STUDENTS' MEANINGS FOR COORDINATE SYSTEMS: CONTINUOUS AND ORDERED-DISCRETE REFERENCE FRAMES

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Graphical representations are commonly used in everyday life and are important in STEM fields. Interpreting graphs entails understanding the underlying structures of graphs, including coordinate systems and reference frames. In this report, we characterize one student's constructions of coordinate systems. These constructions indicate two distinct types of reference frames not currently distinguished in the literature: (a) continuous and (b) ordered-discrete. Using data from a 10-session teaching experiment, we discuss the interplay of a student's perception of tasks, the reference frames she reasoned with, and differences in those reference frames. We consider how the interplay of the aforementioned items may have influenced the quantities she considered as well as the coordinate systems she constructed. We conclude with suggestions for research and teaching that support students' productive graphing activity.

Keywords: Geometry and Spatial Reasoning, Mathematical Representations, Cognition, Middle School Education

Graphs are a powerful way to visualize, explore, and communicate relationships between quantities. In STEM contexts, graphs can be used to mathematize spatial situations or represent relationships between covarying quantities (Paoletti et al., 2020; Glazer, 2011). In our view, students' meanings for graphs should depend on their meanings for coordinate systems, especially if their meanings for graphs are to be productive (Lee et al., 2020). Recent research has focused on differences in the underlying coordinate systems that students construct and how reasoning within these coordinate systems explains their graphing activity (Paoletti et al., 2018, 2022; Parr, 2023). In this report, we offer another contribution to this literature by characterizing two novel types of reference frames that underlie coordinate systems and by describing how students may use these reference frames to reason about quantities represented in coordinate systems. We begin with a theoretical background that defines two different types of coordinate systems and establishes a distinction between types of reference frames. We describe one student's use of both types of references frames within each coordinate system. We conclude with implications for teachers, curriculum designers, and researchers.

Theoretical Background: Two Types of Coordinate Systems and Reference Frames

Researchers (Lee, 2017; Lee et al., 2019, 2020) have distinguished between two kinds of coordinate systems (CSs): spatial and quantitative. Each type of CS is built by coordinating one or more reference frames. *Reference frames* (RFs), which are constructed to gauge relative extents of attributes in phenomena, consist of some orienting reference objects, directionality, and some anticipation of a measurement process that could be carried out (Lee et al., 2020; Joshua et al., 2015). When students consider quantities (Thompson, 2011), there must be at least one RF involved. We provide several examples of abstract quantities, using parentheticals to provide specific examples of situational quantities with explicit RFs: Distance (e.g., number of miles east a person is from school), time (e.g., number of minutes after passing a rest stop), and temperature (e.g., degrees Fahrenheit above 0).

A *spatial CS* involves the mental coordination of one or more RFs and a selection of units of measure which are imposed onto a physical space of interest. In this case, RFs are used to gauge the relative locations of objects within that space. In a spatial CS, locations in the space may be tagged with coordinates guided by these RFs and obtained through carrying out the anticipated measurement. For example, a student might organize the map in Figure 1a by constructing a spatial CS consisting of two RFs that imply the consideration of distinct distances from a reference object (like the star icon). The spatial CS could then be used to describe the X's (or any object's) location in terms of unique pairs of distances from the star icon.

A *quantitative CS* involves the mental coordination of one or more quantities which are activated upon assimilation of a situation, disembedded from it, and inserted into a new representational space through the coordination of their RFs. For example, a person may coordinate the relationship between the time and temperature throughout a day and represent this relationship via a graph in a quantitative CS. Within both spatial and quantitative CSs, locations within the CS are imbued with quantitative extents, which necessarily involve RFs.

In this paper, we add a fourth dimension to thinking within RFs: continuity. In addition to reference object, directionality, and some anticipated measurement process, we have found in our work with students that the notion of directionality and some anticipated measurement process could be established either discretely or continuously. Hence, we distinguish between two kinds of RFs that students indicated when reasoning in both types of CSs: *ordered-discrete RFs* and *continuous RFs*. A *continuous RF* involves understanding a continuum of an attribute's extents relative to the reference object and guides measuring activities that would lead to measurements as continuous quantities. An *ordered-discrete RF* involves segmenting an attribute's extents according to distinct, bounded regions that are arranged in some (implicit or explicit) sequence. Ordinal or directional language can be an indication of an established ordered-discrete RF and guides measuring activities that would lead to measurements in discrete units. For example, an individual who has established an ordered-discrete RF within a designed region might describe sub-regions in ordinal terms (e.g., second row or last circle from the center) or directional terms (e.g., left side or near the middle).

