FLOER HOMOLOGY BEYOND BORDERS

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Abstract. Bordered Floer homology is an invariant for 3-manifolds with boundary, defined by the authors in 2008. It extends the Heegaard Floer homology of closed 3-manifolds, defined in earlier work of Zoltán Szabó and the second author. In addition to its conceptual interest, bordered Floer homology also provides powerful computational tools. This survey outlines the theory, focusing on recent developments and applications.

The goal of this note is to survey some developments in bordered Floer homology, an extension of Heegaard Floer homology. We start with a biased outline of Heegaard Floer homology, focusing somewhat on the aspects relevant to the extension. We then briefly outline the structure of bordered Floer homology, before turning to a discussion of its use for computations, extensions of it, and some recent applications to 4-dimensional topology and its 3-dimensional shadows.

1. Heegaard Floer homology

Heegaard Floer homology [135, 138] is an invariant of 3- and 4-dimensional manifolds defined by Zoltán Szabó and the second author via methods from symplectic geometry (specifically, Lagrangian Floer homology [40]). The construction was inspired by gauge theory, especially the Seiberg-Witten invariants [159], which also give a package of invariants with similar properties [88]. Thanks to work of Çağatay Kutluhan, Yi-Jen Lee, and Clifford H. Taubes [92–96] and Vincent Colin, Paolo Ghiggini, and Ko Honda [21–23], building on earlier work of Michael Hutchings and Taubes [71–73], we now know that, at least for 3-manifolds, these two differently defined invariants agree. The two perspectives have different relative strengths: Seiberg-Witten theory is more directly connected with the differential geometry of the underlying manifold, whereas Heegaard Floer homology is more closely connected to its combinatorial topological properties.

On a formal level, Heegaard Floer homology associates to a 3-manifold Y the chain homotopy type of a chain complex $CF^-(Y)$ over a polynomial algebra in an indeterminant U; in particular, its homology $HF^-(Y)$ is also a 3-manifold invariant. The $U = 0$ specialization of $CF^-(Y)$, denoted $\widehat{CF}(Y)$, and its homology, $\widehat{HF}(Y)$, are also useful invariants. (The invariant $HF^-(Y)$ is analogous to the S^1-equivariant homology of a space, while $\widehat{HF}(Y)$ is like its non-equivariant homology; cf. [27, 85, 88, 120].)

A striking result of Sucharit Sarkar and Jiajun Wang led to a combinatorial scheme for computing $\widehat{HF}(Y)$, via so-called *nice* auxiliary data [146], a notable achievement given the many 3-dimensional applications of $\widehat{HF}(Y)$. However, the construction of invariants of smooth, closed 4-dimensional manifolds requires the use the U-unspecialized version $HF^-(Y)$ [138]. Specifically, a cobordism between two 3-manifolds Y_1 and Y_2 gives rise to a map between their respective Heegaard Floer homology groups. The closed, smooth 4-manifold invariant is constructed using the interaction between these maps with the U-module structure of $HF^-(Y)$. Although the 4-manifold invariant can now be, in principle, described combinatorially [123], the algorithm for computing it is still too unwieldy to be implemented in practice.

1.1. Knot invariants from Heegaard Floer homology.

In 2003, Jacob Rasmussen and, independently, Szabó and the second author constructed a version of Heegaard Floer homology for knots in S^3. The resulting *knot Floer homology* is a bigraded abelian group whose graded Euler characteristic is the Alexander polynomial. This invariant contains much topological information about the underlying knot, including its Seifert genus [133] and whether its complement fibers over the circle [48, 128]; see also [75].

Building on Sarkar's earlier work, Ciprian Manolescu, Sarkar, and the second author gave a combinatorial construction of knot Floer homology, now called *grid homology* [122, 124, 130]. For a knot with n crossings, grid homology can be computed as the homology groups of a chain complex with roughly $n!$ generators. As such, it can be computed explicitly for knots with ≤ 16 or so crossings [8, 13].

