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Supersolvable posets and fiber-type arrangements*

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Abstract. We develop a theory of modularity and supersolvability for chain-finite geometric posets, extending that of Stanley for finite lattices and building a new connection between combinatorics and topology. From a combinatorial point of view, our theory features results about factorizations of the characteristic polynomials, dovetails with established notions on geometric semilattices, and behaves well under quotients by translative group actions. We also establish a topological counterpart in the context of toric and abelian arrangements, akin to Terao's fibration theorem connecting bundles of hyperplane arrangements to supersolvability of their intersection lattice. From this, we obtain a combinatorially determined class of $K(\pi, 1)$ toric arrangements. Moreover, we characterize combinatorially when our toric arrangement bundles are pulled back from Fadell–Neuwirth's bundles of configuration spaces, and establish an analogue of Falk–Randell's formula relating the Poincaré polynomial to the lower central series of the fundamental group.

Keywords: supersolvable lattice, hyperplane arrangement, configuration space

1 Introduction

The theory of **supersolvable lattices** is a cornerstone of enumerative, algebraic and topological combinatorics. Its foundations were laid in work by Stanley [20, 19], motivated by the study of subgroups in supersolvable groups and building on the classical notion of **modular elements** in lattices.

Supersolvable lattices arise in a variety of contexts, e.g., the study of factorizations of characteristic polynomials [16] and of shellable posets [6], as well as in convex geometry [1] and representation theory [8]. More generally, modularity is a key concept in lattice theory, see [5, II.§7]. Several of these connections interact in the context of matroid theory, where modular flats exhibit a rich structure [7].

A seminal result by Terao [21] shows the equivalence between supersolvability of the lattice of flats of a matroid and an inductive fibration property of the complement

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manifold of any **arrangement of hyperplanes** that realizes the given matroid over $\mathbb{C}$. This opens up a powerful bridge between combinatorics and topology.

More precisely, Terao's result states that the intersection lattice of a complex hyperplane arrangement $\mathcal{A}$ is supersolvable if and only if the arrangement $\mathcal{A}$ is "fiber type", i.e., there is a tower of arrangements $\emptyset = \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \ldots \subsetneq \mathcal{A}_d = \mathcal{A}$ such that the natural projection of complements $M(\mathcal{A}_i) \to M(\mathcal{A}_{i-1})$ is a linear fibration. Falk and Randell's study of fiber-type arrangements [14] unveiled a wealth of combinatorial-topological structure echoeing classical features of configuration spaces. This includes for instance a combinatorial formula for the lower central series of the fundamental group of the arrangement's complement [14, Theorem 4.1] and the proof that the fibrations arising in fiber-type arrangements are pullbacks from the classical Fadell-Neuwirth bundle for the configuration space of points in the plane [9].

A major point of interest of fiber-type arrangements is related to the long-standing $K(\pi, 1)$-problem for hyperplane arrangements, asking for a combinatorial characterization of asphericity of the arrangement's complement. Indeed, fiber-type hyperplane arrangements have aspherical complements, and they are characterized by a combinatorial condition: supersolvability of the intersection lattice.

Here we devise a general theory of modular elements and supersolvability for posets beyond lattices – so-called "geometric posets" (see Definition 1). On the combinatorial side we derive some fundamental results about factorization of characteristic polynomials (Theorem 2) and quotients by poset automorphisms (Theorem 1).

When the geometric poset is a semilattice, our definition of supersolvability agrees with that given by Falk and Terao [15] in studying intersection posets of affine hyperplane arrangements. Moreover, we prove that a geometric semilattice is supersolvable if and only if its canonical extension to a geometric lattice is supersolvable [2, Theorem 4.2.4]. This leads to a first topological consequence of our work: an affine analogue of Terao's fibration theorem [2, Theorem 4.3.3].

Just as for classical lattice supersolvability, our theory has a strong topological counterpart in terms of **toric arrangements** (see Definition 6). Indeed, the notion of "geometric poset" Definition 1 seems to provide the right level of generality for studying intersection data of an arrangement of subtori in a complex torus.

