RECTIFIABILITY AND TANGENTS IN A ROUGH RIEMANNIAN SETTING

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Abstract. Characterizing rectifiability of Radon measures in Euclidean space has led to fundamental contributions to geometric measure theory. Conditions involving existence of principal values of certain singular integrals [MP95] and the existence of densities with respect to Euclidean balls [Pre87] have given rise to major breakthroughs. We explore similar questions in a rough elliptic setting where Euclidean balls $B(a,r)$ are replaced by ellipses $B_\Lambda(a,r)$ whose eccentricity and principal axes depend on a.

Precisely, given $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$, we first consider the family of ellipses $B_\Lambda(a,r) = a + \Lambda(a)B(0,r)$ and show that almost everywhere existence of the principal values

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n\setminus B_\Lambda(a,\epsilon)} \frac{\Lambda(a)^{-1}(y-a)}{|\Lambda(a)^{-1}(y-a)|^{m+1}} d\mu(y) \in (0,\infty)$$

implies rectifiability of the measure μ under a positive lower density condition. Second we characterize rectifiability in terms of the almost everywhere existence of

$$\theta^m_{\Lambda(a)}(\mu,a) = \lim_{r\downarrow 0}\frac{\mu(B_\Lambda(a,r))}{r^m} \in (0,\infty).$$

1. Introduction

In this paper, we study two classical questions from geometric measure theory: Does rectifiability of a measure follow from its density properties? Does rectifiability of a measure follow from the existence of principal values of singular integrals?

The origins of Geometric Measure Theory can be traced back to the 1920s and 1930s when Besicovitch began studying the density question for 1-dimensional sets in the plane, [Bes28, Bes38]. A modern formulation of Besicovitch's results is that if we let $m = 1$, $n = 2$, and $\mu = \mathcal{H}^m \llcorner E$ be a Radon measure for some Borel $E \subset \mathbb{R}^n$ then whenever

$$0 < \mu(\mathbb{R}^n) < \infty \quad \text{and} \quad 0 < \lim_{r\downarrow 0}\frac{\mu\left(B(x,r)\right)}{\omega_m r^m} < \infty \quad \mu - \text{a.e. } x, \tag{1.1}$$

it follows E is m-rectifiable. In [MR44], it was shown that when $m = 1$, $n = 2$, and μ is any Radon measure, (1.1) implies rectifiability of μ. The extension $n \geq 2$ was provided in [Moo50]. Federer proved [Fed47] a general converse to Besicovitch's question, i.e., that is m-rectifiable measures have positive and finite density almost everywhere. In [Mar61] the first step to considering m-dimensional sets for $m \geq 2$ was made, proving that 2-dimensional sets in $\mathbb{R}^n$ with density one at almost every point are rectifiable. In [Mar64], Marstrand showed that if the s density of a measure exists on a set of positive measure then s is an integer. Finally the density question for sets in $\mathbb{R}^n$ was resolved in [Mat75], where Mattila proved that if $\mu = \mathcal{H}^m \llcorner E$ has density 1-almost everywhere, then E is rectifiable. Preiss ultimately resolved the density question for measures in Euclidean space in [Pre87], see Theorem 1.5. The introduction of [Pre87] is also a great source for a detailed history of this problem and brief description of the difficulties that needed to be overcome for each subsequent generalization.

Another fundamental problem in geometric measure theory is understanding the relationship between the regularity of a set or measure and the behavior of singular integral operators on that set or measure. In the quantitative setting, David and Semmes [DS91, DS93] showed that the L^2-boundedness of all singular integral operators of Calderon-Zygmund type is equivalent to uniform rectifiability of Ahlfors regular measures. They conjectured that the L^2-boundedness of the Riesz transform should be sufficient to imply rectifiability of Ahlfors regular measures. In [Mat95, MP95] a qualitative version of this conjecture was shown to be true: existence of principal values of m-dimensional Riesz transform implies rectifiability of measures under reasonable density assumptions. The conjecture of David and Semmes was resolved in the codimension one case in [NVT14]. Since then, there has been success in extending this codimension 1 quantitative characterization to the setting of other singular integrals which arise as the gradient of fundamental solutions to divergent form elliptic PDEs with the "frozen coefficeint method" [KS11, CAMT19, PPT21, MMPT23]. We discuss some of the most relevant new results to the current article in Section 1.1.

To study these problems we introduce a generalization of tangent measures called Λ-tangents. This is in line with the generalization of Preiss' tangent measures to metric groups, see [Mat05]. However, without a metric preserving group action, our methods fall outside those previously used.

1.1. Principal values and rectifiability.

To motivate the first main theorem of this article, Theorem 1.1, consider the setting of a symmetric uniformly elliptic, matrix $A \in \mathbb{R}^{n\times n}$ and the associated operator $L_A := -\operatorname{div}(A\nabla\cdot)$. For dimensions $n \geq 2$, the fundamental solution has gradient given by

$$\nabla_1\Theta(x,y;A) = c_n \frac{A^{-1}(y-x)}{\det(A)^{1/2}\left\langle A^{-1}(y-x), y-x\right\rangle^{n/2}} = c_n \frac{(\Lambda^{-1})^2(y-x)}{\det(\Lambda)|\Lambda^{-1}(y-x)|^n}, \tag{1.2}$$

where Λ is the unique positive definite matrix satisfying $\Lambda^2 = A$. See, for instance, [Mit13]. Given a Radon measure μ, the principal value of the gradient of the single layer potential associated to L_A at x is given by

$$T_A\mu(x) = \lim_{\epsilon\downarrow 0}\int_{|\Lambda^{-1}(y-x)|\geq\epsilon} \nabla_x\Theta(x,y;A)d\mu(y) = \lim_{\epsilon\downarrow 0}\int_{|\Lambda^{-1}(x-y)|\geq\epsilon} \frac{\Lambda^{-2}(y-x)}{|\Lambda^{-1}(y-x)|^n}d\mu(y). \tag{1.3}$$