Heegaard Floer homology can also be used to construct link invariants in a different manner. Given a link $L \subset S^3$ there is a closed 3-manifold, the *double cover of* S^3 *branched along* L, which is the unique

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3-manifold $\Sigma(L)$ that admits a map $f\colon \Sigma(L) \to S^3$ which is a 2-to-1 covering space away from L and is modeled on $z \mapsto z^2$ in the normal direction to L. Thus, given a closed 3-manifold invariant, applying that invariant to $\Sigma(L)$ gives a link invariant. In the case of $\widehat{HF}$, the link invariant $\widehat{HF}(\Sigma(L))$ is surprisingly similar to an invariant defined by Mikhail Khovanov, inspired by the Jones polynomial and constructions in representation theory [79]. For an unlink, the (reduced) Khovanov homology and the Heegaard Floer homology of its branched double cover agree. More strikingly, both $\widehat{HF}(\Sigma(L))$ and the Khovanov homology of L satisfy an exact triangle when one resolves a crossing in the two possible ways; for Heegaard Floer homology, this is a special case of the *surgery exact triangle* [134], parallel to one envisaged by Andreas Floer in gauge theory [18, 41]. These properties lead to a precise relationship between $\widehat{HF}(\Sigma(L))$ and Khovanov homology: for every link diagram, there is a spectral sequence whose E_2 term is the (reduced) Khovanov homology of L and which converges to $\widehat{HF}(\Sigma(L))$ (both with mod-2 coefficients) [137]. By now we know that there are a number of other spectral sequences from Khovanov homology to various Floer-homological invariants: sutured Heegaard Floer homology [52], instanton Floer homology [90], monopole Floer homology [16], and knot Floer homology [31]. Because of the topological content of these latter invariants, the spectral sequences have interesting consequences for Khovanov homology.

1.2. **Contact geometry and Heegaard Floer homology.** Gauge theory is closely tied with the symplectic geometry of the underlying manifold. According to a celebrated theorem of Donaldson's [26], his smooth 4-manifold invariant is non-zero for a Kähler manifold. Taubes [150] proved the analogous nonvanishing theorem for Seiberg-Witten invariants of symplectic manifolds. This has a 3-dimensional counterpart: Kronheimer and Mrowka [86,91] constructed an invariant for contact structures over 3-manifolds, which takes values in its Seiberg-Witten Floer homology; moreover, this contact invariant is non-trivial when the contact structure is fillable, in a suitable sense, by a symplectic manifold. These results, combined with work of Yakov Eliashberg and William Thurston [36] relating contact structures and foliations, form a bridge between gauge theory and fundamental topological constructions developed by David Gabai [45, 46]. This bridge was exploited by Kronheimer and Mrowka in their celebrated proof that all knots in S^3 have "Property P" [87].

An analogous bridge between Heegaard Floer homology and contact geometry is built upon the work of Emannuel Giroux [49], who reformulated contact structures over Y as certain equivalence classes of open book decompositions of Y. Using Giroux's correspondence, Szabó and the second author were able to define an invariant for contact structures over Y, analogous to the Kronheimer-Mrowka contact invariant, but taking values in $\widehat{HF}(-Y)$ [136] (where $-Y$ denotes orientation reverse). By work of Taubes [151–155] and Colin-Ghiggini-Honda [21–23], this contact invariant is a refinement of Kronheimer and Mrowka's. (See also [10].)

This contact invariant forms an important technical tool within the theory (used, e.g., to prove that $\widehat{HFK}$ detects the Seifert genus [133]); and it is also a vehicle for studying contact structures in their own right (e.g., [115–117]).

2. Bordered Floer homology

Bordered Floer homology is an invariant for 3-manifolds with boundary [111]. The basic structure is easy to describe. To a surface F, it associates a differential graded algebra $A(F)$. Given an F-bordered 3-manifold, that is, an oriented 3-manifold Y and an orientation-preserving diffeomorphism $\phi\colon F \to \partial Y$, the theory associates a right A_∞-module $\widehat{CFA}(Y,\phi)_{A(F)}$ over the algebra $A(F)$ and a left *dg* module of a particular kind (a twisted complex or, in the internal language of the subject, *type D structure*) ${}^{A(-F)}\widehat{CFD}(Y,\phi)$ over the algebra associated to the orientation-reversed surface. Both $\widehat{CFA}$ and $\widehat{CFD}$ depend on some auxiliary choices, but up to homotopy equivalence are independent of those choices. Given two bordered 3-manifolds $(Y_1, \phi_1\colon F \to \partial Y_1)$ and $(Y_2, \phi_2\colon -F \to \partial Y_2)$, we can recover the Heegaard Floer invariant $\widehat{CF}(Y_1 \cup_{\phi_2\circ\phi_1^{-1}} Y_2)$ as either a tensor product

$$\widehat{CFA}(Y_1)_{A(F)} \boxtimes {}^{A(F)}\widehat{CFD}(Y_2)$$

or as the complex of A_∞-module morphisms

$$\mathrm{Mor}^{A(F)}(\widehat{CFA}(-Y_2), \widehat{CFA}(Y_1))$$

or type D structure morphisms

$$\mathrm{Mor}_{A(F)}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$$

[7, 106, 111]. We call these results *pairing theorems* (to distinguish them from the gluing theorems used in the analysis underlying the subject).

