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On the Lucky and Displacement Statistics of Stirling Permutations

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Abstract

Stirling permutations are parking functions, and we investigate two parking function statistics in the context of these objects: lucky cars and displacement. Among our results, we consider two extreme cases of the lucky statistic, both of which are enumerated by important integer sequences: extremely lucky Stirling permutations (those with maximally many lucky cars) and extremely unlucky Stirling permutations (those with exactly one lucky car). We show that the number of extremely lucky Stirling permutations of order n is the Catalan number C_n, and the number of extremely unlucky Stirling permutations is $(n-1)!$. We also give a selection of results for luck that lies between these two extremes. Further, we establish that the displacement of all the Stirling permutations of order n is n^2, and we prove several results about displacement composition vectors. We conclude with directions for further study.

1 Introduction

Throughout the paper, we let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$, and we write $[i, j] := \{i, i+1, \ldots, j-1, j\}$ for integers $i \leq j$ and $[n] := [1, n]$. A *parking function of length* n is a tuple $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$ whose rearrangement into weakly increasing order $\alpha^{\uparrow} = (b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq i$ for all $i \in [n]$. For example, the sequences $\alpha_1 = (1, 3, 1, 5, 6, 3)$ and $\alpha_2 = (4, 4, 1, 2, 2, 3, 3, 1)$ are parking functions since their rearrangements are $\alpha_1^{\uparrow} = (1, 1, 3, 3, 5, 6)$ and $\alpha_2^{\uparrow} = (1, 1, 2, 2, 3, 3, 4, 4)$, respectively. However, the sequence $\alpha_3 = (2, 2, 3, 1, 6, 6)$ is not a parking function since $\alpha_3^{\uparrow} = (1, 2, 2, 3, 6, 6)$ and $b_5 = 6 \not\leq 5$. We let PF_n denote the set of parking functions of length n.

We recall another equivalent interpretation of parking functions.

Definition 1. Consider a one-way street with exactly n parking spots and a line-up of n cars, each with a preferred spot that we record as a sequence $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$. Car i tries to park in its preferred spot a_i. If spot a_i is available, then car i parks there; otherwise, car i continues down the street until car i finds the first available spot to park and parks there. If all cars park, then the tuple α is a *parking function*.

Throughout the paper, we use the language from the above definition when referring to parking functions. Parking functions were introduced by Konheim and Weiss [21] in their study of hashing functions, and they established that $|\,\mathrm{PF}_n\,| = (n+1)^{n-1}$. Since their introduction, past work studied this family of combinatorial objects, their statistics, and numerous generalizations [3, 4, 6, 7, 9, 11, 23, 25].

One such work, motivating the current, is that of Gessel and Seo [15] on so-called "lucky" cars in parking functions.

Definition 2. In a parking function $\alpha = (a_1, a_2, \ldots, a_n)$, car i is said to be *lucky* if car i parks in its preferred spot a_i. Given $\alpha \in \mathrm{PF}_n$, write $\mathrm{Lucky}(\alpha)$ for the set of lucky cars in α, and write $\mathrm{lucky}(\alpha) := |\mathrm{Lucky}(\alpha)|$ for the number of lucky cars in α.

For example, for $\alpha_1 = (1, 1, \ldots, 1) \in [n]^n$, we know $\mathrm{Lucky}(\alpha_1) = \{1\}$ and α_1 is one of the "unluckiest" parking functions since the only car that parks in its preferred spot is car 1. The lucky cars of a different parking function $\alpha_2 = (n, 1, 1, \ldots, 1) \in [n]^n$ are 1 and 2; that is, we have $\mathrm{Lucky}(\alpha_2) = \{1, 2\}$ and so α_2 is luckier than α_1. Permutations of $[n]$ are the "luckiest" parking functions, as every car gets to park in their preference; that is, if α is a permutation of $[n]$, then $\mathrm{Lucky}(\alpha) = [n]$ and $\mathrm{lucky}(\alpha) = n$.

Gessel and Seo [15] established that the generating function for the lucky car statistic on parking functions is

$$\sum_{\alpha \in \mathrm{PF}_n} q^{\mathrm{lucky}(\alpha)} = q \prod_{i=1}^{n-1} (i + (n - i + 1)q). \tag{1}$$

Their work is related to rooted trees, and the proof relies on ordered set partitions and generating function techniques. More recently, Harris, Kretschmann, and Martínez Mori [18] established that the sequences $\alpha \in [n]^n$ (not necessarily parking functions) with exactly $n-1$ cars parking in their preferred spot is equal to the total number of comparisons performed by the Quicksort algorithm given all possible orderings of an array of size n.

We recall another related statistic for parking functions known as *displacement*.

Definition 3. Let $\alpha = (a_1, \ldots, a_n) \in [n]^n$ be a parking function. If car i parks in spot $p(i) \in [n]$, then the *displacement* of car i is defined by $d(i) = p(i) - a_i$, the difference between where car i parks and its preference a_i. The *total displacement* of α is given by $d(\alpha) = \sum_i d(i)$. In this way, cars with zero displacement are lucky cars, while every car whose displacement is positive is unlucky.

Displacement has been widely studied in the literature. Knuth [20] established a bijection between the total displacement of parking functions and the number of inversions of labeled trees. Beissinger and Peled [2] constructed a bijection from parking functions to labeled trees that carries the displacement of parking functions to the external activity of trees. Yan [3] studied recurrence relations for generalized displacement in u-parking functions, and Aguillon, Alvarenga, Harris, Kotapati, Mori, Monroe, Saylor, Tieu, and Williams II [1] found a bijection with the set of ideal states in the Tower of Hanoi game and parking functions of length n with displacement equal to one. For more on the lucky statistic and displacement, we point the reader to the 2023 work of Stanley and Yin [30]. Researchers have also studied the displacement statistic within a more probabilistic framework, examining its relationship to the area/cost construction statistic for hashing tables with linear probing. They showed that the displacement statistic converges to normal, Poisson, or Airy distributions, depending on the ratio of cars to spots [8, 14, 19].

In this paper, we focus on a subset of parking functions consisting of Stirling permutations.

Definition 4. A permutation of the multiset $\{1, 1, 2, 2, 3, 3, \ldots, n, n\}$ is a *Stirling permutation of order* n if every value j appearing between the two instances of i satisfies $j > i$. We write Q_n to denote the set of Stirling permutations of order n, and we write $w \in Q_n$ as a word $w(1)w(2)\cdots w(2n)$.

For example, we have $123321 \in Q_3$, but $123231 \notin Q_3$ as 2 appears between the two instances of 3.

