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Dynamic spatial price equilibrium, dynamic user equilibrium, and freight transportation in continuous time: A differential variational inequality perspective

Terry L. Friesz *

Harold and Inge Marcus Professor of Industrial Engineering, Penn State, University Park, PA 16802, USA

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ABSTRACT

In this paper we provide a statement of dynamic spatial price equilibrium (DSPE) in continuous time as a basis for modeling freight flows in a network economy. The model presented describes a spatial price equilibrium due to its reliance on the notion that freight movements occur in response to differences between the local and distant prices of goods for which there is excess demand; moreover, local and distant delivered prices are equated at equilibrium. We propose and analyze a differential variational inequality (DVI) associated with dynamic spatial price equilibrium to study the Nash-like aggregate game at the heart of DSPE using the calculus of variations and optimal control theory. Our formulation explicitly considers inventory and the time lag between shipping and demand fulfillment. We stress that such a time lag cannot be readily accommodated in a discrete-time formulation. We provide an in-depth analysis of the DVI's necessary conditions that reveals the dynamic user equilibrium nature of freight flows obtained from the DVI, alongside the role played by freight transport in maintaining equilibrium commodity prices and the delivered-price-equals-local-price property of spatial price equilibrium. By intent, our contribution is wholly theoretical in nature, focusing on a mathematical statement of the defining equations and inequalities for dynamic spatial price equilibrium (DSPE), while also showing there is an associated differential variational inequality (DVI), any solution of which is a DSPE. The model of spatial price equilibrium we present integrates the theory of spatial price equilibrium in a dynamic setting with the path delay operator notion used in the theory of dynamic user equilibrium. It should be noted that the path delay operator used herein is based on LWR theory and fully vetted in the published dynamic user equilibrium literature. This integration is new and constitutes a significant addition to the spatial price equilibrium and freight network equilibrium modeling literatures. Among other things, it points the way for researchers interested in dynamic traffic assignment to become involved in dynamic freight modeling using the technical knowledge they already possess. In particular, it suggests that algorithms developed for dynamic user equilibrium may be adapted to the study of urban freight modelled as a dynamic spatial price equilibrium. As such, our work provides direction for future DSPE algorithmic research and application. However, no computational experiments are reported herein; instead, the computing of dynamic spatial price equilibria is the subject of a separate manuscript.

* Corresponding author.
E-mail address: tfriesz@psu.edu.

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1. Introduction

We are concerned herein with generalization of the notion of spatial price equilibrium from a static to a dynamic setting. Spatial price equilibrium is considered an expression of how the transport of commodities occurs in response to differences between local and distant prices of transportable goods, with an underlying equilibration process that equates local and delivered prices. In particular, spatial price equilibrium is achieved when remote market price equals local market price plus the generalized cost of transport of the good of interest to the remote market. Spatial price equilibrium is often considered the foundation theory of freight transportation. The static theory of spatial price equilibrium has been moved forward by Samuelson (1952), Beckmann et al. (1956), Takayama and Judge (1964), Florian and Los (1982), Friesz et al. (1983), Tobin and Friesz (1983), Smith (1984), Chao and Friesz (1984), Smith and Friesz (1985), and several others over the last 70 years.

It is Beckmann in Chapter 5 of Beckmann et al. (1956) who is the first to give the key insights for an extremal formulation of spatial price equilibrium; this occurs in the same book where Beckmann gives the much studied and widely employed mathematical programming formulation of static user equilibrium of passenger flows. Also found in Chapter 5 of Beckmann et al. (1956) are remarks about how dynamic equilibrium models might be constructed. Although Beckmann's suggestions regarding dynamic equilibrium have been followed with respect to dynamic user equilibrium (Wardrop's first principle), relatively little research has been done on extensions of spatial price equilibrium to a dynamic setting.[a] We do note that Friesz et al. (2006) studied dynamic spatial oligopoly using dynamics like those of this paper but without explicit time shifts accounting for shipment delays.

In this paper, we present a dynamic extension of spatial price equilibrium and, in the process, provide (*i*) a succinct definition of dynamic spatial price equilibrium (DSPE), (*ii*) an associated differential variational inequality (DVI), and (*iii*) an analysis of that DVI. These are provided in the hope of stimulating research on DSPE by other scholars, as well as applications by professionals involved in strategic freight planning. Our contribution is the first spatial price equilibrium formulation that explicitly treats lead times (time shifts) in a continuous time context in the articulation of inventory dynamics, when such lead times are dictated by the transport of goods to distant markets. Intimately tied to our basic formulation is its reformulation as a differential variational inequality (DVI), which immediately provides necessary conditions that must be satisfied by equilibrium solutions.

The work reported herein is meant to be a theoretical contribution centered around formulation and a thorough explanation of the relevant DVI necessary conditions, in which there are noninteger time shifts that capture the intrinsic lead time between shipping and demand fulfillment. The treatment of explicit time shifts is new to the study of spatial price equilibrium and still not common in supply chain modeling. No analysis of time shifts in models of goods transport have used the rigorous continuous-time perspective contained in this paper. This is so despite the fact that treatments of time shifts using a discrete-time modeling perspective are, at best, approximations that may fail to assure time-consistent model solutions.

We employ the calculus of variations to show the differential variational inequality (DVI) associated with DSPE has necessary conditions that are recognizably appropriate time-shifted spatial price equilibrium conditions. Thereby, we establish that our DVI formulation of spatial price equilibrium integrates, in a dynamic setting, the path delay operator notion from the theory of dynamic user equilibrium with the theory of spatial price equilibrium. This integration constitutes a significant addition to the spatial price equilibrium and freight network equilibrium modeling literatures. Among other things, it points the way for researchers interested in dynamic traffic assignment to become involved in freight modeling using the technical knowledge they already possess. In particular, our DVI representation allows algorithms developed for solving dynamic user equilibrium models to be adapted to computing a DSPE.[b]

As such, the DSPE model presented herein may be applied to any circumstance warranting use of a dynamic aggregative freight model when adequate data, including that needed to estimate or derive inverse commodity supply and demand functions, are available. However, the detailed analysis of the existence of solutions to the DVI, as well as algorithms for solving it, are presented in a separate manuscript (Friesz, 2023). However, we do comment that DSPE, like DUE, will suffer from the nonmonotonic nature of the path delay operator. Convergence for some applicable algorithms may, however, be proven by invoking weak monotonicity of path delay (Friesz et al., 2021) simultaneously with strong monotonicity of derived demand, as that notion is defined in Friesz et al. (1983).

The freight traffic we envision is that of trucks traveling over a congested network in a large metropolitan region. Anywhere within the network, the traffic stream contains both passenger vehicles and trucks whose travel characteristics are determined by a 2-class dynamic network loading model that informs the spatial price equilibrium. In particular, we imagine a traffic delay operator of the form $\Phi_p^k(t,h)$, which provides the delay per unit of flow (departure rate) of commodity k relative to path p, where h is a vector of such flows, departing at time t, expressed via a network loading model that has preloaded automobile flows or simultaneously loads both freight and automobile flows. Path delay operators are now widely accepted in the dynamic traffic assignment and dynamic user equilibrium (DUE) modeling literature as a means of capturing congestion phenomena in a fashion consistent with Lighthill-Whitham-Richards (LWR) traffic flow theory and the first-in-first-out (FIFO) queue discipline. Effective means for the numerical calculation and software instantiation of path delay operators via dynamic network loading are now well established, as extensively reviewed in Friesz and Han (2022, 2023b, 2023c). Moreover, LWR-based delay operators cannot be replaced with ad hoc link delay functions based on link congestion functions used in static traffic assignment; doing so runs the risk of violating the first-in-first-out (FIFO) queue discipline, as well as the introduction of other flow propagation errors that render computed DSPE/DUE solutions meaningless.