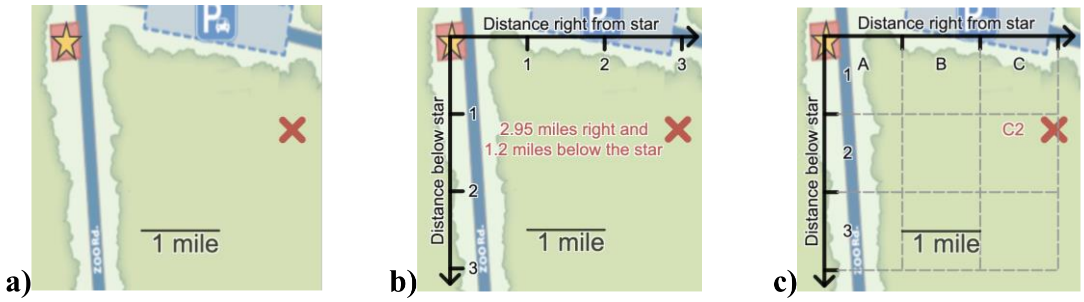

Figure 1: A map indicating: a) no CS, b) a spatial CS constituted by coordinating two continuous RFs c) a spatial CS constituted by a coordination of two ordered-discrete RFs.

We note an individual may understand an attribute as continuous and still construct an ordered-discrete RF; such a construction is dependent on the student's conceived context, goals, or current quantitative constraints in their reasoning. For example, Figure 1b shows how a spatial CS could be constituted by coordinating two continuous RFs. Figure 1c shows how a spatial CS could be constituted by coordinating two ordered-discrete RFs. In this report, we address the research question: *How does a student's construction of continuous and ordered-discrete RFs impact her reasoning in spatial and quantitative CSs?*

Methods

To address our RQ, we report on data from a teaching experiment (Steffe & Thompson, 2000) with three sixth-grade students: Nina (who self-identified as Latina), Tara (who self-identified as a White female), and Jacobi (who self-identified as an African American male). We focus this report on Nina's activity because she provided the strongest indications of the RFs of interest. The teaching experiment took place in a middle school whose population consisted of over 75% students of color. Participants were recruited based on teacher recommendation and student availability. Nina attended 10 teaching experiment sessions each lasting 35–40 minutes (Table 1). We video- and audio-recorded each session to capture utterances and gestures. Student activity on the Desmos platform was screen recorded, and we digitized all written work.

Table 1: Small Group Teaching Experiment Sequence

Session	Students Present	Task	Intended Student Goal
0	Nina	Pre-Interview	Various
1	Nina, Tara, Jacobi	X Marks the Spot – Guess Where	Construct and/or interpret spatial RFs and CSs to describe and/or identify locations in space
2	Nina, Tara	X Marks the Spot – Anywhere	
3		X Marks the Spot – Classmates' Descriptions	
4	Nina, Jacobi	X Marks the Spot – Anywhere	
5	Nina, Tara, Jacobi	North Pole Task	
6 7	Nina, Tara, Jacobi	Zoo Task	Interpret points in a quantitative CS

8	Nina, Tara, Jacobi	Kodiak Task	Interpret graphs in a quantitative CS
9	Nina, Tara		
10	Nina, Tara	Post-Interview (Growing Fruit)	Various