The study of toric arrangements is a recent field of research that has given rise to combinatorial structures such as arithmetic Tutte polynomials, arithmetic matroids, matroid schemes and group actions on semimatroids [18, 10, 4], to name a few. In particular, the poset of intersections of a toric arrangement has been studied from different points of view [17, 12]. Our notion of supersolvability applies to intersection posets of toric arrangements, and has several implications for the topology of the arrangement complement, see §3.

2 Supersolvable posets

We recall basic ideas about posets and supersolvable geometric lattices. We then define M- and TM-ideals (Definition 3) and the corresponding notions of supersolvability (Definition 4).

2.1 Generalities about posets

Let $\mathcal{P}$ be a partially ordered set (or "poset") with partial order relation $\leq$. For $x, y \in \mathcal{P}$ write $x < y$ when $x \leq y$ and $x \neq y$, and $x \lessdot y$ when $x < y$ and $x \leq z < y$ implies $x = z$. Given any $x \in \mathcal{P}$ let $\mathcal{P}_{<x} := \{y \in \mathcal{P} \colon y < x\}$, partially ordered by the restriction of $<$. The posets $\mathcal{P}_{\leq x}$, $\mathcal{P}_{>x}$ and $\mathcal{P}_{\geq x}$ are defined analogously. The *interval* between two elements $x, y \in \mathcal{P}$ is the set $[x, y] := \mathcal{P}_{\geq x} \cap \mathcal{P}_{\leq y}$. We refer to [20] for standard poset terminology and notation. Departing slightly from standard notation, for any two elements $x, y \in \mathcal{P}$, we define $x \vee y$ to be the *set* of minimal upper bounds and $x \wedge y$ to be the set of maximal lower bounds. That is:

$$x \vee y := \min\{z \in \mathcal{P} \colon z \geq x \text{ and } z \geq y\}, \quad x \wedge y := \max\{z \in \mathcal{P} \colon z \leq x \text{ and } z \leq y\}.$$

More generally, denote by $\bigvee T$ and $\bigwedge T$ the sets of minimal upper bounds and maximal lower bounds of a set $T \subseteq \mathcal{P}$.

A *complement* of an element x in a chain-finite poset $\mathcal{P}$ is any $z \in \mathcal{P}$ such that $x \vee z \subseteq \max \mathcal{P}$ and $x \wedge z \subseteq \min \mathcal{P}$. Given a subset $X \subseteq \mathcal{P}$ we say that $z \in \mathcal{P}$ is a complement to X if z is a complement of every $x \in X$.[1]

2.2 Locally geometric posets

Recall that a chain-finite lattice $\mathcal{L}$ is called **geometric** if and only if, for all $x, y \in \mathcal{L}$:

$$x \lessdot y \text{ if and only if there is an atom } a \in A(\mathcal{L}) \text{ with } a \not\leq x,\ y = x \vee a.$$

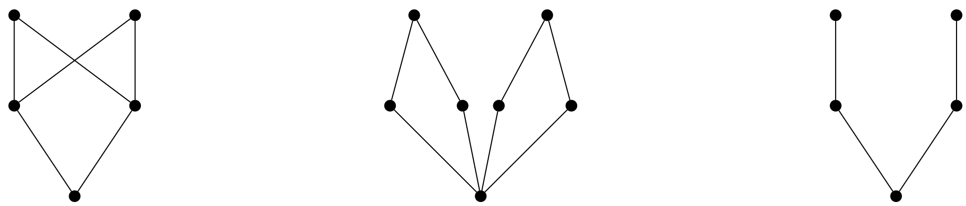
Definition 1 (Locally geometric and geometric posets)**.** A graded, bounded below poset $\mathcal{P}$ is **locally geometric** if, for every $x \in \mathcal{P}$, the subposet $\mathcal{P}_{\leq x}$ is a geometric lattice. A locally geometric poset $\mathcal{P}$ is **geometric** if for all $x, y \in \mathcal{P}$:

(‡‡) if $\mathrm{rk}(x) < \mathrm{rk}(y)$ and $I \subseteq A(\mathcal{P})$ is such that $\bigvee I \ni y$ and $|I| = \mathrm{rk}(y)$, then there is $a \in I$ such that $a \not\leq x$ and $a \vee x \neq \emptyset$.