For the remainder of this paper, $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$, is a matrix valued mapping denoted as $a \mapsto \Lambda(a)$ and $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a uniformly elliptic matrix valued function. Given $m \in \{1, ..., n-1\}$, define

$$T^m_\Lambda\mu(x) := \lim_{\epsilon\downarrow 0}\int_{|\Lambda(x)^{-1}(y-x)|\geq\epsilon} \frac{\Lambda(x)^{-1}(y-x)}{|\Lambda(x)^{-1}(y-x)|^{m+1}}d\mu(y). \tag{1.4}$$

Since multiplication by $\Lambda(x)^{-1}$ is a linear transformation, when $\Lambda(x)^2 = A(x) \equiv A(0)$,

$$\Lambda(x)^{-1}T^{n-1}_\Lambda\mu(x) = T_A\mu(x)$$

recovers the gradient of single layer potential in the constant coefficient codimension 1 setting. Therefore, in this setting, the existence of $T^{n-1}_\Lambda\mu(x)$ and $T_A\mu(x)$ are equivalent. We use $T^m_\Lambda\mu(x)$ as it is more convenient in the geometric setting. Theorem 1.1 states that given lower density bounds on μ, the a.e. existence of $T^m_\Lambda\mu(x)$ implies that a.e. tangents to μ are flat. Assuming upper density bounds on μ, this implies rectifiability. More precisely:

Theorem 1.1. *Suppose $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$ is a measurable function and μ is a finite Borel measure. If $\theta^m_*(\mu,x) > 0$ and $T^m_\Lambda\mu(x)$ exists for μ almost every x, then:*

(1) For μ a.e. x, $\operatorname{Tan}(\mu,x) \subset \mathcal{M}_n$, the space of flat measures in $\mathbb{R}^n$.
(2) If also $\theta^m_*(\mu,x) < \infty$ almost everywhere, then μ is m-rectifiable.

See the definition of $\operatorname{Tan}(\mu,x)$ *and* $\mathcal{M}_n$ *in Sections 2.1 and 2.2.*

We reiterate that the coefficients Λ need not satisfy any continuity assumptions nor have any uniformly controlled eccentricity.

In the case where $\Lambda = I_n$, this theorem was proven in [MP95]. There, the fact that $\mathrm{Tan}(\mu, x) \subset \mathcal{M}_n$ almost everywhere is ultimately the consequence of a (doubly) rotationally-symmetric condition for the tangent measures. In the setting of Theorem 1.1 it is not clear that tangent measures satisfy this type of symmetry. The novelty of our approach is that, taking guidance from what would occur on a Riemannian manifold, we introduce a notion of anisotropic tangent measures called Λ-tangents. They absorb the anisotropy at the level of μ to recover the same symmetry condition for Λ-tangents that was used in [MP95]. After showing this implies a.e. Λ-tangents to μ are flat, we recover a.e. flatness of tangents to μ.

When $m = n-1$, we can additional assume that $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is in $\widetilde{\mathrm{DMO}}$, see Section 4.2, and frame Theorem 1.1 in terms related to the elliptic equation

$$L_A u := -\operatorname{div}(A\nabla u) = 0.$$

Denote the fundamental solution to the equation by Γ_A, that is, $L_A\Gamma_A(\cdot, y) = \delta_y$. When A has sufficiently nice varying coefficients, the expectation is that $\nabla_1\Gamma_A(x, y)$ is close to $\nabla_1\Theta(x, y; A(x))$, but not equal. Still, there is no a priori reason that the existence of principal values of the singular integrals defined with respect to $\nabla_1\Gamma_A(x, y)$ and $\nabla_1\Theta(x, y; A(x))$ are equivalent. Here ∇_1 is used to denote taking the gradient in the first component only. This is a necessary distinction because $\Theta\left(x, y; A(x)\right)$ has multiple entries that depend on x.

However, roughly speaking, estimates from [MMPT23, Lemma 3.12 and 3.13] show that if $A \in \widetilde{\mathrm{DMO}}$ then the principal values in (1.3) and (1.5) converge in an $L^1(\mu)$ sense, see Lemma 4.7 for the formal statement. This yields the following corollary of Theorem 1.1.

Corollary 1.2. *Let μ be a finite Borel measure on $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ a uniformly elliptic matrix-valued function. Suppose $0 < \theta^{n-1}_*(\mu, x) \le \theta^{n-1,*}(\mu, x) < \infty$ and $A \in \widetilde{\mathrm{DMO}}$. If*

$$\lim_{\epsilon\downarrow 0} \int_{|\Lambda(x)^{-1}(y-x)|\ge\epsilon} \nabla_1\Gamma_A(x, y)d\mu(y) < \infty \tag{1.5}$$

for μ a.e. x, then μ is $(n-1)$-rectifiable.

Remark 1.3. Given a positive definite matrix $A \in \mathbb{R}^{n\times n}$, we consider the Finsler p-Laplacian corresponding to the norm $x \mapsto \langle Ax, x\rangle^{1/2}$. When $p = 1 + \frac{n-1}{m}$, that is,

$$L^m_A(\cdot) := -\operatorname{div}\left(\langle A\nabla\cdot, \nabla\cdot\rangle^{\frac{n-1}{2m}} A\nabla\,\cdot\right) \tag{1.6}$$

We note that the function $\widetilde{\Theta}^m(\cdot, y; A) = \langle A^{-1}(x-y), x-y\rangle^{\frac{1-m}{2}}$ solves $L^m_A\widetilde{\Theta}^m(\cdot, y; A) = c_0\delta_y$ for some constant c_0 depending on m, n, A. Therefore, $\Theta^m(\cdot, \cdot; A) = c_0^{-1}\widetilde{\Theta}^m(\cdot, \cdot; A)$ is the fundamental solution for L^m_A. A computation shows that for some $c_1 = c_1(m, n, A)$,

$$c_1\nabla_x\Theta^m(x, y; A) = \frac{A^{-1}(y-x)}{\langle A^{-1}(y-x), y-x\rangle^{\frac{m+1}{2}}} = \Lambda^{-1}\frac{\Lambda^{-1}(x-y)}{|\Lambda^{-1}(x-y)|^{m+1}},$$

where $\Lambda^2 = A$. When A has variable entries, let $\Gamma^m_A(\cdot, y)$ denote the function so that $L^m_A\Gamma^m_A(\cdot, y) = \delta_y$. In analogy to the way Corollary 1.2 is proven, we suspect anytime A has sufficient regularity to ensure that a Finsler p-analog, with $p = 1 + \frac{n-1}{m}$, of Lemma 4.7 holds for some $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$, then under appropriate density assumptions, the existence of principal values of the Finsler p-type single-layer potential should imply rectifiability of μ.