Tasks

We describe Nina's activity across several tasks from Sessions 3, 4, and 10. In the *X Marks the Spot-Anywhere* and *-Classmates' Description* tasks (Sessions 2–4), students took on the roles of Describer and Guesser in Desmos. The Describer was prompted to mark an X on the map and then generate a description of the X's location. The Guesser used that description to mark an X on their own version of the map. Students had access to a set of digital overlays (e.g., vertical lines, horizontal lines, concentric circles anchored at the star icon) that could be activated to potentially support students' location descriptions. For example, in Figure 2, the 'Horizontal' and 'Vertical' overlays are activated. In the *Anywhere* variation of the task, a pair of students take turns as Describer and Guesser for each other. In the *Classmates' Descriptions* variation, the students worked together as Guessers, with hypothetical classmates as Describer. The hypothetical classmates' descriptions were researcher-authored and sequenced to progress from (what we then considered) less precise to more precise descriptions in both polar-like and Cartesian-like CSs. We had yet to distinguish between continuous and ordered-discrete RFs when we authored these descriptions. However, in retrospect, the descriptions that we considered less precise used language indicative of ordered-discrete RFs while the descriptions that we considered more precise used language indicative of continuous RFs. In Session 10 Nina and Tara completed a post-interview together wherein they attempted tasks individually, and the teacher-researcher (TR) facilitated discussion across their responses. The fourth task of the post-interview, *Growing Fruit,* asked students to describe a situation that would be reflected by a given graph representing the relationship between a hypothetical fruit's weight and calorie content, both of which changed over time (Figure 3a).

Analysis

Consistent with teaching experiment methodology, we analyzed the data via conceptual analysis, which entails "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 45). We watched all videos and identified moments that offered insight into the CSs and RFs Nina constructed as she addressed each task. We then created models characterizing whether Nina was constructing quantitative or spatial CSs. As we described Nina's reasoning in each type of CS, we characterized continuous and ordered-discrete RFs as an important distinction in her reasoning; we had not considered this distinction prior to conducting this analysis.

Results

Nina used two distinct types of RFs, *ordered-discrete* and *continuous* to construct and interpret both spatial and quantitative CSs. Further, the RF Nina constructed influenced her reasoning in each CS. Because Nina constructed both types of RFs within both types of CSs, we present these types of reasoning in a two-by-two matrix and detail four examples from the teaching experiment that demonstrate each combination (Table 2).

Table 2: Task and activity in which Nina constructed a CS using each type of RF

	Ordered-discrete RFs	Continuous RFs
Spatial CS	Describing locations in *X-marks the Spot-Anywhere*	Interpreting locations in *X-marks the Spot Anywhere*
Quantitative CS	Interpreting a given graph in the *Growing Fruit task*	Interpreting a modified graph in the *Growing Fruit task*

Continuous Reference Frames in Spatial Coordinate Systems

In Session 3, Nina interpreted continuous RFs in a spatial CS when she followed a hypothetical classmate's description to mark an X in the *X Marks the Spot – Classmates' Descriptions* task. Nina's interpretation of the following description shows her ability to construct continuous RFs in a spatial CS:

> Click the Star (1) and Circles (3) options. Imagine the star is like a clock with the line going straight up being 12 o'clock and the line going straight down being 6 o'clock. The X is 1.25 miles from the star [icon] halfway between 10 and 11 o'clock.

This description is intended to introduce a pseudo-polar, spatial CS in which the continuous RFs are the radial distance (explicitly in 'miles') from the star icon and angle measure (implicitly in 'hours') from the top vertical line. After reading the description, Tara moved the cursor to an approximately correct location. Nina grabbed a measuring device (a wax-covered string bent at a length equivalent to the '1 mile' key on the map), which she used to confirm Tara's approximation. Specifically, Nina placed one end of her measuring device at the center of the star overlay and oriented the other end halfway between the lines representing 10 and 11 o'clock, near where Tara had placed the cursor. Nina reasoned that if the string piece was one mile, then the X must be slightly beyond it. Hence, Nina reasoned about distance from the center as a continuous quantity (i.e., 1.25 miles is slightly more than 1 mile) while also attending to the clock description as a continuous quantity (i.e., a location halfway between 10 and 11 o'clock). Thus, Nina used continuous RFs to generate an exact location in a spatial CS.

