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Dynamical Systems and Mathematical Practices for Future Secondary Teachers

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The relevance of upper division mathematics courses for future secondary teachers is a longstanding thorny issue. Suggested improvements include capstone courses and revised upper division content courses to explicitly address future teachers' relevant secondary mathematics content knowledge, beliefs about teaching and learning, and experience with learning mathematics while engaging in authentic mathematical practices. In this report, we investigate prospective teachers' reflections on their opportunities in an upper division Inquiry-Oriented Dynamical Systems course to engage in the eight Common Core State Standards for Mathematical Practice. Analysis of students' self-reported engagement in the eight Practices revealed five practices that strongly resonated with them and the various ways that their experiences in an inquiry-oriented classroom supported meaningful and powerful engagement in these Mathematical Practices. We conclude with implications for practice.

Keywords: mathematical practices, inquiry, prospective teachers, dynamical systems

The relevance and usefulness of upper division mathematics courses for prospective secondary school mathematics teachers has long been of concern (Begle, 1972; Klein, 1932; 2016; Wasserman et al., 2019). A number of studies document that teachers find their advanced mathematics courses have little relevance to their teaching (e.g., Cofer, 2015; Wasserman, 2017, Zazkis & Leikin, 2010). While these challenges are longstanding and pervasive, professional organizations have outlined possibilities for improving the connection between university and secondary mathematics for prospective teachers (Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of the Mathematical Sciences [CBMS], 2001; 2012). Creating capstone courses is one approach. Another recommendation, and the one taken in our work, is to redesign upper division math content courses so intentionally strong connections to high school mathematics content and teaching are made.

In recent years progress has been made on the university-secondary mathematics connection. For example, a recent issue of *ZDM Mathematics Education* focuses on how the intersectional nature of mathematical and mathematics educational content might be addressed in a wide range of university courses in order to prepare better secondary mathematics teachers (Wasserman et al., 2023). We contribute to this uptick in progress by investigating the opportunities for prospective teachers in an Inquiry-Oriented Dynamical Systems and Modeling (IODSM) course to engage in the eight Common Core State Standards for Mathematical Practice (MP) (Common Core State Standards Initiative [CCSSI], 2010). The eight Standards are: 1) Make sense of problems and persevere in solving them, 2) Reason abstractly and quantitatively, 3) Construct viable arguments and critique the reasoning of others, 4) Model with mathematics, 5) Use appropriate tools strategically, 6) Attend to precision, 7) Look for and make use of structure, and 8) Look for and express regularity in repeated reasoning. In particular, we address the following research question: *How frequently do prospective teachers in an IODSM course report engaging in the eight Standards for MP and how do they describe their engagement in these Standards?*

In related prior work, Apkarian et al. (2023) investigated the impact of an IODSM course on prospective teachers' knowledge of rate of change, their shifting beliefs about learning and teaching, and their self-reported ways in which their emerging beliefs and knowledge would influence their future practice. The work reported here adds a new dimension to this prior work by examining IODSM student-reported connections to the eight Standards for MP.

Theoretical Background

The "inquiry" part of the IODSM course is heavily influenced by the following four pillars of inquiry described by Laursen & Rasmussen (2019): 1) Students engage deeply with coherent and meaningful mathematical tasks, 2) Students collaboratively process mathematical ideas, 3) Instructors inquire into student thinking, and 4) Instructors foster equity in their design and facilitation choices. In the IODSM course, students collaboratively reinvent mathematics by engaging in the kind of work that potentially reflects how mathematicians go about their work and which are embodied in several of the Standards for MP. Our use of reinvention is informed by the instructional design theory of Realistic Mathematics Education (RME), which views mathematical concepts, structures, and ideas as inventions that humans create to organize the phenomena of the physical, social, and mental world (Freudenthal, 1973).

Our work is also informed by the emergent perspective (Cobb & Yackel, 1996), which views learning as both an individual and social process. Of particular relevance for this report are the emergent perspective's constructs of social and sociomathematical norms. Social norms refer to regularities in discourse, such as students routinely explaining their own thinking, listening to and attempting to make sense of others' thinking, asking questions if something is unclear, and indicating their agreement or disagreement with reasons. We conjecture that social norms have considerable overlap with the first and third MPs (Make sense of problems and persevere in solving them and Construct viable arguments and critique the reasoning of others). Also related to these Standards is the sociomathematical norm that justifications be based on underlying concepts as opposed to appeals to procedures or external authorities such as the text or instructor. This particular norm may, for example, relate to the third and sixth Standards for MPs.

In this report we do not examine actual classroom interactions and hence the full power and full set of constructs of the emergent perspective cannot be leveraged. Instead, we use the constructs of social and sociomathematical norms to reflect on the extent to which students' report how often they engage in the various Standards and the nature of that engagement.