Corollary 1.2 is in the reverse direction of the following theorem from [Pul22, Theorem 3.2]. Unfortunately, Corollary 1.2 and Theorem 1.4 do not provide characterization of rectifiable measures because of the different truncations used to define the principal values.

Theorem 1.4. *Let μ be an $(n-1)$-rectifiable measure on $\mathbb{R}^n$ with compact support. Let $A \in C^\alpha$ be uniformly elliptic. Then for every $f \in L^2(\mu)$ the principal value*

$$\lim_{\epsilon \downarrow 0} \int_{|x-y|>\epsilon} \nabla_1 \mathcal{E}(x,y) f(y) d\mu(y) < \infty$$

exists for μ-almost every x.

At first, one may expect that the quantitative nature of "the L^2-boundedness of a singular integral operator" might be stronger than assuming that principal values exist. However, for measures that are not absolutely continuous with respect to the Lebesgue measure this is a difficult question. Intuitively this is because the existence of a principal value is equivalent to some sort of local symmetry of the measure (or a sufficiently small density). This intuition was recently formalized for measures satisfying an upper-density assumptions [JM20a]. There, it is shown that for a measure with an L^2-bounded Calderon-Zygmund type singular integral operator, existence of principal values is equivalent to either the density of the measure being zero or the measure being symmetric in terms of a transport distance to a family of symmetric measures. Previous proofs that L^2-boundedness of the Riesz transform implies existence of principal values relies on the fact that L^2-boundedness implies rectifiability and then proceed with a careful extension of Calderon-Zygmund estimates to Lipschitz graphs [Mat99, Chapter 20].

It was first shown in [Tol08] that one can drop the lower-density assumption in [MP95]. Additionally, a square function for the center of mass has been used to characterize rectifiable measures [MV09, Vil22] and extend some results to an Ω-symmetric setting [Vil21]. Further results on rectifiability and principal values in various settings can be found in [Ver92, MM94, Huo97, JM20b, JM22a].

1.2. **Densities and rectifiability.** In the seminal work [Pre87], Preiss characterized m-rectifiable measures in terms of the existence of positive and finite densities, see Sections 2.2 and 3 for definitions.

Theorem 1.5 ([Pre87])**.** *Let μ be a Radon measure on $\mathbb{R}^n$ and $0 < \theta^m_*(\mu, a)$ for μ a.e. a in $\mathbb{R}^n$. There exists a dimensional constant $\delta_n > 0$ so that the following are equivalent.*

(1) μ is countably m-rectifiable.
(2) For μ a.e. a, any of the following hold:
  - *i) $0 < \theta^m(\mu, a) < \infty$.*
  - *ii) $\mathrm{Tan}(\mu, a) \subset \mathcal{M}_{n,m}$, the set of m-dimensional flat Radon measures on $\mathbb{R}^n$.*
  - *iii) $\theta^m_*(\mu, a) < \infty$ and $\mathrm{Tan}(\mu, a) \subset \mathcal{M}_n$, the space of flat measures on $\mathbb{R}^n$.*
  - *iv)*
$$\frac{\theta^{m,*}(\mu, a)}{\theta^m_*(\mu, a)} - 1 < \delta_n.$$

The following statement summarizes the main results of Section 5 where we show that in the Riemannian setting even with very rough metric, an analogue of Preiss' theorem holds.

Theorem 1.6. *Let μ be a Radon measure on $\mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$. If $\theta^m_*(\mu, a) > 0$ for μ a.e. $a \in \mathbb{R}^n$ and δ_n is as in Theorem 1.5, the following are equivalent:*

(1) μ is countably m-rectifiable.
(2) For μ almost every a, any of the following hold:
  - *i) $0 < \theta^m_{\Lambda(a)}(\mu, a) < \infty$.*
  - *ii) $\mathrm{Tan}_\Lambda(\mu, a) \subset \mathcal{M}_{n,m}$, the set of m-dimensional flat Radon measures on $\mathbb{R}^n$.*
  - *iii) $\theta^m_*(\mu, a) < \infty$ and $\mathrm{Tan}_\Lambda(\mu, a) \subset \mathcal{M}_n$, the space of flat measures on $\mathbb{R}^n$.*

iv) There exists an invertible matrix $\Lambda(a)$, so that

$$\frac{\theta^{m,*}_{\Lambda(a)}(\mu,a)}{\theta^{m}_{\Lambda(a),*}(\mu,a)} - 1 < \delta_n.$$

Remark 1.7. Depending upon eigenvalues of $\Lambda(a)$, there exist positive finite constants c_a, C_a so that $B_\Lambda(0,c_a) \subset B(0,1) \subset B(0,C_a)$. Hence, in Theorem 1.6 the hypothesis $\theta^m_*(\mu,a) > 0$ is equivalent to assuming $\theta^m_{\Lambda,*}(\mu,a) > 0$.

Remark 1.8. Since Theorem 1.6 holds for arbitrary $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$ condition (i) says that the rectifiability of μ is equivalent to the existence at μ almost every a of some choice of $\Lambda(a)$ so that the density $\theta^m_\Lambda(\mu,a)$ exists.