Ordered-Discrete Reference Frames in Spatial Coordinate Systems

In Session 4, Nina primarily used ordered-discrete RFs. For example, in her third turn as Describer in the *Anywhere* variation of the task, Nina established a spatial CS using two ordered-discrete RFs to describe a region in which her X was located. Nina marked an X as in Figure 2a and provided the description "Use horizontal and vertical lines. The lines make squares so count from the left, go all the way to the bottom, and count 6. Then go up 2, the x is in the right corner." To Nina, the combination of the vertical and horizontal overlays created distinct 'squares' (discrete regions) that Jacobi could count (ordering language) to identify which region contained the X (Figure 2a). Reflecting the non-continuous nature of the RFs Nina was constructing, the relative size of these 'squares' was not relevant from her perspective; her description did not distinguish the partial boxes in the bottom-most row and left-most column from the other boxes in the grid. Hence, we infer she was reasoning about discrete, ordered, regions from the bottom left corner. Hence, Nina established a spatial CS to describe a region by coordinating two discrete ordered RFs (The number of boxes to the right starting from the left side of the map and the number of boxes vertically up from the bottom of the map).

a) b)

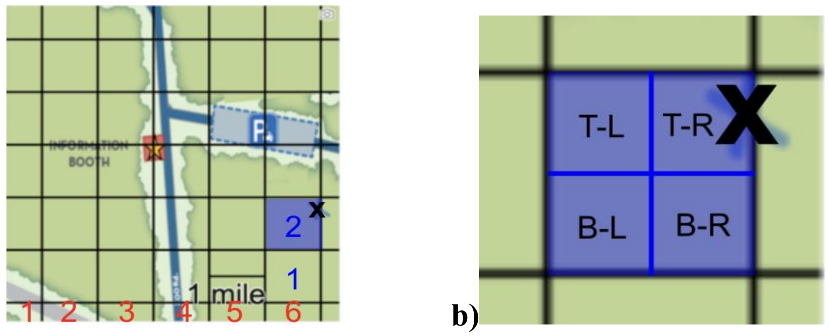

Figure 2. Representation of Nina's ordered-discrete reasoning in space in a) the first part and in b) the second part of the description

Nina's use of ordered-discrete RFs influenced her activity in the spatial CS as it led her to using more than one set of ordered-discrete RFs as she described increasingly narrow regions in which points were located. That is, we infer that Nina's addition of "x is in the right corner" was a second ordered-discrete RF she constructed within the first box she described. (We note Nina did not specify between top or bottom right corner, but we conjecture she meant top-right based on the X's placement.) Our inference is based on her use of "right corner" as a location rather than a reference object (i.e., "in the right corner" as opposed to "1 cm from the right corner"). One possible way she could have done this is by mentally subdividing the 'square' into (at least) four discrete, ordered quadrants (i.e., top-left, top-right, bottom-left, bottom-right; Figure 2b). Thus, we infer that Nina could have coordinated two ordered-discrete RFs (left/right and top/bottom from the midpoint of the box) to describe a narrower region within a particular region of a spatial CS.

Reference Frames in Quantitative Coordinate Systems

In Session 10, Nina addressed the *Growing Fruit* task (Figure 3a). A normative explanation would include a description that at first the fruit gains weight while its calories remain the same and then the fruit's weight and calories increase simultaneously. Based on Nina's activity in Sessions 7-11, we anticipated Nina would use continuous RFs to interpret the given graph and produce a normative explanation. However, we infer that Nina initially reasoned about the quantities using an ordered-discrete RF and shifted to using a continuous RF when the TR added numbers to the horizontal axis. We provide evidence for each claim in the next two sections.

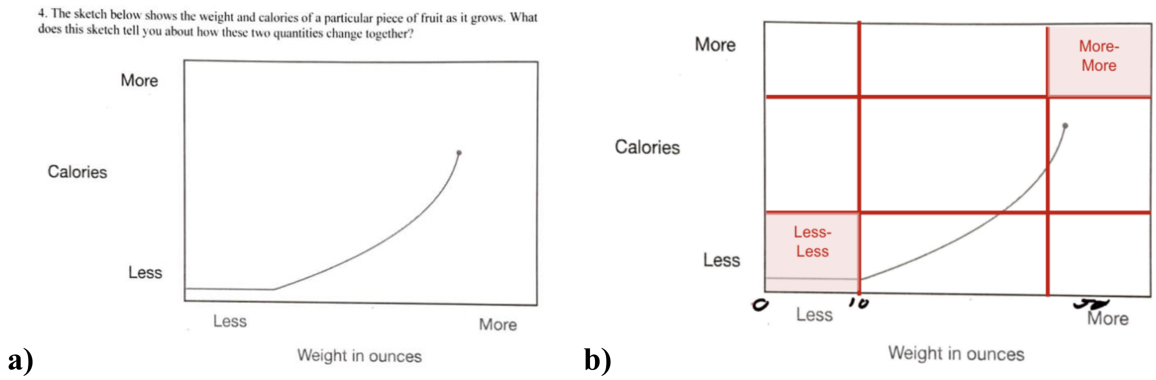

Figure 3: a) Growing Fruit Task as presented and b) a potential depiction of how ordered-discrete RFs could be coordinated to reason about the horizontal segment of the graph.

