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# Diffeomorphisms of 4-manifolds with boundary and exotic embeddings

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**Abstract**
We define family versions of the invariant of 4-manifolds with contact boundary due to Kronheimer and Mrowka and use these to detect exotic diffeomorphisms of 4-manifolds with boundary. Further, we show the existence of the first example of exotic 3-spheres in a smooth closed 4-manifold with diffeomorphic complements.

## 1 Introduction

Let $W$ be a smooth compact 4-manifold with boundary. Denote by $\mathrm{Homeo}(W)$ and $\mathrm{Diff}(W)$ the groups of homeomorphisms and diffeomorphisms of $W$, respectively; and by $\mathrm{Homeo}(W, \partial)$ and $\mathrm{Diff}(W, \partial)$ the groups of homeomorphisms and diffeomorphisms fixing boundary pointwise, respectively. In this paper, we study a comparison between the mapping class groups arising from these groups, through the maps

$$\pi_0(\mathrm{Diff}(W)) \to \pi_0(\mathrm{Homeo}(W)), \quad \pi_0(\mathrm{Diff}(W, \partial)) \to \pi_0(\mathrm{Homeo}(W, \partial)) \qquad (1)$$

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induced from the natural inclusions. A non-zero element of the kernels of maps (1) may be called an *exotic diffeomorphism* of $W$. Such a diffeomorphism is topologically isotopic to the identity, but not smoothly so.

The first example of exotic diffeomorphisms in dimension 4 was given by Ruberman [54]. This example was detected by an invariant for diffeomorphisms based on Yang–Mills gauge theory for families. Later, based on Seiberg–Witten theory for families, Baraglia and the second author [11], Kronheimer and Mrowka [37], and J. Lin [43] gave other examples. However, all of these examples are given for closed 4-manifolds. In this paper, we shall present exotic diffeomorphisms of 4-manifolds with non-trivial (i.e. not $S^3$) boundary. Some of our main results are formulated as an attempt at solving the following conjecture that we propose:

**Conjecture 1.1** *Let $Y$ be a closed, connected, oriented 3-manifold. Then there exists a compact, oriented smooth 4-manifold $W$ with $\partial W = Y$ such that the natural map*

$$\pi_0(\mathrm{Diff}(W, \partial)) \to \pi_0(\mathrm{Homeo}(W, \partial))$$

*is not injective, i.e. there exists an exotic diffeomorphism of $W$ relative to the boundary. More strongly,*

$$\pi_0(\mathrm{Diff}(W)) \to \pi_0(\mathrm{Homeo}(W))$$

*is not injective.*

The problem of finding exotic smooth structures on $W$ was considered in [61, Question 1.4] and [18, Question 1] and finding a non-trivial element in the kernel of the above maps is related to finding a non-trivial loop in the space of smooth structures on $W$, so this conjecture can be seen as a family version of the work in [61, Question 1.4] and [18, Question 1]. Three major difficulties in solving this conjecture are:

- What kind of gauge-theoretic invariants of diffeomorphisms in the case of a 4-manifold with boundary should be considered? One possibility might be to use families Floer theoretic relative invariants such as those in [56], but they are hard to compute in general.
- For what concrete diffeomorphisms can the invariants be calculated?
- In the closed simply-connected case a homeomorphism acting trivially on homology is topologically isotopic to the identity by work of Quinn [53] and Perron [51]. Is there an analog of this theorem for 4-manifolds with boundaries?

The last point was resolved by Orson and Powell [50] very recently, and one of our main contributions in this paper is about the first two points where we define a numerical and computable invariant for diffeomorphisms of 4-manifolds with boundary.

***Remark 1.2*** One may also conjecture that the maps in Conjecture 1.1 are non-surjective. To our knowledge, there is no known counter example to this conjecture. Positive evidence to the non-surjectivity of the maps is given in [34] where non-surjectivity is shown for certain $W$, which is a partial generalization of results for closed 4-manifolds in [10, 15, 21, 22, 33, 45] to 4-manifolds with boundary.

***Remark 1.3*** For higher homotopy groups, it seems also natural to state conjectures similar to Conjecture 1.1. Compared to the $\pi_0$ case, there are few results on the non-injectivity or the non-surjectivity on $\pi_i$ for $i > 0$, even for closed 4-manifolds. For example, a non-injectivity result for $S^4$ is proven by Watanabe in [57] and a non-surjectivity result on $K3$ is proven by Baraglia and the second author in [12]. Our framework also yields invariants, say, $\pi_i(\mathrm{Diff}(W, \partial)) \to \mathbb{Z}$ for $i > 0$, and it may be possible to attack this problem using these invariants.

### 1.1 Families Kronheimer–Mrowka invariants

Our basic strategy is to make use of contact structures on 3-manifolds $Y$ to study diffeomorphisms of 4-manifolds bounded by $Y$. Let us first explain the main tool of this paper, which is a family version of the invariant defined by Kronheimer and Mrowka in [36] for 4-manifolds with contact boundary. This is a rather simple numerical invariant compared to invariants using monopole Floer theory [38].

Let $(W, \mathfrak{s})$ be a compact Spin$^c$ 4-manifold with contact boundary $(Y, \xi)$, here the Spin$^c$ structure on $Y$ induced by $\xi$ agrees with the one induced by $\mathfrak{s}$. Define $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$ to be the group of diffeomorphisms that preserve the isomorphism class $[\mathfrak{s}]$ and $\partial W$ pointwise. Define a subgroup $\mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$ of $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$ by collecting all diffeomorphisms in $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$ that act trivially on the homology of $W$. From these groups, we shall define homomorphisms

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \pi_0(\mathrm{Diff}(W, [\mathfrak{s}], \partial)) \to \mathbb{Z}_2 \tag{2}$$

and

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \pi_0(\mathrm{Diff}_H(W, [\mathfrak{s}], \partial)) \to \mathbb{Z}, \tag{3}$$

which we call the *families Kronheimer–Mrowka invariants* of diffeomorphisms. These invariants are defined by considering a family of Seiberg–Witten equations over a family of 4-manifolds with a cone-like almost Kähler end. This end is associated with the contact structure $\xi$ on $Y$.

The families Kronheimer–Mrowka invariant is defined similarly to the definition of the families Seiberg–Witten invariant for families of closed 4-manifolds [40, 54], but to derive an interesting consequence involving isotopy of absolute diffeomorphisms (property (i) of Theorem 1.4) we need to consider a refined version of the above invariant using information about the topology of the space of contact structures on $Y$. Let $\Xi^{\mathrm{cont}}(Y)$ denote the space of contact structures on $Y$. We shall define a map, which is not a homomorphism in general,

$$FKM(W, \mathfrak{s}, \xi, \bullet, \bullet) : \mathrm{Diff}(W, [\mathfrak{s}], \partial) \times \pi_1(\Xi^{\mathrm{cont}}(Y), \xi) \to \mathbb{Z}_2,$$

or equivalently

$$\overline{FKM}(W, \mathfrak{s}, \xi, \bullet) : \mathrm{Diff}(W, [\mathfrak{s}], \partial) \to (\mathbb{Z}_2)^{\pi_1(\Xi^{\mathrm{cont}}(Y), \xi)}, \tag{4}$$

which is defined by considering families of 4-manifolds with contact boundary, rather than fixing the contact structure on the boundary. The map (4) satisfies the property that, if $f \in \mathrm{Diff}(W, [\mathfrak{s}], \partial)$ is isotopic to the identity in $\mathrm{Diff}(W)$, then some component of the value $\overline{FKM}(W, \mathfrak{s}, \xi, f) \in (\mathbb{Z}_2)^{\pi_1(\Xi^{\mathrm{cont}}(Y), \xi)}$ is trivial. For some class of $f$ and element of $\pi_1(\Xi^{\mathrm{cont}}(Y), \xi)$, we may also extract a $\mathbb{Z}$-valued invariant rather than $\mathbb{Z}_2$.

Now we shall see some applications of these invariants.

### 1.2 Exotic diffeomorphisms

We give evidence for Conjecture 1.1 by showing that many 3-manifolds bound 4-manifolds that admit exotic diffeomorphisms with several other interesting properties.

Let $Y_1, Y_2$ be two closed, oriented 3-manifolds. We say there is a *ribbon homology cobordism* from $Y_1$ to $Y_2$ if there is an integer homology cobordism from $Y_1$ to $Y_2$ which has a handle decomposition with only 1- and 2-handles. Note that ribbon homology cobordism is not a symmetric relation.

**Theorem 1.4** *Given any closed, oriented 3-manifold $Y$ with rational homology of $S^3$ or of $S^1 \times S^2$, there exists a ribbon homology cobordism to a 3-manifold $Y'$ such that $Y'$ bounds a compact oriented smooth 4-manifold $W$ for which there exists a diffeomorphism $f \in \mathrm{Diff}(W, \partial)$ such that the (absolute) mapping class $[f] \in \pi_0(\mathrm{Diff}(W))$ generates an infinite subgroup of the kernel of*

$$\pi_0(\mathrm{Diff}(W)) \to \pi_0(\mathrm{Homeo}(W)).$$

Namely, $\{f^n\}_{n \in \mathbb{Z} \setminus \{0\}}$ are exotic diffeomorphisms that are smoothly non-isotopic to each other in $\mathrm{Diff}(W)$. Moreover, in fact, $\{f^n\}_{n \in \mathbb{Z}}$ are topologically isotopic to the identity in $\mathrm{Homeo}(W, \partial)$. (This follows from a result by Orson and Powell, Theorem 6.2.)

***Remark 1.5*** For a general oriented closed 3-manifold $Y$, we have a statement similar to Theorem 1.4, under replacing $\pi_0(\mathrm{Diff}(W)) \to \pi_0(\mathrm{Homeo}(W))$ with $\pi_0(\mathrm{Diff}(W)) \to \mathrm{Aut}(H_2(W; \mathbb{Z}))$.

Moreover, the diffeomorphism $f$ in Theorem 1.4 yields exotic spheres in 4-manifolds:

**Theorem 1.6** *In the setup of Theorem 1.4, there exists a homologically non-trivial embedded 2-sphere $S$ in $W$ such that the spheres $\{f^n(S)\}_{n \in \mathbb{Z}}$ are mutually exotic in the following sense: if $n \neq n'$, then $f^n(S)$ and $f^{n'}(S)$ are topologically isotopic in* $\mathrm{Homeo}(W, \partial)$*, but not smoothly isotopic in* $\mathrm{Diff}(W, \partial)$.

***Remark 1.7*** There is a great deal of work on exotic surfaces in 4-manifolds that makes use of the diffeomorphism types of their complements, see [1, 5, 19, 20]. Recently, Baraglia [9] gave exotic surfaces in closed 4-manifolds whose complements are diffeomorphic, based on a method closely related to our technique. Also, in [44], J. Lin and the third author gave exotic surfaces in the punctured $K3$ whose complements are diffeomorphic using the 4-dimensional Dehn-twist and the families Bauer–Furuta invariant.

***Remark 1.8*** One motivation to consider 4-manifolds with boundary is to find simple manifolds with exotic diffeomorphisms. At the moment the simplest example of a closed 4-manifold that admits an exotic diffeomorphism has $b_2 = 25$: concretely, $4\mathbb{C}P^2\#21(-\mathbb{C}P^2) \cong K3\#\mathbb{C}P^2\#2(-\mathbb{C}P^2)$. On the other hand, when we consider 4-manifolds with boundary, one may get an exotic diffeomorphism of a compact 4-manifold with $b_2 = 4$, see Remark 6.5 for details.

An interesting point in Theorem 1.4 is that, despite that this theorem shall be shown by the invariants involving contact structures explained in Subsection 1.1, the results can be described without contact structures. Another interesting point is that the $f^n$ are not mutually smoothly isotopic in the *absolute* diffeomorphism group $\mathrm{Diff}(W)$. Of course, this is stronger than the corresponding statement for $\mathrm{Diff}(W, \partial)$, and the proof requires us to use the refined invariant (4), which is less straightforward than the invariants (2) and (3). Not only the gauge-theoretic aspect, the proof of Theorem 1.4 uses some non-trivial techniques of Kirby calculus.

***Remark 1.9*** Under the setup of Theorem 1.4, the mapping class of $f$ in $\mathrm{Diff}(W, \partial)$ generates a direct summand isomorphic to $\mathbb{Z}$ in the abelianization of the kernel of

$$\pi_0(\mathrm{Diff}(W, \partial)) \to \pi_0(\mathrm{Homeo}(W, \partial)),$$

see Remark 6.4. This comes from the fact that the map (3) is a homomorphism. In an upcoming paper [29], under suitable conditions on $Y$, we also prove the existence of a $\mathbb{Z}^\infty$-summand in the abelianization of the kernel of $\pi_0(\mathrm{Diff}(W, \partial)) \to \pi_0(\mathrm{Homeo}(W, \partial))$.

### 1.3 Exotic codimension-1 submanifolds in 4-manifolds

Now we will focus on exotic 3-manifolds in 4-manifolds.

**Definition 1.10** We call two smoothly embedded 3-manifolds $Y_1$ and $Y_2$ in a smooth, oriented 4-manifold $X$ are *exotic* if

(i) there is a topological ambient isotopy $H_t : X \times [0, 1] \to X$ such that $H_0 = Id$ and $H_1(Y_1) = Y_2$,
(ii) there is no such smooth isotopy,
(iii) complements of $Y_1$ and $Y_2$ are diffeomorphic.

***Remark 1.11*** If we ignore (i) in Definition 1.10, recently, Budney–Gabai [13] and independently Watanabe [58] found examples of smoothly embedded 3-balls in $S^1 \times D^3$ that are homotopic but not smoothly isotopic.

***Remark 1.12*** If we do not consider (iii), one can easily construct an example that satisfies (i) and (ii) in Definition 1.10 as follows. Let $W$ and $W'$ be a pair of Mazur corks [2, 25] that are homeomorphic but not diffeomorphic [4, 26]. Then their doubles $D(W)$ and $D(W')$ are diffeomorphic to the standard $S^4$ [48]. Thus we have two embedded 3-manifolds $Y = \partial W$ and $Y' = \partial W'$ that are not smoothly isotopic in $S^4$, since if they were smoothly isotopic. By the isotopy extension theorem, their complements would

be diffeomorphic as well. Now notice that the two embedded 3-manifolds $Y$ and $Y'$ are topologically isotopic since there exists an orientation preserving homeomorphism $f : S^4 \to S^4$ such that $f(Y) = Y'$. Now Quinn's result [52] says that $f$ is topologically isotopic to identity and thus the two embedded 3-manifolds $Y$ and $Y'$ are topologically isotopic.

In this work, to the author's best knowledge, we construct the first example of exotic 3-spheres in a 4-manifold with diffeomorphic complements i.e, satisfying all conditions (i)-(iii):

**Theorem 1.13** *Let $X = 4\mathbb{C}P^2\#21(-\mathbb{C}P^2)$ and $S$ be the 3-sphere embedded in $X$ which gives a connected sum decomposition $X = X_1\#X_2$, with $b_2^+(X_1) = b_2^+(X_2) = 2$. Then there exists infinitely many copies of $S$ with diffeomorphic complements that are topologically isotopic but not smoothly.*

This theorem follows from a more general statement, Theorem 7.8.

***Remark 1.14*** It is not explicitly stated but known to the experts that techniques involving the complexity of h-cobordism defined Morgan–Szabó [46] can be used to show the existence of exotic 3-spheres in 4-manifolds. However, to the authors' knowledge, the techniques can only produce a pair of such $S^3$ and moreover, the complements of such spheres may not be diffeomorphic.

***Remark 1.15*** In [9], Baraglia gave examples of exotic embedded surfaces $\Sigma$ in some 4-manifolds with diffeomorphic complements. It follows from his work that one may find infinitely many exotic embedded $S^1 \times \Sigma$ whose complements are diffeomorphic to each other. Baraglia's argument is based on a family version of the adjunction inequality. Our exotic embedded 3-manifolds have vanishing $b_1$, and use some vanishing results for family versions of connected sum formula for Seiberg–Witten invariants.

In Sect. 7, we generalize Theorem 1.13 to 3-manifolds other than $S^3$, in particular, we establish the following result.

**Theorem 1.16** *Let $Y$ be one of the following 3-manifolds:*

(i) *a connected sum of elliptic 3-manifolds, or*
(ii) *a hyperbolic three-manifold labelled by*

$$0, 2, 3, 8, 12, \ldots, 16, 22, 25, 28, \ldots, 33, 39, 40, 42, 44, 46, 49$$

*in the Hodgson–Weeks census,* [59].

*Then there exists a smooth closed 4-manifold $X$ and infinitely many embeddings $\{i_n : Y \to X\}_{n\in\mathbb{Z}}$ that are exotic in the following sense: topologically isotopic (as embeddings) but not smoothly isotopic (as submanifolds). Moreover, the complements $X\backslash i_n(Y)$ are diffeomorphic to each other.*

***Remark 1.17*** In addition to the use of the families gauge theory, the proof of Theorem 1.16 uses two deep results in 3-manifold theory; the generalized Smale conjecture

for hyperbolic 3-manifolds proven by Gabai [24] and the connectivity of the space of positive scalar curvature metrics for 3-manifolds proven by Bamler and Kleiner [8]. These results are used to obtain the vanishing of certain parameterized moduli space for the Seiberg–Witten equations. The list in (ii) comes from [42, Figure 1] due to F. Lin and Lipnowski, which lists small hyperbolic 3-manifolds having no irreducible Seiberg–Witten solution.

***Remark 1.18*** Recently in [6, 7], Auckly and Ruberman also detected higher-dimensional families of exotic embeddings and diffeomorphisms by using families Yang–Mills gauge theory. Also, they detected exotic embeddings of 3-manifolds into $S^4$.

Drouin also detected exotic lens spaces in some 4-manifolds by modifying the argument by Auckly and Ruberman.

*Organization:* In Sect. 2, we review the Kronheimer–Mrowka invariant for 4-manifolds with contact boundary, which has a sign ambiguity and show that certain auxiliary data fixes the sign. In Sect. 3, we define the families Kronheimer–Mrowka invariants (2) and (2) of 4-manifolds with contact boundary, and the refined families Kronheimer–Mrowka invariant (4) as well. We also establish basic properties of these invariants. In Sect. 4, we show several vanishing results for the families Seiberg–Witten and Kronheimer–Mrokwa invariant which can be regarded as family versions of the connected sum formula, Frøyshov's vanishing result, and adjunction inequality. In Sect. 5, we construct some diffeomorphisms of some 4-manifolds with boundary, for which we shall show the families Kronheimer–Mrowka invariants are non-trivial. In Sect. 6.2, we give the proof of one of our main theorems, Theorem 1.4. In Sect. 7, we prove the results on exotic embeddings of 3-manifolds into 4-manifolds.

## 2 Signed Kronheimer–Mrowka invariants

We first review the Kronheimer–Mrowka invariant introduced in [36]. The Kronheimer–Mrowka invariant

$$\mathfrak{m}(W, \mathfrak{s}, \xi) \in \mathbb{Z}/\{\pm 1\}$$

is an invariant of a 4-manifold $W$ with a contact boundary $(Y, \xi)$ equipped with a 4-dimensional Spin$^c$ structure $\mathfrak{s}$ compatible with the $\xi$.

Usually, the notion of Spin/Spin$^c$ structure on an oriented manifold $W$ of dimension $d$ is defined by fixing a metric on $W$ and as a lift of the structure of the $SO(d)$-frame bundle of $W$ to a principal $Spin(d)$- or $Spin^c(d)$-bundle. Here we note that one can define those notions without using Riemannian metric, which shall be convenient when we consider families of manifolds. Denote by $GL^+(d, \mathbb{R})$ the group of real square matrices of size $d$ of det $> 0$. A Spin structure on an oriented $d$-manifold can be defined as a lift of the structure group of the frame bundle from $GL^+(d, \mathbb{R})$ to the double cover $\widetilde{GL}^+(d, \mathbb{R})$. Similarly, a Spin$^c$ structure is also defined using $(\widetilde{GL}^+(d, \mathbb{R}) \times S^1)/\pm 1$ instead of $Spin^c(d)$.

In the case of the usual Seiberg–Witten invariant for a closed 4-manifold $X$, it is enough to fix a homology orientation of $X$, i.e. an orientation of $H^1(X;\mathbb{R})\oplus H^+(X;\mathbb{R})$ to fix a sign of the invariant. However, in Kronheimer–Mrowka's setting, we cannot use such data to give an orientation of the moduli space. In order to improve this, we introduce a two element set

$$\Lambda(W,\mathfrak{s},\xi)$$

depending on a tuple $(W,\mathfrak{s},\xi)$ whose element gives an orientation of the moduli space in Kronheimer–Mrowka's setting.

