived in a similar manner, which we defer to the full version.

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Framing lattices and flow polytopes

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Abstract. We introduce the framing lattice of a framed graph, a new lattice whose Hasse diagram is the dual graph of a framed triangulation of a flow polytope. We show that every framing lattice is an HH lattice, hence polygonal, semidistributive, and congruence uniform. We also study lattice congruences determined by simple operations called M-moves. Framing lattices provide a unifying framework for the study of many remarkable lattice structures, and several well known results about them are straight forward corollaries of our results.

Keywords: Flow polytope, framed triangulation, Tamari lattice, weak order, Cambrian lattice, cross-Tamari lattice.

1 Introduction

Flow polytopes of acyclic oriented graphs are fundamental objects in the study of combinatorial optimization. In recent years, there has been an explosion of interest in these objects due to their connections with other areas such as representation theory [1], diagonal harmonics [7], and Grothendieck polynomials [8]. From the combinatorial and geometric perspective, a special focus on flow polytopes concentrates on their volumes and triangulations. A novel method for triangulating flow polytopes using a framing of the graph was developed in [4].

Since then, various families of combinatorial objects have revealed tight connections with triangulations of flow polytopes. Examples of this include the Boolean lattice, the Tamari lattice, and the weak order on permutations, each of which is a partially ordered set whose Hasse diagram appears as the dual graphs of a framed triangulation of a flow polytope. On the other hand, flow polytopes serve as a powerful tool to approach open problems about the combinatorial objects involved. For instance, certain framed triangulations of flow polytopes were used in [5] to solve an open conjecture about geometric realizations of s-permutahedra. These recent developments motivate the following question:

Is the dual graph of any framed triangulation the Hasse diagram of a lattice?

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In this paper, we give a positive answer to this question. For any directed graph G and any framing F of G, we define a lattice structure called the *framing lattice* $\mathscr{L}_{G,F}$, whose Hasse diagram is the dual graph of the corresponding framed triangulation. The family of framing lattices captures many important lattices appearing in the literature, including those shown in Figure 1. Four explicit examples are shown in Figure 2, including a new family of lattices that we call cross-Tamari lattices.

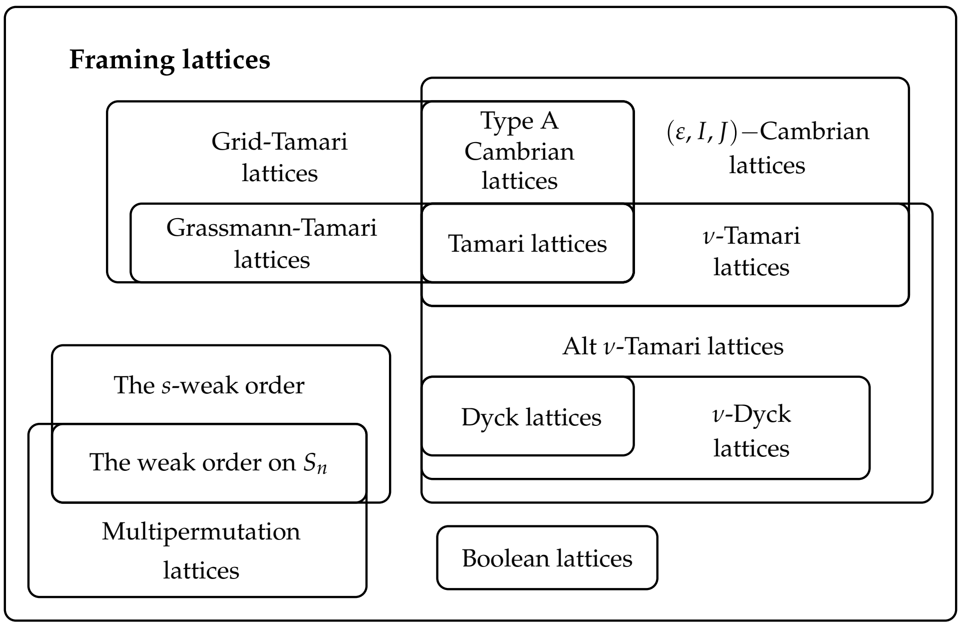

Figure 1: Some lattices captured by the theory of framing lattices.

We prove several structural results about framing lattices. We show that every framing lattice is an HH lattice, hence polygonal, semidistributive, and congruence uniform, and study lattice congruences determined by simple operations on framed graphs called M-moves. We remark that these properties are usually non-trivial results proven in several research works for the special classes outlined in Figure 1; and they all follow from our global uniform results.

2 Framed triangulations of flow polytopes

Let G be a directed acyclic graph on vertex set $V(G) = [n]$ and edge multiset $E(G)$ such that all edges are directed from smaller vertices to larger vertices and G has a unique source $s = 1$ and sink $t = n$. We call such a graph G a ***flow graph***. A path from the source to the sink is said to be a ***route***. For a vertex v in a flow graph G with vertex set $[n]$, let $\mathrm{In}(v)$ and $\mathrm{Out}(v)$ respectively denote the (possibly empty) incoming and outgoing edges at v. A ***unit flow*** on G is then a tuple $(x_e)_{e\in E(G)} \in \mathbb{R}_{\geq 0}^{|E(G)|}$ satisfying

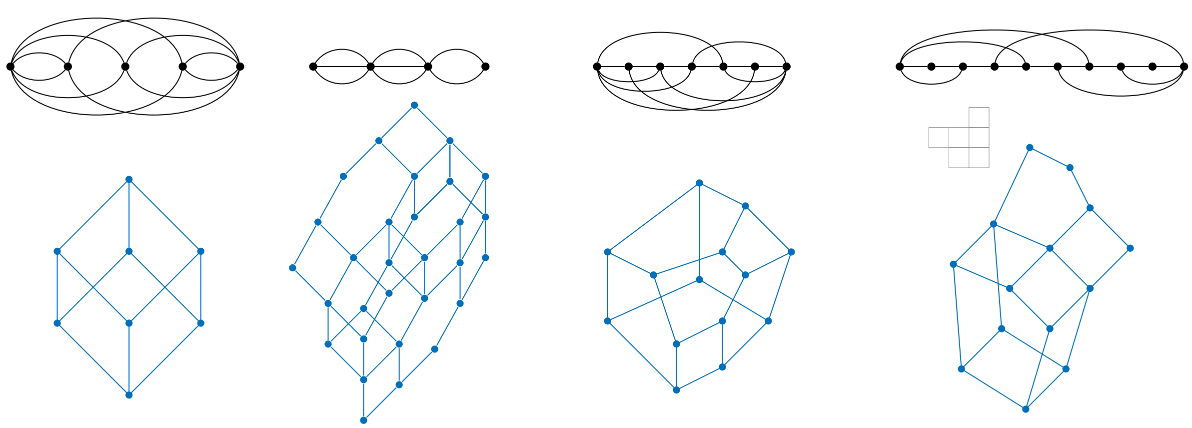

Figure 2: Four framed graphs and the Hasse diagrams of their framing lattices. The first is the Boolean lattice $\mathscr{B}_3$. The second is the lattice of multipermutations of $1^2 2^2 3$. The third is the ε-cambrian lattice with $\varepsilon = - - + -$. The fourth is a cross-Tamari lattice of the cross-shaped grid shown below the right-most graph.

$\sum_{e \in \mathrm{Out}(j)} x_e - \sum_{e \in \mathrm{In}(j)} x_e = u_j$, where $u_1 = 1$, $u_n = -1$, and $u_j = 0$ for $1 < j < n$. The ***flow polytope*** of G is the set $\mathcal{F}_G$ of unit flows on G and its dimension is given by $|E(G)| - |V(G)| + 1$. The vertices of $\mathcal{F}_G$ can be characterized as the unit flows on G which have value one on the edges of a route and value zero on the remaining edges. Thus $\mathcal{F}_G$ can be described as the convex hull of the indicator vectors of the routes of G.

Example 2.1 (The oruga graph and the cube)**.** Let $G_n = \mathrm{Oruga}(n)$ be the ***oruga graph*** on the vertex set $[n+1]$ containing two edges e_{2i-1} and e_{2i} between i and $i+1$ for $i \in [n]$.

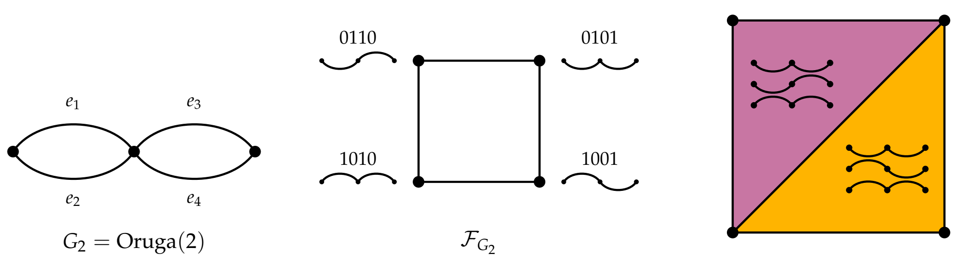

Figure 3: An example of the oruga graph, its flow polytope, and a framed triangulation.

The flow polytope $\mathcal{F}_{G_n}$ is combinatorially a cube of dimension n, whose vertices are of the form $e_{i_1} + \cdots + e_{i_n}$, where $e_i \in \mathbb{R}^{2n}$ denote the standard basis vectors and $i_k = 2k-1$ or $i_k = 2k$, for each value $k \in [n]$. These are the indicator vectors of the routes of G_n.

We now recall the framed triangulations of flow polytopes introduced in [4]. A ***framing*** at the vertex v is a pair of linear orders $(\preceq_{\mathrm{In}(v)}, \preceq_{\mathrm{Out}(v)})$ on the incoming and outgoing edges at v. A ***framed graph***, denoted (G, F), is a flow graph with a framing F at every vertex. An example of a framing of the Oruga(2) graph is shown in Figure 4, where the labels indicate the order of the incoming and outgoing edges at every vertex.

For a path P containing a vertex v, let Pv (resp. vP) denote the maximal subpath of P ending (resp. beginning) at v. Furthermore, let $\mathscr{I}(v)$ (resp. $\mathscr{O}(v)$) denote the set of paths in G ending (resp. beginning) at v. Our notation $\mathscr{I}$ stands for "incoming" and $\mathscr{O}$ for "outgoing". We define the ***relations*** $\leq_{\mathscr{I}(v)}$ and $\leq_{\mathscr{O}(v)}$ on $\mathscr{I}(v)$ and $\mathscr{O}(v)$ as follows.

Given paths $Pv, Qv \in \mathscr{I}(v)$, let $w \leq v$ be the first vertex after which Pv and Qv coincide. If w is the first vertex of Pv or Qv, we say that $Pv =_{\mathscr{I}(v)} Qv$. Otherwise let e_P be the edge of P entering w and let e_Q be the edge of Q entering w. Then $Pv <_{\mathscr{I}(v)} Qv$ if and only if $e_P <_{\mathrm{In}(w)} e_Q$. The relation $\mathscr{O}(v)$ is defined similarly.

Note that if Pv is a subpath of Qv, then $Pv =_{\mathscr{I}(v)} Qv$. But, if they do not start at the same vertex, then they are different paths. Therefore, the relation $\leq_{\mathscr{I}(v)}$ is not even a partial order. However, if we restrict $\leq_{\mathscr{I}(v)}$ (resp. $\leq_{\mathscr{O}(v)}$) to the set of paths starting at the source s (resp. v) and ending at v (resp. the sink t), then it is a *linear order*.

We say that a vertex v of a path P is an ***inner vertex*** if v is not the first or last vertex of the path. If v is an inner vertex of paths P and Q, we say that P and Q are ***incoherent at v*** if $Pv <_{\mathscr{I}(v)} Qv$ and $vQ <_{\mathscr{O}(v)} vP$, or if $Qv <_{\mathscr{I}(v)} Pv$ and $vP <_{\mathscr{O}(v)} vQ$, and we say that they are ***coherent at v*** otherwise. Paths P and Q are then said to be ***coherent*** if they are coherent at each common inner vertex and they are ***incoherent*** otherwise. A set of pairwise coherent routes is called a ***clique***. We denote by $\mathcal{C}$ the ***collection of maximal cliques***. Examples of these concepts are illustrated in Figure 4.

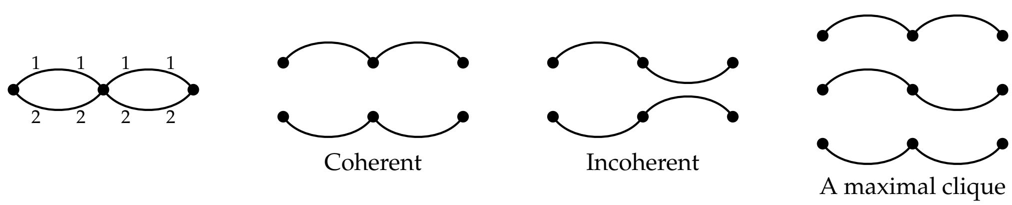

Figure 4: Examples of coherent and incoherent routes, and a maximal clique for the given framing of the Oruga(2) graph.

The motivation for the definition of a framed graph is that the maximal cliques determined by the framing induce a triangulation of the flow polytope. We denote by Δ_C the convex hull of the indicator vectors of the routes in a maximal clique C.

Proposition 2.2 (Danilov et al. [4]). *Let (G, F) be a framed graph. The set $\{\Delta_C \mid C \in \mathcal{C}\}$ is the set of the top-dimensional simplices in a regular unimodular triangulation of $\mathcal{F}_G$.*

A triangulation of $\mathcal{F}_G$ whose facets are the maximal cliques of (G, F) for some framing F is called a ***framed triangulation of $\mathcal{F}_G$***. The framed triangulation of the framing in Figure 4 is shown in Figure 3. The following lemma gives properties about adjacent facets of the triangulation.

Lemma 2.3. *Let $C \neq C'$ be maximal cliques satisfying $C \setminus \{R\} = C' \setminus \{R'\}$. Then,*

(i) The routes R and R' incoherent at some vertex v. Furthermore, they are incoherent at every vertex in the maximal path P_v in $R \cap R'$ that contains v, and coherent everywhere else.

(ii) The routes RvR' and $R'vR$ are contained in $C \cap C'$.

From now on, unless otherwise specified, we draw the framed graphs (G, F) in such a way that the order of the framing of the incoming and outgoing edges at every vertex is ***increasing from top to bottom***. This has two advantages: we do not need to include the labels of a framing for the incoming and outgoing edges to the figure, and the coherence relation becomes very intuitive because two paths are coherent at a vertex v if they "***do not cross***" at v, as illustrated in Figure 5.

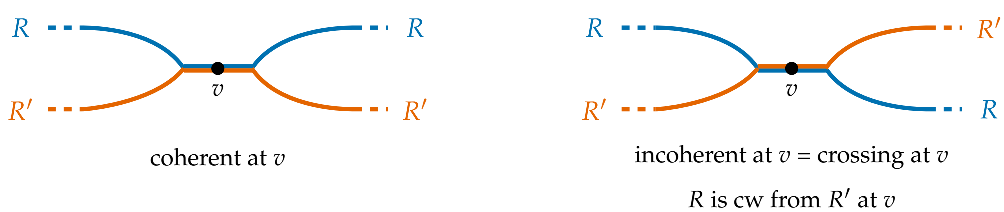

Figure 5: The coherence and cw relation between two routes at v.

This convention motivates the following definition. We say that a route R is ***clockwise (cw)*** from R' at v if $Rv <_{\mathscr{I}(v)} R'v$ and $vR' <_{\mathscr{O}(v)} vR$. We use the ***notation*** $R <_v^{\text{cw}} R'$ when R is cw from R' at v. In particular, R and R' are incoherent at v if and only if $R <_v^{\text{cw}} R'$ or $R' <_v^{\text{cw}} R$. Note also that $<_v^{\text{cw}}$ is a transitive relation, i.e. if $R <_v^{\text{cw}} R'$ and $R' <_v^{\text{cw}} R''$, then $R <_v^{\text{cw}} R''$.

Example 2.4 (A framed triangulation of the oruga graph)**.** Let $G_n = \text{Oruga}(n)$ be the oruga graph from Example 2.1, and let F be the framing that orders the incoming and outgoing edges of G_n from top to bottom. The maximal cliques of (G, F) are in bijective correspondence with permutations of $[n]$ as follows.

Given a permutation $[i_1, \ldots, i_n]$ of $[n]$, construct a maximal clique consisting of $n+1$ routes $R_0, \ldots, R_n$, where R_k is the route containing the top edges e_{2i_j-1} for $1 \leq j \leq k$, and the bottom edges e_{2i_j} for $k < j \leq n$. That is, R_k is the route with top edges at positions $i_1, \ldots, i_k$ and bottom edges at the positions $i_{k+1}, \ldots, i_n$.

The resulting set of routes is a maximal clique, and all the maximal cliques are of this form. Moreover, two facets are adjacent if and only if the corresponding permutations can be obtained from each other by swapping two consecutive numbers. Thus, the dual graph of this framed triangulation of $\mathcal{F}_{G_n}$ is the Hasse diagram of the classical weak order of permutations of $[n]$.

3 Framing lattices

The weak order from the previous example is known to be a lattice. The purpose of this section is to introduce a lattice structure whose Hasse diagram is the dual graph of a framed triangulation of a flow polytope for any framed graph.

Let $C \neq C'$ be maximal cliques satisfying $C \setminus \{R\} = C' \setminus \{R'\}$. By Lemma 2.3, the routes R and R' are incoherent at some point v. If $R <_v^{\mathrm{cw}} R'$, then we say that R' is obtained from R by a ***ccw rotation*** at v. In this case, we say that C' is obtained from C by a ***ccw rotation***. The ***framing poset*** $\mathscr{L}_{G,F} = (\mathcal{C}, \leq_{\mathrm{rot}}^{\mathrm{ccw}})$ is the poset on maximal cliques where $C \leq_{\mathrm{rot}}^{\mathrm{ccw}} C'$ if C' can be obtained from C by a sequence of ccw rotations. We simply write $C \leq C'$ when the partial order is clear from context.

A ***polygon*** in a lattice is an interval $[x, y]$ that is the union of two finite maximal chains from x to y that are disjoint except at x and y. A lattice is said to be ***polygonal*** if the following two conditions hold: (1) If y_1 and y_2 are distinct and cover an element x, then $[x, y_1 \vee y_2]$ is a polygon; and (2) if y_1 and y_2 are distinct and are covered by an element x, then $[y_1 \wedge y_2, x]$ is a polygon.

Theorem 3.1. *If (G, F) be a framed graph then $\mathscr{L}_{G,F}$ is a poset. Moreover, it is a polygonal lattice whose polygons consist of squares, pentagons, or hexagons.*

Given a lattice $\mathscr{L}$, let $E(\mathscr{L})$ denote the set of covering relations of $\mathscr{L}$. We say that $\mathscr{L}$ is an ***$\mathcal{HH}$-lattice*** if it is finite, semidistributive, polygonal, and there exist a labeling function $\ell : E(\mathscr{L}) \to \mathcal{L}$ where $\mathcal{L}$ is a set of labels, and a ranking function $r : \mathcal{L} \to \mathbb{N}$ satisfying the following condition on every polygon $[x, y]$ of $\mathscr{L}$. Let x_1 and x_2 denote the two elements covering x, and let y_1 and y_2 denote the two elements covered by y, such that x_1 and y_1 (resp. x_2 and y_2) belong to the same maximal chain. The labeling ℓ and rank function r must satisfy: (1) $\ell(x, x_1) = \ell(y_2, y)$ and $\ell(x, x_2) = \ell(y_1, y)$; and (2) if $t_1, \ldots, t_k$ is a maximal chain in a polygon, then

$$r(t_1), r(t_k) < r(t_2), r(t_{k-1}) < \cdots < r(t_{\frac{k+1}{2}}) \quad \text{if } k \text{ is odd; and}$$

$$r(t_1), r(t_k) < r(t_2), r(t_{k-1}) < \cdots < r(t_{\frac{k}{2}}), r(t_{\frac{k}{2}+1}) \quad \text{if } k \text{ is even.}$$

It is known that every $\mathcal{HH}$-lattice is ***congruence uniform*** [2], i.e. it can be obtained from the one element lattice by a sequence of *doublings of intervals*, a simple operation introduced by Alan Day in the seventies, see [2] and the references therein.

Theorem 3.2. *The framing poset $\mathscr{L}_{G,F}$ is an HH lattice. In particular, it is semidistributive and congruence uniform.*

The following lemma due to Björner, Edelman, and Ziegler and the tools developed below are central to prove the above results. We skip most of the details due to space constraints.

Lemma 3.3. *(BEZ Lemma [6, Lemma 9-2.2]) Let P be a finite poset with $\widehat{0}$. If the join $x \vee y$ exists for every $x, y \in P$ such that x and y cover a common element z, then P is a lattice.*

To apply the BEZ lemma, we need a characterization of comparability in $\mathscr{L}_{G,F}$. We say that C ***is cw from*** C' if for all $R \in C$, $R' \in C'$, and $v \in R \cap R'$ we have that R and R' are coherent at v or $R <_v^{\mathrm{cw}} R'$.

Proposition 3.4. *Let C and C' be maximal cliques. Then $C \leq C'$ if and only if C is cw from C'.*

Given two maximal cliques covering a common maximal clique, we construct their join algorithmically. Given a set S of pairwise coherent routes, we construct a maximal clique $C_{\max}(S)$ containing S and the ccw-most routes that are coherent with the routes in S. Informally, $C_{\max}(S)$ is obtained by adding the ccw-most routes at each vertex until a maximal clique is formed. The formal construction is described in Algorithm 1, where $\leq_{\mathscr{I}(v)}^{\mathrm{rev}}$ denotes the reverse order of the linear order $\leq_{\mathscr{I}(v)}$. Similarly, we construct a maximal clique $C_{\min}(S)$ containing S whose routes are as clockwise as possible.

Algorithm 1 The construction of $C_{\max}(S)$

$C_{\max}(S) := S$
for $v \in V(G)$ (in increasing order) **do**
    for $Pv \in \mathscr{I}(v)$ (in the order $\leq_{\mathscr{I}(v)}^{\mathrm{rev}}$) **do** ▷ Pv possibly empty
        for $vQ \in \mathscr{O}(v)$ (in the order $\leq_{\mathscr{O}(v)}$) **do** ▷ vQ possibly empty
            if PvQ is coherent with all routes of $C_{\max}(S)$ **then**
                $C_{\max}(S) := C_{\max}(S) \cup \{PvQ\}$
                break ▷ This terminates the innermost loop
            end if
        end for
    end for
end for

Lemma 3.5. *The clique $C_{\max}(S)$ is the unique maximal clique with the following property. If a route $R \notin S$ is coherent with all routes in S, then for any $R' \in C_{\max}(S)$ and $v \in R \cap R'$ either R and R' are coherent at v or $R' <_v^{\mathrm{cw}} R$. The dual statement holds for $C_{\min}(S)$.*

When $S = \emptyset$, we abbreviate $C_{\min} = C_{\min}(\emptyset)$ and $C_{\max} = C_{\max}(\emptyset)$. The maximal cliques $C_{\min}$ and $C_{\max}$ are respectively the $\widehat{0}$ and $\widehat{1}$ of $\mathscr{L}_{G,F}$. The proof of Theorem 3.1 follows from the next lemma together with the BEZ lemma.

Lemma 3.6. *Let C_1 and C_2 be distinct maximal cliques covering a maximal clique Q in $\mathscr{L}_{G,F}$ and let $S = C_1 \cap C_2$. Then, the following statements hold.*

(i) The set of maximal cliques containing S is an interval $I_S = [C_{\min}(S), C_{\max}(S)]$, with $Q = C_{\min}(S)$, $C_1 \neq C_{\max}(S)$, and $C_2 \neq C_{\max}(S)$.

(ii) The interval I_S is a square, pentagon, or a hexagon.

(iii) $C_1 \vee C_2$ exists and is $C_{\max}(S)$.

Remark 3.7. The join of two arbitrary maximal cliques C and C' in $\mathscr{L}_{G,F}$ is not $C_{\max}(S)$ for $S = C \cap C'$. However, it is possible to compute it with a modified version of Algorithm 1.

Our proof of Theorem 3.2 relies on a characterization of semidistributive lattices and HH lattices based on the polygons of the lattice. The congruence uniform property follows from being an HH lattice.

The following result concerns lattice quotients of the framing lattice. It is based on an operation discovered by Yip called an M-move, and was proved independently by González D'León and Yip[1]. Given a framed graph (G, F) and an oriented edge (v, w) such that $v \neq s$ and $w \neq t$, an M-move applied to (v, w) is the framed graph $(G_{v,w}, F_{v,w})$ obtained by replacing the edge (v, w) by the two edges (s, w) and (v, t), while keeping the order of the incoming edges at w and the outgoing edges at v.

Theorem 3.8. *The framing lattice $\mathscr{L}_{G_{v,w},F_{v,w}}$ is a lattice quotient of $\mathscr{L}_{G,F}$.*

We finish this section with the following enumerative conjecture, which is motivated by Section 4.4 and a result in [3], and is supported by computational evidence.

Conjecture 3.9. Let F_1 and F_2 be two framings of G. Then, the framing lattices $\mathscr{L}_{G,F_1}$ and $\mathscr{L}_{G,F_2}$ have the same number of linear intervals of length k for every $k \geq 0$.

4 Examples

4.1 The Boolean lattice

The Boolean lattice $\mathscr{B}_n$ is the lattice on the subsets of $[n]$ ordered by inclusion. We now describe how to obtain $\mathscr{B}_n$ as a framing lattice. Let G_{B_n} be the flow graph with vertex set $\{s, t\} \cup [n]$ and edge set constructed as follows. For each vertex $i \in [n]$ we add a pair of edges (s, i) and $(s, i)'$ and a pair of edges (i, t) and $(i, t)'$. All framing lattices of G_{B_n} will be isomorphic, so the choice of framing does not matter. However, for convenience we choose F to be a framing with $(s, i) <_{\mathscr{I}(i)} (s, i)'$ and $(s, i) <_{\mathscr{O}(i)} (s, i)'$ at each $i \in [n]$. See the left-most graph and lattice in Figure 2 for an example of G_{B_3} and $\mathscr{B}_3$.

A maximal clique of (G_{B_n}, F) contains the routes $\{(s, i), (i, t)\}$ and $\{(s, i)', (i, t)'\}$, and either the route $R_i := \{(s, i), (i, t)'\}$ or the route $R'_i := \{(s, i)', (i, t)\}$ for each $i \in [n]$. For a set $S \subseteq [n]$, define the maximal clique C_S to be the unique maximal clique with routes R'_i with $i \in S$, and R_i with $i \notin S$. The map $S \mapsto C_S$ is an order preserving bijection between $\mathscr{B}_n$ and $\mathscr{L}_{G_{B_n},F}$. Therefore, the framing lattice $\mathscr{L}_{G_{B_n},F}$ is the Boolean lattice $\mathscr{B}_n$.

[1] Personal communication.

4.2 The lattice of multipermutations

Given n positive integers $m_1, \ldots, m_n$, the set of multipermutations $[a_1, \ldots, a_{m_1+\cdots+m_n}]$ of $1^{m_1} \cdots n^{m_n}$ forms a lattice whose cover relations are given by interchanging two consecutive values $a_k < a_{k+1}$. The special case $m_i = 1$ for all i recovers the classical weak order on permutations.

We define the multioruga graph $G_{m_1,\ldots,m_n}$ as the graph on the vertex set $[n+1]$ containing $m_i + 1$ edges $e_{i,0}, \ldots, e_{i,m_i}$ which are drawn from bottom to top between i and $i+1$ for $i \in [n]$. The framing F is induced by this drawing (edges ordered from top to bottom).

The associated flow polytope is a product of simplices $\mathcal{F}_{G_{m_1,\ldots,m_n}} = \Delta_{m_1} \times \cdots \times \Delta_{m_n}$ where $\Delta_{m_i} = \text{conv}\{e_{i,0}, \ldots, e_{i,m_i}\}$. Maximal cliques of the framed triangulation are in bijection with multipermutation as follows.

Given a multipermutation $[a_1, \ldots, a_{m_1+\cdots+m_n}]$ of $1^{m_1} \cdots n^{m_n}$ and an integer k satisfying $0 \le k \le m_1 + \cdots + m_n$, we let R_k be the route consisting of the edges $e_{1,j_1(k)}, \ldots, e_{n,j_n(k)}$, where $j_i(k) := |\{k' \le k : a_{k'} = i\}|$. In other words, $j_i(k)$ counts the number of appearances of i up to position k in the multipermutation. The collection of routes $R_0, \ldots, R_{m_1+\cdots+m_n}$ is a maximal clique, and all maximal cliques are of this form. A counterclockwise rotation of a route R_k in a maximal clique corresponds to interchanging two consecutive values $a_k < a_{k+1}$ in the multipermutation. Thus, the framing lattice $\mathscr{L}_{G_{m_1,\ldots,m_n},F}$ is the lattice of multipermutations of $1^{m_1} \cdots n^{m_n}$. An example is shown in Figure 2.

4.3 The Cambrian lattice

Reading's type A ε-Cambrian lattices [10] are lattices on triangulations of a polygon. The parameter ε is a map $\varepsilon : [n] \to \{\pm\}$ that assigns a positive or negative sign to each element of $[n]$. We define the polygon $P_\varepsilon(n)$ as a convex $(n+2)$-gon with vertices $0, 1, \ldots, n+1$ ordered from left to right, such that 0 and $n+1$ are on a horizontal line and i is above this line if $\varepsilon(i) = +$, or below if $\varepsilon(i) = -$. The ε-Cambrian lattice is the poset on triangulations of $P_\varepsilon(n)$ whose cover relations are increasing slope diagonal flips.

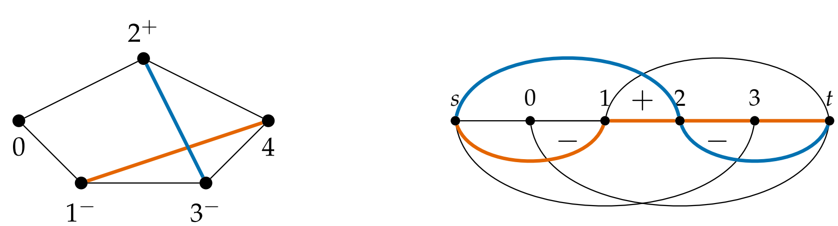

Figure 6: The polygon $P_\varepsilon(3)$ and the Cambrian caracol graph G_ε for $\varepsilon = -+-$.

Let the ***Cambrian caracol graph*** G_ε be the graph with vertex set $\{s, 0, 1, \ldots, n, t\}$ and the following three kinds of edges:

- horizontal edges $(s,0),(0,1),(1,2),\dots,(n-1,n),(n,t)$,
- positive edges $(s,a)^+,(a-1,t)^+$ when $\varepsilon(a)=+$ (above the horizontal line), and
- negative edges $(s,a)^-,(a-1,t)^-$ when $\varepsilon(a)=-$ (below the horizontal line).

The graph G_ε is independent of ε. The framing F_ε is the one induced by the drawing, which depends on ε. The routes of G_ε are in bijection with the diagonals of the polygon $P_\varepsilon(n)$. More precisely, diagonal ij corresponds to the route entering at i exiting at $j-1$. Under this bijection, two routes are coherent if and only if the corresponding diagonals do not cross; see Figure 6. Moreover, the framing lattice $\mathscr{L}_{G_\varepsilon,F_\varepsilon}$ is the ε-Cambrian lattice. An example is shown in Figure 2.

4.4 The cross-Tamari lattice

The cross-Tamari lattice is a new poset structure introduced in this paper which generalizes the alt ν-Tamari lattices of Ceballos and Chenevière [3].

Let D be a set lattice points in $\mathbb{Z}^2$. We say that D is ***horizontally connected*** if for any pair of points (x,y) and (x',y) in D we have $(z,y)\in D$ for all $x<z<x'$. Let $\mathrm{row}_D(z)$ denote the set of points in D with y-coordinate z. We say that D is ***horizontally nested*** if the x-coordinates of the points in $\mathrm{row}_D(v)$ are a subset of the x-coordinates of the points in $\mathrm{row}_D(w)$ whenever $|\mathrm{row}_D(v)|\le|\mathrm{row}_D(w)|$. Similarly, we define ***vertically connected*** and ***vertically nested***. A set of lattice points $D\subseteq\mathbb{Z}^2$ is a ***cross-shaped grid*** if it is both horizontally and vertically connected, and horizontally and vertically nested.

If D has a columns and b rows, it is convenient to assign positions to the points in D according to a relabeling of the columns with the numbers $1,\dots,a$ and the rows with $\overline{1},\dots,\overline{b}$, in some order. We identify a point $p\in D$ with its ***position*** $p=(v,\overline{w})$ where v is the label of column and $\overline{w}$ is the label of the row of the point. We denote by $\ell(v)$ (resp. $\ell(\overline{w})$) the number of elements of D in column v (resp. row $\overline{w}$). A ***proper labeling*** of the rows and columns of D is a labeling satisfying the following conditions:

- the column labels form a unimodular sequence[2] and $\ell(v)<\ell(v')$ implies $v<v'$
- the row labels form a unimodular sequence and $\ell(\overline{w})<\ell(\overline{w}')$ implies $w<w'$

Intuitively, this means that we label the rows and columns from shortest to longest from the outside towards the center. Such a labeling is not unique if D has rows or columns of the same length, but any proper labeling will be good for our purposes. An example of a cross-shaped grid and a proper labeling of its rows and columns is shown in Figure 7. In this example, the bottom-left corner (colored blue) has position $(4,\overline{2})$.

[2] increases and then decreases

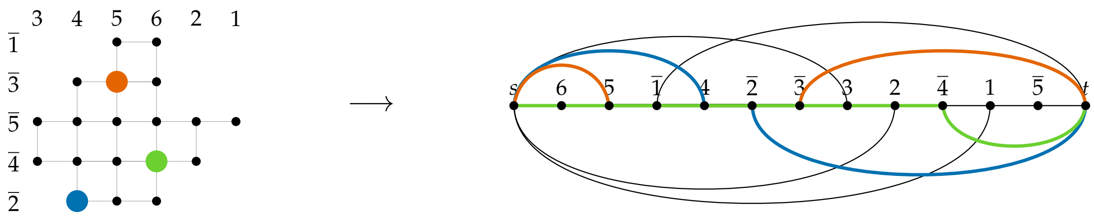

Figure 7: A cross-shaped grid D with a proper labeling L of its rows and columns (left). The (D, L)-caracol graph $G_{D,L}$ with the routes corresponding to the marked points in D highlighted (right).

Let D be a cross-shaped grid. Two distinct points $p, p' \in D$ are ***incompatible*** if one of them is strictly north-east of the other and every lattice point in the smallest rectangle containing p and p' belongs to D. Two points are ***compatible*** if they are not incompatible. A ***maximal filling*** of a cross-shaped grid is a maximal set of pairwise compatible points. If two maximal fillings $M \neq M'$ differ by one single element $M \setminus \{p\} = M' \setminus \{p'\}$ where p' is located strictly north-east of p, then we say the M' is obtainable from M by an ***increasing flip***. The ***cross-Tamari order*** $\mathrm{Tam}(D)$ is the poset of maximal fillings of D where $M \preceq_D M'$ if M' can be obtained from M by a sequence of increasing flips.

The case where D is the set of lattice points weakly above a staircase shape recovers the classical Tamari lattice. If D is the set of lattice points weakly above a given lattice path ν then we recover of ν-Tamari lattice of Préville-Ratelle and Viennot [9]. Cross-Tamari lattices also include the alt ν-Tamari lattices [3] and the ε-Cambrian lattices [10].

Next, we will show that the cross-Tamari order can be obtained as a framing lattice. In particular, this implies that it is a lattice, a non-trivial fact.

Let D be a cross-shaped grid and L be a proper labeling of its columns and rows with the numbers $[a]$ and $[\bar{b}]$. We define the ***(D, L)-caracol graph*** $G_{D,L}$ as the graph on the vertex set $\{s, t\} \sqcup [a] \sqcup [\bar{b}]$, whose edges are given as follows.

First we define a linear order $\prec$ on the vertices, whose minimal element is s, maximal element is t, and the following three relations hold: $i_2 \prec i_1$ when $i_1 < i_2$, $\bar{j}_1 \prec \bar{j}_2$ when $j_1 < j_2$, and $x \prec \bar{y}$ when $(x, \bar{y}) \in D$. The fact that $\prec$ is a linear order follows from the conditions on D and L. We place the vertices $\{s, t\} \sqcup [a] \sqcup [\bar{b}]$ in a horizontal line following the linear order $\prec$ and draw an edge between each pair of consecutive elements. This looks like $s - a - \cdots - \bar{b} - t$. We add additional edges (s, i) and $(\bar{j}, t)$ as follows. For $i \in [a-1]$, we draw an edge (s, i) below the horizontal line if column label i is on the right of column label a, and above if it is on the left. For $j \in [b-1]$, we draw an edge $(\bar{j}, t)$ below the horizontal line if row label $\bar{j}$ is below of row label $\bar{b}$, and above if it is above. The resulting graph is $G_{D,L}$, and the framing $F_{D,L}$ is the framing induced by our drawing; see Figure 7 for an example.

The points in D are in bijection with the routes of $G_{D,L}$. More precisely, the point $(i, \bar{j})$ corresponds to the route entering at i and exiting at $\bar{j}$. Under this bijection, two points in D are incompatible if and only if the corresponding routes are incoherent. Moreover, the framing lattice $\mathscr{L}_{G_{D,L},F_{D,L}}$ is the cross-Tamari lattice $\mathrm{Tam}(D)$. An example is shown in Figure 2.

4.5 Other examples

The previous examples are only a small selection of well studied lattices that appear as examples of framing lattices. Other examples include the Grassmann-Tamari lattices of Santos, Stump, and Welker, the grid Tamari lattices of McConville, the $(\varepsilon, I, \bar{J})$-Cambrian lattices of Pilaud, the permutree lattices of Pilaud and Pons, the s-weak order of Ceballos and Pons, and tau-Tilting posets for certain gentle algebras. The description of these lattices as framing lattices essentially follows from bijections presented in other works, and will be discussed in more detail in a longer version of this manuscript.

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On the sum of the entries in a character table

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Abstract. In 1961, Solomon proved that the sum of all the entries in the character table of a finite group does not exceed the cardinality of the group. We state a different and incomparable property here – this sum is at most twice the sum of dimensions of the irreducible characters. We establish the validity of this property for all finite irreducible Coxeter groups. The main tool we use is that the sum of a column in the character table of such a group is given by the number of square roots of the corresponding conjugacy class representative. We then show that the asymptotics of character table sums is the same as the number of involutions in symmetric, hyperoctahedral and demihyperoctahedral groups. Finally, we derive generating functions for the character table sums for these latter groups as well as generalized symmetric groups as infinite products of continued fractions.