[a] A notable exception is Holguin-Veras et al. (2016) who base their analysis on the tours of freight vehicles rather than point-to-point deliveries as is traditional in spatial price equilibrium modeling.

[b] For a discussion of DUE algorithms, see the review in Friesz and Han (2023a).

Han et al. (2016) and Friesz and Han (2022, 2023b, 2023c) explain how the dynamic network loading (DNL) problem is formulated to assure LWR theory and FIFO-obedient queuing are in effect. They also discuss how a DNL problem is solved to obtain the associated path delay operator. They present examples and cite definitive references demonstrating how the path delay operator is accessible and practical during numerical calculations of equilibria. In fact, the path delay operator concept has been presented, explained, and applied in the following: Han et al. (2013), Han et al. (2015a), Han et al. (2015b), Han et al. (2016), Friesz and Han (2019), and Friesz et al. (2021). In particular, Han et al. (2016) is a complete exposition of the path delay operator (PDO) in a single journal paper. Moreover, it is a paper devoted exclusively to the definition, analysis, and mathematical properties of the PDO. The other works named above establish the merit of the path delay operator (PDO) in terms of its computability and utility for studying traffic networks. These works also show the compatibility of the PDO with key aspects of behavioral modelling—like elastic travel demand, bounded rationality in route and departure time choice, and existence of equilibrium solutions.

In Han et al. (2016), the path delay operator formalism is presented in considerable depth. They begin by describing the pure dynamic user equilibrium (DUE) submodel and the dynamic network loading (DNL) submodel, both of which are needed to compute a dynamic user equilibrium. The DUE submodel determines departure rates given path delays, and the DNL submodel determines path delays given departure rates. They then introduce the concept of a path delay operator (PDO) that maps the departure rate vector to the path delay vector for each instant of continuous time. As such, the PDO is the DNL, since the only inputs to the DNL are departure rates and its most relevant outputs are path delays. The DNL explicitly involves a network-LWR model.

Han et al. (2016) present, as background, a network-LWR model with an associated Riemann solver. This model is the foundation for reformulating the dynamic network loading (DNL) problem as a partial differential algebraic equation (PDAE) system involving LWR link dynamics, boundary conditions, and path disaggregation constraints, as well as merge and diverge junction models. Within the PDAE system, they construct a path delay computational procedure for each departure rate vector; this procedure demonstrates that path delay may be determined from departure rates, for any given instant of time, and that the notion of a PDO is well founded and intrinsic to DNL, although the PDO cannot generally be expressed in a closed form. In practice, it is a numerical operator that is invoked for each instant of time considered in a DUE solution algorithm. The complete PDAE equation set is summarized in Section 3.5 of Han et al. (2016).

We reiterate that, in this paper, we seek a model of spatial price equilibrium that integrates, in a dynamic setting, the path delay operator notion presented in Friesz and Han (2022, 2023b, 2023c) with the foundation theory of goods movement, namely the theory of spatial price equilibrium. This integration influences the essential features of the model explicated herein. In the presentation that follows, we proceed constructively: a model is presented, reformulated to make it tractable, and analyzed to uncover critical properties. A familiarity with optimal control theory and differential variational inequalities at the level of Friesz (2010), Friesz and Han (2019) and/or Friesz and Han (2022, 2023d) will facilitate understanding the necessary conditions of the proposed DSPE DVI.

2. Key notation and assumptions

We posit a freight network that transports several types of commodities between markets with positive excess supply and those with positive excess demand. We use t to denote continuous time. The decision variables of the model correspond to the arabic letters h, S, D, and I. In particular, h will refer to path flow, S to rate of supply, D to demand rate, and I to inventory/backorder level. Furthermore, we will use c to refer to transportation cost per unit of flow. Commodity prices will be π. The relevant subscripts/superscripts for variables, as is meaningful, are p for a specific path, k for a specific commodity, and i or j for a specific node. Other notation will be introduced as needed. Dual variables for the terminal-time constraints will be related to commodity prices, as explained in Section 5.1 and Section 5.2.

We will employ the following sets in discussing spatial price equilibrium and its extension from a static to a dynamic setting:

$\mathscr{N}$ = the set of nodes in the network of interest
$\mathscr{K}$ = the set of commodities
$\mathscr{W}$ = the set of origin-destination pairs
$\mathscr{P}$ = the set of all paths connecting the OD pairs
$\mathscr{P}_{ij}^{k}$ = the subset of paths connecting $(i,j) \in \mathscr{W}$ and suitable for commodity $k \in \mathscr{K}$

We consider continuous time $t \in [t_0, t_1]$ where t_0 is the initial instant of the time interval of interest and t_1 is the final instant. Of course, $t_1 > t_0$. We also have, for every $t \in [t_0, t_1]$, the following vectors:

$$h(t) = \left(h_p^k(t) : p \in \mathscr{P}, k \in \mathscr{K}\right)$$

$$S(t) = \left(S_i^k(t) : i \in \mathscr{N}, k \in \mathscr{K}\right)$$

$$D(t) = \left(D_i^k(t) : i \in \mathscr{N}, k \in \mathscr{K}\right)$$

$$\pi(t) = \left(\pi_i^k(t) : i \in \mathscr{N}, k \in \mathscr{K}\right)$$

Moreover, we assume

$$h \in \left(L_+^2[t_0, t_1]\right)^{|\mathscr{P}||\mathscr{K}|} \qquad \pi : [t_0, t_1] \rightarrow \left(L_+^2[t_0, t_1]\right)^{|\mathscr{N}||\mathscr{K}|}$$

$$S \in \left(L_+^2[t_0, t_1]\right)^{|\mathscr{N}||\mathscr{K}|} \qquad I : [t_0, t_1] \rightarrow \left(W^1[t_0, t_1]\right)^{|\mathscr{N}||\mathscr{K}|}$$

$$D \in \left(L_+^2[t_0, t_1]\right)^{|\mathscr{N}||\mathscr{K}|}$$

where $L_+^2[t_0, t_1]$ is the space of square integrable functions relative to the interval $[t_0, t_1]$ of the real line and $\left(L_+^2[t_0, t_1]\right)^{|\mathscr{N}||\mathscr{K}|}$ is its $|\mathscr{N} \| \mathscr{K}|$-fold product, while $W^1[t_0, t_1]$ is a Sobolev and space and $\left(W^1[t_0, t_1]\right)^{|\mathscr{N}||\mathscr{K}|}$ is its $|\mathscr{N} \| \mathscr{K}|$-fold product. We take the variables (*h, S, D*) to be controls; the inventory variables *I* will be the state variables.

3. Delay operator, minimum shipping latency, and spatial price equilibrium

The notion of a unit path delay operator $\Phi_p^k(t, h)$ for $p \in \mathscr{P}$ gives the delay (latency) per unit of flow for shipments of commodity k over path p when those shipments depart the origin at time t and encounter traffic conditions h. It includes free flow shipping delay.

Throughout our presentation, the following assumptions will be in effect:

Assumption 1. The h variables are piecewise smooth.

Assumption 2. The S and D variables are piecewise smooth.

Assumption 3. Inverse supply and demand functions exist.

Assumption 4. Every path delay, value of time, and freight tariff is strictly positive.

In the DUE literature the unit path delay operator is taken to be measurable; therefore, that assumption, although we shall not refer to it explicitly, applies here implicitly. These very mild regularity conditions are needed for the analyses of and remarks about necessary conditions for the DSPE DVIs presented in Sections 5 and 6.