Let $W$ be a connected compact oriented 4-manifold with connected contact boundary $(Y,\xi)$. Let $\mathfrak{s}$ be a Spin$^c$ structure on $W$ which is compatible with $\xi$. Pick a contact 1-form $\theta$ on $Y$ and a complex structure $J$ of $\xi$ compatible with the orientation. There is now an unique Riemannian metric $g_1$ on $Y$ such that $\theta$ satisfies $|\theta|=1, d\theta = 2*\theta$, and $J$ is an isometry for $g|_\xi$, where $*$ is the Hodge star operator with respect to $g_1$. This can be written explicitly by

$$g_1 = \theta\otimes\theta + \frac{1}{2}d\theta(\cdot, J\cdot)|_\xi.$$

Define a symplectic form $\omega_0$ on $\mathbb{R}^{\geq 1}\times Y$ by the formula $\omega_0 = \frac{1}{2}d(s^2\theta)$, where $s$ is the coordinate of $\mathbb{R}^{\geq 1}$. We define a conical metric on $\mathbb{R}^{\geq 1}\times Y$ by

$$g_0 := ds^2 + s^2 g_1. \tag{5}$$

On $\mathbb{R}^{\geq 1}\times Y$, we have a canonical Spin$^c$ structure $\mathfrak{s}_0$, a canonical Spin$^c$ connection $A_0$, a canonical positive Spinor $\Phi_0$. These are given as follows. The pair $(g_0,\omega_0)$ determines a compatible almost complex structure $J$ on $\mathbb{R}^{\geq 1}\times Y$. The Spin$^c$ structure on $\mathbb{R}^{\geq 1}\times Y$ is given by:

$$\mathfrak{s}_0 := (S^+ = \Lambda_J^{0,0}\oplus\Lambda_J^{0,2}, S^- = \Lambda_J^{0,1}, \rho : \Lambda^1 \to \mathrm{Hom}(S^+,S^-)),$$

where the Clifford multiplication $\rho$ is given by

$$\rho = \sqrt{2}\,\mathrm{Symbol}(\overline{\partial}+\overline{\partial}^*).$$

(See Lemma 2.1 in [36].) The notation $\Phi_0$ denotes

$$(1,0)\in \Omega^{0,0}_{\mathbb{R}^{\geq 1}\times Y}\oplus\Omega^{0,2}_{\mathbb{R}^{\geq 1}\times Y} = \Gamma(S^+|_{\mathbb{R}^{\geq 1}\times Y}).$$

Then the *canonical* Spin$^c$ *connection* $A_0$ on $\mathfrak{s}_0$ is uniquely defined by the equation

$$D^+_{A_0}\Phi_0 = 0 \tag{6}$$

on $\mathbb{R}^{\geq 1}\times Y$. We write the conical part $\mathbb{R}^{\geq 1}\times Y$ by $C^+$.

Let $W^+$ be a non-compact 4-manifold with a conical end

$$W^+ := W \cup_Y (\mathbb{R}^{\geq 1} \times Y).$$

Fix a smooth extension of $(A_0, \Phi_0)$ on $W^+$. On $W^+$ define Sobolev spaces

$$\begin{aligned} \mathcal{C}_{W^+} &= (A_0, \Phi_0) + L^2_{k,A_0}(i\Lambda^1_{W^+} \oplus S^+_{W^+}) \quad \text{and} \\ \mathcal{V}_{W^+} &= L^2_{k-1,A_0}(i\Lambda^+_{W^+} \oplus S^-_{W^+}) \end{aligned}$$

for $k \geq 4$, where $S^+_{W^+}$ and $S^-_{W^+}$ are positive and negative spinor bundles and the Sobolev spaces are given as completions with respect to the following inner products:

$$\langle s_1, s_2 \rangle_{L^2_{k,A_0}} := \sum_{i=0}^{k} \int_{W^+} \langle \nabla^i_{A_0} s_1, \nabla^i_{A_0} s_2 \rangle \, \mathrm{dvol}_{W^+}, \tag{7}$$

where the connection $\nabla^i_{A_0}$ is induced from $A_0$ and the Levi-Civita connection. The gauge group is defined by

$$\mathcal{G}_{W^+} := \left\{ u : W^+ \to U(1) | 1 - u \in L^2_{k+1} \right\}.$$

The action of $\mathcal{G}_{W^+}$ on $\mathcal{C}_{W^+}$ is given by

$$u \cdot (A, \Phi) := (A - u^{-1}du, u\Phi).$$

Set

$$\mathcal{B}_{W^+} := \mathcal{C}_{W^+}/\mathcal{G}_{W^+}$$

and call it *the configuration space*. Note that since $(A_0, \Phi_0)$ is irreducible in the end, one can see every element in $\mathcal{B}_{W^+}$ is irreducible.

We have *the perturbed Seiberg–Witten map*

$$\begin{aligned} &\mathfrak{F} : \mathcal{C}_{W^+} \to \mathcal{V}_{W^+} \\ &(A, \Phi) \mapsto \left( \frac{1}{2} F^+_{A^t} - \rho^{-1}(\Phi\Phi^*)_0 - \frac{1}{2} F^+_{A^t_0} + \rho^{-1}(\Phi_0\Phi_0^*)_0 + \eta, D^+_A \Phi \right). \end{aligned} \tag{8}$$

Here $\eta$ is a generic perturbation decaying $C^r$ exponentially.

We have the infinitesimal action of gauge group at every point $(A, \Phi) \in \mathcal{C}_{W^+}$

$$\delta_{(A,\Phi)} : L^2_{k+1,A_0}(i\Lambda^0_{W^+}) \to L^2_{k,A_0}(i\Lambda^1_{W^+} \oplus S^+)$$

and the linearization of the Seiberg–Witten map at $(A, \Phi) \in \mathcal{C}_{W^+}$

$$\mathcal{D}_{(A,\Phi)}\mathfrak{F} : L^2_{k,A_0}(i\Lambda^1_{W^+} \oplus S^+_{W^+}) \to L^2_{k-1,A_0}(i\Lambda^+_{W^+} \oplus S^-_{W^+}).$$

The sum

$$\mathcal{D}_{(A,\Phi)}\mathfrak{F} + \delta^*_{(A,\Phi)} : L^2_{k,A_0}(i\Lambda^1_{W^+} \oplus S^+_{W^+}) \to L^2_{k-1,A_0}(i\Lambda^0_{W^+} \oplus i\Lambda^+_{W^+} \oplus S^-_{W^+}) \tag{9}$$

is a linear Fredholm operator.

The moduli space is defined to be

$$M(W, \mathfrak{s}, \xi) := \{(A, \Phi) \in \mathcal{B}_{W^+} | \mathfrak{F}(A, \Phi) = 0\}.$$

It is proven in [36], the moduli space $M(W, \mathfrak{s}, \xi)$ is compact. For a suitable class of perturbations, it is proven in [36] that $M(W, \mathfrak{s}, \xi)$ is a smooth manifold of dimension

$$d(W, \mathfrak{s}, \xi) = \langle e(S^+, \Phi_0|_{\partial W}), [W, \partial W] \rangle,$$

where $e(S^+, \Phi_0|_{\partial W}) \in H^4(W, \partial W)$ is the relative Euler class of $S^+$ with respect to the section $\Phi_0|_{\partial W}$ on the boundary.

In order to give orientations of moduli spaces, we need the following lemma:

**Lemma 2.1** [36, Theorem 2.4] *The line bundle*

$$det(\mathcal{D}_{(A,\Phi)}\mathfrak{F} + \delta^*_{(A,\Phi)}) \to \mathcal{B}_{W^+}$$

*is trivial.* □

Here we give data to fix this sign. We first give a definition of an orientation set.

**Definition 2.2** Define the two element set by

$$\Lambda(W, \mathfrak{s}, \xi) := \{\text{orientations of the determinant line bundle of (9) over } \mathcal{B}_{W^+}\}.$$

Note that $\Lambda(W, \mathfrak{s}, \xi)$ does not depend on the choices of elements in $\mathcal{B}_{W^+}$ since $\mathcal{B}_{W^+}$ is connected. Once we fix an element in $\Lambda(W, \mathfrak{s}, \xi)$, we have an induced orientation on the moduli space $M(W, \mathfrak{s}, \xi)$. We also give another description of $\Lambda(W, \mathfrak{s}, \xi)$ using almost complex 4-manifolds bounded by $(Y, \xi)$. We use the following existence result of almost complex 4-manifolds bounded by a given 3-manifold. The proof is written in the proof of Proposition 28.1.2 of [38], for example.

**Lemma 2.3** *Let $Y$ be a closed oriented 3-manifold and $\xi$ be an oriented 2-plane field on $Y$. Then there is an almost complex 4-manifold $(W, J)$ bounded by $(Y, \xi)$, which means $\partial W = Y$ and $JTY \cap TY = \xi$ up to homotopy of 2-plane fields.* □

Using this lemma, we also define another two element set.

**Definition 2.4** For a fixed almost complex 4-manifold $(Z, J)$ bounded by $(-Y, \xi)$, we define

$$\Lambda(W, \mathfrak{s}, \xi, Z, J)$$

to be the two-element set of trivializations of the orientation line bundle for the linearized equation with a slice on the closed Spin$^c$ 4-manifold $(W \cup Z, \mathfrak{s} \cup \mathfrak{s}_J)$, where $\mathfrak{s}_J$ is the Spin$^c$ structure determined by $J$.

By its definition, $\Lambda(W, \mathfrak{s}, \xi, Z, J)$ can be regarded as the set of homology orientations of the closed 4-manifold $W \cup Z$.

We see behavior of $\Lambda(W, \mathfrak{s}, \xi, Z, J)$ under the changes of $(Z, J)$. The excision argument enables us to show the following:

**Lemma 2.5** *For any two choices of almost complex bounds $(Z, J)$ and $(Z', J')$, one has a canonical identification*

$$\Lambda(W, \mathfrak{s}, \xi, Z, J) \cong \Lambda(W, \mathfrak{s}, \xi, Z', J').$$

***Proof*** This follows from an excision argument. Take an almost complex 4-manifold $Z_1$ bounded by $(Y, \xi)$. We apply Theorem A.1 by putting $A_1 = W$, $B_1 = Z$, $A_2 = Z_1$, $B_2 = Z'$,

$D_1 =$ The linearization of the Seiberg–Witten map with slice on $X_1 = W \cup Z$, and
$D_2 =$ The linearization of the Seiberg–Witten map with slice on $X_2 = Z_1 \cup Z'$.

By Theorem A.1, we have an isomorphism

$$\det D_1 \otimes \det D_2 \to \det \widetilde{D}_1 \otimes \det \widetilde{D}_2$$

up to homotopy. Since the Spin$^c$ structures on $X_2$ and $\widetilde{X}_2$ are induced by almost complex structures, $\det D_2$ and $\det \widetilde{D}_2$ has a canonical trivialization. So, we obtain a canonical isomorphism

$$\det D_1 \to \det \widetilde{D}_1.$$

This gives a correspondence between $\Lambda(W, \mathfrak{s}, \xi, Z, J)$ and $\Lambda(W, \mathfrak{s}, \xi, Z', J')$. □

For a Spin$^c$ 4-manifold with countact boundary $(W, \xi)$, we introduced two orientations sets

$$\Lambda(W, \mathfrak{s}, \xi, Z, J) \quad \text{and} \quad \Lambda(W, \mathfrak{s}, \xi).$$

We can define a natural correspondence between these orientation sets. Take an almost complex bound $Z_1$ of $(Y, \xi)$. We again apply Theorem A.1 by putting $A_1 = W$, $B_1 = C^+$, $A_2 = Z_1$ and $B_2 = Z$

$D_1 =$ The linearization of the Seiberg–Witten map with slice on $X_1 = W \cup C^+$, and
$D_2 =$ The linearization of the Seiberg–Witten map with slice on $X_2 = Z_1 \cup Z$.

By Theorem A.1, we have an isomorphism

$$\det D_1 \otimes \det D_2 \to \det \widetilde{D}_1 \otimes \det \widetilde{D}_2.$$

Since the Spin$^c$ structures on $X_2$ and $\widetilde{X}_2$ are induced by almost complex structures, $\det D_2$ and $\det \widetilde{D}_2$ has a canonical trivialization. So, we obtain a canonical isomorphism

$$\det D_1 \to \det \widetilde{D}_1.$$

This gives a bijection

$$\psi : \Lambda(W, \mathfrak{s}, \xi, Z, J) \to \Lambda(W, \mathfrak{s}, \xi).$$

A similar proof enables us to show $\psi$ does not depend on the choices of $Z_1$.

**Lemma 2.6** *There is a canonical one-to-one correspondence*

$$\psi : \Lambda(W, \mathfrak{s}, \xi, Z, J) \to \Lambda(W, \mathfrak{s}, \xi). \tag{10}$$

We use this alternative description of the orientation set when we define signed families Kronheimer–Mrowka invariants.

***Remark 2.7*** For a symplectic filling $(W, \omega)$, one can choose a canonical element in $\Lambda(W, \mathfrak{s}_\omega, \xi)$ by choosing an orientation coming from a compatible almost complex structure, where $\mathfrak{s}_\omega$ is the Spin$^c$ structure coming from $\omega$.

**Definition 2.8** For a fixed element in $\lambda \in \Lambda(W, \mathfrak{s}, \xi)$, we define the *signed Kronheimer–Mrowka invariant* by

$$\mathfrak{m}(W, \mathfrak{s}, \xi, \lambda) := \begin{cases} \#M(W, \mathfrak{s}, \xi) \in \mathbb{Z} \text{ if } \langle e(S^+, \Phi_0|_{\partial W}), [W, \partial W] \rangle = 0 \\ 0 \in \mathbb{Z} \text{ if } \langle e(S^+, \Phi_0|_{\partial W}), [W, \partial W] \rangle \neq 0. \end{cases}$$

We often abbreviate $\mathfrak{m}(W, \mathfrak{s}, \xi, \lambda)$ by $\mathfrak{m}(W, \mathfrak{s}, \xi)$.

The above definition enables us to define a map

$$\mathfrak{m} : \mathrm{Spin}^c(W, \xi) \to \mathbb{Z}$$

for a fixed element in $\Lambda(W, \mathfrak{s}, \xi)$, where $\mathrm{Spin}^c(W, \xi)$ is the set of isomorphism classes of all Spin$^c$ structures which are compatible with $\xi$ on the boundary.

## 3 Families Kronheimer–Mrowka invariant

We introduce families Kronheimer–Mrowka invariants in this section. We follow the construction of families Seiberg–Witten invariants [40, 54].

Let $W$ be a connected compact oriented 4-manifold with contact boundary $Y$. It is possible to consider a version of our invariant for disconnected $Y$: In that case, we need to replace $\pi_1(\Xi^{\mathrm{cont}}(Y), \xi)$ that appeared in the introduction with the direct sum of such fundamental groups for all components of $Y$. For simplicity, we shall suppose that $Y$ is connected in this paper.

As explained in Sect. 2, we define the notion of Spin structure/Spin$^c$ structure without using Riemannian metric, by considering $\widetilde{GL}^+(d, \mathbb{R})$. If a Spin structure or a Spin$^c$ structure $\mathfrak{s}$ is given on $W$, define $\mathrm{Aut}(W, \mathfrak{s})$ to be the group of automorphisms of the Spin$^c$ manifold $(W, \mathfrak{s})$.

Let $\Xi^{\mathrm{cont}}(Y)$ be the space of contact structures on $Y$ equipped with the $C^\infty$-topology, which is an open subset of the space of oriented 2-plane distributions. Let $\tilde{\Gamma} : B \to \Xi^{\mathrm{cont}}(Y)$ be a smooth map. We denote the smooth homotopy class of $\tilde{\Gamma}$ by $\Gamma$. Let $W \to E \to B$ be a fiber bundle over a closed smooth manifold $B$ of dimension $n$. Let $\mathrm{Diff}^+(W, [\mathfrak{s}])$ denote the group of orientation preserving diffeomorphisms fixing the isomorphism class of $\mathfrak{s}$, and let $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$ denote the group of diffeomorphisms fixing boundary pointwise and the isomorphism class of $\mathfrak{s}$. Let $\mathrm{Aut}_\partial(W, \mathfrak{s})$ denote the inverse image of $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$ under the natural surjection

$$\mathrm{Aut}(W, \mathfrak{s}) \to \mathrm{Diff}^+(W, [\mathfrak{s}]).$$

Suppose that the structure group of $E$ reduces to $\mathrm{Aut}_\partial(W, \mathfrak{s})$. Namely, $E$ is a fiber bundle whose restriction to the boundary is a trivial bundle of 3-manifolds, and is equipped with a fiberwise Spin$^c$ structure $\mathfrak{s}_E$. Suppose also that

$$\mathfrak{s}_{\tilde{\Gamma}(b)} = \mathfrak{s}_E|_{E_b}$$

on each fiber.

For these data, we define the families Kronheimer–Mrowka invariant

$$FKM(E, \Gamma) = FKM(E, W, \mathfrak{s}_E, \Gamma) \in \mathbb{Z}_2.$$

This invariant is trivial by definition unless when $\langle e(S^+, \Phi_0), [W, \partial W]\rangle + n = 0$ where $e(S^+, \Phi_0)$ is the relative Euler class with respect to a special section $\Phi_0$, which we introduced in the previous section.

When $n = 1$, we can define a signed family Kronheimer–Mrowka invariant

$$FKM(E) \in \mathbb{Z}$$

under a certain assumption on determinant line bundles.

### 3.1 Notation

Let $(Y, \xi)$ be a closed contact 3-manifold. We use the following geometric objects used in Sect. 2:

- a contact 1-form $\theta$ on $Y$,
- a complex structure $J$ of $\xi$ compatible with the orientation,
- the Riemannian metric $g_1$ on $Y$ such that $\theta$ satisfies $|\theta| = 1$, $d\theta = 2 * \theta$,
- the symplectic form $\omega_0$ on $\mathbb{R}^{\geq 1} \times Y$,
- the conical metric $g_0$ on $\mathbb{R}^{\geq 1} \times Y$,
- the canonical Spin$^c$ structure $\mathfrak{s}_0$ on $\mathbb{R}^{\geq 1} \times Y$,

- the canonical positive Spinor $\Phi_0$ on $\mathbb{R}^{\geq 1} \times Y$, and
- the canonical Spin$^c$ connection $A_0$ on $\mathfrak{s}_0$.

Let $(W, \mathfrak{s})$ be a connected compact oriented Spin$^c$ 4-manifold with connected contact boundary $(Y, \xi)$.

Assume that a trivialization of $E|_\partial$, the fiberwise boundary of $E$, is given. From this assumption, we may further suppose that there is a trivialization of a family of collar neighborhoods of the family of the boundaries $E|_\partial$. This is because the group of diffeomorphisms of $W$ fixing boundary pointwise is weakly homotopy equivalent to the group of diffeomorphisms of $W$ that are the identity near $\partial W$ (see, e.g. [39, Theorem 5.3.1]). Let $W^+$ be a non-compact 4-manifold with a conical end defined in Sect. 2. We define a fiber bundle

$$W^+ \to E^+ \to B$$

whose fiber is Spin$^c$ 4-manifold with conical end obtained from $W \to E \to B$ by considering $W^+ = W \cup_Y (\mathbb{R}^{\geq 1} \times Y)$ on each fiber.

From now on, we will explain auxiliary data that are needed to define the family Kronheimer–Mrowka invariant. These data consist of choices of a contact form, a complex structure on a contact plane, a compatible Riemann metric, an extension of the canonical connection and the canonical spinor, which are denoted by $\mathcal{Q}$. In addition, we also need to fix choices of a weight function and a perturbation, which are again denoted by $\mathcal{R}$. The main point here is that the set of these auxiliary data is non-empty and contractible, and thus the cobordism class of the Seiberg–Witten moduli space does not depend on the choices of such additional data. Although it is not so hard to verify it, for the readers let us carefully write the spaces of such additional data.

Let $\mathcal{Q}(Y, W, \mathfrak{s}, \xi)$ be the set of tuples

$$(\theta, J, g, A_0^W, \Phi_0^W),$$

- $\theta$ is a contact form for the contact structure $\xi$,
- $J$ is an complex structure on the contact structure $\xi$ compatible with orientation.
- $g$ is a smooth extension of the canonical metric for $(\xi, \theta, J)$ on the conical end to the whole manifold $W$,
- $(A_0^W, \Phi_0^W)$ is a smooth extension of the canonical configuration $(A_0, \Phi_0)$ on the conical end to the whole manifold $W$.