Keywords: finite group, irreducible Coxeter group, character table, symmetric group, hyperoctahedral group, demihyperoctahedral group, absolute square roots, generalized symmetric group, asymptotics, continued fractions

1 Introduction

For any finite group, it is natural to consider the sum of the entries of the character table. Solomon [18] proved that this is always a nonnegative integer by proving something stronger, namely that all row sums are nonnegative integers. He did so by showing that the sum of a row indexed by an irreducible representation is the multiplicity of that representation in the group algebra with respect to the conjugacy action. He then deduced that the sum of the entries in the character table of a finite group is at most the cardinality of the group.

In this extended abstract, we take a different approach to estimating the sum of the entries of the character table by considering column sums instead. It is well known that the column sums are always integers, though not necessarily non-negative [10, Proposition 3.14]. However, for groups whose irreducible characters are real, the column sums

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are given by the number of square roots of conjugacy class representatives by a classical result by Frobenius and Schur [11]. Weyl groups are well-known examples of such groups. However, this is not the case for generalized symmetric groups $G(r, 1, n), r \geq 3$. For $G(r, 1, n)$, column sums are given by so-called absolute square roots [2].

From extensive computations, we observe the following upper bound for the sum of the entries of the character table for many but not all groups.

Property S . *The sum of the the entries of the character table of a finite group is at most twice the sum of dimensions of its irreducible representations.*

We know Property S will not hold in general, but it seems to hold for a large class of natural groups. The smallest counterexamples are of order 64. Our main result is for the following important class of finite groups.

Theorem 1.1. *Property S holds for all finite irreducible Coxeter groups.*

Note that this also settles the issue for Weyl groups. The proof of Theorem 1.1 follows by a case analysis. By analysing the square roots, it is easy to prove the result for dihedral groups. We will prove Property S for the symmetric, hyperoctahedral and demihyperoctahedral groups in the later sections. By explicit computations, we have verified the result for exceptional irreducible finite Coxeter groups. Details will appear in [4]. It is tempting to believe that Property S holds for all finite simple groups. We have not yet done a systematic study in that direction, but we certainly believe the following.

Conjecture 1.2. *Property S holds for all alternating groups.*

Property S holds for abelian groups H because the orthogonality of rows in a character table leads to the vanishing of row sums of all representations except the trivial one. Using this fact, we prove that $G \times H$ satisfies the property if it is true for G. It turns out that Property S holds for any finite group whose all irreducible representations have dimensions at most 2. This class includes generalized dihedral groups and generalized quaternion groups.

It is natural to consider the sequence of these sums for the infinite familes of irreducible Coxeter groups. In Section 2, we consider this sum s_n for the symmetric group S_n. We compute its generating function in Section 2.1. In Section 2.2, we sketch the proof of Property S for S_n and show that the asymptotics of s_n is the same as the number of involutions in S_n. We state similar results for the hyperoctahedral groups B_n in Section 3 and for the demihyperoctahedral groups D_n in Section 4. Since the main ideas are similar, we only state the results. We then extend the generating function result to the generalized symmetric groups $G(r, 1, n)$ in two ways. We give generating functions for the sum of the number of square roots as well as column sums for conjugacy class representatives in Section 5. The proofs of these results will appear in an upcoming article [4].

2 Symmetric groups

2.1 Generating function for the total sum of character table

Let S_n be the symmetric group on n letters. The set of irreducible representations and the conjugacy classes of S_n are indexed by the set of integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of n, denoted $\lambda \vdash n$. Write a partition in frequency notation as

$$\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle, \tag{2.1}$$

where m_i denote the number of parts of length i in λ. We are interested in s_n, the sum of the entries of the character table of S_n [17, Sequence A082733]. The first few terms of (s_n) are given by

$$1, 2, 5, 13, 31, 89, 259, 842, 2810, 10020, 37266, 145373.$$

No formula is given for this sequence. Let Γ_λ be the sum of entries of the column indexed by $\lambda \vdash n$ in the character table of S_n. By applying the following classical result of Frobenius and Schur for the symmetric group, we obtain a formula for column sums in terms of square root counting function.

Theorem 2.1 ([11, Theorem 4.5]). *Given a finite group G, let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. Then*

$$|\{x \in G \mid x^2 = g\}| = \sum_{\chi \in \mathrm{Irr}(G)} \sigma(\chi)\chi(g) \quad \text{for each } g \in G,$$

where $\sigma(\chi)$, known as the Frobenius-Schur indicator *of χ, is 1, 0 or -1 if χ is real, complex or quaternionic, respectively.*

Remark 2.2. *It is a standard fact [9, Section 8.10] that all irreducible characters of any Weyl group (for example, symmetric, hyperoctahedral and demihyperoctahedral groups) have Frobenius-Schur indicator 1. Thus, column sums of the character table of any Weyl group are given by the number of square roots of conjugacy class representatives.*

Therefore, $\Gamma_\lambda = |\{x \in S_n : x^2 = w_\lambda\}|$, where w_λ is some fixed element of cycle type λ. Recall that the *double factorial* of an integer n is given by $n!! = n(n-2)\cdots$ ending at either 2 or 1 depending on whether n is even or odd respectively. Define

$$o_r(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} (2k-1)!!\, r^k. \tag{2.2}$$

Proposition 2.3 ([1, Corollary 3.2]). *The column sum Γ_λ is 0 unless m_{2i} is even for all $i \in \{1, \ldots, \lfloor n/2 \rfloor\}$. If that is the case,*

$$\Gamma_\lambda = \prod_{i=1}^{\lfloor n/2 \rfloor} (m_{2i} - 1)!!\, (2i)^{m_{2i}/2} \prod_{j=0}^{\lfloor n/2 \rfloor} o_{2j+1}(m_{2j+1}).$$

Let $\mathcal{S}(x)$ be the (ordinary) generating function of the sequence (s_n), i.e.

$$\mathcal{S}(x) = \sum_{n \geq 0} s_n x^n. \tag{2.3}$$

To give a formula for $\mathcal{S}(x)$, we recall that generating functions which are expressed as continued fractions have a long history beginning with the influential work of Flajolet [8]. There are two kinds of continued fractions which appear commonly. The *Stieltjes continued fraction*, or *S-fraction* has linear terms and the *Jacobi continued fraction*, or *J-fraction* has quadratic terms.

Recall that an *involution* in S_n is a permutation w which squares to the identity. Let i_n be the number of involutions in S_n. A well-known result due to Flajolet [8, Theorem 2(iia)] gives the generating function $\mathcal{I}(x)$ of involutions in S_n as the J-fraction

$$\mathcal{I}(x) = \sum_{n \geq 0} i_n x^n = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{2x^2}{\ddots}}}. \tag{2.4}$$

Flajolet also showed in the same theorem [8, Theorem 2(iib)] that the generating function of odd double factorials is the S-fraction

$$\mathcal{D}(x) = \sum_{n \geq 0} (2n-1)!!\, x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{2x}{\ddots}}}. \tag{2.5}$$

The quantity $o_r(m)$ (defined in (2.2)) and its generalizations have been studied in [12]. Setting $t = 0, m = 0$ and $u_1 = 1$ in the same theorem [8, Theorem 2] we obtain the generating function for $o_r(m)$ as the J-fraction

$$\mathcal{R}_r(x) = \sum_{n \geq 0} o_r(n) x^n = \cfrac{1}{1 - x - \cfrac{rx^2}{1 - x - \cfrac{2rx^2}{\ddots}}}. \tag{2.6}$$

Bessenrodt–Olsson [5] found an explicit bijection between the number of columns in the character table of S_n that have sum zero and the number of partitions of n with at least one part congruent to $2 \pmod 4$. They also computed the generating function for the number partitions whose associated column sum is nonzero.

Let $x, x_1, x_2, \ldots$ be a family of commuting indeterminates. The following result answers a question of Amdeberhan [3].

Theorem 2.4. *The number of square roots of a permutation with cycle type λ written as (2.1) is the coefficient of $x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ in*

$$\prod_{i \geq 1} \mathcal{D}(2i x_{2i}^2) \mathcal{R}_{2i-1}(x_{2i-1}).$$

Consequently, the generating function of the character table sum is

$$\mathcal{S}(x) = \prod_{i \geq 1} \mathcal{D}(2i x^{4i}) \mathcal{R}_{2i-1}(x^{2i-1}).$$

2.2 Proof of Property S for S_n

Recall that *derangements* are permutations without fixed points. We define another sequence (g_n) by

$$g_n := \sum_{\substack{\lambda \vdash n \\ m_1(\lambda)=0}} \Gamma_\lambda, \quad n \geq 1 \quad \text{and} \quad g_0 = 1.$$

Then g_n counts the sum of those columns of the character table of S_n which are indexed by the conjugacy classes corresponding to derangements. The next result is a convolution type statement involving s_n, g_n, and i_n.

Proposition 2.5. *For a positive integer n, we have*

$$s_n = \sum_{k=0}^{n} i_k g_{n-k}.$$

We next prove the following lemma which gives us control over the sequence s_n.

Lemma 2.6. *For $n \geq 2$, we have $2i_{n-1} \leq i_n \leq n i_{n-1}$. Further, for $n \geq 4$, we have $i_k g_{n-k} \leq i_{n-1}/(n-2)$ for all $0 \leq k \leq n-3$.*

Using Lemma 2.6, we show that $s_n \leq i_n + i_{n-1}$, which helps to prove the following:

Theorem 2.7. *Property S holds for all symmetric groups.*

Using the asymptotics of (i_n) derived by Chowla–Herstein–Moore [6, Theorem 8], we confirm the observation of user Lucia [3].

Corollary 2.8. *The total sum sequence (s_n) grows asymptotically as fast as (i_n) and hence*

$$s_n \sim \left(\frac{n}{e}\right)^{n/2} \frac{e^{\sqrt{n}-1/4}}{\sqrt{2}}.$$

3 Weyl groups of type B

The group $\mathbb{Z}_2 \wr S_n$ is called the *hyperoctahedral group* B_n. It can also be written as the generalized symmetric group $G(2,1,n)$. But following [16], we can define it in a more elementary way. But of course, some of these statements here can be seen directly from Section 5 using that language.

Definition 3.1. *Regard S_{2n} as the group of permutations of the set $\{\pm 1, \ldots, \pm n\}$. For an integer $n \geq 2$, the* hyperoctahedral *group of type B_n is defined as*

$$B_n := \{w \in S_{2n} \mid w(i) + w(-i) = 0, \text{ for all } i, \ 1 \leq i \leq n\}$$

Every element $w \in B_n$ can be uniquely expressed as a product of cycles

$$w = w_1\overline{w_1} \cdots w_r\overline{w_r}v_1 \cdots v_s,$$

where for $1 \leq j \leq r$, $w_j\overline{w_j} = (a_1, \ldots, a_{\lambda_j})(-a_1, \ldots, -a_{\lambda_j})$ for some positive integer λ_j and for $1 \leq t \leq s$, $v_t = (b_1, \ldots, b_{\mu_t}, -b_1, \ldots, -b_{\mu_t})$ for some positive integer μ_t. An element $w_j\overline{w_j}$ is called a *positive* cycle of length λ_j and v_t is called a *negative* cycle of length μ_t. This cycle decomposition of w determines a unique pair of partitions $(\lambda \mid \mu)$ called the cycle type of w, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$.

Theorem 3.2 ([16, Theorem 7.2.5])**.** *The set of conjugacy classes of B_n is in natural bijection with the set of ordered pairs of partitions $(\lambda \mid \mu)$ such that $|\lambda| + |\mu| = n$.*

Let s_n^B denote the total sum of the entries of the character table of B_n. The generating function of s_n^B can be obtained from the more general results in Section 5; see Remark 5.9. Let $\Gamma^B_{(\lambda|\mu)}$ be the column sum corresponding to the conjugacy class $(\lambda \mid \mu)$. To find the asymptotics of s_n^B, we define the following

$$g_n^B := \sum_{(\lambda|\mu)}{}' \, \Gamma^B_{(\lambda|\mu)}$$

where the sum runs over all ordered pairs of partitions $(\lambda \mid \mu)$ of total size n such that λ has no part of size 1. Moreover, let i_n^B denote the number of involutions in B_n. Here, we have the following counterpart of Proposition 2.5.

Proposition 3.3. *For positive integers n, we have*

$$s_n^B = \sum_{k=0}^{n} i_k^B g_{n-k}^B.$$

Following similar ideas as in the case of the symmetric group, we prove the next two results, where we use a result of Lin [13, Eq. (5)] for the asymptotics of i_n^B.

Theorem 3.4. *Property S holds for all hyperoctahedral groups.*

Corollary 3.5. *The total sum sequence (s_n^B) grows asymptotically as fast as (i_n^B) and hence*

$$s_n^B \sim \frac{e^{\sqrt{2n}}}{\sqrt{2e}} \left(\frac{2n}{e}\right)^{n/2}.$$

4 Weyl groups of type D

The Weyl group of type D, also known as the *demihyperoctahedral group* D_n, is defined as the following index two subgroup of B_n:

$$D_n := \{w \in B_n \mid w(1) \cdots w(n) > 0\}.$$

Proposition 4.1 ([15, Lemma 2.3]). *Let $\pi \in B_n$ have cycle type $(\lambda \mid \mu)$. Then $\pi \in D_n$ if and only if $\ell(\mu)$ is even.*

The following results gives a description of the conjugacy classes in D_n and characterize the existence of square roots.

Proposition 4.2 ([16, Theorem 8.2.1]). *Given a pair of partitions $(\lambda \mid \mu)$ of n, if an element $\pi \in D_n$ has cycle type $(\lambda \mid \mu)$, the associated conjugacy class $C_{\lambda,\mu}$ in B_n splits into two D_n conjugacy classes if and only if $\mu = \emptyset$ and all the parts of λ are even. The class $C_{\lambda,\mu}$ remains a D_n conjugacy class if and only if either $\mu \neq \emptyset$ or else one of the parts of λ is odd. In particular, for an odd n, any conjugacy class of B_n does not split.*

Proposition 4.3. *A pair of partitions $(\lambda \mid \mu)$ of n (such that $\ell(\mu)$ is even) is the cycle type of a square element of D_n if and only if the following holds:*

1. *all even parts of λ have even multiplicity,*
2. *all parts of μ have even multiplicity, and*
3. *either λ has an odd part or $4 \mid \ell(\mu)$.*

Using Proposition 4.3, we then obtain the following.

Theorem 4.4. *The generating function for the number of conjugacy classes in D_n with non-zero column sum is*

$$\prod_{i=1}^{\infty} \frac{1}{1-q^{4i}} \left[\left(\prod_{j=1}^{\infty} \frac{1}{1-q^{2j}} \right) \left(\prod_{k=0}^{\infty} \frac{1}{1-q^{2k+1}} - 1 \right) + \frac{1}{2} \left(\prod_{j=1}^{\infty} \frac{1}{1-q^{2j}} + \prod_{k=1}^{\infty} \frac{1}{1+q^{2k}} \right) + 1 \right] - 1.$$

Recall the generating function for double factorials in (2.5). To find the generating function for the sum of the entries in the character table of D_n, we generalize $\mathcal{R}_r(x)$ by the J-fraction

$$\mathcal{R}'_r(x,y) = \cfrac{1}{1-(1+y)x - \cfrac{rx^2}{1-(1+y)x-\cfrac{2rx^2}{\ddots}}}. \tag{4.1}$$

Theorem 4.5. *The generating function of the sum of the entries in the character table of D_n is obtained by setting all even powers of y to 1 and odd powers of y to 0 in the formal power series*

$$\prod_{i\geq 0}\Big(\mathcal{D}(4ix^{4i})\mathcal{D}(2iyx^{2i})\mathcal{R}'_{2i+1}(x^{2i+1},y)\Big) + \prod_{i\geq 0}\mathcal{D}(4ix^{4i}) - 1.$$

Let s_n^D denote the sum of the entries of the character table of D_n and i_n^D denote the number of involutions in D_n. The following lemma relates the quantities s_n^D and i_n^D.

Lemma 4.6. *For positive integers n, $s_n^D \leq i_n^D + (s_n^B - i_n^B) + g_n^B$. Moreover, for odd positive integers n, $i_n^D = i_n^B/2$ and $s_n^B = 2s_n^D$. When n is even, $2i_n^D - i_n^B = 2^{n/2}\,(n-1)!!$. Therefore, for all positive integers n, $i_n^D \leq i_n^B \leq 2i_n^D$.*

The main result here follows now from Lemma 4.6.

Theorem 4.7. *Property S holds for all demihyperoctahedral groups.*

Corollary 4.8. *The total sum sequence (s_n^D) grows asymptotically as fast as (i_n^D) and hence*

$$s_n^D \sim \frac{e^{\sqrt{2n}}}{2\sqrt{2e}}\left(\frac{2n}{e}\right)^{n/2}.$$

5 Generalized symmetric groups

We follow [15, Section 2] for the notational background used in this section. For non-negative integers r, n, let $\mathbb{Z}/r\mathbb{Z} \equiv \mathbb{Z}_r = \{\overline{0}, \overline{1}, \ldots, \overline{r-1}\}$ be the additive cyclic group of order r, where we use bars to distinguish these elements from those in the symmetric group. Then define the *generalized symmetric group*

$$G(r,1,n) = \mathbb{Z}_r \wr S_n := \{(z_1,\ldots,z_n;\sigma) \mid z_i \in \mathbb{Z}_r, \sigma \in S_n\}.$$

If $\pi = (z_1, z_2, \ldots, z_n; \sigma)$ and $\pi' = (z'_1, z'_2, \ldots, z'_n; \sigma')$, then their product is given by

$$\pi\,\pi' = (z_1 + z'_{\sigma^{-1}(1)}, \ldots, z_n + z'_{\sigma^{-1}(n)}; \sigma\sigma'),$$

where $\sigma\sigma'$ is the standard product of permutations in S_n.

The group $G(r,1,n)$ can also be realized as a subgroup of the symmetric group S_{rn}. In this interpretation $G(r,1,n)$ consists of all permutations π of the set $\{\bar{k}+i \mid 0 \le k \le r-1, 1 \le i \le n\}$ satisfying $\pi(\bar{k}+i) = \bar{k}+\pi(i)$ for all allowed k and i. For convenience, we identify the letters $\bar{0}+i$ with i for $1 \le i \le n$. Given a permutation $\pi \in G(r,1,n)$, its values at $1,\dots,n$ determine π uniquely.

The two definitions above are identified using the bijective map ϕ defined on the window $[1,\dots,n]$ by

$$\phi((z_1,\dots,z_n;\sigma)) = \begin{bmatrix} 1 & 2 & \cdots & n \\ z_{\sigma(1)}+\sigma(1) & z_{\sigma(2)}+\sigma(2) & \cdots & z_{\sigma(n)}+\sigma(n). \end{bmatrix}$$

This map satisfies $\phi(\pi\,\pi') = \phi(\pi)\circ\phi(\pi')$, where $\circ$ is the usual composition of permutations in S_{rn}.

Let $\pi = (z_1,\dots,z_n;\sigma) \in G(r,1,n)$ and $(u_1),\dots,(u_t)$ be the cycles of σ. Let $(u_i) = (u_{i,1},\dots,u_{i,\ell_i})$ where ℓ_i is the length of the cycle (u_i). Define the *color* of the cycle (u_i) as $z(u_i) := z_{u_{i,1}} + z_{u_{i,2}} + \dots + z_{u_{i,\ell_i}} \in \mathbb{Z}_r$. For $j \in \{0,\dots,r-1\}$, let λ^j be the partition formed by the lengths of cycles of color j of σ. Note that $\sum_j |\lambda^j| = n$. The r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^0 \mid \lambda^1 \mid \dots \mid \lambda^{r-1})$ is called the *cycle type* of π. We refer to such an r-tuple of partitions as an *r-partite partition* of size n, denoted $\lambda \models_r n$. For example, the cycle type of the element

$$(\bar{2},\bar{1},\bar{1},\bar{1},\bar{0},\bar{2};\,(123)(45)(6)) \in G(3,1,6)$$

is $(\emptyset \mid (3,2) \mid (1))$. The following theorem asserts that the conjugacy classes of $G(r,1,n)$ are indexed by r-partite partitions of n.

Theorem 5.1. *[14, p. 170] Two elements π_1 and π_2 in $G(r,1,n)$ are conjugate if and only if their corresponding cycle types are equal.*

Recall the function $\mathcal{D}(x)$ from (2.5) and $\mathcal{R}(x)$ from (2.6). The following result generalizes Theorem 2.4.

Theorem 5.2. *The generating function (in n) for the sum of the number of square roots of all the conjugacy class representatives in $G(r,1,n)$ is*

$$\begin{cases} \prod_{i\ge 0}\left(\mathcal{D}(2irx^{4i})^r\,\mathcal{R}_{r(2i+1)}(x^{2i+1})^r\right) & r \text{ odd}, \\ \prod_{i\ge 0}\left(\mathcal{D}(2irx^{4i})^r\,\mathcal{D}((2i+1)rx^{4i+2})^{r/2}\,\mathcal{R}_{\frac{r(2i+1)}{4}}(2x^{2i+1})^{r/2}\right) & r \text{ even}. \end{cases}$$

In contrast with the case of S_n, the square root function does not give column sums for character table of $G(r,1,3)$, $r>2$ as the group has non-real irreducible characters [2]. Given $\pi = (z_1,z_2,\dots,z_n;\sigma) \in G(r,1,n)$, define the bar operation as $\overline{\pi} := (-z_1,\dots,-z_n;\sigma)$. An element $g \in G(r,1,n)$ is said to have an *absolute square root* if there exists $\pi \in G(r,1,n)$ such that $\pi\overline{\pi} = g$. The next result describes columns sum for $G(r,1,n)$ in terms of absolute square roots.

Theorem 5.3. *[2, Theorem 3.4] Let $\{\chi_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda}$ is a r-partite partition of $n\}$ be the set of irreducible characters of $G(r,1,n)$. Then*

$$\sum_{\boldsymbol{\lambda}} \chi_{\boldsymbol{\lambda}}(g) = |\{\pi \in G(r,1,n) \mid \pi\overline{\pi} = g\}| \quad \forall g \in G(r,1,n),$$

where the sum runs over all r-partite partitions $\boldsymbol{\lambda}$.

By analyzing the absolute square roots we will provide generating functions for number of columns with zero sums and the total sum of the character table of $G(r,1,n)$.

Lemma 5.4. *1. The absolute square of a cycle of odd length d (of any color) is a cycle of the same length of color 0.*

2. The absolute square of a cycle of even length d (of any color) is a product of two cycles, each of length $d/2$, such that sum of their colors is zero.

The following results extend Bessenrodt–Olsson's theorems [5] from S_n to $G(r,1,n)$.

Proposition 5.5. *An r-partite partition $\lambda = (\lambda^0 \mid \lambda^1 \mid \ldots \mid \lambda^{r-1})$ is the cycle-type of an absolute square in $G(r,1,n)$ if and only if the following hold:*

1. *each even part in λ^0 has even multiplicity,*
2. *$\lambda^i = \lambda^{r-i}$ for all $i \geq 1$, and*
3. *each part in $\lambda^{r/2}$ has even multiplicity when r is even.*

Theorem 5.6. *The generating function for r-partite partitions which are cycle-types of absolute squares in $G(r,1,n)$ is:*

$$\begin{cases} \left(\prod_{i=0}^{\infty} \frac{1}{1-q^{2i+1}}\right) \left(\prod_{j=1}^{\infty} \frac{1}{1-q^{4j}}\right) \left(\prod_{k=1}^{\infty} \frac{1}{1-q^{k}}\right)^{(r-1)/2} & r \text{ odd,} \\ \left(\prod_{i=0}^{\infty} \frac{1}{1-q^{2i+1}}\right) \left(\prod_{j=1}^{\infty} \frac{1}{(1-q^{4j})(1-q^{2j})}\right) \left(\prod_{k=1}^{\infty} \frac{1}{1-q^{k}}\right)^{(r-2)/2} & r \text{ even.} \end{cases}$$

Using [2, Observation 4.2], we obtain the number of absolute square roots for cycles of a single length and color.

Proposition 5.7. *Given a positive integer r, the following holds.*

1. *The number of absolute square roots of an element of cycle type $\lambda^0 = ((2k)^{2m_{2k}})$ (and all other λ^i is zero) is $(2m_{2k}-1)!!\,(2kr)^{m_{2k}}$.*

2. *The number of absolute square roots of an element of cycle type* $\lambda^0 = ((2k+1)^{m_{2k+1}})$ *(and all other* λ^i *is zero) is*

$$\sum_{j=0}^{\lfloor \frac{m_{2k+1}}{2} \rfloor} \binom{m_{2k+1}}{2j} (2j-1)!!\,(2k+1)^j\, r^{m_{2k+1}-j}.$$

3. *The number of absolute square roots of an element of cycle type* $\lambda^a = \lambda^{r-a} = (k^{m_k})$ *(and all other* λ^i *is zero) is* $m_k!\,(kr)^{m_k}$.

4. *For even* r*, the number of absolute square roots of an element of cycle type* $\lambda^{r/2} = (k^{2m_k})$ *(and all other* λ^t *is zero) is* $(2m_k-1)!!\,(kr)^{m_k}$.

Adin–Postnikov–Roichman [2, Corollary 4.3] also give a formula to count the number of absolute square roots of any element in $G(r,1,n)$. Using Proposition 5.7, we generalize their result to determine the sum of the character table in terms of generating functions. To do so, we also need the classic generating function for the factorials due to Euler [7] given by

$$\mathcal{F}(x) = \sum_{n\geq 0} n!x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{2x}{\ddots}}}}}. \tag{5.1}$$

Theorem 5.8. *The generating function (in* n*) of the total sum of the character table of* $G(r,1,n)$ *is*

$$\begin{cases} \prod_{i\geq 0} \Big(\mathcal{F}(irx^{2i})^{(r-1)/2}\, \mathcal{D}(2irx^{4i})\, \mathcal{R}_{(2i+1)/r}(rx^{2i+1}) \Big) & r \text{ odd,} \\ \prod_{i\geq 0} \Big(\mathcal{F}(irx^{2i})^{(r-2)/2}\, \mathcal{D}(2irx^{4i})\, \mathcal{D}(rix^{2i})\, \mathcal{R}_{(2i+1)/r}(rx^{2i+1}) \Big) & r \text{ even.} \end{cases}$$

Remark 5.9. *When* $r = 2$*, absolute square roots are exactly the usual square roots. Thus the generating function for* (s_n^B) *can be obtained by setting* $r = 2$ *in either Theorem 5.2 or Theorem 5.8.*

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Vertex models for the product of a Schur and Demazure polynomial

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Abstract. The product of a Schur polynomial and Demazure atom or character expands positively in Demazure atoms or characters, respectively. The structure coefficients in these expansions have known combinatorial rules in terms of skyline tableaux. We develop alternative rules using the theory of integrable vertex models, inspired by a technique introduced by Zinn-Justin. We apply this method to coloured vertex models for atoms and characters obtained from Borodin and Wheeler's models for non-symmetric Macdonald polynomials. The structure coefficients are then obtained as partition functions of vertex models that are compatible with both Schur (uncoloured) and Demazure (coloured) vertex models.

Keywords: Demazure atoms, Demazure characters, Schur polynomials, vertex models, structure coefficients, key polynomials

1 Introduction

Demazure atoms, also called standard bases, are a family of non-symmetric polynomials indexed by weak compositions. Demazure characters, also called key polynomials, are a closely related family of polynomials which are also indexed by weak compositions; they may be written as a sum of Demazure atoms. Denote the Demazure atom and character on a weak composition $\alpha = (\alpha_1, \dots, \alpha_n)$ in the variables $x = (x_1, \dots, x_n)$ by $\mathcal{A}_\alpha(x)$ and $\mathcal{K}_\alpha(x)$, respectively. The set of Demazure atoms or characters over all weak compositions of length n are a basis for $\mathbb{Z}[x_1, \dots, x_n]$. It is known that the products of a Schur polynomial $s_\lambda(x)$ and a Demazure polynomial have positive expansions:

$$s_\lambda(x)\mathcal{A}_\alpha(x) = \sum_\beta c^\beta_{\lambda,\alpha}\mathcal{A}_\beta(x),$$
$$s_\lambda(x)\mathcal{K}_\alpha(x) = \sum_\beta d^\beta_{\lambda,\alpha}\mathcal{K}_\beta(x),$$

where the structure coefficients $c^\beta_{\lambda,\alpha}$ and $d^\beta_{\lambda,\alpha}$ are non-negative integers.

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In [3], Haglund, Luoto, Mason and van Willigenburg give formulas to calculate $c_{\lambda,\alpha}^{\beta}$ and $d_{\lambda,\alpha}^{\beta}$ in terms of skyline tableaux. Here, we use the theory of integrable vertex models to derive alternative rules where the structure coefficients are calculated as the number of fillings of "diamond" vertex models. We emulate the technique developed in [11] where Zinn-Justin reproves the puzzle rule of [4, 5] for the product of two double Schur polynomials. Wheeler and Zinn-Justin later use the same technique to find structure coefficients for double Grothendieck polynomials [10]. Knutson and Zinn-Justin also employ techniques from integrability in a series of papers computing puzzle rules for products of Schubert classes in d-step flag varieties (for $d \leq 4$) [6, 7, 8].

The proofs in [10, 11] are completely combinatorial, gluing vertex models together in two different ways and showing both are equivalent. One side of the equation is manifestly a product and the other side is manifestly a summation. Applying a Yang–Baxter equation to the model for the product transforms it into the model for the summation. Both of these results concern products of symmetric polynomials, whereas our results involve the non-symmetric Demazure polynomials. Our results follow from a variant of the Yang–Baxter equation stated in Lemma 1.

In this extended abstract, we define Demazure atoms and characters as the partition function of a vertex model. Our conventions for atoms match those of Mason [9] who defines $\mathcal{A}_\alpha(x)$ in terms of semi-skyline augmented fillings; reversing the order of the composition and basement in Mason's diagrams yields $\mathcal{K}_\alpha(x)$. Our model is derived from setting $q = t = 0$ in Borodin and Wheeler's [1] vertex model for permuted basement non-symmetric Macdonald polynomials $f_\alpha^\rho(x; q, t)$, where ρ is a permutation. In our conventions, we have $\mathcal{A}_\alpha(x) = f^{\mathrm{id}}_{(\alpha_n,\ldots,\alpha_1)}(x_n, \ldots, x_1; 0, 0)$ and $\mathcal{K}_\alpha(x) = f_\alpha^{w_0}(x_n, \ldots, x_1; 0, 0)$. Significant modifications are made to make the vertices compatible with the Schur polynomial model in [11]. Our model for $\mathcal{A}_\alpha(x)$ bears more resemblance to that of Brubaker, Buciumas, Bump and Gustafsson [2] with differing weights and boundary.

A benefit of this approach is that vertex models may be developed independently and then fit into this framework, allowing one to test rules assuming an analogue of Lemma 1 holds. Our results are suggestive of further applications such as extensions to the Grothendieck model in [10].

2 Vertex models for Schur and Demazure polynomials

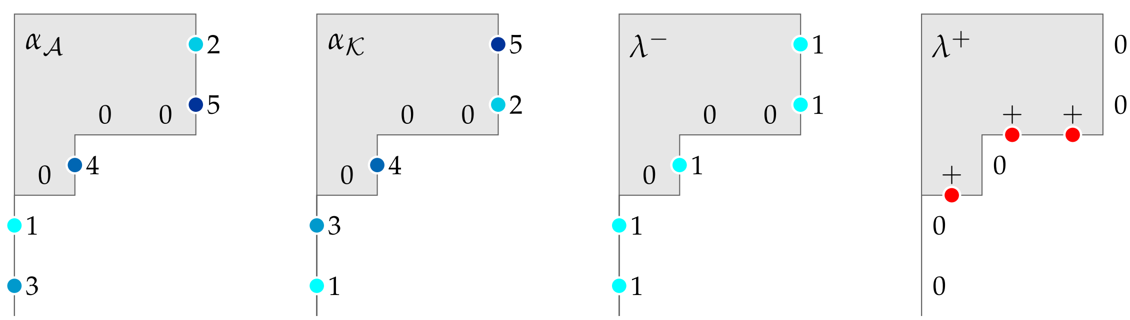
A weak composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a sequence of non-negative integers. The integer α_i is the part of α at index i and the length of α is its number of parts; the largest part in α is denoted $\max(\alpha)$. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a weak composition sorted in descending order. Throughout this extended abstract, α and β are weak compositions, λ is a partition and all weak compositions have length n.

We describe two strings, $\alpha_{\mathcal{A}}$ and $\alpha_{\mathcal{K}}$, that re-encode a weak composition α. Let $\lambda =$

sort(α) be the partition with the same parts as α sorted in descending order. Enclose the Young diagram of λ between the top left corner of a rectangle and a North-East lattice path as depicted in Example 1. East steps are labelled 0 and North steps are labelled with the integers 1 through n so that i occurs after precisely α_i East steps. If North steps occur in the same vertical, then moving North, we label them in descending order for $\alpha_{\mathcal{A}}$ and ascending order for $\alpha_{\mathcal{K}}$. We then obtain either string by reading labels off the lattice path from South-West to North-East.

We also specify two strings, λ^- and λ^+, that re-encode a partition λ. For λ^-, East steps are labelled 0 and North steps are labelled 1. For λ^+, East steps are labelled with the symbol $+$ and North steps are labelled 0. Strings are read off the lattice path as before.

Example 1. *We depict our labelling procedure below with $\alpha = (0,3,0,1,3)$ and $\lambda = \text{sort}(\alpha) = (3,3,1,0,0)$, assigning each label a colour as a visual aid. Darker shades of blue correspond to larger integers.*



Reading the labels from South-West to North-East produces the strings:

$$\alpha_{\mathcal{A}} = ●●○●○○●● = 31040052$$
$$\alpha_{\mathcal{K}} = ●●○●○○●● = 13040025$$
$$\lambda^- = ●●○●○○●● = 11010011$$
$$\lambda^+ = ○○●○●●○○ = 00{+}0{+}{+}00$$

The model for Demazure atoms consists of a lattice filled with the tiles in Figure 1. This is a coloured vertex model where the tiles are "vertices" much like those in [2]. Labels of tiles must match along adjacent edges and along the boundary of the lattice. We label the left boundary with the string $\alpha_{\mathcal{A}}$ and label the bottom edges 1 through n from left to right; the other boundary edges are labelled 0. All tiles in the model have weight 1 except for the tiles of weight x_c where c is the column number where the tile occurs; columns are numbered 1 through n from left to right. A filling's weight is the product of its tile weights and the sum of all filling weights is called the partition function.

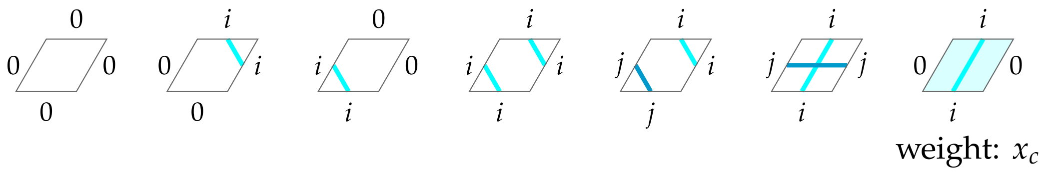
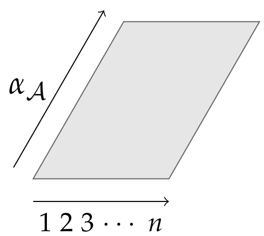
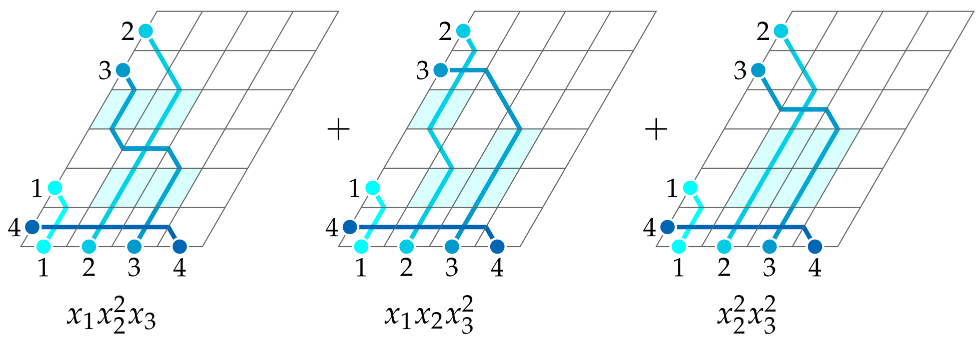


Figure 1: Tiles for the Demazure atom vertex model where $1 \leq i < j$. The rightmost tile has weight x_c where c is the column number where the tile occurs.

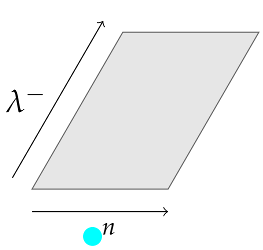
The partition function of this vertex model is the Demazure atom on n variables, depicted diagrammatically:

$\mathcal{A}_\alpha(x) =$ 

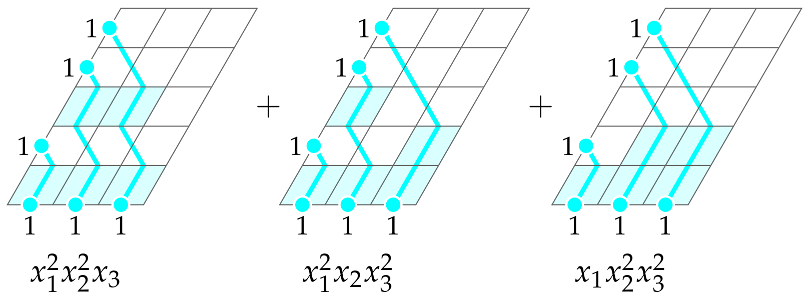
Example 2. *Let* $\alpha = (0, 2, 2, 0)$, *so that* $\alpha_\mathcal{A} = 410032 =$ ● ● ○ ○ ● ● *labels the left boundary. There are three fillings of the atom model, showing* $\mathcal{A}_\alpha(x_1, x_2, x_3, x_4) = x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2$.



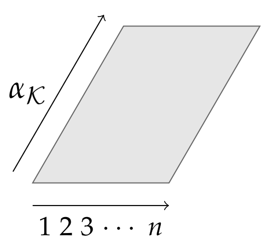
Zinn-Justin considers a similar model for Schur polynomials in [11] which may be thought of as the "uncoloured" version of the model for atoms. Using the same tiles with only colour 1, we label the left boundary with the string λ^- and label all bottom edges with 1, which we denote as ●$^n = 1^n$:

$s_\lambda(x) = \lambda^-$ 

Example 3. *Let* $\lambda = (2,2,1)$. *There are three fillings of the Schur model, showing that* $s_\lambda(x_1,x_2,x_3) = x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2$.