Remark 1. We do not require $\mathcal{P}$ itself to even be a (semi)lattice. If $\mathcal{P}$ is a lattice, then it is locally geometric if and only if it is geometric. A geometric (semi)lattice in the sense of [22] is precisely a (semi)lattice satisfying condition (‡‡). Further note that if $\mathcal{P}$ is locally geometric, then so are $\mathcal{P}_{\leq x}$ and $\mathcal{P}_{\geq x}$ for any $x \in \mathcal{P}$.

[1] Notice that this definition generalizes the usual one for lattices.

Example 1. A classical example of a geometric lattice is a Boolean lattice B_n, the set of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion. A simplicial poset, in which every closed interval is isomorphic to a Boolean lattice, is then a locally geometric poset. One such example is depicted in Figure 1a: this is a geometric poset that is not a lattice nor a semilattice.



(a) A geometric poset. (b) A locally geometric but non-geometric poset (c) A ranked, but not locally geometric, poset.

Figure 1

Example 2. The poset of Figure 1b is locally geometric but not geometric.

2.3 Supersolvable geometric lattices

There are several equivalent definitions for a modular element in a geometric lattice (see eg. [7, Theorem 3.3]). The one we state below is the most useful for our setting and is due to Stanley [19]. We also extend Stanley's definition of supersolvable lattices [20, Corollary 2.3] to the context of *chain-finite* lattices.

Let $\mathcal{L}$ be a chain-finite lattice. Then $\mathcal{L}$ has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Let $x \in \mathcal{L}$. The *complements* of x in $\mathcal{L}$ are the elements $y \in \mathcal{L}$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$.

Definition 2. An element x in a geometric lattice $\mathcal{L}$ is **modular** if the complements of x form an antichain. A geometric lattice $\mathcal{L}$ is **supersolvable** if there is a chain $\hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$ where each y_i is a modular element with $\mathrm{rk}(y_i) = i$.

2.4 Ideals in locally geometric posets

Let $\mathcal{P}$ be a locally geometric poset. An *order ideal* in $\mathcal{P}$ is a downward-closed subset. An order ideal is *pure* if all maximal elements have the same rank. An order ideal $\mathcal{Q}$ is *join-closed* if $T \subseteq \mathcal{Q}$ implies $\bigvee T \subseteq \mathcal{Q}$.

Definition 3 (M-ideals and TM-ideals)**.** An **M-ideal** of a locally geometric poset $\mathcal{P}$ is a pure, join-closed order ideal $\mathcal{Q} \subseteq \mathcal{P}$ such that:

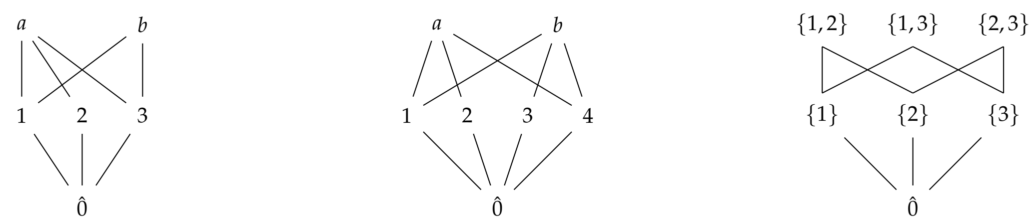
(1) if $y \in \mathcal{Q}$ and $a \in A(\mathcal{P})$ such that $a \vee y = \emptyset$ then $a \in \mathcal{Q}$, and

(2) for every $x \in \max(\mathcal{P})$, there is some $y \in \max(\mathcal{Q})$ such that y is a modular element in the geometric lattice $\mathcal{P}_{\leq x}$.

An M-ideal $\mathcal{Q}$ in a locally geometric poset $\mathcal{P}$ is a **TM-ideal** if $|a \vee y| = 1$ for all $y \in \mathcal{Q}$ and all $a \in A(\mathcal{P}) \setminus A(\mathcal{Q})$.

Remark 2. Our definition of an M-ideal extends Definition 2: An order ideal $\mathcal{Q}$ in a geometric lattice $\mathcal{L}$ is an M-ideal if and only if $\mathcal{Q} = \mathcal{L}_{\leq y}$ for some modular element y.