Varying over $\Xi^{\mathrm{cont}}(Y)$, we obtain a fiber bundle $\mathcal{Q}(Y, W, \mathfrak{s}) \to \Xi^{\mathrm{cont}}(Y)$ with fiber $\mathcal{Q}(Y, W, \mathfrak{s}, \xi)$.

Let $\mathcal{P}(Y, W, \xi, g)$ be the set of pairs $(\sigma, \eta)$, where

- $\sigma$ is a smooth proper extension of the $\mathbb{R}^{\geq 1}$ coordinate of the conical end to the whole manifold $W$, and
- $\eta$ is an imaginary valued $g$-self-dual 2-form that belongs to $e^{-\epsilon_0 \sigma} C^r(i\Lambda^{+_g})$ for some $\epsilon_0 > 0$ and $r \geq k$, where $e^{-\epsilon_0 \sigma} C^r(i\Lambda^{+_g})$ denotes the completion of the vector space of compactly supported smooth sections of $i\Lambda^{+_g}$ with respect to the norm:

$$\|s\| := \|e^{-\epsilon_0\sigma}s\|_{C^r(W^+;i\Lambda^{+_g})}.$$

Varying over the set of $g$, we obtain a fiber bundle $\mathcal{R}(Y, W, \mathfrak{s}, \xi) \to \mathcal{Q}(Y, W, \mathfrak{s}, \xi)$ with fiber $\mathcal{P}(Y, W, \xi, g)$, which is independent from $(A_0^W, \Phi_0^W)$-component. Varying over $\Xi^{\mathrm{cont}}(Y)$, we obtain a fiber bundle $\mathcal{R}(Y, W, \mathfrak{s}) \to \Xi^{\mathrm{cont}}(Y)$ with fiber $\mathcal{R}(Y, W, \mathfrak{s}, \xi)$, which covers $\mathcal{Q}(Y, W, \mathfrak{s}) \to \Xi^{\mathrm{cont}}(Y)$.

Except for $\eta$, we consider the $C^\infty$-topology for the above data. For $\eta$, we equip weighted $C^r$-topology. Since the total space of a fiber bundle with contractible fiber and base is also contractible, we have that $\mathcal{R}(Y, W, \mathfrak{s}, \xi)$ is contractible.

The group $\mathrm{Aut}_\partial(W, \mathfrak{s})$ acts on the total space $\mathcal{R}(Y, W, \mathfrak{s})$ via pull-back. Thus $E$ induces an associated fiber bundle $E_{\mathcal{R}} \to B$ with fiber $\mathcal{R}(Y, W, \mathfrak{s})$. Since the image of $\mathrm{Aut}_\partial(W, \mathfrak{s})$ under the natural map $\mathrm{Aut}(W, \mathfrak{s}) \to \mathrm{Diff}(W, [\mathfrak{s}])$ is contained in $\mathrm{Diff}(W, \partial)$, whose restriction to the boundary acts trivially on $\Xi^{\mathrm{cont}}(Y)$, the map $\mathcal{R}(Y, W, \mathfrak{s}) \to \Xi^{\mathrm{cont}}(Y)$ induces a map $E_{\mathcal{R}} \to \Xi^{\mathrm{cont}}(Y)$. Define $\pi : E_{\mathcal{R}} \to B \times \Xi^{\mathrm{cont}}(Y)$ be the product of these natural maps $E_{\mathcal{R}} \to B$ and $E_{\mathcal{R}} \to \Xi^{\mathrm{cont}}(Y)$. Then each fiber of $\pi$ is homeomorphic to $\mathcal{R}(Y, W, \mathfrak{s}, \xi)$. The fiber bundle

$$E_{\mathcal{R}}^{\tilde{\Gamma}} := (\mathrm{Id}, \tilde{\Gamma})^* E_{\mathcal{R}} \to B$$

the pull back of $\pi : E_{\mathcal{R}} \to B \times \Xi^{\mathrm{cont}}(Y)$ under $(\mathrm{Id}, \tilde{\Gamma}) : B \to B \times \Xi^{\mathrm{cont}}(Y)$, has fiber homeomorphic to $\mathcal{R}(Y, W, \mathfrak{s}, \xi)$. Since $\mathcal{R}(Y, W, \mathfrak{s}, \xi)$ is contractible, the space of sections of $E_{\mathcal{R}}^{\tilde{\Gamma}} \to B$ is non-empty and contractible.

In a similar manner, we can define a fiber bundle $E_{\mathcal{Q}}^{\tilde{\Gamma}} \to B$ with fiber $\mathcal{Q}(Y, W, \mathfrak{s}, \xi)$, associated with $E$ and $\Gamma$. We first fix a section

$$s^{\mathcal{Q}} = (\theta_b, J_b, g_b, A_{0,b}, \Phi_{0,b})_{b\in B} : B \to E_{\mathcal{Q}}^{\tilde{\Gamma}}$$

which determines the following data:

- a fiberwise contact form $\theta_b$,
- a fiberwise complex structure $J_b$,
- a fiberwise Riemann metric $\{g_b\}_{b\in B}$ on $E^+$ such that $g_b|_{\mathbb{R}^{\geq 1}\times Y} = g_{0,b}$, and
- a smooth family of smooth extensions $(A_{0,b}, \Phi_{0,b})$ of $(A_0, \Phi_0)$ on each fiber.

Here $g_{0,b}$ is the metric on $Y$ depending on $J_b$ and $\theta_b$ introduced in the previous section. Consider the following functional spaces on each fiber $E_b^+$:

$$\mathcal{C}_{W^+,b} = (A_{0,b}, \Phi_{0,b}) + L^2_{k,A_{0,b},g_b}(i\Lambda^1_{W^+} \oplus S^+_{W^+}) \quad \text{and}$$
$$\mathcal{V}_{W^+,b} = L^2_{k-1,A_{0,b},g_b}(i\Lambda^+_{W^+} \oplus S^-_{W^+}).$$

These give Hilbert bundles:

$$\mathbb{U}_{E^+}(s^{\mathcal{Q}}) := \bigcup_{b\in B} \mathcal{C}_{W^+,b} \quad \text{and} \quad \mathbb{V}_{E^+}(s^{\mathcal{Q}}) := \bigcup_{b\in B} \mathcal{V}_{W^+,b}.$$

For the precise definitions of these Sobolev spaces see Sect. 2. The gauge group

$$\mathcal{G}_{W^+} := \{\, u : W^+ \to U(1) \,\big|\, 1 - u \in L^2_{k+1} \,\}$$

is defined and it acts on $\mathbb{U}_{E^+}$ preserving fibers in Sect. 2. We define a family version of the configuration space by

$$\mathcal{B}_{E^+}(s^{\mathcal{Q}}) := \mathbb{U}_{E^+}/\mathcal{G}_{W^+}. \tag{11}$$

Now we also choose a section

$$s^{\mathcal{R}} = (\theta_b, J_b, g_b, A_{0,b}, \Phi_{0,b}, \sigma_b, \eta_b)_{b\in B} : B \to E^{\Gamma}_{\mathcal{R}}$$

which is compatible with the fixed section $s^{\mathcal{Q}}$, i.e. the first five components of $s^{\mathcal{R}}$ coincide with these of $s^{\mathcal{S}} = (\theta_b, J_b, g_b, A_{0,b}, \Phi_{0,b})_{b\in B}$. For each fiber $E^+_b$, we have the perturbed Seiberg–Witten map

$$\begin{aligned} &\mathfrak{F}_b : \mathcal{C}_{E^+,b} \to \mathcal{V}_{E^+,b} \\ &(A - A_{0,b}, \Phi - \Phi_{0,b}) \\ &\quad \mapsto \left( \frac{1}{2} F^+_{A^t} - \rho^{-1}(\Phi\Phi^*)_0 - \frac{1}{2} F^+_{A^t_{0,b}} + \rho^{-1}(\Phi_{0,b}\Phi^*_{0,b})_0 + \eta_b,\, D^+_A \Phi \right). \end{aligned} \tag{12}$$

This gives a bundle map

$$\mathfrak{F}(s^{\mathcal{R}}) : \mathbb{U}_{E^+}(s^{\mathcal{Q}}) \to \mathbb{V}_{E^+}(s^{\mathcal{Q}}). \tag{13}$$

**Definition 3.1** We say that $\{\eta_b\}_{b\in B}$ is a *family regular perturbation* if (13) is transverse to zero section of $\mathbb{V}_{E^+}(s^{\mathcal{Q}})$.

For each fiber, we have the infinitesimal action of gauge group at every point $(A_b, \Phi_b) \in \mathcal{C}_{W^+,b}$

$$\delta_{(A_b,\Phi_b)} : L^2_{k+1,A_{0,b}}(i\Lambda^0_{W^+}) \to L^2_{k,A_{0,b}}(i\Lambda^1_{W^+} \oplus S^+)$$

and the linearization of the Seiberg–Witten map at $(A_b, \Phi_b) \in \mathcal{C}_{W^+,b}$

$$\mathcal{D}_{(A_b,\Phi_b)}\mathfrak{F} : L^2_{k,A_b,g_b}(i\Lambda^1_{W^+} \oplus S^+_{W^+}) \to L^2_{k-1,A_b,g_b}(i\Lambda^+_{W^+} \oplus S^-_{W^+})$$

The sum

$$\mathcal{D}_{(A_b,\Phi_b)}\mathfrak{F} + \delta^*_{(A_b,\Phi_b)} : L^2_{k,A_b,g_b}(i\Lambda^1_{W^+} \oplus S^+_{W^+}) \to L^2_{k-1,A_b,g_b}(i\Lambda^+_{W^+} \oplus S^-_{W^+})$$

is a linear Fredholm operator. This gives a fiberwise Fredholm operator

$$\mathbb{L}_{E^+}(s^{\mathcal{Q}}) : T_{\text{fiber}}\mathbb{U}_{E^+}(s^{\mathcal{Q}}) \to \mathbb{V}_{E^+}(s^{\mathcal{Q}}),$$

where $T_{\text{fiber}}$ means the fiberwise tangent bundle of $\mathbb{U}_{E^+}(s^{\mathcal{Q}})$. By taking the determinant line bundle for each fiber, we obtain a line bundle

$$\det(\mathbb{L}_{E^+}(s^{\mathcal{Q}})) \to \mathcal{B}_{E^+}(s^{\mathcal{Q}}). \tag{14}$$

***Remark 3.2*** As a similar study, in [49], Juan defined the families monopole contact invariant for families of contact structures on a fixed 3-manifold. At the moment, we do not know the triviality of the line bundle (14).

### 3.2 Constructions of the invariant

For a fixed section $s^{\mathcal{R}} : B \to E^{\Gamma}_{\mathcal{R}}$ such that $\{\eta_b\}_{b\in B}$ is regular, the parametrized moduli space is defined to be

$$\mathcal{M}(E, \tilde{\Gamma}, s^{\mathcal{R}}) := \{(A, \Phi) = (A_b, \Phi_b)_{b\in B} \in \mathbb{U}_{W^+} | \mathfrak{F}(A, \Phi) = 0\}/\mathcal{G}_{W^+}.$$

Recall the formal dimension

$$d(W, \mathfrak{s}, \xi) = \langle e(S^+, \Phi_0|_{\partial W}), [X, \partial X]\rangle$$

of the (unparametrized) moduli space over the cone-like end 4-manifold $W^+$.

**Proposition 3.3** *For a regular perturbation, $\mathcal{M}(E, \tilde{\Gamma}, s^{\mathcal{R}})$ is a smooth compact manifold of dimension $d(W, \mathfrak{s}, \xi) + n$. If the determinant line bundle*

$$\det(\mathbb{L}_{E^+}(s^{\mathcal{Q}})) \to \mathcal{B}_{E^+}(s^{\mathcal{Q}}),$$

*is trivialized, an orientation of $\mathcal{M}(E, \tilde{\Gamma}, s^{\mathcal{R}})$ is naturally induced by an orientation of $B$ and an orientation of $\det(\mathbb{L}_{E^+}(s^{\mathcal{Q}}))|_b$ on a fiber of $b \in B$.*

***Proof*** The proof is the standard perturbation argument with the compact parameter space $B$. We omit it. □

**Definition 3.4** We define the *families Kronheimer–Mrowka invariant* of $E$ by

$$FKM(E, \tilde{\Gamma}, s^{\mathcal{R}}) := \begin{cases} \#\mathcal{M}(E, \tilde{\Gamma}, s^{\mathcal{R}}) \in \mathbb{Z}_2 & \text{if } d(W, \mathfrak{s}, \xi) + n = 0, \\ 0 \in \mathbb{Z}_2 & \text{if } d(W, \mathfrak{s}, \xi) + n \neq 0 \end{cases}$$

for a fixed section $s^{\mathcal{R}}$.

Since we will see the number $FKM(E, \tilde{\Gamma}, s^{\mathcal{R}})$ does not depend on the choices of sections $s^{\mathcal{R}}$ and $\tilde{\Gamma}$ up to smooth homotopy, we always drop $s^{\mathcal{R}}$ in the notion and write $FKM(E, \Gamma)$.

**Proposition 3.5** *The number $FKM(E, \tilde{\Gamma}, s^{\mathcal{R}})$ is independent of the choices of the following data:*

- *a section* $s^{\mathcal{R}}$ *and*
- *a choice of* $\tilde{\Gamma}$ *which belongs to the homotopy class* $\Gamma$.

*Also* $FKM(E, \Gamma)$ *depends only on the isomorphism class of* $E$ *as* $\mathrm{Aut}((W, \mathfrak{s}), \partial)$*-bundles and* $\Gamma$.

***Proof*** We take a smooth homotopy $\tilde{\Gamma}_t : I \times B \to \Xi^{\mathrm{cont}}(Y)$ between $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ parametrized $t \in [0, 1]$. Take two sections

$$s_0^{\mathcal{R}} : B \to E^{\mathcal{R}}_{\tilde{\Gamma}_0} \quad \text{and} \quad s_1^{\mathcal{R}} : B \to E^{\mathcal{R}}_{\tilde{\Gamma}_1}$$

so that (13) is transverse for $i = 0$ and $i = 1$. Note that a fiber of the bundle

$$\bigcup_{t \in I} E^{\tilde{\Gamma}_t}_{\mathcal{R}} \to I \times B$$

is contractible, we can take a section $s_t^{\mathcal{R}} : I \times B \to \bigcup_{t \in I} E^{\tilde{\Gamma}_t}_{\mathcal{R}}$ connecting $s_0^{\mathcal{R}}$ and $s_1^{\mathcal{R}}$ such that (13) for $s_t^{\mathcal{R}}$ is transverse. So, the moduli space for $s_t^{\mathcal{R}}$ gives a cobordism between $\mathcal{M}(E, \tilde{\Gamma}_0, s_0^{\mathcal{R}})$ and $\mathcal{M}(E, \tilde{\Gamma}_1, s_1^{\mathcal{R}})$. This completes the proof. □

***Remark 3.6*** Note that there is no reducible solution to the monopole equations over the conical end 4-manifold $W^+$ under our boundary condition, and we do not have to impose any condition on $b_2^+(W)$ to ensure the well-definedness of the invariant, as well as the unparametrized Kronheimer–Mrowka invariant.

### 3.3 Invariant of diffeomorphisms

Now suppose that the base space $B$ is $S^1$ and $\tilde{\Gamma}$ is a constant map to $\xi$. In this case, the family $E \to S^1$ is determined by an element of $\mathrm{Aut}_\partial(W, \mathfrak{s})$. An element of $\mathrm{Aut}_\partial(W, \mathfrak{s})$ is given as a pair $(f, \tilde{f})$: $f$ is a diffeomorphism $f : W \to W$ which preserves the isomorphism class of $\mathfrak{s}$ and fix $\partial W$ pointwise, and $\tilde{f}$ is a lift of $f$ to an automorphism on the honest $\mathrm{Spin}^c$ structure $\mathfrak{s}$ acting trivially over $\partial W$. All $E \to S^1$ can be viewed as the mapping torus of $W$ by $(f, \tilde{f})$.

**Lemma 3.7** *Let* $E$ *be the mapping torus of* $W$ *by* $(f, \tilde{f})$. *Then the invariant* $FKM(E, \Gamma)$ *depends only on the diffeomorphism* $f$ *and* $\xi$.

***Proof*** The kernel of the natural surjection

$$\mathrm{Aut}_\partial(W, \mathfrak{s}) \to \mathrm{Diff}(W, [\mathfrak{s}], \partial)$$

is given by the gauge group $\mathcal{G}_W = \{ u : W \to U(1) \,\big|\, u|_{\partial W} \equiv 1 \}$. Now suppose that we have two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of $f$ to $\mathrm{Aut}_\partial(W, \mathfrak{s})$. Let $E_i$ be the mapping torus of $(f, \tilde{f}_i)$. Then the composition $\tilde{f}_1 \circ \tilde{f}_2^{-1}$ is given by a smooth map $u : W \to U(1)$ with $u \equiv 1$ on $\partial W$. Taking an extension of $u$ to a neighborhood of $W$ in $W^+$, and also a partition of unity around $\partial W$, we can extend $u$ to a smooth map $u^+ : W^+ \to U(1)$ with $1 - u \in L^2_{k+1}$. Hence the moduli spaces used in the definition of $FKM(E_1)$ and that of $FKM(E_2)$ are identical to each other. □

**Definition 3.8** For a fixed contact structure $\xi$ on $Y$, we define the *Kronheimer–Mrowka invariant for diffeomorphims* $FKM(W, \mathfrak{s}, \xi, f)$ to be the invariant $FKM(E, \Gamma \equiv \xi)$ of the mapping torus $E$ with fiber $(W, \mathfrak{s})$ defined taking a lift of $f$ to $\mathrm{Aut}_\partial(W, \mathfrak{s})$. Note that, by Lemma 3.7, $FKM(W, \mathfrak{s}, \xi, f)$ is independent of the choice of lift. If $(\mathfrak{s}, \xi)$ is specified, we sometimes abbreviate $FKM(W, \mathfrak{s}, \xi, f)$ to $FKM(W, f)$.

Now we have defined a map

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \mathrm{Diff}(W, [\mathfrak{s}], \partial) \to \mathbb{Z}_2.$$

We will show that this map is a homomorphism and descents to a map

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \pi_0(\mathrm{Diff}(W, [\mathfrak{s}], \partial)) \to \mathbb{Z}_2.$$

### 3.4 A signed refinement of *FKM* for diffeomorphisms

Again, in this subsection, we assume that $\Gamma$ is a constant function to $\xi$. Define a subgroup $\mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$ of the relative diffeomorphism group $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$ as the group of diffeomorphisms that act trivially on homology and preserve the isomorphism class $[\mathfrak{s}]$ and $\partial W$ pointwise. Note that, if $W$ is simply-connected, $\mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$ coincides with the group $\mathrm{Diff}_H(W, \partial)$, the group of diffeomorphisms that act trivially on homology and preserve $\partial W$ pointwise.

For each element of $\Lambda(W, \mathfrak{s}, \xi)$, we shall define a map

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \mathrm{Diff}_H(W, [\mathfrak{s}], \partial) \to \mathbb{Z}.$$

The construction of this map is done essentially by a similar fashion to define $FKM : \mathrm{Diff}(W, [\mathfrak{s}], \partial) \to \mathbb{Z}_2$, but we need to count the parametrized moduli space taking into account its orientation.

For $f \in \mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$, let $\tilde{f}$ be a lift of $f$ to an automorphism of the Spin$^c$ structure. Let $E_{f,\tilde{f}}$ denote the mapping torus of $(W, \mathfrak{t})$ as a fiber bundle of Spin$^c$ 4-manifolds. Take a section

$$s^{\mathcal{Q}} : S^1 \to E^{\Gamma}_{\mathcal{Q}}.$$

Let $\mathcal{B}_{E_{f,\tilde{f}}}(s^{\mathcal{Q}})$ denote the families (quotient) configuration space associated to $E_{f,\tilde{f}}$ introduced in (11).

**Lemma 3.9** *Suppose $f$ is homologically trivial. Each element in $\Lambda(W, \mathfrak{s}, \xi)$ induces a section of the orientation bundle*

$$\Lambda(E_{f,\tilde{f}}) \to \mathcal{B}(E_{f,\tilde{f}}) \tag{15}$$

*over the configuration space $\mathcal{B}(E_{f,\tilde{f}})$ for all $f \in \mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$ and lifts $\tilde{f}$ to the Spin$^c$ structure.*

***Proof*** We first prove the line bundle (15) is trivial for any pair $(f, \tilde{f})$ such that $f$ is homologically trivial.