Ordered-discrete reference frames in quantitative coordinate systems. When initially interpreting the horizontal segment (and possibly the entire graph), Nina employed an ordered-discrete reference frame. Describing a situation that created this graph, Nina explained:

N: The fruit starts off, like, without any calories [*points to horizontal segment*] and doesn't weigh a lot. And then while it grows [*traces curve*] it gains calories and … weighs more.
TR: Gains calories and weighs more?
N: Yeah.
TR: [*referring to the curved part of the graph*] So that's sort of what you [Tara] were saying, too. So, I think you're both in agreement. Now let me ask you [Nina] this question. If we start say, here [*gestures to the vertical intercept*] and just paying attention to this part [*tracing the horizontal portion of the graph*]. What's changing?
N: Nothing.

Considering Nina's argument, Tara disagreed with it. Tara traced the horizontal part of the graph saying, "Well as you go right the weight is getting bigger because it's getting closer to 'more' weight, I guess. But the calories would stay the exact same right here." Nina explicitly disagreed with Tara's argument stating, "I don't think here it's getting bigger [*traces horizontal segment*]. 'cause it's like [*pointing to "Less" markers on each axis*]...[3 second pause] For me it's like not getting bigger cause it's like still at less."

We interpret Nina as reasoning with ordered-discrete RFs as she interpreted the weight and calories of the fruit for the horizontal segment. In particular, she argued the horizontal segment was representing the quantities as both being in a static state of 'small' because the segment was close to the 'Less' label on each axis. Like her activity in *X Marks the Spot - Anywhere,* Nina was reasoning about ordered-discrete RFs on each axis by creating regions based on the 'Less' labels along each axis. We show one potential illustration of the resulting regions Nina may have been reasoning about in Figure 3b.

We note Nina's initial description of the curved segment ("while it grows it gains calories and … weighs more") could be indicative of reasoning with either ordered-discrete RFs or continuous RFs. If Nina understood that the curved graph spanned the (Less, Less) region and the (More, Medium) region, then she might have argued that the weight and calories both increased by some unknown amount as each moved into a higher-ordered region. If, however, Nina understood there to be a continuum of values beyond the (Less, Less) region, then she could have been using a continuous RF to reason about this part of the graph. As the TR was not aware of the distinction between the two types of RFs in the moment, he did not explore this possibility further. However, he did conjecture the 'Less' and 'More' labels on the axes, which were novel relative to quantitative CS used in previous sessions, may have been the catalyst for her reasoning about the straight segment. Hence, he opted to add numbers to the horizontal axis (0, 10, 50; seen in Figure 3b) to see if this change would lead Nina to a different interpretation.

Continuous Reference Frames in Quantitative Coordinate Systems. When the TR added the numbers to the horizontal axis, Nina immediately engaged in reasoning about the horizontal segment using a continuous RF in a quantitative CS and generated a normative interpretation of this part of the graph:

TR: But say if there were numbers here. Say this was like 0, 10, and like 50 [*writes in numbers on horizonal axis as shown in Figure 3b*]
N: Then it would get bigger
TR: Then you think it would-
N: It would weigh more
TR: You think it would weigh more?
N: Yeah
TR: And what about the calories? Would that be changing?
N: No [*shakes head*]
TR: No? Okay so it's sort of like this distinction between sort of like 'less' like we're in this less state-
N: Yeah
TR: -versus if there were numbers, you'd say they were changing? [*Nina nods head*]

When the TR added numbers to the horizontal axis, Nina immediately interpreted the horizontal segment as showing the weight increasing ("It would weigh more") as the calories remained constant. Thus, we infer Nina understood the horizontal axis as a continuous RF representing the weight of a hypothetical fruit. Furthermore, she agreed with the TR that the distinction between viewing the horizontal segment as representing a state of 'less-ness' versus viewing it as a record of change was based on the addition of the numbers. Hence, we infer the addition of numbers to the axes changed Nina's interpretation of the graph (and of the situation) as she shifted from using an ordered-discrete RF to using a continuous RF. Further, she exhibited reasoning compatible with a continuous RF on the next task in the post-interview, which asked her to construct a graph to represent the weight and calories of a novel fruit.