There has been recent work in the literature extending the results of [Mat75] to other settings. It is extended to some homogeneous groups in [JM22b] and to finite-dimensional strictly convex Banach spaces by the third author in [Wil23]. In the codimension 1 Heisenberg and parabolic settings [Mer22, MMP22] show that the existence of appropriate densities for measures implies rectifiability of the measure. The study of density questions in the Heisenberg group was started by [CT15] which demonstrates that Marstrand's density theorem holds for the Heisenberg group and the study of uniform measures in the Heisenberg group was initiated in [CMT20]. It is also known that locally 2-uniform measures in $\mathbb{R}^3$ with respect to the density $\|\cdot\|_{\ell^\infty}$ are rectifiable, [Lor03].

In early drafts of this work, we used Λ-tangents to prove the equivalences of (1) and (i)-(iii) in Theorem 1.6. While writing this paper, Bernd Kirchheim suggested an alternate proof of the equivalence of (1) and (i). His suggestion could be modified to prove the equivalence of (1) and (iv), cf. Theorem 5.3. We chose to include the original proof of the equivalence of (1) and (i) for completeness, cf. Theorem 5.2.

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2. Background and preliminaries

2.1. **Tangent measures and d-cones.** Whenever we say μ is a Radon measure we assume it is a Radon outer measure. We write $B(x,r) = \{|y-x| \le r\}$ and $U(x,r) = \{|y-x| < r\}$. If $x = 0$ we may simply write B_r and U_r. Whenever $E \subset \mathbb{R}^n$ and $r > 0$, we let $rE = \{rx : x \in E\}$. For each $a \in \mathbb{R}^n$ and $r > 0$, define the translation and scaling map

$$T_{a,r}(y) = \frac{y-a}{r} \qquad \forall\, y \in \mathbb{R}^n.$$

Given a Radon measure μ on $\mathbb{R}^n$ and a Borel $T : \mathbb{R}^n \to \mathbb{R}^n$, denote by $T[\mu]$ the image measure of μ by T, namely $T[\mu](E) = \mu(T^{-1}(E))$. In particular, $T_{a,r}[\mu]$ is defined by

$$T_{a,r}[\mu](E) = \mu(T^{-1}_{a,r}(E)) = \mu(a + rE)$$

for all $E \subset \mathbb{R}^n$.

Definition 2.1 (Tangent measures)**.** Let μ be a Radon measure on $\mathbb{R}^n$. We write $\nu \in \mathrm{Tan}(\mu, a)$ and say that ν is a tangent measure to μ at a if ν is a non-zero Radon measure and there exists $c_i > 0$ and $r_i \downarrow 0$ so that

$$c_i T_{a,r_i}[\mu] \overset{*}{\rightharpoonup} \nu,$$

where $\overset{*}{\rightharpoonup}$ denotes convergence in the weak-$*$ sense. Further we write $\mathrm{Tan}[\mu]$ for the weak-$*$ closure of $\cup_{a \in \mathrm{spt}\ \mu} \mathrm{Tan}(\mu, a)$.

For a compact set $K \subset \mathbb{R}^n$, and two Radon measures μ, ν we define

$$F_K(\mu, \nu) = \sup \left\{ \int f d(\mu - \nu) \mid \mathrm{Lip}(f) \leq 1,\ f \in C_c(K) \right\}.$$

If $K = B_r$ we simply write $F_r(\cdot, \cdot)$. We recall, see [Mat99, Lemma 14.13] that for a sequence of Radon measures $\{\mu_k\}$ and a Radon measure μ,

$$\mu_k \overset{*}{\rightharpoonup} \mu \iff \lim_{k \to \infty} F_r(\mu_k, \mu) = 0 \quad \forall r > 0. \tag{2.1}$$

It is well-known, see [Pre87, Proposition 1.12], that

$$F(\mu, \nu) := \sum_{\ell=1}^{\infty} 2^{-\ell} \min\{1, F_\ell(\mu, \nu)\} \tag{2.2}$$

defines a metric on the space of Radon measures. Moreover F generates the topology of weak-$*$ convergence. We denote $F(\mu) = F(\mu, 0)$.

Proposition 2.2. *Let μ be a Radon measure on $\mathbb{R}^n$ and $T, T_i : \mathbb{R}^n \to \mathbb{R}^n$ be proper homeomorphisms, i.e., homeomorphisms so that $T^{-1}(K)$ is compact whenever K is compact. If $\mu_i \overset{*}{\rightharpoonup} \mu$ and T_i, T_i^{-1} converge uniformly on compact subsets to T, T^{-1} respectively, then $T_i[\mu_i] \overset{*}{\rightharpoonup} T[\mu]$.*

Proof. Fix $f \in C_c(\mathbb{R}^n)$ and let $K = \mathrm{spt}\ f$. Since f has compact support, f is uniformly continuous. Let ω denote its modulus of continuity. Since $T_i^{-1} \to T^{-1}$ locally uniformly, if $F_i = T^{-1}(K) \cup T_i^{-1}(K)$, then $F_i \subset F$ for some fixed compact set F. Since $T_i \to T$ locally uniformly, $\delta_i := \|T_i - T\|_{L^\infty(F)} \xrightarrow{i \to \infty} 0$. Therefore,

$$\begin{aligned}
\lim_{i \to \infty} \left| \int f d(T_i[\mu_i] - T[\mu]) \right| &= \lim_{i \to \infty} \left| \int f \circ T_i d\mu_i - f \circ T d\mu \right| \\
&\leq \limsup_i \int |f \circ T_i - f \circ T| d\mu_i + \left| \int f \circ T d(\mu_i - \mu) \right| \\
&\leq \limsup_i \omega(\delta_i) \mu_i(F_i).
\end{aligned}$$

Since $\mu_i \overset{*}{\rightharpoonup} \mu$, and $F_i \subset F$ we know $\limsup_i \mu_i(F_i) \leq \limsup_i \mu_i(F) < \infty$. The proposition follows since $\limsup_i \omega(\delta_i) = 0$. □

The next theorem originates in [Pre87, Theorem 2.12], but our presentation follows [Mat99, Theorem 14.16].

