SUPPORTING LEARNING THROUGH INTERPRETING OTHERS' SOLUTIONS FROM A RADICAL CONSTRUCTIVIST PERSPECTIVE: A THEORETICAL REPORT

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In this theoretical report, we leverage a radical constructivist perspective to explain how designing learning environments in which students work towards making sense of others' mathematical solutions may support learning. We elaborate on radical constructivist constructs—social goals, cognitive perturbations, and reflective abstraction—and use these constructs to model how engagement with others' mathematical solutions may engender learning. We illustrate our model with a task we designed to promote students' meanings for spatial coordinate systems. We conclude with implications for research and teaching.

Keywords: Learning Theory; Cognition; Problem-based Learning

Students working to make sense of worked examples (e.g., Barbieri et al., 2023) or classmates' solutions to mathematical problems (e.g., Webb et al., 2014) have been positively associated with mathematics achievement. Although researchers have described reasons why such activity may translate to achievement gains (Brown et al., 1992; Webb et al., 2023), they have not provided explanatory mechanisms for how such learning occurs for an individual. As a theory of learning focused on ways individuals develop knowledge, radical constructivism can provide such explanations (von Glasersfeld, 1995). In this report, we consider how students' working to make sense of others' mathematical solutions may support learning from a radical constructivist perspective. Additionally, we consider implications of our analysis for task design.

Learning in Radical Constructivism and Connections to Social Interactions

In this section, we present radical constructivist constructs and coordinate them to yield explanatory mechanisms through which a student may learn from others' solutions. First, we conceptualize that in a classroom, a student prompted to interpret a solution can experience a disturbance to their settled cognitive state (i.e., a perturbation). Second, as the student works to understand a solution, they can enact schemes relevant to their understanding of, and goals for, interpreting the solution. If the student experiences a cognitive perturbation, they may modify or reorganize their schemes to neutralize the disturbance. Such reorganizations can result in learning at a higher cognitive level (i.e., reflective abstraction). Next, we offer more detail about each construct and how they relate in the context of students examining others' solutions.

Schemes, Goals, and Perturbations

To begin, as students engage with others' solutions, we posit they would draw upon *schemes*, which entail "a situation, an activity triggered by how the person perceives the situation, and a result of the activity that a person assimilates to her or his expectations" (Hackenberg, 2014, p.

87). Any interaction prompting the student to draw on one or more schemes is a *disturbance* to the student's settled (equilibrated) state. A student uses their schemes as part of their goal-directed activity. Moreover, in classrooms where sharing solutions is prioritized, the student may conceive they are working toward a *social goal* with others (see Steffe & Thompson, 2000, for criteria to determine if a student is working towards a social goal). We note that even if other students are working toward a different goal (i.e., only intending to obtain a correct solution), a student's perception of a social goal can drive their goal-directed activity.

If a student works to interpret classmates' solutions, they have experienced a disturbance to their equilibrium; von Glasersfeld (1980) broadly defined a *perturbation* as any input that creates a disturbance in a student's equilibrium. Not all perturbations are *cognitive perturbations*. Students might be able to neutralize some perturbations with their current schemes without experiencing any discrepancies as they activate and anticipate the results; such perturbations are not cognitive perturbations. Neutralizing other perturbations may involve a student experiencing discrepancies in their use of a scheme (Steffe & Olive, 2009; von Glasersfeld, 1995). Such perturbations are cognitive perturbations. Cognitive perturbations are important because they can lead to a student reorganizing or modifying their existing schemes to achieve an equilibrated state (Steffe, 1991a, 1991b; Tillema & Gatza, 2024; von Glasersfeld, 1995).

When a student experiences a cognitive perturbation through engagement with others' solutions, they may experience a minor or major cognitive perturbation. Many researchers have equated perturbations with major cognitive conflict or the individual experiencing a 'problem' (e.g., Booker, 1996; Lerman, 1996; Simon et al., 2010). Although cognitive conflict is one type of cognitive perturbation, students can also experience minor cognitive perturbations without (consciously) experiencing cognitive conflict or a problem (Steffe, 2011; Steffe & Olive, 2002, 2009). To exemplify this distinction, we again turn to a student working to make sense of others' solutions. A student might experience a minor cognitive perturbation when a solution has some feature or way of reasoning that is novel for the student and the student is able to neutralize the perturbation with minor modifications to their current schemes. If an observer infers that the interpreting student undertakes a major modification or reorganization of their schemes, then the observer could characterize the perturbation as major. Finally, the student may experience a non-neutralizable cognitive perturbation if the student's current schemes do not support them in satisfactorily interpreting (from the student's perspective) the solution.