We first regard $\mathcal{B}_{E_{f,\tilde{f}}}(s^{\mathcal{Q}})$ as a mapping torus of the trivial bundle

$$\underline{\mathcal{B}}_{W,\mathfrak{s},\xi} := I \times \mathcal{B}_{W,\mathfrak{s},\xi} \to I$$

via the map $(f, \tilde{f})$. From Lemma 2.1, we see that the determinant line bundle $\Lambda(W, \mathfrak{s}, \xi)$ over $\underline{\mathcal{B}}_{W,\mathfrak{s},\xi}$ is trivial. So, it is sufficient to prove that the induced map

$$(f, \tilde{f})_* : \Lambda(W, \mathfrak{s}, \xi) \to \Lambda(W, \mathfrak{s}, \xi)$$

preserves a given orientation of $\Lambda(W, \mathfrak{s}, \xi)$. In order to see this, we use the following canonical identification (10). First, we fix an almost complex 4-manifold $(Z_1, J_1)$ bounded by $(-Y, \xi)$. We recall that

$$\Lambda(W, \mathfrak{s}, \xi, Z, J)$$

is defined by the two-element set of trivializations of the orientation line bundle for the linearized equation with a slice on the closed Spin$^c$ 4-manifold $(W \cup Z, \mathfrak{s} \cup \mathfrak{s}_J)$. Then, (10) gives an identification

$$\psi : \Lambda(W, \mathfrak{s}, \xi, Z, J) \to \Lambda(W, \mathfrak{s}, \xi).$$

Note that $(f, \tilde{f})$ also naturally acts on $\Lambda(W, \mathfrak{s}, \xi, Z, J)$.

**Claim 3.10** *The following diagram commutes:*

$$\begin{array}{ccc} \Lambda(W, \mathfrak{s}, \xi, Z, J) & \xrightarrow{\psi} & \Lambda(W, \mathfrak{s}, \xi) \\ {\scriptstyle (f,\tilde{f})_*}\downarrow & & {\scriptstyle (f,\tilde{f})_*}\downarrow \\ \Lambda(W, \mathfrak{s}, \xi, Z, J) & \xrightarrow{\psi} & \Lambda(W, \mathfrak{s}, \xi). \end{array}$$

***Proof of Claim 3.10*** This result follows the construction of $\psi$ based on Theorem A.1. □

Note that the orientation of $\Lambda(W, \mathfrak{s}, \xi, Z, J)$ is determined just by the homology orientation of $W \cup Z$. Since we assumed that $\psi$ is homologically trivial, the induced action

$$(f, \tilde{f})_* : \Lambda(W, \mathfrak{s}, \xi, Z, J) \to \Lambda(W, \mathfrak{s}, \xi, Z, J)$$

is also trivial. Hence, we can see that the bundle $\Lambda(E_{f,\tilde{f}}) \to \mathcal{B}(E_{f,\tilde{f}})$ is trivial. Now, we give an orientation of $\Lambda(E_{f,\tilde{f}})$ from a fixed element in $\Lambda(W, \mathfrak{s}, \xi)$. For a fixed element in $\Lambda(W, \mathfrak{s}, \xi)$, an element $\Lambda(E_{f,\tilde{f}})$ is induced by choosing a point in $\mathcal{B}_{E_{f,\tilde{f}}}$

and restricting the bundle $\Lambda(E_{f,\tilde{f}})$ to the point. Note that such a correspondence does not depend on the choices of lifts $\tilde{f}$. This completes the proof. □

If $E = E_{(f,\tilde{f})}$ is the mapping torus, we can count the parametrized moduli space associated with $E$ over $\mathbb{Z}$ by Lemma 3.9.

**Definition 3.11** For $f \in \mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$ and a lift $\tilde{f}$, let $E = E_{(f,\tilde{f})}$ be the mapping torus of $(W, \mathfrak{s})$ by $(f, \tilde{f})$. we define the *signed families Kronheimer–Mrowka invariant* of $E$ by

$$FKM(E, \xi) := \begin{cases} \#\mathcal{M}(E, \Gamma \equiv \xi, s^{\mathcal{R}}) \in \mathbb{Z} & \text{if } d(W, \mathfrak{s}, \xi) + 1 = 0, \\ 0 \in \mathbb{Z} & \text{if } d(W, \mathfrak{s}, \xi) + 1 \neq 0 \end{cases}$$

for a fixed element in $\Lambda(W, \mathfrak{s}, \xi)$.

Repeating the argument in Lemma 3.7, we obtain:

**Lemma 3.12** *Let $E$ be the mapping torus of $(W, \mathfrak{s})$ by $(f, \tilde{f})$. Then the invariant $FKM(E) \in \mathbb{Z}$ depends only on $f \in \mathrm{Diff}_H(W, [\mathfrak{s}], \partial)$.*

***Proof*** The proof is essentially the same as that of Lemma 3.7. □

**Definition 3.13** We define the *signed Kronheimer–Mrowka invariant for diffeomorphims* $FKM(W, \mathfrak{s}, \xi, f)$ to be the invariant $FKM(E, \xi)$ of the mapping torus $E$ with fiber $(W, \mathfrak{s})$ defined by taking a lift of $f$ to $\mathrm{Aut}_\partial(W, \mathfrak{s})$. Note that, by Lemma 3.12, $FKM(W, \mathfrak{s}, \xi, f)$ is independent of the choice of lift. If $(\mathfrak{s}, \xi)$ is specified, we sometimes abbreviate $FKM(W, \mathfrak{s}, \xi, f)$ to $FKM(W, f)$.

### 3.5 Properties of the families Kronheimer–Mrowka invariant

In this subsection, we prove some basic properties of the families Kronheimer–Mrowka invariant. This is parallel to Ruberman's original argument [54, Subsection 2.3].

Let $(W, \mathfrak{s})$ be a connected compact oriented Spin$^c$ 4-manifold with connected contact boundary $(Y, \xi)$. In this subsection, we fix $(\mathfrak{s}, \xi)$ and we sometimes drop this from our notation of $FKM(W, \mathfrak{s}, \xi, f)$.

First we note the following additivity formula:

**Proposition 3.14** *For diffeomorphisms $f, f'$ of $W$ preserving the isomorphism class of $\mathfrak{s}$ and fixing $\partial W$ pointwise, we have*

$$FKM(W, \mathfrak{s}, \xi, f) + FKM(W, \mathfrak{s}, \xi, f') = FKM(W, \mathfrak{s}, \xi, f' \circ f) \mod 2.$$

*Moreover, when an element of $\Lambda(W, \mathfrak{s}, \xi)$ is fixed, we have*

$$FKM(W, \mathfrak{s}, \xi, f) + FKM(W, \mathfrak{s}, \xi, f') = FKM(W, \mathfrak{s}, \xi, f' \circ f)$$

*as $\mathbb{Z}$-valued invariants for homologically tirivial diffeomorphisms $f, f'$.*

***Proof*** We regard $FKM(W, f' \circ f)$ as the counting of $\mathcal{M}(E_{f' \circ f})$. Note that the moduli space $\mathcal{M}(E_{f' \circ f})$ is equipped with the map $\mathcal{M}(E_{f' \circ f}) \to S^1$. We fix the following data:

- a Riemann metric $g$ on $W$ which coincides with $g_1$ on $\partial W = Y$ and
- a regular perturbation $\eta$ on $W^+$ for the metric $g$.

The invariant $\mathcal{M}(E_f)$ can be seen as the counting of parametrized moduli space over $[0, \frac{1}{2}]$ with regular 1-parameter family of perturbation $\eta_t$ and a 1-parameter family of metrics $g_t$ satisfying

- $g_0 = g$, $g_{\frac{1}{2}} = f^* g$ and
- $\eta_0 = \eta$, $\eta_{\frac{1}{2}} = f^* \eta$.

Also, the invariant $\mathcal{M}_{E_{f'}}$ can be seen as the counting of parametrized moduli space over $[\frac{1}{2}, 1]$ with the regular 1-parameter family of perturbation $\eta_t$ and a 1-parameter family of metrics $g_t$ satisfying

- $g_{\frac{1}{2}} = f^* g$, $g_1 = (f' \circ f)^* g$ and
- $\eta_{\frac{1}{2}} = f^* \eta$, $\eta_1 = (f' \circ f)^* \eta$.

Then we have a decomposition

$$\bigcup_{t \in [0, \frac{1}{2}]} \mathcal{M}(W, g_t, \eta_t) \cup \bigcup_{t \in [\frac{1}{2}, 1]} \mathcal{M}(W, g_t, \eta_t) = \bigcup_{t \in [0,1]} \mathcal{M}(W, g_t, \eta_t).$$

The counting of $\bigcup_{t \in [0,1]} \mathcal{M}(W, g_t, \eta_t)$ is equal to $FKM(W, f' \circ f)$ by the definition. This completes the proof. Once we fix an orientation of $\Lambda(W, \mathfrak{s}, \xi)$, the same argument enables us to prove the equality

$$FKM(W, \mathfrak{s}, \xi, f) + FKM(W, \mathfrak{s}, \xi, f') = FKM(W, \mathfrak{s}, \xi, f' \circ f) \in \mathbb{Z}.$$

□

Proposition 3.14 immediately implies:

**Corollary 3.15** *We have* $FKM(W, \mathfrak{s}, \xi, f) = 0$ *for* $f = \mathrm{Id}$.

**Lemma 3.16** *The number* $FKM(W, \mathfrak{s}, \xi, f)$ *is invariant under smooth isotopy of diffeomorphisms in* $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$.

***Proof*** By Corollary 3.15 and Proposition 3.14, it suffices to check that if $f$ is isotopic to the identity, then we have $FKM(W, f) = 0$. Take a generic unparametrized perturbation $\eta$. Let $f_t$ be a smooth isotopy from Id to $f$. Let $\eta_t = f_t^* \eta$, and $g_t$ be the underlying family of metrics. The $\mathcal{M}(W, g_t, \eta_t)$ is diffeomorphic to $\mathcal{M}(W, g_0, \eta_t)$, which is empty. □

**Corollary 3.17** *If* $FKM(W, \mathfrak{s}, \xi, f) \neq 0$, *then* $f$ *is not isotopic to the identity through* $\mathrm{Diff}(W, [\mathfrak{s}], \partial)$.

***Proof*** This follows from Corollary 3.15 and Lemma 3.16. □

We end up with this subsection by summarizing the above properties:

**Corollary 3.18** *The families Kronheimer–Mrowka invariant defines homomorphisms*

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \pi_0(\mathrm{Diff}(W, [\mathfrak{s}], \partial)) \to \mathbb{Z}_2$$

*and*

$$FKM(W, \mathfrak{s}, \xi, \bullet) : \pi_0(\mathrm{Diff}_H(W, [\mathfrak{s}], \partial)) \to \mathbb{Z}.$$

***Proof*** This follows from Proposition 3.14, Corollaries 3.15, and 3.16. □

### 3.6 Isotopy of absolute diffeomorphisms

We now consider a slight refinement of the families Kronheimer–Mrowka invariant for diffeomorphisms defined until Subsection 3.4 to take into account isotopies of diffeomorphisms that are not necessarily the identity on the boundary. We need to treat a family of contact structures on the boundary in Kronheimer–Mrowka's setting. Such a situation is also treated in [49].

For a contact structure $\xi$ on an oriented closed 3-manifold $Y$, let $[\xi]$ denote the isotopy class of $\xi$. Let $W$ be a compact oriented smooth 4-manifold bounded by $Y$. Let $f \in \mathrm{Diff}(W, [\mathfrak{s}], \partial)$ and $\gamma$ be a homotopy class of a loop in $\pi_1(\Xi^{\mathrm{cont}}(Y), \xi)$. Henceforth we fix $\xi$ and abbreviate $\pi_1(\Xi^{\mathrm{cont}}(Y), \xi)$ as $\pi_1(\Xi^{\mathrm{cont}}(Y))$. Pick a representative $\tilde{\gamma} : S^1 \to \Xi^{\mathrm{cont}}(Y)$ of $\gamma$ and a section $s^{\mathcal{R}} : S^1 \to E^{\tilde{\gamma}}_{\mathcal{R}}$. Then we defined a $\mathbb{Z}_2$-valued invariant

$$FKM(W, [\mathfrak{s}], \xi, f, \gamma) \in \mathbb{Z}_2.$$

For $f$ and $\gamma$, we can define the monodromy action on $\Lambda(W, \mathfrak{s}, \xi)$. If this action is trivial, we may count the parametrized moduli space over $\mathbb{Z}$, and thus can define

$$FKM(W, [\mathfrak{s}], \xi, f, \gamma) \in \mathbb{Z},$$

whose sign is fixed once we choose an element of $\Lambda(W, \mathfrak{s}, \xi)$. Henceforth we fix an element of $\Lambda(W, \mathfrak{s}, \xi)$. It is useful to note that, for a pair admitting a square root, say $(f^2, \gamma^2) \in \mathrm{Diff}(W, [\mathfrak{s}], \partial) \times \pi_1(\Xi^{\mathrm{cont}}(Y), \xi)$, the corresponding monodromy action is trivial. Let us summarize the situation in the following diagram:

$$\begin{array}{ccc} \mathrm{Diff}(W, [\mathfrak{s}], \partial) \times \pi_1(\Xi^{\mathrm{cont}}(Y), \xi) & \xrightarrow{FKM(W,[\mathfrak{s}],\xi,\bullet,\bullet)} & \mathbb{Z}_2 \\ \uparrow & & \uparrow \mathrm{mod}\ 2 \\ \{ f^2 \mid f \in \mathrm{Diff}(W, [\mathfrak{s}], \partial)\} \times \{ \gamma^2 \mid \gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y), \xi) \} & \xrightarrow[FKM(W,[\mathfrak{s}],\xi,\bullet,\bullet)]{} & \mathbb{Z}. \end{array}$$

The cobordism argument as in Proposition 3.5 enables us to prove the invariance of the signed and refined families Kronheimer–Mrowka's invariant.

**Proposition 3.19** *Let $f \in \mathrm{Diff}(W, [\mathfrak{s}], \partial)$. If $f$ is isotopic to the identity through $\mathrm{Diff}(W)$, then there exists $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$ such that*

$$FKM(W, [\mathfrak{s}], \xi, f, \gamma) = 0 \in \mathbb{Z}_2 \tag{16}$$

*and*

$$FKM(W, [\mathfrak{s}], \xi, f^2, \gamma^2) = 0 \in \mathbb{Z}. \tag{17}$$

***Proof*** We may suppose that $d(W, \mathfrak{s}, \xi) + 1 = 0$. Let $f_t$ be a path in $\mathrm{Diff}(W)$ between $f$ and the identity. Define a path $\tilde{\gamma} : [0, 1] \to \Xi^{\mathrm{cont}}(Y)$ by $\tilde{\gamma}(t) = f_t^*\xi$, and set $\gamma = [\tilde{\gamma}] \in \pi_1(\Xi^{\mathrm{cont}}(Y))$. Pick a generic element $a$ of the fiber of $\mathcal{R}(Y, W, \mathfrak{s}) \to \Xi^{\mathrm{cont}}(Y)$ over the $\xi$. The pull-back $s(t) = f_t^*a$ gives rise to a section $s : [0, 1] \to \tilde{\gamma}^*\mathcal{R}(Y, W, \mathfrak{s})$. By formal-dimensional reason, the moduli space for $a$ is empty. Moreover, the pull-back under $f$ induces a homeomorphism between the moduli space for $a$ and that for $f^*a$, and hence the parametrized moduli space for $s$ is empty. This completes the proof of (16).

Next we prove (17). Let $\tilde{\gamma}_{\#} : [0, 1] \to \Xi^{\mathrm{cont}}(Y)$ denote the concatenated path of two copies of $\tilde{\gamma}$. The path $\tilde{\gamma}_{\#}$ is a representative of $\gamma^2 \in \pi_1(\Xi^{\mathrm{cont}}(Y))$. Define a section $s' : [0, 1] \to \tilde{\gamma}^*\mathcal{R}(Y, W, \mathfrak{s})$ by $s'(t) = f^*(f_t^*a)$. Namely, $s'$ is the pull-back section of $s$ under $f$. By concatenating $s$ with $s'$, we obtain a section $s \cup s' : [0, 1] \to \tilde{\gamma}_{\#}^*\mathcal{R}(Y, W, \mathfrak{s})$. The left-hand side of (17) is the signed counting of the parametrized moduli space for $s \cup s'$, but again the moduli space is empty. Thus we have (17). □

Lemma 3.19 can be generalized more:

**Proposition 3.20** *Let $f, g \in \mathrm{Diff}(W, [\mathfrak{s}], \partial)$. If $f$ and $g$ are isotopic to each other through $\mathrm{Diff}(W)$, then there exists $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$ such that*

$$FKM(W, [\mathfrak{s}], \xi, f, \gamma) = FKM(W, [\mathfrak{s}], \xi, g, \gamma) \in \mathbb{Z}_2 \tag{18}$$

*and*

$$FKM(W, [\mathfrak{s}], \xi, f^2, \gamma^2) = FKM(W, [\mathfrak{s}], \xi, g^2, \gamma^2) \in \mathbb{Z}. \tag{19}$$

***Proof*** We may suppose that $d(W, \mathfrak{s}, \xi) + 1 = 0$. Let $h_t$ be a path in $\mathrm{Diff}(W)$ between $f$ and $g$. Define a path $\tilde{\gamma} : [0, 1] \to \Xi^{\mathrm{cont}}(Y)$ by $\tilde{\gamma}(t) = h_t^*\xi$, and set $\gamma = [\tilde{\gamma}] \in \pi_1(\Xi^{\mathrm{cont}}(Y))$. Take a section $s_f^{\mathcal{R}} : [0, 1] \to \tilde{\gamma}^*\mathcal{R}(Y, W, \mathfrak{s})$ so that $s_f^{\mathcal{R}}(1) = f^*s_f^{\mathcal{R}}(0)$. The quantity $FKM(W, [\mathfrak{s}], \xi, f, \gamma)$ is the signed counting of the parametrized moduli space for $s_f^{\mathcal{R}}$. Let $s_g^{\mathcal{R}}$ denote the pull-back of the section $s_f^{\mathcal{R}}$ under $\sqcup_{t\in[0,1]}(h_t^* f^{-1})$. This section satisfies that $s_g^{\mathcal{R}}(1) = g^*s_g^{\mathcal{R}}(0)$, and $FKM(W, [\mathfrak{s}], \xi, g, \gamma)$ can be calculated by the signed counting of the parametrized moduli space for this section $s_g^{\mathcal{R}}$. However, the pull-back under $\sqcup_{t\in[0,1]}(h_t^*(f^{-1})^*)$

gives rise to a diffeomorphism between these moduli spaces corresponding to $s_f^{\mathcal{R}}$ and $s_g^{\mathcal{R}}$, and this implies (18).