Theorem 2.3. *Let μ be a Radon measure on $\mathbb{R}^n$. Then at μ almost all $a \in \mathbb{R}^n$ every $\nu \in \mathrm{Tan}(\mu, a)$ has the following two properties:*

(1) $T_{x,r}[\nu] \in \mathrm{Tan}(\mu, a)$ for all $x \in \mathrm{spt}\ \nu, r > 0$.
(2) $\mathrm{Tan}(\nu, x) \subset \mathrm{Tan}(\mu, a)$ for all $x \in \mathrm{spt}\ \nu$.

A useful tool for quantifying properties of tangent measures is their distance to d-cones.

Definition 2.4 (Cones, d-cones, and basis)**.** A collection of non-zero Radon measures $\mathcal{M}$ is called a *cone* if $\mu \in \mathcal{M} \implies c\mu \in \mathcal{M}$ for all $c > 0$. A cone of Radon measures is called a *d-cone* if $\mu \in \mathcal{M} \implies T_{0,r}[\mu] \in \mathcal{M}$ for all $r > 0$. The *basis* of a d-cone is the collection of $\mu \in \mathcal{M}$ so that $F_1(\mu) = 1$. We let $\mathcal{M}_B$ denote the basis of $\mathcal{M}$. A d-cone $\mathcal{M}$ is said to have a *closed* (respectively *compact*) *basis* if the basis is closed (respectively compact) with respect to the weak-$*$ topology.

Proposition 2.5. [Pre87, Proposition 2.2] *If a d-cone $\mathcal{M}$ of Radon measures has closed basis, then $\mathcal{M}$ has a compact basis if and only if for every $\lambda \geq 1$ there is a $\tau = \tau(\lambda) > 1$ so that*

$$F_{\tau r}(\mu) \leq \lambda F_r(\mu) \quad \forall \mu \in \mathcal{M} \quad \forall r > 0. \tag{2.3}$$

In this case, $0 \in \operatorname{spt} \mu$ *for all* $\mu \in \mathcal{M}$.

Let $\mathcal{M}$ be a d-cone and ν a Radon measure in $\mathbb{R}^n$. If $s > 0$ and $0 < F_s(\nu) < \infty$ we define the *distance* between ν and $\mathcal{M}$ at scale s by

$$d_s(\nu, \mathcal{M}) = \inf \left\{ F_s\left(\frac{\nu}{F_s(\nu)}, \mu\right) \mid \mu \in \mathcal{M} \text{ and } F_s(\mu) = 1 \right\}. \tag{2.4}$$

If $F_s(\nu) \in \{0, \infty\}$ we define $d_s(\nu, \mathcal{M}) = 1$.

Proposition 2.6. [KPT09, Remark 2.1 and 2.2] *If μ, ν are Radon measures,*

$$F_r(\mu, \nu) = r F_1(T_{0,r}[\mu], T_{0,r}[\nu]). \tag{2.5}$$

If $\mathcal{M}$ is a d-cone and ν a Radon measure,

i) $d_s(\nu, \mathcal{M}) \leq 1$ *for all* $s > 0$.
ii) $d_s(\nu, \mathcal{M}) = d_1\left(T_{0,s}[\nu], \mathcal{M}\right)$ *for all* $s > 0$.
iii) If $\nu_i \overset{*}{\rightharpoonup} \nu$ *and* $F_s(\nu) > 0$, *then* $d_s(\nu, \mathcal{M}) = \lim_{i\to\infty} d_s(\nu_i, \mathcal{M})$.

The ideas behind this next theorem originate in [Pre87, Theorem 2.6], but our presentation is a combination of those in [Pre87, Theorem 2.6] and [KPT09, Theorem 2.1].

Theorem 2.7. *Suppose $\mathcal{F}$ is a closed d-cone with compact basis, μ is a Radon measure, and* $r_0 > 0$.
(1) If there exists $\widetilde{\nu} \in \operatorname{Tan}(\mu, a) \cap \mathcal{F}$, $0 < \epsilon < 1$, and $\nu \in \operatorname{Tan}(\mu, a)$ so that $0 < \epsilon < d_{r_0}(\nu, \mathcal{F})$, then there exists $\nu_\epsilon \in \operatorname{Tan}(\mu, a)$ satisfying

$$\begin{cases} d_{r_0}(\nu_\epsilon, \mathcal{F}) = \epsilon \\ d_r(\nu_\epsilon, \mathcal{F}) \leq \epsilon \quad\quad r > r_0. \end{cases}$$

(2) Suppose $\mathcal{M}$ is a d-cone with closed basis and the property

$$\begin{cases} \exists \epsilon_0 > 0 \textit{ such that } \forall\ \epsilon \in (0, \epsilon_0) \textit{ there exists no } \nu \in \mathcal{M} \\ \textit{satisfying } d_r(\nu, \mathcal{F}) \leq \epsilon\ \forall r \geq r_0 > 0 \textit{ and } d_{r_0}(\nu, \mathcal{F}) = \epsilon. \end{cases} \tag{P}$$

Whenever $a \in \mathbb{R}^n$ is so that

$$\operatorname{Tan}(\mu, a) \subset \mathcal{M} \textit{ and } \operatorname{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset,$$

then $\operatorname{Tan}(\mu, a) \subset \mathcal{F}$.

2.2. **Remarkable d-cones.** Several specific examples of d-cones will play an important role in this article. We introduce here, the space of m-dimensional flat measures in $\mathbb{R}^n$,

$$\mathcal{M}_{n,m} = \{c\mathcal{H}^m \llcorner V : V \in G(n,m) \text{ and } 0 < c < \infty\},$$

where $G(n,m)$ is the space of m-dimensional planes in $\mathbb{R}^n$. We denote the space of flat measures in $\mathbb{R}^n$,

$$\mathcal{M}_n = \bigcup_{m=0}^{n} \mathcal{M}_{n,m}.$$

We also consider the space of uniform measures on $\mathbb{R}^n$,

$$\mathcal{U}(\mathbb{R}^n) = \{\nu : 0 \in \operatorname{spt} \nu \text{ and } \nu(B(x,r)) = \nu(B(y,r)) \quad \forall x, y \in \operatorname{spt} \nu, \quad \forall r > 0\},$$

and the space of m-uniform measures

$$\mathcal{U}^m(\mathbb{R}^n) = \{\nu \in \mathcal{U}(\mathbb{R}^n) : \exists c > 0 \text{ so that } \nu(B(x,r)) = cr^m \ \forall x \in \operatorname{spt} \nu, \ \forall r > 0\}.$$

The next lemma is a remark in [Pre87, Section 3.7(2)].

