Quadratic Hessian equation

YU YUAN

In memory of my teacher, Ding Wei-Yue laoshi (1945–2014)

1. Introduction

The elementary symmetric quadratic or σ_2 equation

$$\sigma_2\left(D^2u\right) = \sum_{1\leqslant i_1<i_2\leqslant n} \lambda_{i_1}\lambda_{i_2} = \frac{(\triangle u)^2 - \left|D^2u\right|^2}{2} = 1 \tag{1.1}$$

with $\lambda_i's$ being the eigenvalues of the Hessian D^2u of scalar function u, is a nonlinear Hessian dependence equation of the lowest integer order, and is called fully nonlinear equation, because the nonlinearity is on the highest order derivatives of the solutions. The σ_2 equation sits in between the (linear) Laplace equation $\sigma_1\left(D^2u\right) = \lambda_1 + \cdots + \lambda_n = \triangle u = 1$ and the (fully nonlinear) Monge-Ampère equation $\sigma_n\left(D^2u\right) = \lambda_1\lambda_2\cdots\lambda_n = \det D^2u = 1$. The 2-sheet hyperboloid level set of the equation

$$\left\{\lambda \in \mathbb{R}^n : \ \lambda_1 + \cdots + \lambda_n \ = \pm\sqrt{2 + |\lambda|^2}\right\}$$

is rotationally symmetric, unlike all the other σ_k equations with $3 \leqslant k \leqslant n$.

To make those equations elliptic, or monotone dependence on Hessian along positive definite symmetric matrices, we require the linearized operator positive definite. Equivalently, the normal of the level set has positive sign for all components in the eigenvalue space, respectively positive definite in the matrix space. For example, among all four branches of level set $\lambda_1\lambda_2\lambda_3 = 1$, only one is elliptic; the same is true for $\lambda_1\lambda_2\lambda_3 = -1$. The negative definite or all negative component case is also considered as elliptic. In particular, the two symmetric branches of $\sigma_2(\lambda) = 1$ are both elliptic. The choice of branch is made automatically by C^2

This work is partially supported by an NSF grant.

solutions. For less smooth solutions such as continuous viscosity ones, the positive $\triangle u > 0$ or negative $\triangle u < 0$ branch has to be specified. Afterward, $-D^2 u$ is on the other branch in viscosity sense.

Replacing the flat eigenvalues $\lambda_i' s$ with the principal curvatures $\kappa_i' s$ of graph $(x, u(x))$ in Euclid space $\mathbb{R}^n \times \mathbb{R}^1$, one has the scalar curvature equation

$$\sigma_2(\kappa_1, \cdots, \kappa_n) = 1.$$

Recall $\kappa_i' s$ are the eigenvalues of the normalized second fundamental form II by the induced metric g or shape matrix

$$IIg^{-1} = \frac{D^2 u}{\sqrt{1 + |Du|^2}} \left[I - \frac{Du \otimes Du}{1 + |Du|^2} \right] = \left[\partial_{x_i} A_{p_j}(Du) \right],$$

where $II = D^2 u / \sqrt{1 + |Du|^2}$, $g = I + Du \otimes Du$, and $A(p) = \sqrt{1 + |p|^2}$. Replacing the flat eigenvalues $\lambda_i' s$ with the eigenvalues of Schouten tensor of a conformal metric $g = u^{-2} g_0$, one has σ_2-type Yamabe equation in conformal geometry, which simplifies to

$$\sigma_2 \left(u D^2 u - \frac{1}{2} |Du|^2 I \right) = 1$$

for flat metric g_0. Replacing the flat eigenvalues $\lambda_i' s$ with the eigenvalues of Hermitian Hessian $\partial \bar{\partial} u$, one has σ_2-type equation arising from complex geometry. Lastly, in three dimensions $\sigma_2(D^2 u) = 1$ or equivalently $\arctan \lambda_1 + \cdots + \arctan \lambda_3 = \pm \pi/2$ is the potential equation of minimal Lagrangian graph $(x, Du(x))$ with phase $\pm \pi/2$ in Euclid space $\mathbb{R}^3 \times \mathbb{R}^3$.

2. Results

2.1. Outline. Once an equation is given, the first question to answer is the existence of solutions. Smooth ones cannot be obtained immediately, in general; worse, they may not even exist. The typical approach is to first seek weak solutions, in the integral sense if the equation has divergence structure, or in the "pointwise integration by parts sense", namely, in the viscosity sense if the equation enjoys a comparison principle. After obtaining those weak solutions, one studies the regularity and other properties of the solutions, such as Liouville or Bernstein type rigidity for entire solutions. All these hinge on *a priori* estimates of derivatives of solutions:

$$\|D^2u\|_{L^\infty(B_1)} \leqslant C(\|Du\|_{L^\infty(B_2)}) \leqslant C(\|u\|_{L^\infty(B_3)}).$$

Having the L^∞ bound of the Hessian available, the ellipticity of the above fully nonlinear equations becomes uniform, we can apply the Evans-Krylov-Safonov theory (for the ones with convexity/concavity, possibly without divergence structure) or the Evans-Krylov-De Giorgi-Nash theory (for the ones with convexity/concavity and divergence structure) to obtain $C^{2,\alpha}$ estimates of solutions. Either theory can handle the quadratic Hessian equation along with all other σ_k equations, because they all share the divergence structure. In the σ_2 equation (1.1) case, the linearized operator is readily seen in divergence form

$$\triangle_F = \sum_{i,j=1}^{n} F_{ij}\partial_{ij} = \sum_{i,j=1}^{n} \partial_i \left(F_{ij}\partial_j\right),$$

with

$$(F_{ij}) = \triangle u \; I - D^2u = \sqrt{2+|D^2u|^2} \; I - D^2u > 0.$$

Here and in the remaining, for certainty, and without loss of generality, we assume $\triangle u > 0$. The concavity of the equation is evident in an equivalent form

$$\triangle \, u - \sqrt{2+|D^2u|^2} = 0, \tag{1.2}$$

whose linearized operator

$$I - \frac{D^2u}{\sqrt{2+|D^2u|^2}}$$

also reveals the uniform ellipticity of the equation with uniformly bounded Hessian solutions.