The unit cost $c_p^k(t, h)$ for shipping k over p is given by the sum of a fixed freight tariff r_{ij}^k and the economic loss due to congestion:

$$c_p^k(t, h) = r_{ij}^k + \zeta_k \Phi_p^k(t, h) \tag{1}$$

where ζ_k is the value of time for commodity $k \in K$. We will also have need for the definition

$$\tau_{ij}^k \equiv \min \Phi_p^k(t, h) : h \in \Omega(S^*, D^*), t \in [t_0, t_1], \tag{2}$$

which is the minimum shipping time (delay or latency) for commodity $k \in \mathscr{K}$ and path $p \in \mathscr{P}_{ij}^k$ connecting OD pair $(i,j) \in \mathscr{W}$, where $\Omega(S^*, D^*)$ is a set of feasible departure rate solutions corresponding to equilibrium supply S^* and equilibrium demand D^*. The cost vector is an operator in an infinite dimensional vector space:

$$c(h) \equiv \left(c_p^k(\cdot, h) : p \in \mathscr{P}, k \in \mathscr{K}\right) \in \left(L_+^2[t_0, t_1]\right)^{|\mathscr{P}||\mathscr{K}|}$$

where $L_+^2[t_0, t_1]$ and $\left(L_+^2[t_0, t_1]\right)^{|\mathscr{P}||\mathscr{K}|}$ have been previously defined.

We may sometimes, when context prevents misunderstanding, we may take time dependency to be implicit and drop direct reference to t. Also, when the spatial price equilibrium conditions are stated in terms of time shifts, we may drop the dependence on unshifted time, again taking that dependency to be implicit.

3.1. The extension of static spatial price equilibrium to DSPE

The essential characteristic of a spatial price equilibrium is that, if the shipping rate between a pair of supply and demand nodes is positive, the delivered price equals the local price. Moreover, if the delivered price exceeds local price, the shipping rate is zero. These properties are easily expressed for a dynamic setting like ours in the following way that is familiar from the study of static spatial price equilibrium, as presented in Friesz et al. (1983) and the works of other scholars:

$$h_p^k > 0, p \in \mathscr{P}_{ij}^k \Rightarrow \pi_i^k(t) + c_p^k(t, h) = \pi_j^k\left(t + \tau_{ij}^k\right) \tag{3}$$

$$\pi_i^k(t) + c_p^k(t, h) > \pi_j^k\left(t + \tau_{ij}^k\right), p \in \mathscr{P}_{ij}^k \Rightarrow h_p^k = 0 \tag{4}$$

where each h_p^k refers to the flow (departure rate) associated with path $p \in \mathscr{P}_{ij}^k$ of commodity $k \in \mathscr{K}$; similarly, each π_i^k refers to the price of commodity $k \in \mathscr{K}$ produced at node (market) $i \in \mathscr{N}$. In (3) and (4), we have indicated that prices $\pi_i^k(t)$ and $\pi_j^k\left(t + \tau_{ij}^k\right)$ are compared to establish equilibrium because of the delay intrinsic to shipping goods from market i to market j. Note also that

$$\pi_i^k(t) + c_p^k(t,h) \geq \pi_j^k\left(t + \tau_{ij}^k\right) \ \forall (i,j) \in \mathscr{W}, k \in \mathscr{K}, p \in \mathscr{P}_{ij}^k \tag{5}$$

If (5) did not obtain, then, for some $(i,j) \in \mathscr{W}$, we would have

$$\pi_i^k(t) + c_p^k(t,h) < \pi_j^k\left(t + \tau_{ij}^k\right)$$

and need to consider two cases:

(i) $h_p > 0 \Rightarrow \pi_i^k(t) + c_p^k(t,h) = \pi_j^k\left(t + \tau_{ij}^k\right)$, which is a contradiction; or

$(ii) h_p = 0$, which is a failure to take advantage of an apparent spatial arbitrage (delivered price strictly less than local price). Since the presence of such arbitrage opportunities is inconsistent with equilibrium, we enforce (5). We also note that, in light of the non-negativity of departure rates ($h \geq 0$), expression (4) is redundant since it is implied by (3).

At the same time S_i^k and D_i^k, which respectively refer to supply and demand rates associated with the production and consumption of commodity $k \in \mathscr{K}$ at node $i \in \mathscr{N}$, must satisfy:

$$\pi_i^k = \psi_i^k(S) \text{ when } S_i^k > 0 \tag{6}$$

$$\pi_i^k = \Theta_i^k(D) \text{ when } D_i^k > 0 \tag{7}$$

where $\Psi_i^k(.)$ and $\Theta_i^k(.)$ are, respectively, inverse supply and inverse demand functions. We also expect that

$$\Psi_i^k(S) > \pi_i^k \ \Rightarrow S_i^k = 0 \tag{8}$$

to reflect that there will be no production when the price computed from the inverse supply function exceeds the market price. Similarly, we expect that

$$\pi_i^k > \Theta_i^k(D) \Rightarrow D_i^k = 0 \tag{9}$$

to reflect that there will be no consumption when price computed from the inverse demand function is less than the market price. The upshot of (6), (7), (8) and (9) is

$$\begin{aligned} \Psi_i^k(S) - \pi_i^k &\geq 0 \ \forall k \in \mathscr{K}, i \in \mathscr{N} \\ -\Theta_i^k(D) + \pi_i^k &\geq 0 \ \forall k \in \mathscr{K}, i \in \mathscr{N} \end{aligned}$$

Naturally, we have the vectors

$$\begin{aligned} \Psi(S) &= \left(\Psi_i^k(S) : i \in \mathscr{N}\right) \\ \Theta(D) &= \left(\Theta_i^k(D) : i \in \mathscr{N}\right) \end{aligned}$$

where time is implicit.

3.2. Causation of time shifts in DSPE

When first articulating (3), (4), and (5), it seems that one is stipulating the conditions of price equilibrium before presenting the dynamics that govern the evolution of commodity flows and prices, whatever those might be. Do the time shifts appearing in those expressions arise from a philosophy of pricing, such as cash on delivery (COD), or do they devolve from the physics of freight transport? In Section 4, we introduce dynamics describing the formation of inventories from considerations of production, consumption, export and import at each node (market). Such inventory dynamics involve time shifts for path flows (departure rates) to reflect the intrinsic physical delay associated with the movement of tangible goods but make no direct reference to prices. In Section 5, we will see how it is possible to depict DSPE as a differential variational inequality (DVI) without the a priori assumption of (3), (4), and (5), the spatial price equilibrium conditions with time shifts. Moreover, we show in Sections 5 and 6 that (3), (4), and (5) arise from consideration of the dual variables of the inventory dynamics that constrain the DVI. As such the DVI explains (3), (4), and (5) as the consequence of inventory dynamics with time shifts that reflect the physics of transport. That is, the inventory dynamics do not presume any particular pricing policy yet force obedience to equilibrium conditions (3), (4), and (5), as is systematically explained in Section 5.3.

4. The unembellished DSPE model

In this section we present the mathematical statement of our basic dynamic spatial price equilibrium (DSPE) model under the assumption that there are prescribed initial and final inventory levels for each commodity, but there are no constraints on inventory other than the inventory (flow conservation) dynamics to be articulated below.