Lemma 2.8. *The following d-cones have compact basis: $\mathcal{M}_n$, $\mathcal{M}_{n,m}$, $\mathcal{U}^m(\mathbb{R}^n)$, and $\mathcal{U}(\mathbb{R}^n)$.*

2.2.1. *Symmetric measures.* In this section we define the d-cone of symmetric measures and review some of their properties. The information from this section is contained within [MP95], but included here in a condensed fashion for the readers convenience.

Definition 2.9. [MP95, Definition 3.4] Let ν be a non-zero locally finite measure over $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a point of symmetry of ν if

$$\int_{B(x,r)} \langle z - x, y \rangle \, d\nu(z) = 0$$

for every $y \in \mathbb{R}^n$ and every $r > 0$. The measure ν is said to be symmetric if every point in $\operatorname{spt} \nu$ is a point of symmetry. We denote the d-cone of all symmetric measures on $\mathbb{R}^n$ whose support contains $\{0\}$ by $\mathcal{S}_n$.

Lemma 2.10. [MP95, Lemma 3.5] *Let ν be a non-zero locally finite measure over $\mathbb{R}^n$, $s > 0$, and $x \in \mathbb{R}^n$. Then the following three conditions are equivalent.*

(1) x is a point of symmetry of ν.
(2) There exists an $m \in \{1, \dots, n\}$ so that

$$\int_{r \le |x-z| \le R} \frac{x - z}{|x - z|^{m+1}} \, d\nu(z) = 0$$

for all $0 < r < R < \infty$.
(3) For all continuous $g : \mathbb{R} \to \mathbb{R}$ with compact support in $\mathbb{R} \setminus \{0\}$,

$$\int_{\mathbb{R}^n} (x - z) g(|x - z|) \, d\nu(z) = 0.$$

The next lemma states $\mathcal{S}_n$ has two properties which are the hypothesis (iv) and conclusion (b) of [MP95, Lemma 3.2]. The fact that $\mathcal{S}_n$ satisfies the hypotheses of that lemma is verified across [MP95, Lemma 3.6, 3.9, 3.11].

Lemma 2.11. *$\mathcal{S}_n$ has the following properties*

(1) There is $\epsilon_0 > 0$ such that whenever $\nu \in \mathcal{S}_n$ satisfies

$$\limsup_{r \to \infty} d_r \left(\nu, \mathcal{M}_{n,m}\right) < \epsilon_0$$

for some $m = 0, 1, \dots, n$, then the linear span of $\operatorname{spt} \nu$ has dimension at most m.
(2) Suppose $d = \dim V$, $V = \operatorname{span} \operatorname{spt} \nu$, and $\nu \in \mathcal{S}_n$. Then either there exists some $c > 0$ so that $\nu = c\mathcal{H}^d \llcorner V$ or else $\operatorname{Tan}[\nu] \cap \cup_{i=1}^{d-1} \mathcal{M}_{n,i} \neq \emptyset$.

2.2.2. *Uniform measures.* In this section we recall some information about uniform measures. While all the ideas originate in [Pre87],[1] our presentation is heavily influenced by [DL08, Section 6].

Lemma 2.12. [Pre87, Lemma 3.9] *If ν is a uniform measure on $\mathbb{R}^n$, then $\mathcal{M}_n \cap \mathrm{Tan}(\nu, x) \neq \emptyset$ for ν almost every $x \in \mathbb{R}^n$.*

Proposition 2.13. [DL08, Proposition 6.16] *If ν is m-uniform then there exists some m-uniform λ so that, for any sequence $\{r_i\}$ with $r_i \to \infty$,*

$$\lim_{i\to\infty} r_i^{-m} T_{0,r_i}[\nu] = \lambda.$$

Definition 2.14. For $\nu \in \mathcal{U}^m(\mathbb{R}^n)$ we define $\mathrm{Tan}_\infty(\nu) = \{\lambda\}$ where λ is the measure from Proposition 2.13. We call λ the tangent at infinity. Moreover, we say that ν is flat at infinity if $\lambda \in \mathcal{M}_{n,m}$.

Proposition 2.15. [DL08, Propositions 6.18, 6.19] *There exists a constant $\epsilon_0 = \epsilon(m,n)$ so that if $\nu \in \mathcal{U}^m(\mathbb{R}^n)$, $\{\lambda\} = \mathrm{Tan}_\infty(\nu)$, and*

$$d_1(\lambda, \mathcal{M}_{n,m}) \leq \epsilon_0,$$

then $\lambda \in \mathcal{M}_{n,m}$. Moreover, in this case $\nu = \lambda$.

We now show that when $\mathcal{F} = \mathcal{M}_{n,m}$ and $\mathcal{M} = \mathcal{U}^m(\mathbb{R}^n)$ property (P) holds.