Considering the minimal surface structure of the quadratic Hessian equation in three dimensions, the $C^{2,\alpha}$ estimate can also be achieved via geometric measure theory. For the σ_n or Monge-Ampère equation, back in the 1950s, Calabi attained C^3 estimates by interpreting the cubic derivatives in terms of the curvature of the corresponding Hessian metric $g = D^2u$. Further, iterating the classic Schauder estimates, one obtains smoothness of the solutions, and even analyticity, if the smooth equations such as all the σ_k equations are also analytic.

2.2. Rigidity of entire solutions. The classic Liouville type theorem asserts every entire harmonic function bounded from below or above is a constant by the Harnack inequality. Thus every semiconvex harmonic function is a quadratic one, as its double derivatives are all harmonic with lower bounds, hence constants. Similarly, every entire (convex) solution to the Monge-Ampère equation $\det D^2 u = 1$ is quadratic. This was first proved in two dimensional case by Jörgens, later in dimension up to five by Calabi, and in all dimensions by Pogorelov. Also, Cheng-Yau had a geometric proof.

Recently, Shankar-Yuan [SY3] proved that every entire semiconvex solution to the quadratic Hessian equation $\sigma_2\left(D^2u\right) = 1$ is quadratic. In dimension two, it is the above classic Jörgens's theorem without any extra condition (not even convexity) on the entire solutions, thus a Bernstein type result. In three dimensions, this was proved in [Y] earlier, as a by-product of rigidity for the special Lagrangian equation.

Under an almost convexity condition on entire solutions to $\sigma_2\left(D^2u\right) = 1$ in general dimension, Chang-Yuan derived the rigidity [ChY]. Under a general semiconvexity and an additional quadratic growth assumption on entire solutions in general dimension, Shankar-Yuan showed the rigidity in [SY1]. Assuming only quadratic growth on entire solutions to $\sigma_2\left(D^2u\right) = 1$ in three and four dimensions, the same rigidity result was proved in the joint work with Warren [WY] and Shankar [SY4] respectively. Assuming a super quadratic growth condition, Bao-Chen-Ji-Guan [BCGJ] demonstrated that all convex entire solutions to $\sigma_2\left(D^2u\right) = 1$ along with other $\sigma_k\left(D^2u\right) = 1$ are quadratic polynomials; and Chen-Xiang [CX][showed that all "super quadratic" entire solutions to $\sigma_2\left(D^2u\right) = 1$ with $\sigma_1\left(D^2u\right) > 0$ and $\sigma_3\left(D^2u\right) \geqslant -K$ are also quadratic polynomials.

Warren's rare saddle entire solution $u\left(x_1, \cdots, x_n\right) = \left(x_1^2 + x_2^2 - 1\right) e^{x_3} + \frac{1}{4} e^{-x_3}$ to $\sigma_2\left(D^2u\right) = 1$ in dimension three and above [W], confirms the necessity of the semiconvexity or the quadratic growth assumption. C.-Y. Li [L] followed with "non-degenerate" entire solution $u\left(x\right) = \left(x_1^2 + x_2^2 - 1\right) e^{x_n} + \frac{n-2}{4} e^{-x_n} + \left(x_3 + \cdots + x_{n-1}\right) x_n$ in dimension n and above for $n \geqslant 4$.

2.2.1. Two dimensions. In the following, we recap Nitsche's idea in showing the rigidity of entire solutions in two dimensions.

Given a C^2 solution u to $\sigma_2\left(D^2u\right) = 1$, up to negation, we assume D^2u is on the positive branch of the hyperbola $\lambda_1\lambda_2 = 1$, in turn, u is convex. Let w be the Legendre-Lewy transform of $u\left(x\right)$, that is, the Legendre transform of

$u(x)+|x|^2/2$. Geometrically their "gradient" graphs satisfy $(x, Du(x)+x) = (Dw(y), y) \in \mathbb{R}^2\times\mathbb{R}^2$, and the "slopes" of graphs satisfy

$$\left(I, D^2u(x)+I\right) = \left(D^2w(y)\frac{\partial y}{\partial x}, \frac{\partial y}{\partial x}\right).$$

It follows that

$$I < D^2u(x)+I = \left(D^2w(y)\right)^{-1} \quad \text{or} \quad D^2u(x) = \left(D^2w(y)\right)^{-1} - I.$$

Taking determinants yields

$$1 = \sigma_2\left(D^2u\right) = \det\left[\left(D^2w(y)\right)^{-1} - I\right] = \mu_1^{-1}\mu_2^{-1} - \mu_1^{-1} - \mu_2^{-1} + 1.$$

or an equation for the eigenvalues $\mu_i's$ of D^2w

$$1 = \mu_1 + \mu_2 = \triangle w.$$

Noticing the boundedness of Hessian D^2w

$$0 < D^2w < I,$$

we see the constancy by Liouville. Consequently from the flatness of graphs $(x, Du(x)+x) = (Dw(y), y)$ or constancy of $\left(D^2w\right)^{-1} - I = D^2u$, it follows that u is quadratic. Note that Jörgens' original "involved" proof made use of a partial Legendre transformation.

Going further, this Legendre-Lewy transformation proof of Jörgens' theorem, coupled with Heinz transformation, led Nitsche [N] to his elementary proof of the original Bernstein theorem: every entire solution to the minimal surface equation

$$\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = 0$$

in two dimensions is linear.

The remaining proof goes as follows. The mean curvature vector $\vec{H}$ of the graph $(x_1, x_2, f(x))$ is

$$\begin{aligned}\vec{H} &= \triangle_g(x_1, x_2, f(x)) = \sum_{i,j}\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)(x_1, x_2, f(x)) \\ &= \left(\operatorname{div}\frac{\left(1+f_2^2, -f_1f_2\right)}{\sqrt{1+|Df|^2}}, \operatorname{div}\frac{\left(-f_1f_2, 1+f_1^2\right)}{\sqrt{1+|Df|^2}}, \operatorname{div}\frac{(f_1, f_2)}{\sqrt{1+|Df|^2}}\right),\end{aligned}$$

where $\sqrt{g}g^{-1} = \begin{bmatrix} 1+f_2^2 & -f_1f_2 \\ -f_2f_1 & 1+f_1^2 \end{bmatrix} / \sqrt{1+|Df|^2}$ was used in the last equality. The magnitude H of the mean curvature vector is, as noted before

$$H = Tr\left[\partial_{x_i} A_{p_j}(Df)\right] = \operatorname{div} \frac{(f_1, f_2)}{\sqrt{1+|Df|^2}} = 0.$$

Hence, $\triangle_g (x_1, x_2, f(x)) = (0,0,0)$.