Methods

The participants were 30 students enrolled in an upper division IODSM course at a large, Hispanic-serving institution in the southwestern United States. This course fulfills an upper division math elective requirement and it was designed specifically for prospective secondary teachers by infusing content related to high school mathematics. We collected qualitative data from a survey taken at the beginning of the semester, a detailed homework assignment where students explored the Standards for MP (CCSSI, 2010) and their connection to their experiences in the IODSM course, and an hour-long interview with a subset of students. In this report, we only discuss findings from their written homework assignment.

On this homework assignment, students reflected on how their IODSM classroom experiences relate to the Standards for MP. Students were asked to read the eight practices and categorize each practice into one of three bins based on how often they experienced the MP in class, and to explain why they placed each practice into the Bin that they did. Bin 1 was the most opportunities, Bin 2 was some opportunities, and Bin 3 was the least opportunities. Students also provided an example from classwork to support their justification. The assignment provided insight into students' understanding and engagement with the Standards for MP. It allowed for an in-depth exploration of students' perceptions and experiences, contributing valuable qualitative data to the field. By linking their experiences to specific Standards, the students provided a rich and detailed view of their interaction with the mathematical concepts laid out in the Standards, helping us gain a nuanced understanding of their learning experiences. We note that the Standards for MP were never discussed in class. Therefore, the responses from students reflect their own interpretation of the MPs.

To analyze the data, we used a thematic analysis approach, as described by Braun and Clarke (2006), to identify, analyze, and interpret patterns within the data. Students' responses were separated by MP into a spreadsheet that included their bin classification, justification, and the example. Note, if students mentioned uncertainty in placing an MP between two bins, we coded them as the less often bin. This happened only two times and both were deciding between Bin 1 and 2; and thus, they were placed in Bin 2. After data were organized, two researchers read students' responses and took notes according to their interpretation of the students' explanation to identify interesting aspects and patterns. Then all researchers met to discuss meaningful ways to organize and code the data. We discussed interesting trends students demonstrated as a response to their classification of bins and possible explanations for them. Initial codes were created for each MP and highlighted specific aspects of the MP description. Common themes helped to identify what parts of the MP students considered to experience the most in the class and why they placed each practice in the corresponding bin. After collapsing overlapping themes, re-working and refining codes, all authors agreed on the coding of all of the MPs. Lastly, we found the frequency of common themes. We also calculated the standard deviation for each practice. This allowed us to see which practices students agreed on more about engaging in and those that they did not. In this report, we focus only on practices where there was more agreement (low standard deviation) or modest agreement (medium standard deviation).

Results

The average number of the bin placement for each mathematical practice ranged from 1.032 to 2.000 (i.e., for some practices nearly all students selected Bin 1 [the most often bin] while for some practices the average selection was Bin 2 [the second most often bin]). Standard deviations were between 0.1796 and 0.7878. Three natural groupings of the practices emerged by standard deviation (SD), with three practices having SD less than 0.5 (most agreement), two practices with SD between 0.5 and 0.75 (modest agreement) and three practices with SD between 0.75 and 1 (least agreement). As mentioned, we only report on the practices with the most or modest agreement. We hypothesize that practices that received a higher SD can be partly attributed to the ambiguity of wording of the practices and/or students' personal interpretations. Table 1 lists the mathematical practices by SD group, as well as the themes identified across student responses and the frequency of occurrence for each theme. Only themes that were identified in the responses of at least one third of the class are listed in Table 1.

Practices with Most Agreement (Low Standard Deviation)

MP 1. The first theme for MP 1 was *highlighting the emphasis on making sense of problems* which was when students mentioned the importance of taking a step back to read and understand the problem to make sense of things. The second theme was *the importance of working hard and* *not giving up* which was when students reflected on attempting problems and their perseverance towards the correct outcome. This student response exemplifies making sense of problems:

> ... for a lot of problems in the class, you just cannot look at the given information and just solve for one variable, and that's it. There has to be more meaning to it, what exactly is the solution we are looking for, how do we work for the problem in question, what can we do and then we can attempt a method and change up our attempt if needed.

The students addressed the importance of making sense of a problem by stating that when working on a problem their solution approach cannot be deciphered by just looking at "one variable" or applying a single technique and solving for it, instead they dive further into finding meaning and value in the text, and only then, continue working towards a solution. That is, students in the class described that solving problems requires more than purely manipulating variables, they said that the solution has to make logical sense and they need to question themselves continually to check if their solution approach is right.