Example 3. Consider the poset $\mathcal{P}$ in Figure 2b. The subposet $\{\hat{0}, 2, 3\}$ is a TM-ideal; the subposet $\{\hat{0}, 4\}$ is an M-ideal that is not a TM-ideal. Note that in every locally geometric poset $\mathcal{P}$, both $\mathcal{P}$ and $\{\hat{0}\}$ are M-ideals.



(a) A supersolvable, but not strictly supersolvable, poset.

(b) A strictly supersolvable poset.

(c) A "locally" supersolvable, but not supersolvable, poset.

Figure 2

2.5 Supersolvability in geometric posets

We are now prepared to present our definition of a supersolvable locally geometric poset, which extends the definition of a supersolvable geometric lattice (cf. Definition 2).

Definition 4. A locally geometric poset $\mathcal{P}$ is **supersolvable** if there is a chain

$$\hat{0} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_n = \mathcal{P}$$

where each $\mathcal{Q}_i$ is an M-ideal of $\mathcal{Q}_{i+1}$ with $\mathrm{rk}(\mathcal{Q}_i) = i$. If moreover every $\mathcal{Q}_i$ is a TM-ideal of $\mathcal{Q}_{i+1}$ we call $\mathcal{P}$ **strictly supersolvable**.

Example 4. Any rank-one locally geometric poset is strictly supersolvable. The poset $\mathcal{P}$ from Example 3 is strictly supersolvable via the chain $\hat{0} \subset \{\hat{0}, 2, 3\} \subset \mathcal{P}$.

Example 5. The poset $\mathcal{P}$ from Figure 2a is not strictly supersolvable: its only proper M-ideals are $\mathcal{P}_{\leq 1}$ and $\mathcal{P}_{\leq 3}$, and the fact that $|1 \vee 3| = 2$ means neither is a TM-ideal.

If a locally geometric poset is supersolvable, then every closed interval $\mathcal{P}_{\leq x}$ is a supersolvable geometric lattice. However, this "local" supersolvability is not enough for $\mathcal{P}$ itself to be supersolvable, as demonstrated in the following example.

Example 6. Consider the poset $\mathcal{P}$ depicted in Figure 2c. Every closed interval in $\mathcal{P}$ is supersolvable (since every Boolean lattice is), however it is not itself supersolvable. Indeed, the only proper order ideals which are pure and join-closed are principal, that is, $\mathcal{P}_{\leq x}$ for some rank-one element x. However, such an order ideal cannot satisfy Definition 3.(2) since no single element is covered by all maximal elements.

Remark 3. A geometric lattice $\mathcal{L}$ satisfies Definition 4 if and only if it satisfies the supersolvability criterion of Definition 2. In a geometric semilattice $\mathcal{L}$, M-ideals and TM-ideals are equivalent, thus $\mathcal{L}$ is supersolvable if and only if it is strictly supersolvable.

For geometric posets, M-ideals can be characterized using partitions of atoms [2, Theorem 4.1.2.], providing the following characterization of supersolvability, reminiscent of [15, Remark 2.6] for geometric semilattices.

Proposition 1. *([2, Corollary 4.1.3]) Let $\mathcal{P}$ be a geometric poset. Then $\mathcal{P}$ is supersolvable if and only if there is a chain $\{\hat{0}\} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \ldots \subset \mathcal{Q}_n = \mathcal{P}$ of pure, join-closed order ideals of $\mathcal{P}$ with $\operatorname{rk}(\mathcal{Q}_i) = i$ and so that for every $i = 1, \ldots, n$ and any two distinct $a_1, a_2 \in A(\mathcal{Q}_i) \setminus A(\mathcal{Q}_{i-1})$ and every $x \in a_1 \vee a_2$ there is $a_3 \in A(\mathcal{Q}_{i-1})$ with $x > a_3$.*

Example 7. Dowling posets [3] form a class of locally geometric posets that generalize partition lattices and Dowling lattices, which are known to be supersolvable geometric lattices [11, 20]. We can show [2, Proposition 2.6.1.] that, for any positive integer n, finite group G, and finite G-set S, the Dowling poset $D_n(G, S)$ is strictly supersolvable.

2.6 Group actions

Let G be a group. An *action* of G on a poset $\mathcal{P}$ is any group homomorphism $G \to \operatorname{Aut}(\mathcal{P})$ from G to the group of automorphisms of $\mathcal{P}$. Given a group element $g \in G$ it is customary to denote the associated automorphism by $g : \mathcal{P} \to \mathcal{P}$. For $x \in \mathcal{P}$ we will write gx for $g(x)$. Following [10], we focus on the following special type of action.