Considering the first component equation $\triangle_g x_1 = 0$, the conjugate function of x_1 is defined as

$$x_1^*(x_1, x_2) = \int^{(x_1,x_2)} \frac{f_1f_2dx_1 + \left(1+f_2^2\right)dx_2}{\sqrt{1+|Df|^2}}.$$

Similarly, the conjugate function of x_2 is also defined. Together, they represent the "normalized" metric

$$\frac{1}{\sqrt{1+|Df|^2}} \begin{bmatrix} 1+f_1^2 & f_1f_2 \\ f_2f_1 & 1+f_2^2 \end{bmatrix} = \begin{bmatrix} Dx_2^* \\ Dx_1^* \end{bmatrix}.$$

By symmetry of the left-side matrix, $\partial_2 x_2^* = \partial_1 x_1^*$, then there exists a double potential u so that $Du = (x_2^*, x_1^*)$. Thus, one has Heinz transformation u of the height function f of a minimal graph satisfying

$$\frac{1}{\sqrt{1+|Df|^2}} \begin{bmatrix} 1+f_1^2 & f_1f_2 \\ f_2f_1 & 1+f_2^2 \end{bmatrix} = D^2u \quad \text{and} \quad \det D^2u = 1 \quad \text{on } \mathbb{R}^2.$$

As just obtained, the Hessian D^2u of entire solution u is a constant matrix, and in turn, Df is a constant vector. The original Bernstein theorem is reached.

In passing, let us note the conjugate function of f

$$f^*(x_1, x_2) = \int^{(x_1,x_2)} \frac{-f_2dx_1 + f_1dx_2}{\sqrt{1+|Df|^2}}$$

satisfies

$$Df^* = \frac{(-f_2, f_1)}{\sqrt{1+|Df|^2}} \quad \text{and} \quad \sqrt{1-|Df^*|^2} = \frac{1}{\sqrt{1+|Df|^2}} \in (0,1)$$

or

$$\frac{Df^*}{\sqrt{1-|Df^*|^2}} = (-f_2, f_1).$$

Consequently

$$\operatorname{div}\left(\frac{Df^*}{\sqrt{1-|Df^*|^2}}\right)=0.$$

Observe that the above conjugation process from minimal surface to maximal surface is reversible. "Incidentally" we have obtained a two dimensional Bernstein type result: every entire solution to the maximal surface equation div $\left(Df/\sqrt{1-|Df|^2}\right)=0$ is linear, which was first proved up to four dimensions by Calabi, and in general dimension by Cheng-Yau.

2.2.2. General dimensions. Next, we outline the argument toward rigidity for semiconvex entire solutions to $\sigma_2\left(D^2u\right)=1$ in general dimensions.

The Legendre-Lewy transform of a general semiconvex solution satisfies a uniformly elliptic, saddle equation with bounded Hessian. In the almost convex case, the new equation becomes concave, thus the Evans-Krylov-Safonov theory yields the constancy of the bounded new Hessian, and in turn, the old one. To beat the saddle case, one has to be "lucky", only one time. Recall that, in general the Evans-Krylov-Safonov fails as shown by the saddle counterexamples of Nadirashvili-Vladuts. Our earlier trace Jacobi inequality, as an alternative log-convex vehicle, other than the maximum eigenvalue Jacobi inequality, in deriving the Hessian estimates for general semiconvex solutions [SY1], could rescue the saddleness. But the trace Jacobi only holds for large enough trace of the Hessian. It turns out that the trace added by a large enough constant satisfies the elusive Jacobi inequality.

Equivalently, the reciprocal of the shifted trace Jacobi quantity is superharmonic, and it remains so in the new vertical coordinates under the Legendre-Lewy transformation by a transformation rule. Then the iteration arguments developed in our joint work with Caffarelli show the "vertical" solution is close to a "harmonic" quadratic at one small scale, "luckily" (two steps in the execution: the superharmonic quantity concentrates to a constant in measure by applying Krylov-Safonov's weak Harnack; a variant of the superharmonic quantity, as a quotient of symmetric Hessian functions of the new potential, is very pleasantly concave and uniformly elliptic, consequently, closeness to a "harmonic" quadratic is possible by the Evans-Krylov-Safonov theory), and the closeness improves increasingly as we rescale (this is a self-improving feature of elliptic equations, no concavity/convexity needed). Thus a Hölder estimate for the bounded Hessian is realized, and consequently so is the constancy of the new and then the old Hessian.

Note that, in three dimensions, our proof provides a "pure" PDE way to establish the rigidity, distinct from the geometric measure theory way used in our earlier work on the rigidity for special Lagrangian equations two decades ago.

The details are in the following.

Step 1. Bounded Hessian and uniform ellipticity after Legendre-Lewy transform.

The Legendre-Lewy transform $w(y) = \mathcal{L}\mathcal{L}\left[u(x) + K|x|^2/2\right]$ of a general semi-convex solution $u(x)$ with $D^2 u \geqslant (\delta - K) I$ satisfies a uniformly elliptic, saddle equation with bounded Hessian:

$$\begin{gathered}(x, Du(x) + Kx) = (Dw, y), \\ 0 < D^2 w = \left(D^2 u + K\right)^{-1} < \delta^{-1} \quad \text{or } \lambda_i = \mu_i^{-1} - K \geqslant \delta - K, \\ g(\mu) = -f\left(\mu^{-1} - K\right) = -\sigma_2\left(\mu^{-1} - K\right) = -1.\end{gathered}$$

By Lin-Trudinger [LT], and also Chang-Yuan [ChY]

$$\lambda_1^{-1} \lessapprox f_{\lambda_1} \lessapprox \lambda_1, \quad f_{\lambda_{k\geqslant 2}} \approx \lambda_1 \quad \text{for } \lambda_1 \geqslant \cdots \geqslant \lambda_n;$$

for $\sigma_2(\lambda) = 1$ with $\lambda_i \geqslant \delta - K$, all but one eigenvalues are bounded, $|\lambda_{k\geqslant 2}| \leqslant C(K)$ and $f_{\lambda_1}\lambda_1 \approx 1$, then

$$g_{\mu_i} = f_{\lambda_i}\mu_i^{-2} = f_{\lambda_i}\ (\lambda_i + K)^2 \approx C(n, K)\ \lambda_1\ .$$