4.1. Inventory dynamics and related constraints

For all $k \in \mathscr{K}$ and $i \in \mathscr{N}$, the inventory dynamics we shall consider are the following:

$$\frac{dI_i^k(t)}{dt} = S_i^k(t) + \sum_{j\in:(j,i)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ji}^k} h_p^k\left(t-\tau_{ji}^k\right) - \sum_{j:(i,j)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ij}^k} h_p^k(t) - D_i^k(t) \tag{10}$$

$$I_i^k(t_0) = A_i^k \tag{11}$$

$$I_i^k(t_1) = B_i^k \tag{12}$$

where there is an adjoint variable[c] λ_i^k associated with each inventory differential Eq. (10). We do not yet know the relationship of the adjoint variables to prices. The constants A_i^k and B_i^k give the initial and final values of inventory for commodity k and market i. For commodity $k \in \mathscr{K}$, the flows inbound to market (node) i are time shifted by an amount τ_{ji}^k that represents the travel time between $(j, i) \in \mathscr{W}$.

In expressions (10)-(12), t_0 is the initial instant of the time interval $[t_0,t_1]$ of interest, for which the final instant is t_1. Of course, $t_1 > t_0$. In order for a shipment from market j to influence inventory at market i at the time t for which the inventory dynamics (10) (which are flow conservation constraints) are expressed, that shipment must depart at $t-\tau_{ji}$, where τ_{ji} is the shipment time between those markets. Furthermore, each h_p^k refers to the departure rate of commodity k along path p, while S_i^k and D_i^k refer to supply and demand rates at node i of commodity k, respectively.

Nonnegativity restrictions on the path departure, supply and demand rates must be imposed:

$$-h_p^k \le 0 \qquad \left(\mu_p^k\right) \qquad \forall k \in \mathscr{K}, p \in \mathscr{P} \tag{13}$$

$$-S_i^k \le 0 \qquad \left(\alpha_i^k\right) \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{14}$$

$$-D_i^k \le 0 \qquad \left(\beta_i^k\right) \qquad \forall k \in \mathscr{K}, i \in \mathscr{N}, \tag{15}$$

where the variables in parentheses are dual variables associated with their nonnegativity constraints (13)-(15). We rewrite the terminal time constraints on inventory in implicit form:

$$B_i^k - I_i^k(t_1) = 0 \qquad \left(\pi_i^k\right) \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{16}$$

where the B_i^k are fixed. The π_i^k are dual variables for the stipulated final value constraints (16); we shall see for the unembellished model that they are also commodity prices.

4.2. The model in summary

For all $t \in [t_0,t_1]$ we seek a solution to this system, each aspect of which has been presented above:

$$h_p^k > 0, p \in \mathscr{P}_{ij}^k \Rightarrow \pi_i^k(t) + c_p^k(t,h) = \pi_j^k\left(t+\tau_{ij}^k\right) \tag{17}$$

$$\pi_i^k(t) + c_p^k(t,h) \ge \pi_j^k\left(t+\tau_{ij}^k\right) \qquad \forall (i,j) \in \mathscr{W}, k \in \mathscr{K}, p \in \mathscr{P}_{ij}^k \tag{18}$$

$$\frac{dI_i^k(t)}{dt} = S_i^k(t) + \sum_{j\in:(j,i)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ji}^k} h_p^k\left(t-\tau_{ji}^k\right) - \sum_{j:(i,j)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ij}^k} h_p^k(t) - D_i^k(t) \tag{19}$$

$$I_i^k(t_0) = A_i^k \; B_i^k - I_i^k(t_1) = 0 \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{20}$$

$$S_i^k > 0 \Rightarrow \pi_i^k = \Psi_i^k(S) \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{21}$$

$$D_i^k > 0 \Rightarrow \pi_i^k = \Theta_i^k(D) \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{22}$$

$$\Psi_i^k(S) - \pi_i^k \ge 0 \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{23}$$

[c] Also sometimes called a costate variable. Such variables are dynamic dual variables describing the sensitivity an agent's performance functional to changes in state.

$$\pi_i^k - \Theta_i^k(D) \geq 0 \quad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{24}$$

$$h \geq 0 \quad S \geq 0 \quad D \geq 0 \tag{25}$$

Formulation (17) through (25) may be restated as a nonlinear complementarity problem when an appropriate inexpensive goods condition is imposed, as discussed in Friesz (2023), as an aid to computation. However, our interest in this paper is in restating (17)-(25) as a differential variational inequality (DVI) in order to illustrate the dynamic user equilibrium nature of its solutions as revealed by a thorough variational analysis of DSPE.

4.3. *The differential variational inequality*

Familiarity with differential variational inequalities allows us to conjecture that the DSPE problem may be solved by $(h^*, S^*, D^*) \in \Omega$ that solves

$$\int_{t_0}^{t_1} \left[c(t,h^*)^T(h-h^*) - \Theta(D^*)^T(D-D^*) + \Psi(S^*)^T(S-S^*) \right] dt \geq 0 \qquad \forall (h,S,D) \in \Omega \tag{26}$$

where Ω is the set of feasible controls, defined as

$$\Omega = \{(h,S,D)\} : (17) - (25) \ hold\}$$

If the application of appropriate necessary conditions for differential variational inequality (26) yields the DSPE conditions (17) and (18), along with the other relevant considerations described above, we will have established that we have a correct DVI formulation.

4.4. *Recasting the problem*

It is helpful to briefly consider this abstract variational inequality:

$$F(x^*)^T(x-x^*) \geq 0 \qquad x, x^* \in \Lambda,$$

which is equivalent to

$$F(x^*)^T x \geq F(x^*)^T x^* \qquad x, x^* \in \Lambda,$$

which is a statement that $x^* \in \Lambda$ solves

$$\min F(x^*)^T x \quad \text{s.t.} \quad x \in \Lambda \tag{27}$$

Note that program (27) is a mathematical construct whose objective function presumes knowledge of the equilibrium solution $x^* \in \Lambda$, making it useful only for analysis not computation.

We study a reformulation of DVI (26) using the ideas presented in creating (27) to generate the following optimal control problem:

$$min \sum_{k \in \mathscr{K}} \sum_{i \in \mathscr{N}} \pi_i^k \left[B_i^k - I_i^k(t_1) \right] + \int_{t_0}^{t_1} L(h,S,D,t) dt \ s.t. \ (h,S,D) \in \Omega \tag{28}$$

where

$$L = \sum_{k \in \mathscr{K}} \sum_{(i,j) \in \mathscr{W}} \sum_{p \in \mathscr{P}_{ij}} c_p^k(t,h^*) h_p^k - \sum_{k \in \mathscr{K}} \sum_{i \in \mathscr{N}} \Theta_i^k(D^*)^T D_i^k + \sum_{k \in \mathscr{K}} \sum_{j \in \mathscr{N}} \Psi_j^k(S^*) S_j^k \tag{29}$$

We reiterate that this optimal control problem may be used for analysis but not for computation since it is parametric in the equilibrium solution (h^*, S^*, D^*). We will use it to derive differential spatial price equilibrium conditions, thereby showing our DVI formulation is valid. That is, any solution of the DVI will be a DSPE.