The analytic proof of the pairing theorem involves a neck-stretching argument, inspired by the proofs of product formulas for gauge-theoretic invariants [28, 88]. (See also [17,35].) We start by taking a Heegaard diagram which is adapted to the splitting $Y_1 \cup_F Y_2$

and degenerating the Heegaard surface along a corresponding circle. The counts of holomorphic disks that constitute the Heegaard Floer differential degenerate into counts of pairs of holomorphic disks on the two sides, satisfying a matching condition along F. So far, the counts are not algebraic: because of the matching condition, high-dimensional moduli spaces on the two sides contribute to the count. The next step is to deform the matching condition. Formally, this deformation can be realized as a cellular approximation to the diagonal in James Stasheff's associahedra [147]. After the deformation, we obtain algebraic counts that correspond to the tensor product description from the pairing theorem.

The dependence of $\widehat{CFA}(Y_1,\psi)$ on the choice of boundary parameterization is governed by certain bimodules [109]. Given a surface diffeomorphism $\phi\colon F\to F$, there is a bimodule $\widehat{CFDA}(\phi)$, with the property that for any bordered 3-manifold $(Y_1,\psi\colon F\to Y_1)$

$$\widehat{CFA}(Y_1,\psi)\boxtimes\widehat{CFDA}(\phi)\simeq\widehat{CFA}(Y_1,\psi\circ\phi). \tag{2.1}$$

Moreover, the tensor product of the bimodules associated to ϕ_1 and ϕ_2 coincides with the bimodule associated to the composite $\phi_2\circ\phi_1$.

The algebra $A(F)$ is defined combinatorially, from a handle decomposition for F. (Different handle decompositions lead to derived equivalent, but non-quasi-isomorphic, algebras [109].) By contrast, $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ are defined by counting J-holomorphic curves. For suitable (*nice*, extending [146]) choices of auxiliary data, these curve counts are combinatorial, and there is also a combinatorial proof of the tensor-product pairing theorem. However, there is no known combinatorial proof of invariance for $\widehat{CFA}$ and $\widehat{CFD}$ along these lines (but see [168]), and nice auxiliary choices result in enormous complexes. So, in practice—and for most of the computational applications below—one often ends up counting J-holomorphic curves, not using nice auxiliary choices.

3. Computations from bordered Floer homology

Every closed 3-manifold Y^3 admits a *Heegaard decomposition* $Y=H_1\cup_F H_2$ into two handlebodies glued together along their boundaries. If we identify each H_i with some standard (bordered) handlebody, then Y is determined by the gluing diffeomorphism, which is a map $\phi\colon F\to F$. For an appropriate choice of standard handlebody, it is easy to compute the bordered invariants $\widehat{CFD}(H_i)$ and $\widehat{CFA}(H_i)$. So, in view of Equation (2.1), to compute $\widehat{HF}(Y)$ for general Y, it suffices to compute suitable bimodules associated to surface diffeomorphisms—or, in fact, for any set of generators of the mapping class group. It turns out that there are particularly simple generators, called *arcslides*, for a groupoid extension of the mapping class group [5,14], for which the bimodules are determined by a few simple curve counts and the structure equation $\partial^2=0$ [108]. A computer implementation of this algorithm [166] (and refinements of it [167]) is practical for many manifolds.

The bordered description arising from factoring mapping classes contains information beyond $\widehat{HF}$ of the underlying 3-manifold. When $Y=\Sigma(L)$ is a branched double cover of a link L in S^3, we can factor its gluing map as a product of Dehn twists along an explicit set of curves which correspond to crossings in a projection of L. The bimodules for these Dehn twists can be expressed as mapping cones between pairs of bimodules, corresponding to the two resolutions of the crossing. This description induces a filtration on $\widehat{CF}(\Sigma(L))$ [107], which can be shown [110] to induce the aforementioned spectral sequence [137] from the Khovanov homology of L to $\widehat{HF}(\Sigma(L))$. Like the arcslide bimodules, these resolution bimodules and the maps between them can be described explicitly, using only a few simple curve counts and the structure equation; this gives a combinatorial formula for the spectral sequence [107]. (Another conjectural combinatorial description of the spectral sequence was given by Szabó [149]; it would be interesting to relate the two.) The higher differentials in the original spectral sequence count holomorphic polygons, and the key step in showing the spectral sequences agree is an extension of the pairing theorem to polygons [110].