Gessel and Stanley [16] studied Stirling permutations. The Stirling polynomial $Q_n(t) = \sum_{w \in Q_n} t^{\mathrm{des}(w)}$, where $\mathrm{des}(w)$ refers to the descent statistic in permutations, arises as the numerator for the generating function

$$\sum_{m \geq 0} S(m+n,n)t^m = \frac{Q_n(t)}{(1-t)^{2n+1}},$$

where $S(m+n,n)$ are the Stirling numbers of the second kind [24, A008277]. The Stirling numbers of the second kind $S(n,k)$ count the number of set partitions of $[n]$ consisting of exactly k blocks [28, p. 81]. Gessel and Stanley [16] showed $|Q_n| = (2n-1)!!$. Since then, others have studied Stirling permutations to include work on enumerating descents, mesa sets, and other statistics related to generalized permutations [5, 12, 13, 17, 26].

Stirling permutations of order n can be interpreted as parking functions of length $2n$. Given a Stirling permutation $w = w(1)w(2)\cdots w(2n) \in Q_n$, the weakly increasing rearrangement of w is always $(1,1,2,2,3,3,\ldots,n,n)$, which satisfies the required inequality condition characterizing parking functions. Thus, we have $w \in \mathrm{PF}_{2n}$. Henceforth, we abuse notation and drop parenthesis and commas when talking about parking functions. We instead adopt the one-line notation used in Stirling permutations.

Since Stirling permutations can be viewed as a subset of parking functions, it is natural to explore the behavior of parking function statistics within this subset. Our focus is on the displacement and lucky statistics of Stirling permutations. Furthermore, we investigate their admissible sets in the context of lucky statistics. Given a subset $S \subset [2n]$, we say S is an *admissible lucky set* (or just *admissible*) if there exists $w \in Q_n$ such that $\mathrm{Lucky}(w) = S$. Otherwise, we say that S is *not admissible*.

Our paper is organized in the following way. In Section 2, we consider two extreme cases and prove the following:

- The so-called "extremely unlucky" Stirling permutations (those with exactly one lucky car) are enumerated by $(n-1)!$ (Theorem 13).
- The so-called "extremely lucky" Stirling permutations (those with exactly n lucky cars) are enumerated by Catalan numbers (Theorem 18).

In Sections 3 and 4, we look between these extremes by considering possible admissible sets of Stirling permutations. As a foray into this broad topic, our results are as follows:

- If S is an admissible lucky set for Q_n, then S is also admissible for all Q_m with $m > n$ (Theorem 23).
- We provide conditions under which there are exactly two lucky cars (Theorem 32). The set $\{1, n-1, 2n-2\}$ is admissible if and only if n is even (Proposition 35).

In Section 5, we consider the displacement statistic of Stirling permutations and prove the following:

- The displacement of every Stirling permutation is n^2 (Corollary 38).
- For every Stirling permutation, the number of cars displaced is between (inclusively) n and $2n-1$ (Corollary 41).
- We show that extremely lucky Stirling permutations are uniquely determined by their displacement composition, a tuple that describes the displacement of the unlucky cars (Theorem 43 and Corollary 45).

We conclude the paper in Section 6 with a selection of open problems.

2 Extremely unlucky/lucky Stirling permutations

The lucky statistic of Stirling permutations in Q_n has a well-defined range, as we show in the upcoming Lemma 5. It is natural to look at the Stirling permutations that achieve the extreme values in this range. The main results of this section are characterizations and enumerations of those two families of Stirling permutations – one enumerated by $(n-1)!$, and the other enumerated by the nth Catalan number.

Lemma 5. *Let $w = w(1)\cdots w(2n) \in Q_n$.*

(a) Car 1 is always lucky.

(b) If car i is the first car with $w(i) = 1$, then $i \in \mathrm{Lucky}(w)$.

(c) Then $1 \leq \mathrm{lucky}(w) \leq n$.

Proof.

(a) Car 1 is always lucky in each $w \in Q_n$, as car 1 is the first car to park when all the spots (including the desired spot by car 1) are available. Thus, we have $1 \in \mathrm{Lucky}(w)$ for every $w \in Q_n$.

(b) Let $i \in [n]$ be the smallest i such that $w(i) = 1$, i.e., car i is the first car to prefer spot 1. Then car i parks in spot 1. Hence, we know $i \in \mathrm{Lucky}(w)$.

(c) It follows from (a) that $1 \leq \mathrm{lucky}(w)$ for every $w \in Q_n$. For the upper bound, note that each car prefers one of the first n parking spots by the definition of Stirling permutations. Thus, at most n cars get to park in their desired spots, meaning $\mathrm{lucky}(w) \leq n$.

□

In the context of unlucky cars, the following complementary observation is immediate yet useful.

Remark 6. Every car whose preference is the second instance of a value in $w \in Q_n$ is unlucky. In particular, the last car $2n$ is always unlucky.

The main results of the next two subsections consider the extremes described by Lemma 5(c). Namely, we establish that the number of "extremely unlucky" Stirling permutations (those with only one lucky car) is $(n-1)!$ (see Theorem 13) and that "extremely lucky" Stirling permutations (those with n lucky cars) are counted by Catalan numbers (see Corollary 20).

2.1 Stirling permutations with extremely bad luck

The lower bound in Lemma 5(c) suggests that having exactly one lucky car is of "extremely bad luck" for Stirling permutations.

Definition 7. A Stirling permutation w with $\mathrm{Lucky}(w) = \{1\}$ is an *extremely unlucky* (Stirling) permutation.

One necessary condition for having exactly one lucky car is that $w(1) = 1$. Similarly, we must also have $w(2) = 1$, since otherwise $w(2)$ would be lucky.

Example 8. The only extremely unlucky Stirling permutations of order 3 are 112233 and 112332.

We now prove a technical result that helps us characterize extremely unlucky Stirling permutations. The upshot of this result is that if the first x values of w are all less than x, then x is not lucky.

Lemma 9. *Let $w = w(1)w(2)\cdots w(2n) \in Q_n$ and $x \in [2n]$. If $w(i) < x$ for all $i \leq x$, then $x \notin \mathrm{Lucky}(w)$.*

Proof. Assume that $w(i) < x$ for all $i \leq x$, for some $x \geq 2$ (the statement is meaningless for $x < 2$). Thus, the first x cars all want to park in the first $x - 1$ spaces. By the pigeonhole principle, at least one car is unlucky and has to take the next available space. Consequently, one of these initial x cars parks in spot x. By the assumption, the first appearance of $w(j) = x$ occurs for $j > x$. Thus, car j does not get to park in its preferred spot because spot x has already been claimed. Hence, we have $x \notin \mathrm{Lucky}(w)$. □

Example 10. Let $w = 122133 \in Q_3$. Consider $x = 3$, and note that $\{w(1), w(2), w(3)\} = \{1, 2\}$. Indeed, we have $3 \notin \mathrm{Lucky}(w)$, confirming the result of Lemma 9. Now consider $x = 2$. Given $\{w(1), w(2)\} = \{1, 2\}$, the assumption of Lemma 9 does not hold. This still leaves open the possibility for car 2 to be unlucky. However, car 2 parks in its preferred spot 2, indicating that car 2 is not unlucky after all.