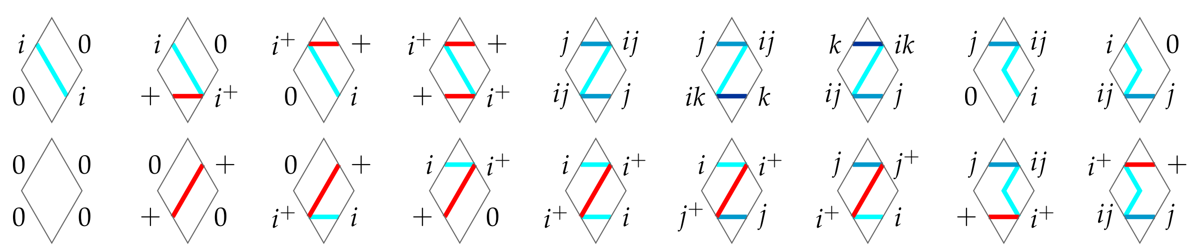


Lastly, as noted in [2], Remark 4.5, the model for Demazure characters uses the same tiles rotated 180 degrees, which only alters the fifth tile in Figure 1. We can obtain Demazure characters as a partition function for the following vertex model filled with these new tiles:

$\mathcal{K}_\alpha(x) =$ 

3 Vertex models for $c^\beta_{\lambda,\alpha}$ and $d^\beta_{\lambda,\alpha}$

In this section we build two "diamond" vertex models filled with the tiles below where $1 \leq i < j < k$. If a blue line of shade b shares an edge with a red line, the edge is labelled b^+. Two shades of blue a and b with $a < b$ may share an edge labelled ab. All tiles have weight 1.

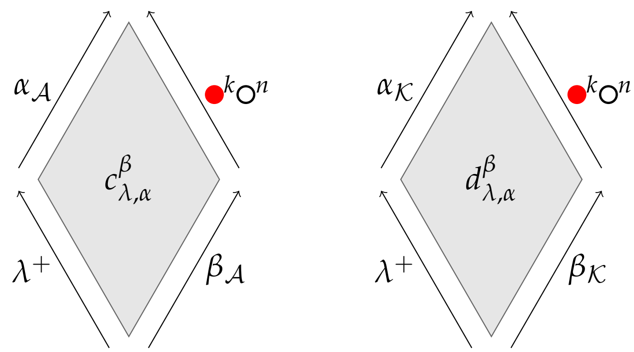


Further, we do not allow two adjacent tiles to form an internal banned rhombus as depicted in Figure 2. These restrictions are still local and can be imposed with additional labels, but we exclude them to avoid clutter. We may now state our theorem.

(a) Banned rhombi for atom model. (b) Banned rhombi for character model.

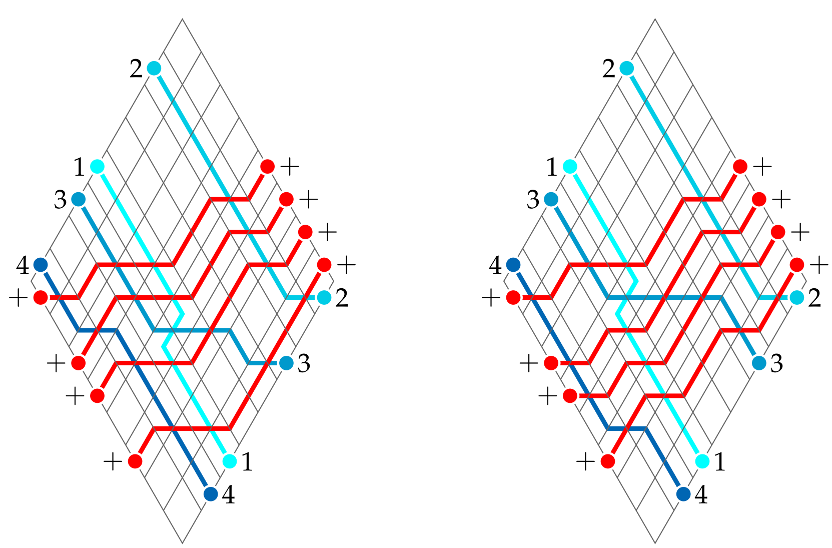
Figure 2: Restrictions on adjacent diamond tiles where $1 \le i < j$.

Theorem 1. *The structure coefficients $c^{\beta}_{\lambda,\alpha}$ and $d^{\beta}_{\lambda,\alpha}$ respectively count the number of fillings of the vertex models*



where $k = \max(\beta)$ and the restrictions in Figure 2a and Figure 2b apply respectively within each model.

Example 4. *For $\alpha = (1,3,1,0)$, $\lambda = (3,1,0,0)$ and $\beta = (1,4,3,1)$, we have that $k = \max(\beta) = 4$. There are two fillings of the corresponding Schur–atom model and thus $c^{\beta}_{\lambda,\alpha} = 2$.*



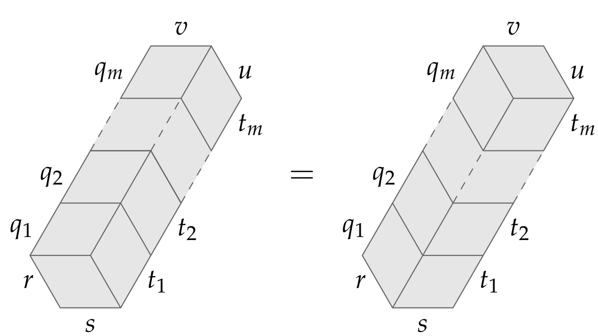
We call the vertex model for $c^{\beta}_{\lambda,\alpha}$ the Schur–atom model and the vertex model for $d^{\beta}_{\lambda,\alpha}$ the Schur–character model. Recall that we assume α, β and λ all have length n, but we may append zeros to make their lengths match if needed. Similarly, we may append zeros to the end of the strings $\alpha_{\mathcal{A}}$, $\alpha_{\mathcal{K}}$ and λ^- and append $+$ symbols to the end of λ^+ so that these strings all have length $n+k$ and fit in the diagram.

4 Proof of Theorem 1

In this section, we only explain the proof of the Schur–atom model, but the proof of the Schur–character model is analogous. In Figure 3, we have tiles in three orientations with a new orientation in the second row containing a tile of weight $-x_c$. We call the tiles in the first row right-sheared and the tiles in the second row left-sheared. We again depict vertex models with grey diagrams where tiles must have the same orientation as the grey region they are placed within. In our configurations, right- and left-sheared tiles are in the same column, say c, if one is on top of the other; hence these tiles may have weight x_c or $-x_c$, respectively.

Note red lines may now share the same path as blue lines; these tiles facilitate the proof and do not appear in the final Schur–atom model. We still ban rhombi between diamond tiles as in Figure 2a, but there are no such restrictions between tiles that are not both diamonds. The key to the proof is the following lemma equating columns of tiles.

Lemma 1. *Let $q_1, \ldots, q_m$, r, s, $t_1, \ldots, t_m$, u and v be fixed labels where u and r are in $\{○, ●\} = \{0, +\}$. The following column configurations have the same weight:*



Proof. Proving this lemma is the main difficulty of this work. The proof is by induction with manual checking of several edge cases. □

Remark 1. *In [10, 11] the authors proceed similarly with a Yang–Baxter equation that equates unit hexagons with unrestricted boundaries. In contrast, our equation requires that we restrict the labels on the South-West and North-East edges, suggesting a more general framework to explore.*

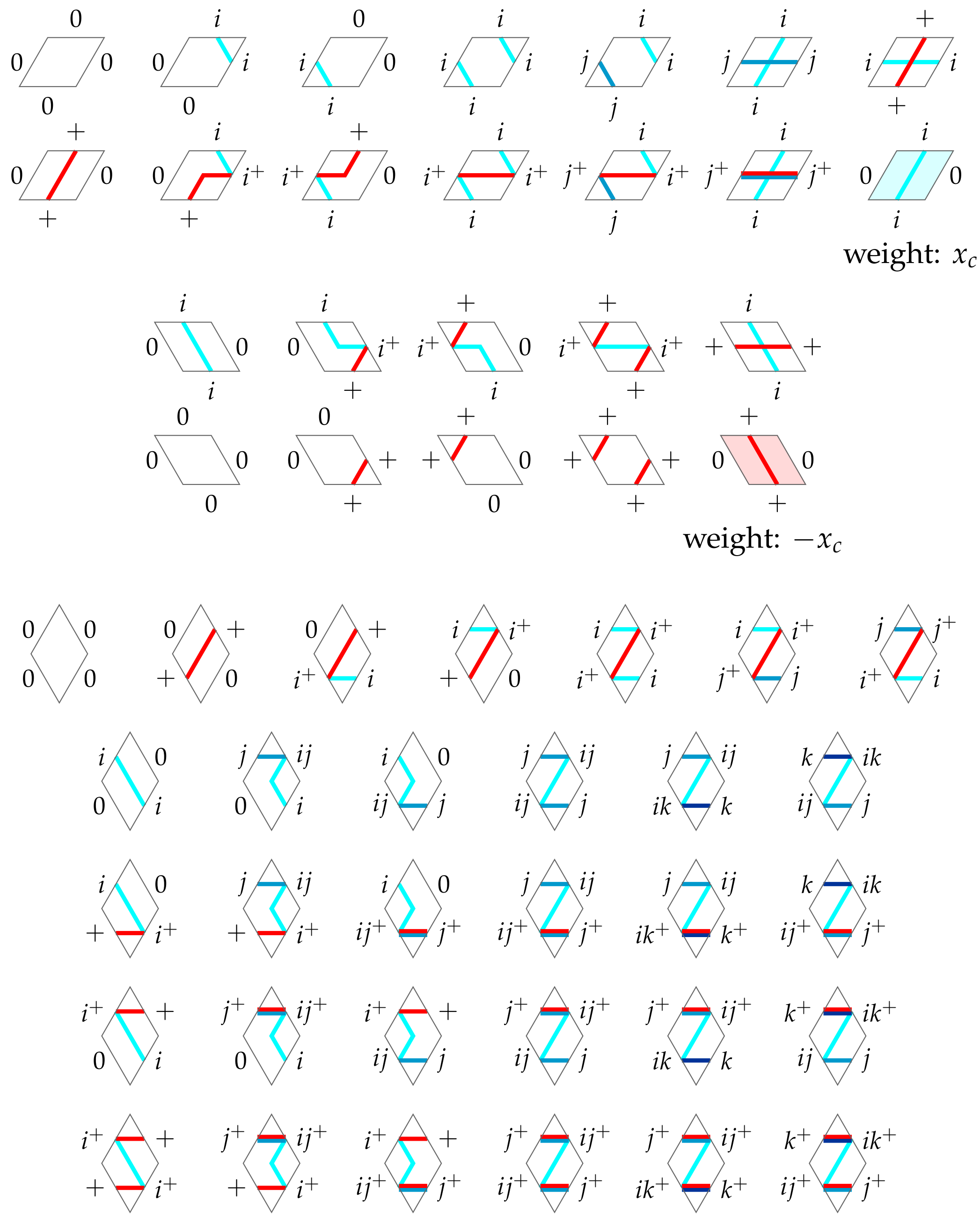
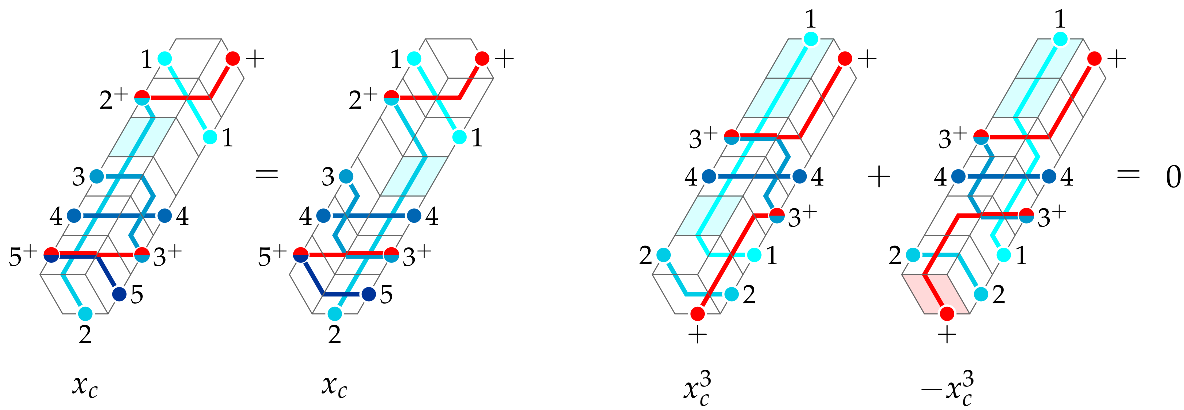


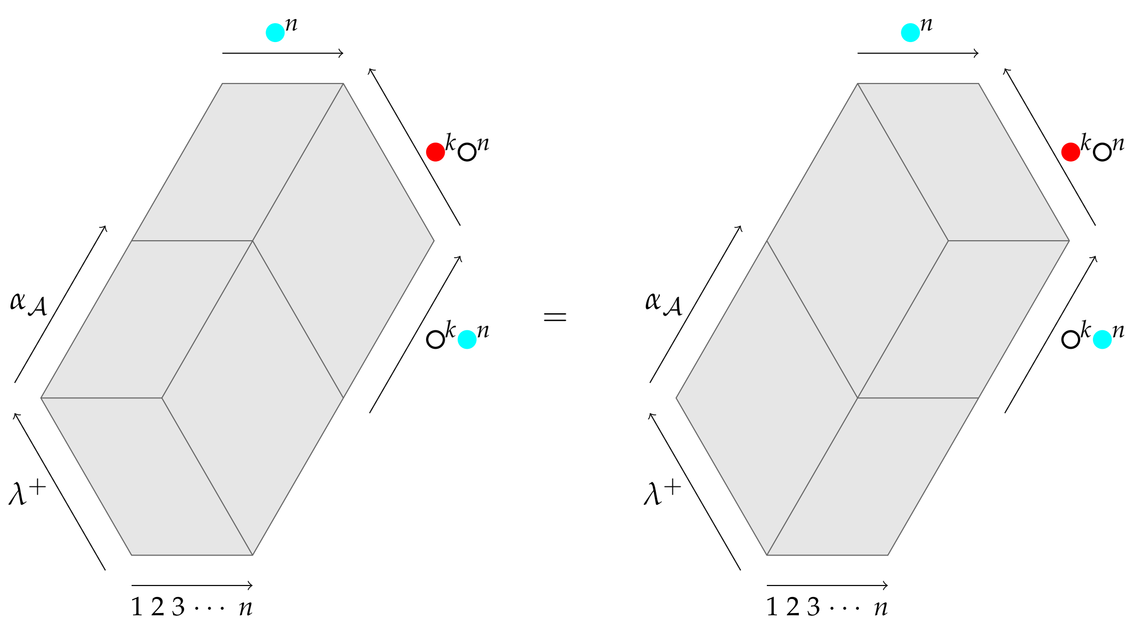
Figure 3: The full set of tiles where $1 \leq i < j < k$.

Example 5. *We consider two examples of Lemma 1. In the first example, both sides of the equation have weight x_c. In the second example, there are two ways to fill the column on the left-hand side which sum to a weight of* 0 *and there is no way to fill the column on the right-hand side.*



Note that rotating columns 180 degrees gives an analogous column lemma used to prove correctness of the Schur–character model. Interpreting the next lemma proves our result.

Lemma 2. *Set $k = \max(\alpha) + \max(\lambda)$. The configurations below have the same weight:*



Proof. Repeatedly applying Lemma 1 to internal columns transforms the left-hand side into the right-hand side. First apply the lemma to the column of length $2(k+n)$ containing the left-sheared tile most to the North-East and then repeat with the next left-sheared

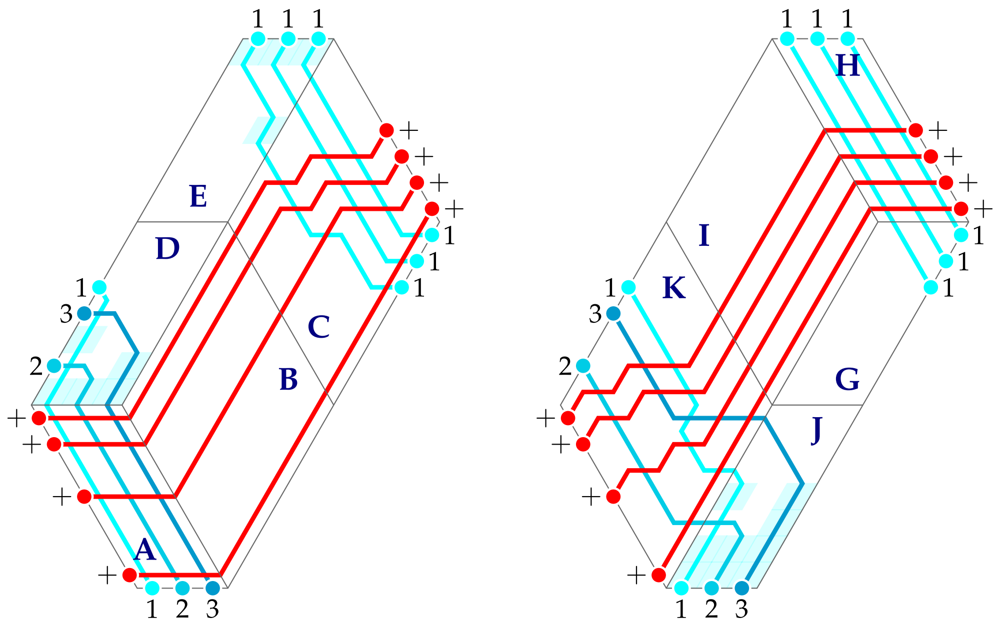


Figure 4: Two fillings of weight $x_1^3x_2^2x_3^4$ from the configurations in Lemma 2 where $\lambda = (2,2,1)$ and $\alpha = (2,1,2)$, so that $n = 3$ and $k = 4$. Regions are labelled to facilitate exposition.

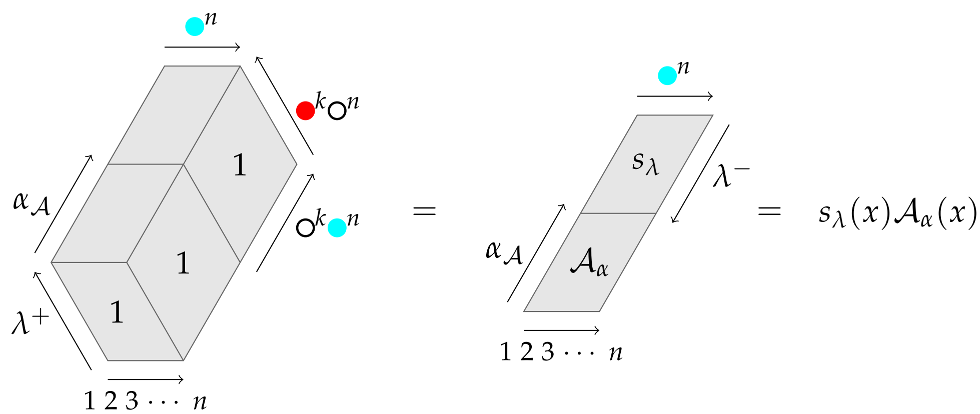
tile, moving right-to-left and top-to-bottom. The boundary conditions ensure that the South-West and North-East labels of columns we equate are always in $\{\circ, \bullet\}$ at every stage in this process. $\square$

The proof now follows from examining both sides of the equation in Lemma 2. In short, the left-hand side is manifestly the product $s_\lambda(x)\mathcal{A}_\alpha(x)$ and the right-hand side is manifestly the summation $\sum_\beta c^\beta_{\lambda,\alpha}\mathcal{A}_\alpha(x)$ where $c^\beta_{\lambda,\alpha}$ counts fillings of the Schur–atom model.

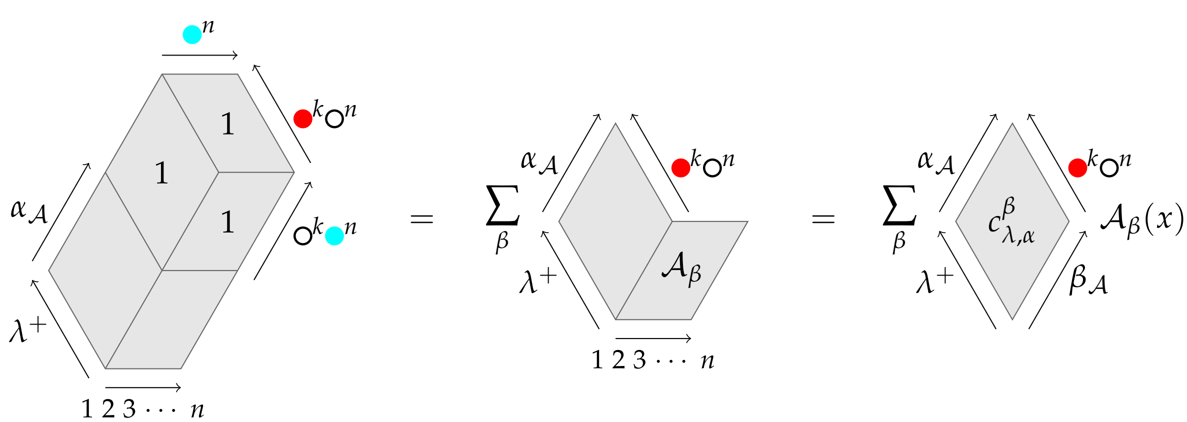
Proof of Theorem 1. We examine both sides of the equation in Lemma 2, which is better illustrated with the example fillings in Figure 4. Within region **A**, all red lines must move East and blue lines must move North-West. Those red lines must move straight North-East through **B**, transmitting the string λ^+ to the South-West boundary of **C**. From Lemma 9 in [11], there is only one way to fill **C**, which forces the shared boundary between **C** and **E** to be the string λ^- upside-down. Thus, regions **A**, **B** and **C** have weight 1.

Next, we recognize region **D** as our vertex model for the Demazure atom $\mathcal{A}_\alpha(x)$. From the previous paragraph, we have that λ^- is upside-down on the South-East boundary of **E**. Rotating region **E** by 180 degrees, we see that it is the vertex model for $s_\lambda(x)$

with the variables in reverse order; since $s_\lambda(x)$ is symmetric, the weight of the left-hand side is the product $s_\lambda(x)\mathcal{A}_\alpha(x)$. We summarize pictorially:



Considering the right-hand side, we have that regions **G**, **H** and **I** always have weight 1 and follow the same pattern as in Figure 4, transmitting the string ●^{k}○^{n} to the North-East boundary of region **K**. Within region **J**, all blue lines must exit the North-West boundary if they are to reach the North-West boundary of **K**. This follows from our choice of $k = \max(\lambda) + \max(\alpha)$. The blue lines then travel through the boundary between **J** and **K**, varying over strings $\beta_{\mathcal{A}}$ that encode weak compositions. Fixing a particular $\beta_{\mathcal{A}}$ along this boundary, we recognize region **J** as $\mathcal{A}_\beta(x)$ and region **K** as the Schur–atom model from Theorem 1. Thus, the right-hand side is a summation over compositions β where each summand is a product of our Schur–atom model and $\mathcal{A}_\beta(x)$. We give another pictorial summary:



By Lemma 2, we can equate both sides, completing the proof. As a final note, we used that $k = \max(\lambda) + \max(\alpha)$ in our proof, but when considering the filling of a particular diamond where β is given, it suffices to set $k = \max(\beta)$ as we do in Theorem 1. $\square$

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On the f-vectors of flow polytopes for the complete graph

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Abstract. The Chan-Robbins-Yuen polytope (CRY_n) of order n is a face of the Birkhoff polytope of doubly stochastic matrices that is also a flow polytope of the directed complete graph K_{n+1} with netflow $(1, 0, 0, \ldots, 0, -1)$. The volume and lattice points of this polytope have been actively studied, however its face structure has received less attention. We give generating functions and explicit formulas for computing the f-vector by using Hille's (2003) result bijecting faces of a flow polytope to certain graphs, as well as Andresen-Kjeldsen's (1976) result that enumerates certain subgraphs of the directed complete graph. We extend our results to flow polytopes over the complete graph having arbitrary (non-negative) netflow vectors and recover the f-vector of the Tesler polytope of Mészáros–Morales–Rhoades (2017).

Keywords: Chan-Robbins-Yuen polytope, flow polytopes, complete graphs, Fishburn matrices

1 Introduction

The Chan-Robbins-Yuen polytope (CRY_n) of order n is defined as the convex hull of n by n permutation matrices π for which $\pi_{i,j} = 0$ for $j \geq i + 2$ [6]. This polytope has been the object of much interest in the research community, as it possesses many interesting traits. For example, Zeilberger proved in [17] using a variation of the Morris constant term identity that CRY_n has normalized volume equal to the product of the first $n-2$ Catalan numbers. A second algebraic proof was provided in [2], though a combinatorial proof of this fact remains elusive. CRY_n is also a face of the Birkhoff polytope of doubly stochastic matrices having dimension $\binom{n}{2}$ and 2^{n-1} vertices [6].

CRY_n is also an example of a more general family of polytopes, namely those which are flow polytopes of the complete (transitively directed) graph K_{n+1} on vertex set $\{v_1, \ldots, v_{n+1}\}$, which include the family of Tesler polytopes [11].

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Definition 1.1. For $n \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{N}^n$, we denote the flow polytope $\mathcal{F}_{K_{n+1}}(\mathbf{a}, -\sum_{i=1}^n a_i)$ as $\mathbf{Flow}_n(\mathbf{a})$. We will denote the f-vector of $\mathcal{F}_{K_{n+1}}(\mathbf{a}, -\sum_{i=1}^n a_i)$ by $f^{(n)}(\mathbf{a})$ or ($f^{(n)}(\mathbf{a}; x)$ if written as a Laurent polynomial, where the coefficient of x^i gives the number of i-dimensional faces for $i \geq -1$).

In particular, CRY_n is realized as an instance of $\mathbf{Flow}_n(\mathbf{a})$ by setting $\mathbf{a} = (1, 0, \ldots, 0)$. $\mathbf{Flow}_n(\mathbf{a})$ has also been studied by Mészáros–Morales–Rhoades [11] in the context of **Tesler polytopes**, in which they show that the case of all $a_i > 0$, such as $\mathbf{a} = (1, 1, \ldots, 1)$, is combinatorially equivalent to a product of simplices $\Delta_n \times \Delta_{n-1} \times \ldots \times \Delta_1$. This was later generalized to other graphs by Mészáros–Simpson–Wellner [12]. Part of the difficulty in obtaining the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ for more general $\mathbf{a}$ arises from the fact that $\mathbf{Flow}_n(1, 1, \ldots, 1)$ is simple, whereas general instances of $\mathbf{Flow}_n(\mathbf{a})$ (including the case of CRY_n) are not.

In this manuscript, we give an explicit formula for the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ for any non-negative $\mathbf{a}$ as a sum over certain compositions. Namely, given a netflow vector $\mathbf{a}$, let $\mathrm{revcomp}(\mathbf{a})$ be the composition obtained by reading the entries of $\mathbf{a}$ from right to left, inductively creating blocks whenever a new nonzero entry is encountered, and recording the tuple of sizes coming from the list of blocks (see Example 2.10). Furthermore, let $\geq$ be the partial order of refinement on compositions, and let $\ell(\alpha)$ be the number of parts of composition α.

Theorem 1.2. *Given a netflow vector $(\mathbf{a}, -\sum_{i=1}^n a_i) = (a_1, \ldots a_n, -\sum_{i=1}^n a_i)$ with $a_i \in \mathbb{N}$, let α be the integer composition of n given by $\alpha = \mathrm{revcomp}(\mathbf{a})$. Then the f-vector Laurent polynomial of $\boldsymbol{Flow}_n(\mathbf{a})$ is given by:*

$$f(\mathbf{a}; x) = \frac{1}{x} + \frac{1}{x^n} \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \pi_{\ell(\beta)}(x) \mathbf{x}^{\beta - \mathbf{1}}|_{x_i = (x+1)^i - (x+1)} \tag{1.1}$$

where $\pi_n(x) := x^n [n]_{x+1}! = \prod_{i=1}^n ((x+1)^i - 1)$.

The reader may notice that equation (1.1) looks almost like an evaluation of a quasisymmetric function. We will discuss this viewpoint in Section 2.2.

Note that in the case of $a_i > 0$ for all i, we recover the results of [11, Thm 1.7] that $f(\mathbf{a}; x) = [n]_{x+1}!$, a consequence of $\mathbf{Flow}_n(\mathbf{a})$ being combinatorially equivalent to a product of simplices $\Delta_n \times \Delta_{n-1} \times \ldots \times \Delta_1$ as referenced above. In the case that $\mathbf{a} = (1, 0, \ldots, 0)$, we obtain a succinct formula for the previously-unknown f-vector of CRY_n as a sum over complete homogeneous symmetric functions $h_m(\mathbf{x}) := \sum_{1 \leq i_1 \leq \ldots \leq i_n} x_{i_1} \cdots x_{i_n}$.

Corollary 1.3. *Let $f^{(n)}(x)$ be the f-vector of $CRY_n = \boldsymbol{Flow}_n(1, 0, \ldots, 0)$ written as a Laurent polynomial. Then for all $n \geq 1$:*

$$f^{(n)}(x) = \frac{1}{x} + \frac{1}{x^n} \sum_{m=0}^{n-2} (-1)^m (1+x)^m \pi_{n-m}(x) \cdot h_m((x+1)^1 - 1, (x+1)^2 - 1, \ldots, (x+1)^{n-m-1} - 1) \tag{1.2}$$

This is a direct generalization of a theorem due to Andresen–Kjeldsen [1, Prop. 3.3] (which is recovered by setting $x = 1$) enumerating certain subgraphs of K_{n+1}. In their paper, the authors of [1] study two families of subgraphs originating from their prior work in automata theory:

$$\Omega_n := \{H \subseteq K_{n+1} \mid \text{every } v \in V(H) \text{ lies along a direct path from } v_1 \text{ to } v_{n+1}\}$$

and the following set of **primitive** subgraphs:

$$\Omega'_n := \{H \in \Omega_n \mid V(H) = \{v_1, \dots, v_{n+1}\}\}.$$

They then give formulas for the cardinalities $\psi_n := |\Omega_n|$ (c.f. [16, A005016]) and $\xi_n := |\Omega'_n|$ (c.f. [16, A005321]). For example, they show that:

$$\psi_n = \sum_{m=0}^{n-2} (-2)^m \pi_{n-m} \cdot h_m(2^1 - 1, 2^2 - 1, \dots, 2^{n-m-1} - 1) \tag{1.3}$$

where $\pi_n := \prod_{i=1}^{n}(2^i - 1)$. One may actually recover ψ_n from ξ_n (and vice versa), as shown in [1, eq. 1], which is a special case of our Corollary 3.2.

The connection between Corollary 1.3 and equation (1.3) is made explicit via the following powerful theorem of Hille [8], originally introduced in the context of quivers. Here a subgraph $H \subseteq G$ is **$\mathbf{a}$-valid** if H is the support of an $\mathbf{a}$-flow on G, and the **first Betti number of** H is $\beta_1(H) := |E(H)| - |V(H)| + c(H)$, where $c(H)$ is the number of connected components of H. See also [7].

Theorem 1.4 ([8]). *Let $\mathcal{F}_G(\mathbf{a})$ be a flow polytope such that $a_i \geq 0$ for all i. Then for $d \geq 0$, the d-dimensional faces of $\mathcal{F}_G(\mathbf{a})$ are in one-to-one correspondence with subgraphs $H \subseteq G$ such that H is $\mathbf{a}$-valid and $\beta_1(H) = d$. The empty face of $\mathcal{F}_G(\mathbf{a})$ corresponds to the empty subgraph of G.*

In this way, we see that the f-vector of CRY_n is exactly a generating function over Ω_n, where variable x keeps track of the first Betti number of $H \in \Omega_n$. This connection leads us to define the new notion of primitive f-vector of $\mathbf{Flow}_n(\mathbf{a})$ as follows.

Definition 1.5. The **primitive f-vector** of $\mathbf{Flow}_n(\mathbf{a})$, denoted $\widetilde{f}^{(n)}(\mathbf{a})$ (or as $\widetilde{f}^{(n)}(\mathbf{a}; x)$ if written as a polynomial) is a generating function over the set of $\mathbf{a}$-valid subgraphs of K_{n+1} that are primitive (use the entire vertex set) keeping track of the first Betti number.

Note that it follows immediately from the definition that $\widetilde{f}^{(n)}(1, 0, \dots, 0; x)|_{x=1} = \xi_n$ in the same way that $f^{(n)}(1, 0, \dots, 0; x)|_{x=1} = \psi_n$ from Theorem 1.4. See also Figure 1.

Later in the text, we describe closed-form expressions for the primitive f-vector of $\mathbf{Flow}_n(\mathbf{a})$ (Lemma 2.5 and Lemma 2.11) and describe a relationship between $f^{(n)}(\mathbf{a})$ and $\widetilde{f}^{(n)}(\mathbf{a})$ for arbitrary (non-negative) $\mathbf{a}$ (Lemma 2.6), as well as the special case of CRY_n (Corollary 3.2). Data for $f^{(n)}(1, 0 \dots, 0)$ and $\widetilde{f}^{(n)}(1, 0 \dots, 0)$ are included in Table 1 and Table 2, respectively.

A second, special relationship exists between the f-vector and primitive f-vector in the case of CRY_n, and specializes to [1, Prop. 4.1] of Andresen–Kjeldsen by setting $x = 1$:

Lemma 1.6. *For all $n \geq 1$, the f-vector and primitive f-vector of CRY_n are related as:*

$$xf^{(n)}(x) = (1+x)^{n-1}\widetilde{f}^{(n)}(x). \tag{1.4}$$

Finally, we remark that Jelínek [10] observed that Ω'_n is in fact in bijection with the set of **primitive Fishburn matrices** (upper triangular, 0-1 matrices such that no row nor column is the zero vector) , and consequently is related to the enumeration of interval orders [5], by interpreting $H \in \Omega'_n$ as the upper-triangular matrix determined by its edges. As discussed in [9], the bijections continue, as the more general notion of Fishburn matrices are in bijection with Stoimenow matchings, ascent sequences, and more [5, 15]. See [9, 10] for a more comprehensive list of related combinatorial objects.

Either from Corollary 1.3 or from a multivariate generating function of Fishburn matrices due to Jelínek [10, Thm. 2.1] one obtains the following nice generating function for d-dimensional faces of CRY_n for varying d and n.

Corollary 1.7. *The number of d-dimensional faces of CRY_n is given by the coefficient $f_d^{(n)} = [t^n x^d]F(t,x)$, where $F(t,x)$ is defined by:*

$$F(t,x) := \frac{1}{x-xt} + \sum_{n=0}^{\infty} t^n x^{-n} \prod_{i=1}^{n} \frac{(1+x)^i - 1}{1+((1+x)^i - 1 - x)tx^{-1}}. \tag{1.5}$$

The rest of this paper is organized as follows: In Section 2, we derive our main result Theorem 1.2 as well as results for general primitive f-vectors (Lemma 2.5 and Lemma 2.11) needed in the proof. We conclude in Section 3 by specializing our results to CRY_n.

2 Main Results

Remark 2.1. Notations and conventions: Our vector $\mathbf{a}$ used in this paper is often denoted $\widetilde{\mathbf{a}}$ in the flow polytope literature, as it does not account for the last vertex whose netflow is predetermined by the first n entries. Moreover, we note that in the case of $a_i \geq 0$ for all i as we are assuming in this manuscript, a consequence of Theorem 1.4 is that the combinatorial equivalence class of $\mathcal{F}_G\left(\mathbf{a}, -\sum_{i=1}^n a_i\right)$ is completely determined by the support of $\mathbf{a}$. Hence we may assume for the rest of the paper that $\mathbf{a} \in \{0,1\}^n$. An excellent source for any other unexplained terms and notation is [3].

We now describe various results that build towards Theorem 1.2.

2.1 Formulas as sums over subsets

In [1], the authors define certain sequences of numbers which prove useful for the exact enumeration of the sets Ω_n and Ω'_n (here we require a change of convention to non-increasing sequences instead of non-decreasing sequences).

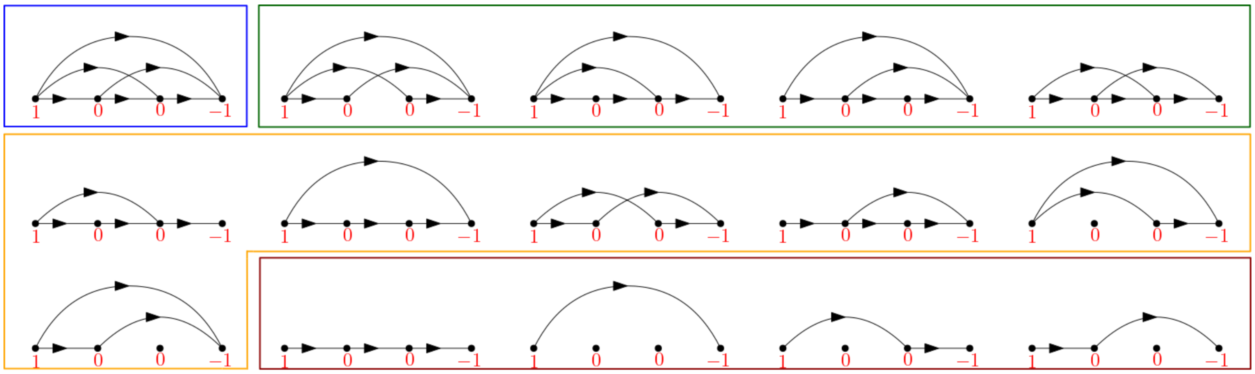

Figure 1: The elements of Ω_3 grouped by first Betti number, corresponding to the f-vector $(1,4,6,4,1)$ of CRY_3 (but excluding the empty face which would correspond to the empty graph). The primitive f-vector $(0,1,4,4,1)$ corresponds to the number of graphs in each grouping which use all vertices.

n	f-vector of CRY_n
1	1, 1
2	1, 2, 1
3	1, 4, 6, 4, 1
4	1, 8, 26, 45, 45, 26, 8, 1
5	1, 16, 98, 327, 681, 944, 897, 588, 262, 76, 13, 1
6	1, 32, 342, 1943, 6982, 17326, 31236, 42198, 43521, 34601, 21249, 10020, 3571, 933, 169, 19, 1

Table 1: The first few f-vectors of CRY_n.

Definition 2.2 ([1]). Let $S_{n,m}$ be the set of all sequences $(i_1, \ldots i_n)$ having length n such that:

(i) $i_1 = n - m$, (ii) $i_n = 1$, (iii) $i_j \geq i_{j+1} \geq i_j - 1$ for all $j < n$.

For our purposes, it will be simpler to think of the sequences in $S_{n,m}$ as subsets of $[n] := \{1, \ldots, n\}$ through the following correspondence, of which we omit the proof.