3.1. Satellite knots.

Suppose K is a knot in S^3, say, and P is another knot, called the *pattern*, in the solid torus $S^1\times D^2$. Choose a framing for K or, equivalently, an identification of $\partial(S^3\setminus\mathrm{nbd}(K))$ with $\partial(S^1\times D^2)$. Replacing $\mathrm{nbd}(K)$ with the copy of $S^1\times D^2$ containing P using this framing gives a new knot K_P in S^3, called a *satellite knot*.

It is natural to wonder how invariants of K, P, and K_P relate. For the Alexander polynomial, there is a simple formula:

$$\Delta_{K_P}(t)=\Delta_K(t^n)\Delta_P(t)$$

where Δ_P is the Alexander polynomial of $P\subset S^3$, and n is the winding number of the pattern around the solid torus. For knot Floer homology, an answer comes from bordered Floer homology. First, one can express the bordered invariants of the exterior of K in terms of the knot Floer homology of

K [111] (see also [64, 67]). Then, the knot Floer homology of K_P is a tensor product of this bordered invariant with an appropriate module. In fact, Ina Petkova showed that, in a precise sense, this construction lifts the classical formula for the Alexander polynomial [143]. Satellite formulas from bordered Floer homology have been used by many authors for computations and applications; some applications of these will be mentioned below. We also note that there is important earlier work on satellite operations in knot Floer homology by Matthew Hedden [60, 61] and Eaman Eftekhary [32].

4. Extensions of bordered Floer homology

4.1. **Bordered-sutured Floer homology.** Inspired by Gabai's notion of sutured manifold hierarchies [45, 46], András Juhász introduced a different extension [74] of Heegaard Floer homology to 3-manifolds with sutures on their boundary. This invariant can be seen as a natural generalization of the knot Floer homology $\widehat{HFK}$, and its behavior under sutured decompositions can be used to prove or re-prove important properties of knot Floer homology [75]. (See also [89] for an analogous construction in gauge theory.)

Ruman Zarev realized both bordered Floer homology and sutured Floer homology as special cases of a more general theory, bordered-sutured Floer homology [161, 162]. His framework unifies many different objects appearing in bordered Floer homology, and the geometric objects it allows are important for proving properties of the theory, like the formulation of the pairing theorem in terms of morphism spaces described above, or John Etnyre, Shea Vela-Vick, and Zarev's description of HFK^- in terms of $\widehat{HFK}$ [38] (see also [97]). Bordered-sutured Floer homology also gives an invariant of tangles (strong enough, for instance, to detect trivial tangles [4]); see also Section 4.4.

4.2. **Cornered Floer homology.** Since the Seiberg-Witten invariant was shown to admit extensions to 3-manifolds (Heegaard Floer homology, monopole Floer homology) and then 2-manifolds (bordered Floer homology), it is natural to ask if it can be extended further. Christopher Douglas and Manolescu proposed the next step, an extension to closed 1-manifolds, surfaces with boundary, and 3-manifolds with corners, which they called *cornered Floer homology* [30]. The invariant of a circle is a kind of algebra with both horizontal and vertical multiplications, which they called a *sequential 2-algebra*. To a surface with boundary they associate an algebra-module over this (which behaves like an algebra with respect to horizontal multiplication, say, and a module with respect to vertical multiplication), and to a 3-manifold with a corner, a 2-module. Suitable tensor products recover bordered Floer homology. Their construction of the 3-manifold invariants uses Sarkar-Wang's notion of nice diagrams, which makes the construction combinatorial but means they were not able to prove invariance directly. A variant of their construction, with somewhat more complicated structures but where it was possible to prove invariance, was given by Douglas, Manolescu, and the first author [29]. A key idea is that the invariants of a 3-manifold Y with boundary $F_1 \cup_{S^1} F_2$ and a corner at S^1 can be obtained from the bordered invariant of the smoothing of Y by taking a tensor product with an invariant of $[0,1] \times (F_1 \cup_{S^1} F_2)$, viewed as smooth on one side and with a corner on the other. Invariance then follows from the invariance theorem for bordered Floer homology, and the pairing theorem follows from a computation for certain specific tri-modules and the bordered Floer pairing theorem.

Work of Andrew Manion and Raphael Rouquier has revealed surprising connections between the original Douglas-Manolescu construction and representation theory [119].