Reorganization of Schemes and Reflective Abstraction

To further describe the reorganization of schemes that may occur after a perturbation, we use Piaget's (2001) notion of abstraction. Abstraction is a mechanism explaining an individual's modification of their schemes toward greater cohesion and generality. In this report, we use the concept of *reflective abstraction*. In broad strokes, reflective abstraction entails two processes: a projection of actions or schemes to a higher level of thought and a reorganization that occurs at this higher level (Ellis et al., 2024; Piaget, 2001; Steffe, 2024; Tallman & O'Bryan, 2024; Tallman & Uscanga, 2020; von Glasersfeld, 1995). The reorganization can involve the creation of a coherent relationship or network of relationships between existing schemes as well as with new schemes (Piaget, 2001; Tallman & Uscanga, 2020). Such a reorganization involves taking prior meanings as input for further operating and thus can be considered a "higher" level. We note the cognizing subject need not be consciously aware of any reorganization.

Other researchers have argued for the importance of supporting reflective abstraction and have provided suggestions for doing just that. First, offering students repeated opportunities to develop schemes relevant to particular meanings can support their connecting their schemes and reasoning across similar (and different) contexts (Tallman & O'Bryan, 2024; Thompson, 2013). Second, offering explicit occasions for students to compare activity across tasks can support reflective abstraction (Ellis et al., 2024; Piaget, 1976, 2001; Tallman & O'Bryan, 2024). Taken together, we conjecture students' repeated opportunities to create solutions, consider others' solutions, and to explicitly reflect on solutions can also create opportunities for reflective abstraction. We illustrate these considerations with the following task design.

Exemplifying the Constructs: X-marks the Spot

We designed the *X-Marks the Spot Task* leveraging the above radical constructivist constructs to support students' work with spatial coordinate systems (described below). Our conjecture was that multiple rounds of describing (to classmates) and interpreting descriptions (written by classmates) of locations in space could occasion major or minor cognitive perturbations. Further, we offered deliberate opportunities for students to reflect on location descriptions at a higher level of thought. We intended these experiences to support students in engaging in reflective abstraction as they reorganize their meanings for organizing space.

Task Background: Spatial Coordinate Systems and Conventions

In this report, we focus on a task designed to support students' developing meanings for spatial coordinate systems (Lee, 2017; Lee & Hardison, 2016; Lee et al., 2020; Paoletti et al., 2022). A *spatial coordinate system* is a coordinate system (CS) that entails either mentally overlaying a CS onto some perceived space or overlaying a space onto an already established CS. In either case, objects within the space can be located via coordinates. Radar on a ship and GPS are different examples of spatial CSs (i.e., polar and Cartesian CSs, respectively).