4.5. *Optimal control with time shifts*

In this section, we consider optimal control with time-shifted state and control variables in order to treat the time shifts appearing in the spatial price equilibrium conditions introduced earlier. Our presentation is based on Budelis and Bryson (1970), Friesz et al. (2001), and Friesz (2010). Let us address the following abstract optimal control problem:

$$\min J(u) = \nu^T \psi[x(t_1), t_1] + \int_{t_0}^{t_1} f_0(x, u, t)dt \tag{30}$$

subject to

$$\frac{dx}{dt} = f(x, x_\tau, u, u_\tau, t) \qquad t \in [t_0, t_1] \tag{31}$$

$$x(t_0) = x_0 \tag{32}$$

$$x(t) = x_0(t) \qquad t \in [t_0 - \tau, t_0) \tag{33}$$

$$g(x, u, t) \le 0 \quad (\mu) \tag{34}$$

$$\psi[x(t_1), t_1] = 0 \quad (\nu) \tag{35}$$

where t is continuous time, x is the state vector, u is the control vector, $x_\tau \equiv x(t-\tau)$, $u_\tau \equiv u(t-\tau)$, $\tau > 0$, ν is a dual variable, and τ is a constant time shift. In the discussion that follows, H is the Hamiltonian given by

$$H = f_0(x, u, t) + \lambda^T f(x, x_\tau, u, u_\tau, t) + \mu^T g(x, u, t) \tag{36}$$

and λ is the adjoint vector obeying the following:

$$\frac{d\lambda}{dt} = -H_x \qquad t \in [t_0, t_1 - \tau] \tag{37}$$

$$\frac{d\lambda}{dt} = -H_x - H_{x_\tau}\big|^{t+\tau} \qquad t \in (t_1 - \tau, t_1] \tag{38}$$

$$\lambda(t_1) = \nu \frac{\partial \psi(x(t_1), t_1)}{\partial x(t_1)}, \tag{39}$$

where ν is a dual variable for the terminal time constraint $\psi[x(t_1), t_1] = 0$. Budelis and Bryson (1970), Friesz et al. (2001), and Friesz (2010) give the following necessary conditions:

$$H_u + [H_{u_\tau}]^{t+\tau} = 0 \qquad t \in [t_0, t_1 - \tau] \tag{40}$$

$$H_u = 0 \qquad t \in (t_1 - \tau, t_1] \tag{41}$$

$$\mu^T g = 0 \tag{42}$$

$$\mu \ge 0 \tag{43}$$

Expressions (40) and (41) are the minimum principle, while (42) and (43) are complementary slackness conditions.

5. Analyzing the necessary conditions

As we have reviewed in Section 4.5, necessary conditions for optimal control problems require the minimization of the Hamiltonian, which is comprised of the integrand of the objective and priced out constraints. The optimal control problem (28)-(29) has the following Hamiltonian:

$$\begin{aligned} H_0 = &\sum_{k\in\mathscr{K}} \sum_{(i,j)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ij}^k} c_p^k(t, h^*) h_p^k - \sum_{k\in\mathscr{K}} \sum_{i\in\mathscr{N}} \Theta_i^k(D^*) D_i^k + \sum_{k\in\mathscr{K}} \sum_{i\in\mathscr{N}} \Psi_i^k(S^*) S_i^k \\ &+ \sum_{k\in\mathscr{K}} \sum_{i\in\mathscr{N}} \lambda_i^k(t) \left[S_i^k(t) + \sum_{j:(j,i)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ji}^k} h_p^k\left(t - \tau_{ji}^k\right) - \sum_{j:(i,j)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ij}^k} h_p^k(t) - D_i^k(t) \right] \\ &+ \sum_{k\in\mathscr{K}} \sum_{i\in\mathscr{N}} \left[\rho_p^k\left(-h_p^k\right) + \rho_{p\tau}^k\left(-h_{p\tau}^k\right) + \alpha_i^k\left(-S_i^k\right) + \beta_i^k\left(-D_i^k\right) \right] \end{aligned} \tag{44}$$

Note that in (44) the following term

$$\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\lambda_i^k(t)\left[-\sum_{j:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k(t)+\sum_{j:(j,i)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ji}^k}h_p^k\left(t-\tau_{ji}^k\right)\right]=-\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\lambda_i^k(t)\sum_{j:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k(t)+\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\lambda_i^k(t)\sum_{j:(j,i)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ji}^k}h_p^k\left(t-\tau_{ji}^k\right) \tag{45}$$

is evident. One may, by exchanging the roles of i and j in the second term, rewrite the immediately preceding expression as

$$-\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\lambda_i^k(t)\sum_{j:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k(t)+\sum_{k\in\mathscr{K}}\sum_{j\in\mathscr{N}}\lambda_j^k(t)\sum_{i:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k\left(t-\tau_{ij}^k\right) \tag{46}$$

Using $\pi_i^k=-\lambda_i^k$ for all $i\in\mathscr{N}$ and $k\in\mathscr{K}$, a relationship that will be established subsequently, (46) becomes

$$\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\pi_i^k(t)\sum_{j:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k(t)-\sum_{k\in\mathscr{K}}\sum_{j\in\mathscr{N}}\pi_j^k(t)\sum_{i:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k\left(t-\tau_{ij}^k\right) \tag{47}$$

As a consequence, we may state the Hamiltonian for the DVI as

$$\begin{aligned}H_0=&\sum_{k\in\mathscr{K}}\sum_{(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}c_p^k(t,h^*)h_p^k-\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\Theta_i^k(D^*)D_i^k+\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\Psi_i^k(S^*)S_i^k\\&+\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\pi_i^k(t)\left[D_i^k(t)-S_i^k(t)\right]\\&+\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\pi_i^k(t)\sum_{j:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k(t)-\sum_{k\in\mathscr{K}}\sum_{j\in\mathscr{N}}\pi_j^k(t)\sum_{i:(i,j)\in\mathscr{W}}\sum_{p\in\mathscr{P}_{ij}^k}h_p^k\left(t-\tau_{ij}^k\right)\\&+\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{N}}\left[\rho_p^k\left(-h_p^k\right)+\rho_{p\tau}^k\left(-h_{p\tau}^k\right)+\alpha_i^k\left(-S_i^k\right)+\beta_i^k\left(-D_i^k\right)\right]\end{aligned} \tag{48}$$

5.1. *Transversality conditions and adjoint equations*

The transversality conditions (39) yield

$$\lambda_i^k(t_1)=\pi_i^k\frac{\partial\left[B_i^k-I_i^k(t_1)\right]}{\partial I_i^k(t_1)}=-\pi_i^k \tag{49}$$

Thus, the dual variables π_i^k assigned to the terminal inventory constraints are constant and equal to the negative of the adjoint variables at the terminal time t_1. That is, we confirm our prior assumption that $\pi_i^k=-\lambda_i^k$ for all $i\in\mathscr{N}$ and $k\in\mathscr{K}$. In Section 5.2, we show the dual variables are valid prices. Moreover, commodity prices are constant over time for the unembellished model. However, demonstration of these features requires the detailed analysis presented below in Sections 5.2 and 5.3.

We next show the adjoint variables are constant for the purposely simplified unembellished model summarized in Section 4.2. Because the only state variables are inventories and such variables do not appear in the Hamiltonian, the analysis of the adjoint equations is especially easy. We obtain the following result from the adjoint Eqs. (37) and (38):

$$\frac{d\lambda_i^k}{dt}=-\frac{\partial H_0}{\partial I_i^k}=0\qquad\forall(i,k),t\in[t_0,t_1], \tag{50}$$

which implies that each λ_i^k is a constant. Moreover, (49) and (50) establish that

$$\pi_i^k=-\lambda_i^k=a\ constant\qquad\forall(i,k) \tag{51}$$

5.2. *The minimum principle: supply and demand variables*

In this section we establish that prices follow the demand and supply functions and the dual variables on terminal inventory are the commodity prices we seek in spatial price equilibrium.

The following are first-order conditions associated with commodity supply and demand that arise from the minimum principle:

$$\frac{\partial H_0}{\partial S_i^k}=\Psi_i^k(S^*)-\pi_i^k-\alpha_i^k=0 \tag{52}$$

$$\frac{\partial H_0}{\partial D_i^k}=(-1)\cdot\theta_i^k(D^*)+\pi_i^k-\beta_i^k=0 \tag{53}$$

$$\alpha_i^kS_i^k=0\ \alpha_p^k\geq0 \tag{54}$$

$$\beta_i^k D_i^k = 0 \quad \beta_p^k \geq 0 \tag{55}$$

Conditions (52)-(54) assure the following:

$$S_i^k > 0 \Rightarrow \Psi_i^k(S^*) = \pi_i^k \tag{56}$$

$$D_i^k > 0 \Rightarrow \theta_i^k(D^*) = \pi_i^k \tag{57}$$

In other words, nontrivial supplies and demands are consistent with inverse supply and inverse demand functions, respectively, for all markets and commodities. Results (56) and (57) confirm that the dual variables π_i^k are prices. That is, we have established that the dual vector $\pi = \left(\pi_i^k : \ k \in \mathscr{K}, \ i \in \mathscr{N}\right)$ is comprised of prices at individual markets (nodes) of the network economy. Moreover, because prices are constant by virtue of (51), supplies and demands are likewise constant.