Students' responses coded as *importance of working hard and not giving up* referred to the importance of being perseverant when solving a problem and if they committed a mistake that they could step back and try a different approach. For example,

> Some of the problems that I encounter in class can be particularly challenging and require a great deal of patience and perseverance to solve. It is important to remain focused and persistent in working through these problems, breaking them down into smaller parts and utilizing any available resources or strategies to find a solution.

Students also described that due to the nature of the course more challenging problems were constantly being encountered which forced them to persevere in working out solutions. Also, students recognize that mathematics is not about getting the correct answer at the first try, but that it is important to persevere and change methods as necessary.

Table 1. *Student themes from Mathematical Practices.*

<table>
<tr><th>SD</th><th>Mathematical Practice</th><th>Themes Identified</th><th>Freq.</th></tr>
<tr><td rowspan="6">Low</td><td rowspan="2">(MP1) Make sense of problems and persevere in solving them</td><td>Highlights the emphasis on making sense of problems</td><td>30</td></tr>
<tr><td>Importance of working hard and not giving up</td><td>11</td></tr>
<tr><td rowspan="2">(MP2) Reason abstractly and quantitatively</td><td>Contextualization or decontextualization</td><td>20</td></tr>
<tr><td>Doing mathematics with meaning</td><td>16</td></tr>
<tr><td rowspan="2">(MP4) Model with mathematics</td><td>Mathematics applications</td><td>16</td></tr>
<tr><td>Mathematical techniques</td><td>10</td></tr>
<tr><td rowspan="5">Medium</td><td rowspan="3">(MP3) Construct viable arguments and critique the reasoning of others</td><td>Critiquing and revision of ideas</td><td>22</td></tr>
<tr><td>Group work and collaborative learning</td><td>22</td></tr>
<tr><td>Constructing Arguments and Justifying Answers</td><td>13</td></tr>
<tr><td rowspan="2">(MP6) Attend to precision</td><td>Precision in communication ideas</td><td>28</td></tr>
<tr><td>Group work and collaborative learning</td><td>13</td></tr>
</table>

MP 2. Themes identified for MP 2 included *contextualization or decontextualization* and *doing mathematics with meaning.* An example student response for the first theme is "A lot of times we are given problems with context, then we create mathematical models of the scenarios, solve the problem using math, then relate our solutions back to the context/scenario" which highlights the transition between abstract mathematics and contextual understanding of the problem and vice versa. Additionally, some students talked about giving meaning to the variables when working with equations and functions in class and such responses were coded under the second theme of *doing mathematics with meaning*. For example, a student reported,

> This standard most directly applies to our use of putting differential equations into words. We explored each piece of a differential equation individually, and then we worked on putting a differential equation into a sentence. We spoke of a differential equation with meaning, rather than speaking out the signs, numbers, and letters as they are. We have used these statements of meaning in most of our work in this class thus far.

In the excerpt above the student mentioned that it is important to be aware of the symbolic representations of models in regard to the context, emphasizing the need to provide meaning to them. The student expressed the necessity to read equations with intention and comprehension. In addition, students described their experience in the class with differential equations as translating them into meaningful sentences rather than mere symbol recitation, which exemplifies the practice's aim to ensure students can contextualize symbols and equations in real-world problems.

MP 4. The fourth practice revealed that students were seeing differential equations as a powerful tool to model real world phenomena and the two main themes identified were *mathematics applications* and *mathematics techniques.* The first theme was when students mentioned real-world scenarios in which mathematical concepts were applied. For example,

> …there have been many times in the class where our problem is a model of a real-world situation, with examples of the salty tank problem, the helicopter problem, and the list problem just to name a few. I also think that we have been given the opportunity to think about whether the model fits the situation as for example when we were asked if the model of the fish population matched the mathematical model…

Student responses highlighted the application of differential equations to model real-world systems, which aligns with the practice's focus on using mathematical knowledge to address real-life situations. The second theme in MP4 was the use of *mathematical techniques* in facilitating problem solving. Responses which discussed the use of Euler's Method (or as we called it, the Tip-to-Tail method), graphs, slope fields, phase lines, and other math techniques were coded in this theme.

Practices with Modest Agreement (Medium Standard Deviation)

MP 3. Three main themes were identified for the third practice. These included *critiquing and revision of ideas, group work and collaborative learning and constructing arguments and justifying answers.* For the first theme students acknowledged the importance of peer review and critiquing of ideas when developing their understanding of mathematics and underscored the value of the collaborative nature of the learning experience. For example,

> ... We are asked to come up with our own explanations for certain answers or approaches. We have to share them with the rest of the class and also take critiques if others don't agree until everyone has a satisfactory answer and explanation. Many examples where groups would have to come up with a graph tend to cause some disagreement thus leading to more discussion and ultimately to a well-backed understanding.