To prove (19), as in the proof of Lemma 3.19, let $\tilde{\gamma}_{\#} : [0, 1] \to \Xi^{\mathrm{cont}}(Y)$ denote the concatenated path of two copies of $\tilde{\gamma}$, which represents $\gamma^2 \in \pi_1(\Xi^{\mathrm{cont}}(Y))$. Define a section $(s')_f^{\mathcal{R}} : [0, 1] \to \tilde{\gamma}^*\mathcal{R}(Y, W, \mathfrak{s})$ as the pull-back section of $s_f^{\mathcal{R}}$ under $f$. Similarly, define $(s')_g^{\mathcal{R}}$ as the pull-back section of $s_g^{\mathcal{R}}$ under $g$. The signed counting of the concatenated section $s_f^{\mathcal{R}} \cup (s')_f^{\mathcal{R}}$ is the left-hand side of (19), and similarly for $g$. Again the pull-back under $\sqcup_{t\in[0,1]}(h_t^*(f^{-1})^*)$ gives rise to a diffeomorphism between these moduli spaces corresponding to $(s')_f^{\mathcal{R}}$ and $(s')_g^{\mathcal{R}}$, and together with the diffeomorphism for $s_f^{\mathcal{R}}$ and $s_g^{\mathcal{R}}$ considered above, we obtain a diffeomorphism between these moduli spaces corresponding to $s_f^{\mathcal{R}} \cup (s')_f^{\mathcal{R}}$ and $s_g^{\mathcal{R}} \cup (s')_g^{\mathcal{R}}$. Thus we have (19). □

**Proposition 3.21** *Let $f$, $f' \in \mathrm{Diff}(W, [\mathfrak{s}], \partial)$ and $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$. Then we have*

$$\begin{aligned} FKM(W, [\mathfrak{s}], \xi, f \circ f', \gamma) &= FKM(W, [\mathfrak{s}], \xi, f) + FKM(W, [\mathfrak{s}], \xi, f', \gamma) \\ &= FKM(W, [\mathfrak{s}], \xi, f, \gamma) + FKM(W, [\mathfrak{s}], \xi, f') \end{aligned} \tag{20}$$

*in $\mathbb{Z}_2$. Moreover, if all of the diffeomorphisms in (20) and $\gamma$ induce the trivial monodromy on $\Lambda(W, \mathfrak{s}, \xi)$, then the equalities (20) hold over $\mathbb{Z}$.*

***Proof*** Denote by $\tilde{\gamma}_{\mathrm{const}}$ the constant path at $\xi$ in $\Xi^{\mathrm{cont}}(Y)$ and set $\gamma_{\mathrm{const}} = [\tilde{\gamma}_{\mathrm{const}}] \in \pi_1(\Xi^{\mathrm{cont}}(Y))$. By definition, we have

$$FKM(W, [\mathfrak{s}], \xi, f, \gamma_{\mathrm{const}}) = FKM(W, [\mathfrak{s}], \xi, f).$$

Pick a generic element $a \in \mathcal{P}(Y, W, \mathfrak{s}, \xi)$. Take a path $a$ between $a$ and $f^*a$ in $\mathcal{P}(Y, W, \mathfrak{s}, \xi)$. Such a path can be thought of as a section

$$s : [0, 1] \to \tilde{\gamma}_{\mathrm{const}}^*\mathcal{P}(Y, W, \mathfrak{s}) \cong [0, 1] \times \mathcal{P}(Y, W, \mathfrak{s}, \xi).$$

Pick a loop $\tilde{\gamma}$ that represents $\gamma$, and take a generic section

$$s' : [0, 1] \to \tilde{\gamma}^*\mathcal{P}(Y, W, \mathfrak{s})$$

so that $s'(0) = f^*a$ and $s'(1) = (f \circ f')^*a = f'^*(f^*a)$. Then $FKM(W, [\mathfrak{s}], \xi, f, \gamma_{\mathrm{const}})$ is the algebraic count of the parametrized moduli space for $s$, and $FKM(W, [\mathfrak{s}], \xi, f', \gamma)$ is that for $s'$. Thus the algebraic count of the parameterized moduli space for the path that is obtained by concatenating $s$ and $s'$ is the right-hand side of (20).

On the other hand, parametrizing intervals, we can regard the concatenation of the sections $s$ and $s'$ as a section

$$s \cup s' : [0, 1] \to \tilde{\gamma}^*\mathcal{P}(Y, W, \mathfrak{s})$$

such that $(s \cup s')(0) = a$ and $(s \cup s')(0) = (f \circ f')^* a$. Therefore the algebraic count of the parameterized moduli space for $s \cup s'$ is the left-hand side of (20). This completes the proof of the first equality. The second equality follows just by a similar argument. □

## 4 Several vanishing results

In this section, we prove several vanishing results for both of the families Seiberg–Witten invariant and Kronheimer–Mrokwa invariant.

Before stating several vanishing results, let us introduce a notion of *strong L-space* for convenience.

**Definition 4.1** A rational homology 3-sphere $Y$ is a *strong L-space* if there exists a Riemann metric $g$ on $Y$ such that there is no irreducible solutions to the Seiberg–Witten equation on $(Y, g, \mathfrak{s})$ for a Spin$^c$ structure $\mathfrak{s}$ on $Y$.

Note that all strong L-spaces are L-space. However, the authors do not know whether the converse is true or not.

We also recall the families Seiberg–Witten invariant for diffeomorphisms following [54]. Let $X$ be a closed oriented smooth 4-manifold with $b_2^+(X) > 2$ and $\mathfrak{s}$ be a Spin$^c$ structure on $X$. Fix a homology orientation of $X$. Let $f : X \to X$ be an orientation-preserving diffeomorphism of $X$ such that $f^*\mathfrak{s} = \mathfrak{s}$ (precisely, $f^*\mathfrak{s}$ is isomorphic to $\mathfrak{s}$). Then we can define a numerical invariant $FSW(X, \mathfrak{s}, f)$, which takes value in $\mathbb{Z}$ if $f$ preserves the homology orientation, and which takes value in $\mathbb{Z}_2$ if $f$ reverses the homology orientation. It is valid essentially only when the formal dimension of $\mathfrak{s}$ is $-1$: otherwise the invariant $FSW(X, \mathfrak{s}, f)$ is just defined to be zero.

### 4.1 A family version of the vanishing result for embedded submanifolds

#### 4.1.1 Embeddings of 3-manifolds

We first prove a family version of the vanishing result for embedded 3-manifolds. For the original version, see [23] for example.

**Theorem 4.2** *Let $(X, \mathfrak{s})$ be a closed Spin$^c$ 4-manifold with $b_2^+(X) > 0$ and $Y$ be a closed oriented 3-manifold.*

(i) *Suppose there is a smooth embedding $i : Y \to X$. Let $(Y, g)$ be a strong L-space for some metric $g$ on $Y$ and $f$ be an orientation-preserving self-diffeomorphism of $X$ such that*

- $f^*\mathfrak{s} = \mathfrak{s}$,
- $f(i(Y)) = i(Y)$ *as subsets of $X$, and*
- $f : (i(Y), i_*g) \to (i(Y), i_*g)$ *is an isometry.*

*We also impose that the map*

$$H^2(X; \mathbb{Q}) \to H^2(Y; \mathbb{Q})$$

*is non-zero. Then we have*

$$FSW(X, \mathfrak{s}, f) = \begin{cases} 0 \in \mathbb{Z} \text{ if } f \text{ does not flip the homology orientation of } X, \\ 0 \in \mathbb{Z}_2 \text{ if } f \text{ flips the homology orientation of } X. \end{cases}$$

(ii) *Suppose $X$ contains an essentially embedded smooth surface $S$ with non-zero genus and zero self-intersection that violates the adjunction inequality, i.e. we have*

$$2g(S) - 2 < |\langle c_1(\mathfrak{s}), [S] \rangle|.$$

*Also, the normal sphere bundle $\partial \nu(S)$ is supposed to be $f|_{\partial \nu(S)} = \mathrm{Id}_{\partial \nu(S)}$. Then, we have*

$$FSW(X, \mathfrak{s}, f) = \begin{cases} 0 \in \mathbb{Z} \text{ if } f \text{ does not flip the homology orientation of } X, \\ 0 \in \mathbb{Z}_2 \text{ if } f \text{ flips the homology orientation of } X. \end{cases}$$

Note that Theorem 4.2(ii) also follows from [9, Theorem 1.2]. In the proof, we mainly follow the original Frøyshov's argument which uses a neck stretching argument and non-exact perturbations and Kronheimer–Mrowka's proof of the Thom conjecture [35].

***Proof*** We first prove (i). Since the proof of (ii) is similar to that of (i), we only write a sketch of proof of (ii). Because the family Seiberg–Witten invariant is an isotopy invariant, we can assume that $f$ can be a product in a neighborhood $N$ of $Y'$ which is isometry with respect to $g$. Now, we consider the family version of the moduli space

$$\mathcal{PM}(X, \mathfrak{s}, f) \to S^1$$

as a mapping torus of the moduli space over $I = [0, 1]$:

$$\bigcup_{t \in [0,1]} \mathcal{M}(X, \mathfrak{s}, g_t) \to I,$$

where $g_t$ is a smooth 1-parameter family of metrics such that

- $g_1 = f^* g_0$ and
- for any $t \in [0, 1]$, we have $g_t|_N = g + dt^2$.

Here we take a metric $g$ so that there is no irreducible solution to the Seiberg–Witten equation on $Y$. By assumption, we have a trivialization of the family $E_f \to S^1$ obtained as the mapping torus of $f$ near $N$:

$$E_{f|_{i(Y)}} \subset E_f,$$

where $E_{f|_{i(Y)}}$ us the mapping torus of $f|_{i(Y)}$.

Now, near $E_{f|_{i(Y)}}$, we consider a neck stretching argument. We consider a family of metrics $g_{t,s}$ parametrized by $s \in [0, \infty)$ satisfying the following conditions:

- $g_{t,0} = g_t$,
- as Riemann manifolds, $(E_{f|_{i(Y)}}, g_{t,s}|_N) = ([0, s+1] \times Y, g + dt^2)$,
- outside of $E_{f(i(Y))}$ in $E_f$, the metric $g_{t,s}$ coincides with $g_t$.

By the assumption $\mathrm{Im}(H^2(X;\mathbb{Q}) \to H^2(Y;\mathbb{Q})) \neq 0$, we take a closed 2-form $\eta$ on $X$ such that

$$0 \neq [\eta|_Y] \in H^2(Y;\mathbb{R}).$$

Then, we consider the perturbation of the family Seiberg–Witten equation on $E_f$ using $\eta$:

$$\begin{cases} F^+_{A_t^t} + \sigma(\Phi_t, \Phi_t) = \epsilon\eta^+ \\ D_{A_t}\Phi_t = 0 \end{cases}$$

for a small $\epsilon > 0$. We take the $\epsilon$ so that there is no solution to the $\epsilon\eta$-perturbed Seiberg–Witten equation on $Y$ with respect to $(g', \mathfrak{s}|_Y)$:

$$\begin{cases} F_{B^t} + \sigma(\phi, \phi) = \epsilon\eta|_Y \\ D_B\phi = 0. \end{cases}$$

Suppose $FSW(X, \mathfrak{s}, f) \neq 0$. Put $\eta_0 := \epsilon\eta$. Now we take an increasing sequence $s_i \to \infty$. Then, since $FSW(X, \mathfrak{s}, f) \neq 0$, there is a sequence of solutions $(A_i, \Phi_i)$ to the Seiberg–Witten equation with respect to $g_{t_i,s_i}$ for some $t_i \in [0, 1]$.

**Claim 4.3** *We claim that*

$$\sup_{i \in \mathbb{Z}_{>0}} \mathcal{E}^{top}_{\eta_0, g_{t_i,s_i}}((A_i, \Phi_i)|_{E_{f|_{i(Y)}}}) < \infty,$$

*where, for a Spin$^c$ 4-manifold $W$ with boundary, the topological energy perturbed by $\eta_0$ is defined to be*

$$\mathcal{E}^{top}_{\eta_0}(A, \Phi) := \frac{1}{4}\int_W (F_{A^t} - 4\eta_0) \wedge (F_{A^t} - 4\eta_0) - \int_{\partial W} \langle \Phi|_Y, D_B\Phi|_Y \rangle.$$

***Proof of Claim 4.3*** For a Spin$^c$ 4-manifold $W$ with boundary, define the perturbed analytical energy by

$$\begin{aligned} \mathcal{E}^{an}_{\eta_0}(A, \Phi) := &\frac{1}{4}\int_W |F_{A^t} - 4\eta_0|^2 + \int_W |\nabla_A\Phi|^2 + \frac{1}{4}\int_W (|\Phi|^2 + (s/2))^2 \\ &- \int_W \frac{s^2}{16} + 2\int_W \langle \Phi, \rho(\eta_0)\Phi \rangle - \int_{\partial W} (H/2)|\Phi|^2, \end{aligned}$$

where $s$ is the scalar curvature and $H$ is the mean curvature. It is proven that if $(A, \Phi)$ is a solution to the Seiberg–Witten equation perturbed by $\eta_0$, the equality

$$\mathcal{E}^{an}_{\eta_0}(A, \Phi) = \mathcal{E}^{top}_{\eta_0}(A, \Phi).$$

holds ( [38, (29.6), Page 593]). Since $X$ is closed, we know that, on $X$,

$$\sup_{i\in\mathbb{Z}_{>0}} \mathcal{E}^{an}_{\eta_0}(A_i, \Phi_i) = \sup_{i\in\mathbb{Z}_{>0}} \mathcal{E}^{top}_{\eta_0}(A_i, \Phi_i) < \infty.$$

On the other hand, we have

$$\mathcal{E}^{an}_{\eta_0}(A_i, \Phi_i) = \mathcal{E}^{an}_{\eta_0}((A_i, \Phi_i)|_{E_{f|_{i(Y)}}}) + \mathcal{E}^{an}_{\eta_0}((A_i, \Phi_i)|_{(E_{f|_{i(Y)}})^c}).$$

Since $X$ is compact, $\eta_0$ is bounded and $g_{t,s}$ is a compact family of metrics, we have lower bounds

$$-\infty < \inf_{i\in\mathbb{Z}_{>0}} \mathcal{E}^{an}_{\eta_0}((A_i, \Phi_i)|_{(E_{f|_{i(Y)}})^c}(A_i, \Phi_i).$$

So, we see

$$\sup_{i\in\mathbb{Z}_{>0}} \mathcal{E}^{top}_{\eta_0, g_{t_i,s_i}}((A_i, \Phi_i)|_{E_{f|_{i(Y)}}}) = \sup_{i\in\mathbb{Z}_{>0}} \mathcal{E}^{an}_{\eta_0, g_{t_i,s_i}}((A_i, \Phi_i)|_{E_{f|_{i(Y)}}}) < \infty.$$

□

By taking a subsequence, we can suppose $t_i \to t_\infty \in [0, 1]$. As in the proof of [36], we can also take a subsequence of $(A_i, \Phi_i)$ so that

$$\mathcal{E}^{top}_{g_{t_i,s_i}}((A_i, \Phi_i)|_{[l_i,l_i+1]\times Y\subset E_{f|_{i(Y)}}}) \to 0.$$

So, as the limit of $(A_i, \Phi_i)|_{[l_i,l_i+1]\times Y\subset E_{f|_{i(Y)}}}$, we obtain an (perturbed) energy zero solution $(A_\infty, \Phi_\infty)$ on $[0, 1] \times Y$. By considering the temporal gauge, this gives a solution to (4.1.1). This gives a contradiction.

The proof of (ii) is similar to (i). Let $Y$ be the product $S^1 \times S$. We take a Riemann metric $g$ on $Y$ forms

$$g = dt^2 + g_0,$$

where $t$ is a coordinate of $S^1$ and $g_0$ has a constant scalar curvature. Kronheimer–Mrowka's argument in [38] implies that if there is a solution to the Seiberg–Witten equation on $Y$ with respect to $(g, \mathfrak{s}|_Y)$, then the adjunction inequality

$$|\langle c_1(\mathfrak{s}), [S]\rangle| \leq 2g(S) - 2$$

holds ([38, Proposition 40.1.1]) when $g(S) > 0$. Since we now are assuming the opposite inequality $|\langle c_1(\mathfrak{s}), [S]\rangle| < 2g(S) - 2$, it is sufficient to get a solution to the Seiberg–Witten equation on $Y$ with respect to $(g, \mathfrak{s}|_Y)$. The remaining part of the proof is the same as that of (i), i.e. we consider the neck stretching argument near $\partial(\nu(S))$. This completes the proof. □

We also provide a version of Theorem 4.2 for the Kronheimer–Mrowka invariant.

**Theorem 4.4** *Let $(W, \mathfrak{s})$ be a compact Spin$^c$ 4-manifold with contact boundary $(Y, \xi)$.*

(i) *Suppose there is a smooth embedding $i : Y \to W$ where $(Y, g)$ is a strong L-space for some metric $g$ on $Y$. Let $f$ be a self-diffeomorphism of $W$ such that*

- *$f|_{\partial W}$ is the identity,*
- *$f^*\mathfrak{s} = \mathfrak{s}$,*
- *$i(Y)$ separates $W$,*
- *$f(i(Y)) = i(Y)$ as subsets of $W$, and*
- *$f : (i(Y), i_*g) \to (i(Y), i_*g)$ is an isometry.*

*We also impose that the map*

$$H^2(W; \mathbb{Q}) \to H^2(Y; \mathbb{Q})$$

*is non-zero. Then we have*

$$FKM(X, \mathfrak{s}, \xi, f) = \begin{cases} 0 \in \mathbb{Z} \text{ if the action of } f \text{ on } \Lambda(Y, \mathfrak{s}, \xi) \text{ is trivial,} \\ 0 \in \mathbb{Z}_2 \text{ if the action of } f \text{ on } \Lambda(Y, \mathfrak{s}, \xi) \text{ is non-trivial.} \end{cases}$$

(ii) *Suppose $W$ includes an essentially embedded smooth surface $S$ with non-zero genus that violating adjunction inequality and $[S]^2 = 0$. Then for a homologically trivial diffeomorphism $f$ on $W$ fixing the boundary pointwise such that $f(\partial(\nu(S)))$ is smoothly isotopic (rel $\partial$) to $\partial(\nu(S))$, we have*

$$FKM(X, \mathfrak{s}, \xi, f) = \begin{cases} 0 \in \mathbb{Z} \text{ if the action of } f \text{ on } \Lambda(Y, \mathfrak{s}, \xi) \text{ is trivial,} \\ 0 \in \mathbb{Z}_2 \text{ if the action of } f \text{ on } \Lambda(Y, \mathfrak{s}, \xi) \text{ is non-trivial.} \end{cases}$$

***Proof*** The proof is essentially parallel to that of Theorem 4.2 An unparametrized version of Theorem 4.4 is proven in [30, Theorem 1.19 (i)]. We use the same perturbations used in the proof of [30, Theorem 1.19 (i)]. The only difference between the proofs of Theorem 4.2 and Theorem 4.4 appears in the proof of Claim 4.3. Note that when we consider the Seiberg–Witten equation on 4-manifolds with conical end, there is no global notion of energy: on the interior, we have usual topological and analytical energies, and on the cone, we have the symplectic energy. We combine these two energies to show Claim 4.3. For that part, see that the proof of [30, Lemma 4.6]. It is easy to see the proof of [30, Lemma 4.6] can be also applied to our family case. □

### 4.1.2 Embeddings of surfaces

We also prove Theorem 4.4 for embedded surfaces to find exotic embeddings of surfaces into 4-manifolds with boundary. Let $\Sigma_g$ denote a closed oriented surface of genus $g$.

**Theorem 4.5** *Let $(W, \mathfrak{s})$ be a compact Spin$^c$ 4-manifold with contact boundary $(Y, \xi)$. Let $i : \Sigma_g \to W$ be a smooth embedding satisfying one of the following two conditions:*

(i) – *$i_*[\Sigma_g]$ is non-torsion, and*

– $g = 0$.

(ii) – $g > 0$ *and*
– *the adjunction inequality for* $(i(\Sigma_g), \mathfrak{s})$ *is violated.*

*For any diffeomorphism* $f$ *on* $W$ *fixing the boundary pointwise and preserving the Spin*$^c$ *structure* $\mathfrak{s}$ *such that* $i$ *is smoothly isotopic (rel* $\partial$*) to* $f \circ i$*, we have*

$$FKM(X, \mathfrak{s}, \xi, f) = \begin{cases} 0 \in \mathbb{Z} \text{ if the action of } f \text{ on } \Lambda(Y, \mathfrak{s}, \xi) \text{ is trivial}, \\ 0 \in \mathbb{Z}_2 \text{ if the action of } f \text{ on } \Lambda(Y, \mathfrak{s}, \xi) \text{ is non-trivial}. \end{cases}$$

*Moreover, in the case (i), we can replace the assumption that* $i$ *is isotopic to* $f \circ i$ *with the assumption that the image* $i(\Sigma_g)$ *is smoothly isotopic to the image of* $f^2 \circ i(\Sigma_g)$.

***Proof*** The proof is also based on the neck stretching argument near a normal neighborhood of $i(\Sigma_g)$. The closed case is treated in [9]. Since there is no big difference between the proof of Theorem 4.5 and [9, Theorem 1.2], we omit the proof. □

### 4.2 A fiberwise connected sum formula

We first review a fiberwise connected sum formula which is first proven in [32, Theorem 7.1]. Note that in [32, Theorem 7.1], the case that the connected sum along $S^3$ is treated. We generalize the vanishing result in [32, Theorem 7.1] to the result on the connected sums along any strong L-spaces with $b_1 = 0$. In the context of Donaldson's theory, the connected sum result is written in [55, Theorem 3.3]. For a 4-manifold $X$, $X^\circ$ denotes a compact punctured 4-dimensional submanifold of $X$.