Consequently, level set

$$\left\{\mu \mid g(\mu) = -\sigma_2\left(\mu^{-1} - K\right) = -1\right\} \text{ is a uniformly elliptic surface.} \tag{1.3}$$

The new equation also takes the form

$$\frac{\sigma_{n-2}(\mu)}{\sigma_n(\mu)} - (n-1)K\frac{\sigma_{n-1}(\mu)}{\sigma_n(\mu)} + \frac{n(n-1)}{2}K^2 = 1.$$

Remark. In the almost convex case $K = \sqrt{2/n(n-1)}$, the new equation becomes

$$\frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}(\mu)} = \left[(n-1)K\right]^{-1}\ , \tag{1.4}$$

thus concave. Then the Evans-Krylov-Safonov theory yields the constancy of the bounded new Hessian

$$\left[D^2 w\right]_{C^\alpha(B_R)} \leqslant \frac{C(n)}{R^\alpha}\left\|D^2 w\right\|_{L^\infty(B_{2R})} \leqslant \frac{C(n)}{R^\alpha} \to 0 \quad \text{as } R \to \infty,$$

and in turn, the old one.

The uniformly elliptic level set $\left\{g(\mu)=-\sigma_2\left(\mu^{-1}-K\right)=-1\right\}$ is saddle for large K. There are $C^{1,\alpha}$ singular solutions to uniformly elliptic saddle equation in five dimensions by Nadirashvili-Tkachev-Vladuts.

In the next two steps, we really use the following equivalent uniformly elliptic saddle equation

$$H\left(D^2 w\right)=\sigma_n(\mu)\left[1-\sigma_2\left(\mu^{-1}-K\right)\right]=0,$$

which in three dimensions, becomes

$$\begin{aligned} -\sigma_1(\mu)+2K\sigma_2(\mu)-\left(3K^2-1\right)\sigma_3(\mu)=0 \quad \text{or} \\ \frac{\sigma_2(\mu)}{\sigma_1(\mu)}=\left[2K-\left(3K^2-1\right)\frac{\sigma_3(\mu)}{\sigma_2(\mu)}\right]^{-1}. \end{aligned}$$

Step 2. Shifted trace Jacobi inequality to rescue saddleness.

For $F\left(D^2 u\right)=\sigma_2(\lambda)=1$ with $D^2 u \geqslant -KI$, $b(x)=\ln(\triangle u+nK)$ satisfies the elusive strong subharmonicity

$$\triangle_F b=F_{ij}\partial_{ij} b \geqslant F_{ij}\partial_i b\ \partial_j b=\left|\nabla_F b\right|^2$$

which is equivalent to the superharmonicity, by a transformation rule in [SY1].

$$\begin{aligned} \triangle_H a(y) \leqslant 0 \\ \text{with } a(y)=\frac{1}{\lambda_1+K+\cdots+\lambda_n+K}=\frac{\sigma_n(\mu)}{\sigma_{n-1}(\mu)} \overset{n=3}{=} \frac{\sigma_3(\mu)}{\sigma_2(\mu)}. \end{aligned}$$

Remark. For log-convex function $b=\ln\lambda_{\max}$ or $\ln(\lambda_{\max}+K)$, Jacobi inequality $\triangle_F b \geqslant \left|\nabla_F b\right|^2$ holds in three dimensions without any restriction, in general dimensions with necessary semiconvexity condition. The reciprocal $\mu_{\min}(y)=(\lambda_{\max}+K)^{-1}$ is superharmonic $\triangle_H \mu_{\min}(y) \leqslant 0$. But $\mu_{\min}\left(D^2 w\right)$ is not a uniformly elliptic function/operator on $D^2 w$, though concave. Therefore, it is not adequate via $\mu_{\min}(y)$ to run the Caffarelli-Yuan procedure for Hölder of Hessian $D^2 w$.

For log-linear $b=\ln\triangle u$, Qiu showed $\triangle_F b \overset{\text{3-d}}{\geqslant} \left|\nabla_F b\right|^2$ in [Q]. It is indeed another log-convex function

$$\ln\triangle u=\ln\sqrt{|\lambda|^2+2} \mod \sigma_2(\lambda)=1$$

satisfying $\triangle_F \ln \triangle u \geqslant |\nabla_F \ln \triangle u|^2$ for large enough $\triangle u \geqslant C(K)$ under semi-convexity $D^2 u \geqslant -K$. It is important to have the above shifted Jacobi quantity $b(x) = \ln(\triangle u + nK)$ valid without assuming $\triangle u$ large enough, to execute our argument toward constancy of the Hessian.

Remark. The above Jacobi inequality is actually an equality on minimal surface $(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^1$ for $\operatorname{div}\left(Df/\sqrt{1+|Df|^2}\right) = 0$:

$$\triangle_g b = |\nabla_g b|^2 + \left|IIg^{-1}\right|^2 \quad \text{or} \quad \triangle_g \, \omega = -\omega \left|IIg^{-1}\right|^2 \leqslant 0,$$

where $b = \ln \sqrt{1+|Df|^2}$, and $\omega = \langle (0, \cdots, 0, 1), N \rangle = 1/\sqrt{1+|Df|^2}$ is the effective deformation, while varying the minimal surface along a Jacobi vector field $J = (0, \cdots, 0, 1)$.

Step 3. Hölder estimate of new Hessian on saddle equation.

We illustrate the argument for Hölder estimates for Hessian on uniformly elliptic saddle equation in three dimensions, where the idea is not lost, but the gain is a better understanding of the idea.

Again, the new/"vertical" uniformly elliptic saddle equation is

$$\frac{\sigma_2(\mu)}{\sigma_1(\mu)} = \left[2K - \left(3K^2 - 1\right) \frac{\sigma_3(\mu)}{\sigma_2(\mu)}\right]^{-1}$$
$$\text{for } 1 \geqslant \mu_{i \geqslant 2} \geqslant c(K) > 0 \text{ and } 1 \geqslant \mu_1 > 0.$$

Apply Krylov-Safonov's weak Harnack to the bounded superharmonic $a(y) = \frac{\sigma_3(\mu)}{\sigma_2(\mu)}$ from Step 2, $a(y)$ concentrates to a level $l = \min_{B_\varepsilon} a(y)$ in one small (enough) ball $B_\varepsilon(0)$, that is $\frac{\sigma_3(\mu)}{\sigma_2(\mu)} \approx l$ in 99.99999% of B_ε. Note the concave $\frac{\sigma_3(\mu)}{\sigma_2(\mu)}$ is not uniformly elliptic, because μ_1 could be close 0.