5.3. Minimum principle: departure rates

The Hamiltonian H_0 expressed in the form (48) sets the stage for applying the minimum principle (40) for the time-shifted departure rates intrinsic to DSPE; doing so, we find

$$\frac{\partial H}{\partial h_p^k} + \left[\frac{\partial H}{\partial h_{p\tau}^k}\right]^{t+\tau} = c_p^k(t,h^*) + \pi_i^k(t) - \pi_j^k\left(t+\tau_{ij}^k\right) - \mu_p^k - \mu_{p\tau}^k = 0$$

when

$$t \in \left[t_0, t_1 - \tau_{ij}^k\right] \tag{58}$$

We also know

$$\rho_p^k h_p^k = 0 \qquad \rho_p^k \geq 0 \tag{59}$$

$$\rho_{p\tau}^k h_{p\tau}^k = 0 \qquad \rho_{p\tau}^k \geq 0 \tag{60}$$

where

$$h_{p\tau}^k \equiv h_{p\tau}^k\left(t - \tau_{ij}^k\right)$$

From (58), (59), and (60) we have

$$c_p^k(t,h^*) + \pi_i^k(t) - \pi_j^k\left(t+\tau_{ij}^k\right) \geq 0 \tag{61}$$

From the same conditions, we see that the following result reminiscent of static spatial price equilibrium holds for solutions of DVI (26):

$$\begin{aligned} h_p^k \quad &> \quad 0 \text{ and } h_{p\tau}^k > 0, p \in \mathscr{P}_{ij}^k \Rightarrow \rho_p^k = \rho_{p\tau}^k = 0 \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right] \\ &\Rightarrow \qquad c_p^k(t,h^*) + \pi_i^k(t) - \pi_j^k\left(t+\tau_{ij}^k\right) = 0 \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right] \end{aligned} \tag{62}$$

Moreover, we see that

$$c_p^k(t,h^*) + \pi_i^k(t) > \pi_j^k\left(t+\tau_{ij}^k\right), \ p \in \mathscr{P}_{ij}^k \ \Rightarrow h_p^k = 0 \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right] \tag{63}$$

since the alternative, $h_p^k > 0$, requires $c_p^k(t,h^*) + \pi_i^k \ = \pi_j^k$, which is a contradiction.

The minimum principle (58) gives

$$c_p^k(t,h^*) + \pi_i^k(t) \geq \rho_p^k, \ p \in \mathscr{P}_{ij}^k \qquad t \in \left(t_1 - \tau_{ij}^k, t_1\right]$$

from which we obtain

$$c_p^k(t,h^*) + \pi_i^k(t) > 0, \ p \in \mathscr{P}_{ij}^k \ \Rightarrow h_p^k = 0 \qquad t \in \left(t_1 - \tau_{ij}^k, t_1\right] \tag{64}$$

as well as

$$h_p^k > 0, \ p \in \mathscr{P}_{ij}^k \Rightarrow c_p^k(t,h^*) + \pi_i^k(t) = 0 \qquad t \in \left(t_1 - \tau_{ij}^k, t_1\right] \tag{65}$$

Note that, in (64), $c_p^k(t,h^*) + \pi_i^k(t) = 0$ cannot occur because $c_p^k(t,h^*)$ is strictly positive for physical reasons. Hence, the circumstance of positive flow for $t \in \left(t_1 - \tau_{ij}^k, t_1\right]$ is impossible, so long as the π_i^k are nonnegative and verified as commodity prices, as they are in (56) and (57). The impossibility of (65) is fitting since there is insufficient time for a shipment to reach its destination. Therefore, conditions (62), (64) and (65) are valid statements of spatial price equilibrium; consequently, any solution of (26) is a DSPE.

6. Interpretation of DSPE and connection to due

For the unembellished model's specification, as a consequence of (61) and the definition

$$c_p^k(t,h) \equiv r_{ij}^k + \zeta_k \Phi_p^k(t,h),$$

we have that

$$\Phi_p^k(t,h) \geq \frac{\pi_j^k - \pi_i^k - r_{ij}^k}{\zeta_k} \tag{66}$$

Moreover, we know from (62) that at equilibrium (66) holds as an equality; thus

$$\min_{\substack{h\in\Omega \\ t\in[t_0,t_1]}} \Phi_p^k(t,h) = \frac{\pi_j^k - \pi_i^k - r_{ij}^k}{\zeta_k} \equiv \tau_{ij}^k, \tag{67}$$

which confirms that each τ_{ij}^k is a constant since the adjoint equations assure commodity prices are constant and the tariff r_{ij}^k is constant by stipulation. This justifies using the necessary conditions for fixed time shifts put forward in Section 4.5. Moreover, because delay must be strictly positive for physical reasons, each $\tau_{ij}^k > 0$.

Although perhaps already clear, we want to emphasize that observation (67) means that the path flows of our unembellished dynamic spatial price equilibrium model constitute a dynamic user equilibrium with minimum travel time τ_{ij}^k for every OD pair and commodity, as that notion is presented in Friesz et al. (1993), Friesz and Han (2019), and Friesz and Han (2022, 2023d). In particular, we note from (51), (62) and (67) that

$$\begin{aligned}
h_p^k &> 0, h_{p\tau}^k > 0 \Rightarrow c_p^k(t,h) + \pi_i^k - \pi_j^k = 0 \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right] \\
&\Rightarrow \quad r_{ij}^k + \zeta_k \Phi_p^k(t,h) + \pi_i^k - \pi_j^k = 0 \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right] \\
&\Rightarrow \quad \Phi_p^k(t,h) = \frac{\pi_j^k - \pi_i^k - r_{ij}^k}{\zeta_k} \equiv \tau_{ij}^k \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right]
\end{aligned}$$

for all $k \in \mathscr{K}$, $(i,j) \in \mathscr{W}$, and $p \in \mathscr{P}_{ij}^k$. That is, positive departure rates mandate that path delay is minimized, which is recognized as the defining characteristic of dynamic user equilibrium. Furthermore, as revealed by our analysis, equilibrium commodity prices are fixed in our unembellished model, freight departure (shipping) rates h adjust dynamically to assure those prices are respected, and the generalized shipping cost $c_p^k(t,h) \equiv r_{ij}^k + \zeta_k \Phi_p^k(t,h)$ maintains the delivered-price-equals-local-price property of spatial price equilibrium for all $k \in \mathscr{K}$, $(i,j) \in \mathscr{W}$, and $p \in \mathscr{P}_{ij}^k$. These are exactly the properties one wishes of a dynamic freight model that responds to transactions within a network economy characterized by established commodity prices, production schedules, and consumption patterns. Our preceding analysis has established the following:

Theorem 1. *For the unembellished DSPE, commodity prices are constant, and its solutions are not only spatial price equilibria relative to the defining relationships of* Section 4.2*, but they also are dynamic user equilibria relative to path flow h and the delay operator* $\Phi(t, h)$*in the sense that*

$$h_p^k(t) > 0, p \in \mathscr{P}_{ij}^k \Rightarrow \Phi_p^k(t,h) = \tau_{ij}^k \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right]$$