**Theorem 4.6** *Let* $(Y, h)$ *be a strong L-space with* $b_1(Y) = 0$*. Let* $(X, \mathfrak{s})$, $(X', \mathfrak{s}')$ *be two compact Spin*$^c$ *4-manifolds with* $b_2^+ > 1$, $\partial X = -Y$, $\partial X' = Y$*, and suppose there is an isomorphism* $\mathfrak{s}|_{\partial X} = \mathfrak{s}'|_{-\partial X'}$*. Let* $f$ *be an orientation-preserving diffeomorphism on the closed 4-manifold* $X^\# := X \cup_Y X'$ *such that*

(i) *the diffeomorphism* $f$ *is smoothly isotopic to a connected sum* $f' \#_Y g'$ *on* $X \cup_Y X'$ *of diffeomorphisms* $f'$ *and* $g'$ *on* $X$ *and* $X'$ *which are, near the boundary of* $X$ *and* $X'$*, the product of an isometry of* $(Y, h)$ *with the identity on* $[0, 1]$*, and*
(ii) *the diffeomorphism* $f$ *preserves the Spin*$^c$*-structure* $\mathfrak{s}\#\mathfrak{s}'$ *on* $X^\#$.

*Then we have*

$$FSW(X^\#, \mathfrak{s}\#\mathfrak{s}', f) = \begin{cases} 0 \in \mathbb{Z} \text{ if } f \text{ does not flip the homology orientation of } X, \\ 0 \in \mathbb{Z}_2 \text{ if } f \text{ flips the homology orientation of } X. \end{cases}$$

***Proof*** For completeness, we give a sketch of the proof. Let $Y$ be a rational homology 3-sphere with a Riemannian metric $h$ such that there is no irreducible solution to the Seiberg–Witten equation on $Y$ with respect to $(\mathfrak{s}|_Y, h)$. Suppose $X^\circ$ and $(X')^\circ$ are 4-manifolds with boundary $-Y$ and $Y$ and define

$$X \# X' = X^\circ \cup_Y (X')^\circ.$$

Since the family Seiberg–Witten invariant $FSW$ is an isotopy invariant, we can assume that $f$ is described as the connected sum $f'\#g'$ on $X\#_Y X'$ of diffeomorphisms $f'$ and $g'$ on $X^\circ$ and $(X')^\circ$ which are the identity near the boundary of $X^\circ$ and $(X')^\circ$. With respect to the Spin$^c$ structure $\mathfrak{s}\#\mathfrak{s}'$, the gluing theory enables us to construct a diffeomorphism

$$\begin{aligned}\mathcal{PM}((X^\circ \cup [0,\infty)\times Y),\, f') \times_{U(1)} \mathcal{PM}(((-\infty,0]\times Y)\\ \cup (X')^\circ,\, g') \to \mathcal{PM}(X\#X',\, f'\#g'),\end{aligned} \tag{21}$$

where the spaces $\mathcal{PM}(X^\circ \cup [0,\infty)\times Y),\, f')$ and $\mathcal{PM}(((-\infty,0]\times Y)\cup (X')^\circ,\, g')$ are parametrized moduli spaces on the cylindrical-end Riemannian 4-manifolds $X^\circ \cup [0,\infty)\times Y$ and $((-\infty,0]\times Y)\cup (X')^\circ$ asymptotically the flat reducible solutions. Here we used the product metric $h + dt^2$ on the cylinder part. Precisely, we need to use weighted $L^2_k$ norms to obtain Fredholm properties of linearized Seiberg–Witten equations with slice. From index calculations, we have

$$\dim \mathcal{PM}(X^\circ \cup [0,\infty)\times Y,\, f') = \dim \mathcal{PM}(X,\, f')$$

and

$$\dim \mathcal{PM}((-\infty,0]\times Y \cup (X')^\circ),\, g') = \dim \mathcal{PM}(X',\, g').$$

So we have

$$\dim \mathcal{PM}(X,\, f') + \dim \mathcal{PM}(X',\, g') + 1 = \dim \mathcal{PM}(X\#X',\, f).$$

Note that $\dim \mathcal{PM}(X\#X',\, f)$ is 0 if we assume the family Seiberg–Witten invariant of $(X\#X',\, \mathfrak{s}\#\mathfrak{s}',\, f)$ is non-zero. (Otherwise, we define 0 as the family Seiberg–Witten invariant in this paper. ) Thus, one of $\dim \mathcal{PM}(X,\, f')$ and $\dim \mathcal{PM}(X',\, g')$ is negative. This implies one of $\mathcal{PM}(X,\, f')$ and $\mathcal{PM}(X',\, g')$ is empty since they are assumed to be regular. This completes the proof. □

In Theorem 4.6, we assumed that $f$ is isotoped into $f'$ so that $f'|_Y$ is an isometry. However, if $Y$ admits a positive scalar curvature metric, then the following stronger result holds:

**Theorem 4.7** *Let $Y$ be a rational homology 3-sphere with a positive scalar curvature metric and $b_1(Y) = 0$. Let $(X, \mathfrak{s}), (X', \mathfrak{s}')$ be two compact Spin$^c$ 4-manifolds with $b_2^+ > 1$, $\partial X = -Y$ and $\partial X' = Y$. Suppose $\mathfrak{s}|_{\partial X} = \mathfrak{s}'|_{-\partial X'}$. Let $f$ be an orientation-preserving diffeomorphism on the closed 4-manifold $X^\# := X \cup_Y X'$ such that*

(i) *the diffeomorphism $f$ is smoothly isotopic to a connected sum $f'\#_Y g'$ on $X \cup_Y X'$ of diffeomorphisms $f'$ and $g'$ on $X$ and $X'$ so that $f(Y) = Y$,*
(ii) *the diffeomorphism $f$ preserves the Spin$^c$-structure $\mathfrak{s}\#\mathfrak{s}'$ obtained as the connected sum of $\mathfrak{s}$ and $\mathfrak{s}'$ along $Y$.*

*Then we have*

$$FSW(X^{\#}, \mathfrak{s}\#\mathfrak{s}', f) = \begin{cases} 0 \in \mathbb{Z} \text{ if } f \text{ does not flip the homology orientation of} X, \\ 0 \in \mathbb{Z}_2 \text{ if } f \text{ flips the homology orientation of } X. \end{cases}$$

***Proof*** The proof is similar to that of Theorem 4.6. Instead of assuming the isometric property of diffeomorphisms, we use the contractivity of the space of positive scalar curvature metrics proven in [8]. Let us explain how to take a fiberwise Riemann metric to obtain a diffeomorphism corresponding to (21). Because the family Seiberg–Witten invariant is an isotopy invariant, we can assume that $f$ is a product in a neighborhood $N$ of $Y$ preserving the level of $N = [0, 1] \times Y$. Now, we shall consider the parametrized moduli space

$$\mathcal{PM}(X, \mathfrak{s}, f) \to S^1,$$

regarded as the quotient the moduli space over $[0, 1]$:

$$\bigcup_{t \in [0,1]} \mathcal{M}(X, \mathfrak{s}, g_t) \to [0, 1],$$

where $g_t$ is a smooth 1-parameter family of metrics such that

- $g_1 = f^* g_0$,
- for any $t \in [0, 1]$, we have $g_t|_N = h_t + dt^2$ for a smooth 1-parameter family of metrics $h_t$ on $Y$,
- $h_0$ is a positive scalar curvature metric on $Y$.

By the connectivity of the space of positive scalar curvature metrics proven in [8], we can take $h_t$ so that $h_t$ is a positive scalar curvature metric for every $t \in [0, 1]$. Under these settings, we obtain a diffeomorphism between moduli spaces of the form (21), and the rest of the proof is the same as that of Theorem 4.6. □

We next show a vanishing result of the Kronheimer–Mrowka invariant similar to Theorem 4.6.

**Theorem 4.8** *Let $Y$ be a strong L-space with $b_1(Y) = 0$. Let $(W, \mathfrak{s})$, $(X', \mathfrak{s}')$ be Spin$^c$ 4-manifolds with $b_2^+ > 1$, $\partial W = -Y \cup Y'$ and $\partial X' = Y$ for an oriented 3-manifold $Y'$. Suppose $Y'$ is equipped with a contact structure $\xi$ such that $\mathfrak{s}_\xi = \mathfrak{s}|_{Y'}$ and $\mathfrak{s}|_Y = \mathfrak{s}'|_Y$. Let $f$ be a diffeomorphism on the closed 4-manifold $W^{\#} := W \cup_Y X'$ such that*

(i) *the diffeomorphism $f$ is smoothly isotopic to $f' \cup g'$ on $W \cup_Y X'$ of diffeomorphisms $f'$ and $g'$ on $X$ and $X'$ which are isometry (not necessarily identity) near the boundary of $W$ and $X'$ with respect to metric $h$ on $Y$, and*
(ii) *the diffeomorphism $f$ preserves the Spin$^c$-structure $\mathfrak{s}\#\mathfrak{s}'$ on $W^{\#}$.*

*Then we have*

$$FKM(W^{\#}, \mathfrak{s}\#\mathfrak{s}', f) = \begin{cases} 0 \in \mathbb{Z} \text{ if } f \text{ does not flip the elements in } \Lambda(W^{\#}, \mathfrak{s}\#\mathfrak{s}', \xi), \\ 0 \in \mathbb{Z}_2 \text{ if } f \text{ flips the homology orientation of } X. \end{cases}$$

***Proof*** The proof is completely the same as that of Theorem 4.6. □

### 4.3 A fiberwise blow up formula

We first review a fiberwise blow up formula proven in [11] for the family Seiberg–Witten invariant.

**Theorem 4.9** ([11]) *Let $(X, \mathfrak{s})$, $(X', \mathfrak{s}')$ be closed Spin$^c$ 4-manifolds with $b_2^+(X) > 1$ and $b_2^+(X') = 1$. Let $f$ be an orientation-preserving diffeomorphism of the closed 4-manifold $X\#X'$. Suppose that the formal dimension of the Seiberg–Witten moduli spaces for $(X, \mathfrak{s})$ and $(X', \mathfrak{s}')$ are 0 and $-2$ respectively.*

(i) *Let $f$ be a homologically trivial diffeomorphism on $X\#X'$ such that $f$ is smoothly isotopic to a connected sum $f'\#g'$ of $f' = \mathrm{Id}_X$ and a homologically trivial diffeomorphism $g'$ of $X'$, which is the identity near boundary. Then we have*

$$FSW(X\#X', \mathfrak{s}\#\mathfrak{s}', f) = 0 \in \mathbb{Z}.$$

(ii) *Let $f$ be a self-diffeomorphism on $X\#X'$ such that*

- *$f$ preserves the Spin$^c$ structure $\mathfrak{s}\#\mathfrak{s}'$,*
- *$f$ is smoothly isotopic to a union $f' \cup g'$ on $X\#X'$ of the identity $f' = \mathrm{Id}_X$ of $X$ and a diffeomorphism $g'$ on $X'$ which is the identity near $\partial X'$, and*
- *$g'$ reverses the homology orientation of $X'$.*

*Then we have*

$$FSW(X\#X', \mathfrak{s}\#\mathfrak{s}', f) = SW(X, \mathfrak{s}) \in \mathbb{Z}_2,$$

*where the right-hand side is the mod 2 Seiberg–Witten invariant.*

***Remark 4.10*** Note that [11, Theorem 1.1] only treats the connected sum along $S^3$, but there is no essential change when we extend their result to the case of the sums along any strong L-space with $b_1 = 0$.

Now, we state a fiberwise blow up formula for the families Kronheimer–Mrowka invariant.

**Theorem 4.11** *Let $(W, \mathfrak{s})$, $(X, \mathfrak{s}')$ be Spin$^c$ 4-manifolds with $b_2^+(X) = 1$, $\partial W = -Y$ and $\partial X = \emptyset$ for an oriented 3-manifold $Y$. Suppose $Y$ is equipped with a contact structure $\xi$ such that $\mathfrak{s}_\xi = \mathfrak{s}|_Y$, $d(W, \mathfrak{s}, \xi) = 0$ and $d(X, \mathfrak{s}') = -2$, where $d(X, \mathfrak{s}')$ is the virtual dimension of the Seiberg–Witten moduli space for $(X, \mathfrak{s}')$. Let $f$ be a diffeomorphism on the 4-manifold $X\#W$ such that $f$ preserves the Spin$^c$ structure $\mathfrak{s}\#\mathfrak{s}'$ obtained as the connected sum of $\mathfrak{s}$ and $\mathfrak{s}'$.*

(i) *Suppose $f$ is smoothly isotopic (rel $\partial$) to a union $f' \cup g'$ on $X\#W$ of the identity $f' = \mathrm{Id}_X$ of $X$ and a homologically trivial diffeomorphism $g'$ of $W$ which is the identity near boundary. Then we have*

$$FKM(X\#W, \mathfrak{s}\#\mathfrak{s}', \xi, f) = 0 \in \mathbb{Z}.$$

(ii) *Suppose $f$ is smoothly isotopic to a union $f' \cup g'$ of the identity $f' = \mathrm{Id}_X$ of $X$ and a diffeomorphism $g'$ of $W$ which is the identity near boundary and $g'$ reverses the homology orientation of $X$.*
*Then we have*

$$FKM(X\#W, \mathfrak{s}\#\mathfrak{s}', \xi, f) = \mathfrak{m}(W, \mathfrak{s}, \xi) \in \mathbb{Z}_2.$$

***Proof*** The proof is essentially the same as that for a similar gluing formula for the families Seiberg–Witten invariant of families of closed 4-manifolds [11, Theorem 1.1], and we omit the proof. □

***Remark 4.12*** We also remark that Theorem 4.11 can easily be generalized to the case of the connected sum along a strong L-space.

As a special case, we give a certain fiberwise connected sum formula for a certain class of 4-manifolds parametrized by $S^1$.

Let $W$ be an oriented compact smooth 4-manifold with contact boundary $(Y, \xi)$ and with $b_2^+ > 1$. Let $\mathfrak{s}$ be a Spin$^c$ structure on $W$ of formal dimension 0. Let $N, \mathfrak{t}, f_N$ be as in Sect. 5. Let us consider the manifold $W\#N$ obtained as the connected sum along $D^4$ in $N$ and the diffeomorphism $\mathrm{Id}_W \# f_N$.

**Proposition 4.13** *Under the above notation, one has*

$$FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \mathrm{Id}\,\#f_N) = \mathfrak{m}(W, \mathfrak{s}, \xi) \in \mathbb{Z}_2$$

***Proof*** This is a corollary of Theorem 4.11. □

## 5 Construction of diffeomorphisms and non-vanishing results

In this section, we construct an ingredient of desired exotic diffeomorphisms in our main theorems.

First, we describe the setting we work on. Set

$$N := \mathbb{C}P^2\#2(-\mathbb{C}P^2) = \mathbb{C}P^2\#(-\mathbb{C}P_1^2)\#(-\mathbb{C}P_2^2).$$

Let $\mathfrak{t}$ be a Spin$^c$ structure on $N$ such that each component of

$$c_1(\mathfrak{t}) \in H^2(N) = H^2(\mathbb{C}P^2) \oplus H^2(-\mathbb{C}P_1^2) \oplus H^2(-\mathbb{C}P_2^2)$$

is a generator of $H^2(\mathbb{C}P^2)$, $H^2(-\mathbb{C}P_1^2)$, and $H^2(-\mathbb{C}P_2^2)$. Let $H, E_1, E_2$ be the generators of $H^2(\mathbb{C}P^2)$, $H^2(-\mathbb{C}P_1^2)$, $H^2(-\mathbb{C}P_2^2)$, namely

$$c_1(\mathfrak{t}) = H + E_1 + E_2.$$

By abuse of notation, let $H, E_1, E_2$ denote also representing spheres of the Poincaré duals of these classes. A diffeomorphism $f_N : N \to N$ satisfying the following

properties is constructed in [31, Proof of Theorem 3.2] (where the diffeomorphism is denoted by $f_0'$):

- $f_N$ fixes the isomorphism class of $\mathfrak{t}$, and
- $f_N$ reverses orientation of $H^+(N)$.

By isotopy, we may suppose also that $f_N$ fixes a 4-dimensional small disk $D^4$ in $N$.

We consider simply connected compact oriented 4-manifolds $W$ and $W'$ with common contact boundary $(Y, \xi)$. Assume also that we have a diffeomorphism

$$\psi : W\#N \to W'\#N$$

that satisfies the following: If we decompose $H^2(W'\#N; \mathbb{Z})$ into

$$H^2(W'\#N; \mathbb{Z}) = H^2(W'; \mathbb{Z}) \oplus H^2(N; \mathbb{Z}),$$

the induced action $\psi^* : H^2(W'\#N; \mathbb{Z}) \to H^2(W'\#N; \mathbb{Z})$ on $a + b \in H^2(W'; \mathbb{Z}) \oplus H^2(N; \mathbb{Z})$ is of the form

$$\psi^*(a + b) = h'(a) + b, \tag{22}$$

where

$$h' : H^2(W'; \mathbb{Z}) \to H^2(W'; \mathbb{Z})$$

is an isomorphism.

Let $\mathfrak{s}$ be a Spin$^c$ structure on $W$, and let $\mathfrak{s}'$ be the Spin$^c$ structure on $W'$ determined by $h'(c_1(\mathfrak{s}')) = c_1(\mathfrak{s})$. Then it follows from (22) that

$$(\psi^{-1})^*(c_1(\mathfrak{s}) + c_1(\mathfrak{t})) = c_1(\mathfrak{s}') + c_1(\mathfrak{t}). \tag{23}$$

Define a self-diffeomorphism $f$ of $W\#N$ by

$$f := (\mathrm{Id}_W \# f_N) \circ \psi^{-1} \circ (\mathrm{Id}_{W'} \# f_N^{-1}) \circ \psi. \tag{24}$$

Note that $f$ is the identity on the boundary, while $\psi$ might not be. Note also that $f$ acts trivially on the (co)homology of $W\#N$. Indeed, we obtain from (22) that

$$\begin{aligned} f^* &= \psi^* \circ \mathrm{diag}(\mathrm{Id}_{H^2(W')}, (f_N^{-1})^*) \circ (\psi^{-1})^* \circ \mathrm{diag}(\mathrm{Id}_{H^2(W)}, f_N^*) \\ &= \mathrm{diag}(h' \circ h'^{-1}, (f_N^{-1})^* \circ f_N^*) = \mathrm{Id}\,. \end{aligned}$$

**Proposition 5.1** *Suppose that*

$$(\mathfrak{m}(W, \mathfrak{s}, \xi), \mathfrak{m}(W', \mathfrak{s}', \psi_*\xi)) \equiv (1, 0) \text{ or } (0, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$

*Then we have*

$$FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f) \equiv 1 \in \mathbb{Z}_2$$

*for the above diffeomorphism* $f$.

***Proof*** It follows from a combination of the gluing formula Proposition 4.13, the additivity formula Proposition 3.14, and (23) that

$$\begin{aligned}
&FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f)\\
&= FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \mathrm{Id}_W \# f_N) + FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \psi^{-1} \circ (\mathrm{Id}_{W'} \# f_N^{-1}) \circ \psi)\\
&= FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \mathrm{Id}_W \# f_N) + FKM(W'\#N, \mathfrak{s}'\#\mathfrak{t}, \psi_*\xi, \mathrm{Id}_{W'} \# f_N^{-1})\\
&= \mathfrak{m}(W, \mathfrak{s}, \xi) + \mathfrak{m}(W', \mathfrak{s}', \psi_*\xi)\\
&= 0 + 1 = 1 \text{ in } \mathbb{Z}_2.
\end{aligned}$$

This completes the proof. □

**Corollary 5.2** *For every non-zero integer* $n \in \mathbb{Z}$, *we have*

$$FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^n) \neq 0 \in \mathbb{Z}. \tag{25}$$

*Moreover, the mapping class of* $f$ *above generates a* $\mathbb{Z}$*-summand of the abelianization of*

$$\mathrm{Ker}(\pi_0(\mathrm{Diff}(W\#N, \partial) \to \mathrm{Aut}(H_2(W\#N; \mathbb{Z}))).$$

***Proof*** This follows from Proposition 5.1 and Corollary 3.18. □

We can modify the above argument for the generalized families Kronheimer–Mrowka invariant with a loop in $\Xi^{\mathrm{cont}}(\Xi(Y))$.

**Lemma 5.3** *Let* $W$ *be a compact oriented 4-manifold and with boundary* $Y = \partial W$. *Let* $\xi$ *be a contact structure on* $Y$. *Let* $\Sigma$ *be an embedded 2-sphere in the interior of* $W$ *whose self-intersection is non-negative and whose homology class is non-torsion. Let* $N = \mathbb{C}P^2\#2(-\mathbb{C}P^2)$, $\mathfrak{s} \in Spin^c(W\#N, \xi)$, *and set* $\mathfrak{t} := \mathfrak{s}|_N$. *Let* $f_0$ *be a self-diffeomorphism of* $N$ *which preserves* $\mathfrak{t}$. *Then, for every* $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$, *we have*

$$FKM(W\#N, [\mathfrak{s}], \xi, \mathrm{Id}_W \# f_0, \gamma) = 0$$

*in* $\mathbb{Z}_2$.