Lemma 2.3. *The map* $\operatorname{desc} : S_{n,m} \to \binom{[n-1]}{m}$ *mapping a sequence in* $S_{n,m}$ *to the set of indices of its descents is a bijection.*

From here on we will be more interested in the inverse bijection of Lemma 2.3, and hence will denote by $\operatorname{seq}_n : [n] \to \bigsqcup_{m=0}^{n} S_{n+1,m}$ the map that takes takes a subset of $[n]$ to its corresponding non-increasing sequence of length $n+1$.

Example 2.4. The following is an example of seq_4 applied to subsets of the set $[4]$ of cardinality 2:

$$\operatorname{seq}_4(\{1,2\}) = (3,2,1,1,1), \quad \operatorname{seq}_4(\{1,3\}) = (3,2,2,1,1), \quad \operatorname{seq}_4(\{1,4\}) = (3,2,2,2,1),$$
$$\operatorname{seq}_4(\{2,3\}) = (3,3,2,1,1), \quad \operatorname{seq}_4(\{2,4\}) = (3,3,2,2,1), \quad \operatorname{seq}_4(\{3,4\}) = (3,3,3,2,1).$$

n	$\widetilde{f}$-vector of CRY_n
1	$0,1$
2	$0,1,1$
3	$0,1,4,4,1$
4	$0,1,11,33,42,26,8,1$
5	$0,1,26,171,507,840,865,584,262,76,13,1$
6	$0,1,57,718,4017,12866,26831,39268,42211,34221,21184,10015,3571,933,169,19,1$

Table 2: The first few primitive f-vectors of CRY_n.

These are all the ingredients we need to write down a first formula for $\widetilde{f}^{(n)}(\mathbf{a};x)$.

Lemma 2.5. *For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ of length n, a formula for $\widetilde{f}^{(n)}(\mathbf{a};x)$ (that is, the primitive f-vector of $\boldsymbol{Flow}_n(\mathbf{a})$ written as a polynomial in x) is given by:*

$$\widetilde{f}^{(n)}(\mathbf{a};x) = \frac{1}{x^n} \sum_{S \in [\operatorname{supp}(\mathbf{a}'),[n-1]]} (-1)^{|S|+n+1} \prod_{j \in [n]} ((x+1)^{\operatorname{seq}_{n-1}(S)_j} - 1) \tag{2.1}$$

where $\mathbf{a}' = (a_2, a_3, \ldots a_n)$, supp *is the support function (namely* $\operatorname{supp}(\mathbf{a}')$ *returns the set of indices j such that $a_{j+1} \neq 0$), and $[\operatorname{supp}(\mathbf{a}'),[n-1]]$ is the interval of the Boolean lattice from* $\operatorname{supp}(\mathbf{a}')$ *to $[n-1]$.*

Proof sketch. The idea of the proof is to start with the set of all primitive subgraphs of K_{n+1} (not just $\mathbf{a}$-valid ones) and apply the principle of inclusion and exclusion in order to obtain the set of primitive subgraphs that are also $\mathbf{a}$-valid.

Associate to $T \subseteq \{v_2, \ldots v_n\}$ its indicator set $S_T \subseteq [n-1]$ in the canonical way (namely $i \in S_T$ if and only if $v_{i+1} \in T$). For each such S, define R_S to be the set of primitive subgraphs of K_{n+1} such that $i \in S^c$ implies $indeg(v_{i+1}) = 0$, where $indeg(v_{i+1})$ is the in-degree of vertex v_{i+1}. Then $S_1 \subseteq S_2$ implies $R_{S_1} \subseteq R_{S_2}$, and so the set $Prim_{\mathbf{a}}$ of $\mathbf{a}$-valid primitive subgraphs of K_{n+1} may be found via inclusion-exclusion:

$$|Prim_{\mathbf{a}}| = \sum_{S \in [\operatorname{supp}(\mathbf{a}'),[n-1]]} (-1)^{|S|+n+1} |R_S|, \tag{2.2}$$

where the lowest set in the interval is $\operatorname{supp}(\mathbf{a}')$ since the elements of any subset of $R_{\operatorname{supp}(\mathbf{a}')}$ are $\mathbf{a}$-valid. Finally, if we let $r_S(x)$ be the generating function over the set R_S that keeps track of the sum of all outdegrees of each graph in R_S, then a modified argument as that appearing in the proof of [1, Prop 3.2] gives that:

$$r_S(x) = \prod_{j \in [n]} ((x+1)^{\operatorname{seq}_{n-1}(S)_j} - 1). \tag{2.3}$$

Combining equations (2.2) and (2.3) gives a generating function over the set $Prim_{\mathbf{a}}$ keeping track of the sum of all outdegrees of each graph. Finally, since our graphs are primitive, the first Betti number of each graph is exactly the sum of all outdegrees minus n, from which the final formula follows. □

The next result describes how the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ may be obtained easily as a sum of primitive f-vectors.

Lemma 2.6. *For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ of length n:*

$$f^{(n)}(\mathbf{a};x) = \frac{1}{x} + \sum_{\mathbf{b} \leq \mathbf{a}} \widetilde{f}^{(|\mathbf{b}|)}(\mathbf{b};x) \tag{2.4}$$

where $\mathbf{b} \leq \mathbf{a}$ if $\mathbf{b}$ can be obtained from $\mathbf{a}$ by deleting some subset (possibly empty) of the zeros in $\mathbf{a}$ and where $|\mathbf{b}|$ is the length of $\mathbf{b}$.

Proof sketch. Let F be a face of $\mathbf{Flow}_n(\mathbf{a})$. If F is the empty face, then it does not correspond to a primitive graph and hence contributes a term of $\frac{1}{x}$ to $f^{(n)}(\mathbf{a};x)$. Otherwise, F is non-empty and hence corresponds to an $\mathbf{a}$-valid subgraph $H \subseteq K_{n+1}$ by Theorem 1.4. Let $S_H \subseteq \{v_1,\ldots,v_{n+1}\}$ be the set of vertices which are part of the support of a flow determining H. Then H is a primitive graph when restricted to the vertex set S_H, hence is counted by $\widetilde{f}^{(|b|)}(\mathbf{b};x)$ for some $\mathbf{b}$ determined by S_H. The possible $\mathbf{b}$'s that can appear are exactly those described in the lemma statement. □

Example 2.7. As an example, Lemma 2.6 would give us the following equivalence:

$$\begin{aligned} f^{(6)}(1,0,0,1,1,0;x) = \frac{1}{x} + \widetilde{f}^{(6)}(1,0,0,1,1,0;x) + 2\widetilde{f}^{(5)}(1,0,1,1,0;x) + \widetilde{f}^{(5)}(1,0,0,1,1;x) \\ + \widetilde{f}^{(4)}(1,1,1,0;x) + 2\widetilde{f}^{(4)}(1,0,1,1;x) + \widetilde{f}^{(3)}(1,1,1;x) \end{aligned}$$

where the coefficient 2 arises in front of $\widetilde{f}^{(5)}(1,0,1,1,0;x)$, for example, as there are two ways to delete zeros that result in this input vector.

2.2 Formulas as evaluations of sums of quasisymmetric polynomials

We can rewrite Lemma 2.5 as an evaluation of a certain polynomial by using the standard bijection of subsets of $[n-1]$ with integer compositions of n. Indeed, given a composition α and corresponding set S_α we define the multivariate polynomial:

$$P_\alpha(x_1,\ldots,x_n) := \sum_{\beta \geq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} \mathbf{x}^\beta \tag{2.5}$$

where $\mathbf{x}^\beta := x_1^{\beta_1} \cdots x_{\ell(\beta)}^{\beta_{\ell(\beta)}}$, and where the relation $\geq$ is the standard relation of *refinement* on compositions.

Remark 2.8. The polynomial P_α may look familiar to the reader. Indeed, we recall that the monomial quasisymmetric functions, M_α, and Gessel's fundamental quasisymmetric functions, F_α, are defined in infinitely many variables x_i respectively via:

$$M_\alpha := \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}, \qquad F_\alpha := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_k \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

A standard result of quasisymmetric functions describes how to write the monomial quasisymmetric functions in terms of Gessel's fundamental quasisymmetric functions and vice versa. Namely we have the equations (c.f. [14, 13]):

$$F_\alpha = \sum_{\beta \ge \alpha} M_\beta, \qquad M_\alpha = \sum_{\beta \ge \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta. \tag{2.6}$$

Hence, the polynomial $P_\alpha(x_1, \ldots x_n)$ from above is exactly the expansion of M_α into the fundamental basis, except that we only keep the first term of each F_β; that is:

$$P_\alpha(x_1, \ldots, x_n) = \sum_{\beta \ge \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta(x_1, \ldots, x_{\ell(\beta)}). \tag{2.7}$$

The polynomials P_α capture all of the data needed to compute the primitive f-vector $\widetilde{f}_n(\mathbf{a}; x)$.

Definition 2.9. For a subset $S \subseteq [n-1]$, let the **reverse** of S, denoted $\mathrm{rev}(S)$, be defined as:

$$\mathrm{rev}(S) = \{n - i \,|\, i \in S\} \tag{2.8}$$

For a natural number vector $\mathbf{a} := (a_1, \ldots, a_n)$, we define the **reverse composition**, denoted $\mathrm{revcomp}(\mathbf{a})$, as the composition corresponding to the set $\mathrm{rev}(\mathrm{supp}(\mathbf{a}))$. Computationally, $\mathrm{revcomp}(\mathbf{a})$ may be obtained quickly by reading the entries of $\mathbf{a}$ from right to left, inductively creating blocks whenever a new nonzero entry is encountered, and recording the tuple of sizes coming from the list of blocks.

Example 2.10. If $\mathbf{a} = (1,1,0,0,1,0,1,0)$, then when we read $\mathbf{a}$ right to left, we first encounter the block $(0,1)$, followed by $(0,1)$, followed by $(0,0,1)$ and finally (1). The reverse composition of $\mathbf{a}$ is then obtained by writing down the sizes of these blocks, hence $\mathrm{revcomp}(\mathbf{a}) = (2,2,3,1)$. For a non 0-1 vector, we may first replace every nonzero entry with a 1 and then perform the same procedure described here.

Lemma 2.11. *For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ of length n, let α be the composition of n given by $\alpha = \mathrm{revcomp}(\mathbf{a})$. Then the primitive f-vector of $\boldsymbol{Flow}_n(\mathbf{a})$ written as a polynomial is given by:*

$$\widetilde{f}^{(n)}(\mathbf{a}; x) = \frac{1}{x^n} P_\alpha(x, (x+1)^2 - 1, \ldots, (x+1)^n - 1) \tag{2.9}$$

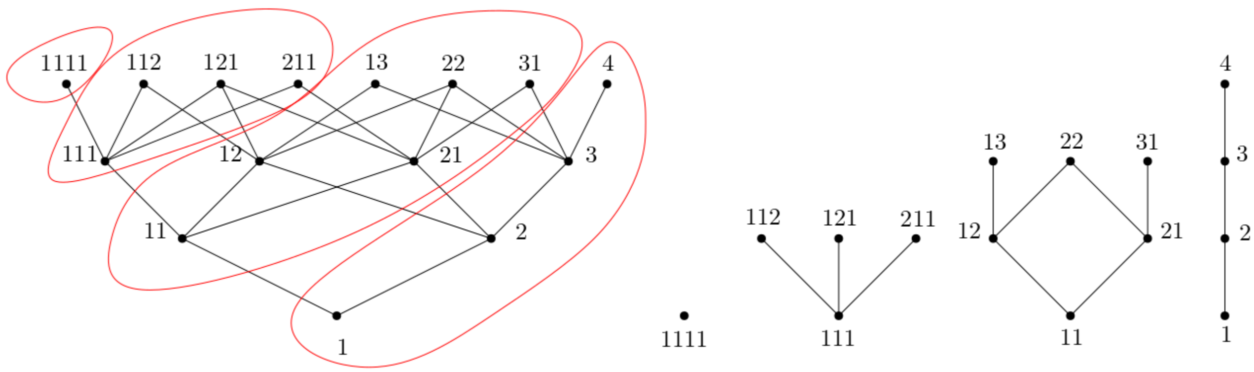

Figure 2: The composition poset $(\mathcal{C}, \leq)$ of [4] with lassos indicating the downsets determined by $\lessdot_1$ (left) and the coarsening $(\mathcal{C}, \leq_1)$ (right).

Proof sketch. The proof follows from Lemma 2.5 by applying the standard bijection between sets of size $n-1$ and compositions of n and interpreting all quantities involved. Each term of equation (2.1) translates to a term of P, and subset inclusion translates under this bijection to refinement of compositions. □

We may combine Lemma 2.6 and Lemma 2.11 to obtain an explicit formula for the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ for $\mathbf{a}$ non-negative, but first we need one more definition.

Aside from the refinement partial order on the set of integer compositions for some fixed n, recall that the set of *all* compositions (of all positive integers) forms a poset $\mathcal{C} := (\mathcal{C}, \leq)$ where there are two types of cover relations (which we will denote $\lessdot_1$ and $\lessdot_2$). For compositions α and β, these are described by ([4]):

- $\alpha \lessdot_1 \beta$ if β can be obtained from α by adding 1 to a part, and
- $\alpha \lessdot_2 \beta$ if β can be obtained from α by adding 1 to a part and then splitting this part into two parts.

Definition 2.12. Define $\widetilde{\mathcal{C}} := (\mathcal{C}, \leq_1)$ to be the coarsening of $\mathcal{C}$ by taking only the transitive closure of $\lessdot_1$. See Figure 2 for an example.

Lemma 2.13. *For all $n \in \mathbb{N}$ and non-negative $\mathbf{a} \in \mathbb{N}^n$, let α be the composition of n given by $\alpha = \operatorname{revcomp}(\mathbf{a})$. Then the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ written as a Laurent polynomial is given by:*

$$f^{(n)}(\mathbf{a}; x) = \frac{1}{x} + \frac{1}{x^n} \sum_{\beta \leq_1 \alpha} x^{|\alpha| - |\beta|} \left(\prod_{i=1}^{\ell(\alpha)} \binom{\alpha_i - 1}{\alpha_i - \beta_i - 1} \right) P_\beta(x, (x+1)^2 - 1, \ldots, (x+1)^{|\beta|} - 1) \quad (2.10)$$

Proof sketch. Combining the results of Lemma 2.6 and Lemma 2.11 we obtain:

$$f^{(n)}(\mathbf{a}; x) = \frac{1}{x} + \sum_{\mathbf{b} \leq \mathbf{a}} \frac{1}{x^{|\mathbf{b}|}} P_\beta(x, (x+1)^2 - 1, \ldots, (x+1)^{|\beta| - 1})$$

where $\leq$ is the partial order on 0-1 vectors described in Lemma 2.6 and $\beta = \mathrm{revcomp}(\mathbf{b})$. Translating both $\mathbf{a}$ and $\mathbf{b}$ into compositions via revcomp, we find that the resulting partial order is exactly that of $(\mathcal{C}, \leq_1)$ (see Figure 2). Indeed, the number of parts of $\mathrm{revcomp}(\mathbf{a})$ corresponds to the number of 1's appearing in $\mathbf{a}$, and deleting a 0 in $\mathbf{a}$ corresponds to decreasing the corresponding part of $\mathrm{revcomp}(\mathbf{a})$ by 1. In Lemma 2.6 we are only able to delete 0's and not 1's, hence the number of parts of $\mathrm{revcomp}(\mathbf{b})$ must be the same as the number of parts of $\mathrm{revcomp}(\mathbf{a})$. Finally, the product of binomial coefficients $\left(\prod_{i=1}^{\ell(\alpha)} \binom{\alpha_i - 1}{\alpha_i - \beta_i - 1}\right)$ keeps track of the number of ways of deleting 0's from $\mathbf{a}$ that result in the same $\mathbf{b}$, and the factor of $x^{|\alpha|-|\beta|}$ arises as a result of taking the common denominator of all terms. □

All of the work has now been done in order to prove our main result, Theorem 1.2.

Proof sketch of Theorem 1.2. Lemma 2.13 gives $f^{(n)}(\mathbf{a}; x)$ as a linear combination of P_β's coming from downsets of the poset $(\mathcal{C}, \leq_1)$. However, each P_β is also a sum over compositions (equation (2.5)). The result follows from expanding the P_β's, keeping track of all indices, and cancelling sums that telescope; see the upcoming full version of this text for more details. □

3 Formulas for CRY_n

Given the significance of CRY_n in the research community, we dedicate this section to the explicit formulas for CRY_n obtained by specializing the results in the previous section. We first obtain the following result by setting $\mathbf{a} = (1, 0, \ldots, 0)$ in Lemma 2.11. We remark that it is a generalization of [1, Prop. 3.2] which one can recover by setting $x = 1$.

Corollary 3.1. *Let $\widetilde{f}^{(n)}(x)$ be the primitive f-vector of CRY_n written as a polynomial. Then for all $n \geq 1$:*

$$\widetilde{f}^{(n)}(x) = \frac{1}{x^n} \sum_{m=0}^{n-1} (-1)^m \pi_{n-m}(x) \cdot h_m((x+1)^1 - 1, (x+1)^2 - 1, \ldots, (x+1)^{n-m} - 1). \tag{3.1}$$

Proof sketch. In the case of CRY_n, $\mathbf{a} = (1, 0, \ldots, 0)$, hence $\mathrm{revcomp}(\mathbf{a}) = (n)$. Hence $P_\alpha(x_1, \ldots, x_n)$ in Lemma 2.11 has a term for every integer composition of n. Factoring out $x_1 \cdots x_{n-m}$ from the terms coming from level $n - m$ in the poset of compositions by refinement leaves $h_m(x_1, \ldots, x_{n-m})$. We then evaluate each x_i in the same way as Lemma 2.11. □

We omit the proof of Corollary 1.3, as it follows similarly, except by specializing to $\mathbf{a} = (1, 0, \ldots, 0)$ in Theorem 1.2 instead of Lemma 2.11.

The following result is a specialization of Lemma 2.6 to the case of $\mathbf{a} = (1, 0, \ldots, 0)$ and further gives [1, eq. (1)] by summing over all d:

Corollary 3.2. *The f-vector and primitive f-vector of CRY_n satisfy* $f_d^{(n)} = \sum_{i=0}^{n-1} \binom{n-1}{i} \widetilde{f}_d^{(n-i)}$.

Proof. By Lemma 2.6, we can obtain $f^{(n)}(\mathbf{a};x)$ from $\widetilde{f}^{(n)}(\mathbf{a};x)$ by summing over all subsets of zeros in $\mathbf{a}$. For CRY_n, $\mathbf{a} = (1,0,\dots,0)$, so all possible subsets of 0's occur. $\square$

We conclude with a proof sketch of the intriguing relationship between $f^{(n)}$ and $\widetilde{f}^{(n)}$ described in Lemma 1.6.

Proof sketch of Lemma 1.6. The proof is analogous to that of [1, Prop. 4.1]. We have:

$$\begin{aligned}\frac{(1+x)^n}{x}\widetilde{f}^{(n-1)} - f^{(n)}(x) = \frac{(1+x)^n}{x} \cdot \frac{1}{x^{n-1}} \sum_{m=0}^{n-2} (-1)^m \pi_{n-m-1}(x) h_m((x+1)^1 - 1, \dots, (x+1)^{n-m-1} - 1)) \\ -\frac{1}{x} - \frac{1}{x^n} \sum_{m=0}^{n-2} (-1)^m (1+x)^m \pi_{n-m}(x) h_m((x+1)^1 - 1, \dots, (x+1)^{n-m-1} - 1).\end{aligned}$$

Which after algebraic manipulations simplifies to:

$$\frac{(1+x)^n}{x}\widetilde{f}^{(n-1)} - f^{(n}(x) = -\frac{1}{x} + \frac{1}{x^n} \sum_{m=0}^{n-2} \pi_{n-m-1}(x) h_m(-(x+1)^2 + x + 1, \dots, -(x+1)^{n-m} + x + 1)$$

We may use the path model for the complete homogeneous symmetric functions to rewrite this expression as:

$$\frac{(1+x)^n}{x}\widetilde{f}^{(n-1)} - f^{(n}(x) = -\frac{1}{x} + \frac{1}{x^n} \sum_{i=0}^{\infty} (N(x))_{1,i}^{n-1} \tag{3.2}$$

where $N(x)$ is the weighted adjacency matrix for the infinite path graph having a self loop at each vertex, with loop at vertex i ($i \geq 2$) having weight $(x+1)^i - 1$ and with edge $(i, i+1)$ having weight $-(x+1)^{i+1} + (x+1)$. In other words, $N(x)$ has the following form:

$$N(x) := \begin{bmatrix} 0 & x & 0 & \cdots & 0 & \\ 0 & -(x+1)^2 + (x+1) & (x+1)^2 - 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & 0 & \cdots \\ 0 & 0 & 0 & -(x+1)^i + (x+1) & (x+1)^i - 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{bmatrix}.$$

A simple induction shows that $\sum_{i=0}^{\infty} (N(x))_{1,i}^{n-1} = x^{n-1}$, and after plugging into equation (3.2), gives the result. $\square$

Acknowledgements

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Counting unicellular maps under cyclic symmetries

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Abstract. We count unicellular maps (matchings of the edges of a $2n$-gon) of arbitrary genus with respect to the $2n$-rotation symmetries of the polygon. An associated generating function that keeps track of the number of symmetric vertices of the resulting map generalizes the celebrated Harer-Zagier formula.

The answer to this enumerative question is not in the form of the usual cyclic sieving phenomenon (CSP), but does recover in the leading terms (genus-0 maps) a well known CSP for the Catalan numbers. The approach is representation theoretic, in that we relate symmetric unicellular maps with factorizations of the Coxeter element in a reflection group of type $G(m,1,n)$.

Keywords: Harer-Zagier formula, unicellular maps, reflection groups, cyclic sieving

1 Introduction

Unicellular maps are the 3-constellations of the form $\sigma\alpha c = \mathbf{1}$ where $\sigma, \alpha, c \in S_{2n}$, σ is a fixed point free involution, α an arbitrary permutation, and $c := (1, 2, \ldots, 2n)$ the long cycle. This corresponds to gluing the edges of a $2n$-gon (the gluing pattern is encoded in the involution σ).

The *genus* g of a unicellular map is given as $2g = n + 1 - \mathrm{cyc}(\alpha)$ (see also [6, p. 23]). The Harer-Zagier numbers $\varepsilon_g(n)$ count the unicellular maps with n edges and genus g and they have a very nice generating function formula:

$$\frac{1}{(2n-1)!!}\sum_g \varepsilon_g(n)\Phi_{n+1-2g}(X) = \frac{(1+X)^n}{(1-X)^{n+2}}, \tag{1.1}$$

where the polynomials $\Phi_n(X)$ are essentially the Eulerian polynomials; they are defined as follows:

$$\Phi_n(X) = \frac{\sum_{k=0}^{n-1} A(n,k)X^k}{(1-X)^{n+1}} \qquad \text{or equivalently} \qquad \Phi_n(X) = \sum_{k=0}^{\infty}(k+1)^n X^k, \tag{1.2}$$

where $A(n,k)$ is an Eulerian number (i.e., the number of permutations in S_n with k descents).

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Definition 1.1 (Rotation of constellations)**.** There is a natural cyclic action Ψ of order $2n$ on unicellular maps that corresponds to rotating the polygon. In terms of the constellation, the action is given as

$$\Psi\big[(\sigma,\alpha,c)\big] = (c^{-1}\sigma c, c^{-1}\alpha c, c).$$

To count symmetric 3-constellations, we essentially need to count the factorizations $\sigma\alpha c = \mathbf{1}$ that are fixed by *simultaneous* conjugation by some power c^N of c. Equivalently this means counting factorizations $\sigma\alpha c = \mathbf{1}$ in S_{2n} all of whose factors σ,α,c also belong to the centralizer $Z_{S_{2n}}(c^N)$. Now, the centralizer $Z_{S_{2n}}(c^N)$ is just the reflection group[1] $G(m,1,2n/m)$ where m is the order of c^N (i.e. $m = 2n/\gcd(2n,N)$). From now on, we will always assume that N divides $2n$ and we will always have $mN = 2n$.

That is, the problem of counting 3-constellations fixed under Ψ^r is equivalent to counting factorizations $\sigma\alpha c = \mathbf{1}$ in $G(m,1,N) = Z_{S_{2n}}(c^r)$ where σ belongs to the conjugacy classes of $G(m,1,N) \leq S_{2n}$ into which the class S_{2n} of fixed point free involutions has been decomposed. This problem turns out to be particularly easy because $c = (1,2,\ldots,2n)$ is a Coxeter element *also* in $G(m,1,N)$.

There is however a caveat: In the Harer-Zagier formula (1.1), the genus is directly related to the reflection length of α so we can keep track of it with representation theory. Here, the genus of a symmetric constellation is related to the length of α as an element in S_{2n} but this is not the same as (or a multiple of) its length as an element in $G(m,1,N)$. There are two natural approaches here; track the length as an element in $G(m,1,N)$ and interpret it as a *combinatorial* statistic on the map (this succeeds with Theorem 3.8) or define a new length function to track the genus and attempt to express it representation-theoretically (a first attempt here fails; we discuss it in Section 4).

We present the first approach in Section 3, where we interpret the usual length function for $G(m,1,N)$ as a combinatorial (but sadly not topological) statistic on the maps. Then, Zagier's proof [14] of the Harer-Zagier formula (1.1) generalizes essentially out of the box; we have existing theorems that replace all the ingredients of the proof and we prove Theorem 3.8 which is a direct generalization of (1.1).

In Section 4 we define a new length function for $G(m,1,N)$ that corresponds to the topological genus; it is a class invariant and is even somewhat compatible with a factorization in the group algebra of $G(m,1,N)$ which gives us some control over the formulas coming from the Frobenius lemma. It is not clear though what the analog of the Eulerian polynomials $\Phi_n(X)$ of (1.2) should be in this case (nor whether such an analog should a priori exist!).

We first start with a mini review of Zagier's proof of the Harer-Zagier formula (1.1) to set up a pattern of how the proofs would go in these two approaches.

[1]Note that the reflections of $G(m,1,N)$ do not come from transpositions of S_{2n}; they come from some elements of type $\big(2^m, 1^{2n-2m}\big)$ (the transposition-like ones) and some other ones –multiple cycle types– for the diagonal-like reflections; see Example 3.3 and Remark 3.4.

2 Main ingredients of Zagier's proof of the Harer-Zagier formula

We give in this section the main ingredients in Zagier's proof (or a re-imagining of Zagier's proof relying more on Jucys-Murphy elements). We will generalize each of them in the next section.

The first is a direct application of the Frobenius lemma from representation theory (recall: $n+1-2g = \mathrm{cyc}(\alpha) = 2n - \ell_R(\alpha)$).

$$\sum_g \varepsilon_g(n) X^{n+1-2g} = \frac{(2n-1)!!}{(2n)!} \cdot \sum_{\chi \in \widehat{S_{2n}}} \chi(\sigma)\chi(c) \cdot \widetilde{\chi}\left(\sum_{w \in S_{2n}} w X^{2n-\ell_R(w)}\right), \tag{A}$$

where σ is any fixed point free involution in S_{2n}, c any fixed long cycle, and $\widetilde{\chi}$ denotes the normalized character χ (i.e. $\widetilde{\chi}(a) := \chi(a)/\chi(1)$ for an element $a \in \mathbb{C}[S_{2n}]$).

The second ingredient is a well known factorization in the symmetric group algebra:

$$\sum_{w \in S_{2n}} w X^{2n-\ell_R(w)} = X(X+J_2)(X+J_3)\cdots(X+J_{2n}), \tag{B1}$$

where $J_i := (1i) + \cdots + (i-1i)$ is the i-th Jucys-Murphy element. As an application of this factorization we know for instance that the normalized traces appearing in (A) are just binomials:

$$\frac{1}{(2n)!} \cdot \widetilde{\chi}_k\left(\sum_{w \in S_{2n}} w X^{2n-\ell_R(w)}\right) = \binom{X+2n-1-k}{2n}, \tag{B2}$$

where χ_k is the k-th exterior power of the reflection representation of S_{2n} (it is a direct application of the Murnaghan-Nakayama rule that only these irreducible characters are non-zero on the long cycle c).

The third ingredient is that the eulerian polynomials of (1.2) give exactly the change-of-basis between the binomials in X that appear above and the monomials X^n:

$$\sum_{k=1}^{n} \varepsilon_k X^k = \sum_{k=1}^{n} b_k \binom{X+n-k}{k} \quad \text{if and only if} \quad (1-X)^{n+1} \sum_{k=1}^{n} \varepsilon_k \Phi_k(X) = \sum_{k=1}^{n} b_k X^{k-1}. \tag{C}$$

This has many proofs but it is very conveniently stated in Theorems 2.5 and 2.10 in [8].

The final ingredient is the usual relation (as in [2] or [3]) between the characters χ such that $\chi(c) \neq 0$, the Coxeter numbers $c_\chi = k(2n)$, the exterior powers χ_k, and hence the matrix of an element in the reflection representation of S_{2n}:

$$\sum_{\chi \in \widehat{S_{2n}}} \chi(\sigma)\chi(c) X^{\frac{c_\chi}{2n}} = \sum_{k=0}^{2n-1} \chi_k(\sigma)(-1)^k X^k = \frac{\mathfrak{p}(\sigma; X)}{1-X}, \tag{D}$$

where $\mathfrak{p}(\sigma; X)$ is the characteristic polynomial of σ in the *standard* $(2n)$-dimensional representation of S_{2n}. Together (A),(B2),(C),(D) give us the Harer-Zagier formula (1.1) because $\mathfrak{p}(\sigma; X) = (1 - X^2)^n$.

3 Counting symmetric maps keeping track of $G(m, 1, N)$-length

In this section we generalize the Harer-Zagier formula (1.1) in a way that has all of the ingredients of Zagier's proof from the previous section working out of the box. To have a *meaningful interpretation* of the theorem however we will give first a combinatorial interpretation of the $G(m, 1, N)$-length.

Recall that the for the 3-constellation $\boldsymbol{\pi} = (\sigma, \alpha, c)$ the number $\mathrm{cyc}(\alpha)$ of cycles of α equals the number of vertices $\boldsymbol{v}(\boldsymbol{\pi})$ of the combinatorial map $\boldsymbol{\pi}$ and also that

$$n + 1 - 2g = 2n - \ell_{S_{2n}}(\alpha) = \mathrm{cyc}(\alpha) = \boldsymbol{v}(\boldsymbol{\pi}).$$

So, then the Harer-Zagier formula (1.1) can be rephrased as

$$\frac{1}{(2n-1)!!} \sum_v \mathcal{E}_v(n) \Phi_v(X) = \frac{(1+X)^n}{(1-X)^{n+2}}, \tag{3.1}$$

where $\mathcal{E}_v(n) = \varepsilon_{(n+1-2v)/2}(n)$ counts the number of unicellular maps $\boldsymbol{\pi}$ witn n edges and v vertices.

Now, we will give an explicit definition of unicellular maps with rotational symmetry at least m:

Definition 3.1. Let n, m, N be positive integers such that $mN = 2n$. We denote by $C^m(N)$ the number of 3-constellations $\boldsymbol{\pi} = (\sigma, \alpha, c)$ with factors from S_{2n} that are fixed by the operation Ψ^N (i.e. have symmetry at least m):

$$C^m(N) = \Big\{(\sigma, \alpha) \in S_{2n}^2 \mid \sigma\alpha c = \sigma^2 = \mathbf{1},\ \ell_{S_{2n}}(\sigma) = n,\ c^{-N}\sigma c^N = \sigma,\ c^{-N}\alpha c = \alpha\Big\}.$$

As we mentioned earlier, we can enumerate $C^m(N)$ by counting certain factorizations in $G(m, 1, N)$. The factors σ, α, c are still elements of $G(m, 1, N)$ and c is its Coxeter element, but the class in S_{2n} of fixed point free involutions σ breaks into multiple conjugacy classes (see Remark 3.4) and the new length $\ell_{G(m,1,N)}(\alpha)$ is not a function of g (or equivalently $\boldsymbol{v}(\boldsymbol{\pi})$). For this reason we define these two statistics:

Definition 3.2. Let n, m, N be positive integers such that $mN = 2n$ and let σ be a fixed point free involution of S_{2n} such that $c^{-N}\sigma c^N = \sigma$. We write $\boldsymbol{d_m(\sigma)}$ for the number of Ψ^N-orbits of *centrally symmetric* 2-cycles of σ. (A *centrally symmetric* transposition is one of the form $(i, n+i)$.)

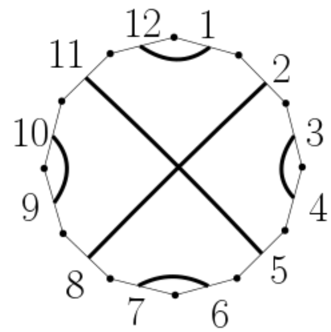

Figure 1: For the involution σ of the figure, we have $d_4(\sigma) = 1$ but $d_2(\sigma) = 2$.

Example 3.3. Consider the involution $\sigma := (1, 12)(2, 8)(3, 4)(5, 11)(6, 7)(9, 10)$ of S_{12}. There are two *centrally symmetric* 2-cycles: $(2, 8)$ and $(5, 11)$. The involution is symmetric both under Ψ^3 (conjugation by c^3 or rotation of order $m = 4$) and under Ψ^6 (conjugation by c^6 or rotation of order $m = 2$). But the cycles $(2, 8)$ and $(5, 11)$ form two orbits under Ψ^6 but only one orbit under Ψ^3. See Figure 1.

Remark 3.4 (d_m detects conjugacy class in $G(m, 1, N)$)**.** The point of this definition is that it detects the conjugacy class of the involution σ as an element of $G(m, 1, N)$. **The number $d_m(\sigma)$ counts on how many indices from 1 to N the involution σ acts diagonally-like (maps i to $-i$).** For Example 3.3 above, the centralizer $Z_{S_{2n}}(c^3)$ is isomorphic to the group $G(4, 1, 3)$ where the coordinates of the (3-dimensional ambient space) correspond to the three sets $\{1, 4, 7, 10\}$, $\{2, 5, 8, 11\}$, $\{3, 6, 9, 12\}$. In this case σ becomes $(1, 3^{-i})(2, \bar{2})$: the first 2-cycle $(1, 3^{-i})$ corresponds to the part $(1, 12)(4, 3)(7, 6)(10, 9)$ and the 2-cycle $(2, \bar{2})$ corresponds to the part $(2, 8)(5, 11)$. Then, the d_4 value here is $d_4(\sigma) = 1$ because the involution σ has a single *diagonal* position in $G(4, 1, 3)$.

Similarly the centralizer $Z_{S_{2n}}(c^6)$ is isomorphic to the group $G(2, 1, 6)$ with coordinates corresponding to the three sets $\{1, 7\}$, $\{2, 8\}$, $\{3, 9\}$, $\{4, 10\}$, $\{5, 11\}$, $\{6, 12\}$. In this case σ becomes $(1, \bar{6})(2, \bar{2})(3, 4)(5, \bar{5})$ and thus $d_2(\sigma) = 2$ since σ has two *diagonal* positions in $G(2, 1, 6)$.

We need to also replace the quantity $v(\boldsymbol{\pi})$ (the number of vertices of the map $\boldsymbol{\pi}$) with a new object that keeps track of the rotational symmetry of the vertices of the polygon that were identified into vertices of the map.

Definition 3.5. For any 3-constellation $\boldsymbol{\pi} = (\sigma, \alpha, c)$ in S_{2n}, and any numbers m, N such that $mN = 2n$, we define $v^m_{\text{free}}(\boldsymbol{\pi})$ to be the number of vertices of $\boldsymbol{\pi}$ (equivalently cycles of α) that are not fixed by *any* power of Ψ^N (apart from of course $\Psi^{Nm} = \text{Id}$).

Proposition 3.6. *If a 3-constellation $\boldsymbol{\pi} = (\sigma, \alpha, c)$ in S_{2n} is fixed under some power Ψ^N, then if m is such that $mN = 2n$,*

$$\ell_{G(m,1,N)}(\alpha) = \frac{2n - v^m_{\text{free}}(\boldsymbol{\pi})}{m}.$$

Before finally stating the main theorem of this section, we need to define the generalizations of the polynomials $\Phi_n(X)$ of (1.2). We will be using a well known generalization of Eulerian polynomials for $G(m,1,N)$ that encodes the notion of descent due to Steingrímsson [12].

Definition 3.7. For any two positive integers m, N we define the polynomials

$$\Phi_{m,N}(X) = \frac{\sum_{k=0}^{N} A(m,N,k)X^k}{(1-X)^{N+1}} \quad \text{or equivalently} \quad \Phi_{m,N}(X) = \sum_{k=0}^{\infty}(mk+1)^N X^k,$$

where $A(m,N,k)$ is the number of elements in $G(m,1,N)$ with k descents, see [12, Thm. 17].

With these interpretations, we are ready to state and give a (sketch of the) proof of the following generalization of the Harer-Zagier theorem (1.1) that counts maps that remain invariant under a given rotation of the initial polygon.

Theorem 3.8. *For any $n, m, N, k \in \mathbb{Z}_{>0}$ such that $2n = mN$, the numbers $\mathcal{E}_{k,v}(m,N)$ of 3-constellations $\boldsymbol{\pi} = (\sigma, \alpha, c)$ in S_{2n} with $d_m(\sigma) = k$ and $v^m_{\text{free}}(\boldsymbol{\pi}) = mv$ (see Defn. 3.2 and Defn. 3.5) such that $\Psi^N(\boldsymbol{\pi}) = \boldsymbol{\pi}$ (see Defn. 1.1) can be calculated via:*

$$\frac{1}{\binom{N}{k}\cdot(N-k-1)!!\cdot m^{\frac{N-k}{2}}}\sum_v \mathcal{E}_{k,v}(m,N)\cdot\Phi_{m,v}(X) = \frac{1}{1-X}\cdot\left(\frac{1+X}{1-X}\right)^{\frac{N-k}{2}},$$

where the polynomials $\Phi_{m,v}(X)$ are as in Defn. 3.7.

Sketch. All the ingredients (A),(B2),(C),(D) are readily available. (A) is just the Frobenius lemma. For (B2) see [8, Prop. 3.2] but it can also be shown using the following version of (B1):

$$\sum_{w\in G(m,1,N)} wX^{N-\ell_{G(m,1,N)}(w)} = (X+J_1)(X+J_2)\cdots(X+J_n),$$

where $J_i = (1,i) + \cdots + (i-1, i^{\bar{\xi}}) + (i, i^{\xi}) + \ldots + (i, i^{\bar{\xi}})$ are a version of the JM elements. The approach of [10, Prop. 4.8] expresses the character values on these generalized Jucys-Murphy elements as certain content calculations, see also [9, Section 4.2] or [15].