Step 3 Continued: Remarkably, μ is approximately on the uniformly elliptic ($1 \geqslant \mu_3, \mu_2 \geqslant c(K) > 0$), concave (level set $\frac{\sigma_2(\mu)}{\sigma_1(\mu)} = l$ is a concave surface) equation

$$\frac{\sigma_2(\mu)}{\sigma_1(\mu)} = \left[2K - \left(3K^2 - 1\right) \; l \;\right]^{-1} \quad \text{in 99.99999\% of } B_\varepsilon$$

By existence of Dirichlet problem via Evans-Krylov-Safonov and Alexandrov maximum principle in measure,

$$w(y) = \text{quadratic } Q(y) \pm 0.00000000001 \text{ in } B_{\varepsilon/2}.$$

By self-improving property of the smooth uniformly elliptic equation (no concavity needed)

$$-\sigma_1(\mu) + 2K\sigma_2(\mu) - \left(3K^2 - 1\right)\sigma_3(\mu) = 0,$$

through iterative procedure $w(y) \approx y^{2+\alpha}$ near $y=0$. Similarly near everywhere in $B_{\varepsilon/4}$. Thus Hölder estimates for $D^2 w$.

Finally, by quadratic scaling

$$\left[D^2 w\right]_{C^\alpha(B_R)} \leqslant \frac{C(n,K)}{R^\alpha} \quad \overset{R\to\infty}{\longrightarrow} \quad 0.$$

Thus $D^2 w$ is constant matrix, in turn, so is $D^2 u$.

Remark. In the original Caffarelli-Yuan procedure, the superharmonic quantity $a = \frac{\sigma_3(\mu)}{\sigma_2(\mu)}$ is $1 - e^{K\triangle u}$ for saddle uniformly elliptic equation $F\left(D^2 u\right) = 0$ with convex level setS along trace trM of the level set $\{F(M) = 0\}$. Once, only one lucky chance needed, $\triangle u$ concentrates in measure, solution u is close to a quadratic, by solving the Laplace equation and applying the Alexandrov maximum principle in measure. Afterwards, the self-improving machinery of smooth uniformly elliptic equation takes over for Hölder estimate of Hessian.

2.3. Regularity for viscosity solutions. Any reasonable solution to the σ_1 or Laplace equation $\triangle u = 1$ such as continuous viscosity in pointwise sense or distributional solution in integral sense is analytic. The same is not true for $\sigma_{k\geqslant 3}$ equations, and unknown for σ_2 equation in dimension five and higher currently. In fact, by now there are $C^{1,\varepsilon}$ and Lipschitz Pogorelov-like singular viscosity solutions to $\sigma_{k\geqslant 3}$ equations in dimension three and higher. In the σ_n or Monge-Ampère equation case, those viscosity solutions are also singular solutions in the Alexandrov integral sense.

The advance in the joint work with Chen-Shankar [CSY] also led us to obtain interior regularity (analyticity) for almost convex viscosity solutions to the quadratic Hessian equation (1.1), in the joint work with Shankar [SY2]. Due to similar conceptual and technical challenges–smooth approximations may not preserve those semiconvexity constraints–we cannot invoke our available Hessian estimates with Shankar [SY1] for general semiconvex solutions or with McGonagle and Song [MSY] for almost convex solutions, while taking the limit and deduce interior regularity. A key observation is that the Legendre-Lewy transform of any semiconvex viscosity solution to the equivalent concave equation (1.2) stays viscosity solution to a new concave (1.4) (only for the original almost convex solution) and uniformly elliptic equation (1.3) (for all original semiconvex solutions). In passing, let us note that, for a general fully nonlinear second order elliptic equation, Alvarez-Lasry-Lions showed that the Legendre transform of any strictly convex C^2 solution is a convex viscosity solution of a conjugate equation. Moreover, the "striking" role of the C^2 regularity of the original solution in their arguments was pointed out [ALL, p.281]. It follows that the transformed $C^{1,1}$ solution is smooth by the Evans-Krylov-Safonov theory. Then the boundedness of the original solutions combined with the constant rank theorem by Caffarelli-Guan-Ma [CGM] implies that the original viscosity solution is smooth.

Shortly after, Mooney [M] provided a different proof of the interior regularity for convex viscosity solutions: every such convex solution is strictly 2-convex, then all smooth approximated solutions enjoy uniform Pogorelov-type $C^{1,1}$ and higher derivative estimates by Chou-Wang [CW], in turn, the interior regularity by taking limit.

In two dimensions, the above regularity result (now the convexity condition is automatic) actually also follows from Heinz's famous Hessian estimate earlier. In three dimensions, the regularity for continuous viscosity solutions to (1.1) follows from the Hessian estimate in our joint work with Warren [WY] and smooth existence with smooth boundary value by Caffarelli-Nirenberg-Spruck, and also Trudinger. Our most recent joint work with Shankar [SY4] on Hessian estimates for the quadratic Hessian equation (1.1) in four dimensions yields up the same regularity in four dimensions. There, a direct way to interior regularity without first deriving the Hessian estimates is also provided. Consequently, a compactness argument leads to an implicit Hessian estimate.

2.4. A priori Hessian estimates. In our long investigation, culminating in the most recent joint work with Shankar [SY4], we obtained an implicit Hessian estimate and interior regularity (analyticity) for the quadratic Hessian equation $\sigma_2\left(D^2u\right) = 1$ in four dimensions. Our compactness method (almost Jacobi inequality–doubling–twice differentiability–small perturbation) also provides respectively a Hessian estimate for smooth solutions satisfying a dynamic semiconvexity condition in higher dimensions, which includes convexity, almost convexity, and semiconvexity conditions appeared in the recent papers on Hessian estimates, and a non-minimal surface proof for the corresponding three dimensional results in our earlier joint work with Warren [WY].

Other consequence is a rigidity result for entire solutions to the σ_2 equation with quadratic growth, namely all such solutions must be quadratic, provided the smooth solutions in dimension $n \geqslant 5$ also satisfying the dynamic semiconvex assumption.