In fact, as the above example suggests, the property described in Lemma 9 is a necessary and sufficient condition to characterize the extremely unlucky Stirling permutations.

Corollary 11. *A Stirling permutation $w \in Q_n$ is extremely unlucky if and only if for all $x \in [2, n]$, we have $w(i) < x$ for all $i \leq x$.*

Proof. Let $w \in Q_n$ be a Stirling permutation. If for all $x \in [2, n]$, we have $w(i) < x$ for all $i \leq x$, then w is extremely unlucky by Lemma 9.

Now, suppose that w is extremely unlucky. Recall that $2 \notin \text{Lucky}(w)$ and this means that $w(2) = 1$, giving the result for $x = 2$. Fix $x \in [3, n]$ and suppose, inductively, that for all $y \in [2, x-1]$, we have $w(i) < y$ for all $i \leq y$. Suppose, for the purpose of obtaining a contradiction, that $w(x) \geq x$. For w to be extremely unlucky, for each $y \in [2, x-1]$, the first y cars must have parked in spots $[1, y]$. Thus, the first $x - 1$ cars have parked in spots $[1, x-1]$, meaning that spot $w(x)$ is available for car x to park in. Thus, car x is lucky, which contradicts the assumption that w is extremely unlucky. □

The following set is useful in our proof of the enumeration of the extremely unlucky Stirling permutations.

Definition 12. For a finite, nonempty set S of integer numbers, we define $S^{\triangledown}$ to be the subset of S consisting of everything but the largest element of S; that is,

$$S^{\triangledown} \coloneqq S \setminus \{\max(S)\}.$$

Next, we enumerate extremely unlucky Stirling permutations via a constructive proof. In this proof we build extremely unlucky Stirling permutations of order n by first positioning the two copies of n. Then we position the two copies of $n-1$, and so on. At each step, we always ensure that the only lucky car is the first car whose preferred spot is 1. The construction is demonstrated after the proof of the theorem, in Example 14.

Theorem 13. *Given an integer $n \geq 1$, there are $(n-1)!$ extremely unlucky Stirling permutations of order n.*

Proof. Following Corollary 11, we can construct the extremely unlucky Stirling permutations of order n as follows. First, we position two (necessarily consecutive) copies of n, with the leftmost of those appearing as $w(i_n)$ for some $i_n \in [n+1, 2n]^{\triangledown}$. There are $2n-(n+1)+1-1 = n-1$ ways to do this. The rightmost n necessarily appears at $w(i_n^+)$, where i_n^+ is the smallest element larger than i_n in the set $[n+1, 2n]$; that is, we have $i_n^+ = i_n + 1$.

Next, we position the two copies of $n-1$. Because we are building a Stirling permutation, these are either consecutive or they immediately surround the two copies of n. Thus, by Corollary 11, the leftmost of these appears as $w(i_{n-1})$ for some

$$i_{n-1} \in \Big([n, 2n] \ \setminus \ \{i_n, i_n^+\}\Big)^{\triangledown},$$

and the rightmost $n-1$ necessarily have position $w(i_{n-1}^+)$, where i_{n-1}^+ is the smallest element larger than i_{n-1} in the set $[n, 2n] \setminus \{i_n, i_n^+\}$. There are $2n - n + 1 - 2 - 1 = n - 2$ choices for i_{n-1}.

We continue this process until we find two available positions for the leftmost 3, one available position for the leftmost 2, and finally insert the two 1s as $w(1)$ and $w(2)$. For example, the leftmost x appears as $w(i_x)$ for some

$$i_x \in \Big([x+1, 2n] \ \setminus \ \{i_n, i_n^+\} \ \setminus \ \{i_{n-1}, i_{n-1}^+\} \ \setminus \ \cdots \ \setminus \ \{i_{x+1}, i_{x+1}^+\}\Big)^{\triangledown},$$

and thus there are $2n - (x+1) + 1 - 2(n-x) - 1 = x - 1$ choices for i_x. The second appearance of x is at $w(i_x^+)$, where i_x^+ is the smallest element larger than i_x in the set

$$[x+1, 2n] \ \setminus \ \{i_n, i_n^+\} \ \setminus \ \{i_{n-1}, i_{n-1}^+\} \ \setminus \ \cdots \ \setminus \ \{i_{x+1}, i_{x+1}^+\}.$$

This process has constructed $(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = (n-1)!$ distinct Stirling permutations, which are exactly the extremely unlucky Stirling permutations of order n. □

Therefore extremely unlucky Stirling permutations are enumerated, with a small shift of index, by [24, A000142].

We demonstrate the construction in the proof of Theorem 13 with an example.

Example 14. Let $n = 6$.

- The index i_6 can be all of the five values in $[7, 12]^{\triangledown} = \{7, 8, 9, 10, 11\}$. Suppose $i_6 = 9$, and so $i_6^+ = 10$.
- The index i_5 can be all of the four values in $\big([6, 12] \setminus \{9, 10\}\big)^{\triangledown} = \{6, 7, 8, 11\}$. Suppose $i_5 = 6$, and so $i_5^+ = 7$.
- The index i_4 can be all of the three values in $\big([5, 12] \setminus \{9, 10\} \setminus \{6, 7\}\big)^{\triangledown} = \{5, 8, 11\}$. Suppose $i_4 = 8$, and so $i_4^+ = 11$.
- The index i_3 can be all of the two values in $\big([4, 12] \setminus \{9, 10\} \setminus \{6, 7\} \setminus \{8, 11\}\big)^{\triangledown} = \{4, 5\}$. Suppose $i_3 = 5$, and so $i_3^+ = 12$.
- The index i_2 must be the one value in $\big([3, 12] \setminus \{9, 10\} \setminus \{6, 7\} \setminus \{8, 11\} \setminus \{5, 12\}\big)^{\triangledown} = \{3\}$. Thus, we must have $i_2 = 3$ (in fact, the value of i_2 is always 3), and hence $i_2^+ = 4$.

Thus, we have produced the Stirling permutation

$$1\ 1\ 2\ 2\ 3\ 5\ 5\ 4\ 6\ 6\ 4\ 3,$$

which is, indeed, extremely unlucky.