7. DSPE with time-dependent commodity prices

It is the Hamiltonian's lack of dependence on inventory that makes the adjoint variables and, hence, commodity prices constant, in the unembellished model studied in previous sections, as can be seen in expressions (49)-(51). We now introduce an additional term pertaining to deterioration and pilferage of inventory in the DSPE inventory dynamics that gives rise to the following state dynamics:

$$\frac{dI_i^k(t)}{dt} = S_i^k(t) + \sum_{j\in:(j,i)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ji}^k} h_p^k\left(t - \tau_{ji}^k\right) - \sum_{j:(i,j)\in\mathscr{W}} \sum_{p\in\mathscr{P}_{ij}^k} h_p^k(t) - D_i^k(t) - \alpha_i^k I_i^k(t) \tag{68}$$

$$I_i^k(t_0) = A_i^k \tag{69}$$

$$I_i^k(t_1) = B_i^k \tag{70}$$

where α_i^k is the fixed rate of deterioration and pilferage of inventory, for all $k \in \mathscr{K}$ and $i \in \mathscr{N}$. This means that our original Hamiltonian H_0 given by (48) is replaced by the new Hamiltonian

$$H_1 = H_0 - \sum_{k \in \mathscr{K}} \sum_{i \in \mathscr{N}} \alpha_i^k \lambda_i^k I_i^k \tag{71}$$

7.1. Deriving dynamic commodity prices

The minimum principle continues to give the same spatial price equilibrium conditions (61), (62), and (63), as well as these first order conditions, presented previously, for all $k \in \mathscr{K}$ and $i \in \mathscr{N}$:

$$\begin{aligned} \frac{\partial L}{\partial S_i^k} &= \Psi_i^k(S^*) + \lambda_i^k - \alpha_i^k = 0 \\ \frac{\partial L}{\partial D_i^k} &= (-1) \cdot \Theta_i^k(D^*) - \lambda_i^k - \beta_i^k = 0 \\ \alpha_i^k S_i^k &= 0 \ \alpha_p^k \geq 0 \\ \beta_i{}^k D_i^k &= 0 \ \beta_p{}^k \geq 0 \end{aligned}$$

It, of course, follows that

$$\begin{aligned} S_i^k &> 0 \Rightarrow \Psi_i^k(S^*) = -\lambda_i^k \\ D_i^k &> 0 \Rightarrow \Theta_i^k(D^*) = -\lambda_i^k \end{aligned}$$

Due to the very meaning of inverse commodity supply and demand functions, these expressions assure that commodity prices obey

$$\pi_i^k = -\lambda_i^k \qquad \forall k \in \mathscr{K}, i \in \mathscr{N}, \tag{72}$$

although the π_i^k are no longer dual variables associated with the terminal inventory constraints; rather, they are now merely the adjoint variables multiplied by -1. This makes sense because adjoint variables are generally dynamic rather than fixed dual variables; if dynamic prices are to be found, they are intuitively going to be related to nonconstant adjoint variables. This means that the relevant Hamiltonian is like (44) when the alternative foundation of identity (72) is understood to be implicit and consideration of inventory deterioration and pilferage is incorporated, as in (71).

To preserve our previous notation for commodity prices π_i^k, we need to rename the dual variables for the terminal constraints

$$B_i^k - I_i^k(t_1) = 0 \ \left(\widetilde{\pi}_i^k\right) \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \tag{73}$$

In (73), the $\widetilde{\pi}_i^k$ are now dual variables of the terminal inventory constraints $B_i^k - I_i^k(t_1) = 0$. As such, the adjoint equations and transversality conditions for $t \in [t_0, t_1]$ now become

$$\begin{aligned} \frac{d\lambda_i^k}{dt} &= -\frac{\partial H_1}{\partial I_i^k} = \alpha_i^k \lambda_i^k \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \\ \lambda_i^k(t_1) &= \frac{\partial \sum_{i \in \mathscr{N}} \sum_{k \in \mathscr{K}} \widetilde{\pi}_i^k \left[B_i^k - I_i^k(t_1)\right]}{\partial I_i^k(t_1)} = -\widetilde{\pi}_i^k \qquad \forall k \in \mathscr{K}, i \in \mathscr{N} \end{aligned}$$

Thus, because $\pi_i^k(t) = -\lambda_i^k(t)$, we are led to the price function

$$\pi_i^k(t) = \widetilde{\pi}_i^k \exp\left[\alpha_i^k (t - t_1)\right] \tag{74}$$

7.2. Finding shipping latencies

For the unembellished model, we have previously used this ratio:

$$\frac{\pi_j^k\left(t + \tau_{ij}^k\right) - \pi_i^k(t) - r_{ij}^k}{\zeta_k}$$

to define the minimum shipping time; this was possible because it lacked any time dependence due to the provably constant nature of commodity prices. Now we will need to explicitly minimize over time, and accordingly write this expression:

$$\tau_{ij}^k = \frac{\pi_j^k\left(t_d + \tau_{ij}^k\right) - \pi_i^k(t_d) - r_{ij}^k}{\zeta_k} = \min\left\{\frac{\pi_j^k\left(t + \tau_{ij}^k\right) - \pi_i^k(t) - r_{ij}^k}{\zeta_k} : \ t \in \left[t_0, t_1 - \tau_{ij}^k\right]\right\}, \tag{75}$$

where t_d is the delay minimizing departure time. Indeed, for a general problem, there may be multiple departure times or even a dense arc of time during which departures occur and the same minimum shipping delay is realized. Substituting (74) into (75) gives

$$\tau_{ij}^k = \frac{\widetilde{\pi}_j^k \exp\left[\alpha_j^k\left(t_d - \tau_{ij}^k - t_1\right)\right] - \widetilde{\pi}_i^k \exp\left[\alpha_i^k(t_d - t_1)\right] - r_{ij}^k}{\zeta_k} \tag{76}$$

Therefore

$$\tau_{ij}^k \zeta_k = \widetilde{\pi}_j^k \exp\left(- \alpha_j^k \tau_{ij}^k\right) \exp\left[\alpha_j^k(t_d - t_1)\right] - \widetilde{\pi}_i^k \exp\left[\alpha_i^k(t_d - t_1)\right] - r_{ij}^k$$

For small α_j^k, the first two terms of a Taylor series expansion will yield an accurate approximation of $\exp\left(- \alpha_j^k \tau_{ij}^k\right)$. That is, we use

$$\exp\left(- \alpha_j^k \tau_{ij}^k\right) \approx 1 - \alpha_j^k \tau_{ij}^k$$

for small $\alpha_j^k > 0$. Therefore, (76) may be placed in this form:

$$\tau_{ij}^k \zeta_k = \widetilde{\pi}_j^k \left(1 - \alpha_j^k \tau_{ij}^k\right) \exp\left[\alpha_j^k(t_d - t_1)\right] - \widetilde{\pi}_i^k \exp\left[\alpha_i^k(t_d - t_1)\right] - r_{ij}^k \tag{77}$$

It follows that

$$\tau_{ij}^k \zeta_k + \alpha_j^k \widetilde{\pi}_j^k \tau_{ij}^k \exp\left[\alpha_j^k(t_d - t_1)\right] = \widetilde{\pi}_j^k \exp\left[\alpha_j^k(t_d - t_1)\right] - \widetilde{\pi}_i^k \exp\left[\alpha_i^k(t_d - t_1)\right] - r_{ij}^k$$