The change-of-basis (C) is in Theorems 3.17 and 3.18 of [8]. The final ingredient (D) comes from our previous work, joint with Chapuy, in [2, Section 9.5.2] where we prove an *equality* in $G(m,1,N)$ between $\sum\chi(c)\chi$ and a virtual character that involves the exterior powers of certain N-dimensional representations that are analogues of the standard representation of S_N. □

Remark 3.9. The genus 0 case, or equivalently $\text{cyc}(\alpha) = n+1$, appears only if $v_{\text{free}}(\boldsymbol{\pi}) = n+1$ (no symmetry) or $v_{\text{free}}(\boldsymbol{\pi}) = n$ ($\boldsymbol{\pi}$ has some symmetry). In this way, Theorem 3.8 recovers the known symmetry count in the form of a CSP [11, §7] in the genus-0 case (there the matchings must be non-crossing and determine a (different) noncrossing partition of the odd vertices $1, 3, \ldots 2n-1$; it is this object that is studied in [11]).

Remark 3.10. The approach described above can give a complete version of Zagier's main theorem from [14] (i.e. for any conjugacy class of $G(m,1,N)$ not just the fixed point free involutions).

Remark 3.11. The approach of this section can be generalized to other factorization counting questions, where conjugation by the long cycle is a natural symmetry. For instance, in works of Goupil-Schaeffer [4] and Bernardi-Morales [1], one could try to count symmetric factorizations by transfering the question to some $G(m,1,n)$ group. Factorizations of the Coxeter element $c \in G(m,1,n)$ have been extensively studied by Lewis-Morales, where the authors also observe [7, §8.2] in their setting that the $G(m,1,n)$-factorizations cannot keep track of the topological genus of a corresponding map.

4 Counting symmetric maps keeping track of genus

The main disadvantage of Theorem 3.8 is that the enumeration cannot keep track of the topological genus of the map $\boldsymbol{\pi}$. We discuss in this section a partial attempt to resolve this. We define a new length function in $G(m,1,N)$ given as

$$\ell_{sp}(w) := \ell_{S_{mN}}(w),$$

that is the *symmetric length* of $w \in G(m,1,N)$ is its length as an element of S_{mN}. Notice that this is a class function since if two elements are conjugate in $G(m,1,N)$ then they are also conjugate in S_{mN} hence have the same length.

Then, a generalization of (1.1) in the spirit of Theorem 3.8 but using $\ell_{sp}(w)$ instead of $\ell_{G(m,1,N)}(w)$ would rely on understanding

$$\sum_{\chi \in \widehat{G(m,1,N)}} \chi(\sigma_k)\chi(c) \cdot \sum_{w \in G(m,1,N)} \frac{\chi(w)}{\chi(1)} X^{\ell_{sp}(w)},$$

where σ_k is any involution with $d_m(\sigma_k) = k$.

It is not difficult to see that there is a factorization

$$\begin{aligned}\sum_{w \in G(m,1,N)} wX^{\ell_{sp}(w)} = &\big[\mathbf{1} + X^{m-1}(11^{\xi}) + \ldots + X^{m-1}(11^{\bar{\xi}})\big] \times \\ &\times \big[\mathbf{1} + X^{m}(12) + \ldots + X^{m}(12^{\xi}) + \ldots + X^{m-1}(22^{\bar{\xi}})\big] \cdots\end{aligned}$$

where each reflection τ contributes the term $X^{\ell_{sp}}(\tau)$.

This factorization might be seen as an analogue of $(B1)$ and we can certainly calculate the corresponding traces for irreducible characters (either manually or by the techniques of [13, Lemma 3.7], or even by following Jucys original argument [5, Section 4] and relying on existing determinations of the eigenvalues of these generalized Jucys-Murphy elements on eigenvectors indexed by tuples of Young tableau as for instance [10]).

However, we have no analogue of $(B2)$: Even though Sage experiments suggest that we always have nice formulas for $\sum_{w\in G(m,1,N)} \widetilde{\chi}(w)X^{\ell_{sp}}(w)$, it is not clear that there exists a change of basis analogous to (C) (or even that one *might exist*: we need to transform more than n polynomials; the corresponding polynomials with $X^{\ell_{G(m,1,N)}}$ depend only on the Coxeter number of χ when $\chi(c)\neq 0$ but with ℓ_{sp} this is no longer true.

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Weak Bruhat interval 0-Hecke modules in finite type

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Abstract. We extend the weak Bruhat interval modules, defined for 0-Hecke algebras in type A, to arbitrary finite types. We determine structural properties, with a main focus on projective covers and injective hulls, for certain general families of these modules in a type-independent way. As an application, we recover a number of results for type A 0-Hecke modules in a uniform manner. We also obtain some further results relating to recently-introduced type A 0-Hecke modules.

Keywords: 0-Hecke algebra, Coxeter group, projective cover, quasisymmetric functions

1 Introduction

The 0-Hecke algebra $H_W(0)$ associated to a finite Coxeter group W is a certain deformation of the group algebra of W. Norton [20] classified the projective indecomposable modules and the simple modules over $H_W(0)$ up to isomorphism. Subsequently, Fayers [12] proved that $H_W(0)$ is a Frobenius algebra, and Huang [14] provided combinatorial interpretations of the projective indecomposable modules for each classical type.

In type A, the quasisymmetric characteristic [11] provides an isomorphism between the Grothendieck group of type A 0-Hecke modules and the ring of quasisymmetric functions. Due to this connection, the past decade has seen significant activity related to constructing 0-Hecke modules in type A that correspond to various notable bases of quasisymmetric functions; examples include [2, 5, 19, 21, 22]. There has also been a focus on understanding the structure of such modules, especially regarding indecomposability, projective covers and injective hulls, for example, Choi, Kim, Nam and Oh [8] applied the ribbon tableau model of [14] to obtain the projective covers of modules in [5, 21, 22, 23]. A notable development in this regard was the introduction of *weak Bruhat interval modules* by Jung, Kim, Lee and Oh [15]. These modules, defined in terms of intervals in the left weak Bruhat order on symmetric groups, provided a uniform approach to understanding modules associated to quasisymmetric functions. It was also determined

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in [15] how these modules behave under certain equivalences of categories introduced in [12]. This was applied in [15] to determine structural properties of certain families of modules by realising them as images under these functors of other families of modules for which corresponding properties were known.

In this extended abstract, we summarise results from [3]. We extend the definition of weak Bruhat interval modules to all Coxeter types, and we show the projective indecomposable $H_W(0)$-modules are weak Bruhat interval modules, as proven in type A in [15]. Throughout, we use a type-independent description of the projective indecomposable $H_W(0)$-modules in terms of right descents of elements of W. We extend certain results of [15] regarding equivalences of categories to all types and to quotients and submodules of weak Bruhat interval modules, and identify a type-independent indecomposability criterion that covers a number of the type A modules associated to quasisymmetric functions. We determine projective covers of a significant family of $H_W(0)$-modules in a type-independent way, and use our results concerning the equivalences of categories to obtain injective hulls of a related family of $H_W(0)$-modules. Our approach works directly with elements of W, and our results are stated in terms of right descent sets. Finally, we apply this approach to recover a number of results on indecomposability, projective covers and injective hulls for type A families of $H_W(0)$-modules in a uniform manner. We also obtain some new results for certain modules recently introduced in [19].

2 0-Hecke algebras and weak Bruhat interval modules

A *finite Coxeter system* (W, S) is a finite group W with generating set S satisfying the relations $s^2 = 1$ for all $s \in S$, and $(st)_{m(s,t)} = (ts)_{m(s,t)}$ for all pairs of distinct elements $s, t \in S$, where $m(s,t) = m(t,s) \in \mathbb{Z}_{\geq 2}$ and $(st)_{m(s,t)}$ denotes the alternating product of s and t with $m(s,t)$ factors. For $w \in W$, the length $\ell(w)$ of w is the minimal number of terms appearing in a product of elements of S equal to w; any such product with minimal number of terms is a *reduced word* for W.

An element $s \in S$ is a *left descent* of $w \in W$ if $\ell(sw) = \ell(w) - 1$, and a *right descent* of w if $\ell(ws) = \ell(w) - 1$. Let $\mathrm{D}_L(w)$ denote the set of left descents of w, and $\mathrm{D}_R(w)$ the set of right descents of w. For $I \subseteq S$, the *right descent class* $\mathcal{D}_I$ consists of the elements $w \in W$ such that $\mathrm{D}_R(w) = I$. Denote the union of right descent classes $\mathcal{D}_X$ such that $I \subseteq X \subseteq J$ by $\mathcal{D}_I^J$.

The *parabolic subgroup* W_I is the subgroup of W generated by I. Let $w_0(I)$ denote the longest element in W_I, that is, $\ell(w) < \ell(w_0(I))$ for all $w \in W_I \setminus \{w_0(I)\}$. Let w_0 denote the longest element in W, i.e., $w_0 = w_0(S)$.

2.1 0-Hecke algebras

Let $\mathbb{K}$ be any field. The *0-Hecke algebra* $H_W(0)$ of a finite Coxeter system (W, S) is the associative $\mathbb{K}$-algebra generated by $\{\pi_s : s \in S\}$ with relations

$$\pi_s^2 = \pi_s \quad \text{and} \quad (\pi_s \pi_t)_{m(s,t)} = (\pi_t \pi_s)_{m(s,t)}$$

for all distinct $s, t \in S$.

Let $\overline{\pi}_s$ denote $\pi_s - 1$. The algebra $H_W(0)$ is also generated by $\{\overline{\pi}_s : s \in S\}$. Given $w \in W$ with reduced word $w = s_1 \ldots s_k$, define π_w to be the product $\pi_{s_1} \cdots \pi_{s_k}$, and define $\overline{\pi}_w$ to be $\overline{\pi}_{s_1} \cdots \overline{\pi}_{s_k}$; note π_w and $\overline{\pi}_w$ are well-defined. The projective indecomposable $H_W(0)$-modules have the following description due to Norton [20].

Theorem 2.1. *[20, Theorem 4.12(2)] Let (W, S) be a finite Coxeter system and let $I \subseteq S$. The left ideal $\mathcal{P}_I := H_W(0)\pi_{w_0(I)}\overline{\pi}_{w_0(S\setminus I)}$ is a projective indecomposable $H_W(0)$-module with $\mathbb{K}$-basis $\{\pi_w \overline{\pi}_{w_0(S\setminus I)} : w \in \mathcal{D}_I\}$.*

The set $\{\mathcal{P}_I : I \subseteq S\}$ is a complete list of non-isomorphic projective indecomposable $H_W(0)$-modules. For $I \subseteq J \subseteq S$, let $\mathcal{P}_I^J$ denote the $H_W(0)$-module $H_W(0)\pi_{w_0(I)}\overline{\pi}_{w_0(S\setminus J)}$. The following result on $\mathcal{P}_I^J$ is analogous to [14, Theorem 3.2], and proved similarly. Modules isomorphic to $\mathcal{P}_I^J$ will play a significant role in our work.

Theorem 2.2. *Let $I \subseteq J \subseteq S$. Then $\mathcal{P}_I^J$ has a $\mathbb{K}$-basis*

$$\{\pi_w \overline{\pi}_{w_0(S\setminus J)} : w \in W \text{ with } I \subseteq D_R(w) \subseteq J\},$$

and decomposes as a direct sum of projective indecomposable modules via the formula

$$\mathcal{P}_I^J \cong \bigoplus_{I \subseteq X \subseteq J} \mathcal{P}_X.$$

2.2 Weak Bruhat interval modules

The *left weak Bruhat order* $\leq_L$ on W is the partial order defined by $u \leq_L v$ if some reduced word for u appears as a terminal segment in some reduced word for v. Given $u, v \in W$ with $u \leq_L v$, the *left weak Bruhat interval* is the set $[u, v]_L = \{w \in W : u \leq_L w \leq_L v\}$.

Definition 2.3. Let $[u, v]_L \subseteq W$. The *weak Bruhat interval module* $\mathrm{B}(u, v)$ is the vector space $\mathbb{K}[u, v]_L$ with $H_W(0)$-action defined by

$$\pi_s w = \begin{cases} w & \text{if } s \in \mathrm{D}_L(w), \\ sw & \text{if } s \notin \mathrm{D}_L(w) \text{ and } sw \in [u, v]_L, \\ 0 & \text{if } s \notin \mathrm{D}_L(w) \text{ and } sw \notin [u, v]_L \end{cases} \tag{2.1}$$

for all $s \in S$ and $w \in [u, v]_L$.

The type A case of Definition 2.3 is precisely [15, Definition 1(1)]. That (2.1) defines an action of $H_W(0)$ follows from Theorems 3.1 and 3.3 in [10].

Example 2.4. Let $W = \mathfrak{S}_4$; we write elements of symmetric groups in one-line/list notation, e.g. $s_2 = 1324$ and $s_2s_3s_2 = 1432$. Figure 1 shows the action of π_1, π_2 and π_3 (where π_i denotes π_{s_i}) on the basis $[1324, 1432]_L$ of $\mathrm{B}(1324, 1432)$ and the basis $[3142, 4231]_L$ of $\mathrm{B}(3142, 4231)$. Following [15] we draw Hasse diagrams from top to bottom, so the 0-Hecke operators move elements downwards (or send them to zero).

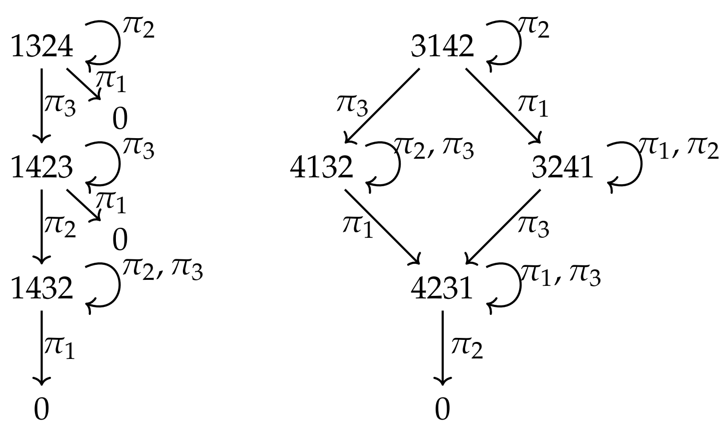

Figure 1: The $H_{\mathfrak{S}_4}(0)$-action on $\mathbb{K}$-bases for $\mathrm{B}(1324, 1432)$ and $\mathrm{B}(3142, 4231)$.

For $I \subseteq J \subseteq S$, the union $\mathcal{D}_I^J$ of right descent classes is an interval in left weak Bruhat order [6, Theorem 6.2]. In particular, each right descent class $\mathcal{D}_I$ itself is an interval in left weak Bruhat order.

Example 2.5. Figure 2 shows the poset $(\mathfrak{S}_4, \leq_L)$, in which we colour $\mathcal{D}_I$ using distinct colours for each I other than $I = \varnothing$ and $I = \{1, 2, 3\}$.

We now realise the $H_W(0)$-modules $\mathcal{P}_I$ and $\mathcal{P}_I^J$ as weak Bruhat interval modules. Denote the shortest element in $\mathcal{D}_I$ by u_I and the longest element in $\mathcal{D}_I$ by v_I. Note that $u_I = w_0(I)$ and $v_I = w_0\, w_0(S \setminus I)$.

Theorem 2.6. *Let $I \subseteq J \subseteq S$. Then $\mathcal{P}_I^J \cong \mathrm{B}(u_I, v_J)$.*

We henceforth denote $\mathrm{B}(u_I, v_J)$ by P_I^J and $\mathrm{B}(u_I, v_I)$ by P_I, to emphasise their nature as (direct sums of) projective indecomposable $H_W(0)$-modules.

Example 2.7. Consider the $H_{\mathfrak{S}_4}(0)$-module $\mathrm{B}(2134, 4231)$, and let i denote s_i. Since $2134 = u_{\{1\}}$ and $4231 = v_{\{1,3\}}$, we have $\mathcal{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{B}(2134, 4231) = \mathrm{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{P}_{\{1\}} \oplus \mathrm{P}_{\{1,3\}}$ by Theorems 2.2 and 2.6. Figure 3 depicts the basis elements for $\mathrm{P}_{\{1\}}^{\{1,3\}}$; the orange/pink

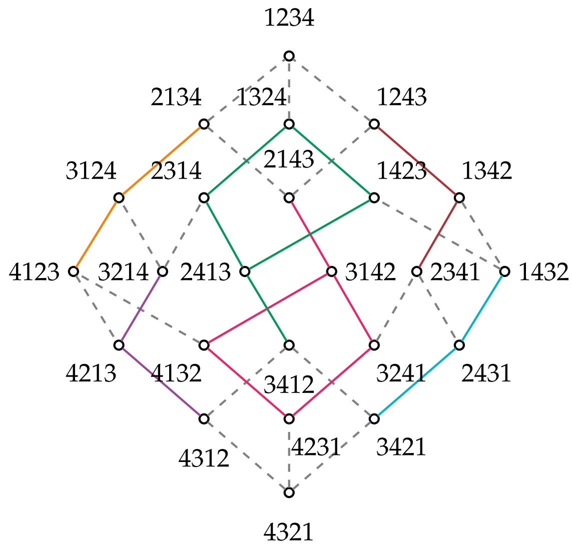

Figure 2: The poset $(\mathfrak{S}_4, \leq_L)$ and the right descent classes $\mathcal{D}_I$.

colour (cf. Figure 2) indicates $\mathrm{P}_{\{1\}}^{\{1,3\}}$ is isomorphic to the direct sum of $\mathrm{P}_{\{1\}}$ and respectively $\mathrm{P}_{\{1,3\}}$. Note however the basis elements of $\mathrm{P}_{\{1\}}$ do not span a submodule of $\mathrm{P}_{\{1\}}^{\{1,3\}}$.

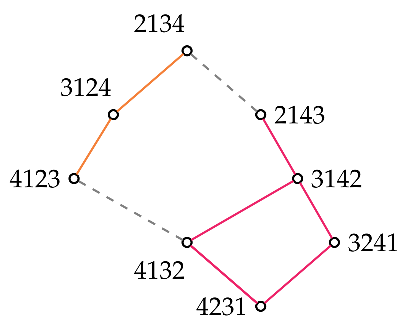

Figure 3: The $H_{\mathfrak{S}_4}(0)$-module $\mathrm{B}(2134, 4231) = \mathrm{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{P}_{\{1\}} \oplus \mathrm{P}_{\{1,3\}}$.

The following indecomposability criterion follows from the algebraic structure of $H_W(0)$; in particular since $H_W(0)$ is a Frobenius algebra [12] it is self-injective, and so the projective indecomposable modules are also injective indecomposable.

Proposition 2.8. *Every submodule and quotient of* P_I *is indecomposable.*

Specialising Proposition 2.8 to weak Bruhat interval modules obtains the following.

Proposition 2.9. *The weak Bruhat interval modules* $\mathrm{B}(w, v_I)$ *and* $\mathrm{B}(u_I, w)$ *are indecomposable for all* $w \in \mathcal{D}_I$*, and all submodules of* $\mathrm{B}(w, v_I)$ *and quotients of* $\mathrm{B}(u_I, w)$ *are also indecomposable.*

Several of the families of 0-Hecke modules for bases of quasisymmetric functions are isomorphic to weak Bruhat interval modules that are either submodules or quotient modules of some P_I. Applications of Proposition 2.9 will be given in Section 4.

2.3 Equivalences of categories

Fayers [12] introduced certain (dual) equivalences of the category $H_W(0)$-mod, as follows. Define involutions ϕ, θ and an anti-involution χ on $H_W(0)$ by

$$\phi : \pi_s \mapsto \pi_{w_0 s w_0}, \qquad \theta : \pi_s \mapsto 1 - \pi_s, \qquad \chi : \pi_s \mapsto \pi_s.$$

Let M be a $H_W(0)$-module. Define $\phi[M]$ and $\theta[M]$ to be the $H_W(0)$-modules whose underlying space is M, and whose actions $\cdot_\phi$ and $\cdot_\theta$ are given by $\pi_s \cdot_\phi m = \phi(\pi_s) \cdot m$ and $\pi_s \cdot_\theta m = \theta(\pi_s) \cdot m$, for $m \in M$. Define $\chi[M]$ to be the $H_W(0)$-module whose underlying space is the dual space M^* of M, with action given by $(\pi_s \cdot^\chi f)(m) = f(\chi(\pi_s) \cdot m)$, for $f \in M^*$ and $m \in M$. The functors $M \mapsto \phi[M]$ and $M \mapsto \theta[M]$ are self-equivalences of $H_W(0)$-mod, and the functor $M \mapsto \chi[M]$ is a dual equivalence of $H_W(0)$-mod.

Jung, Kim, Lee and Oh [15] determined the images of type A weak Bruhat interval modules under ϕ, θ and χ and their compositions; see [15, Table 1] for a summary. We extend this result on ϕ, $\hat{\theta} := \theta \circ \chi$, and $\hat{\omega} := \phi \circ \theta \circ \chi$ to arbitrary finite type, and moreover to quotients and submodules of weak Bruhat interval modules defined by upper order ideals in intervals in weak Bruhat order. The cases for the other compositions can be extended similarly by introducing *negative weak Bruhat interval modules* in arbitrary type: the type A definition is given in [15, Definition 1(2)], and the natural extension of this to finite type is well-defined by [10]. In this work, we do not use the negative analogue of weak Bruhat interval modules. For $Y \subseteq W$, let $w_0 Y w_0$ denote the set $\{w_0 y w_0 : y \in Y\}$. Similarly, $Y w_0 := \{y w_0 : y \in Y\}$, and $w_0 Y := \{w_0 y : y \in Y\}$. Note that if Y is an upper order ideal in $[u, v]_L$, then $\mathbb{K}Y$ is a submodule of $\mathrm{B}(u, v)$.

Theorem 2.10. *Let Y be an upper order ideal in $[u, v]_L$. Then we have the following isomorphisms of $H_W(0)$-modules.*

$$\phi[\mathrm{B}(u, v)/\mathbb{K}Y] \cong \mathbb{K}([w_0 u w_0, w_0 v w_0]_L \setminus w_0 Y w_0),$$
$$\hat{\theta}[\mathrm{B}(u, v)/\mathbb{K}Y] \cong \mathbb{K}([v w_0, u w_0]_L \setminus Y w_0),$$
$$\hat{\omega}[\mathrm{B}(u, v)/\mathbb{K}Y] \cong \mathbb{K}([w_0 v, w_0 u]_L \setminus w_0 Y).$$

We use Theorem 2.10 to determine the images of the modules P_I^J.

Corollary 2.11. *Let $I \subseteq J \subseteq S$. Then*

$$\phi[\mathrm{P}_I^J] \cong \mathrm{P}_{w_0 I w_0}^{w_0 J w_0}, \quad \hat{\theta}[\mathrm{P}_I^J] \cong \mathrm{P}_{S \setminus w_0 J w_0}^{S \setminus w_0 I w_0} \quad \text{and} \quad \hat{\omega}[\mathrm{P}_I^J] \cong \mathrm{P}_{S \setminus J}^{S \setminus I}.$$

Corollary 2.11 will be applied in Sections 3 and 4.

3 Projective covers and injective hulls

In this section, we determine the projective covers and injective hulls for significant families of $H_W(0)$-modules. A *projective cover* of a $H_W(0)$-module M is a projective $H_W(0)$-module P such that there is an epimorphism $f : P \to M$ whose kernel is contained in the radical of P. Projective covers exist since $H_W(0)$ is Artinian, and the projective module P is unique up to isomorphism. In [8, Section 5], Choi, Kim, Nam and Oh constructed projective covers for the 0-Hecke modules introduced by Tewari and van Willigenburg in [23], in terms of *generalised compositions*, using the ribbon tableau model of [14]. Our approach, similarly to [8], involves directly establishing radical membership; we work with and state results in terms of right descent sets.

The morphism $f : \mathrm{P}_I^J \to \mathrm{P}_I^J/\mathbb{K}Y$ given by $f(w) = w + \mathbb{K}Y$ is an epimorphism with kernel equal to $\mathbb{K}Y$.

Theorem 3.1. *Let Y be an upper order ideal in $\mathcal{D}_I^J$ with $u_J \notin Y$. Then P_I^J is the projective cover of $\mathrm{P}_I^J/\mathbb{K}Y$.*

Specialising Theorem 3.1 to weak Bruhat interval modules obtains the following.

Corollary 3.2. *Let $I \subseteq J$ and $w \in \mathcal{D}_J$. Then P_I^J is the projective cover of $\mathrm{B}(u_I, w)$.*

Remark 3.3. The type A case of Corollary 3.2 has been obtained independently, in the language of generalised compositions, by Kim, Lee and Oh in [16, Lemma 5.2].

Example 3.4. Consider the $H_{\mathfrak{S}_4}(0)$-module $\mathrm{B}(2134, 4132)$, and let i denote s_i. Since $2134 = u_{\{1\}}$ and $4132 \in \mathcal{D}_{\{1,3\}}$, by Corollary 3.2 we have that $\mathrm{P}_{\{1\}}^{\{1,3\}}$ is the projective cover of $\mathrm{B}(2134, 4132)$. The projective module $\mathrm{P}_{\{1\}}^{\{1,3\}}$ is depicted in Figure 3; note the appearance of the interval $[2134, 4132]_L$ in this figure.

An *injective hull* of a $H_W(0)$-module M is an injective $H_W(0)$-module Q together with a monomorphism $g : M \to Q$ such that the image of g has nontrivial intersection with every non-zero submodule of Q. The injective module Q is unique up to isomorphism.

Since $M \mapsto \hat{\omega}[M]$ is a dual equivalence of categories, P is the projective cover of M if and only if $\hat{\omega}[P]$ is the injective hull of $\hat{\omega}[M]$. The analogous statement holds for $M \mapsto \hat{\theta}[M]$. As an application of Theorem 2.10, we obtain the injective hulls of another significant family of $H_W(0)$-modules from Theorem 3.1.

Theorem 3.5. *Let Y be an upper order ideal in $\mathcal{D}_I^J$ with $v_I \in Y$. Then P_I^J is the injective hull of $\mathbb{K}Y$.*

The specialisation of Theorem 3.5 to weak Bruhat interval modules is as follows.

Corollary 3.6. *Let $I \subseteq J$ and $w \in \mathcal{D}_I$. Then P_I^J is the injective hull of $\mathrm{B}(w, v_J)$.*

4 Applications to modules for quasisymmetric functions

Much recent work has been devoted to constructing $H_{\mathfrak{S}_n}(0)$-modules whose images under the quasisymmetric characteristic [11] are important families of quasisymmetric functions. In this section, we apply results from Sections 2 and 3 to uniformly recover a number of results on indecomposability, projective covers, and injective hulls for various such modules. We also obtain new results concerning the modules associated to the recently-introduced row-strict dual immaculate functions and row-strict extended Schur functions of Niese, Sundaram, van Willigenburg, Vega, and Wang [18].

The $H_{\mathfrak{S}_n}(0)$-modules associated to quasisymmetric functions are usually stated in terms of *compositions of n*: sequences of positive integers that sum to n. Compositions of n are in bijection with subsets of $[n-1]$: if $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of n, then the associated subset $\mathrm{set}(\alpha)$ is $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. We denote the complement of $\mathrm{set}(\alpha)$ by $\mathrm{set}(\alpha)^c$. The *reversal* of α, denoted by α^r, is the composition obtained by reversing the sequence α.

Example 4.1. Let $\alpha = (1,3,2)$. Then $\mathrm{set}(\alpha) = \{1,4\}$ and $\alpha^r = (2,3,1)$.

As done in previous examples, we index projective indecomposable $H_{\mathfrak{S}_n}(0)$-modules by subsets of $[n-1]$, where i is understood to denote s_i. We first consider modules for the dual immaculate [4] and extended Schur [1] bases of quasisymmetric functions, and their row-strict analogues [19]. These modules are defined in terms of certain families of tableaux of shape α.

The *diagram* $D(\alpha)$ associated to a composition α is the left-justified array of boxes with α_i boxes in the ith row from the top. A *standard immaculate tableau* of shape α is a labelling of the boxes of $D(\alpha)$ by the integers $1, \ldots, n$, each used once, such that entries increase from left to right along rows and from top to bottom in the first column. A standard immaculate tableau is a *standard extended tableau* if the entries increase from top to bottom in every column. The set of standard immaculate tableaux of shape α, and its subset of standard extended tableaux, are denoted by $\mathrm{SIT}(\alpha)$ and $\mathrm{SET}(\alpha)$ respectively.

For $T \in \mathrm{SIT}(\alpha)$, the *reading word* $\mathrm{rw}(T)$ of T is the permutation obtained from reading the entries in each row in T from right to left, starting with the topmost row and iterating downwards. Let T_0^α and T_1^α be the elements of $\mathrm{SIT}(\alpha)$ with shortest, respectively, longest (Coxeter length) reading words, and let $\mathscr{T}_1^\alpha$ be the element of $\mathrm{SET}(\alpha)$ with longest reading word.

Example 4.2. The standard immaculate tableaux $\mathrm{SIT}(2,2)$ are shown in Figure 4. The standard extended tableaux $\mathrm{SET}(2,2)$ are the middle and rightmost tableaux. The leftmost tableau is T_1^α, the middle tableau is $\mathscr{T}_1^\alpha$, and the rightmost tableau is T_0^α. Their reading words, from left to right, are 4132, 3142, and 2143.

In [5], Berg, Bergeron, Saliola, Serrano and Zabrocki define a $H_{\mathfrak{S}_n}(0)$-action on the $\mathbb{K}$-span of $\mathrm{SIT}(\alpha)$, and show the quasisymmetric characteristics of the resulting modules

1	4		1	3		1	2
2	3		2	4		3	4

Figure 4: The three standard immaculate tableaux of shape $(2, 2)$.

$\mathcal{V}_\alpha$ are the dual immaculate functions of [4]. In [21], Searles defines an $H_{\mathfrak{S}_n}(0)$-action on the $\mathbb{K}$-span of $\mathrm{SET}(\alpha)$, and shows the quasisymmetric characteristics of the resulting modules X_α are the extended Schur functions of [1].

In [15, Theorem 5], Jung, Kim, Lee and Oh prove the isomorphisms

$$\mathcal{V}_\alpha \cong \mathrm{B}(\mathrm{rw}(T_0^\alpha), \mathrm{rw}(T_1^\alpha)) \quad \text{and} \quad X_\alpha \cong \mathrm{B}(\mathrm{rw}(T_0^\alpha), \mathrm{rw}(\mathscr{T}_1^\alpha)). \tag{4.1}$$

It is also shown in the proof of [15, Theorem 5] that reading words of $\mathrm{SIT}(\alpha)$ belong to $\mathcal{D}_{\mathrm{set}(\alpha)^c}$, and that $\mathrm{rw}(T_0^\alpha)$ is the shortest element of $\mathcal{D}_{\mathrm{set}(\alpha)^c}$.

Indecomposability of $\mathcal{V}_\alpha$ and X_α were proved in [5, Theorem 3.12] and, respectively, [21, Theorem 3.13]. Combining (4.1) and Proposition 2.9 we recover these results, and additionally that all quotients of these modules are also indecomposable.

Theorem 4.3. *For any composition α, the modules $\mathcal{V}_\alpha$, X_α, and all quotients of these modules are indecomposable.*

The projective covers for $\mathcal{V}_\alpha$ and X_α were established in [8, Theorem 3.2] and [8, Theorem 3.5]. One can recover these results by combining (4.1) and Corollary 3.2.

Theorem 4.4. *For any composition α, the projective cover of $\mathcal{V}_\alpha$ and of X_α is $\mathrm{P}_{\mathrm{set}(\alpha)^c}$.*

In [19], Niese, Sundaram, van Willigenburg, Vega and Wang define a $H_{\mathfrak{S}_n}(0)$-action on the $\mathbb{K}$-span of $\mathrm{SIT}(\alpha)$ (different from that of [5]), and obtain $H_{\mathfrak{S}_n}(0)$-modules $\mathcal{W}_\alpha$ whose quasisymmetric characteristics are the row-strict dual immaculate functions of [18]. The same action is defined on the $\mathbb{K}$-span of $\mathrm{SET}(\alpha)$ in [19], obtaining $H_{\mathfrak{S}_n}(0)$-modules $\mathcal{Z}_\alpha$ whose quasisymmetric characteristics are the row-strict extended Schur functions of [18].

Remark 4.5. We use $\mathcal{V}_\alpha$ to denote the modules for dual immaculate functions, following [5] and [15]. On the other hand, in [19], these modules are denoted $\mathcal{W}_\alpha$ and the modules for row-strict dual immaculate functions are denoted $\mathcal{V}_\alpha$. Therefore, our use of $\mathcal{V}_\alpha$ and $\mathcal{W}_\alpha$ is the reverse of that in [19].

To apply the results of Sections 2 and 3, we need to identify $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ as weak Bruhat interval modules. For $T \in \mathrm{SIT}(\alpha)$, define the *row-strict reading word* $\mathrm{rw}_{\mathcal{R}}(T)$ of T to be the permutation obtained by reading the entries of T from left to right along rows, beginning at the bottom row and proceeding to the top row. For example, the row-strict reading words of the tableaux in Example 4.2 from left to right are $2314, 2413$ and 3412.

Theorem 4.6. *For any composition α,*

$$\mathcal{W}_\alpha \cong \mathrm{B}(\mathrm{rw}_{\mathcal{R}}(T_1^\alpha), \mathrm{rw}_{\mathcal{R}}(T_0^\alpha)) \quad \text{and} \quad \mathcal{Z}_\alpha \cong \mathrm{B}(\mathrm{rw}_{\mathcal{R}}(\mathscr{T}_1^\alpha), \mathrm{rw}_{\mathcal{R}}(T_0^\alpha)),$$

and these Bruhat interval modules are submodules of $\mathrm{P}_{\mathrm{set}(\alpha^r)}$.

The indecomposability of $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ were proved in [19, Theorem 6.15] and [19, Theorem 7.13]. One can recover this via Theorem 4.6 together with Proposition 2.9, which additionally shows that any submodule of these modules is indecomposable.

Corollary 4.7. *For any composition α, the modules $\mathcal{W}_\alpha$, $\mathcal{Z}_\alpha$, and all submodules of these modules are indecomposable.*

Using Corollary 3.2, we determine the injective hulls of $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$.

Corollary 4.8. *For any composition α, the injective hull of $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ is* $\mathrm{P}_{\mathrm{set}(\alpha^r)}$.

Remark 4.9. In [3], we prove Theorem 4.6 by showing directly that $\{\mathrm{rw}_{\mathcal{R}}(T) : T \in \mathrm{SIT}(\alpha)\}$ is an interval in left weak Bruhat order, and then showing the action on $\mathrm{SIT}(\alpha)$ defined in [19] agrees with the action on this weak Bruhat interval module. Alternatively one can show these permutations form an interval in left weak Bruhat order by noting that $\mathrm{rw}_{\mathcal{R}}(T) = \mathrm{rw}(T)w_0$ and appealing to (4.1). It also follows that $\mathcal{W}_\alpha \cong \hat{\theta}[\mathcal{V}_\alpha]$ and $\mathcal{Z}_\alpha \cong \hat{\theta}[\mathcal{X}_\alpha]$. We note the fact that the modules $\mathcal{W}_\alpha$ and $\mathcal{Z}_\alpha$ can be obtained by applying $\hat{\theta}$ to $\mathcal{V}_\alpha$ and $\mathcal{X}_\alpha$ is observed in [9, Table 1].

For completeness, we also provide the projective cover of $\mathcal{W}_\alpha$. Choi, Kim, Nam, and Oh showed that the injective hull of $\mathcal{V}_\alpha$ is $\oplus_{\beta\in[\underline{\boldsymbol{\alpha}}]}\mathrm{P}_{\mathrm{set}(\beta)^c}$ [7, Theorem 4.1], where $[\underline{\boldsymbol{\alpha}}]$ is a particular set of compositions obtained from α; see [7, Section 4] for a full definition of $[\underline{\boldsymbol{\alpha}}]$. Applying $\hat{\theta}$ to this formula obtains the following.

Theorem 4.10. *For any composition α, the projective cover of $\mathcal{W}_\alpha$ is* $\oplus_{\beta\in[\underline{\boldsymbol{\alpha}}]}\mathrm{P}_{\mathrm{set}(\beta^r)}$.

As far as we are aware, the injective hull of X_α and projective cover of $\mathcal{Z}_\alpha$ are not currently known.

Finally, we consider a family of modules defined on *standard permuted composition tableaux* by Tewari and van Willigenburg in [23]. These modules are denoted $\mathbf{S}_\alpha^\sigma$, where α is a composition and σ a permutation (see [23, Section 3] for a full definition); and when σ is the identity it was shown in [22] that these correspond to the quasisymmetric Schur functions of [13]. These modules have a direct sum decomposition $\mathbf{S}_\alpha^\sigma = \oplus_E \mathbf{S}_{\alpha,E}^\sigma$, where each E is an equivalence class of standard permuted composition tableaux. Jung, Kim, Lee, and Oh define a reading word $\mathrm{rw}_{\mathcal{S}}$ on the standard permuted composition tableaux ([15, Definition 6]). Let τ_E (respectively, τ_E') denote the standard permuted composition tableau in E that has shortest (respectively, longest) reading word. It is proved in [15, Theorem 6] that

$$\mathbf{S}_{\alpha,E}^\sigma \cong \mathrm{B}(\mathrm{rw}_{\mathcal{S}}(\tau_E), \mathrm{rw}_{\mathcal{S}}(\tau_E')), \tag{4.2}$$

and that $\mathrm{rw}_{\mathcal{S}}(\tau_E)$ is the shortest element of some right descent class.

The projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$ was determined in [8, Theorem 5.11] in terms of a generalised composition associated to E. Combining (4.2) with Corollary 3.2 recovers this result, with a different statement in terms of right descent sets.

Theorem 4.11. *Let $rw_{\mathcal{S}}(\tau_E) \in \mathcal{D}_I$ and $rw_{\mathcal{S}}(\tau'_E) \in \mathcal{D}_J$. Then P^J_I is the projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$.*

The images of the modules $\mathbf{S}^{\sigma}_{\alpha}$ and $\mathbf{S}^{\sigma}_{\alpha,E}$ under $\hat{\omega}$ are a family of modules that generalise the modules introduced in [2] for the Young row-strict dual immaculate functions of [17]. Specifically, denoting these modules by $\mathbf{R}^{\sigma}_{\alpha}$ and $\mathbf{R}^{\sigma}_{\alpha,E}$, one has $\mathbf{R}^{\sigma}_{\alpha} \cong \hat{\omega}[\mathbf{S}^{w_0\sigma w_0}_{\alpha^r}]$, and for E an equivalence class of standard permuted composition tableaux corresponding to α_r and $w_0\sigma w_0$, $\mathbf{R}^{\sigma}_{\alpha,E} \cong \hat{\omega}[\mathbf{S}^{w_0\sigma w_0}_{\alpha^r,E}]$ ([15, Proposition 1]). The injective hull of $\mathbf{R}^{\sigma}_{\alpha,E}$ was determined in [15, Corollary 2], using $\hat{\omega}$. Applying $\hat{\omega}$ to Theorem 4.11 gives a description of the injective hull in terms of right descent sets.