Again, the Hessian estimate for the σ_2 or Monge-Ampère equation in dimension two was achieved by Heinz in the 1950s. Hessian estimates fail for the Monge-Ampère equation in dimension three and higher, as illustrated by the famous counterexamples of Pogorelov in the 1970s; those irregular solutions also serve as counterexamples for cubic and higher order symmetric $\sigma_{k\geqslant 3}$ equations, for example by Urbas. Hessian estimates for solutions with certain strict convexity constraints to the Monge-Ampère and $\sigma_{k\geqslant 2}$ equations were derived by Pogorelov and later Chou-Wang respectively using the Pogorelov technique; some (pointwise) Hessian estimates in terms of certain integrals of the Hessian were obtained by Urbas in the early 2000s. The gradient estimates for σ_k equations were derived by Trudinger, Chou-Wang in the mid 1990s.

The compactness proof toward an implicit Hessian estimates for almost convex solutions in [MSY] is based on the concavity of uniformly elliptic equation (1.4) under the Legendre-Lewy transformation, a constant rank theorem by Caffarelli-Guan-Ma [CGM], on the vertical side, and a strip argument on the horizontal side.

The proof toward an explicit Hessian estimate for semiconvex solutions is based on an elusive-Jacobi inequality-satisfying quantity, the maximum eigenvalue of the Hessian of the solutions, envisioned to be true in 2012. Another essential new device is a mean value inequality for the strongly subharmonic maximum eigenvalue under the Legendre-Lewy transformation with uniformly elliptic equation (1.3), and its weighted version converted back to the original variables or horizontal side.

The new idea for Hessian estimates, under a dynamic semiconvexity condition in dimension five and higher, and consequently interior regularity in dimension four, is first to get a doubling, or a "three-sphere" inequality for the Hessian bound on the middle ball, in terms of Hessian bound on a small inner ball and gradient bound on the outer large ball:

$$\max_{B_2(0)} \triangle u \leqslant C\left(r, \|u\|_{Lip(B_3(0))}\right) \max_{B_r(0)} \triangle u.$$

Using a Jacobi inequality, true with a lower σ_3-bound condition for Hessian, satisfied by convex solutions, Guan-Qiu [GQ] reached their Hessian estimate for the quadratic Hessian equation. Qiu [Q] followed with his doubling in three dimensions, where the Jacobi inequality was unconditionally available since [WY]. But the maximum of Guan-Qiu test function

$$\begin{array}{c} P=2\ln\left(9-|x|^2\right)+\alpha\,|Du|^2/2+\beta\ \ (x\cdot Du-u) \\ +\ln\max\left\{\ln\frac{\triangle u}{\max_{B_1(0)}\triangle u},\gamma^{-1}\right\} \\ \text{with small } \gamma=\gamma(n)>0,\ \text{ smaller } \beta=\beta\left(\gamma,\|u\|_{Lip(B_3(0))}\right)>0 \\ \text{and smallest } \alpha=\alpha\left(\gamma,\|u\|_{Lip(B_3(0))}\right)>0, \end{array}$$

could not be ruled out from happening on the small inner ball without the σ_3-lower bound assumption, thus Qiu's "three-sphere" inequality.

Now only an almost Jacobi inequality is available in dimension four. In fact, as observed in 2012, there is no Jacobi inequality in dimension four, and worse, even no subharmonicity of the Laplace of the log of Hessian in dimension five, thus the added dynamic semiconvexity condition in higher dimensions for an almost Jacobi inequality:

$$\begin{array}{c} \triangle_F b\overset{n=4}{\geqslant}\left(\dfrac{1}{2}+\dfrac{\lambda_{\min}}{\triangle u}\right)|\nabla_F b|^2\geqslant 0; \\ \triangle_F b\overset{n\geqslant 5}{\geqslant}\left(c_n+\frac{\lambda_{\min}}{\triangle u}\right)|\nabla_F b|^2\geqslant 0,\quad \textbf{IF}\quad c_n+\frac{\lambda_{\min}}{\triangle u}\geqslant 0 \\ \text{with}\quad c_n=\frac{\sqrt{3n^2+1}-n+1}{2n}\quad\text{and}\quad b=\ln\triangle u \end{array}.$$

Note that for $\sigma_2(\lambda)=1$, we have $\frac{\lambda_{\min}}{\triangle u}>-\frac{n-2}{n}$, and at extreme configuration $\lambda=\left(K,\cdots,K,-\frac{n-2}{2}K+\frac{1}{(n-1)K}\right)$, one has $\frac{\lambda_{\min}}{\triangle u}\underset{\to}{>}-\frac{n-2}{n}$. In fact Jacobi inequality holds $\triangle_F b\overset{n=3}{\geqslant}\frac{1}{3}|\nabla_F b|^2\geqslant\left(\frac{1}{2}-\frac{1}{3}\right)|\nabla_F b|^2$ unconditionally in three dimensions.

But the almost Jacobi inequality is really a regular one away from the extreme configuration of the equation, where the equation is conformally uniformly elliptic. Qiu's doubling argument can be pushed through.

Now to find a small inner ball where the Hessian is bounded, we first show the almost everywhere twice differentiability of continuous viscosity solutions,

$$u(x)-Q_y(x)=o\left(|x-y|^2\right)$$

by adapting Chauder-Trudinger's argument [CT] for k-convex functions with $k>n/2$, with the gradient estimates $\|Du\|_{L^\infty(B_1)}\leqslant C(n)\|u\|_{L^\infty(B_2)}$, actually its integral form of control (a Hölder substitute for k-convex function was used in [CT]) for σ_k equations in [T] and also [CW], and the fact that D^2u is a bounded Borel measure for solutions of $\sigma_2\left(D^2u\right)=1$, as

$$\int_{B_1}\left|D^2u\right|dx \overset{\triangle u=\sqrt{2+|D^2u|^2}}{<} \int_{B_1}\triangle u dx \leqslant C(n)\,\|Du\|_{L^\infty(B_1)}.$$

Then Savin's small perturbation (from the quadratic polynomial at a twice differentiable point) [S] guarantees the small inner ball with bounded Hessian.

In our most recent follow-up paper with Shankar [SY5], a new proof of regularity for strictly convex solutions to $\det D^2u=1$ is found, using similar doubling methods, instead of Euclid distance, now in terms of an extrinsic distance on the maximal Lagrangian submanifold determined by the potential Monge-Ampère equation. This "strict convex" regularity was achieved originally by Pogorelov in the 1960s and 1970s, and generalized by Urbas and Caffarelli in the late 1980s.