2.2 Stirling permutations with extremely good luck

In contrast to the extremely unlucky Stirling permutations of Theorem 13, we now consider the very fortunate ones. To begin, note that by Lemma 5(c), if $w \in Q_n$, then $\text{Lucky}(w) \subseteq [2n]$ with $\text{lucky}(w) \leq n$. In other words, at most half of the cars are lucky.

Definition 15. A Stirling permutation $w \in Q_n$ for which $\text{lucky}(w) = n$ is an *extremely lucky* (Stirling) permutation.

Note, in particular, that $w \in Q_n$ is extremely lucky if and only if lucky cars park in each of the first n available parking spots. In particular, extremely lucky Stirling permutations are *Catalan objects*; that is, they are enumerated by the Catalan numbers [24, A000108].

Definition 16. The *Catalan numbers* are the sequence $(C_n)_{n\geq 0}$ defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Many families of combinatorial objects are Catalan objects, and we refer the interested reader to Stanley's work [29] for a comprehensive book on the subject. Our proof that extremely lucky permutations are another family of Catalan objects is established by giving a bijection to valid parenthesizations of n pairs of parentheses; that is, expressions that contain n pairs of parentheses that are correctly matched. It is well-known that valid parenthesizations are Catalan objects [27, Prop. 6.2.1, p. 169], a result which was proven by Eugène Charles Catalan, although the sequence was known much earlier by Mingantu, a Mongolian mathematician [22]. In preparation for that main enumerative result, we first characterize extremely lucky Stirling permutations.

Proposition 17. *A Stirling permutation $w \in Q_n$ is extremely lucky if and only if the subword of $w(1)\cdots w(2n)$ consisting of the second appearances of each number is the word $n\cdots 321$; that is, the second appearances of each number in w occur in decreasing order.*

Proof. Suppose first that there is some $x < n$ for which the second appearance $x = w(j)$ appears to the left of the second appearance of $x+1$ in $w(1)\cdots w(2n)$. Since w is a Stirling permutation, we must therefore have

$$w = \cdots x \cdots x \cdots (x+1) \cdots (x+1) \cdots .$$

But then $x+1$ would not be the parking spot of a lucky car because either car j would park in spot $x+1$ or some car h with $h < j$ and $w(h) < w(j)$ would have parked in spot $x+1$, before the cars preferring spot $x+1$ have had a chance to do so. Thus, the Stirling permutation w is not extremely lucky.

For the other direction of the argument, suppose that the second appearances of each number in w occur in decreasing order. Then certainly the first appearance of a value $x \in [1, n]$ in $w(1)\cdots w(2n)$ also appears to the left of the second appearance of all numbers in $[1, x-1]$. In fact, all the entries to the left of the first appearance of x are either larger than x or the first appearance of a number in $[1, x-1]$. Thus, spot x is available for the first car preferring spot x for each $x \in [1, n]$. In other words, each car whose preference is the first appearance of a number in $[1, n]$ is a lucky car, and hence $\text{lucky}(w) = n$. □

Proposition 17 characterizes extremely lucky Stirling permutations and allows us to give a bijective correspondence between extremely lucky Stirling permutations of order n and valid parenthesizations of n pairs of parentheses. For example, the $C_3 = 5$ valid parenthesizations of 3 pairs of parentheses are

$$((( ))),\quad (( ) ( )),\quad (( )) ( ),\quad ( ) (( )),\quad \text{and}\quad ( ) ( ) ( ).$$

Theorem 18. *Extremely lucky Stirling permutations of order n are in bijection with valid parenthesizations of n pairs of parentheses.*

Proof. Given an extremely lucky $w \in Q_n$, replace the first occurrence of each x by "(" and the second by ")." Because w is a Stirling permutation, this produces a valid parenthesization.

Now, given a valid pairing of n pairs of parentheses, replace each ")" by $n, \ldots, 1$ in decreasing order from left to right. For each ")" replaced by x, replace its partner "(" by x too. This construction necessarily produces a Stirling permutation, and Proposition 17 means that we get exactly those Stirling permutations that are extremely lucky. □

We demonstrate the argument in the proof of Theorem 18 with the following example.

Example 19. By the bijection described above, the valid parenthesization

$$( ) ( ) (( ) (( )))$$

is matched with the extremely lucky Stirling permutation

$$6\ 6\ 5\ 5\ 1\ 4\ 4\ 2\ 3\ 3\ 2\ 1 \in Q_6.$$

It is important to note that, as in the proof of Theorem 18, every Stirling permutation can be paired with a valid parenthesization. Thus, many Stirling permutations could describe the same string of parentheses, and among those, only one is extremely lucky. For instance, the Stirling permutations 112235546643 and 665514423321 both correspond to ( ) ( ) (( ) (( ))), but only the latter permutation is extremely lucky (cf. Examples 14 and 19).

Corollary 20. *For every positive integer n, the number of extremely lucky Stirling permutations of order n is the Catalan number C_n.*

3 Admissible lucky sets

In this section, we give initial results towards characterizing the subsets of cars that can be lucky for Stirling permutations. To this end, we recall the following definition from the introduction.

Definition 21. A subset $S \subseteq [2n]$ is an *n-admissible lucky set* (or just *admissible* if context is clear) if there exists $w \in Q_n$ such that $\text{Lucky}(w) = S$. Otherwise, we say that S is *not n-admissible* (or *not admissible*).

In what follows, we characterize when a set with two elements is an admissible lucky set and prove several properties of three-element admissible lucky sets. It remains an open problem to give a complete characterization of all admissible lucky sets.

Example 22. For $n = 4$, the set $S = \{1, 3, 6\}$ is a 4-admissible lucky set since $\text{Lucky}(w) = S$ for $w = 33144221 \in Q_4$. In contrast, the set $S' = \{1, 4, 8\}$ is not 4-admissible since there is no $w \in Q_4$ such that $\text{Lucky}(w) = S'$. (Indeed, recall from Remark 6 that 8 is unlucky for all $w \in Q_4$.) Table 1 was generated via exhaustive research and lists all 4-admissible lucky sets.

{1}	{1, 2}	{1, 2, 3}	{1, 2, 3, 4}
	{1, 3}	{1, 2, 4}	{1, 2, 3, 5}
	{1, 4}	{1, 2, 5}	{1, 2, 3, 6}
	{1, 5}	{1, 2, 6}	{1, 2, 3, 7}
	{1, 7}	{1, 2, 7}	{1, 2, 4, 5}
		{1, 3, 4}	{1, 2, 4, 6}
		{1, 3, 5}	{1, 2, 4, 7}
		{1, 3, 6}	{1, 2, 5, 6}
		{1, 3, 7}	{1, 2, 5, 7}
		{1, 5, 6}	{1, 3, 4, 5}
		{1, 5, 7}	{1, 3, 4, 6}
			{1, 3, 4, 7}
			{1, 3, 5, 6}
			{1, 3, 5, 7}

Table 1: The thirty-one admissible lucky sets for Stirling permutations of order 4, arranged by size.