***Proof*** Let us first consider the case with $[\Sigma]^2 = 0$. General case can be reduced to this case by standard argument using the blow-up formula Theorem B.3 and the connected sum formula Theorem 4.11. Neck stretching argument along the boundary $S^1 \times S^2$ of a tubular neighbourhood of $\Sigma$, as in Theorem 4.4, gives the conclusion. □

**Proposition 5.4** *Let $W$ be a compact oriented 4-manifold and with boundary $Y = \partial W$. Let $\xi$ be a contact structure on $Y$. Let $\Sigma$ be an embedded 2-sphere in the interior of $W$ whose self-intersection is non-negative and whose homology class is non-torsion. Suppose that*

$$\mathfrak{m}(W', \mathfrak{s}', \psi_* \xi) \equiv 1 \in \mathbb{Z}_2.$$

*Then, for the above diffeomorphism $f$ Eq. (24), we have*

$$FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f, \gamma) \neq 0 \in \mathbb{Z}_2 \tag{26}$$

*for every $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$, and*

$$FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^{2n}, \gamma^2) \neq FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^{2n'}, \gamma^2) \in \mathbb{Z} \tag{27}$$

*for every $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$ and every distinct $n, n' \in \mathbb{Z}$.*

***Proof*** For every $\gamma$, it follows from the gluing formula Proposition 4.13, the additivity formula Lemma 3.21, and (23) that

$$\begin{aligned}
&FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f, \gamma) \\
&\quad = FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \mathrm{Id}_W \# f_N, \gamma) + FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \psi^{-1} \circ (\mathrm{Id}_{W'} \# f_N^{-1}) \circ \psi) \\
&\quad = FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \mathrm{Id}_W \# f_N, \gamma) + FKM(W'\#N, \mathfrak{s}'\#\mathfrak{t}, \psi_* \xi, \mathrm{Id}_{W'} \# f_N^{-1}) \\
&\quad = 0 + \mathfrak{m}(W', \mathfrak{s}', \psi_* \xi) = 1 \in \mathbb{Z}_2.
\end{aligned}$$

Thus we have (26).

Next, applying Lemma 3.21 inductively, for $n > 0$ and every $\gamma$, we obtain that

$$\begin{aligned}
&FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^{2n}, \gamma^2) \\
&\quad = FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, (f^2)^{n-1}, \gamma^2) + FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^2) \\
&\quad = \cdots \\
&\quad = FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, \mathrm{Id}, \gamma^2) + nFKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^2)
\end{aligned}$$

This combined with (25) implies (27) for $n, n' \geq 0$ with $n \neq n'$. A similar argument works for $n, n' \leq 0$ just by considering $f^{-1}$ in place of $f$. □

## 6 Proof of Theorem 1.4

Before proving Theorem 1.4, we review several definitions and theorems which are used in the proof of Theorem 1.4.

### 6.1 Contact topology

Let $\xi$ be a contact structure on an oriented 3-manifold. A knot $K \subset (Y, \xi)$ is called *Legendrian* if $T_p K \subset \xi_p$ for $p \in K$. A Legendrian knot $K$ in a contact manifold $(Y, \xi)$ has a standard neighborhood $N$ and a framing $fr_\xi$ given by the contact planes. If $K$ is null-homologous, then $fr_\xi$ relative to the Seifert framing is the *Thurston–Bennequin* number of $K$, which is denoted by $tb(K)$. If one does $fr_\xi - 1$-surgery on $K$ by removing $N$ and gluing back a solid torus so as to effect the desired surgery, then there is a unique way to extend $\xi|_{Y-N}$ over the surgery torus so that it is tight on the surgery torus. The resulting contact manifold is said to be obtained from $(Y, \xi)$ by *Legendrian surgery* on $K$. Also, for a knot $K$ in $(S^3, \xi_{\text{std}})$, the *maximal Thurston–Bennequin number* is defined as the maximal value of all Thurston–Bennequin numbers for all Legendrian representations of $K$.

A *symplectic cobordism* from the contact manifold $(Y_-, \xi_-)$ to $(Y_+, \xi_+)$ is a compact symplectic manifold $(W, \omega)$ with boundary $-Y_- \cup Y_+$ where $Y_-$ is a *concave* boundary component and $Y_+$ is *convex*, this means that there is a vector field $v$ near $\partial W$ which points transversally inwards at $Y_-$ and transversally outwards at $Y_+$, $\mathcal{L}_v \omega = \omega$ and $\iota_v \omega|_{Y_\pm}$ is a contact form of $\xi_\pm$. If $Y_-$ is empty, $(W, \omega)$ is called a *symplectic filling*.

We mainly follow a technique to construct symplectic cobordisms called *Weinstein handle attachment* [60]. One may attach a 1-, or 2-handle to the convex end of a symplectic cobordism to get a new symplectic cobordism with the new convex end described as follows. For a 1-handle attachment, the convex boundary undergoes, possibly internal, a connected sum. A 2-handle is attached along a Legendrian knot $L$ with framing one less than the contact framing, and the convex boundary undergoes a Legendrian surgery.

**Theorem 6.1** *Given a contact 3-manifold $(Y, \xi = \operatorname{Ker}\theta)$ let $W$ be a part of its symplectization, that is $(W = [0, 1] \times Y, \omega = d(e^t\theta))$. Let $L$ be a Legendrian knot in $(Y, \xi)$ where we think of $Y$ as $Y \times \{1\}$. If $W'$ is obtained from $W$ by attaching a 2-handle along $L$ with framing one less than the contact framing, then the upper boundary $(Y', \xi')$ is still a convex boundary. Moreover, if the 2-handle is attached to a symplectic filling of $(Y, \xi)$ then the resultant manifold would be a strong symplectic filling of $(Y'\xi')$.*

The theorem for Stein fillings was proven by Eliashberg [16], for strong fillings by Weinstein [60], and was first stated for weak fillings by Etnyre and Honda [17].

### 6.2 Proof of Theorem 1.4

In this section, we will show the existence of exotic diffeomorphisms of 4-manifolds with boundary. First, we need the following result to guarantee the existence of topological isotopy between diffeomorphisms of 4-manifolds with boundary.

**Theorem 6.2** (Orson–Powell, [50, Corollary C]) *Let $M$ be a smooth, connected, simply connected, compact 4-manifold with boundary of rational homology of $S^3$ or of $S^1 \times S^2$. Let $f : M \to M$ be a diffeomorphism such that $f|_{\partial M} = Id_{\partial M}$ and $f_* = Id : H_2(M; \mathbb{Z}) \to H_2(M; \mathbb{Z})$. Then $f$ is topologically isotopic rel. $\partial M$ to*

*$Id_M : M \to M$, i.e. there is a topological isotopy $F_t : M \to M$ with $F_0 = f$, $F_1 = Id_M$ such that $F_t|_{\partial M} = Id_{\partial M}$ for all $t \in [0, 1]$.*

**Lemma 6.3** *Every closed oriented connected 3-manifold $Y$ bounds a simply-connected 4-manifold $W$ that can be decomposed as $W_0 \cup$ (a 2-handle), where $W_0$ has a Stein structure and the 2-handle is attached along an unknot with framing $-1$.*

***Proof*** For a general $Y$, the third author constructed such a manifold $W$ in [47, Proof of Theorem 1.1]. □

We shall now prove Theorem 1.4 where we show the existence of exotic diffeomorphisms of 4-manifolds with boundary.

***Proof of Theorem 1.4*** Let $N, \mathfrak{t}, f$ be as in Sect. 5. For a 3-manifold $Y$, we consider the associated 4-manifold $W_1 = W_0 \cup h_1$ as constructed in Lemma 6.3. If necessary, we can attach a 2-handle $h_2$ along the unknot with framing $+1$ on $W_1$. Note that this process does not change the upper boundary $Y$. Let us call this modification $W_1$ as well. Thus we can write $W_1 = W_0 \cup h_1 \cup h_2$. As a core of $h_2$, we have an embedded 2-sphere $S$ whose self-intersection is non-negative and whose homology class is non-torsion.

Attach an Akbulut–Mazur cork $(A, \tau)$, i.e, a pair of algebraically canceling 1- and 2-handle as shown in the Fig. 1 on $W_1$ along $Y$ such that the 2-handle of $A$ linked the unknotted 2-handle $h_2$ algebraically thrice and with $h_1$ algebraically once. Thus we get a manifold $W = W_1 \cup A$ and denote the boundary by $Y' = \partial W$. In particular, by applying a cork-twist one can get a manifold $W' = (W - \mathrm{int}(A)) \cup_\tau A$ with boundary $Y'$. Notice that the cork-twist changed the dotted 1-handle with the 0 framed 2-handle in the Fig. 1. Since, in $W'$, the two 2-handles $h_1, h_2$ are passing over the 1-handle, this process increases their Thurston–Bennequin numbers without changing the smooth framing with respect to the standard contact structure on $S^3$, see Fig. 2.

By construction and the use of Theorem 6.1, $W'$ has a Stein structure and $\mathfrak{m}(W', \mathfrak{s}', \xi) = 1$, where $\mathfrak{s}'$ is the Spin$^c$ structure corresponding to the Stein structure and $\xi$ is the induced contact structure on the boundary.

Notice that $W$ and $W'$ are related by the cork-twist of $(A, \tau)$ and the cork-twist $\tau$ on $\partial A$ extends over $A\#\mathbb{CP}^2$ [3]. This gives a diffeomorphism $W\#\mathbb{CP}^2 \to W'\#\mathbb{CP}^2$ that is the identity map on $W - \mathrm{int}(A)$ and is the extension of the cork-twist to $A\#\mathbb{CP}^2$ on the rest. Thus we get a diffeomorphism $\psi : W\#N \to W'\#N$ as the one in Sect. 5 (if necessary, we need to precompose $\psi$ with an involution of $\mathbb{CP}^2$ to get the map that acts by identity on homology, a careful proof has been written by Auckly–Kim–Melvin–Ruberman [5]), and construct a self-diffeomorphism $f : W\#N \to W\#N$ along the procedure in Sect. 5. We claim that this diffeomorphism $f$ is the desired diffeomorphism.

Adopt the canonical Spin$^c$ structure on $W$ as $\mathfrak{s}$ in Sect. 5. As noted above, $W$ contains an embedded 2-sphere whose self-intersection is non-negative and whose homology class is non-torsion. Moreover, we have that $\mathfrak{m}(W', \mathfrak{s}', \xi) = 1$. Thus it follows from Proposition 5.4 that

$$FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^{2n}, \gamma^2) \neq FKM(W\#N, \mathfrak{s}\#\mathfrak{t}, \xi, f^{2n'}, \gamma^2) \in \mathbb{Z}$$

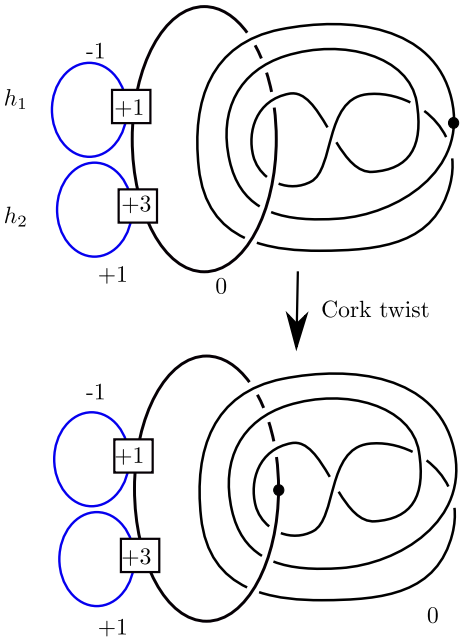


**Fig. 1** Attach the Akbulut–Mazur cork which is linking with with $h_1$ and $h_2$

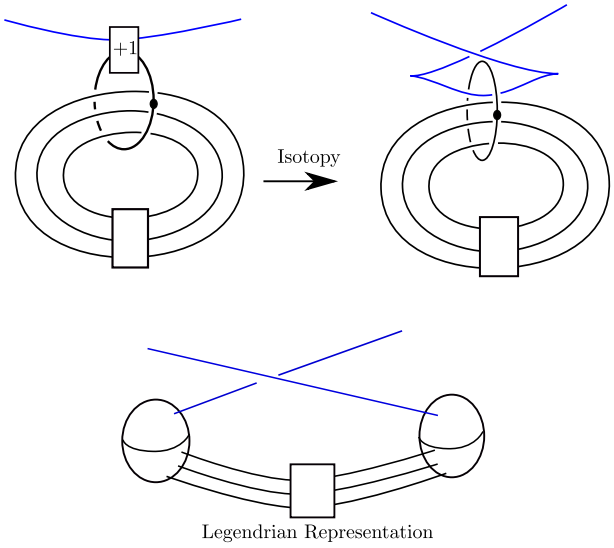


**Fig. 2** Contact framing of the blue knot is increased by 1 when it passes through the 1-handle

for every $\gamma \in \pi_1(\Xi^{\mathrm{cont}}(Y))$ and every distinct $n, n' \in \mathbb{Z}$. Therefore, by Proposition 3.20, $f^n$ and $f^{n'}$ are not smoothly isotopic to each other through $\mathrm{Diff}(W)$. On the other hand, it follows from Theorem 6.2 that all $f^n$ are topologically isotopic to the identity through $\mathrm{Homeo}(W, \partial)$. This completes the proof. □

***Proof of Theorem 1.6*** First, note that all $f^n$ are mutually topologically isotopic as in the proof of Theorem 1.4 above. Thus all $f^n(S)$ are mutually topologically isotopic. To show that $f^n(S)$ and $f^{n'}(S)$ are not smoothly isotopic if $n \neq n'$, it suffices to show that $f^{n-n'}(S)$ is not smoothly isotopic to $S$. This follows from Corollary 5.2 combined with Theorem 4.5, which we shall prove in Sect. 4. □

***Remark 6.4*** In the setup of Theorem 1.4, the mapping class of $f$ in $\mathrm{Diff}(W, \partial)$ generates a direct summand isomorphic to $\mathbb{Z}$ in the abelianization of the kernel of

$$\pi_0(\mathrm{Diff}(W, \partial)) \to \pi_0(\mathrm{Homeo}(W, \partial)).$$

This is a direct consequence of Corollary 5.2.

***Remark 6.5*** We will construct an explicit example of "small" 4-manifolds with boundary that admits exotic diffeomorphism by following the strategy of the Proof of Theorem 1.4. We start with a 4-manifold $W$ which is obtained by attaching a 2-handle $h$ on $B^4$ along an unknot with framing $+1$. Now we will attach a pair of canceling 1- and 2-handle such that the 2-handle of the canceling pair is linked positively with $h$ algebraically thrice, let this be called $W$ and $X = W \# N$ where $b_2(X) = 4$. Now if we apply the cork-twist on $W$, that process will increase the maximum Thurston–Bennequin number of $h$ by 3 and thus the resultant manifold $W'$ will have a Stein structure. Note that $W \# N$ is diffeomorphic to $W' \# N$, and thus by the previous proof, we can construct an exotic diffeomorphism on $X$.

## 7 Exotic embeddings of 3-manifolds in 4-manifolds

Before introducing the results on exotic embeddings, we first review a relation between generalized Smale conjecture and exotic embeddings.

### 7.1 Generalized Smale conjecture and exotic embeddings

Let $Y$ be a closed 3-manifold with the Riemann metric $g$ whose sectional curvature is $\pm 1$. The generalized Smale conjecture says that the inclusion $\mathrm{Isom}(Y, g) \to \mathrm{Diff}(Y)$ is a homotopy equivalence, where $\mathrm{Isom}(Y, g)$ denotes the group of isometries on $(Y, g)$. Some examples of $(Y, g)$ which satisfy this property are known and some examples of $(Y, g)$ which do not satisfy it are known. For example, the hyperbolic 3-manifolds satisfy the generalized Smale conjecture [24]. We use some results related to the generalized Smale conjecture. However, we do not need to restrict ourselves to a Riemannian metric with sectional curvature $\pm 1$ for our purpose. Thus, we consider a closed Riemannian 3-manifold more generally in this paper.

**Definition 7.1** We say a Riemannian 3-manifold $(Y, g)$ is *admissible*, if the cokernel of the induced map on

$$\pi_0(\mathrm{Isom}^+(Y, g)) \to \pi_0(\mathrm{Diff}^+(Y))$$

is trivial, where $\mathrm{Isom}^+(Y, g)$ and $\mathrm{Diff}^+(Y)$ are the groups of orientation preserving isometries and diffeomorphisms of $(Y, g)$ and $Y$ respectively.

Our techniques to detect exotic embeddings can be applied for a Riemannian 3-manifold $(Y, g)$ with finite cokernel of $\pi_0(\mathrm{Isom}^+(Y, g)) \to \pi_0(\mathrm{Diff}^+(Y))$.

We use the following lemma for embeddings of $Y$ into a 4-manifold $X$.

**Lemma 7.2** *Suppose $(Y, g)$ is an admissible 3-manifold. Let $i : Y \to X$ be a smooth embedding and $f$ an orientation-preserving self-diffeomorphism of $X$ satisfying that $f(i(Y)) = i(Y)$. Then, for every $n \in \mathbb{Z}$, we can deform $f^n$ by a smooth isotopy of diffeomorphisms of $X$ fixing $i(Y)$ setwise so that*

$$f^n|_{i(Y)} \in \mathrm{Isom}(i(Y), i_*g).$$

***Proof*** We regard $f$ as a self-diffeomorphism of $i(Y)$. Note that $f|_{i(Y)}$ is orientation preserving. Since by the assumption on $(Y, g)$, the map $\pi_0(\mathrm{Isom}^+(i(Y), i_*g)) \to \pi_0(\mathrm{Diff}^+(i(Y)))$ is sujective, and so the diffeomorphism $f^n$ lies in the image of the map $\pi_0(\mathrm{Isom}^+(i(Y), i_*g)) \to \pi_0(\mathrm{Diff}^+(i(Y)))$. By isotopy extension lemma, we complete the proof. □

**Lemma 7.3** *Let $(Y, g)$ be one of the following 3-manifolds:*

(i) *elliptic 3-manifolds and*
(ii) *hyperbolic 3-manifolds.*

*Then $(Y, g)$ is admissible.*

***Proof*** For elliptic 3-manifolds, the $\pi_0$-Smale conjecture is proven by combining several works, for more details see [27, Theorem 1.2.1]. Also, the Smale conjecture for hyperbolic 3-manifolds is solved by Gabai [24]. □

In order to prove several vanishing results, we also need the following property:

**Lemma 7.4** *Let $(Y, g)$ be one of the following geometric 3-manifolds:*

(i) *3-manifolds having positive scalar curvature metric and*
(ii) *the hyperbolic three-manifolds labeled by*

$$0, 2, 3, 8, 12, \ldots, 16, 22, 25, 28, \ldots, 33, 39, 40, 42, 44, 46, 49$$

*in the Hodgson–Weeks census which correspond to 3-manifolds in* [41, Table 1].

*Then, for any Spin$^c$-structure $\mathfrak{t}$ on $Y$, there is no irreducible Seiberg–Witten solution to $(Y, \mathfrak{t}, g)$, i.e. $Y$ is a strong L-space.*

***Proof*** Since 3-manifolds listed in (i) have positive scalar curvature, by the Weitzenböck formula, we see that there is no irreducible Seiberg–Witten solution to $(Y, \mathfrak{t}, g)$. For hyperbolic 3-manifolds in (iv), [41, Theorem 1.1] implies the conclusion. □

### 7.2 Results on exotic embeddings

In this section, we prove Theorem 1.16.

Let $Y$ be a closed, oriented, connected 3-manifold, and let $X$ be a smooth 4-manifold possibly with boundary. We first construct a 4-manifold for a given 3-manifold.