Definition 5. Let $\mathcal{P}$ be a poset with an action of a group G. We call the action **translative** if $x \vee gx \neq \emptyset$ implies $x = gx$ for every $x \in \mathcal{P}$ and every $g \in G$.

Write $Gx = \{gx \colon g \in G\}$ for the orbit of an $x \in \mathcal{P}$ under G. On the set of orbits $\mathcal{P}/G := \{Gx \colon x \in \mathcal{P}\}$ we consider the relation given by $Gx \leq Gy$ if there is $g \in G$ with $x \leq gy$. If the action is translative, this is a partial order relation on $\mathcal{P}/G$.

Theorem 1. *Let $\mathcal{P}$ be a locally geometric poset with a translative action of a group G and let $\mathcal{Q}$ be a G-invariant subposet of $\mathcal{P}$. If $\mathcal{Q}$ is an M-ideal in $\mathcal{P}$, then $\mathcal{Q}/G$ is an M-ideal in $\mathcal{P}/G$. Moreover, if $\mathcal{P}$ satisfies (‡‡), the converse also holds.*

2.7 Characteristic polynomial

The *characteristic polynomial* of any bounded-below poset $\mathcal{P}$ with a rank function rk is

$$\chi_{\mathcal{P}}(t) := \sum_{x \in \mathcal{P}} \mu_{\mathcal{P}}(x) t^{\mathrm{rk}(\mathcal{P}) - \mathrm{rk}(x)},$$

where $\mu_{\mathcal{P}}$ is the Möbius function of $\mathcal{P}$. A feature of supersolvable geometric lattices is that their characteristic polynomial decomposes into linear factors over $\mathbb{Z}$. We show that this is true also for *strictly* supersolvable posets.

Theorem 2. *Let $\mathcal{Q}$ be a TM-ideal of a locally geometric poset $\mathcal{P}$ with $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{P}) - 1$, and let $a = |A(\mathcal{P}) \setminus A(\mathcal{Q})|$. Then*

$$\chi_{\mathcal{P}}(t) = \chi_{\mathcal{Q}}(t) \cdot (t - a).$$

In particular, if $\mathcal{P}$ is strictly supersolvable via the chain of TM-ideals $\hat{0} = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_n = \mathcal{P}$, and $a_i = |A(\mathcal{Q}_i) \setminus A(\mathcal{Q}_{i-1})|$ for each i, then

$$\chi_{\mathcal{P}}(t) = \prod_{i=1}^{n} (t - a_i).$$

Remark 4. The assumption that $\mathcal{Q}$ is a TM-ideal in Theorem 2 is necessary, as demonstrated in the following examples. Accordingly, a poset being supersolvable is not enough for its characteristic polynomial to factor completely over $\mathbb{Z}$.

Example 8. Consider the poset $\mathcal{P}$ depicted in Figure 2 (see also Example 12). Its characteristic polynomial is

$$\chi_{\mathcal{P}}(t) = t^2 - 4t + 4 = (t-2)(t-2).$$

This agrees with the fact that the TM-ideal $\mathcal{Q} = \{\hat{0}, 2, 3\}$ in Figure 2b has $\chi_{\mathcal{Q}}(t) = t - 2$ and $|A(\mathcal{P}) \setminus A(\mathcal{Q})| = 2$.

Example 9. Consider again the poset $\mathcal{P}$ in Figure 2a. It is supersolvable, with $\{\hat{0}, 1\}$ and $\{\hat{0}, 3\}$ both M-ideals. However, it is not strictly supersolvable and its characteristic polynomial $\chi_{\mathcal{P}}(t) = t^2 - 3t + 3$ does not factor over the integers.

3 Toric Arrangement Bundles

3.1 Toric Arrangements

Fix a finitely generated free abelian group $\Gamma \cong \mathbb{Z}^d$ and let $T = \mathrm{Hom}(\Gamma, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^d$ be the complex torus.