Corollary 4.12. *Let E be an equivalence class of standard permuted composition tableaux corresponding to α^r and $w_0\sigma w_0$. Suppose $rw_{\mathcal{S}}(\tau_E) \in \mathcal{D}_I$ and $rw_{\mathcal{S}}(\tau'_E) \in \mathcal{D}_J$. Then $\mathrm{P}^{S\setminus I}_{S\setminus J}$ is the injective hull of $\boldsymbol{R}^{\sigma}_{\alpha,E}$.*

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The monoid representation of upho posets and total positivity

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Abstract. In this paper, we establish a bijection between finitary colored upho posets and left-cancellative invertible-free monoids. This bijection maps $\mathbb{N}$-graded colored upho posets to left-cancellative homogeneous monoids. We use this bijection and the new concept of semi-upho posets to prove that every totally positive power series is the rank-generating function of some upho poset, resolving a conjecture of Gao et al.

Keywords: upho posets, left-cancellative monoids, rank-generating functions, totally positive, log-concavity

1 Introduction

A poset is called *upper homogeneous*, abbreviated as *upho*, if each principal order filter is isomorphic to the poset itself. This concept was introduced by Richard Stanley during his research on the enumeration properties of Stern's triangle and its poset [13, 14].

The study of Stern's poset, particularly regarding enumeration problems, has been a focal point of numerous research efforts [15, 10]. Upho posets, as a generalization of Stern's poset, preserve the attribute of self-similarity, and hence exhibit many intriguing structural and enumeration properties. For example, in [4], Gao et al. give a concise characterization of $\mathbb{N}$-graded planar upho posets using their rank-generating functions; in [5], Hopkins proves that the characteristic generating functions of upho posets are the inverse of their rank-generating functions. Moreover, the breadth of applications for upho posets spans several domains, including lattice theory [5, 6], commutative algebra [4, Conjecture 1.1], and finite geometry [6, Theorem 1.3].

For simplicity, we say a formal power series is an *upho function* if it is the rank-generating function of some upho poset. A fundamental problem is:

Is there a criterion to determine if a formal power series is an upho function?

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In [4], Gao et al. prove that there are uncountably many different upho functions, and a straightforward corollary is that almost all upho functions are not rational functions. Hence, the complete characterization of upho functions is anticipated to be challenging.

In this paper, we prove the main conjecture proposed by Gao et al. in [4].

Theorem 1 ([4, Conjecture 3.3])**.** *A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is the rank-generating function of an upho poset P whose Ehrenborg quasi-symmetric function is a Schur-positive symmetric function if and only if $f(x)$ is totally positive.*

In Section 3, we introduce the notion of *colored upho posets* and establish the following correspondence, which is explained in detail later.

Theorem 2. *There is a bijection between finitary colored upho posets and left-cancellative invertible-free monoids. Moreover, this bijection maps $\mathbb{N}$-graded colored upho posets to left-cancellative homogeneous monoids, and maps finite type $\mathbb{N}$-graded colored upho posets to finitely generated left-cancellative homogeneous monoids.*

This correspondence plays an important role in understanding the self-similarity of upho posets: On one hand, we can do concrete calculations on upho posets using monoids from an algebraic perspective; on the other hand, we can get an intuition for the enumeration problems of left-cancellative monoids from a combinatorial perspective.

In Section 4, we explore the relationship between upho functions and total positivity.

Definition 1 ([8])**.** *A formal power series $\sum_{n=0}^{\infty} a_n x^n$ is* totally positive *if all finite minors of the infinite Toeplitz matrix*

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are nonnegative.

Our working definition of totally positive function follows from Theorem 3.

Theorem 3 ([1, 9])**.** *A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is totally positive if and only if $f(x)$ is of the form of $\frac{g(x)}{h(x)}$, where $g(x), h(x) \in 1 + x\mathbb{Z}[x]$ such that all the complex roots of $g(x)$ are real and negative, and all the complex roots of $h(x)$ are real and positive.*

By employing Theorem 2 and analyzing the rank-generating function of the newly defined concept of *semi-upho posets*, we prove the following theorem.

Theorem 4. *Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be totally positive. Then $f(x)$ is an upho function.*

And finally, we show that Theorem 1 is a corollary of Theorem 4.

The paper is structured as follows: Section 2 provides necessary background on upho posets and introduces the concept of semi-upho posets. Section 3 and Section 4 are dedicated to the exposition and proof of the aforementioned results.

2 Background on Upho Posets

2.1 Upho Posets

In this paper, we employ the standard terminology used in order theory. For a more detailed exposition, readers are referred to [12, Chapter 3]. In this subsection, we introduce several concepts related to upho posets, along with some illustrative examples.

Definition 2. *A poset P is* upper homogeneous, *abbreviated as* upho, *if for every $s \in P$ we have $V_{P,s} \cong P$, where $V_{P,s} := \{p \in P \mid s \leq_P p\}$ is the* principal order filter *generated by s.*

Note that each principal order filter has a unique minimal element, so every upho poset P has a unique minimal element, denoted $\hat{0}_P$. We abbreviate $V_{P,s}$ to V_s and $\hat{0}_P$ to $\hat{0}$ when the poset P referred to is clear.

A poset P is said to be $\mathbb{N}$-*graded* if P can be written as a disjoint union $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \dots$ such that every maximal chain has the form $p_0 \lessdot_P p_1 \lessdot_P p_2 \lessdot_P \cdots$, where $p_i \in P_i$ for all $i \in \mathbb{N}$. The *rank function* $\rho : P \to \mathbb{N}$ of P is defined by $\rho(p) = i$ for all $p \in P_i$. We refer to P_i as the *i-th layer* of P.

An $\mathbb{N}$-graded poset P is said to be of *finite type* if $|P_i|$ is finite for all $i \in \mathbb{N}$. The *rank-generating function* of a finite type $\mathbb{N}$-graded poset P is defined to be $F_P(x) := \sum_{k=0}^{\infty} |P_k|\, x^k$.

The *Ehrenborg quasi-symmetric function* [3] of a finite type $\mathbb{N}$-graded poset P is defined to be $E_P := \sum_{n\geq 0} E_{P,n}$, where $E_{P,0} = 1$, and

$$E_{P,n}\left(x_1, x_2, \cdots, x_n\right) := \sum_{\substack{\hat{0}=t_0 \leq_P t_1 \leq_P \cdots \leq_P t_{k-1} <_P t_k \\ \rho(t_k)=n}} x_1^{\rho(t_1)-\rho(t_0)} x_2^{\rho(t_2)-\rho(t_1)} \cdots x_k^{\rho(t_k)-\rho(t_{k-1})},\ n \geq 1.$$

In this paper, we define the following two finiteness conditions.

Definition 3. *A poset P is* Noetherian *if its principal order filters satisfy* ascending chain condition, *that is, for every element $s \in P$, there is no infinite strictly ascending chain $V_s \subseteq V_{s_1} \subseteq V_{s_2} \subseteq \cdots$ (or equivalently, $s >_P s_1 >_P s_2 >_P \cdots$).*

Definition 4. *The* height *of an element s in a poset P is the maximal length of chains in P with s as its maximum. A poset P is* finitary *if every element in P has finite height.*

All $\mathbb{N}$-graded posets are finitary, and all finitary posets are Noetherian. Moreover, in a Noetherian upho poset P, a finitary upho poset $P' \subseteq P$ can be obtained by selecting all elements of finite height in P, with P' inheriting the order of P.

In a poset P, we define $\mathcal{E}_P := \{(r,s) \mid r,s \in P, r \lessdot s\}$, which corresponds to the edges in the Hasse diagram of P if P is finitary. If a poset P has a unique minimum $\hat{0}$, we define $\mathcal{A}_P := \{s \in P \mid \hat{0} \lessdot s\}$, and elements in $\mathcal{A}_P$ are called *atoms* of P. In a Noetherian poset, each maximal chain includes exactly one atom. However, both $\mathcal{A}_P$ and $\mathcal{E}_P$ can be empty if P is non-Noetherian. See Example 1.

A formal power series $f(x) \in 1 + x\mathbb{Z}[[x]]$ is said to be an *upho function* if it is the rank-generating function of an upho poset. An important property of upho function is:

Lemma 1 ([4, Lemma 2.3]). *Let P and Q be upho posets. Then $P \times Q$ is an upho poset. Furthermore, $F_{P\times Q} = F_P F_Q$, and $E_{P\times Q} = E_P E_Q$.*

We list below some examples of upho posets and upho functions.

Example 1. *Nonnegative real numbers $\mathbb{R}_{\geq 0}$ with usual order is a non-Noetherian upho poset, and there are no atoms in $\mathbb{R}_{\geq 0}$. Let $\mathbb{R}_{\geq 0} \times \mathbb{N}$ be the poset product of $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, both with usual order. Then $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is also a non-Noetherian upho poset, and $(0, 1)$ is its only atom.*

Example 2. *The poset $\mathbb{N} \times \mathbb{N}$ with lexicographical order forms a Noetherian upho poset which is not finitary. For every $(m, n) \in \mathbb{N} \times \mathbb{N}$, the isomorphism $\tau : V_{(m,n)} \cong \mathbb{N} \times \mathbb{N}$ is given by $\tau(m, n + s) = (0, s), \tau(m + t, s) = (t, s)$ for all $s \in \mathbb{N}, t \in \mathbb{N}_{>0}$.*

Example 3. *The upho poset P_M is defined by the following data: The elements in P_M are those in the monoid $M = \langle x_1, x_2 \mid x_1^3 = x_2 x_1 \rangle$; The partial order of P_M is defined by left divisibility in M, that is, $a \leq_{P_M} b$ if and only if there exists $c \in M$ such that $ac = b$ in M.*

It can be shown that P_M is a finitary upho poset, yet not $\mathbb{N}$-graded.

Example 4. *The upho poset P_{M_1} and P_{M_2} are defined as follow: P_{M_1} consists of elements in the free monoid M_1 generated by $[0, 1] \subseteq \mathbb{R}$, with a partial order defined by left divisibility. Similarly, P_{M_2} consists of elements in the free commutative monoid M_2 generated by $[0, 1] \subseteq \mathbb{R}$, with its partial order also defined by left divisibility.*

Both P_{M_1} and P_{M_2} are $\mathbb{N}$-graded upho posets that are not of finite type. The d-th layer of P_{M_1} can be thought of as a cube of dimension d, while the d-th layer of P_{M_2} can be thought of as a simplex of dimension d.

Example 5. *Figure 1 shows the Hasse diagrams of $\mathbb{N}$ (with usual order), full binary tree, Stern's poset, and bowtie poset from left to right. Their rank-generating functions are $\frac{1}{1-x}$, $\frac{1}{1-2x}$, $\frac{1}{(1-x)(1-2x)}$, and $\frac{1+x}{1-x}$, respectively. All of them are $\mathbb{N}$-graded upho posets of finite type.*

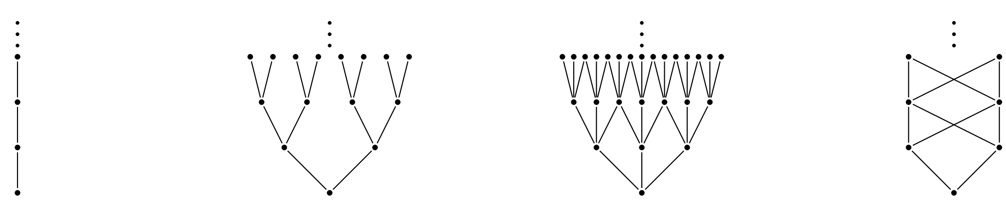

Figure 1: The Hasse diagrams of Example 5.

Example 6. *Fei's poset $\mathcal{F}$ is the upho poset that satisfies the generating rules depicted in the left two components of Figure 2. This poset is a finite type $\mathbb{N}$-graded upho poset, with its rank-generating function given by $F_{\mathcal{F}}(x) = \frac{1-x}{1-2x-x^2}$. The first three layers of the Hasse diagram of Fei's poset are shown in the rightmost component of Figure 2.*

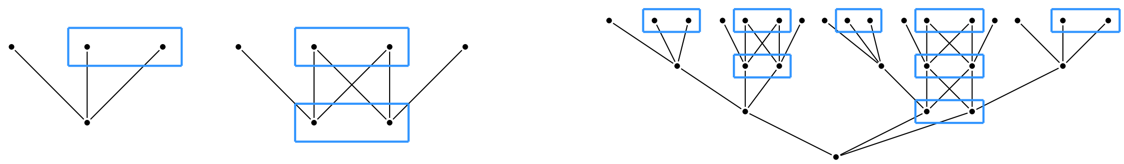

Figure 2: The generating rules and the first several layers of Fei's poset.

On one hand, Example 6 shows that not all upho functions are log-concave, as $|\mathcal{F}_1| \cdot |\mathcal{F}_3| = 51 > 49 = |\mathcal{F}_2|^2$. On the other hand, it illustrates potential connections between upper homogeneity and linear recurrence, inspiring our proof of Proposition 4.

2.2 Semi-upho Posets

In this subsection, we define *semi-upho posets* which have partial self-similarity. The motivation for introducing semi-upho posets is to explore formal power series $g(x)$ that, when multiplied by any upho function $f(x)$, yield an upho function. As shown in Lemma 1, upho functions are optimal choices for $g(x)$. Moreover, in Theorem 6, we generalize this result to the rank-generating functions of *tree-like semi-upho posets*.

Given posets P' and P with unique minima $\hat{0}_{P'}$ and $\hat{0}_P$ respectively, an injection $\eta : P' \hookrightarrow P$ is said to be an *induced saturated order embedding*, abbreviated as *isoembedding*, if $\eta(\hat{0}_{P'}) = \hat{0}_P$, and furthermore, for every chain C in P' with given maximum a and minimum b, C is a maximal chain with given maximum a and minimum b if and only if $\eta(C)$ is a maximal chain with maximum $\eta(a)$ and minimum $\eta(b)$.

Definition 5. *A poset S is* semi-upho *if for every $s \in S$, there exists an isoembedding $V_s \hookrightarrow S$.*

Upho posets are semi-upho posets. Moreover, a semi-upho poset can be thought of as an upho poset with some parts cut off. Figure 3 is an example of *tree-like semi-upho posets*, defined as finitary semi-upho posets whose Hasse diagrams are trees.

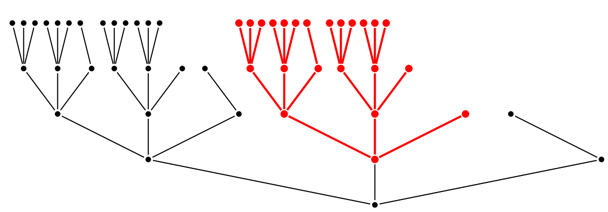

Figure 3: A tree-like semi-upho poset. The red part is a principal order filter that can be isoembedded into the poset itself.

3 Monoid Representation

In this section, we establish a bijection between finitary *colored upho posets* and *left-cancellative invertible-free monoids*, and associate the bijection to upho posets.

3.1 Colored Upho Posets

The motivation for introducing *colored upho posets* is to designate a unique isomorphism between each given principal order filter and the upho poset itself.

Definition 6. *A colored upho poset $\tilde{P}$ consists of the data (P, col_P): The poset P is an upho poset, and the color mapping $\mathrm{col}_P : \mathcal{E}_P \to \mathcal{A}_P$ satisfies the following conditions:*

- $\mathrm{col}_P(\hat{0}, t) = t$ *for all* $t \in \mathcal{A}_P$*;*
- *For every $s \in P$, there exists an isomorphism $\phi_s : V_s \xrightarrow{\sim} P$ such that* $\mathrm{col}_P(u, v) = \mathrm{col}_P(\phi_s(u), \phi_s(v))$ *for all* $(u, v) \in \mathcal{E}_{V_s}$*.*

It can be shown that such ϕ_s is unique for a given $s \in P$.

Finitary colored upho posets can be conceptualized as structures in which each edge extending upward from the same vertex in their Hasse diagrams is assigned a distinct color. Moreover, there exists a unique isomorphism between each principal order filter and the poset itself, which maps edges to identically colored ones.

Similarly, we introduce the concept of colored semi-upho posets.

Definition 7. *A colored semi-upho poset $\tilde{S}$ consists of the data (S, col_S): The poset S is a semi-upho poset, and the color mapping $\mathrm{col}_S : \mathcal{E}_S \to \mathcal{A}_S$ satisfies the following conditions:*

- $\mathrm{col}_S(\hat{0}, t) = t$ *for all* $t \in \mathcal{A}_S$*;*
- *For every $s \in S$, there exists an isoembedding $\psi_s : V_s \hookrightarrow S$ such that* $\mathrm{col}_S(u, v) = \mathrm{col}_S(\psi_s(u), \psi_s(v))$ *for all* $(u, v) \in \mathcal{E}_{V_s}$*.*

It can also be shown that such ψ_s is unique for a given $s \in S$.

3.2 Correspondence with Monoids

In this subsection, we build the bijection between finitary colored upho posets and *left-cancellative invertible-free monoids* explicitly. First, we recall some terminology of monoids, and readers may see [7] for a more detailed exposition.

The *identity element* of a monoid M, denoted e, is an element satisfying $ex = xe = x$ for every $x \in M$. A *zero element* of a monoid M, denoted 0, is an element satisfying $0x = x0 = 0$ for every $x \in M$. An element $x \in M$ is said to be *left-invertible* if there exists an element $y \in M$ such that $yx = e$, and *right-invertible* if there exists an element $y \in M$ such that $xy = e$. An element $x \in M$ is said to be *inveritible* if it is both left-invertible and right-invertible. A monoid M is said to be *invertible-free* if it has no left-invertible or right-invertible elements other than e; equivalently, $ab = e$ implies $a = b = e$. An element a of M is said to be *irreducible* if it is not invertible and is not the product of any two non-invertible elements. The set of all irreducible elements of M is denoted $\mathcal{I}_M$. A monoid M is said to be *left-cancellative* if for every $a, x, y \in M$, $ax = ay$ implies $x = y$. An invertible-free monoid M is said to be *homogeneous* if for every $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m \in M$, where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in \mathcal{I}_M$, we have $n = m$. We define the *length* of this element to be n, and denote the set of all distinct elements of length n by W_n^M.

We define a mapping $\mathcal{M}$ which maps finitary colored upho posets to left-cancellative invertible-free monoids (abbreviated as *LCIF monoids*) by the following rule. For a given finitary colored upho poset $\tilde{P} = (P, \mathrm{col}_P)$, the elements of $\mathcal{M}(\tilde{P})$ are the elements of P. For every $s, t \in P$, we define the multiplication st by $st := \phi_s^{-1}(t)$. It can be verified that such $\mathcal{M}(\tilde{P})$ is a well-defined LCIF monoid, and $\mathcal{A}_{\mathcal{P}} = \mathcal{I}_{\mathcal{M}(\tilde{P})}$.

Conversely, we define a mapping $\tilde{\mathcal{P}} = (\mathcal{P}, \mathcal{C})$ which maps LCIF monoids to finitary colored upho posets by the following rule. The elements of $\mathcal{P}(M)$ are the elements of M. The partial order $\leq_P$ in $\mathcal{P}(M)$ is defined by the left divisibility in M, as is explained in Example 3. Then we have for every $a, b \in M$, $a \lessdot_P b$ if and only if there exists $c \in \mathcal{I}_M$ such that $ac = b$. So the unique minimum in $\mathcal{P}(M)$ is e, and $\mathcal{A}_{\mathcal{P}(M)} = \mathcal{I}_M$. For every ordered pair $(a, b) \in \mathcal{E}_{\mathcal{P}(M)}$, define $\mathcal{C}(M)(a, b)$ to be $c \in \mathcal{I}_M$ such that $ac = b$ (such c is unique by the left cancellative property of M). It can be verified that such $\tilde{\mathcal{P}}(\mathcal{M}) := (\mathcal{P}(M), \mathcal{C}(M))$ is a well-defined finitary colored upho posets.

Furthermore, $\tilde{\mathcal{P}}(\mathcal{M}(\tilde{P})) \cong \tilde{P}$, and $\mathcal{M}(\tilde{\mathcal{P}}(M)) \cong M$. Hence we have:

Theorem 5. *The mutually inverse mappings $\mathcal{M}$ and $\tilde{\mathcal{P}}$ give a bijection between finitary colored upho posets and left-cancellative invertible-free monoids.*

Restrict to $\mathbb{N}$-graded colored upho posets, we have:

Corollary 1. *The mutually inverse mappings $\mathcal{M}$ and $\tilde{\mathcal{P}}$ give a bijection between $\mathbb{N}$-graded colored upho posets and left-cancellative homogeneous monoids (abbreviated as* LCH monoids*). Moreover, finite type $\mathbb{N}$-graded colored upho posets correspond to finitely generated LCH monoids.*

Furthermore, the bijection can be generalized to semi-upho posets.

An *LCIF 0-monoid* is defined to be an LCIF monoid with an additional zero element 0 and some relations $X_i = 0$, where X_i is an element of the original LCIF monoid. We denote these relations $X_i = 0$ as *0-defining relations*. Similarly, an *LCH 0-monoid* is defined to be an LCH monoid added by a zero element 0 and some 0-defining relations. It is worth mentioning that, in general, LCIF 0-monoids are not left-cancellative, and LCH 0-monoids are neither left-cancellative nor homogeneous. The mutually inverse mappings $\mathcal{M}_0$ and $\tilde{\mathcal{S}}$ between finitary colored semi-upho posets and LCIF 0-monoids are defined similarly to $\mathcal{M}$ and $\tilde{\mathcal{P}}$, respectively. The only difference is that the elements in the posets now correspond to *non-zero* elements in the monoids.

Corollary 2. *The mutually inverse mappings $\mathcal{M}_0$ and $\tilde{\mathcal{S}}$ give a bijection between finitary colored semi-upho posets and LCIF* 0*-monoids. Moreover,* $\mathbb{N}$*-graded colored semi-upho posets correspond to LCH* 0*-monoids, and finite type* $\mathbb{N}$*-graded colored semi-upho posets correspond to finitely generated LCH* 0*-monoids.*

3.3 The Forgetful Mapping and Regularity of Upho Posets

Through the *forgetful mapping*, we establish an association between monoids and finitary colored upho posets with finitary upho posets.

The *forgetful mapping* $\mathfrak{F}$ maps a finitary colored semi-upho poset $\tilde{S} = (S, \mathrm{col}_S)$ to S. This mapping is well-defined on upho posets since upho posets are semi-upho.

Figure 4 demonstrates that $\mathfrak{F}$ is not injective, while the surjectivity of $\mathfrak{F}$ remains open.

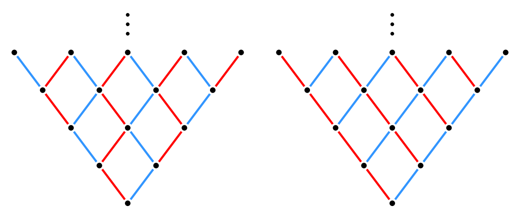

Figure 4: The forgetful mapping is not injective.

Now we define *regular semi-upho posets* to be the semi-upho posets in $\mathbf{im}\mathfrak{F}$, and *regular upho posets* to be the upho posets in $\mathbf{im}\mathfrak{F}$. Moreover, an upho function is said to be *regular upho* if it is the rank-generating function of a regular upho poset.

By the correspondence established in Section 3.2, on one hand, the properties of upho posets can be used to address enumeration problems in left-cancellative monoids. For instance, in an upho poset P, it is straightforward to show that $|P_k| = |P_{k+1}|$ implies $|P_n| = |P_k|$ for all $n \geq k$. Converting this fact into monoids, we then prove that in an LCH monoid M, $|W_k^M| = |W_{k+1}^M|$ implies $|W_n^M| = |W_k^M|$ for all $n \geq k$.

On the other hand, we can use monoids to construct a variety of well-defined upho posets and semi-upho posets, as illustrated in Example 3 and Example 4. Furthermore, our proof of Theorem 4 in Section 4 is entirely based on this method.

4 Totally Positive Upho Functions

In this section, we use Theorem 3 to characterize total positivity, and we split our proof of Theorem 4 into three parts. In Section 4.1, we address the case where the numerator is 1 and the denominator has two roots that are not less than 1. In Section 4.2, we address the case where the numerator is 1 and the denominator has only one root not less than 1. In Section 4.3, we prove the remaining parts of Theorem 4.

4.1 Type I Unitary Totally Positive Functions

We first give a simple yet useful recast of semi-upho posets.

Lemma 2. *A poset P is semi-upho if and only if for every atom $s \in \mathcal{A}_P$, there exists an isoembedding $V_s \hookrightarrow P$.*

In the following text, we say a formal power series $g(x) = \sum b_i x^i$ is *log-concave* if the coefficient sequence $\{b_i\}_{i=0}^{\infty}$ consists solely of nonnegative numbers and contains no internal zeros, moreover, $b_{i+1}^2 \geq b_i b_{i+2}$ for every $i \in \mathbb{N}$. Using Lemma 2, we have:

Proposition 1. *Let $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be a log-concave function, then there exists a tree-like semi-upho poset Q whose rank-generating function equals $g(x)$.*

Sketch of proof. Given a log-concave function $g(x) = \sum_{i=0}^{\infty} b_i x^i$, where $b_0 = 1$, we construct the semi-upho poset layer by layer. We construct the $(k+1)$-th layer from the first k layers by first constructing a canonical maximum case, and then deleting points from right to left. It can be shown that $b_{k+1}^{max} \geq \frac{b_k^2}{b_{k-1}} \geq b_{k+1}$. □

Example 7. *Given $g(x) = 1 + 3x^2 + 7x^3 + 13x^3 + b_4^{max} x^4$, Figure 3 is the Hasse diagram of the poset we construct in the procedure above.*

We further use monoids to show that multiplying a regular upho function by the rank-generating function of a tree-like semi-upho poset yields a regular upho function.

Proposition 2. *Let $M_1 = \langle \mathcal{I}_{M_1} \mid R_{M_1} \rangle$ be a finitely generated LCH monoid, where R_{M_1} is its defining set of relations. Let $M_2 = \langle \{0\} \cup \mathcal{I}_{M_2} \mid R_{M_2} \rangle$ be a finitely generated LCH 0-monoid, where R_{M_2} only has 0-defining relations. We define $M := \langle \mathcal{I}_{M_1} \cup \mathcal{I}_{M_2} \mid R_M \rangle$, where R_M consists of the following relations:*

- *$R_{M_1} \subseteq R_M$;*
- *If $y_i \in \mathcal{I}_{M_2}, Y_j \in M_2$, and $y_i Y_j = 0$ is in R_{M_2}, then $y_i Y_j = x_1 Y_j$ is in R_M;*
- *$y_i x_j = x_1 x_j \in R_M$ for all elements $x_j \in \mathcal{I}_{M_1}, y_i \in \mathcal{I}_{M_2}$.*

Then M is a finitely generated LCH monoid. Moreover, $F_{\mathcal{P}(M)} = F_{\mathcal{P}(M_1)} F_{\mathcal{P}(M_2)}$.

Figure 5 depicts how to "convolve" a tree-like semi-upho poset and an upho poset.

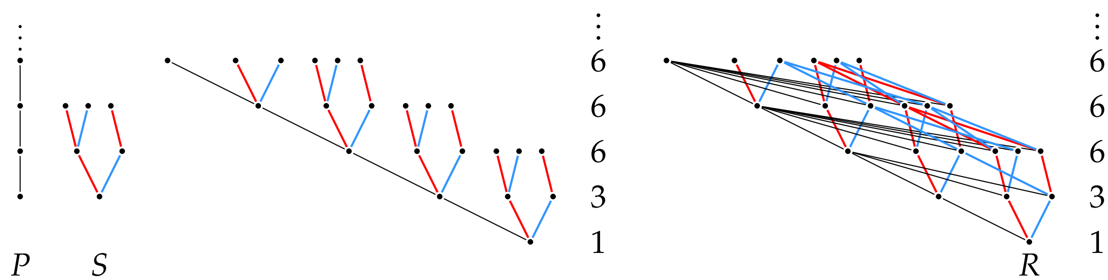

Figure 5: Construction of an upho poset R by "convolving" a regular upho poset P and a tree-like semi-upho poset S.

Rewriting Proposition 2 in terms of formal power series yields the following result:

Theorem 6. *Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be regular upho and $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be the rank-generating function of a tree-like semi-upho poset, then $f(x)g(x)$ is regular upho.*

By employing Proposition 1 and Theorem 6, we obtain the following corollary:

Corollary 3. *Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be regular upho and $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be log-concave, then $f(x)g(x)$ is regular upho.*

Notice that multiplication preserves log-concavity [11, Proposition 2], hence we have:

Proposition 3. *If a formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is of type I, that is:*

$$f(x) = \frac{1}{h(x)} = \prod_{i=1}^{n} \frac{1}{(1-\lambda_i x)},\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

where $h(x)$ is irreducible over $\mathbb{Z}[X]$, $\deg h(x) \geq 2$, and $1 \leq \lambda_{n-1} \leq \lambda_n$, then $f(x)$ is a regular upho function.

Proof. Note that $\frac{1}{1-\lambda_i x}$ for $1 \leq i \leq n-2$ and $\frac{1-x}{(1-\lambda_{n-1}x)(1-\lambda_n x)}$ are log-concave, and $\frac{1}{1-x}$ is a regular upho function, so by Lemma 1 and Corollary 3, $f(x)$ is an upho function. □

4.2 Type II Unitary Totally Positive Functions

In this subsection, we consider formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ of type II:

$$f(x) = \frac{1}{h(x)} = \frac{1}{1+\sum_{i=1}^{\infty} h_i x^i} = \prod_{i=1}^{n} \frac{1}{(1-\lambda_i x)} = 1 + \sum_{i=1}^{\infty} c_i x^i,\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

where $h(x)$ is irreducible over $\mathbb{Z}[X]$, $\deg h(x) \geq 2$, and λ_{n-1}, λ_n satisfies $\lambda_{n-1} < 1 \leq \lambda_n$. We prove that $f(x)$ of type II is regular upho by explicitly constructing an LCH monoid whose rank-generating function equals $f(x)$. A technical lemma we use is:

Lemma 3. *There exist $l_i \in \mathbb{Z}$ for $1 \leq i \leq n$ with $l_1 \geq l_2 \geq \cdots l_n \geq 0$ such that*

$$c_i = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 & \cdots & l_n \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}^i \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Example 8. *For $n = 5$, we have $l_1 = h_1 - 4$, $l_2 = h_1 - h_5 - 4$, $l_3 = h_1 - h_4 + 3h_5 - 3$, $l_4 = h_1 - h_3 + 2h_4 - 3h_5 - 2$ and $l_5 = h_1 - h_2 + h_3 - h_4 + h_5 - 1$.*

Proposition 4. *A formal power series $f(x)$ of type II is a regular upho function.*

Sketch of proof. Let M be a monoid generated by $x_i^j, 1 \leq j \leq n, 1 \leq i \leq r_j$, where $r_1 = l_1$ and $r_i = 1$ for $i > 1$. And we let the defining relations of M be:

$$x_1^j x_k^1 = x_1^1 x_k^1, \quad 2 \leq j \leq n, 1 \leq k \leq l_1 - l_j;$$
$$x_1^j x_1^i = x_1^1 x_1^i, \quad 2 \leq i < j \leq n.$$

By Proposition 2 and Lemma 3, we then prove that M is a well-defined LCH monoid and it rank-generating function equals $f(x)$. □

4.3 Total Positivity Implies Upho

In this subsection, we use the results obtained earlier to prove Theorem 4 and Theorem 1.

We first divide the totally positive functions into the "denominator part" and the "numerator part" using Theorem 3. The result on the "denominator part" can be obtained directly from combining Lemma 1, Proposition 3, Proposition 4, and the fact that $\frac{1}{1-nx}$ is a regular upho function for all $n \in \mathbb{N}$.

Proposition 5. *If a formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is of the form $f(x) = \frac{1}{h(x)}$, where all the complex roots of $h(x) \in 1 + x\mathbb{Z}_{\geq 0}[x]$ are real and positive, then $f(x)$ is regular upho.*

To deal with the "numerator part", we just need to use the lemma below.

Lemma 4. *$g(x) \in 1 + x\mathbb{Z}[x]$ is log-concave if all its complex roots are real and negative.*

Proof of Theorem 4. Combine Corollary 3, Theorem 3, Proposition 5, and Lemma 4. □

Before proving Theorem 1, we state the Thoma–Kerov–Vershik theorem as follows.

Theorem 7 ([2]). *A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is totally positive if and only if $f(t_1)f(t_2)\cdots$ is Schur positive.*

Proof of Theorem 1. For a finite type $\mathbb{N}$-graded upho poset P, according to [4, Lemma 2.2], the Ehrenborg quasi-symmetric function $E_P(x_1, x_2, \dots) = \prod_{i=1}^{\infty} F_P(x_i)$. Combining Theorem 4 and Theorem 7, the proof of Theorem 1 is completed. □

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Shards for the affine symmetric group

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Abstract. The poset of "biclosed sets" in a root system has received attention as a natural extension of the weak Bruhat order on the associated Coxeter group. We will discuss these ideas in the context of the simplest infinite Coxeter groups, the affine symmetric groups. Using the combinatorial model introduced in (Barkley–Speyer 2022), we show that many constructions for the weak order on the symmetric group have analogs for the extended weak order on the affine symmetric group. In particular, shards in the affine braid arrangement biject with completely join-irreducible elements of the extended weak order, and there is a parametrization of both objects by "type-$\widetilde{A}$ arc diagrams".

Keywords: Coxeter groups, weak order, shards

1 Introduction

The weak Bruhat order is a partial order on a Coxeter group which is studied for its connections to generalized permutahedra [6], Coxeter arrangements [4], pattern avoidance [3], preprojective algebras [5], and Catalan combinatorics [10]. Some of these connections fail to be complete or do not make sense when applied to infinite Coxeter groups. Motivated by this failure in the context of Hecke algebras [8], Matthew Dyer introduced a different but related poset associated to each Coxeter group, which is now called the **extended weak order**. In general the extended weak order strictly contains the weak Bruhat order as an order ideal, but for finite Coxeter groups the two posets coincide. There are many fascinating conjectures [7] suggesting that, often, the extended weak order is a more natural object than the usual weak Bruhat order. For example, weak Bruhat order is a lattice for any finite Coxeter group, but is never a lattice for an infinite Coxeter group (of finite rank). In contrast, the extended weak order is conjectured to always be a complete lattice. This conjecture has recently been proven for affine Coxeter groups in [2].

In this extended abstract, we will focus on the combinatorics and geometry of extended weak order in type $\widetilde{A}$, using combinatorial models introduced in [1]. Our focus will be on **shards**, certain cones in a Coxeter arrangement which, for finite Coxeter groups, govern the combinatorics of lattice quotients of the weak order. Our main theorem is the following.

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Theorem 1. *There is a canonical bijection between shards in the Coxeter arrangement of type $\widetilde{A}_n$ and completely join-irreducible elements in the extended weak order of type $\widetilde{A}_n$.*

The analogous result is known to be true for all finite Coxeter groups [9]. Importantly, the result is **false** for the weak Bruhat order of type $\widetilde{A}_n$: there is an injection from completely join-irreducible elements of weak Bruhat order to the set of shards, but it is not a bijection. The weak Bruhat order is "missing" some join-irreducibles, and the extended weak order provides them.

We prove this theorem by parametrizing both shards and complete join-irreducibles by **cyclic arc diagrams**. Arc diagrams were introduced by Nathan Reading [11] as a way of encoding the combinatorics of shards in type A. Our results can be viewed as a type-$\widetilde{A}$ analog of his. Analogs in type B and type D have also been recently introduced by Ashley Tharp in her thesis [12].

The Coxeter group of type A_n is the symmetric group S_{n+1}, and the Coxeter group of type $\widetilde{A}_n$ is the affine symmetric group $\widetilde{S}_{n+1}$. The corresponding Coxeter arrangements are the braid and affine braid arrangements, respectively. Because we are working in a context where we have explicit combinatorial models, we have attempted to avoid Coxeter-theoretic language in the body of the paper, and have included an introduction to the relationship between arc diagrams, weak order, and the geometry of these arrangements for the unfamiliar reader.

2 Weak order and the symmetric group

We begin by recalling the combinatorics of the weak Bruhat order on the symmetric group. Let S_n denote the group of permutations of the set $\{1,\ldots,n\}$. We say that the pair (a,b) is an **inversion** of π if $a<b$ and $\pi^{-1}(a)>\pi^{-1}(b)$. If we write a permutation in one-line notation, then the inversions are the pairs which are out of order. For instance, the inversions of the permutation 51423 are $(1,5)$, $(4,5)$, $(2,5)$, $(3,5)$, $(2,4)$, $(3,4)$. Write $N(\pi)$ for the set of inversions of π. This set determines π uniquely. The **weak order** on S_n is the partial ordering such that $u \le v$ if and only if $N(u) \subseteq N(v)$. Figure 1 depicts the weak order on S_3.

2.1 The poset of regions

In this section, we will consider the relationship between the weak order and convex geometry. To see this, consider the **braid arrangement** $\mathcal{B}_n$. The braid arrangement consists of hyperplanes H_{ab} in $\mathbb{R}^n$, for $1 \le a < b \le n$, where $H_{ab} := \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_a = x_b\}$. In Figure 2, we've depicted (a slice through) $\mathcal{B}_3$. As illustrated in the figure, two points are in the same region (connected component of $\mathbb{R}^n \setminus \bigcup_{a<b} H_{ab}$) if and only if their coordinates are in the same order. Hence regions correspond to total orderings of

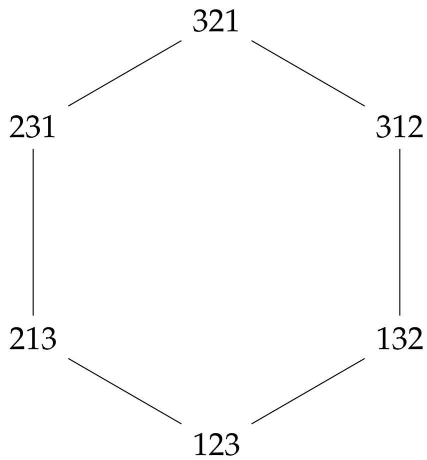

Figure 1: The Hasse diagram of weak order on S_3.