Lemma 2.16. *There exists $\epsilon_0 = \epsilon(m,n) > 0$ so that for all $\epsilon \in (0, \epsilon_0]$ there exists no $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ satisfying*

$$\begin{cases} d_r(\mu, \mathcal{M}_{n,m}) \leq \epsilon & \forall r \geq 1 \\ d_1(\mu, \mathcal{M}_{n,m}) = \epsilon. \end{cases} \tag{2.6}$$

Proof. Let ϵ_0 be as in Proposition 2.15. Suppose $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ satisfies (2.6). Let $\lambda = \mathrm{Tan}_\infty(\mu)$. By Propositions 2.6 and 2.13,

$$d_1(\lambda, \mathcal{M}_{n,m}) = \lim_{j\to\infty} d_1(2^{-jm} T_{0,2^j}[\mu], \mathcal{M}_{n,m}) = \lim_{j\to\infty} d_{2^j}(\mu, \mathcal{M}_{n,m}) \leq \epsilon.$$

So, Proposition 2.15 implies $\lambda, \mu \in \mathcal{M}_{n,m}$. This contradicts (2.6). $\square$

3. Λ-Tangents

Consider a mapping $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$ and the ellipse

$$B_\Lambda(a, r) = a + \Lambda(a) B(0, r),$$

whose eccentricity depends on the point a. For a Radon measure μ we define the m-dimensional upper and lower Λ-densities of μ by

$$\theta_\Lambda^{m,*}(\mu, a) = \limsup_{r\downarrow 0} \frac{\mu\left(B_\Lambda(a,r)\right)}{r^m} \text{ and } \theta_{\Lambda,*}^m(\mu, a) = \liminf_{r\downarrow 0} \frac{\mu\left(B_\Lambda(a,r)\right)}{r^m}. \tag{3.1}$$

In the case these two quantities agree, their common value is the m-dimensional Λ-density, denoted $\theta_\Lambda^m(\mu, a)$. When $\Lambda = \mathrm{Id}$, we suppress the dependence on Λ and recover the usual densities with respect to Euclidean balls $\theta^m(\mu, a)$, $\theta^{m,*}(\mu, a)$, and $\theta_*^m(\mu, a)$.

From a PDE perspective, one would assume the mapping Λ should be uniformly elliptic. At the level of rectifiability, geometry is more flexible and allows us to only require that for each a the matrix $\Lambda(a)$ is invertible. Invertibility is necessary since the geometry of a measure near a can be lost if $\Lambda(a)$ collapses $\mathbb{R}^n$ into a lower dimensional space.

[1] See, for instance [Pre87, Proposition 2.11]

We next define the rescaling

$$T^{\Lambda}_{a,r}(y) = \Lambda(a)^{-1}\left(\frac{y-a}{r}\right),$$

and denote image measures under this rescaling by $T^{\Lambda}_{a,r}[\mu]$. That is,

$$T^{\Lambda}_{a,r}[\mu](E) = \mu(a + r\Lambda(a)E).$$

In particular

$$T^{\Lambda}_{a,r}[\mu](B_1)) = T_{a,r}[\mu](B_{\Lambda}(0,1)) = \mu\left(B_{\Lambda}(a,r)\right).$$

Definition 3.1 (Λ-tangents)**.** If μ is a Radon measure, we define

$$\operatorname{Tan}_{\Lambda}(\mu,a) = \left\{\nu \text{ Radon s.t. } \nu = \lim_i c_i T^{\Lambda}_{a,r_i}[\mu] : c_i > 0,\ r_i \downarrow 0,\ \nu \neq 0\right\}. \tag{3.2}$$

Remark 3.2. Given $\nu \in \operatorname{Tan}_{\Lambda}(\mu,a)$ with $c_i T^{\Lambda}_{a,r_i}[\mu] \xrightarrow{*} \nu$ it is easy to check that $cT_{0,r}[\nu] = \lim_i cc_i T^{\Lambda}_{a,rr_i}[\mu]$ for any $c, r > 0$. In particular $\operatorname{Tan}_{\Lambda}(\mu,a)$ is a d-cone.

We will prove that Λ-tangents have a property that implies tangents to Λ-tangents are Λ-tangents.

Theorem 3.3. *Let μ be a Radon measure on $\mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$. Then at μ almost all $a \in \mathbb{R}^n$ every $\nu \in \operatorname{Tan}_{\Lambda}$ has the following two properties:*

(1) $T_{x,r}[\nu] \in \operatorname{Tan}_{\Lambda}(\mu,a)$ *for all* $x \in \operatorname{spt}\nu, r > 0$.
(2) $\operatorname{Tan}(\nu,x) \subset \operatorname{Tan}_{\Lambda}(\mu,a)$ *for all* $x \in \operatorname{spt}\nu$.

One can directly prove Theorem 3.3 by making several modifications to the original proof of Theorem 2.3. Some of these modifications are showcased in the proof of Theorem 5.1. Instead, we will make use of the following lemma, where $\Lambda(a)_{\sharp}\nu$ will be used to denote the image measure $T[\nu]$ when $T(x) = \Lambda(a)x$.

Lemma 3.4. *Let μ be a Radon measure on $\mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$. For a Radon measure ν the following are equivalent:*

(1) $\nu \in \operatorname{Tan}_{\Lambda}(\mu,a)$
(2) $\Lambda(a)_{\sharp}\nu \in \operatorname{Tan}(\mu,a)$
(3) $\nu \in \operatorname{Tan}((\Lambda(a)^{-1})_{\sharp}\mu, \Lambda(a)^{-1}a)$

Lemma 3.4 provides geometric intuition about Λ-tangents. The equivalence of (1) and (2) says that any Λ tangent could equivalently be generated by applying a fixed linear transformation to a Euclidean tangent measure. The equivalence of (1) and (3) says Λ-tangents are a Euclidean tangent of a linear transformation of the original measure. Each perspective serves its own purpose:

The equivalence of (1) and (2) says that $\Lambda(\cdot)_{\sharp}$ is an isomorphism between $\operatorname{Tan}_{\Lambda}(\mu,\cdot)$ and $\operatorname{Tan}(\mu,\cdot)$. Therefore, any statement about $\operatorname{Tan}(\mu,\cdot)$ that holds almost everywhere has an equivalent statement for $\operatorname{Tan}_{\Lambda}(\mu,\cdot)$ that holds almost everywhere, after unwinding what effect the isomorphism $\Lambda(\cdot)_{\sharp}$ has. This will be used to prove Theorem 3.3.