If we introduce this simplifying definition

$$A_\ell^k \equiv \widetilde{\pi}_\ell^k \exp\left(-\alpha_\ell^k t_1\right),$$

we obtain the following expression for shipping latency:

$$\tau_{ij}^k = \frac{A_j^k\left(1 + \alpha_j^k t_d\right) - A_i^k\left(1 + \alpha_i^k t_d\right) - r_{ij}^k}{\zeta_k + \alpha_j^k A_j^k\left(1 + \alpha_j^k t_d\right)} \tag{78}$$

7.3. Finding departure times

We may find the departure time t_d appearing in (78) by setting the derivative of that expression with respect to t_d to zero. We define

$$\begin{aligned} a &= A_j^k\left(1 + \alpha_j^k t_d\right) - A_i^k\left(1 + \alpha_i^k t_d\right) - r_{ij}^k \\ b &= \zeta_k + \alpha_j^k A_j^k\left(1 + \alpha_j^k t_d\right) \end{aligned}$$

In other words

$$\begin{aligned} \dot{a}b &= \left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right]\left[\zeta_k + \alpha_j^k A_j^k\left(1 + \alpha_j^k t_d\right)\right] \\ a\dot{b} &= \left[A_j^k\left(1 + \alpha_j^k t_d\right) - A_i^k\left(1 + \alpha_i^k t_d\right) - r_{ij}^k\right]\left[\left(\alpha_j^k\right)^2 A_j^k\right] \end{aligned}$$

Therefore

$$\dot{\tau}_{ij}^k \equiv \frac{d\tau_{ij}^k}{dt_d} = \frac{\dot{a}b - a\dot{b}}{b^2}$$

which leads to

$$\dot{\tau}_{ij}^k = \frac{\left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right]\left[\zeta_k + \alpha_j^k A_j^k\left(1 + \alpha_j^k t_d\right)\right] - \left[A_j^k\left(1 + \alpha_j^k t_d\right) - A_i^k\left(1 + \alpha_i^k t_d\right) - r_{ij}^k\right]\left[\left(\alpha_j^k\right)^2 A_j^k\right]}{\left[\zeta_k + \alpha_j^k A_j^k\left(1 + \alpha_j^k t_d\right)\right]^2}$$

The numerator of the expression immediately above may be written as

$$\begin{aligned} N_{ij}^k &= \left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right]\left[\zeta_k + \alpha_j^k A_j^k\right] - \left[A_j^k - A_i^k - r_{ij}^k\right]\left[\left(\alpha_j^k\right)^2 A_j^k\right] \\ &\quad + \left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right]\left[\left(\alpha_j^k\right)^2 A_j^k t_d\right] - \left[A_j^k \alpha_j^k t_d - A_i^k \left(\alpha_i^k t_d\right)\right] \end{aligned} \tag{79}$$

Next let

$$\Gamma_{ij}^k \equiv \left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right]\left[\zeta_k + \alpha_j^k A_j^k\right] - \left(\alpha_j^k\right)^2 \left[A_j^k - A_i^k - r_{ij}^k\right] A_j^k$$

so that

$$\begin{aligned} N_{ij}^k &= \Gamma_{ij}^k + \left[\left(\alpha_j^k\right)^2 A_j^k\right]\left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right] t_d - \left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right] t_d \\ &= \Gamma_{ij}^k + \left[\left(\alpha_j^k\right)^2 A_j^k - 1\right]\left[\alpha_j^k A_j^k - \alpha_i^k A_i^k\right] t_d \end{aligned}$$

We are ready to find the dispatch time for shipping commodity $k \in \mathscr{K}$ from $i \in \mathscr{N}$ to $j \in \mathscr{N}$ by setting $N_{ij}^k = 0$. Doing so leads to the following expression

$$t_d = \frac{\Gamma_{ij}^k}{\left(\alpha_j^k A_j^k - \alpha_i^k A_i^k\right)\left[1 - \left(\alpha_j^k\right)^2 A_j^k\right]} \tag{80}$$

Use of (80) in expression (78) makes clear that there is a constant and well-defined minimum shipping latency.

7.4. Summary of our dynamic commodity price example

Thus, we have shown there are well defined, constant minimum shipping latencies for DSPE with fixed rates of inventory deterioration and pilferage and dynamic commodity prices. We again conclude that the path flows solving the associated DVI form a user equilibrium. That is, our preceding analysis has established the following:

Theorem 2. *For the DSPE with a fixed rate of inventory deterioration and pilferage in each market (node) throughout the planning horizon, there are explicitly dynamic commodity prices, and its solutions are spatial price equilibria relative to the defining relationships of Section 4.2, modified to reflect the inventory dynamics (68). For the approximations made in the preceding example there are also dynamic user equilibria relative to path flow h and the delay operator* $\Phi(t, h)$ *in the sense that*

$$h_p^k(t) > 0, p \in \mathscr{P}_{ij}^k \Rightarrow \Phi_p^k(t,h) = \tau_{ij}^k \qquad t \in \left[t_0, t_1 - \tau_{ij}^k\right]$$

for the constant minimum shipping latency found by substituting (*80*) into (*78*).

There are other modifications of our basic model that will also yield dynamic commodity prices. These include but are not limited to the inclusion of inventory carrying costs, upper and/or lower bounds on inventory, and any type of mixed constraints that include both control variables *h, S, D* and state variables *I*. Any of these modifications would be handled by application of suitably adapted necessary conditions for DVIs. However, the introduction of constraints involving state variables involves jump conditions that will severely compromise the derivation of formulae for constant shipping latencies and are best left for computational study.

8. The issue of uniqueness

It is now widely known via counter examples that path delay in DUE is not reliably strictly monotone increasing. Owing to the generality of the PDO presented in Han et al. (2016), this, in effect, means any delays computed from the perspective of LWR theory are generally nonmonotone and equilibria based on such delays will not generally be unique. However, the alternative of assuming path delay is strictly monotone increasing, in order to assure uniqueness, would be widely repudiated by the DTA community.

There is no principle of nature or social science that says there must be a single, knowable dynamic user or dynamic spatial price equilibrium when nonmonotonic delay is present. That is, nonunique equilibria are generally linked to model properties that realism commands us to acknowledge, and available mathematics provides no remedy in the form of a convergence proof.

9. Concluding remarks

By intent, our contribution is wholly theoretical in nature. We have presented a mathematical statement of the defining equations and inequalities for dynamic spatial price equilibrium (DSPE) and shown it to have an associated differential variational inequality (DVI), any solution of which is a DSPE. The model of spatial price equilibrium we have presented integrates, in a dynamic setting, the path delay operator notion from the theory of dynamic user equilibrium with the theory of spatial price equilibrium. This integration is original and constitutes a significant addition to the spatial price equilibrium and freight network equilibrium modeling literatures. Among other things, it points the way for researchers interested in dynamic traffic assignment to become involved in freight modeling using the technical knowledge they already possess.

In particular, the DVI representation we have presented allows algorithms developed for solving dynamic user equilibrium models to be adapted to computing a DSPE. [For a discussion of DUE algorithms, see the review of them in Friesz and Han (2022, 2023a).] As such, the DSPE models presented herein are ready to apply to any circumstance warranting use of a dynamic aggregative freight model and possessing adequate data, including that needed to estimate or derive inverse commodity supply and demand functions. It is also possible to extend the framework of this paper to consider oligopoly as in Friesz et al. (2006).

CRediT authorship contribution statement

Terry L. Friesz: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Author statement

Dynamic Spatial Price Equilibrium, Dynamic User Equilibrium, and Freight Transportation in Continuous Time: A Differential Variational Inequality Perspective by Terry L. Friesz

This paper was written solely by me, and its creation does not involve any application of generative AI. The content and findings of the paper have not previously been published and are not being considered for publication in any other journal or book.

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