Definition 6. A **toric arrangement** is a collection $\{H_\alpha \colon \alpha \in \mathcal{A}\}$ for some finite set $\mathcal{A} \subseteq \Gamma$, where

$$H_\alpha := \{t \in T \colon t(\alpha) = 0\}.$$

We will often refer to an arrangement $\{H_\alpha \colon \alpha \in \mathcal{A}\}$ simply by $\mathcal{A}$ when there is no confusion. The **complement** of $\mathcal{A}$ is denoted by

$$M(\mathcal{A}) := T \setminus \bigcup_{\alpha \in \mathcal{A}} H_\alpha.$$

We only consider toric arrangements that are **essential**, i.e., where $\mathcal{A}$ generates a full subgroup of Γ.

Example 10. Let $\Gamma = \mathbb{Z}^2$. The arrangement $\mathcal{A} = \{\alpha_1 = (1,0), \alpha_2 = (0,1), \alpha_3 = (1,2)\}$ yields three subtori in $\mathbb{C}^\times \times \mathbb{C}^\times$, cut out by equations $x = 1$, $y = 1$, and $xy^2 = 1$. The real part, in $S^1 \times S^1$, is depicted in Figure 3a.

(a) The arrangement $\mathcal{A}$ from Example 10 is depicted in $S^1 \times S^1$, with H_1 in green, H_2 in red, and H_3 in blue.

(b) The poset of layers $\mathcal{P}(\mathcal{A})$.

Figure 3

3.2 Poset of layers

The intersection data of a toric arrangement may be encoded in a geometric poset.

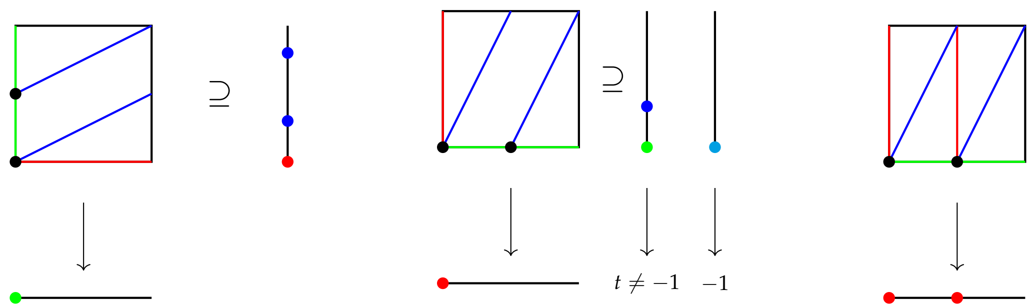
Definition 7. The **poset of layers** of an arrangement $\mathcal{A}$ is the set $\mathcal{P}(\mathcal{A})$ whose elements are the nonempty connected components of intersections $\bigcap_{\alpha \in S} H_\alpha$ where $S \subseteq \mathcal{A}$, partially ordered by reverse inclusion.

By convention, T is the unique minimum element of $\mathcal{P}(\mathcal{A})$. The atoms of $\mathcal{P}(\mathcal{A})$ are precisely the connected components of the H_α, where $\alpha \in \mathcal{A}$.

Remark 5. The poset of layers for a toric arrangement is indeed a geometric poset. The lift of all H_α, $\alpha \in \mathcal{A}$, to the universal cover of T is an arrangement $\mathcal{A}^{\upharpoonleft}$ of affine subspaces in $\mathbb{C}^d$. Its poset of layers $\mathcal{P}(\mathcal{A}^{\upharpoonleft})$ is a geometric semilattice and the action on $\mathcal{A}^{\upharpoonleft}$ of the group of deck transformations induces a translative action of $\mathbb{Z}^d$ on $\mathcal{P}(\mathcal{A}^{\upharpoonleft})$. Then $\mathcal{P}(\mathcal{A})$ is isomorphic to the quotient $\mathcal{P}(\mathcal{A}^{\upharpoonleft})/\mathbb{Z}^d$ (see [10, Lemma 9.8]), and thus it is geometric (via Theorem 1).

Example 11. Let $\mathcal{A}$ be the toric arrangement from Example 10. The Hasse diagram for its poset of layers $\mathcal{P}(\mathcal{A})$ is depicted in Figure 3b. Notice that this poset was seen in Figure 2a and is supersolvable, with M-ideal given by $\mathcal{P}_{\leq H_1}$ or $\mathcal{P}_{\leq H_3}$ (see Example 5).