Determining the properties of admissible lucky sets is the focus of the remainder of this section. We begin with a stability result, which establishes that if a set is n-admissible, then the set is m-admissible for all $m \geq n$.

Theorem 23. *If S is an n-admissible lucky set, then the sets S and $S \cup \{2n+1\}$ are both $(n+1)$-admissible lucky sets.*

Proof. Suppose that S is an n-admissible lucky set and that $w \in Q_n$ satisfies $\text{Lucky}(w) = S$. Then the Stirling permutation $w' = w(n+1)(n+1) \in Q_{n+1}$, created by appending the value $(n+1)$ twice to the end of w, ensures that the last two cars whose preference is $n+1$ end up parking in spots $2n+1$ and $2n+2$, respectively. Thus, we have $\text{Lucky}(w') = S$ as well.

Now consider the Stirling permutation $w'' \in Q_{n+1}$ defined by $w''(i) := w(i) + 1$ for all $1 \leq i \leq 2n$ and $w''(2n+1) = w''(2n+2) := 1$. By construction, we have $\text{Lucky}(w'') = \text{Lucky}(w) \cup \{2n+1\}$, meaning that $S \cup \{2n+1\}$ is $(n+1)$-admissible, as desired. □

Next, we recall a consequence of Lemma 5(a) and Remark 6, which involves elements of admissible lucky sets.

Corollary 24. *If S is n-admissible lucky set, then $1 \in S$ and $2n \notin S$.*

In fact, the preferred parking spot of the first lucky car (car 1) determines the preferences of the initial unlucky cars.

Lemma 25. *Let $w \in Q_n$ with $\text{lucky}(w) \geq 2$, and let x be the second-smallest element of $\text{Lucky}(w)$. Then the first set of unlucky cars $2, 3, \ldots, x-1$ park, in order, in spots*

$$w(1) + 1,\ w(1) + 2,\ \ldots,\ w(1) + x - 2.$$

Proof. In order for cars $2, 3, \ldots, x-1$ to be unlucky, they must each prefer spots that have already been occupied when cars $2, 3, \ldots, x-1$ attempt to park. This implies that the preference of car 2 must be $w(2) = w(1)$, ensuring that car 2 parks in spot $w(1) + 1$. As the value $w(1)$ has now appeared twice, the value $w(3)$ (that is, the preference for car 3) can only be $w(1) + 1$, and thus car 3 parks in spot $w(1) + 2$. Similarly, regardless of its preference, car 4 ends up parking in spot $w(1) + 3$. Continuing in this manner, car y parks in spot $w(1) + y - 1$ for all $y \in [2, x-1]$. □

Remark 26. Under the assumptions of Lemma 25, note that $w(i) \geq w(1)$ for $i \in [2, x-1]$.

The nature of Stirling permutations can impact the parity of certain relevant values, as we see in the following lemma.

Lemma 27. *Let $w \in Q_n$ and $k \in [1, n]$. Let j be minimal such that $w(j) = k$; that is, this j is the first car to prefer spot k in this parking function. If $w(i) > w(j)$ for all $i \in [1, j-1]$, then j must be odd.*

Proof. Let w, k, and j be as described in the assumptions of the statement. Suppose, for the purpose of obtaining a contradiction, that j is even. Then $j - 1$ is odd, so there exist some h and h' with $w(h) = w(h')$, and $h < j < h'$ as each number appears with 2 copies in w. We have $w(h) > w(j)$ by assumption, which contradicts the fact that w is a Stirling permutation. □

In particular, the first car to prefer spot 1 must be an odd-indexed car. In the spirit of the previous lemmas, we can deduce the parity of the second lucky car if the second lucky car is in the second half of the queue.

Proposition 28. *Let $w \in Q_n$ with $\text{lucky}(w) \geq 2$, and let x be the second-smallest element of $\text{Lucky}(w)$. If $x > n$, then x is odd.*

Proof. Suppose that $x > n$. By Lemma 25, after the first $x - 1$ cars have parked, the only unused parking spots are $[1, w(1)-1] \cup [w(1)+x-1, 2n]$. Because $w(1)+x-1 > 1+n-1 = n$, and cars can only prefer the first n spots, the only way for car x to be lucky would be if car x prefers to park in spot y for some $y < w(1)$. From this, Remark 26 and Lemma 27 ensure that x must be odd. □

Having established several important foundational properties of admissible lucky sets, we spend the remainder of this section developing some of the more technical properties of these sets, which becomes relevant in our later examinations of admissible lucky sets of small cardinality. Many of these upcoming results focus on the second-smallest element of an admissible lucky set.

Lemma 29. *Let $w \in Q_n$ with $\text{lucky}(w) \geq 2$, and let x be the second-smallest element of $\text{Lucky}(w)$. If $w(1) = 1$, then $x \leq w(x) \leq n$.*

Proof. Suppose that $w(1) = 1$ and that car x is the second lucky car. By Lemma 25, cars $2, 3, \ldots, x-1$ park to the right of car 1, in spots $2, 3, \ldots, x-1$, respectively. Thus, if car x is to be lucky, then car x must be in the first half of the queue and prefer one of the spots in the set $\{x, \ldots, n\}$. In other words, $x \leq w(x) \leq n$. □

When the second-smallest element of $\text{Lucky}(w)$ is sufficiently large, the preference of other lucky cars must be suitably small.

Lemma 30. *Let $w \in Q_n$ with $\text{lucky}(w) \geq 3$, and let x be the second-smallest element of $\text{Lucky}(w)$. If $x \geq n$, then $w(1) \geq \text{lucky}(w)$. Moreover, the lucky cars park among the first $w(1)$ spots.*

Proof. By Lemma 25, for all $i \in [2, x-1]$, car i parks in spot $w(1) + i - 1$. Because $x \geq n$, spots $w(1)$ through n are all occupied before car x parks. Thus, to ensure there are $\text{lucky}(w) - 1$ unused parking spots for the remaining lucky cars to park in, we must have that $w(1) - 1 \geq \text{lucky}(w) - 1$; that is, we have $w(1) \geq \text{lucky}(w)$. Finally, because spots $w(1)+1$ through n are occupied by unlucky cars, all lucky cars must park among spots $[1, w(1)]$. □

Next, we characterize when the penultimate car in a Stirling permutation is lucky.