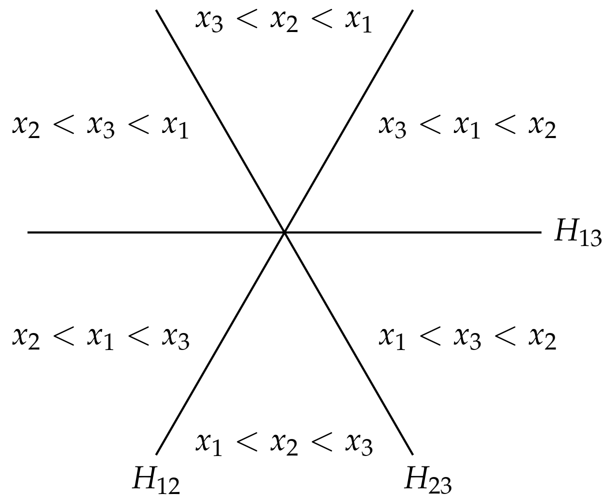

Figure 2: The intersection of $\mathcal{B}_3$ with a two-dimensional subspace of $\mathbb{R}^3$.

the coordinates $x_1, \ldots, x_n$. We can think of these total orderings as the one-line notation of a permutation, so that for instance the permutation 231 corresponds to the region whose points have coordinates satisfying $x_2 < x_3 < x_1$. This gives a bijection between elements of S_n and regions of $\mathcal{B}_n$.

There's another perspective on where this bijection comes from: there is a group action of S_n on $\mathbb{R}^n$, where π acts via $(x_1, \ldots, x_n) \mapsto (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$. This induces an action on the regions of $\mathcal{B}_n$, and this action is simply transitive. So if we fix a "base region", then the group action induces a bijection between regions and elements of S_n. If we use as a base region the region of points such that $x_1 < \cdots < x_n$, then we get the bijection outlined above.

For our purposes, the weak order is more naturally viewed as a partial order on regions of the braid arrangement than it is as a partial order on S_n. Given regions R_1 and R_2, their **separating set** is

$$\mathcal{S}(R_1, R_2) := \{H_{ab} \mid R_1 \text{ and } R_2 \text{ are in different components of } \mathbb{R}^n \setminus H_{ab}\}.$$

Now if B denotes the region with $x_1 < \cdots < x_n$, then πB is the region associated to π via the bijection above. In this case, $\mathcal{S}(B, \pi B) = \{H_{ab} \mid (a, b) \text{ is an inversion of } \pi\}$. Hence weak order can be identified with the order on regions of the braid arrangement so that $R_1 \leq R_2$ if $\mathcal{S}(B, R_1) \subseteq \mathcal{S}(B, R_2)$. This is called the **poset of regions** of $\mathcal{B}_n$.

2.2 Lattice structure

The weak order on S_n is a **complete lattice**, meaning that it admits meets (greatest lower bounds) and joins (least upper bounds) for any collection of elements. To compute the join of a list of permutations, we introduce a closure operator on sets of inversions. Write $T := \{(a,b) \mid 1 \le a < b \le n\}$. If $N \subseteq T$, then we define the **closure** of N to be the minimal set $\overline{N} \supseteq N$ such that if $a < b < c$ and (a,b), (b,c) are both in $\overline{N}$, then (a,c) is in $\overline{N}$. The join of two permutations π_1 and π_2, denoted $\pi_1 \vee \pi_2$, is the unique permutation with inversion set $\overline{N(\pi_1) \cup N(\pi_2)}$. More generally, the join of a family $\{\pi_i\}_{i \in I}$ has inversion set $\overline{\bigcup_{i \in I} N(\pi_i)}$. One can compute the meet of a family $\{\pi_i\}_{i \in I}$ dually: it has inversion set $(\bigcap_{i \in I} N(\pi_i))^\circ$, where $N^\circ := T \setminus \overline{T \setminus N}$ is the **interior** of N.

For an example, let's compute the join of 213 and 132. Then $N(213) = \{(1,2)\}$ and $N(132) = \{(2,3)\}$. We need to compute the closure of $N(213) \cup N(132) = \{(1,2),(2,3)\}$. The closure is forced to contain $(1,3)$, since $(1,2)$ and $(2,3)$ are both elements. Hence $\overline{N(213) \cup N(132)} = \{(1,2),(2,3),(1,3)\}$. The join $213 \vee 132$ should be the unique permutation with this inversion set, which is the permutation 321. As can be seen in Figure 1, we indeed have $213 \vee 132 = 321$.

2.3 Shards and arc diagrams

A permutation π is **join-irreducible** if it cannot be written as a join of elements strictly below π. Equivalently, π covers a unique element π_* in the weak order. The join-irreducible elements (JIs) in S_3 are 213, 132, 231, and 312. Each JI has a unique **lower wall**: an inversion (a,b) such that $\pi^{-1}(a) = 1 + \pi^{-1}(b)$. If (a,b) is the lower wall of a JI π, then $(a,b) \cdot \pi = \pi_*$. The lower walls of the JIs listed above are $(1,2)$, $(2,3)$, $(1,3)$, and $(1,3)$, respectively. Nathan Reading introduced in [11] an elegant way of parametrizing the JIs in S_n via arc diagrams.

Definition 1. A **shard arc** for S_n is the data of:

- an initial value i and a terminal value j, such that $1 \le i < j \le n$, and
- for each intermediate value k with $i < k < j$, a choice of "left" or "right".

We will depict shard arcs using arc diagrams, where an arc is drawn connecting the initial value to the terminal value, and where "left" or "right" at k indicates whether the arc passes over or under k, respectively. (There are many ways to draw such a diagram; we use diagrams only as an abbreviation for the data of a shard arc, so different diagrams indicating the same shard arc should be treated as equivalent.) The four shard arcs in S_4 with initial value 1 and terminal value 4 are shown in Figure 3. For space purposes, we have drawn the arcs horizontally, though the "left/right" terminology more clearly applies to arcs drawn vertically. To each shard arc for S_n, we assign a JI of the weak order on S_n. This JI is the unique permutation π with the following properties:

- If the shard arc has initial value i and terminal value j, then the unique lower wall of π is (i,j), and

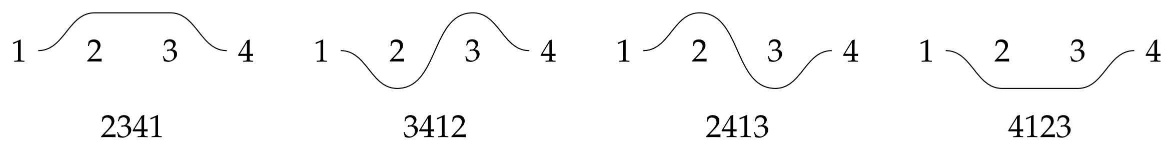

Figure 3: The arc diagrams for shard arcs in S_4 with initial value 1 and terminal value 4. Below each diagram, we have indicated the associated JI in S_4.

- For each intermediate value k, if we have chosen "left" at k, then $\pi^{-1}(k) < \pi^{-1}(j)$, and if we have chosen "right" at k, then $\pi^{-1}(k) > \pi^{-1}(i)$.

In other words, the pair j, i should appear consecutively in the one-line notation for π, and an intermediate value k should appear to the left or right of j, i according to whether we have chosen "left" or "right", respectively. The positions of non-intermediate values, and the relative ordering of intermediate values, are determined by the requirement that (i, j) is the unique lower wall of π. In Figure 3, we have indicated the JI associated to each shard arc.

Shard arcs are also related to the geometry of the braid arrangement. Given a shard arc for S_n, we will associate a convex polyhedral cone in the braid arrangement $\mathcal{B}_n$. To do so, define the half-spaces

$$H_{ab}^+ := \{(x_1, \ldots, x_n) \mid x_a \leq x_b\} \qquad H_{ab}^- := \{(x_1, \ldots, x_n) \mid x_a \geq x_b\}.$$

Consider a shard arc with initial value i and terminal value j. For each intermediate value k, we pick a sign for H_{ik} and for H_{kj}, as follows:

$$\begin{array}{rcl} \text{"left" at } k & \Rightarrow & H_{ik}^- \text{ and } H_{kj}^+ \\ \text{"right" at } k & \Rightarrow & H_{ik}^+ \text{ and } H_{kj}^- \end{array}.$$

The polyhedral cone is defined to be the intersection of H_{ij} with the correctly signed $H_{ik}^{\pm}$ for all intermediate k. The resulting cone Σ is called a **shard** of $\mathcal{B}_n$. Shards are characterized as (the closures of) the connected components of $H_{ij} \setminus \bigcup_{i<k<j} H_{ik}$. In $\mathcal{B}_3$, there are four total shards: H_{12} and H_{23} are themselves shards, and H_{13} is the union of two shards. In Figure 2, the two shards in H_{13} are the left and right halves of H_{13}, which intersect at the origin.

We have constructed bijections shards $\Leftrightarrow$ shard arcs $\Leftrightarrow$ JIs. Let's discuss how to go directly between JIs and shards. Given any permutation π, we can consider the region of the braid arrangement πB. The lower walls (a, b) of π correspond to the hyperplanes H_{ab} in $\mathcal{S}(B, \pi B)$ which are incident to πB. (Hence the term "wall".) We can refine this further: if (a, b) is a lower wall of π, then there is a unique shard Σ contained in H_{ab} which is incident to the region πB. We say that Σ is a **lower shard** of πB. Hence there is a bijection between the lower walls of π and the lower shards of πB. If π is a JI, then there is a unique lower wall of π, and the corresponding lower shard of πB comes from the shard $\Leftrightarrow$ JI bijection. As an example, consider the JI $\pi = 231$. Then the unique lower wall of π is $(1,3)$, so the unique lower shard of πB should be contained in H_{13}. Examining Figure 2, we see that the unique lower shard of πB is the left half of H_{13}, which has shard arc 1 2 3. As expected, this is the shard arc associated to 231.

We have focused on JIs, but the notion of lower shards makes sense for any region of the braid arrangement. For a general region, there will be multiple lower shards. We can record the set of lower shards of πB in a diagram by overlaying the arc diagrams for each shard arc. For instance, the permutation 321 has diagram 1−2−3 and the permutation 123 has the empty diagram 1 2 3. Reading showed [11] that any permutation can be recovered from its arc diagram, and that the collections of shard arcs arising from this construction are exactly the **non-crossing arc diagrams**: those collections that can be drawn so no two shard arcs intersect or share an initial or terminal value.

3 Extended weak order and the affine symmetric group

Definition 2. The **affine symmetric group** $\widetilde{S}_n$ is the group of bijections $\widetilde{\pi} : \mathbb{Z} \to \mathbb{Z}$ satisfying:

(a) $\widetilde{\pi}(i+n) = \widetilde{\pi}(i) + n$ for all $i \in \mathbb{Z}$, and

(b) $\displaystyle\sum_{i=1}^{n} \widetilde{\pi}(i) = \sum_{i=1}^{n} i.$

Elements of $\widetilde{S}_n$ are **affine permutations**. The one-line notation of an affine permutation is defined similarly to a usual permutation, so that for instance the one-line notation of the identity is: $\ldots, -1, 0, 1, 2, 3, 4, 5, \ldots$. Condition (b) in Definition 2 lets us recover an affine permutation from its one-line notation. We abbreviate affine permutations via **window notation**: given a sequence of n integers $x_1, \ldots, x_n$ which have distinct residue classes mod n, we write $[x_1 x_2 \cdots x_n]$ for the unique affine permutation whose one-line notation contains $x_1, \ldots, x_n$ as a consecutive subsequence. For instance, in $\widetilde{S}_3$, the windows [123] and [012] both represent the identity permutation, whereas [102] represents the permutation $\ldots, -3, -1, 1, 0, 2, 4, 3, 5, \ldots$.

The window notations for the elements of $\widetilde{S}_2$ are shown in black in Figure 4.

3.1 Extended weak order

Let $(\prec)$ be a total ordering of the integers (a relation which is transitive, asymmetric, irreflexive, and so that for all distinct $a, b \in \mathbb{Z}$, either $a \prec b$ or $a \succ b$). The symbol $<$ will always denote the usual total ordering on the integers. If $a, b \in \mathbb{Z}$ are distinct modulo n, then we say that the pair (a, b) is an **inversion** of $(\prec)$ if $a < b$ and $b \prec a$. We write $N(\prec)$ for the set of inversions of $(\prec)$.

Definition 3 ([1]). The **extended weak order** for $\widetilde{S}_n$ is the poset whose elements are the total orders $(\prec)$ of $\mathbb{Z}$ satisfying the following properties:

- For all $i, j \in \mathbb{Z}$, we have $i \prec j$ if and only if $i + n \prec j + n$, and
- For all $i \in \mathbb{Z}$, if $i + n \prec i$ then there exists a k with $i + n \prec k \prec i$.

We say that $(\prec_1) \leq (\prec_2)$ in extended weak order if $N(\prec_1) \subseteq N(\prec_2)$.

We call an element of extended weak order a **translationally invariant total order** (TITO). Because we do not count the pair $(0, n)$ as an inversion, the second condition in Definition 3 is necessary to make it so any TITO is determined by its inversion set. To see the issue, consider the following two total orderings which satisfy the first condition of Definition 3 with $n = 2$:

$$\cdots \prec 0 \prec 2 \prec 4 \prec \cdots \prec \cdots \prec -1 \prec 1 \prec 3 \prec \cdots \tag{3.1}$$

$$\cdots \prec 4 \prec 2 \prec 0 \prec \cdots \prec \cdots \prec -1 \prec 1 \prec 3 \prec \cdots .$$

These two total orders have the same inversion set. We resolve this by declaring the first to be a TITO and the second to be not a TITO; alternatively, we could declare the two total orders equivalent, and the resulting theory would be the same.

Because (a, b) is an inversion of $(\prec)$ if and only if $(a + n, b + n)$ is an inversion of $(\prec)$, we will identify the pairs (a, b) and $(a + n, b + n)$. Hence

$$\cdots \prec -3 \prec -4 \prec -1 \prec -2 \prec 1 \prec 0 \prec 3 \prec 2 \prec 5 \prec 4 \prec 7 \prec 6 \prec \cdots \tag{3.2}$$

is a TITO for $\widetilde{S}_4$ with two inversions, $(0, 1)$ and $(2, 3)$. We note that using the window notation $[1, 0, 3, 2]$ is a reasonable way to encode this TITO. We will now extend window notation to allow us to encode any TITO.

Observe that any TITO $(\prec)$ splits up into **blocks**: subintervals which are order-isomorphic to the usual ordering on $\mathbb{Z}$. The blocks of (3.1) are $\cdots \prec 0 \prec 2 \prec 4 \prec \cdots$ and $\cdots \prec -1 \prec 1 \prec 3 \prec \cdots$, while there is a unique block for (3.2). The residue classes mod n of integers appearing in distinct blocks are necessarily distinct. If a block contains k residue classes, then we will use a window listing any k consecutive entries of the block. We give each block its own window and separate them by the symbol $\prec$. So, for instance, (3.1) has window notation $[2] \prec [1]$ and (3.2) has window notation $[1, 0, 3, 2]$.

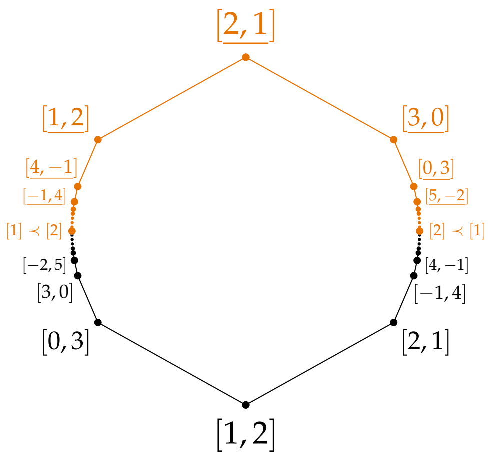

Figure 4: The Hasse diagram for extended weak order. Elements of weak Bruhat order are shown in black, and new elements from extended weak order are in orange.

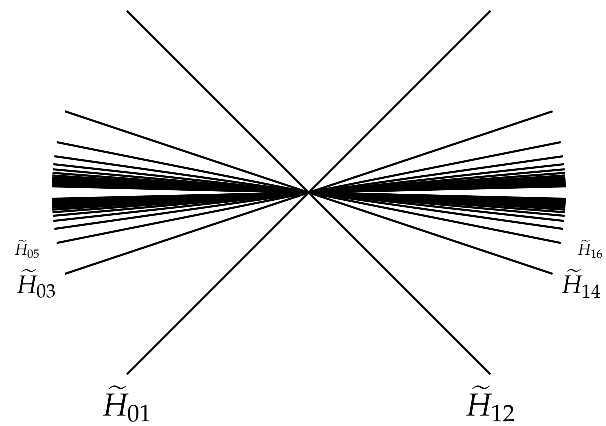

Figure 5: The intersection of the affine braid arrangement $\widetilde{\mathcal{B}}_2$ with a two-dimensional subspace of $\mathbb{R}^3$.

There is one subtlety we haven't yet addressed, which is blocks appearing "in reverse order". For example, consider the following TITO for $\widetilde{S}_4$:

$$\cdots \prec -3 \prec -1 \prec 1 \prec 3 \prec 5 \prec 7 \prec \cdots \prec \cdots \prec 6 \prec 4 \prec 2 \prec 0 \prec -2 \prec -4 \prec \cdots . \tag{3.3}$$

Based on what we have stated so far, the window notation of this TITO would be $[1,3] \prec [2,0]$. However, this does not distinguish (3.3) from the TITO

$$\cdots \prec -3 \prec -1 \prec 1 \prec 3 \prec 5 \prec 7 \prec \cdots \prec \cdots \prec -2 \prec -4 \prec 2 \prec 0 \prec 6 \prec 4 \prec \cdots .$$

To distinguish these, we will write the window notation for (3.3) as $[1,3] \prec [\underline{2,0}]$. What's going on here? It turns out there are exactly two ways to extend the consecutive sequence $2 \prec 0$ to a TITO block: either 0 is covered by $2+4$, or 0 is covered by $2-4$. Once we make that choice, the rest of the block is uniquely determined. In general, we underline a window to indicate that elements i of its block satisfy $i \prec i-n$. If a window is not underlined, then elements of its block satisfy $i \prec i+n$. The TITOs for $\widetilde{S}_2$ are shown in Figure 4.

3.2 The affine braid arrangement

The **affine braid arrangement** $\widetilde{\mathcal{B}}_n$ consists of hyperplanes $\widetilde{H}_{ab}$ in $\mathbb{R}^{n+1}$, for $a < b$ integers that are distinct modulo n. We write a general element of $\mathbb{R}^{n+1}$ as $(y, x_1, \ldots, x_n)$. Then

$$\widetilde{H}_{ab} := \{(y, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_a = x_b\},$$

where we take the convention that $x_{a+kn} = x_a + ky$ for any $k \in \mathbb{Z}$. So for instance, in $\widetilde{\mathcal{B}}_3$, we have $\widetilde{H}_{-1,9} = \{(y, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_2 - y = x_3 + 2y\}$.

When we were studying the the symmetric group, there was a bijection between elements of S_n and regions of $\mathcal{B}_n$. This is no longer true for $\widetilde{S}_n$: we can see by comparing Figure 4 and Figure 5 that there are more regions than elements of $\widetilde{S}_n$. One might wonder if elements of the extended weak order biject with regions of $\widetilde{\mathcal{B}}_n$. This fails in general. To see the problem, let's introduce the half-spaces

$$\widetilde{H}^+_{ab} := \{(y, x_1, \ldots, x_n) \mid x_a \leq x_b\} \qquad \widetilde{H}^-_{ab} := \{(y, x_1, \ldots, x_n) \mid x_a \geq x_b\}.$$

Given a TITO $(\prec)$, we say that $\widetilde{H}^+_{ab}$ **contains** $(\prec)$ if $a \prec b$, and similarly we say $\widetilde{H}^-_{ab}$ **contains** $(\prec)$ if $a \succ b$. We write $\mathcal{H}(\prec)$ for the collection of half-spaces containing $(\prec)$. The geometry in Figures 4 and 5 suggests that the region associated to $(\prec)$ should be the intersection of all the half-spaces in $\mathcal{H}(\prec)$. When this intersection has points in its interior, then this is a reasonable definition. But this is not always the case: for instance, the $\widetilde{S}_2$ TITO with window notation $[1] \prec [2]$ is contained in the half-spaces $\widetilde{H}^-_{01}, \widetilde{H}^-_{03}, \widetilde{H}^-_{05}, \ldots$ and the half-spaces $\widetilde{H}^+_{12}, \widetilde{H}^+_{14}, \ldots$. The intersection of these half-spaces is the line $y = 0$, which has empty interior. However, in this case every finite subset of $\mathcal{H}(\prec)$ has intersection with nonempty interior. (We say the TITO is **weakly separable** in this case; see [1].) A more serious problem arises for the $\widetilde{S}_4$ TITO $[0,1] \prec [\underline{3,2}]$. This is contained in the half-spaces $\widetilde{H}^+_{01}, \widetilde{H}^+_{14}, \widetilde{H}^-_{23}, \widetilde{H}^-_{36}$, whose intersection is contained in the hyperplane $y = 0$. This TITO is not weakly separable.

We see that not every TITO has an associated region. However, every region does have an associated TITO. Given a region R of $\widetilde{\mathcal{B}}_n$, let $\mathcal{H}(R)$ be the collection of half-spaces $\widetilde{H}^\pm_{ab}$ such that $R \subseteq \widetilde{H}^\pm_{ab}$. Then $\mathcal{H}(R)$ is equal to $\mathcal{H}(\prec)$ for a unique TITO $(\prec)$. Uniqueness follows since we can recover the inversion set of $(\prec)$ from its set of containing hyperplanes: $N(\prec) = \{(a,b) \mid \widetilde{H}^-_{ab} \in \mathcal{H}(\prec)\}$. This is the analog of the fact that we can recover the inversion set of a permutation π from the separating set $\mathcal{S}(B, \pi B)$.

3.3 Lattice structure

Like the weak order on S_n, the $\widetilde{S}_n$ extended weak order is a complete lattice [1, 2]. We can compute the join of a collection of TITOs in a similar fashion. Write $\widetilde{T} := \{(a,b) \mid a < b,\ a \not\equiv b \mod n\}$ and $\widetilde{T}_{\text{aug}} := \{(a,b) \mid a < b\}$. If $N \subseteq \widetilde{T}_{\text{aug}}$, then we define the **augmented closure** of N to be the minimal set $\overline{N}_{\text{aug}} \supseteq N$ such that if $a < b < c$ and $(a,b), (b,c)$ are both in $\overline{N}_{\text{aug}}$, then (a,c) is in $\overline{N}_{\text{aug}}$. If $N \subseteq \widetilde{T}$ is a union of inversion sets, then the **closure** of N is the set $\overline{N} := \overline{N}_{\text{aug}} \cap \widetilde{T}$. Now, the join of a family of TITOs $\{\prec_i\}_{i \in I}$ is the unique TITO with inversion set $\overline{\bigcup_{i \in I} N(\prec_i)}$. Analogously, the meet of $\{\prec_i\}_{i \in I}$ has inversion set $(\bigcap_{i \in I} N(\prec_i))^\circ$, where $N^\circ := \widetilde{T} \setminus \overline{(\widetilde{T} \setminus N)}$ is the **interior** of N.

For example, let's compute the join of $[0,3]$ and $[2,1]$ in the extended weak order of $\widetilde{S}_2$. We have $N([0,3]) = \{(0,1)\}$ and $N([2,1]) = \{(1,2)\}$. The augmented closure of the union $\{(0,1),(1,2)\}$ contains $(0,2)$ since $(0,1),(1,2)$ are both elements, and it contains $(1,3)$ since $(1,2),(2,3)$ are both elements. (Recall our convention that $(a,b) = (a+n, b+n)$.) Hence the augmented closure contains every pair, since it contains $(0,1),(1,3),(3,5),\ldots$ and $(1,2),(2,4),(4,6),\ldots$.

It follows that the closure $\overline{N([0,3]) \cup N([2,1])}$ is $\widetilde{T}$. Hence the join $[0,3] \vee [2,1]$ is the unique TITO with inversion set $\widetilde{T}$, which is $[\underline{2,1}]$.

3.4 Shards and arc diagrams

A TITO $(\prec)$ is **completely join-irreducible** if it cannot be written as a join of elements strictly below $(\prec)$. This implies that $(\prec)$ covers a unique element, but is a stronger condition in general. The only TITOs for $\widetilde{S}_2$ which are **not** completely join-irreducible are $[12]$, $[1] \prec [2]$, $[2] \prec [1]$, and $[\underline{2,1}]$. In this section JI will abbreviate "completely join-irreducible element".

The **lower walls** of a TITO $(\prec)$ are the inversions (a,b) so that b and a are consecutive in the total order $\prec$. There exist TITOs, like $[1] \prec [2]$, which have no lower walls. However, each JI has a unique lower wall. The goal of this section is to describe the analog of arc diagrams which parametrizes the JIs.

Definition 4. A **shard arc** for $\widetilde{S}_n$ is the data of:

- an initial value i and a terminal value j, such that $1 \le i \le n$ and $i < j$ and $i \not\equiv j \mod n$, and
- for each intermediate value k with $i < k < j$, a choice of "left" or "right".

These data are required to satisfy a condition which will be explained below.

We can depict these shard arcs in two ways. One is to simply draw an arc diagram for S_j, where j is the terminal value of the arc. This would fully encode the data of the shard arc. However, the conditions on the data are more well-motivated by drawing a **cyclic arc diagram**: we arrange the numbers $1,\ldots,n$ in a circle, and draw an arc starting at i, proceeding clockwise around the circle until it is of length $j-i$, then terminating (at a value congruent to j modulo n). At each intermediate value k, the arc passes k on the outside or inside of the circle depending on whether we have chosen "left" or "right", respectively. Now we can state the condition on $\widetilde{S}_n$ shard arc data: we must be able to draw the cyclic arc diagram in this way without self-crossing.

The JI associated to a shard arc is the unique weak order-minimal TITO with lower wall (i,j) and such that each intermediate value k satisfies $k \prec j$ if we chose "left" and satisfies $i \prec k$ if we chose "right". The TITOs associated to the shard arcs in Figure 6 are

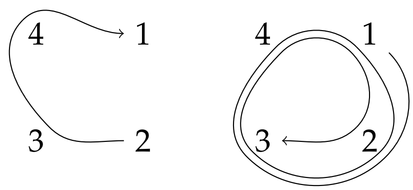

Figure 6: Two shard arcs. The arc on the left starts at 2, ends at 5, and passes 3 and 4 on the inside and outside, respectively. The arc on the right starts at 1 and ends at 11.

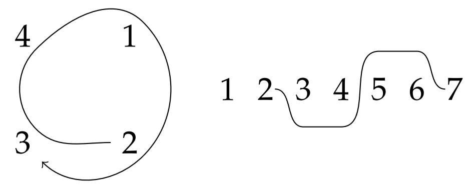

Figure 7: On the left, a cyclic arc diagram. On the right, a "straightened" version of the diagram encoding the same data. (Note that not all S_j arc diagrams give valid $\widetilde{S}_n$ shard arcs.)

$$\cdots \prec 0 \prec 1 \prec -2 \prec -1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 8 \prec 9 \prec 6 \prec 7 \prec \cdots$$
$$\cdots \prec 2 \prec 7 \prec -3 \prec 0 \prec 6 \prec 11 \prec 1 \prec 4 \prec 10 \prec 15 \prec 5 \prec \cdots$$

and the TITO for the shard arc in Figure 7 is

$$\cdots - 3 \prec 1 \prec 5 \prec 9 \prec \cdots \prec 6 \prec 7 \prec 2 \prec 3 \prec -2 \prec -1 \prec \cdots \prec -4 \prec 0 \prec 4 \prec 8 \prec \cdots .$$

To construct the shard associated to a shard arc, for each intermediate value k, we pick a sign for $\widetilde{H}_{ik}$ and for $\widetilde{H}_{kj}$ as follows:

$$\begin{array}{rcl} \text{“left” at } k & \Rightarrow & \widetilde{H}_{ik}^{-} \text{ and } \widetilde{H}_{kj}^{+} \\ \text{“right” at } k & \Rightarrow & \widetilde{H}_{ik}^{+} \text{ and } \widetilde{H}_{kj}^{-} \end{array} .$$

The shard associated to the shard arc is then defined to be the cone Σ which is the intersection of $\widetilde{H}_{ij}$ with $\widetilde{H}_{ik}^{\pm}$ for all intermediate k. Shards are characterized as (the closures of) the connected components of $\widetilde{H}_{ij} \setminus \bigcup_{i<k<j} \widetilde{H}_{ik}$.

The map sending a JI to its associated shard has a geometric description. Let $(\prec)$ be a JI with lower wall (a, b). If $\mathcal{H}(\prec) = \mathcal{H}(R)$ for some region R of $\mathcal{H}$, then the hyperplane $\widetilde{H}_{ab}$ is incident to R. The shard Σ associated to $(\prec)$ is the unique shard contained in $\widetilde{H}_{ab}$ which is incident to R: we say Σ is a **lower shard** of R. However, there exist JIs which do not come from regions, such as $[1, 2] \prec [\underline{3, 4}]$. Even in this case, Σ is characterized as the unique shard of $\widetilde{H}_{ab}$ which, for each intermediate k, is contained in $\widetilde{H}_{ak}^{-}$ if and only if $(\prec)$ is contained in $\widetilde{H}_{ak}^{-}$. Hence, despite the lack of a literal region to go with $(\prec)$, the geometry still behaves as if Σ is the lower shard of a "quasi-region" associated to $(\prec)$. The existence of such exotic JIs makes the following result even more surprising.

Theorem 2. *These correspondences set up bijections*

$$\textit{shards} \Leftrightarrow \textit{shard arcs} \Leftrightarrow \textit{JIs}.$$

Let i and j be the initial and terminal values of a shard arc datum. The JIs which are elements of weak Bruhat order are those with shard arc data having either $i < j < i + n$, or else $j > i + n$ and we have chosen "right" at $i + n$.

In particular, the shard arc in Figure 7, or any shard with arc data having a choice of "left" at $i + n$, does not have an associated join-irreducible element of weak Bruhat order: we truly need to go to the extended weak order to explain these shards.

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Kromatic quasisymmetric functions

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Abstract. We provide a construction for the kromatic symmetric function $\overline{X}_G$ of a graph introduced by Crew, Pechenik, and Spirkl using combinatorial (linearly compact) Hopf algebras. As an application, we show that $\overline{X}_G$ has a positive expansion into multifundamental quasisymmetric functions. We also study two related quasisymmetric q-analogues of $\overline{X}_G$, which are K-theoretic generalizations of the quasisymmetric chromatic function of Shareshian and Wachs. We classify exactly when one of these analogues is symmetric. For the other, we derive a positive expansion into symmetric Grothendieck functions for graphs G that are natural unit interval orders.

Keywords: Chromatic symmetric functions, combinatorial Hopf algebras, linearly compact modules, multifundamental quasisymmetric functions

1 Introduction

The purpose of this note is to re-examine the algebraic origins of the *kromatic symmetric function* of a graph that was recently introduced by Crew, Pechenik, and Spirkl [3], and to study two quasisymmetric analogues of this power series.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{P} = \{1, 2, 3, \dots\}$, and $[n] = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. All graphs are undirected by default, and are assumed to be simple with a finite set of vertices. We do not distinguish between isomorphic graphs.

If G is any graph then we write $V(G)$ for its set of vertices and $E(G)$ its set of edges. A *proper coloring* of G is a map $\kappa : V(G) \to \mathbb{P}$ with $\kappa(u) \neq \kappa(v)$ for all $\{u, v\} \in E(G)$. For maps $\kappa : V \to \mathbb{P}$ let $x^\kappa = \prod_{i \in V} x_{\kappa(i)}$ where $x_1, x_2, \dots$ are commuting variables.

Definition 1.1 (Stanley [12])**.** The *chromatic symmetric function* of G is the symmetric power series $X_G := \sum_\kappa x^\kappa$ where the sum is over all proper colorings κ of G.

Example 1.2. If $G = K_n$ is the *complete graph* with $V(G) = [n]$ then $X_G = n! e_n$ for the *elementary symmetric function* $e_n := \sum_{1 \le i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$.

A poset is *$(3+1)$-free* if it does not contain a 3-element chain $a < b < c$ whose elements are all incomparable to some fourth element d. The *Stanley–Stembridge conjecture* [13] proposes that if G is the incomparability graph of a $(3+1)$-free poset then X_G has a

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positive expansion into elementary symmetric functions. This conjecture has several refinements and generalizations, and has been resolved in a number of interesting special cases, but remains open in general.

Let G be an *ordered graph*, that is, a graph with a total order $<$ on its vertex set $V(G)$. An *ascent* (resp., *descent*) of a map $\kappa : V(G) \to \mathbb{P}$ is an edge $\{u, v\} \in E(G)$ with $u < v$ and $\kappa(u) < \kappa(v)$ (resp., $\kappa(u) > \kappa(v)$). Let $\mathrm{asc}_G(\kappa)$ and $\mathrm{des}_G(\kappa)$ be the number of ascents and descents. Shareshian and Wachs [10] introduced the following q-analogue of X_G:

Definition 1.3 ([10])**.** The *chromatic quasisymmetric function* of an ordered graph G is $X_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa \in \mathbb{N}[q][\![x_1, x_2, \dots]\!]$ where the sum is over all proper colorings.

Example 1.4. If $G = K_n$ then $X_G(q) = [n]_q! e_n$ where $[i]_q = \frac{1-q^i}{1-q}$ and $[n]_q! = \prod_{i=1}^n [i]_q$.

Let $\mathsf{Set}(\mathbb{P})$ be the set of finite nonempty subsets of $\mathbb{P}$. For a map $\kappa : V \to \mathsf{Set}(\mathbb{P})$ define $x^\kappa = \prod_{i \in V} \prod_{j \in \kappa(i)} x_j$. A *proper set-valued coloring* is a map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \cap \kappa(v) = \varnothing$ for all $\{u, v\} \in E(G)$. There is also a "K-theoretic" analogue of X_G:

Definition 1.5 (Crew, Pechenik, and Spirkl [3])**.** The *kromatic symmetric function* of a graph G is the sum $\overline{X}_G = \sum_\kappa x^\kappa \in \mathbb{Z}[\![x_1, x_2, \dots]\!]$ over all proper set-valued colorings of G.

Example 1.6. $\overline{X}_{K_n} = n! \sum_{r=n}^\infty \left\{ {r \atop n} \right\} e_r$ where $\left\{ {r \atop n} \right\}$ is the Stirling number of the second kind.

Remark 1.7. Given $\alpha : V \to \mathbb{N}$, let $\mathrm{Cl}_\alpha(V)$ be the set of pairs (v, i) with $v \in V$ and $i \in [\alpha(v)]$. If G is a graph and $\alpha : V(G) \to \mathbb{N}$ is any map, then the *α-clan graph* $\mathrm{Cl}_\alpha(G)$ has vertex set $\mathrm{Cl}_\alpha(V(G))$ and edges $\{(v, i), (w, j)\}$ whenever $\{v, w\} \in E(G)$ or both $v = w$ and $i \neq j$. As observed in [3], one has $\overline{X}_G = \sum_{\alpha : V(G) \to \mathbb{P}} \frac{1}{\alpha!} X_{\mathrm{Cl}_\alpha(G)}$ where $\alpha! := \prod_v \alpha(v)!$. Many properties of X_G extend to $\overline{X}_G$ via this identity, but some interesting features of $\overline{X}_G$ cannot be explained by this formula alone.

Our main results provide a natural construction for $\overline{X}_G$ using the theory of *combinatorial Hopf algebras*. This approach requires some care, as $\overline{X}_G$ is not a symmetric function of bounded degree. We explain things precisely in terms of *linearly compact Hopf algebras* after reviewing a similar, simpler construction of X_G in Section 2, following [1].

As an application of our approach, we show that $\overline{X}_G$ has a positive expansion into *multifundamental quasisymmetric functions*. We also study two related q-analogues of $\overline{X}_G$, which give K-theoretic generalizations of $X_G(q)$. We classify exactly when one of these analogues is symmetric. For the other, we extend a theorem of Crew, Pechenik, and Spirkl (also lifting a theorem of Shareshian and Wachs) to derive a positive expansion into symmetric Grothendieck functions for graphs G that are natural unit interval orders.

2 Background

Let $\mathbb{K}$ be an integral domain; in practice, one can assume this is $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}[q]$, or $\mathbb{Q}(q)$.

2.1 Hopf algebras

Write $\otimes = \otimes_{\mathbb{K}}$ for the tensor product over $\mathbb{K}$. A *$\mathbb{K}$-algebra* is a $\mathbb{K}$-module A with $\mathbb{K}$-linear product $\nabla : A \otimes A \to A$ and unit $\iota : \mathbb{K} \to A$ maps. Dually, a *$\mathbb{K}$-coalgebra* is a $\mathbb{K}$-module A with $\mathbb{K}$-linear coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{K}$ maps. The (co)product and (co)unit maps must satisfy several associativity axioms; see [5, §1].

A $\mathbb{K}$-module A that is both a $\mathbb{K}$-algebra and a $\mathbb{K}$-coalgebra is a *$\mathbb{K}$-bialgebra* if the coproduct and counit maps are algebra morphisms. A bialgebra $A = \bigoplus_{n\in\mathbb{N}} A_n$ is *graded* if its (co)product and (co)unit are graded maps; in this case A is *connected* if $A_0 = \mathbb{K}$.

Let $\mathrm{End}(A)$ denote the set of $\mathbb{K}$-linear maps $A \to A$. This set is a $\mathbb{K}$-algebra for the product $f * g := \nabla \circ (f \otimes g) \circ \Delta$. The unit of this *convolution algebra* is the composition $\iota \circ \epsilon$ of the unit and counit of A. A bialgebra A is a *Hopf algebra* if $\mathrm{id} : A \to A$ has a two-sided inverse $\mathsf{S} : A \to A$ in $\mathrm{End}(A)$. When it exists, we call S the *antipode* of A.

Example 2.1. Let Graphs_n for $n \in \mathbb{N}$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of undirected graphs with n vertices, and set $\mathsf{Graphs} = \bigoplus_{n\in\mathbb{N}} \mathsf{Graphs}_n$. One views Graphs as a connected, graded Hopf algebra with product $\nabla(G \otimes H) = G \sqcup H$ and coproduct $\Delta(G) = \sum_{S\sqcup T=V(G)} G|_S \otimes G|_T$ for graphs G and H, where $\sqcup$ denotes disjoint union and $G|_S$ denotes the subgraph of G induced on S.

A *lower set* in a directed acyclic graph $D = (V, E)$ is a set $S \subseteq V$ such that if a directed path connects $v \in V$ to $s \in S$ then $v \in S$. An *upper set* is the complement of a lower set.

Example 2.2. Let DAGs_n for $n \in \mathbb{N}$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of directed acyclic graphs with n vertices, and set $\mathsf{DAGs} = \bigoplus_{n\in\mathbb{N}} \mathsf{DAGs}_n$. One views DAGs as a connected, graded Hopf algebra with product $\nabla(C \otimes D) = C \sqcup D$ and coproduct $\Delta(D) = \sum D|_S \otimes D|_T$ for directed acyclic graphs graphs C and D, where the sum is over all disjoint unions $S \sqcup T = V(D)$ with S a lower set and T an upper set.