4.3. **Invariants of contact manifolds.** As mentioned above, a contact structure ξ on a closed 3-manifold Y induces a class $c(\xi) \in \widehat{HF}(-Y)$. It is natural to guess that if $-Y$ is decomposed as $Y_1 \cup_F Y_2$ then there should be classes $c(\xi_1) \in \widehat{CFA}(Y_1)$ and $c(\xi_2) \in \widehat{CFD}(Y_2)$ so that, under the pairing theorem, $c(\xi) = c(\xi_1) \otimes c(\xi_2)$. Akram Alishahi, Viktória Földvári, Kristen Hendricks, Joan Licata, Petkova, and Vera Vértesi [2] showed that this is, in fact the case. Key to their definition is a precise formulation of how ξ and F should interact or, equivalently, what structure ξ_i should induce on the boundary of Y_i. The answer is given by the notion of a *foliated open book*, introduced by Licata and Vértesi [101]. A second ingredient is Honda, William Kazez, and Gordana Matic's definition of $c(\xi)$ [69], which is tied more directly to a Heegaard diagram than the original definition, and a third is the flexibility provided by Zarev's bordered-sutured theory. A key property of $c(\xi)$ is that $c(\xi) = 0$ if ξ is overtwisted (the class of contact structures satisfying an h-principle [34]). The bordered contact invariant leads to a satisfying new proof of this fact: it reduces to a local computation of the contact invariant near an overtwisted disk [3].

There are also other hints of connections between bordered Floer homology and contact topology. For instance, in unpublished work Honda associated a triangulated category to a surface, which he called the *contact category*; objects encode contact structures near a surface and the morphism spaces are generated by bypass attachments. Benjamin Cooper showed that a version of the contact category maps to the category of modules over the bordered-sutured algebras, and at least in special cases this map is an equivalence [24]. Daniel Mathews gave another result along these lines [125]; as he notes, this implies there are A_∞-style operations on the set of contact structures, which can be computed but are not yet understood geometrically [126].

4.4. **Invariants of tangles.** Bordered Floer homology was partly inspired by extensions of Khovanov homology to tangles (by Khovanov [80] and Dror Bar-Natan [11]), and attempts to give an extension of $\widehat{HFK}$ to tangles led to many of the basic definitions in bordered Floer homology [105]. Nonetheless, technical difficulties relating to invariance prevented us from carrying out this construction. Since then, three other extensions of knot Floer homology, in the spirit of bordered Floer homology, have emerged. The first was mentioned above: tangle exteriors are a special case of Zarev's bordered-sutured Floer homology. A second construction was given by Petkova-Vértesi [144]. Like our earlier attempt, they start from a variant of Manolescu-Ozsváth-Sarkar's grid diagrams. In particular, the definition of their invariants is combinatorial and their relationship to knot Floer homology is immediate. By contrast, their proof of invariance uses holomorphic curves. (Indeed, for the most elaborate version of their construction, invariance remains a conjecture: proving it requires overcoming analytic obstacles similar to those for bordered HF^- discussed in Section 5 below.) A third tangle invariant is due to Szabó and the second author. It starts from a standard knot diagram, and the Heegaard diagram it induces, and associates A_∞-bimodules over certain algebras to cups, caps, and crossings. Tensoring these bimodules together, one obtains chain complexes associated to knot diagrams. Two papers [140, 142] give algebraic definitions of these bimodules and prove that the homology groups of the resulting chain complexes are indeed knot invariants. In another paper [141], it is shown that these operations correspond to holomorphic curve counts and, using this, show that the theory indeed recovers knot Floer homology. One remarkable feature of this extension is how efficient it is for computation: while the previous algorithm for computing $\widehat{HFK}$ using grid diagrams is effective only up to 16 crossings or so, this tangle invariant allows computations for many knots of 80 or more crossings [148].

4.5. **Fukaya-categorical invariants.** If F is a torus T^2, the algebra $A(T^2)$ is an explicit, 8-dimensional algebra over $\mathbb{F}_2$ with trivial differential. The invariant of a manifold with torus boundary is a differential module over this algebra. Jonathan Hanselman, Rasmussen, and Liam Watson [57] showed that this invariant is equivalent to a simple, geometric object: an immersed 1-manifold in T^2, perhaps equipped with a local system (over $\mathbb{F}_2$). The tensor product in the pairing theorem turns into taking Floer homology of curves in the torus, an entirely combinatorial construction. The dependence of the invariant on the parametrization of the boundary becomes transparent: if the immersed curve associated to Y is viewed as lying in ∂Y, no parametrization of the boundary is needed. (So, in a sense, the theory becomes borderless.)