Theorem 31. *For $w \in Q_n$, we have $2n - 1 \in \text{Lucky}(w)$ if and only if $w(2n-1) = 1$.*

Proof. First suppose that $w(2n-1) = 1$. If $w(2n) > 1$, then the value 1 would appear between the two instances of $w(2n)$, meaning that w is not a Stirling permutation. Thus, we have $w(2n) = 1$, and so car $2n-1$ is the first car to prefer spot 1. Therefore, car $2n-1$ parks in spot 1, and hence car $2n-1$ is lucky.

Now, suppose that $2n-1 \in \text{Lucky}(w)$. Thus, car $2n-1$ is the first car to prefer $w(2n-1)$, and so $w(2n-1) = w(2n)$. If $w(2n-1) > 1$, then consider the two cars, say cars i and j with $i < j$, with preferred spot $w(2n-1) - 1$. If car i is lucky, then either car j or car h for some $i < h < j$ parks in spot $w(2n-1)$. On the other hand, if car i is not lucky, then for some $h \leq i$, car h parks in spot $w(2n-1)$. In either case, this contradicts the assumption that $2n-1 \in \text{Lucky}(w)$. Therefore, we must have $w(2n-1) = 1$. □

4 Admissible lucky sets of small cardinality

In this section, we fully characterize and enumerate admissible lucky sets of size two. We also provide a first result toward a characterization for admissible lucky sets of size three.

4.1 Admissible lucky sets of size two

As discussed earlier $1 \in S$ for all the admissible sets (see Corollary 24). In order to characterize admissible sets of size two, one needs to identify which cars can be the second lucky car.

Theorem 32. *A set $S = \{h, i\}$ with $h < i$ is an n-admissible lucky set if and only if $h = 1$ and either*

(a) both $i > n$ and i is odd, or

(b) we have $i \leq n$.

Proof. By Proposition 28, the sets described in the statement are the only possible lucky sets. For the other direction, we provide Stirling permutations to demonstrate that all such $\{1, i\}$ can be obtained as lucky sets. Let $w = 112233 \cdots (n-1)(n-1)nn \in Q_n$.

- Suppose i is odd, with $i = 2k+1$. Let α be the Stirling permutation obtained from w by inserting 11 in between $(k+1)(k+1)$ and $(k+2)(k+2)$, namely,

$$\alpha = 2\,2\,3\,3\,\cdots\,(k+1)\,(k+1)\,1\,1\,(k+2)\,(k+2)\,(k+3)\,(k+3)\,\cdots\,n\,n.$$

  Regardless of whether $i \leq n$ or $i > n$, one can verify that $\text{Lucky}(\alpha) = \{1, i\}$.

- Suppose $i \leq n$ and i is even, with $i = 2k$. Let α' be the Stirling permutation obtained from w by inserting $k(2k)(2k)$ in between $(k-1)(k-1)$ and $(k+1)(k+1)$, and placing the remaining copy of k at the end after nn, namely,

$$\alpha' = 1\,1\,\cdots\,(k-1)\,(k-1)\,k\,(2k)\,(2k)\,(k+1)\,(k+1)\,\cdots \\ \cdots\,(2k-1)\,(2k-1)(2k+1)\,(2k+1)\,\cdots\,n\,n\,k.$$

  One can verify that $\text{Lucky}(\alpha') = \{1, i\}$. □

Enumeration of n-admissible lucky sets of size two follows from the above characterization.

Corollary 33. *For $n \geq 1$, there are $n - 1 + \left\lceil \frac{n-1}{2} \right\rceil$ n-admissible lucky sets of size two.*

Next, we conclude that if n is the second-smallest element in $\text{Lucky}(w)$ for $w \in Q_n$, then $\text{Lucky}(w) = \{1, n\}$.

Lemma 34. *Let S be an n-admissible lucky set with second-smallest element n.*

(a) If n is even, then $w(1) = 1, w(n) = n$, and $|S| = 2$.

(b) If n is odd and $w(1) = 1$, then $w(n) = n$ and $|S| = 2$.

Proof. Let $w \in Q_n$ with $\mathrm{Lucky}(w) = S$ and such that 1 and n are the two smallest elements in S.

We first show that for n even, we have $w(1) = 1$. Suppose that n is even and assume, for the purpose of obtaining a contradiction, that $w(1) > 1$. Then, by Lemma 25, cars 2 through $n-1$ park in spots $w(1)+1, w(1)+2, \ldots, w(1)+n-2$ where $w(1)+n-2 \geq n$. So, lucky car n must prefer a spot in the set $[1, w(1)-1]$. Moreover, the value $w(n)$ must be smaller than every element of $\{w(1), \ldots, w(n-1)\}$. By Lemma 27, the index n must be odd. This is a contradiction. Thus, if n is even, then $w(1) = 1$.

Now assume that $w(1) = 1$ and make no assumptions on the parity of n. By Lemma 25, the first $n-1$ cars park in the first $n-1$ spots, so the only way car n can be lucky is if car n prefers to park in spot n; i.e., if $w(n) = n$. There can be no additional lucky cars because the first n spots are occupied by the first n cars. Thus, we have $|S| = 2$. □

4.2 On admissible lucky sets of size three

For $n = 4$, Table 1 gives the 11 distinct 4-admissible lucky sets of cardinality three. In this section, we characterize certain families of three-element admissible lucky sets depending on the parity of n. It remains an open problem to characterize all three-element admissible lucky sets.

Proposition 35. *The set $\{1, n-1, 2n-2\}$ is an n-admissible lucky set if and only if n is even.*

Proof. First suppose that $w \in Q_n$ with $\mathrm{Lucky}(w) = \{1, n-1, 2n-2\}$. Assume $n > 3$, with the result being easy to check in small cases.

Recall that the first car to prefer spot 1 is always lucky, and, by Lemma 27, that first car with preference for spot 1 must be an odd-indexed car. Since $2n-2$ is always even, either car 1 or car $n-1$, with n even, must be the first car to prefer spot 1.

If $w(1) = 1$, by Lemma 25, the only possible spots for the lucky car $n-1$ to prefer are $n-1$ or n. Since car n is unlucky, car n must have the same preference as a previous car, or its preferred spot is occupied. In order to satisfy the Stirling condition, we have $w(n-1) = w(n)$. If $w(n-1) = n-1$, then the first n cars park in the first n spots, meaning that car $2n-2$ cannot possibly be lucky. On the other hand, if $w(n-1) = n$, then the only available spot for lucky car $2n-2$ to park is spot $n-1$. However, the preference of car $n+1$ is at most $n-1$. So, car $n+1$ ends up parking in the next available spot $n-1$, preventing car $2n-2$ from being lucky. Thus, we know $w(1) = 1$ is not possible, and car $n-1$, with n even, is the first car to prefer spot 1. In other words, the set $\{1, n-1, 2n-2\}$ is n-admissible implies n is even.