Example 12. Let $\Gamma = \mathbb{Z}^2$ and $\mathcal{A} = \{\alpha_1 = (1,0), \alpha_2 = (0,2), \alpha_3 = (1,2)\}$. Figure 4c depicts the corresponding H_1, H_2, and H_3 in $S^1 \times S^1$ and Figure 2b depicts the Hasse diagram for its poset of layers. As seen in Example 3 this poset is strictly supersolvable; the maximal elements of its proper TM-ideal are the two connected components of H_2.



(a) A fibration whose fibers are homeomorphic to S^1 with three punctures.

(b) This is not a fibration, as indicated by the two non-homeomorphic fibers.

(c) A fibration whose fibers are homeomorphic to S^1 with two punctures.

Figure 4: Each figure represents a restriction of the projection $S^1 \times S^1 \to S^1$.

3.3 Characterization of fibrations

A subgroup Y of T will be called **admissible** if there is a rank-one direct summand $\Gamma' \subseteq \Gamma$ such that Y is the image of the injection $\varepsilon^* : \mathrm{Hom}(\Gamma', \mathbb{C}^\times) \to \mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ induced by the projection $\varepsilon : \Gamma \to \Gamma'$. When Y is admissible, the corresponding projection

$$p : T \to T/Y \cong \mathrm{Hom}(\Gamma/\Gamma', \mathbb{C}^\times)$$

is a section of the map induced by the quotient $q : \Gamma \to \Gamma/\Gamma'$. This allows us to define toric arrangements

$$\mathcal{A}_Y := \{\alpha \in \mathcal{A} \colon H_\alpha \supseteq Y\} \qquad\qquad \mathcal{A}/Y := q(\mathcal{A}_Y) \subseteq \Gamma/\Gamma',$$

in T and in T/Y, respectively. Then the projection $p : T \to T/Y$ restricts to a map on arrangement complements $\bar{p} : M(\mathcal{A}) \to M(\mathcal{A}/Y)$ and induces an isomorphism of posets $\mathcal{P}(\mathcal{A}_Y) \cong \mathcal{P}(\mathcal{A}/Y)$.

Definition 8. An arrangement $\mathcal{A}$ is **fiber-type** if there is a chain of subgroups $Y_1 \subseteq \ldots Y_{d-1} \subseteq Y_d = (\mathbb{C}^\times)^d$ such that for every projection $p_i : Y_i \to Y_i/Y_{i-1}$ the induced map $\bar{p}_i$ is a fibration on arrangement complements whose fiber is homeomorphic to $\mathbb{C}^\times$ minus a finite set of points.

Remark 6. The poset of layers $\mathcal{P}(\mathcal{A}_Y)$ may be viewed as a subposet of $\mathcal{P}(\mathcal{A})$. Its atoms are the atoms of $\mathcal{A}$ that either contain Y or are disjoint from it. For any $\alpha \notin \mathcal{A}_Y$, every connected component of H_α will intersect Y nontrivially. Moreover, if $Y \in \mathcal{P}(\mathcal{A})$, then the maximal elements of $\mathcal{P}(\mathcal{A}_Y)$ are cosets of Y.

We prove in [2, Theorem 3.3.1.] that $\mathcal{P}(\mathcal{A}_Y)$ is an M-ideal of $\mathcal{P}(\mathcal{A})$ if and only if there is an integer ℓ such that the fibers of the projection $M(\mathcal{A}) \to M(\mathcal{A}/Y)$ are all homeomorphic to $\mathbb{C}^\times$ with ℓ points removed. The number of punctures can be counted by examining how the hypersurfaces not in $\mathcal{P}(\mathcal{A}_Y)$ intersect Y or its translates. When $\mathcal{P}(\mathcal{A}_Y)$ is an M-ideal, the map is moreover locally trivial. Iterating this then yields:

Theorem 3 (Fibration Theorem [2, Theorem A]). *An essential toric arrangement is fiber-type if and only if its poset of layers is supersolvable.*

Example 13. Consider the arrangement from Example 10 (see also Figure 2a). Letting $Y = H_1$, the projection $M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is depicted in Figure 4a. As the picture suggests, this map is a fibration with fiber homeomorphic to T with three points removed. On the other hand, letting $Y = H_2$ the projection $M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is not a fibration. This is evident in Figure 4b, which depicts two non-homeomorphic fibers.