Discussion, Implications, Limitations, and Concluding Remarks

Addressing our RQ, we have shown how Nina used ordered-discrete and continuous RFs to reason within both spatial and quantitative CSs. Within a spatial CS, Nina's use of different RFs led to different strategies to mark or describe a location. With continuous RFs, Nina could identify an exact location, but she reasoned about increasingly narrow regions when using ordered-discrete RFs. In a quantitative CS, Nina's use of RFs impacted her interpretation of a situation represented graphically. When engaging with ordered-discrete RFs, Nina treated a segment of the graph as a single object, representing a static condition (i.e., Less-Less). However, with a minor alteration to the task, Nina considered the weight RF as continuous, thereby interpreting the segment as representing a record of change.

We note that Nina's construction of different RFs was influenced by her interpretation of and/or goals in the task. Although she was capable of reasoning with continuous RFs in both spatial and quantitative CSs, she opted to use ordered-discrete RFs when they satisfied the demands of a given task as she perceived it. We have observed other students who, like Nina, are capable of reasoning about continuous RFs in a spatial CS but opt to use ordered-discrete RFs to satisfy their perceived demands of a given task.

Implications for Curriculum and Instruction

We consider it likely that a continuous RF supersedes an ordered-discrete RF. Our hypothesis is that individuals who have constructed a continuous RF in a context would necessarily be able to construct an ordered-discrete RF in the same context, whereas an individual who constructs an ordered-discrete RF may not yet be able to construct a continuous RF in that context.

However, we emphasize that one type of RF is not inherently preferred; rather, their utility is determined by an activity's (or student's) context and goals. In spatial CSs, regions can be described using continuous RFs (e.g., Webb & Abels, 2011), but there may be instances in which ordered-discrete RFs are sufficient or even more appropriate. Although continuous RFs are more commonly used when constructing quantitative CSs, there are situations in which ordered-discrete RFs are useful. For instance, Webb and Abels (2011) describe using combination charts to describe the relationship between three quantities, such as cost of a number of pencils (represented along a horizontal axis), cost of a certain quantity of erasers (represented on the vertical axis), and total cost of n-pencils and m-erasers (represented in the cell (n, m)). Such a combination chart is an example of a quantitative coordinate system made up of two ordered-discrete RFs, in which number of pencils and number of erasers are discrete quantities.

It is important to be aware of the distinctions between these types of RFs, as their conflation can lead to unintended graphical interpretations. For instance, Figure 1c depicts a spatial coordinate system, but it is ambiguous whether each RF should be treated as continuous or ordered-discrete. On one hand, the vertical numeric labels suggest that students could describe the X's position using a continuum, but the use of letters as labels on the horizontal axis limits the ability to refer to non-discrete positions. Further, the positioning of the labels between tick marks rather than on tick marks may promote the creation of regions rather than a continuum. Depending on how an activity using a similar map is enacted, students may not conceive a distinction between the two types of RFs. Teachers and curriculum designers should be deliberate in crafting tasks and graphs such that students are prompted to engage with both types of RFs and explore the affordances and limitations of each in a variety of spatial and quantitative contexts.

Limitations and Concluding Remarks

This report is limited in that we only analyzed the activity of one student in a particular set of tasks. Future researchers may be interested in exploring how a wider range of students spontaneously construct and utilize both continuous and ordered-discrete RFs in quantitative and spatial CSs. Such research can support the field's understanding about how students reason about graphs and how such reasoning can be supported towards more normative graphing meanings.

Understanding the ways in which students interpret and construct the fundamental components of graphs, such as RFs, is crucial to supporting students' developing meanings for graphs, which are ubiquitous in STEM contexts.

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