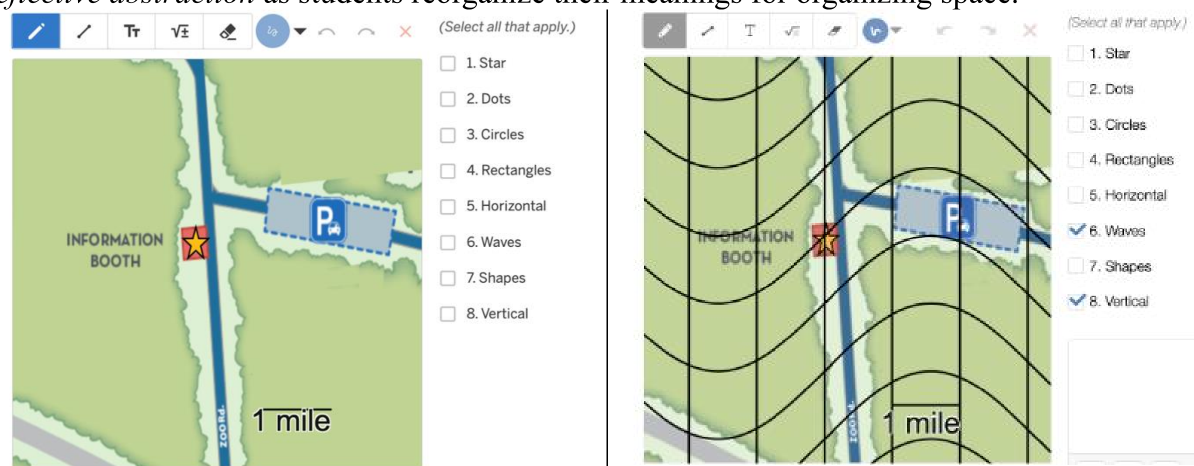
We note conventional coordinate systems involve choices often developed or adopted for the purposes of efficiency and communication (Moore et al., 2019; Zazkis, 2008). Given the communicative value of such conventions, we conjectured that we could support students in developing a *social goal* by offering them repeated prompts to describe locations in space, with the anticipation of classmates' interpreting it, and to interpret classmates' descriptions of locations. This goal could lead to activity that supported students in reorganizing their schemes for organizing space towards more clear and efficient strategies.

The Design of the X Marks the Spot Task

In the *X Marks the Spot Task* (Figure 1), we provide students with the map and buttons shown in Figure 1a with the prompt, "Play with the different overlays. In the next slides you will use the overlays either (a) to describe the location of an X or (b) interpret classmates' descriptions." Students can try each button and observe different overlays.

After exploring the different overlays, we task students (individually or in groups) with marking an X on the map. To support the creation of a *social goal*, we prompt students to use the overlays to provide a written description for the location of their marked X that classmates will interpret to determine the location of the X. In particular, a student may assume their classmates have the same shared goal of marking two Xs in the same location. To achieve this *social goal*, a student may try to write a description that is clear enough for their classmates to follow while also being descriptive enough to mark a precise location for the X. The student assumes their classmates will try to interpret the description and mark an X in the described location.

The act of interpreting real and hypothetical classmates' responses may result in a student experiencing a *cognitive perturbation*. We conjecture students would be able to interpret many descriptions using their current schemes (i.e., *without cognitive perturbation*). We also conjecture interpreting vague descriptions or observing discrepancies in the location of marked Xs for the same description could create *cognitive perturbations* for students. Further, we task students to provide feedback to the author of each description as we conjecture such activity can engender *reflective abstraction* as students reorganize their meanings for organizing space.



(a) (b)

Figure 1: (a) The initial map and (b) the map with Wave and Vertical overlays in the *X Marks the Spot Task.*

Contribution, Implications, and Areas for Future Research

In this theoretical report, we described how radical constructivist constructs can explain how a student's engagement with others' solutions can support their learning. We elaborated on our understanding of social goals, cognitive perturbations, and reflective abstraction. Given the emphasis on collaborative group work in mathematics education, such learning is likely to occur in classrooms where students interpret others' responses positively and reorganize their own meanings as a result of these interpretations.

We described how we designed the *X Marks the Spot Tas*k to support students in creating a social goal that could occasion cognitive perturbations. We conjecture that the use of classmates' descriptions to potentially provoke cognitive perturbations can be productive in this task due to the communicative nature of the mathematics at hand. We conjecture offering students repeated opportunities to first generate their own descriptions and then interpret (real and hypothetical) descriptions that communicate more or less effectively and efficiently increases the chances students would experience cognitive perturbations that could result in reorganizations of their schemes for organizing space. We conjecture there are other mathematical concepts that rely heavily on communicative goals such as conventions, in which students could be supported in learning via the use of (real or hypothetical) student solutions. We call for additional research exploring this possibility. This and other research could build on prior work showing how examining others' solutions to mathematical problems (Barbieri et al., 2023; Webb et al., 2014) can support learning. Further, such research could leverage the constructs outlined in this theoretical report to provide explanations for *how* examining others' solutions can lead to learning.

Acknowledgments

We are grateful to Les Steffe for comments on an earlier version of this report. This material is based upon work supported by the NSF under Grant No. DRL-2200778 and DRL-2142000.

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