The student described the active engagement in constructing explanations, presenting them to peers, and refining them through critique until a shared understanding is achieved. This not only embodies MP 3's focus on constructing arguments and critiquing reasoning but also echoes the collaborative nature of mathematical exploration emphasized in MP 3. Other students' described the collaborative nature of the course. The next most common theme for MP 3 was *group work and collaborative learning* which was attributed to responses mentioning the value of discussing problems with classmates, sharing ideas, and collectively analyzing and working on problems. The following excerpt by students mentions the use of group work daily in class which leads to discussions and support from teammates:

> Since this class consists of almost entirely group work, constructing clear and viable arguments is crucial as we are always explaining our thought process to everyone else in our group, and sometimes the rest of the class. In order to do this successfully we must fully understand what we are doing and be able to explain why each step was made … This class is a team sport, and that quality of respect is crucial as we are all in a learning environment where mistakes are welcomed as long as we work through them together and help each other out along the way…

The student above describes daily group work and emphasizes the process of understanding and exploring mathematical problems and the "team sport" reference signifies the collaborative nature of learning in class. Finally, the third theme was attributed to students who emphasized reasoning and justifying their approach to problems. For example, "although critiquing the reasoning of others is very much one of the more common things that happens during class, the construction of arguments is done in a way more informal which is the reason it is in bin 2". Other student responses expressed this sentiment, or the feeling that there was not always enough time available in class to construct careful arguments.

MP 6. Two main themes emerged for the sixth practice including *precision in communicating ideas and group work and collaborative learning*. Note the second theme around group work also emerged for MP 3. The first theme was attributed to responses which emphasized the clarity and exactness in mathematical communication, including a need to refine mathematical language and avoiding vague terms. For example,

> ....I feel that I have had the most opportunity in this class to engage in this mathematical practice because communicating in a precise manner underpins all the work that we do in this class. Whenever I ask a question, present a result, or draw a graph, I strive to be accurate with my spoken words and written statements….

The students who brought up this theme addressed the importance of communicating mathematical ideas and concepts to others with precise language which relates to the second theme identified for *group work and collaborative learning*. For example,

> Communication is a huge part of this class. Group work and class discussion is what makes this class impactful. If we were to do things on our own all the time, there is a low chance that if we were to get something wrong, we'd understand why and how to find the right answer. Communication with others keeps each individual on track when it comes to using the correct definitions and meanings, symbols, math processes.

Students who mentioned group work and collaborative learning in relation to MP 6 mainly pointed out communication with others and being precise in their language in doing so to get their ideas across. Additionally, students mentioned the importance of working with others as time to get constructive criticism on delivering their ideas.

Conclusion

The AMTE (2017) Standards state that effective mathematics teacher preparation programs should provide opportunities for prospective teachers to learn mathematics that enable them to engage in mathematical practices, and that mathematics content should be taught using teaching methods that serve as models of effective teaching (AMTE, 2017). Consistent with this call, we investigated prospective teachers' reflections on their opportunities in an upper division inquiry-oriented mathematics course to engage in the eight Common Core State Standards for Mathematical Practice. We found that students' self-reported engagement centered five practices (MP 1, 2, 3, 4, 6) as strongly resonating with them. There were three practices (MP 5, 7, 8) that had higher standard deviations in terms of which bins students placed them in. This meant there was not as much agreement on how these practices were reflected, from the students' points of view, in class. We posit high SD may have been because these practices appeared in the latter half of the assignment (so perhaps not read as carefully) and/or that students may not have understood aspects of the educational terms in these practices. Recall that the Standards were never discussed in class.

Collaboratively processing ideas showed up in more than practice (MP3 and 6). This relates to social norms (Cobb & Yackel, 1996) in that central to students engagement in class was the time to collaborative process ideas. Students discussed how sometimes mathematical concepts did not make sense to them until another student explained something or provided more information. Relatedly was the concept of being precise with language. A sociomathematical norm in the class was speaking with meaning (e.g., avoid saying "it"). Being precise in language is not only critical as an MP but also important for future teachers to be precise in their language when engaging with their future students. Lastly, the idea of critiquing was often discussed. Importantly, some students took a negative connotation to critiquing in that they argued that they did not *critique* but they went back and forth discussing mathematics until concepts were agreed upon. Whereas some students fully embraced what it means to critique in their IODSM class. To them, it was important to critique because it meant that ideas were only getting better when the class critiqued reasoning to improve upon said reasoning.

Our next steps are to analyze the interviews already conducted which investigated student perceptions of the MPs in deeper detail, their beliefs about learning and teaching mathematics, and the connections they made between the upper division college mathematics content and secondary school mathematics. We also intend to conduct additional iterations of this course, at the same and different universities, expanding our focus to approximations of practice.

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