3. Problems

Problem 1. Are there singular (Lipschitz) viscosity solutions, $W^{2,1}$ regularity, and any better partial regularity $\sigma_2\left(D^2u\right)=1$ in dimension five or higher?

Given the Jacobi inequality is exhausted in our argument for Hessian estimates and regularity in four dimensions, it is time to look for singular viscosity solutions and better partial regularity for possible singular viscosity solutions in dimension five or higher. For example, a dimension estimate on the singular set of possible singular viscosity solutions. Note that by the gradient estimate, and then smooth approximations in Lipschitz norm, all continuous viscosity solutions are Lipschtiz, and by our almost everywhere twice differentiability [SY4] and Savin's small perturbation [S], the possible singular set is closed and with zero Lebesgue measure (Also true for viscosity solutions to $\sigma_{k\geqslant 3}$ equation in dim $n\geqslant 3$).

From $\int_{B_l}\left|D^2u\right|dx \overset{\sigma_2=1}{<} \int_{B_l}\triangle u dx \leqslant \|Du\|_{L^\infty(\partial B_l)}\,|\partial B_l|$ and the gradient estimate, D^2u is a bounded Borel measure. It is reasonable to expect a $W^{2,1}$ regularity in dim $n\geqslant 5$. Recall that, unlike function $|x_1|$, all (convex) viscosity solutions to $\sigma_n\left(D^2u\right)=1$ have been shown to be $W^{2,1}$.

Problem 2. Regularity for semiconvex viscosity solutions to $\sigma_2\left(D^2u\right)=1$ in dimension five or higher.

It is still unclear to us whether semiconvex viscosity solutions are regular, if only $D^2u\geqslant -KI$ for large $K>0$. The Legendre-Lewy transform is still a $C^{1,1}$ viscosity solution of a new uniformly elliptic equation (1.3), for any semiconvex viscosity solution. However, as the new equation no longer has convex level set, for large K, we are unable to deduce smoothness for the transformed solution at this point. Without the smoothness, we are currently unable to obtain a $C^{1,1}$ version of the constant rank theorem to gain positive definiteness of the semi-positive Hessian, for the $C^{1,1}$ solution of a uniformly elliptic and inversely convex equation on the vertical side. Otherwise, the interior regularity for such semiconvex viscosity solutions would be justified. At this point, it appears a far stretch to reach regularity for dymanic semiconvex viscosity solutions in dimension five or higher.

One follow-up of our Hessian estimates for three and the very recent four dimension σ_2 equation would be

Problem 3. Derive Schauder and Calderón-Zygmund estimates for variable-right-hand-side equation $\sigma_2\left(D^2u\right)=f\left(x\right)$ in dimension four.

With a $C^{1,1}$ assumption on $f\left(x\right)$, Qiu [Q] has generalized the arguments in [WY] to obtain Hessian estimates in dimension three. With an almost sharp Lipschitz assumption on $f\left(x\right)$, very recently, Zhou [Z] reached the Hessian estimate in three dimensions, along his Hessian estimates for the "twist" special Lagrangian equation $\sum_{i=1}^{n}\arctan\lambda_i/f\left(x\right)=c\in\left[\left(n-2\right)\pi/2,n\pi/2\right)$. Consequently $C^{2,\alpha}$ estimates follow. Under a small enough Hölder seminorm assumption on f, Xu [X] derived interior $C^{2,\alpha}$ estimates in dimension three. Notice that the interior gradient estimates in [T] and [CW] needs Lipschitz assumption on f. The subtle small seminorm constraint is due to the non-uniform elliptic nature of the equation. The method works in general dimensions such as the recent four dimension, as long as the interior Hessian estimate is available for the quadratic Hessian equation with constant right-hand side.

Problem 4. Any "elementary" pointwise argument toward Hessian estimates for $\sigma_2\left(D^2u\right)=1$ in dimension three and four, in general higher dimension with the dynamic semiconvexity condition, as in the two dimensional case by Chen-Han-Ou [CHO], and convex case by Guan-Qiu [GQ]?

Forthermore, any explicit Hessian bound in terms of the gradient, as the quadratic exponential dependence in dimension three and general semiconvex case?

To gain more understanding of σ_2 equation, one distinct double divergence structure of σ_2 from $\sigma_{k\geqslant 3}$ is worth studying.

Problem 5. Under what additional condition on $u\in W^{1,2}\left(\Omega\right)$ does the equation

$\sigma_2\left(D^2u\right) = 1$ in the very weak sense (Iwaniec [I]):

$$\int_\Omega \sum_{1\leqslant i<j\leqslant n}\left(\varphi_{ij}u_iu_j - \frac{1}{2}\varphi_{ii}u_j^2 - \frac{1}{2}\varphi_{jj}u_i^2\right)dx = \int_\Omega \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

become $\sigma_2\left(D^2u\right) = 1$, say, and $\triangle u > 0$, in the viscosity sense?

(The double divergence structure is readily seen from the well-known Gauss curvature formula for graph $(x_1, x_2, u(x)) \subset \mathbb{R}^3$ with induced metric $g : K = \left(-\frac{1}{2}\partial_{11}g_{22} + \partial_{12}g_{12} - \frac{1}{2}\partial_{22}g_{11}\right) / (\det g)^2 = \det D^2u / \left(1 + |Du|^2\right)^2$.)

In dimension two, a (necessary) convexity condition should suffice, also for $\sigma_2\left(D^2u\right) = 1$ in the equivalent Alexandrov sense. In general dimensions, what about $\triangle u > 0$ in distribution sense for $u \in C^{1,2^+/3}$? The answer is yes in two dimensions, as shown by Pakzad [P]. Moreover, in two dimensions, no better than $C^{1,1/3}$ "very weak" solution with sign changing $\triangle u$ have been constructed; see the work of Lewicka-Pakzad [LP] and [CS] [CHI]. It is worth noting that the singular solution to $\sigma_2\left(D^2u\right) = 1$ constructed by C.Y. Li [L], $u(x) = \left(x_1^2 + \cdots + x_7^2\right)x_8^{7/5} - \frac{25}{84}x_8^{3/5} - \frac{25}{28}x_8^{14/5} \in W_{loc}^{1,2} \cap C^{3/5}$ jumps branches, because $\triangle u \approx -x_8^{-7/5} = \pm\infty$ near $x_8 = 0$.