While there are reasons one might expect such a result [6, 54, 98], their construction is complicated, surprising—and useful. In particular, in addition to obvious computational applications, it also has theoretical ones. One of the most remarkable is to Cameron Gordon's Cosmetic Surgery Conjecture [51], which states that distinct Dehn surgeries on a knot $K \subset S^3$ never give the same oriented 3-manifold. Although there had been earlier progress on this problem using Heegaard Floer techniques [47, 129, 139, 158, 160], Hanselman's approach provided a powerful new framework for studying this topological problem. Using this theory of immersed curves, Hanselman showed that if r-surgery and s-surgery on K are orientation-preserving homeomorphic then $\{r, s\}$ is either $\{\pm 2\}$ or $\{\pm 1/q\}$ for a particular integer q, as well as a number of further restrictions (e.g., that in the first case the knot has genus 2) [55]. Konstantinos Varvarezos has shown that if one drops the requirement that the homeomorphism be orientation-preserving (in which case there are many known examples of cosmetic surgery, such as [15]), the immersed curve theory still gives obstructions [156].

There are also bordered-style invariants via immersed curves in other settings. Claudius Zibrowius introduced an immersed curve invariant for 4-ended tangles extending knot Floer homology and used it to show that a version of knot Floer homology is invariant under mutation [169], verifying a conjecture formulated by John Baldwin and Adam Levine [9]. Tye Lidman, Allison Moore, and Zibrowius used this invariant to show that so-called L-space knots have no essential Conway spheres [103], verifying a conjecture Lidman and Moore formulated seven years earlier [102]. (See [33, 57, 64] for analogous results for closed 3-manifolds.) In a different direction, Artem Kotelskiy, Watson, and Zibrowius showed that, for

4-ended tangles, Bar-Natan's extension of Khovanov homology to tangles can also be interpreted as an immersed curve in a 4-punctured sphere [83], and this immersed curve in fact agrees [84] with an invariant introduced by Hedden, Christopher Herald, Matthew Hogancamp, and Paul Kirk, inspired by instanton link homology [62] (see also [63, 82]).

5. Bordered HF^-

Extending bordered Floer homology to the HF^- variant of Heegaard Floer homology presents a set of interesting algebraic, geometric, and analytical challenges. To date, most work has focused on the case where the boundary is T^2.

In bordered Floer homology, the operations on the (bordered) modules are defined by counting rigid holomorphic curves with boundary on certain Lagrangian submanifolds. These submanifolds are non-compact; elements of the bordered algebra record the possible asymptotics of curves, while operations on the algebra correspond to (most) codimension-1 degenerations of in the moduli spaces. Passing from $\widehat{HF}$ to HF^- introduces two new complications. First, there are complicated, new codimension-1 degenerations; philosophically, these correspond to non-empty moduli spaces of holomorphic disks with boundary on a single Lagrangian (cf. [43, 44]). Second, there are new possible asymptotics of curves, coming from orbits going off to the boundary. The result is that the algebra $A(T^2)$ must be replaced by a formal 1-parameter deformation of an A_∞-algebra, or a *weighted A_∞-algebra* [113]. The higher operations correspond to the new kinds of degenerations, and the deformation parameter corresponds to the number of orbits. We denote this formal deformation by $\mathcal{A}_-$. Similarly, the module $\widehat{CFA}(Y_1)$ is replaced by a *weighted A_∞-module* $CFA^-(Y_1)$ over $\mathcal{A}_-$ [114].

The module $\widehat{CFD}(Y_2)$ can be defined as

$$\widehat{CFD}(Y_2) = \widehat{CFA}(Y_2) \otimes \widehat{CFDD}(\mathbb{I})$$

where $\widehat{CFDD}(\mathbb{I})$ is a type DD structure (bimodule version of a twisted complex) associated to the identity cobordism of the surface F. For the case of HF^-, the analogous definition has a subtlety: to formulate the notion of a type DD structure over a pair of (weighted) A_∞-algebras requires a well-behaved tensor product of (weighted) A_∞-algebras, and taking one-sided tensor products requires some further variants of this notion. In the unweighted case, the (surprisingly subtle) construction of the tensor product of two A_∞ algebras was given by Samson Saneblidze and Ronald Umble [145]; with some effort, their notion extends to the versions needed for bordered Floer theory [112]. With this algebra in hand, it is easy to construct an appropriate type DD bimodule $CFDD^-(\mathbb{I})$ and define

$$CFD^-(Y_2) = CFA^-(Y_2) \otimes CFDD^-(\mathbb{I}).$$

The next step is to establish a pairing theorem for the type A and the type D modules associated to bordered manifolds. Once again, this pairing theorem can be viewed as a deformation of the diagonal. In the case of $\widehat{HF}$, this deformation took place in the product of associahedra, which is a compactification of a configuration of points in $\mathbb{R}$ relevant to A_∞-algebras; in the case of bordered HF^-, the associahedron is replaced by a suitable compactification of the configuration of points in the interior and boundary of a disk, which we call the *associaplex* [112]. Details of this pairing theorem are forthcoming [104].