**Lemma 7.5** *Given a closed, connected, oriented 3-manifold $Y$, there exists a closed simply-connected 4-manifold $X$ such that $X = X_1 \cup_Y X_2$ with $b_2^+(X_i) > 1$ for $i = 1, 2$. Moreover, we can construct $X$ in such a way that there exists a diffeomorphism $f : X \to X$ which is topologically isotopic to the identity but $FSW(X, \mathfrak{s}, f^n) \neq 0$ for every $n \in \mathbb{Z}\backslash\{0\}$ and for some Spin$^c$ structure $\mathfrak{s}$ on $X$, and thus not smoothly isotopic to the identity.*

***Proof*** We will follow the strategy of Proof of the Theorem 1.4, where we showed that given a 3-manifold $Y$ we can construct a compact simply connected 4-manifold $W\#N$ with boundary $Y'$ (where there is a ribbon homology cobordism from $Y$ to $Y'$) and there exists a self-diffeomorphism $f : W\#N \to W\#N$ which is topologically isotopic to the identity rel to the boundary. By construction, $Y$ is smoothly embedded in $W$ and $Y$ bounds a submanifold $W_1$ with $b_2^+(W_1) > 1$. Now by attaching a simply connected symplectic cap on $(Y', \xi')$ with $b_2^+ > 1$ (existence of such caps are shown in [18]) we can get our desired simply-connected 4-manifold $X$. Let $\mathfrak{s}$ be the Spin$^c$ structure on $X$ obtained as the connected sum of the canonical Spin$^c$ structure and the Spin$^c$ structure $\mathfrak{t}$ on $N$ considered in Sect. 5. Also, we can extend $f$ on the symplectic cap as the identity and get our desired diffeomorphism $f$. Now $FSW(X, \mathfrak{s}, f^n) \neq 0$ for $n \neq 0$ follows from [11, Corollary 9.6, Proof of Theorem 9.7]. □

***Proof of Theorem 1.16*** Let $Y$ be a hyperbolic 3-manifold listed in Theorem 1.16 and let $X$ and $f$ be a corresponding 4-manifold and diffeomorphism constructed in Lemma 7.5. For $n \in \mathbb{Z}$, define smooth embeddings $i_n : Y \to X$ by $i_n(y) = f^n(y)$ for $y \in Y$. Since $f : X \to X$ is topologically isotopic to the identity, we see that all the embeddings $i_n$'s are topologically isotopic to each other. It is now enough to prove that the image of $i_0$ is not smoothly isotopic to the image of $i_n$ for every $n \neq 0$. If the image of $i_0$ and the image of $i_n$ are smoothly isotopic, then we can further deform $f^n$ by smooth isotopy so that $f^n(Y) = i_n(Y) = Y$. Moreover, using Lemma 7.2, we may also assume $f_n^{-1}|_Y$ is isometry with respect to the metric $g$ considered in Lemma 7.4. Moreover, there is no irreducible solution to the Seiberg–Witten equation with respect to $(Y, \mathfrak{t}, g)$ from Lemma 7.4. But then the vanishing theorem Theorem 4.6 implies that $FSW(X, f^n) = 0$ which is a contradiction. For a connected sum of elliptic 3-manifolds, we use Theorem 4.7 instead of Theorem 4.6. □

***Remark 7.6*** The proof of Theorem 1.16 is a constructive proof. One thing that we are not sure about is how to control the second Betti number of $X$. So one may ask the following question.

**Question 7.7** *Given an oriented, connected 3-manifold $Y$, what would be the small second Betti number for $X$ such that $Y$ admits an exotic embedding in $X$.*

More generally, we can find exotic embeddings when the family gauge theoretic invariants do not vanish:

**Theorem 7.8** *Let $(X, \mathfrak{s})$ be a compact simply-connected Spin$^c$ 4-manifold with or without boundary. If $\partial X \neq \emptyset$, we equip $\partial X$ a contact structure $\xi$. Let $f : X \to X$ be a self-diffeomorphism which is the identity on $\partial$ if $\partial X \neq \emptyset$. Suppose that $FSW(X, \mathfrak{s}, f) \neq 0$ if $\partial X = \emptyset$, and that $FKM(X, \mathfrak{s}, \xi, f) \neq 0$ if $\partial X \neq \emptyset$. Let $Y$ be one of the following 3-manifolds:*

(i) *the connected sum of elliptic 3-manifolds, and*
(ii) *the hyperbolic three-manifolds labelled by*

$$0, 2, 3, 8, 12, \ldots, 16, 22, 25, 28, \ldots, 33, 39, 40, 42, 44, 46, 49$$

*in the Hodgson–Weeks census which correspond to 3-manifolds in [41, Table 1].*

(1) *If $\partial X = \emptyset$ and $X$ has the decomposition $X = X_1 \cup_Y X_2$ such that $b_2^+(X_i) > 1$ for $i = 1, 2$, where $X_1$ and $X_2$ are compact 4-manifold with boundary $Y$ and $-Y$ respectively. Then there exist infinitely many embedded 3-manifolds $\{f^n(Y)\}_{n\in\mathbb{Z}}$ that are mutually not smoothly isotopic.*
(2) *If $\partial X \neq \emptyset$ and $X$ has the decomposition $X = X_1 \cup_Y X_2$ such that $b_2^+(X_2) > 1$, where $X_1$ and $X_2$ are compact 4-manifold with boundary $(\partial X) \sqcup Y$ and $-Y$ respectively. Then there exist infinitely many embedded 3-manifolds $\{f^n(Y)\}_{n\in\mathbb{Z}}$ that are mutually not smoothly isotopic.*

***Proof*** When $\partial X = \emptyset$, the result follows from the proof of Theorem 1.16. When $\partial X \neq \emptyset$, we just use the vanishing result Theorem 4.8 on the families Kronheimer–Mrokwa's invariant instead. □

***Remark 7.9*** Notice that, given any closed, connected 3-manifold $Y$, we can always find a closed simply-connected 4-manifold $X$ where $Y$ is smoothly embedded and a self-diffeomorphism $f : X \to X$ which is topologically isotopic but not smoothly. So it is very natural to think that the set $\{f^n(Y)\}_{n\in\mathbb{Z}}$ contains all exotically embedded pairs of $Y$, i.e. topologically isotopic as a pair but not smoothly. However, we cannot conclude that at this point because our vanishing result doesn't hold for all 3-manifolds. So we can ask the following question:

**Question 7.10** *Given a closed, connected 3-manifold $Y$ how does one construct a closed 4-manifold $X$ such that there exists a pair of smooth embeddings $i_1, i_2 : Y \to X$ that are topologically isotopic but not smoothly?*

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## Declarations

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.



## Appendix A Excision for determinant line bundles

In this section, we explain the excision principle that is used to give signs to variants of Kronheimer–Mrowka's invariant for 4-manifolds with contact boundary. This argument is well-known for experts and essentially done in Appendix B of [14].

For $i = 1, 2$ let $X_i$ be a Riemannian 4-manifold and $A_i, B_i$ be codimension 0 submanifold of $X_i$. Here we assume $X_1$ and $X_2$ are closed for simplicity, but this assumption is not essential. For example, we can apply similar arguments to manifolds with conical ends under suitable Sobolev completion. Assume $A_i \cap B_i \subset X_i$ is a compact codimension-0 submanifold and also an isometry between $A_1 \cap B_1$ and $A_2 \cap B_2$ is fixed. We will identify them by this isometry. For $i = 1, 2$, suppose we are given vector bundles $E_i$, $F_i$ on $X_i$ and elliptic differential operators of order $l \in \mathbb{Z}_{\geq 1}$

$$D_i : \Gamma(X_i; E_i) \to \Gamma(X_i; F_i)$$

which are identical on $A_1 \cap B_1 = A_2 \cap B_2$.

Using the identification given above, we form Riemannian 4-manifolds $\widetilde{X}_1 = A_1 \cup B_2$, $\widetilde{X}_2 = A_2 \cup B_1$ and vector bundles $\widetilde{E}_1 = E_1|_{A_1} \cup E_2|_{B_2}$, $\widetilde{E}_2 = E_2|_{A_1} \cup E_1|_{B_1}$, $\widetilde{F}_1 = F_1|_{A_1} \cup F_2|_{B_2}$ $\widetilde{F}_2 = F_2|_{A_1} \cup F_1|_{B_1}$.

Define elliptic operator

$$\widetilde{D}_1 = \begin{cases} D_1 & \text{on } A_1 \\ D_2 & \text{on } B_2 \end{cases}$$

$$\widetilde{D}_2 = \begin{cases} D_2 & \text{on } A_2 \\ D_1 & \text{on } B_1 \end{cases}$$

$D_1, D_2, \widetilde{D}_1, \widetilde{D}_2$ define Fredholm operators under Sobolev completions

$$L^2_{k+l} \to L^2_k$$

for $k \in \mathbb{R}$. In general, for a Fredholm operator $D$, we will define the 1-dimensional real vector space $\det D$ by

$$\det D = \Lambda^{\max} \operatorname{Ker} D \otimes \Lambda^{\max} \operatorname{Cok} D^*.$$

**Theorem A.1** *We can associate a linear isomorphism of 1-dimensional real vector space*

$$\det D_1 \otimes \det D_2 \to \det \widetilde{D}_1 \otimes \det \widetilde{D}_2, \tag{28}$$

*which is independent of data used in the construction up to homotopy.*

As remarked after the proof, this can be easily adapted to the case with conical ends as considered in this paper. Note that in order to fix a sign of the unparametrized Kronheimer–Mrowka invariant, considering families of operators, since it is enough to give an orientation of one fiber of the determinant line bundle.

***Proof*** Choose square roots of partition of unity

$$\phi_1^2 + \psi_1^2 = 1$$
$$\phi_2^2 + \psi_2^2 = 1$$

subordinate to $(A_1, B_1)$ and $(A_2, B_2)$ such that

$$\phi_1 = \phi_2 \quad \text{and} \quad \psi_1 = \psi_2$$

on $A_1 \cap B_1 = A_2 \cap B_2$. Define

$$\Phi : \Gamma(X_1; E_1) \oplus \Gamma(X_2; E_2) \to \Gamma(\widetilde{X}_1; \widetilde{E}_1) \oplus \Gamma(\widetilde{X}_2; \widetilde{E}_2)$$

and

$$\Psi : \Gamma(\widetilde{X}_1; \widetilde{F}_1) \oplus \Gamma(\widetilde{X}_2; \widetilde{F}_2) \to \Gamma(X_1; F_1) \oplus \Gamma(X_2; F_2)$$

by

$$\Phi = \begin{bmatrix} \phi_1 & \psi_2 \\ -\psi_1 & \phi_2 \end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} \phi_1 & -\psi_1 \\ \psi_2 & \phi_2. \end{bmatrix}$$

Using the fact that $\psi_1\phi_1 = \psi_2\phi_2$ holds on the whole manifold, we can see $\Phi$ and $\Psi$ are inverse of each other (See [14]). Set $D = D_1 \oplus D_2$ and $\widetilde{D} = \widetilde{D}_1 \oplus \widetilde{D}_2$. Then we

have

$$\det(D) = \det(D_1) \otimes \det(D_2)$$

and

$$\det(\widetilde{D}) = \det(\widetilde{D}_1) \otimes \det(\widetilde{D}_2).$$

We have

$$\begin{aligned}
&\Psi \widetilde{D} \Phi \\
&= \begin{bmatrix} \phi_1 & -\psi_1 \\ \psi_2 & \phi_2 \end{bmatrix} \begin{bmatrix} \widetilde{D}_1 & \\ & \widetilde{D}_2 \end{bmatrix} \begin{bmatrix} \phi_1 & \psi_2 \\ -\psi_1 & \phi_2 \end{bmatrix} \\
&= \begin{bmatrix} \phi_1 \circ D_1 \circ \phi_1 + \psi_1 \circ D_1 \circ \psi_1 & \phi_1[D_2, \psi_2] - \psi_1[D_2, \phi_2] \\ \psi_2[D_1, \phi_1] - \phi_2[D_1, \psi_1] & \psi_2 \circ D_2 \circ \psi_2 + \phi_2 \circ D_2 \circ \phi_2 \end{bmatrix}.
\end{aligned}$$

Here, we used obvious relations

$$\begin{aligned}
\widetilde{D}_1 \circ \phi_1 = D_1 \circ \phi_1, \quad \widetilde{D}_2 \circ \phi_2 = D_2 \circ \phi_2 \\
\widetilde{D}_2 \circ \psi_1 = D_1 \circ \psi_1, \quad \widetilde{D}_1 \circ \psi_2 = D_2 \circ \psi_2.
\end{aligned}$$

and

$$\psi_1 \phi_1 = \phi_2 \psi_1 \quad \phi_1 \psi_2 = \psi_1 \phi_2.$$

On the other hand, we have

$$D_i = D_i \circ (\phi_i^2 + \psi_i^2) = ([D_i, \phi_i] + \phi_i \circ D_i) \circ \phi_i + ([D_i, \psi_i] + \psi_i \circ D_i) \circ \psi_i$$

for $i = 1, 2$. Thus

$$K := \Psi \widetilde{D} \Phi - D : L^2_{k+l}(X) \to L^2_k(X)$$

is calculated as

$$K = \begin{bmatrix} -[D_1, \phi_1] \circ \phi_1 - [D_1, \psi_1] \circ \psi_1 & \phi_1[D_2, \psi_2] - \psi_1[D_2, \phi_2] \\ \psi_2[D_1, \phi_1] - \phi_2[D_1, \psi_1] & -[D_2, \phi_2] \circ \phi_2 - [D_2, \psi_2] \circ \psi_2. \end{bmatrix}$$

The order of this operator is strictly smaller than $l$, so

$$K : L^2_{k+l}(X) \to L^2_k(X)$$

is a compact operator. Thus the family of operators

$$\{D_t = D + tK\}_{t \in [0,1]}$$

gives a desired isomorphism between $\det(D)$ and $\det(\widetilde{D})$. □

Note that each entry of $K$ is supported on $(A_1 \cap B_1) \times (A_2 \cap B_2)$, so the same conclusion holds even if $X_1$ and $X_2$ have conical ends as considered in this paper, as long as suitable Sobolev completions are used and $A_1 \cap B_1$, $A_2 \cap B_2$ are relatively compact.

## Appendix B Blow up formula for Kronheimer–Mrowka's invariant

**Definition B.1** Fix an element of $\Lambda(W, \mathfrak{s}, \xi)$. For a pair $(W, \xi)$ and a fixed reference Spin$^c$ structure $\mathfrak{s}_0 \in \mathrm{Spin}^c(W, \xi)$, we define two functions

$$KM(W, \mathfrak{s}_0, \xi) := \sum_{e \in H^2(W, \partial W; \mathbb{Z})} \mathfrak{m}(W, \mathfrak{s}_0 + e, \xi) \exp(\langle 2e, - \rangle) : H_2(W, \partial W; \mathbb{R}) \to \mathbb{R}.$$

and

$$\widetilde{KM}(W, \xi)(\nu) := e^{\frac{i}{2\pi} \int_\nu F_{A_0^t}} KM(W, \mathfrak{s}_0, \xi)(\nu) : H_2(W, \partial W; \mathbb{R}) \to \mathbb{R}.$$

Here, $A_0$ is a Spin$^c$ connection for $\mathfrak{s}_0$ extending the canonical Spin$^c$ connection on the conical end.

Note that $KM(W, \mathfrak{s}_0, \xi)$ depends on the fixed Spin$^c$ structure $\mathfrak{s}_0$. On the other hand, we can see the following:

**Lemma B.2** *The function $\widetilde{KM}(W, \xi)$ does not depends on a fixed Spin$^c$ structure $\mathfrak{s}_0$.*

***Proof*** Indeed, for $l \in H^2(X; \mathbb{Z})$, $\mathfrak{s}_0' := \mathfrak{s}_0 + l$, we have

$$2l = c_1(\mathfrak{s}_0') - c_1(\mathfrak{s}_0) = \frac{i}{2\pi} \int_\nu (F_{A_0''} - F_{A_0^t}),$$

so

$$
\begin{aligned}
&e^{\frac{i}{2\pi}\int_\nu F_{A_0''^t}} KM(W,\mathfrak{s}_0',\xi)(\nu)\\
&\quad= e^{\frac{i}{2\pi}\int_\nu F_{A_0''^t}} \sum_{e'\in H^2(W,\partial W;\mathbb{Z})} \mathfrak{m}(W,\mathfrak{s}_0'+e',\xi)\exp(\langle 2e',\nu\rangle)\\
&\quad= e^{\frac{i}{2\pi}\int_\nu F_{A_0''^t}} \sum_{e'\in H^2(W,\partial W;\mathbb{Z})} \mathfrak{m}(W,\mathfrak{s}_0+l+e',\xi,o)\exp(\langle 2e',\nu\rangle)\\
&\quad= e^{\frac{i}{2\pi}\int_\nu F_{A_0''^t}} \sum_{e\in H^2(W,\partial W;\mathbb{Z})} \mathfrak{m}(W,\mathfrak{s}_0+e,\xi,o)\exp(\langle 2(e-l),\nu\rangle)\\
&\quad= e^{\frac{i}{2\pi}\int_\nu F_{A_0''^t}} \sum_{e\in H^2(W,\partial W;\mathbb{Z})} \mathfrak{m}(W,\mathfrak{s}_0+e,\xi,o)\exp(\langle 2e,\nu\rangle)\exp\left(-\frac{i}{2\pi}\int_\nu F_{A_0''^t}-F_{A_0^t}\right)\\
&\quad= e^{\frac{i}{2\pi}\int_\nu F_{A_0^t}} \sum_{e\in H^2(W,\partial W;\mathbb{Z})} \mathfrak{m}(W,\mathfrak{s}_0+e,\xi,o)\exp(\langle 2e,\nu\rangle)\\
&\quad= e^{\frac{i}{2\pi}\int_\nu F_{A_0^t}} \sum_{e\in H^2(W,\partial W;\mathbb{Z})} KM(W,\mathfrak{s}_0,\xi)(\nu)
\end{aligned}
$$

Here we changed the variables by

$$e = l + e'.$$

□

We establish the blow-up formula for Kronheimer–Mrowka's invariant for 4-manifold with boundary, using the pairing formula and the formal $(3+1)$-TQFT property of the monopole Floer homology. As a partial result, [28, Theorem 1.3] combined with a computation of the Bauer–Furuta invariant for $-\mathbb{C}P^2$ gives the blow up formula for Kronheimer–Mrowka's invariant for 4-manifold with boundary under the condition $b_3 = 0$. In this section, we remove the condition $b_3 = 0$ by following the discussion given in Section 39.3 of [38].

**Theorem B.3** *Let $X$ be a compact oriented 4-manifold equipped with a contact structure $\xi$ on the boundary $Y = \partial X$. Denote by the blow up of $X$ at an interior point by*

$$\hat{X} = X\#(-\mathbb{C}P^2)$$

*and the exceptional sphere by $E$. Then for $\nu \in H_2(X,\partial X;\mathbb{R})$ and $\lambda \in \mathbb{R}$,*

$$\widetilde{KM}(\hat{X},\xi)(\nu+\lambda E) = 2\cosh(\lambda)\widetilde{KM}(X,\xi)(\nu)$$

*holds.*

***Proof*** Let us denote by

$$\overset{\circ}{X} : S^3 \to Y$$

and

$$\overset{\circ}{\hat{X}} : S^3 \to Y$$

the cobordism obtained by removing small open disk from $X$ and $\hat{X}$ respectively and let

$$N : S^3 \to S^3$$

be the cobordism obtained by removing two small open disks from $-\mathbb{C}P^2$. By the pairing formula which is proves in [29] and the composition law Proposition 26.1.2 of [38], we have

$$\begin{aligned}\widetilde{KM}(\hat{X}, \xi)(h + \lambda E) &= \langle \widehat{HM}(\overset{\circ}{\hat{X}}, \nu + \lambda E)(1), \Psi_{\partial\nu}(\xi)\rangle \\ &= \langle \widehat{HM}(N, \lambda E) \circ \widehat{HM}(\overset{\circ}{X}, \nu)(1), \Psi_{\partial\nu}(\xi)\rangle,\end{aligned}$$

where we denote the local system $\Gamma_\nu$ just by $\nu$. Now, as explained in the proof of Theorem 39.3.2 of [38], we have

$$\widehat{HM}(N, \lambda E) = \sum_{m\in\mathbb{Z}} U^{m(m+1)/2} e^{-(2m+1)\lambda},$$

which can in fact be expressed using the Jacobi eta function. Kronheimer–Mrowka's invariant is defined to be zero when formal dimension is non-zero, so all terms including higher powers of $U$ disappear. Thus we have

$$\begin{aligned}\widetilde{KM}(\hat{X}, \xi)(\nu + \lambda E) &= (e^{\lambda} + e^{-\lambda})\langle \widehat{HM}(\overset{\circ}{X}, \nu)(1), \Psi_{\partial\nu}(\xi)\rangle \\ &= 2\cosh(\lambda)\widetilde{KM}(X, \xi)(\nu).\end{aligned}$$

□

In particular, if Kronheimer–Mrowka's invariant is non-trivial for some element of $\mathrm{Spin}^c(X, \xi)$, then Kronheimer–Mrowka's invariant is non-trivial for some element of $\mathrm{Spin}^c(\hat{X}, \xi)$.

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