References

[ALL] Alvarez, Olivier; Lasry, Jean-Michel; Lions, Pierre-Louis, *Convex viscosity solutions and state constraints.* J. Math. Pures Appl. (9) **76** (1997), no. 3, 265-288.

[BCGJ] Bao, Jiguang; Chen, Jingyi; Guan, Bo; Ji, Min, *Liouville property and regularity of a Hessian quotient equation.* Amer. J. Math. **125** (2003), no. 2, 301–316.

[CGM] Caffarelli, Luis; Guan, Pengfei; Ma, Xi-Nan, *A constant rank theorem for solutions of fully nonlinear elliptic equations.* Comm. Pure Appl. Math. **60** (2007), no. 12, 1769–1791.

[CS] Cao, Wentao; Székelyhidi, László, Jr., *Very weak solutions to the two-dimensional Monge-Ampére equation.* Sci. China Math. **62** (2019), no. 6, 1041–1056.

[CHI] Cao, Wentao; Hirsch, Jonas; Inauen, Dominik, *$C^{1,1/3-}$ very weak solutions to the two dimensional Monge-Ampére equation.* arXiv:2310.06693.

[ChY] Chang, Sun-Yung Alice; Yuan, Yu, *A Liouville problem for the sigma-2 equation.* Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 659–664.

[CT] Chaudhuri, Nirmalendu; Trudinger, Neil S., *An Alexsandrov type theorem for k-convex functions.* Bull. Austral. Math. Soc. **71** (2005), no. 2, 305–314.

[CHO] Chen, Chuanqiang; Han, Fei; Ou, Qiangzhong, *The interior C2 estimate for Monge-Ampere equation in dimension n=2.* Anal. PDE **9** (2016), no. 6, 1419–1432.

[CSY] Chen, Jingyi; Shankar, Ravi; Yuan, Yu, *Regularity for convex viscosity solutions of special Lagrangian Equation.* Comm. Pure Appl. Math. **76** (2023) 4075–4086.

[CX] Chen, Li; Xiang, Ni, *Rigidity theorems for the entire solutions of 2-Hessian equation.* J. Differential Equations **267** (2019), no. 9, 5202–5219.

[CW] Chou, Kai-Seng; Wang, Xu-Jia, *A variational theory of the Hessian equation.* Comm. Pure Appl. Math. **54** (2001), 1029–1064.

[GQ] Guan, Pengfei; Qiu, Guohuan, *Interior C^2 regularity of convex solutions to prescribing scalar curvature equations.* Duke Math. J. **168** (2019), no. 9, 1641–1663.

[I] Iwaniec,Tadeusz, *On the concept of the weak Jacobian and Hessian.* Papers on analysis, 181–205, Rep. Univ. Jyväskylä Dep. Math. Stat., **83**, Univ. Jyväskylä, Jyväskylä, 2001.

[LP] Lewicka, Marta; Pakzad, Mohammad Reza, *Convex integration for the Monge-Ampère equation in two dimensions.* Anal. PDE **10** (2017), no. 3, 695–727.

[L] Li, Caiyan, *Non-polynomial solutions to Hessian equations.* Calc. Var. Partial Differential Equations **60** (2021), no. 4, Paper No. 123, 6 pp..

[LT] Lin, Min; Trudinger, Neil S., *On some inequalities for elementary symmetric functions.* Bull. Austral. Math. Soc. **50** (1994), 317–326.

[MSY] McGonagle, Matt; Song, Chong; Yuan, Yu, *Hessian estimates for convex solutions to quadratic Hessian equation.* Ann. Inst. H. Poincaré Anal. Non Linéire **36** (2019), no. 2, 451–454.

[M] Mooney, Connor, *Strict 2-convexity of convex solutions to the quadratic Hessian equation.* Proc. Amer. Math. Soc. **149** (2021), no. 6, 2473–2477.

[N] Nitsche, Johannes C. C., *Elementary proof of Bernstein's theorem on minimal surfaces.* Ann. of Math. **66** (1957), 543–544.

[P] Pakzad, Mohammad Reza, *Convexity of weakly regular surfaces of distributional nonnegative intrinsic curvature.* arXiv:2206.09224.

[Q] Qiu, Guohuan, *Interior Hessian estimates for Sigma-2 equations in dimension three.* arXiv:1711.00948.

[S] Savin, Ovidiu, *Small perturbation solutions for elliptic equations.* Comm. Partial Differential Equations **32** (2007), no. 4-6, 557–578.

[SY1] Shankar, Ravi; Yuan, Yu, *Hessian estimate for semiconvex solutions to the sigma-2 equation.* Calc. Var. Partial Differential Equations **59** (2020), no. 1, Paper No. 30, 12 pp.

[SY2] Shankar, Ravi; Yuan, Yu, *Regularity for almost convex viscosity solutions of the sigma-2 equation.* J. Math. Study, **54** (2021), no. 2, 164–170.

[SY3] Shankar, Ravi; Yuan, Yu, *Rigidity for general semiconvex entire solutions to the sigma-2 equation.* Duke Math. J. **171** (2022), no. 15, 3201–3214.

[SY4] Shankar, Ravi; Yuan, Yu, *Hessian estimates for the sigma-2 equation in dimension four.* submitted, arXiv:2305.12587.

[SY5] Shankar, Ravi; Yuan, Yu, *Regularity for the Monge-Ampère equation by doubling.* preprint.

[T] Trudinger, Neil S., *Weak solutions of Hessian equations.* Comm. Partial Differential Equations **22** (1997), no. 7-8, 1251–1261.

[W] Warren, Micah, *Nonpolynomial entire solutions to σ_k equations.* Comm. Partial Differential Equations **41** (2016), no. 5, 848–853.

[WY] Warren, Micah; Yuan, Yu, *Hessian estimates for the sigma-2 equation in dimension 3.* Comm. Pure Appl. Math. **62** (2009), no. 3, 305–321.

[X] Xu, Yingfeng, *Interior estimates for solutions of the quadratic Hessian equations in dimension three.* J. Math. Anal. Appl. **489** (2020), no. 2, 124179, 10 pp.

[Y] Yuan, Yu, *A Bernstein problem for special Lagrangian equations.* Invent. Math. **150** (2002), 117–125.

[Z] Zhou, Xingchen, *Notes on generalized special Lagrangian equation.* submitted, 2023.

University of Washington, Department of Mathematics, Box 354350 Seattle, WA 98195
E-mail: yuan@math.washington.edu