A different construction of an extension of HF^- to bordered 3-manifolds with (perhaps several) torus boundary components has been given by Ian Zemke [163, 164], using the link surgery formula introduced by Manolescu and the second author [121]. A module over the algebra Zemke associates to a torus captures, in a succinct way, the data needed for the surgery formula. To construct the module associated to a 3-manifold Y with torus boundary, he presents Y as a Dehn filling of a link complement in S^3. The link surgery formula can be applied further to prove a version of the pairing theorem. At the time of writing, invariance of his modules remains conjectural, but he was already able to use them to give a sought-after proof of a conjecture [164] about the Heegaard Floer homology of plumbed 3-manifolds [127, 132].

A third construction of a bordered extension of HF^- was recently given by Hanselman [56], as an extension of the immersed curve invariants discussed in Section 4.5. In this case, the invariants take the form of immersed curves equipped with bounding cochains (and some additional data). He proves that, up to appropriate equivalence, this data depends only on the bordered 3-manifold (which he thinks of as a 3-manifold with a specified knot in it), and that it determines the HF^- invariant of any Dehn filling of the 3-manifold. At the time of writing, a formula for the behavior when one changes the parameterization of the boundary, a general pairing theorem for this invariant, and its relationship to the earlier immersed curve invariants, remain conjectural, but the underlying techniques are instrumental in his results on the Cosmetic Surgery Conjecture mentioned in Section 4.5.

6. Applications to 4-dimensional topology

6.1. **Knot concordance.** Heegaard Floer homology is a useful topological tool for studying the interplay between knot theory and smooth 4-dimensional topology. One place where this interplay is particularly visible is the study of *knot concordance*. We review this subject briefly; for a more thorough overview, see [118].

Recall that knots K_0 and K_1 are said to be *smoothly concordant* if there is an embedded annulus A in $[0,1]\times\mathbb{R}^3$ which, for $i=0,1$, meets the corresponding boundary component $\{i\}\times\mathbb{R}^3$ along K_i. The set of equivalence classes of knots can be made into an abelian group, so that addition corresponds to connected sum. This group $\mathcal{C}$ is called the *smooth concordance group*, and knots that are smoothly concordant to the unknot are called *smoothly slice knots*.

Surprisingly little is known about this concordance group. Currently, the only known source of torsion in $\mathcal{C}$ is knots which are the same as their mirror images (i.e., are *negatively amphicheiral*). In particular, it is unknown whether $\mathcal{C}$ contains elements of odd order; indeed, at the time of writing, it is conceivable that $\mathcal{C}\cong\mathbb{Z}^\infty\oplus(\mathbb{Z}/2\mathbb{Z})^\infty$.

There is a weaker notion of concordance, where the annulus A, rather than being required to be smoothly embedded, is merely topologically flat (is covered by neighborhoods homeomorphic as pairs to $(\mathbb{R}^4,\mathbb{R}^2)$). There is a corresponding group $\mathcal{C}^{top}$, the *topological concordance group*. Once again, knots that are topologically flatly concordant to the unknot are called *topologically slice knots*.

There is a natural quotient map $\mathcal{C}\to\mathcal{C}^{top}$. The kernel consists of knots which are topologically but not smoothly slice. A deep theorem of Freedman [42] states that any knot with trivial Alexander polynomial is topologically slice. Since the advent of gauge theory [50], it has been known that there are knots which are topologically but not smoothly slice; indeed, using gauge theory, Hisaaki Endo demonstrated an infinite set of linearly independent knots in $\mathcal{C}$ which are topologically slice [37], an especially tangible manifestation of the richness of smooth 4-manifold topology.

Knot Floer homology can also be used to study concordance phenomena. In [68], Jennifer Hom constructs a different infinite set of topologically slice knots that are linearly independent in the smooth concordance group. Her knots are built via satellite constructions (cabling, Whitehead doubling, and connected sums). In [131], András Stipsicz, Szabó, and the second author construct infinitely many homomorphisms from the smooth concordance group to $\mathbb{Z}$, demonstrating an infinite rank free direct summand in the concordance group of topologically slice knots; compare also [25]. The homomorphisms are constructed via knot Floer homology, and the computations of their invariants rest on bordered techniques for computing knot Floer homology of satellite knots.