A *labeled poset* is a pair (D, γ) consisting of a directed acyclic graph D and an injective map $\gamma : V(D) \to \mathbb{Z}$. We consider $(D, \gamma) = (D', \gamma')$ if there is an isomorphism $D \xrightarrow{\sim} D'$, written $v \mapsto v'$, such that $\gamma(u) - \gamma(v)$ and $\gamma'(u') - \gamma'(v')$ have the same sign for all edges $u \to v \in E(D)$. If (D_1, γ_1) and (D_2, γ_2) are labeled posets then let $\gamma_1 \sqcup \gamma_2 : V(D_1 \sqcup D_2) \to \mathbb{Z}$ be any injective map such that $(\gamma_1 \sqcup \gamma_2)(u) - (\gamma_1 \sqcup \gamma_2)(v)$ has the same sign as $\gamma_i(u) - \gamma_i(v)$ for all $u, v \in V(D_i)$.

Example 2.3. Let $\mathsf{LPosets}_n$ be the free $\mathbb{K}$-module spanned by all labeled poset with n vertices, and set $\mathsf{LPosets} = \bigoplus_{n\in\mathbb{N}} \mathsf{LPosets}_n$. This is a connected, graded Hopf algebra with product $\nabla((D_1, \gamma_1) \otimes (D_2, \gamma_2)) = (D_1 \sqcup D_2, \gamma_1 \sqcup \gamma_2)$ and coproduct $\Delta((D, \gamma)) = \sum (D|_S, \gamma|_S) \otimes (D|_T, \gamma|_T)$ where the sum is over all disjoint decompositions $S \sqcup T = V(D)$ with S a lower set and T an upper set.

A *(strict) composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a finite sequence of positive integers, called its *parts*. We say that α is a composition of $|\alpha| := \sum_i \alpha_i \in \mathbb{N}$.

Example 2.4. Fix a composition α and let $x_1, x_2, \ldots$ be a countable sequence of commuting variables. The *monomial quasisymmetric function* of α is the power series $M_\alpha = \sum_{1\leq i_1<i_2<\cdots<i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}$. Let $\mathsf{QSym} = \mathbb{K}\text{-span}\{M_\alpha : \alpha \text{ any composition}\}$ be the ring of quasisymmetric functions of bounded degree. This ring is a graded connected Hopf algebras for the coproduct $\Delta(M_\alpha) = \sum_{\alpha=\alpha'\alpha''} M_{\alpha'} \otimes M_{\alpha''}$ where $\alpha'\alpha''$ denotes concatenation of compositions, and the counit that acts on power series by setting $x_1 = x_2 = \cdots = 0$.

A *partition* is a composition sorted into decreasing order. We write $\lambda = 1^{m_1}2^{m_2}\cdots$ to denote the partition with exactly m_i parts equal to i.

Example 2.5. The *elementary symmetric function* of a partition λ is the product $e_\lambda := e_{\lambda_1}e_{\lambda_2}\cdots$ where $e_n := M_{1^n}$. These power series are a basis for the Hopf subalgebra $\mathsf{Sym} \subset \mathsf{QSym}$ of symmetric functions of bounded degree.

2.2 Combinatorial Hopf algebras

Following [1], a *combinatorial Hopf algebra* (H, ζ) is a graded, connected Hopf algebra H of finite graded dimension with an algebra homomorphism $\zeta : H \to \mathbb{K}$.

Example 2.6. The pair $(\mathsf{QSym}, \zeta_\mathsf{Q})$ is an example of a combinatorial Hopf algebra, where $\zeta_\mathsf{Q} : \mathsf{QSym} \to \mathbb{K}$ is the map $\zeta_\mathsf{Q}(f) = f(1,0,0,\ldots)$, which sends $M_{(n)} \mapsto 1$ and $M_\alpha \mapsto 0$ for all α with at least two parts.

For a graph G define $\zeta_{\mathsf{Graphs}}(G) = 0^{|E(G)|}$ where throughout we interpret $0^0 := 1$. For a directed acyclic graph D likewise set $\zeta_{\mathsf{DAGs}}(D) = 0^{|E(D)|}$ for each directed acyclic graph D. These formulas extend to linear maps on Graphs and DAGs. Finally let $\zeta_{\mathsf{LPosets}} : \mathsf{LPosets} \to \mathbb{K}$ be the linear map with $\zeta_{\mathsf{LPosets}}((D,\gamma)) = 1$ if $\gamma(u) < \gamma(v)$ for all edges $u \to v \in E(D)$ with $\zeta_{\mathsf{LPosets}}((D,\gamma)) = 0$ otherwise.

Example 2.7. The pairs $(\mathsf{Graphs}, \zeta_{\mathsf{Graphs}})$, $(\mathsf{DAGs}, \zeta_{\mathsf{DAGs}})$, and $(\mathsf{LPosets}, \zeta_{\mathsf{LPosets}})$ are all combinatorial Hopf algebras.

A morphism $\Psi : (H, \zeta) \to (H', \zeta')$ is a graded Hopf algebra morphism $\Psi : H \to H'$ with $\zeta = \zeta' \circ \Psi$. Results in [1] show that there exists a unique morphism from any combinatorial Hopf algebra to $(\mathsf{QSym}, \zeta_\mathsf{Q})$. Moreover, the image of Ψ is contained in the Hopf subalgebra $\mathsf{Sym} \subset \mathsf{QSym}$ if H is cocommutative. There is an explicit formula for this morphism in [1], which translates to the following maps for our examples above.

For a graph G, let $\mathsf{AO}(G)$ be its set of acyclic orientations. For a directed acyclic graph D, let $(D, \gamma^{\mathsf{op}})$ be the labeled poset with $\gamma^{\mathsf{op}}(u) > \gamma^{\mathsf{op}}(v)$ for all edges $u \to v \in E(D)$. Also set $\Gamma(D) = \sum_\kappa x^\kappa \in \mathbb{N}[\![x_1, x_2, \ldots]\!]$ where the sum is over all maps $\kappa : V(D) \to \mathbb{P}$ with $\kappa(u) < \kappa(v)$ whenever $u \to v \in E(D)$.

More generally, for a labeled poset (D, γ) define $\Gamma(D, \gamma) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathbb{P}$ with $\kappa(u) \leq \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) < \gamma(v)$, and with $\kappa(u) < \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) > \gamma(v)$. Such maps κ are called *P-partitions* for $P = (D, \gamma)$ [11].

Proposition 2.8. There is a commutative diagram of combinatorial Hopf algebras

$$\begin{tikzcd}
(\mathsf{Graphs}, \zeta_{\mathsf{Graphs}}) \arrow[r, hook] \arrow[dr] & (\mathsf{DAGs}, \zeta_{\mathsf{DAGs}}) \arrow[r, hook] \arrow[d] & (\mathsf{LPosets}, \zeta_{\mathsf{LPosets}}) \arrow[dl] \\
& (\mathsf{QSym}, \zeta_{\mathsf{Q}}) &
\end{tikzcd}$$

in which the horizontal maps send $G \mapsto \sum_{D \in \mathsf{AO}(G)} D$ and $D \mapsto (D, \gamma^{\mathsf{op}})$, and the QSym-valued maps send $G \mapsto X_G$, $D \mapsto \Gamma(D)$, and $(D, \gamma) \mapsto \Gamma(D, \gamma)$, respectively.

3 K-theoretic generalizations

We now explain how the results in the previous can be extended "K-theoretically" to construct interesting quasisymmetric functions of unbounded degree, including $\overline{X}_G$. This requires a brief discussion of monoidal structures on *linearly compact modules*.

3.1 Linearly compact modules

Let A and B be $\mathbb{K}$-modules with a $\mathbb{K}$-bilinear form $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{K}$. Assume A is free and $\langle \cdot, \cdot \rangle$ is *nondegenerate* in the sense that $b \mapsto \langle \cdot, b \rangle$ is a bijection $B \to \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$.

Fix a basis $\{a_i\}_{i \in I}$ for A. For each $i \in I$, there exists a unique $b_i \in B$ with $\langle a_i, b_j \rangle = \delta_{ij}$ for all $i, j \in I$, and we identify $b \in B$ with the formal linear combination $\sum_{i \in I} \langle a_i, b \rangle b_i$. We call $\{b_i\}_{i \in I}$ a *pseudobasis* for B.

We give $\mathbb{K}$ the discrete topology. Then the *linearly compact topology* [4, §I.2] on B is the coarsest topology in which the maps $\langle a_i, \cdot \rangle : B \to \mathbb{K}$ are all continuous. This topology depends on $\langle \cdot, \cdot \rangle$ but not on the choice of basis for A. For a basis of open sets in the linearly compact topology, see [9, Eq. (3.1)].

Definition 3.1. A *linearly compact* (or *LC* for short) $\mathbb{K}$-module is a $\mathbb{K}$-module B with a nondegenerate bilinear form $A \times B \to \mathbb{K}$ for some free $\mathbb{K}$-module A, given the linearly compact topology; in this case we say that B is the *dual* of A. Morphisms between such modules are continuous $\mathbb{K}$-linear maps.

Let B and B' be linearly compact $\mathbb{K}$-modules dual to free $\mathbb{K}$-modules A and A'. Let $\langle \cdot, \cdot \rangle$ denote both of the associated forms. Every linear map $\phi : A' \to A$ has a unique adjoint $\psi : B \to B'$ such that $\langle \phi(a), b \rangle = \langle a, \psi(b) \rangle$. A linear map $B \to B'$ is continuous when it is the adjoint of some linear map $A' \to A$.

Definition 3.2. Define $B \,\overline{\otimes}\, B' := \mathrm{Hom}_{\mathbb{K}}(A \otimes A', \mathbb{K})$ and give this the LC-topology from the pairing $(A \otimes A') \times \mathrm{Hom}_{\mathbb{K}}(A \otimes A', \mathbb{K}) \to \mathbb{K}$.

If $\{b_i\}_{i\in I}$ and $\{b'_j\}_{j\in J}$ are pseudobases for B and B', then we can realize the *completed tensor product* $B \mathbin{\overline{\otimes}} B'$ concretely as the linearly compact $\mathbb{K}$-module with the set of tensors $\{b_i \otimes b'_j\}_{(i,j)\in I\times J}$ as a pseudobasis.

Suppose $\nabla : B \mathbin{\overline{\otimes}} B \to B$ and $\iota : B \to \mathbb{K}$ are continuous linear maps which are the adjoints of linear maps $\epsilon : \mathbb{K} \to A$ and $\Delta : A \to A \otimes A$. We say that (B, ∇, ι) is an *LC-algebra* if (A, Δ, ϵ) is a $\mathbb{K}$-coalgebra. Similarly, we say that $\Delta : B \to B \mathbin{\overline{\otimes}} B$ and $\epsilon : B \to \mathbb{K}$ make B into an *LC-coalgebra* if Δ and ϵ are the adjoints of the product and unit maps of a $\mathbb{K}$-algebra on A. We define *LC-bialgebras* and *LC-Hopf algebras* analogously; see [9]. If B is an LC-Hopf algebra then its *antipode* is the adjoint of the antipode of A.

3.2 Combinatorial LC-Hopf algebras

Following [9], we define a *combinatorial LC-Hopf algebra* to be a pair (H, ζ) consisting of an LC-Hopf algebra H with a continuous linear map $\zeta : H \to \mathbb{K}[\![t]\!]$ such that $\zeta(\cdot)|_{t\mapsto 0}$ is the counit of H. A morphism of combinatorial LC-Hopf algebras $\Psi : (H, \zeta) \to (H', \zeta')$ is a LC-Hopf algebra morphism $\Psi : H \to H'$ with $\zeta = \zeta' \circ \Psi$.

Example 3.3. Let $\mathfrak{m}\mathsf{QSym}$ be the set of all quasisymmetric power series in $\mathbb{K}[\![x_1, x_2, \dots,]\!]$ of possibly unbounded degree. The (co)product, (co)unit, and antipode QSym all extend to continuous $\mathbb{K}$-linear maps that make $\mathfrak{m}\mathsf{QSym}$ into an LC-Hopf algebra, with $\{M_\alpha\}$ as a pseudobasis. Then $(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ is a combinatorial LC-Hopf algebra when $\overline{\zeta}_{\mathsf{Q}}$ is the map $\overline{\zeta}_{\mathsf{Q}} : f \mapsto f(t, 0, 0, \dots)$.

The preceding example is an instance of a general construction. If A is a free $\mathbb{K}$-module with basis S, then its *completion* $\overline{A}$ is the set of arbitrary $\mathbb{K}$-linear combinations of S. We view $\overline{A}$ as a linearly compact $\mathbb{K}$-module with S as a pseudobasis, relative to the nondegenerate bilinear form $A \times \overline{A} \to \mathbb{K}$ making S orthonormal.

If (H, ζ) is a combinatorial Hopf algebra then then there is a unique way of extending its (co)unit and (co)product to continuous linear maps on $\overline{H}$. As the Hopf algebra $H = \bigoplus_{n\in\mathbb{N}}$ is graded, we can also extend $\zeta : H \to \mathbb{K}$ to a continuous linear map $\overline{\zeta} : \overline{H} \to \mathbb{K}[\![t]\!]$ by the formula $\overline{\zeta}(h) = \zeta(h)t^n$ for $n \in \mathbb{N}$ and $h \in H_n$.

Proposition 3.4. If (H, ζ) is combinatorial Hopf algebra then the extended structures just given make $(\overline{H}, \overline{\zeta})$ into a combinatorial LC-Hopf algebra, and the unique morphism $(H, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ extends to a morphism $(\overline{H}, \overline{\zeta}) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$.

The pair $(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ is a final object in the category of combinatorial LC-Hopf algebras, meaning there is a unique morphism $(H, \zeta) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ for any combinatorial LC-Hopf algebra. More specifically, if H has coproduct Δ, then define $\Delta^{(0)} = \mathrm{id}_H$ and $\Delta^{(k)} = (\Delta^{(k-1)} \mathbin{\overline{\otimes}} \mathrm{id}) \circ \Delta : H \to H^{\overline{\otimes}(k+1)}$ for $k \in \mathbb{P}$. For compositions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, let $\zeta_\alpha : H \to \mathbb{K}$ be the map sending $h \in H$ to the coefficient of $t^{\alpha_1} \otimes t^{\alpha_2} \otimes \cdots \otimes t^{\alpha_k}$ in $\zeta^{\otimes k} \circ \Delta^{(k-1)}(h) \in \mathbb{K}[\![t]\!]$. When $\alpha = \varnothing$ is empty let $\zeta_\varnothing = \zeta(\cdot)|_{t\mapsto 0}$ be the counit of H.

Theorem 3.5 ([8])**.** If (H,ζ) is a combinatorial LC-Hopf algebra then the map $\Psi_{H,\zeta} : h \mapsto \sum_\alpha \zeta_\alpha(h) M_\alpha$ is the unique morphism $(H,\zeta) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$.

Let $\mathfrak{m}\mathsf{Sym}$ be the LC-Hopf subalgebra of symmetric functions in $\mathfrak{m}\mathsf{QSym}$. When H cocommutative, the morphism $\Psi_{H,\zeta}$ evidently has its image in $\mathfrak{m}\mathsf{Sym}$.

3.3 Set-valued P-partitions

For a directed acyclic graph D, let $\overline{\Gamma}(D) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \prec \kappa(v)$ whenever $u \to v \in E(D)$.

Example 3.6. If $D = (1 \to 2 \to 3 \to \cdots \to n)$ is an n-element chain then define $\bar{e}_n := \overline{\Gamma}(D) = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} e_{n+k}$. For each partition λ let $\bar{e}_\lambda := \bar{e}_{\lambda_1}\bar{e}_{\lambda_2}\cdots$. These functions are a pseudobasis for $\mathfrak{m}\mathsf{Sym}$.

For a labeled poset (D,γ) define $\overline{\Gamma}(D,\gamma) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \preceq \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) < \gamma(v)$, and with $\kappa(u) \prec \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) > \gamma(v)$. Such maps κ are called *set-valued P-partitions* for $P = (D,\gamma)$ in [7, 8].

Example 3.7. If $D = (1 \to 2 \to 3 \to \cdots \to n)$ is an n-element chain and S is the set of $i \in [n-1]$ with $\gamma(i) > \gamma(i+1)$ then the we define $\overline{L}_{n,S} := \overline{\Gamma}(D,\gamma)$. Following [7], the *multifundamental quasisymmetric function* of a composition α is defined by $\overline{L}_\alpha := \overline{L}_{n,S}$ where $n = |\alpha|$ and $S = I(\alpha) := \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \dots\} \setminus \{n\}$. These power series form another pseudobasis for $\mathfrak{m}\mathsf{QSym}$ [7]. An element of $\mathfrak{m}\mathsf{QSym}$ is *multifundamental positive* if its expansion in this pseudobasis involves only nonnegative coefficients.

A *multilinear extension* of a directed acyclic graph D with n vertices is a sequence $w = (w_1, w_2, \dots, w_N)$ with $V(D) = \{w_1, w_2, \dots, w_N\}$ such that $i < j$ whenever $w_i \to w_j \in E(D)$, and $w_i \neq w_{i+1}$ for all $i \in [N-1]$. If $\mathcal{M}(D)$ is the set of all multilinear extensions of D and $\gamma : V(D) \to \mathbb{Z}$ is injective, then $\overline{\Gamma}(D,\gamma) = \sum_{w \in \mathcal{M}(D)} \overline{L}_{\ell(w),\mathrm{Des}(w,\gamma)}$ where $\mathrm{Des}(w,\gamma) := \{i \in [\ell(w)-1] : \gamma(w_i) > \gamma(w_{i+1})\}$ for $w \in \mathcal{M}(D)$ [7].

3.4 Acyclic multi-orientations

Let G be a graph. An *acyclic multi-orientation* of G is an acyclic orientation of the α-clan graph $\mathrm{Cl}_\alpha(G)$ from Remark 1.7 for some $\alpha : V(G) \to \mathbb{P}$, such that for each $v \in V(G)$ both (a) if $i, j \in [\alpha(v)]$ have $i > j$ then $(v,i) \to (v,j)$ is a directed edge; and (b) if $i \in [\alpha(v)-1]$ then there exists a directed path involving no edges of the form $(v,j) \to (v,k)$ that connects $(v,i+1)$ to (v,i). Let $\mathfrak{m}\mathsf{AO}(G)$ be the set of all acyclic multi-orientations of G.

One can relate the $\bar{e}$-expansion of the symmetric function $\overline{X}_G$ to the source counts of its acyclic multi-orientations, generalizing a result of Stanley [12, Thm. 3.3].

Theorem 3.8. Let G be a graph and suppose $\overline{X}_G = \sum_\lambda c_\lambda \overline{e}_\lambda$ for some coefficients $c_\lambda \in \mathbb{Z}$. Then the number of acyclic multi-orientations of G with exactly j sources and k vertices is $\sum_{\ell(\lambda)=j, |\lambda|=k} c_\lambda \in \mathbb{N}$.

As noted in [3], in general, the coefficients c_λ appearing in $\overline{X}_G = \sum_\lambda c_\lambda \overline{e}_\lambda$ can be negative, even when $G = \mathsf{inc}(P)$ is the *incomparability graph* of a $(3+1)$-free poset P.

3.5 Morphisms

For each graph G let $\blacktriangle(G) = \sum_{S \cup T = V(G)} G|_S \otimes G|_T$. This only differs from our other coproduct in allowing vertex decompositions that are not disjoint. Likewise, for each directed acyclic graph D and labeled poset $P = (D, \Gamma)$, define $\blacktriangle(D) = \sum D|_S \otimes D|_T$ and $\blacktriangle(P) = \sum (D|_S, \gamma|_S) \otimes (D|_T, \gamma|_T)$, where both sums are over all (not necessarily disjoint) vertex decompositions $S \cup T = V(D)$ in which S is a lower set, T is an upper set, and $S \cap T$ is an antichain.

Use the continuous linear extensions of these operations to replace the coproducts in the completions of Graphs, DAGs, and LPosets, and denote the resulting structures as $\mathfrak{m}$Graphs, $\mathfrak{m}$DAGs, and $\mathfrak{m}$LPosets to distinguish them from $\overline{\text{Graphs}}$, $\overline{\text{DAGs}}$, and $\overline{\text{LPosets}}$.

Theorem 3.9. The pairs $(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}})$, $(\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}})$, and $(\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}})$ are all combinatorial LC-Hopf algebras, and there is a commutative diagram

$$\begin{tikzcd}
(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) \arrow[r, hook] \arrow[dr] & (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}) \arrow[r, hook] \arrow[d] & (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}}) \arrow[dl] \\
& (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}}) &
\end{tikzcd}$$

in which the horizontal maps send $G \mapsto \sum_{D \in \mathfrak{m}\mathsf{AO}(G)} D$ and $D \mapsto (D, \gamma^{\mathsf{op}})$, and the $\mathfrak{m}$QSym-valued maps send $G \mapsto \overline{X}_G$, $D \mapsto \overline{\Gamma}(D)$, and $(D,\gamma) \mapsto \overline{\Gamma}(D,\gamma)$.

Corollary 3.10. The unique morphism $(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ assigns a graph G to its kromatic symmetric function, which is symmetric as $\mathfrak{m}$Graphs is cocommutative. One can express $\overline{X}_G = \sum_{D \in \mathfrak{m}\mathsf{AO}(G)} \overline{\Gamma}(D)$ and thus $\overline{X}_G$ is multifundamental positive.

Fix a directed acyclic graph D. When $\alpha : V(D) \to \mathbb{N}$ is any map, define $\mathrm{Cl}^{\mathsf{dag}}_\alpha(D)$ to be the directed acyclic graph with vertices $\mathrm{Cl}_\alpha(V(D))$ and with edges $(v,i) \to (w,j)$ whenever $v \to w \in E(D)$ or both $v = w$ and $i < j$. When $\gamma : V(D) \to \mathbb{Z}$ is injective, so that (D,γ) is a labeled poset, define $\mathrm{Cl}^{\mathsf{dag}}_\alpha(D,\gamma) = (\mathrm{Cl}^{\mathsf{dag}}_\alpha(D), \tilde{\gamma})$ to be the labeled poset where $\tilde{\gamma}(v,i) < \tilde{\gamma}(w,j)$ if and only if $\gamma(v) < \gamma(w)$ or both $v = w$ and $i > j$.

Theorem 3.11. Assume $\mathbb{Q} \subseteq \mathbb{K}$. Then there is a commutative diagram

$$\begin{tikzcd}
(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) \arrow[r, hook] \arrow[d, "\cong"] & (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}) \arrow[r, hook] \arrow[d, "\cong"] & (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}}) \arrow[d, "\cong"] \\
(\overline{\mathsf{Graphs}}, \overline{\zeta}_{\mathsf{Graphs}}) \arrow[r, hook] & (\overline{\mathsf{DAGs}}, \overline{\zeta}_{\mathsf{DAGs}}) \arrow[r, hook] & (\overline{\mathsf{LPosets}}, \overline{\zeta}_{\mathsf{LPosets}})
\end{tikzcd}$$

with horizontal maps extending Proposition 2.8 and Theorem 3.9, where the vertical isomorphisms are the continuous linear maps sending $G \mapsto \sum_{\alpha:V(G)\to\mathbb{P}} \frac{1}{\alpha!}\mathrm{Cl}_\alpha(G)$, $D \mapsto \sum_{\alpha:V(D)\to\mathbb{P}} \mathrm{Cl}^{\mathsf{dag}}_\alpha(D)$, and $(D,\gamma) \mapsto \sum_{\alpha:V(D)\to\mathbb{P}} \mathrm{Cl}^{\mathsf{dag}}_\alpha(D,\gamma)$, respectively.

3.6 Kromatic quasisymmetric functions

For the rest of this note we assume $\mathbb{K} \supseteq \mathbb{Z}$ and let q be a formal parameter. We will consider the polynomial and power series rings $\mathsf{Sym}[q] \subset \mathfrak{m}\mathsf{QSym}[q] \subset \mathfrak{m}\mathsf{QSym}[\![q]\!]$.

Let G be an *ordered graph*, that is, a graph with a total order $<$ on its vertex set $V(G)$. One can think of the ordering on $V(G)$ as defining an acyclic orientation on the edges of G, and we do not distinguish between G and another ordered graph H if the corresponding directed acyclic graphs are isomorphic. The following power series is a K-theoretic generalization of $X_G(q)$ and q-analogue of $\overline{X}_G$:

Definition 3.12. For an ordered graph G define $\overline{L}_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\max\circ\kappa)} x^\kappa \in \mathfrak{m}\mathsf{QSym}[q]$ where the sum is over all proper set-valued colorings.

Example 3.13. If $G = K_n$ is the complete graph on the vertex set $[n]$ then $\overline{L}_G(q) = [n]_q! \sum_{r=n}^\infty \left\{ {r \atop n} \right\} e_r = [n]_q! \sum_{r=n}^\infty \left\{ {r-1 \atop n-1} \right\} \overline{e}_r$ where $\left\{ {r \atop n} \right\}$ is the Stirling number of the second kind.

Let us clarify the apparent asymmetry in Definition 3.12. Define $\overline{L}^{\mathrm{des,min}}_G(q)$ by replacing "asc" by "des" and "max" by "min" in Definition 3.12. Construct $\overline{L}^{\mathrm{asc,min}}_G(q)$ and $\overline{L}^{\mathrm{des,max}}_G(q)$ analogously. Let ρ be the continuous involution of $\mathfrak{m}\mathsf{QSym}[q]$ sending $M_{(\alpha_1,\dots,\alpha_k)} \mapsto M_{(\alpha_k,\dots,\alpha_1)}$. Let τ be the involution of $\mathfrak{m}\mathsf{QSym}[q]$ sending $f \mapsto q^{\deg_q(f)} f(q^{-1})$.

Proposition 3.14. We have $\overline{L}_G(q) = \rho\left(\overline{L}^{\mathrm{des,min}}_G(q)\right) = \tau\left(\overline{L}^{\mathrm{des,max}}_G(q)\right) = \rho\circ\tau\left(\overline{L}^{\mathrm{asc,min}}_G(q)\right)$.

Recall that a *cluster graph* is a disjoint union of complete graphs.

Theorem 3.15. We have $\overline{L}_G(q) \in \mathfrak{m}\mathsf{Sym}[q]$ if and only if G is a cluster graph.

Fix $D \in \mathfrak{m}\mathsf{AO}(G)$. Each vertex in D has the form (v,i) for some $v \in V(G)$ and $i \in \mathbb{P}$. Define $\mathsf{align}(D) := |\{(u,i) \to (v,j) \in E(D) : u < v \text{ and } i = j = 1\}|$.

Proposition 3.16. If G is an ordered graph then $\overline{L}_G(q) = \sum_{D\in\mathfrak{m}\mathsf{AO}(G)} q^{\mathsf{align}(D)}\overline{\Gamma}(D)$. This power series is multifundamental positive in the sense of being a possibly infinite $\mathbb{N}[q]$-linear combination of multifundamental quasisymmetric functions.

We can make this more explicit, generalizing a result in [10]. Following [7], a *multipermutation* of $n \in \mathbb{N}$ is a word $w = w_1w_2\cdots w_m$ with $\{w_1,w_2,\dots,w_m\} = \{1,2,\dots,n\}$ and $w_i \neq w_{i+1}$ for all $i \in [m-1]$. Let $\overline{S}_n$ be the set of all multipermutations of n.

For each $w = w_1w_2\cdots w_m \in \overline{S}_n$ let $\mathrm{Inv}(w)$ be the set of pairs (w_i,w_j) with $i < j$ and $w_i > w_j$ and $\{w_1,w_2,\dots,w_{i-1}\} \cap \{w_i\} = \{w_1,w_2,\dots,w_{j-1}\} \cap \{w_j\} = \varnothing$. If P is a poset on $[n]$ and $G = \mathsf{inc}(P)$ is its incomparability graph, then we set $\mathrm{inv}_G(w) := |\{(a,b) \in \mathrm{Inv}(w) : \{a,b\} \in E(G)\}|$ and $S(w,P) := \{m-i : i \in [m-1] \text{ and } w_i \not>_P w_{i+1}\}$.

Theorem 3.17. If $G = \mathsf{inc}(P)$ for a poset P on $[n]$ then $\overline{L}_G(q) = \sum_{w \in \overline{S}_n} q^{\mathrm{inv}_G(w)} \overline{L}_{\ell(w),S(w,P)}$.

The homogeneous component of $\overline{L}_G(q)$ of lowest x-degree recovers $X_G(q)$. The latter power series, like X_G, naturally arises as the image of a morphism of combinatorial Hopf algebras. In detail, assume $\mathbb{K} = \mathbb{Z}[q]$ and let $\mathsf{OGraphs}_n$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of ordered graphs with n vertices. Then the direct sum $\mathsf{OGraphs} := \bigoplus_{n \in \mathbb{N}} \mathsf{OGraphs}_n$ has a graded connected Hopf algebra structure in which the product is disjoint union and the coproduct Δ_q satisfies

$$\Delta_q(G) = \textstyle\sum_{S \sqcup T = V(G)} q^{\mathrm{asc}_G(S,T)} G|_S \otimes G|_T \quad \text{for each ordered graph } G, \tag{3.1}$$

where $\mathrm{asc}_G(S,T) := |\{(s,t) \in S \times T : \{s < t\} \in E(G)\}|$. If ζ_{OGraphs} is the algebra morphism $\mathsf{OGraphs} \to \mathbb{K}$ sending $G \mapsto 0^{|E(G)|}$, then $(\mathsf{OGraphs}, \zeta_{\mathsf{OGraphs}})$ is a combinatorial Hopf algebra and the morphism $(\mathsf{OGraphs}, \zeta_{\mathsf{OGraphs}}) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ sends $G \mapsto X_G(q)$.

We do not know how to give the completion $\mathfrak{m}\mathsf{OGraphs} \supset \mathsf{OGraphs}$ a combinatorial LC-Hopf algebra structure that lets us construct $\overline{L}_G(q)$ in a similar way. In particular, we have not been able to find a K-theoretic generalization of the coproduct Δ_q. Unlike the $q = 1$ case, simply replacing $\sqcup$ in (3.1) by arbitrary union $\cup$ does not lead to a coassociative map. This problem remains if we change the q-power exponent $\mathrm{asc}_G(S,T)$ to other forms like $\mathrm{asc}_G(S-T,T)$, $\mathrm{asc}_G(S,T-S)$, or $\mathrm{asc}_G(S-T,T-S)$.

3.7 Another quasisymmetric analogue

The preceding results indicate that $\overline{L}_G(q)$ is an interesting quasisymmetric q-analogue of $\overline{X}_G$ and K-theoretic extension of $X_G(q)$. However, there is another natural candidate for such a generalization. Continue to let G be an ordered graph. Following [6], an *ascent* of a set-valued map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$ is a tuple (u,v,i,j) with $\{u,v\} \in E(G)$, $i \in \kappa(u)$, $j \in \kappa(v)$, and both $u < v$ and $i < j$. Let $\mathrm{asc}_G(\kappa)$ denote the number of such ascents.

Definition 3.18. For an ordered graph G, define $\overline{X}_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa \in \mathfrak{m}\mathsf{QSym}[\![q]\!]$ where the sum is over all proper set-valued colorings $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$.

This definition is closely related to the quasisymmetric functions $X_G(\mathbf{x}, q, \mu)$ studied in [6]. For each map $\mu : V(G) \to \mathbb{N}$, Hwang [6] defines $X_G(\mathbf{x}, q, \mu) := \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa$ where the sum is over all proper set-valued colorings κ of G with $|\kappa(v)| = \mu(v)$. Evidently $\overline{X}_G(q) = \sum_{\mu : V(G) \to \mathbb{P}} X_G(\mathbf{x}, q, \mu)$, and as noted in [6, Rem. 2.2] one has $X_G(\mathbf{x}, q, \mu) = \frac{1}{[\mu]_q!} X_{\mathrm{Cl}_\mu(G)}(q)$ where $[\mu]_q! := \prod_{v \in V(G)} [\mu(v)]_q!$. Here, we view $\mathrm{Cl}_\mu(G)$ as an ordered graph with $(v,i) < (w,j)$ if either $v < w$ or $v = w$ and $i < j$.

Using these observations, various positive or alternating expansions of $X_G(q)$ (e.g., into fundamental quasisymmetric functions [10, Thm. 3.1], Schur functions [10, Thm. 6.3], power sum symmetric functions [2, Thm. 3.1], or elementary symmetric functions [10, Conj. 5.1]) can be extended in a straightforward way to $X_G(\mathbf{x}, q, \mu)$ and $\overline{X}_G(q)$. See Hwang's results [6, Thms. 3.3, 4.10, and 4.19] and his conjecture [6, Conj. 3.10].

Some of these statements require G to be isomorphic to the incomparability graph of a *natural unit interval order*, meaning a poset P on a finite subset of $\mathbb{P}$ such that if $x <_P z$ then $x < z$ and every y incomparable in P to both x and z has $x < y < z$ [10, Prop. 4.1].[1] If G has this property, then so do all of its α-clans. Therefore $\overline{X}_G(q)$ is symmetric if G is the incomparability graph of a natural unit order interval [6, Thm. 3.8].

Example 3.19. If K_n is the complete graph on $[n]$ then $\overline{X}_{K_n}(q) = \sum_{r=n}^{\infty} F_r^{(n)} e_r$ for $F_r^{(n)} := \sum_{\substack{k_1,k_2,\dots,k_n \in \mathbb{P} \\ k_1+k_2+\dots+k_n=r}} \binom{r}{k_1,k_2,\dots,k_n}_q$ where $(q)_n := \prod_{i\in[n]}(1-q^i)$ and $\binom{r}{k_1,k_2,\dots,k_n}_q = \frac{(q)_r}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_n}}$.

When q is a prime power, $F_r^{(n)}$ counts the *strictly increasing flags* of $\mathbb{F}_q$-subspaces $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{F}_q^r$. Vinroot [14] derived a recurrence for the *generalized Galois numbers* $G_r^{(n)} := \sum_{i=0}^{n} \binom{n}{i} F_r^{(i)}$. This can be used to show (setting $F_r^{(n)} = 0$ if $r < 0$) that:

Proposition 3.20. One has $F_{r+1}^{(n)} = \sum_{i=0}^{n-1} \sum_{j=n-1-i}^{n} \binom{n}{j}\binom{j}{n-1-i}(-1)^i \frac{(q)_r}{(q)_{r-i}} F_{r-i}^{(j)}$.

Like $\overline{L}_G(q)$, the power series $\overline{X}_G(q)$ also does not seem to arise naturally as the image in $\mathfrak{m}\mathsf{QSym}$ of a combinatorial LC-Hopf algebra. Unlike $\overline{L}_G(q)$, however, $\overline{X}_G(q)$ is not generally multifundamental-positive (or $\bar{e}$-positive). However, $\overline{X}_G(q)$ does have a nontrivial positivity property that is not shared by $X_G(\mathbf{x}, q, \mu)$ or $\overline{L}_G(q)$.

A *set-valued tableau* T of shape λ is an assignment of sets $T_{ij} \in \mathsf{Set}(\mathbb{P})$ to the cells (i,j) in $\mathsf{D}_\lambda = \{(i,j) \in \mathbb{P}\times\mathbb{P} : 1 \le j \le \lambda_i\}$ of a partition λ. We write $(i,j) \in T$ to indicate that (i,j) belongs to the shape of T. A set-valued tableau T is *semistandard* if $T_{ij} \preceq T_{i,j+1}$ and $T_{ij} \prec T_{i+1,j}$ for all relevant positions. Let $x^T := \prod_{(i,j)\in T} \prod_{k\in T_{ij}} x_k$ and $|T| := \sum_{(i,j)\in T} |T_{ij}|$.

Definition 3.21. The *symmetric Grothendieck function* of a partition λ is the power series $\bar{s}_\lambda := \sum_{T\in\mathsf{SetSSYT}(\lambda)} (-1)^{|T|-|\lambda|} x^T \in \mathbb{Z}[[x_1, x_2, \dots]]$ where $\mathsf{SetSSYT}(\lambda)$ is the set of all semistandard set-valued tableaux of shape λ.

Each $\bar{s}_\lambda$ is in $\mathfrak{m}\mathsf{Sym}$ and the set of all symmetric Grothendieck functions is another pseudobasis for $\mathfrak{m}\mathsf{Sym}$. We write $\mu \subseteq \lambda$ for two partitions with $\mathsf{D}_\mu \subseteq \mathsf{D}_\lambda$ and set $\mathsf{D}_{\lambda/\mu} := \mathsf{D}_\lambda \setminus \mathsf{D}_\mu$. A *semistandard tableau* of shape λ/μ is a filling of $\mathsf{D}_{\lambda/\mu}$ by positive integers such that each row is weakly increasing and each column is strict increasing.

Definition 3.22 ([3, Def. 3.8])**.** Suppose P is a finite poset and λ is a partition. A *Grothendieck P-tableau* of shape λ is a pair $T = (U, V)$ with these two properties: (a) U is a filling of D_μ by elements of P for some partition $\mu \subseteq \lambda$, such that each element of P is in at least one cell, and for each $(i,j) \in \mathsf{D}_\mu$ one has $U_{ij} <_P U_{i,j+1}$ if $(i, j+1) \in \mathsf{D}_\mu$ and $U_{ij} \not>_P U_{i+1,j}$ if $(i+1, j) \in \mathsf{D}_\mu$; and (b) V is a semistandard tableau of shape λ/μ, whose entries in each row i are all less than i (so $\mathsf{D}_{\lambda/\mu}$ must have no cells in the first row).

[1] A finite poset is isomorphic to one with these properties iff it is $(3+1)$- and $(2+2)$-free [10, §4].

Let $\mathscr{G}_P$ be the set of Grothendieck P-tableaux. Let $\lambda(T)$ be the shape of $T \in \mathscr{G}_P$. One of the main results of [3] establishes that if $G = \mathsf{inc}(P)$ is the incomparability graph a $(3+1)$-free poset P then $\overline{X}_G = \sum_{T\in\mathscr{G}_P} \bar{s}_{\lambda(T)}$. This theorem has a q-analogue.

Suppose P is a finite poset on a subset of $\mathbb{P}$, and let $G = \mathsf{inc}(P)$. Choose some $T = (U, V) \in \mathscr{G}_P$ and let μ be the partition shape of the tableau U. Define a *G-inversion* of T to be a pair of cells $(i, j), (k, l) \in \mathsf{D}_\mu$ with $i > k$ such that $U_{ij} < U_{kl}$ but $U_{ij} \not<_P U_{kl}$ and $U_{ij} \not>_P U_{kl}$. Finally, let $\mathrm{inv}_G(T)$ be the number of all G-inversions of T.

Theorem 3.23. If P is a natural unit interval order then $\overline{X}_G = \sum_{T\in\mathscr{G}_P} q^{\mathrm{inv}_G(T)} \bar{s}_{\lambda(T)}$.

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