To prove the other direction of the result, it suffices to produce a Stirling permutation $w \in Q_{2k}$ for which $\mathrm{Lucky}(w) = \{1, 2k-1, 4k-2\}$. Such a permutation is

$$3\,3\,4\,4\,\cdots\,k\,k\,(k+1)\,(k+1)\,1\,(k+2)\,(k+2)\,\cdots\,(2k)\,(2k)\,2\,2\,1 \in Q_{2k}.$$ □

5 Displacement statistic

In this section, we study the displacement statistics of Stirling permutations. As discussed in the introduction, this statistic measures how far cars park from their preferred parking spots in a parking function.

Definition 36. For every $w \in Q_n$, if car i parks in spot $p(i) \in [2n]$, then the *displacement* of car i is $d(i) \coloneqq p(i) - w(i)$. The *(total) displacement* of $w \in Q_n$ is

$$d(w) = \sum_{i=1}^{2n} d(i).$$

Because every Stirling permutation is a parking function, the displacement of a car is always nonnegative. Moreover, by definition, cars with displacement equal to zero are lucky cars, while cars whose displacement is positive are unlucky.

We begin by recalling a result from Elder, Harris, Kretschmann, and Martínez Mori [10] that if $w \in Q_n$, then the displacement of w is invariant under permutations of the entries of w, even though the resulting strings are not necessarily Stirling permutations. This result is a special case of [10, Lemma 3.1], hence we omit its proof but state the result formally below.

Lemma 37. *[10, Lemma 3.1] For every $w \in Q_n$, the displacement $d(w)$ is invariant under permuting the numbers in the one-line notation of w. In other words, for every permutation σ of $[1, 2n]$ and every Stirling permutation $w \in Q_n$, let $\sigma(w) = w(\sigma(1))w(\sigma(2)) \cdots w(\sigma(2n))$. Then $d(\sigma(w)) = d(w)$.*

We now determine the displacement of Stirling permutations.

Corollary 38. *For $w \in Q_n$, we have $d(w) = n^2$.*

Proof. For every parking function $\pi \in \mathrm{PF}_m$, the total displacement can be written as

$$d(\pi) = \binom{m+1}{2} - \sum_{k=1}^{m} \pi_m.$$

Thus, for $w \in Q_n$, we have

$$d(w) = \binom{2n+1}{2} - 2\sum_{k=1}^{n} k = n^2. \qquad \square$$

Definition 36 provides two versions of displacement. One version is for each car and the other version is for the entire Stirling permutation. Corollary 38 addresses the latter of these for Stirling permutations. Now we turn our attention to the former and investigate "compositions" of n^2 that record the displacement data for each car when w is in Q_n.

Definition 39. For $w \in Q_n$, the *displacement composition* of w is the $2n$-tuple encoding the displacements of each car, which is

$$(d(1), d(2), \ldots, d(2n)) \in \mathbb{N}^{2n}.$$

Let $\text{dis}(w)$ denote the displacement composition of w.

As discussed before, the displacement $d(i) = 0$ if and only if car i is lucky. In particular, the first entry of $\text{dis}(w)$ is always 0.

Example 40. For the Stirling permutation $w = 11244233$, the displacement composition is $\text{dis}(w) = (0, 1, 1, 0, 1, 4, 4, 5)$.

Recall that $1 \leq \text{lucky}(w) \leq n$ by Lemma 5(c). Thus, we can bound the number of nonzero entries in $\text{dis}(w)$, whenever $w \in Q_n$.

Corollary 41. *For every $w \in Q_n$, we have $1 \leq |\{i \in [2n] : d(i) = 0\}| \leq n$.*

Two distinct Stirling permutations may share the same displacement composition, indicating that displacement compositions do not uniquely identify a Stirling permutation. For instance, the displacement composition $(0, 1, 0, 1, 3, 4)$ corresponds to both 113322 and 331122. However, extremely lucky Stirling permutations are uniquely determined by their displacement compositions. That is, no two distinct extremely lucky Stirling permutations have the same displacement composition. This is the main result in this section, and we start with a technical lemma used in the proof of that result.

Lemma 42. *Let $w \in Q_n$ be an extremely lucky permutation. For $i \in [n]$, the following properties hold.*

(a) The ith unlucky car of w prefers spot $n - i + 1$.

(b) The ith unlucky car of w has displacement $2i - 1$.

Proof. Let $w \in Q_n$ be an extremely lucky permutation. Hence, we have $\text{lucky}(w) = n$ and there are n unlucky cars.

(a) This is an immediate consequence of Proposition 17.

(b) We proceed by induction. Suppose that car j is the first unlucky car. By (a) we have that car j must prefer spot n. Since car j is the second car to prefer spot n, and no other cars have yet been unlucky, then car j parks in spot $n+1$. Thus, its displacement is $(n + 1) - n = 2(1) - 1$, as claimed.

Now fix $i \geq 1$ and assume that the result holds for all $j \leq i$. Thus, the jth unlucky car parks in spot $(n - j + 1) + (2j - 1) = n + j$ for all $j \leq i$. Let car k be the $(i + 1)$th unlucky car. By (a), this car k must prefer spot $n - i$. Since car k is unlucky, and by Proposition 17, we know that spots in $[n - i, n + i]$ are already occupied and the next available spot is $n + i + 1$. Thus, car k parks in spot $n + i + 1$. Hence, its displacement is $(n + i + 1) - (n - i) = 2i + 1 = 2(i + 1) - 1$, completing the proof. □

We are now ready to establish the injectivity of the map $w \mapsto \mathrm{dis}(w)$ where w belongs to the set of extremely lucky Stirling permutations of order n.

Theorem 43. *Let w be an extremely lucky Stirling permutation. The displacement composition $\mathrm{dis}(w)$ of w uniquely determines w.*

Proof. Suppose that $w \in Q_n$ is an extremely lucky Stirling permutation, and consider its displacement composition $\mathrm{dis}(w)$. Suppose that there exists another $w' \in Q_n$ such that $\mathrm{dis}(w') = \mathrm{dis}(w)$.

The 0 entries in a displacement composition correspond to the lucky cars in the parking function, so we must have $\mathrm{Lucky}(w) = \mathrm{Lucky}(w')$, and hence w' must be extremely lucky as well. By Proposition 17, the values of w' in each of the positions of nonzero entries of $\mathrm{dis}(w)$ must be $n, n-1, \ldots, 1$, in order from left to right. Moreover, these must be the second appearances of each of these numbers in the one-line notation of w'.