This agrees with our observation in Example 3 that in the poset of layers $\mathcal{P} = \mathcal{P}(\mathcal{A})$, the order ideal $\mathcal{P}_{\leq H_1}$ is an M-ideal while $\mathcal{P}_{\leq H_2}$ is not.

From Theorem 3, Falk and Randell's arguments in [14] can be adapted to prove the following results.

Theorem 4 (Asphericity, [2, Corollary B]). *If the poset of layers of a toric arrangement is supersolvable, then the arrangement complement is a $K(\pi, 1)$ space. If the poset is strictly supersolvable, then the fundamental group is an iterated semidirect product of free groups.*

Theorem 5 (Lower Central Series Formula, [2, Theorem D]). *Let $\mathcal{A}$ be a strictly supersolvable toric arrangement with complement $M(\mathcal{A})$, let $\mathcal{A}_0 = \emptyset \subsetneq \mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_n$ be the associated tower of arrangements and set $a_i := |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|$ for $i = 1, \ldots, n-1$. For $j \geq 1$, let φ_j be the rank of the jth successive quotient in the lower central series of the fundamental group $\pi_1(M(\mathcal{A}))$. Then*

$$\prod_{j=1}^{\infty}(1 - t^j)^{\varphi_j} = \prod_{i=1}^{n}(1 - (a_i + 1)t). \tag{3.1}$$

The right-hand side of (3.1) encodes the Betti numbers, and is a specialization of the characteristic polynomial for the associated poset of layers (see Theorem 2).

3.4 Pullback of Fadell–Neuwirth bundles

Suppose that $\bar{p}\colon M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is a toric arrangement bundle arising from a TM-ideal $\mathcal{Q} = \mathcal{P}(\mathcal{A}_Y)$, and fix an order $H_1, H_2, \dots, H_\ell$ of the atoms in $\mathcal{P}(\mathcal{A})$ that are not in $\mathcal{Q}$. The definition of a TM-ideal implies that for any $x \in M(\mathcal{A}/Y)$, and any $1 \leq i \leq \ell$, there is a unique point in $H_i \cap p^{-1}(x)$. Through the identification $p^{-1}(x) \cong \mathbb{C}^\times \subseteq \mathbb{C}$, this defines a continuous map $g_i\colon M(\mathcal{A}) \to \mathbb{C}$. Via Proposition 1, the points $g_1(x), \dots, g_\ell(x)$ must be distinct and nonzero, thus determining a point in the ordered configuration space $\mathrm{Conf}_{\ell+1}(\mathbb{C}) = \{(z_0, \dots, z_\ell) \in \mathbb{C}^{\ell+1}\colon z_i \neq z_j \text{ when } i \neq j\}$.

In fact, the bundle $\bar{p}$ is pulled back from Fadell–Neuwirth's bundle of configuration spaces (as in [13]) through this map g. Consequently, properties of Fadell–Neuwirth's bundles (eg. existence of a section, trivial homological monodromy) may thus be pulled back to obtain properties of toric arrangement bundles.

Theorem 6 ([2, Theorem 5.3.1])**.** *The map $g : M(\mathcal{A}) \to \mathrm{Conf}_{\ell+1}(\mathbb{C})$ given by $g(x) = (0, g_1(x), \dots, g_\ell(x))$ is continuous and yields the following pullback diagram.*

$$\begin{array}{ccc} M(\mathcal{A}) & \xrightarrow{\ h\ } & \mathrm{Conf}_{\ell+2}(\mathbb{C}) \\ \Big\downarrow{\scriptstyle \bar{p}} & & \Big\downarrow{\scriptstyle \pi} \\ M(\mathcal{A}/Y) & \xrightarrow{\ g\ } & \mathrm{Conf}_{\ell+1}(\mathbb{C}) \end{array}$$

Example 14. Let $\mathcal{A}$ be the arrangement of Example 12. The fiber bundle depicted in Figure 4c is pulled back through the map $g : M(\mathcal{A}/Y) \to \mathrm{Conf}_3(\mathbb{C})$, $g(x) = (0, 1, x^2)$.

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