Another exciting application of satellite computations was given by Levine [99]. Using bordered Floer homology, he showed that the Heegaard Floer τ and ϵ invariants behave in a predictable way under Mazur-pattern satellites. Using this, he showed that most of these satellites do not bound disks in any rational homology ball (resolving [81, Problem 1.45]), and that there are knots in homology 3-spheres that do not bound PL disks in any homology 4-balls (resolving [81, Problem 1.31]).

6.2. **Exotic phenomena.** Suppose that Y_1 and Y_2 are bordered 3-manifolds with boundary parameterized by F and W is a 4-manifold with boundary identified with $(-Y_1)\cup[0,1]\times F\cup Y_2$. It is natural to expect that to this data, bordered Floer homology would associate a map

$$F_W\colon \widehat{CFD}(Y_1)\to\widehat{CFD}(Y_2)$$

(or similarly for $\widehat{CFA}$ or the minus variants); and that these maps would satisfy the obvious analogue of the pairing theorem. While it is straightforward to associate a map F_W to W, independence of the map from the choices in its construction has in general not been verified (but see [19]).

Nonetheless, bordered Floer homology has been used recently for dramatic 4-dimensional applications. For instance, Gary Guth showed that, for any n, there are surfaces $(S_1,\partial S_1)$ and $(S_2,\partial S_2)$ in (B^4,S^3), which are topologically isotopic rel boundary but which remain smoothly non-isotopic even after attaching n 1-handles [53]. (Any pair of surfaces become isotopic after attaching enough 1-handles [12, 70].) His construction starts from an example of Kyle Hayden's, who gave a pair of disks in the 4-ball which are smoothly but not topologically isotopic [58, 59], and uses a lower bound on stabilization distance coming from Heegaard Floer homology, discovered by Juhász and Zemke [76]. Subsequently, Sungkyung Kang gave a pair of contractible 4-manifolds which remain nondiffeomorphic after taking the connected sum with $S^2\times S^2$ [78]. (Several other groundbreaking "one-is-not-enough" results have also appeared recently, using other techniques.)

Both of these results require complicated computations of Heegaard Floer invariants. The disks in Guth's result are cables of Hayden's disks. To compute their Heegaard Floer homology, he notes that if $F\colon K_1 \to K_2$ is a concordance, P is a pattern in the solid torus, and F_P is the satellite of F, then there is some map $F_\#\colon \widehat{CFA}(K_1) \to \widehat{CFA}(K_2)$ so that the Heegaard Floer map $(F_P)_*\colon HFK^-((K_1)_P) \to HFK^-((K_2)_P)$ [1, 165] is induced by

$$F_\# \otimes \mathbb{I}_{\widehat{CFD}(S^1\times D^2, P)}$$

and the pairing theorem. The key to proving this is the pairing theorem for triangles [110]. It is not needed for this computation that $F_\#$ be an invariant of F, just that some map with this property exists. Kang's elaborate computation uses similar ideas, as well as work of Hendricks and the first author [65] and Kang's [77] on a bordered extension of Hendricks-Manolescu's involutive Floer homology [66].

Both Guth's and Kang's results can be seen as resolving relative versions of an old question of Wall's: how many stabilizations (connected sums with $S^2 \times S^2$) are required to make a pair of homeomorphic (or homotopy equivalent), simply-connected, closed 4-manifolds diffeomorphic [157].

Other applications of Heegaard Floer homology to 4-manifolds seem around the corner. For instance, Jesse Cohen recently gave an algorithm for computing the maps on $\widehat{HF}$ associated to cobordisms using bordered Floer homology [20], extending our algorithm for computing $\widehat{HF}$ itself [108]. (The pairing theorem for triangles [110] is again a key step; another is a rigidity result for the modules associated to handlebodies [65].) As noted above, these maps on $\widehat{HF}$ do not provide interesting invariants of closed 4-manifolds, but for 4-manifolds with boundary, which have seen an explosion of interest, they do.

Most recently, Levine, Tye Lidman, and Lisa Piccirillo have used the immersed curve formulation of bordered Heegaard Floer homology to give many exotic 4-manifolds with $b_1 = 1$ [100]. In fact, they show that Fintushel-Stern knot surgery [39] for any nontrivial knot in S^3—including knots with trivial Alexander polynomial—can be used to produce exotic manifolds.

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