Now, to obey the Stirling property, the first occurrence of n in w' must be in the rightmost unclaimed (lucky) position appearing to the left of the (already determined) second appearance of n in w'. Similarly, the first occurrence of $n-1$ in w' must be in the rightmost unclaimed position appearing to the left of the second appearance of $n-1$, and so on.

Each of these positions is entirely determined by $\mathrm{dis}(w)$, and so in fact we must have $w = w'$. ☐

We demonstrate the result above in the following example.

Example 44. The displacement composition $(0,0,0,1,3,0,0,5,7,9)$ for $w \in Q_5$ tells us that $\mathrm{Lucky}(w) = \{1,2,3,6,7\}$. Moreover, Proposition 17 tells us that

$$w = \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \ 5\ \ 4\ \ \underline{\quad}\ \underline{\quad}\ \ 3\ \ 2\ \ 1.$$

The Stirling condition tells us that the first appearance of 5 in w must be in position 3. Then we see that the first appearance of 4 in w must be in position 2. Similarly, the first position of 3 in w must be $w(7)$, the first position of 2 must be $w(6)$, and the first position of 1 must be $w(1)$. Therefore, we have

$$w = \underline{\ 1\ }\ \underline{\ 4\ }\ \underline{\ 5\ }\ \ 5\ \ 4\ \ \underline{\ 2\ }\ \underline{\ 3\ }\ \ 3\ \ 2\ \ 1.$$

To conclude this section, we give a restatement of Corollary 20 in terms of displacement compositions.

Corollary 45. *The number of Stirling permutations whose displacement composition has exactly n nonzero parts is C_n, the nth Catalan number.*

6 Open problems

Finally, we propose a selection of open problems related to lucky cars and the displacement statistic on Stirling permutations.

To begin with, recall that in Section 4, we gave several properties of three-element admissible lucky sets. The following problems remain unanswered.

Problem 46. Characterize and enumerate the three-element n-admissible lucky sets. More generally, for fixed n, how many n-admissible lucky sets are there?

Equation (1) gives a generating function for the lucky statistic on parking functions. Table 2 was generated via an exhaustive search and provides analogous data for the generating function defined by the lucky statistic on Stirling permutations defined by

$$T_n(q) = \sum_{w \in Q_n} q^{\mathrm{lucky}(w)}.$$

The sequences of coefficients in Table 2 appear to be new, as a search on the OEIS [24] did not yield these sequences at the time of publication. We do not have conjecture for formulas for these coefficients. Corollary 20 implies that, for all $n \geq 1$, the leading coefficient of $T_n(q)$ is the Catalan number C_n. Theorem 13 shows that the coefficient of q in $T_n(q)$ is given by $(n-1)!$. We thus pose the following questions.

n	$T_n(q)$
2	$2q^2 + q$
3	$5q^3 + 8q^2 + 2q$
4	$14q^4 + 49q^3 + 36q^2 + 6q$
5	$42q^5 + 268q^4 + 417q^3 + 194q^2 + 24q$
6	$132q^6 + 1374q^5 + 3876q^4 + 3665q^3 + 1228q^2 + 120q$
7	$429q^7 + 6752q^6 + 31231q^5 + 52353q^4 + 34675q^3 + 8975q^2 + 720q$

Table 2: The polynomials $T_n(q)$ for $2 \leq n \leq 7$.

Problem 47. Determine a formula for $T_n(q)$ for all $n \geq 1$. Are the polynomials $T_n(q)$ unimodal? Are they real-rooted?

The bound determined in Corollary 41, suggests the following avenue of research.

Problem 48. Determine if, for $i \in [n, 2n-1]$, there exists $w \in Q_n$ such that $\mathrm{dis}(w)$ has exactly i nonzero entries. Equivalently, can we always find a Stirling permutation with i unlucky cars?

Table 3 provides some data in the case where $n = 3$, listing all possible displacement compositions and the Stirling permutations that give rise to those displacement compositions.

Theorem 13 establishes that there are $(n-1)!$ Stirling permutations in Q_n whose displacement composition has exactly 1 zero entry. Moreover, by Corollary 45, there are C_n Stirling permutations in Q_n whose displacement composition has exactly n nonzero parts. As a follow-up question to Problem 48, we can thus try to interpolate between these two extreme cases, enumerating the Stirling permutations whose displacement compositions have k zero parts.

Problem 49. For $k \in [2, n-1]$, determine

$$\left|\{w \in Q_n : \mathrm{dis}(w) \text{ has } k \text{ zero parts}\}\right|.$$

Equivalently, how many $w \in Q_n$ have exactly k lucky cars?

It is also of interest to understand displacement compositions arising from general Stirling permutations.

Problem 50. Let $\mathcal{D}_n \coloneqq \{\mathrm{dis}(w) : w \in Q_n\}$. Characterize and enumerate the elements of $\mathcal{D}_n$.

On a more specific level, one can try to understand the fibers of the map $w \mapsto \mathrm{dis}(w)$. In that direction, a related problem is to determine the number of Stirling permutations with a fixed displacement composition.

Problem 51. Fix $\mathbf{m} \in \mathcal{D}_n$. Characterize and enumerate the elements of the set

$$\mathcal{S}_n(\mathbf{m}) \coloneqq \{w \in Q_n : \mathrm{dis}(w) = \mathbf{m}\}.$$

Put another way, what properties define the collection of Stirling permutations whose displacement compositions are equal to a given vector? Also, are there operations on Stirling permutations whose orbits define the fibers of that map?

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k	$\text{dis}(w)$, where $w \in Q_3$ has k nonzero parts
3	$\text{dis}(123321) = (0,0,0,1,3,5)$
	$\text{dis}(133221) = (0,0,1,0,3,5)$
	$\text{dis}(233211) = (0,0,1,3,0,5)$
	$\text{dis}(331221) = (0,1,0,0,3,5)$
	$\text{dis}(332211) = (0,1,0,3,0,5)$
4	$\text{dis}(122331) = (0,0,1,1,2,5)$
	$\text{dis}(122133) = (0,0,1,3,2,3)$
	$\text{dis}(133122) = (0,0,1,1,3,4)$
	$\text{dis}(221331) = (0,1,0,1,2,5)$
	$\text{dis}(113322) = (0,1,0,1,3,4)$
	$\text{dis}(331122) = (0,1,0,1,3,4)$
	$\text{dis}(221133) = (0,1,0,3,2,3)$
	$\text{dis}(223311) = (0,1,1,2,0,5)$
5	$\text{dis}(112233) = (0,1,1,2,2,3)$
	$\text{dis}(112332) = (0,1,1,1,2,4)$

Table 3: Displacement compositions arising from Stirling permutations in Q_3, aggregated by the number of nonzero parts; i.e., by the number of unlucky cars.

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