The equivalence of (1) and (3) states that the d-cone $\operatorname{Tan}((\Lambda(a)^{-1})_{\sharp}\mu, \Lambda(a)^{-1}a)$ and the d-cone $\operatorname{Tan}_{\Lambda}(\mu,a)$ are the same. Hence, properties about tangent measures derived from the fact that $\operatorname{Tan}(\mu,\cdot)$ forms a d-cone are also valid for Λ-tangents. In this case, no unwinding the effects of an isomorphism is required. This will be used to prove Theorem 3.5.

Proof of Lemma 3.4. To prove the equivalence of (1) and (2) observe

$$\Lambda(a)T^{\Lambda}_{a,r}(y) = T_{a,r}(y).$$

Therefore, by Proposition 2.2, $\nu = \lim_i c_i T^{\Lambda}_{a,r_i}[\mu] \in \mathrm{Tan}_{\Lambda}(\mu, a)$ if and only if

$$\Lambda(a)_\sharp \nu = \lim_i c_i T_{a,r_i}[\mu] = \lim_i c_i T_{a,r_i}[\mu] \in \mathrm{Tan}(\mu, a).$$

To prove the equivalence of (1) and (3) observe

$$T^{\Lambda}_{a,r}(y) = T_{\Lambda(a)^{-1}a,r}\left(\Lambda(a)^{-1}y\right). \tag{3.3}$$

So Proposition 2.2 guarantees $\nu = \lim_i c_i T^{\Lambda}_{a,r_i}[\mu]$ if and only if

$$\nu = \lim_i c_i T_{\Lambda(a)^{-1}a,r_i}[(\Lambda(a)^{-1})_\sharp \mu] \in \mathrm{Tan}((\Lambda(a)^{-1})_\sharp \mu, \Lambda(a)^{-1}a).$$

□

Proof of Theorem 3.3. Let $A \subset \mathbb{R}^n$ be the set of full measure satisfying the conclusion of Theorem 2.3. Fix some $a \in A$ and $\nu \in \mathrm{Tan}_{\Lambda}(\mu, a)$ and $x \in \mathrm{spt}\ \nu$. By Lemma 3.4, $\nu_0 := \Lambda(a)_\sharp \nu \in \mathrm{Tan}(\mu, a)$. On the other hand, $x \in \mathrm{spt}\ \nu \implies \Lambda(a)x \in \mathrm{spt}\ \nu_0$. Since $a \in A$, it follows $\nu_1 := T_{\Lambda(a)x,r}[\nu_0] \in \mathrm{Tan}(\mu, a)$ and a final application of Lemma 3.4 implies $(\Lambda(a)^{-1})_\sharp \nu_1 \in \mathrm{Tan}_{\Lambda}(\mu, a)$. To confirm $T_{x,r}[\nu] \in \mathrm{Tan}_{\Lambda}(\mu, a)$ we check $(\Lambda(a)^{-1})_\sharp \nu_1 = T_{x,r}[\nu]$. Indeed, from the identity $T_{x,r}(y) = \Lambda(a)^{-1} \circ T_{\Lambda(a)x,r} \circ \Lambda(a)(y)$, it follows

$$(\Lambda(a)^{-1})_\sharp \nu_1 = \left(\Lambda(a)^{-1} \circ T_{\Lambda(a)x,r} \circ \Lambda(a)\right)_\sharp \nu = T_{x,r}[\nu].$$

□

We now state and prove the analog of Theorem 2.7(1) for Λ-tangents.

Theorem 3.5. *Fix $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$ and $r_0 > 0$. Suppose $\mathcal{F}$ is a closed d-cone with compact basis and μ is a Radon measure.*

If there exists $\widetilde{\nu} \in \mathrm{Tan}_{\Lambda}(\mu, a) \cap \mathcal{F}$, $0 < \epsilon < 1$, and $\nu \in \mathrm{Tan}_{\Lambda}(\mu, a)$ so that $0 < \epsilon < d_{r_0}(\nu, \mathcal{F})$, then there exists $\nu_\epsilon \in \mathrm{Tan}_{\Lambda}(\mu, a)$ satisfying

$$\begin{cases} d_{r_0}(\nu_\epsilon, \mathcal{F}) = \epsilon \\ d_r(\nu_\epsilon, \mathcal{F}) \leq \epsilon \qquad r > r_0. \end{cases} \tag{3.4}$$

Proof. Let $\nu, \widetilde{\nu} \in \mathrm{Tan}_{\Lambda}(\mu, a)$. It follows $\nu, \widetilde{\nu} \in \mathrm{Tan}((\Lambda(a)^{-1})_\sharp \mu, \Lambda(a)^{-1}a)$ due to Lemma 3.4 (1) and (3). Therefore, Theorem 2.7(1) implies there exists $\nu_\epsilon \in \mathrm{Tan}((\Lambda(a)^{-1})_\sharp \mu, \Lambda(a)^{-1}a)$ satisfying (3.4). By Lemma 3.4(1) and (3), $\nu_\epsilon \in \mathrm{Tan}_{\Lambda}(\mu, a)$ proving Theorem 3.5. □

The next corollary is a slight extension of Theorem 2.7(2) in the setting of Λ-tangents. It is a succinct summary of how [MP95] proves that symmetric tangents implies flat tangents.

Corollary 3.6. *Suppose $\mathcal{F} = \cup_{i=1}^{\infty} \mathcal{F}_i$, each $\mathcal{F}_i$ is a d-cone with compact basis and there exists $\epsilon_i > 0$ and $\mathcal{M}$ a d-cone with closed basis so that for each i: $\mathcal{F}_i \subset \mathcal{M}$ and*

$$\begin{cases} \exists \epsilon_i > 0,\ R_i > 0 \text{ such that } \forall\, \epsilon \in (0, \epsilon_i) \text{ there exists no } \nu \in \mathcal{M} \setminus \cup_{j=1}^{i-1} \mathcal{F}_j \\ \text{satisfying } d_r(\nu, \mathcal{F}_i) \leq \epsilon\ \forall\, r \geq R_i > 0 \ \text{ and } d_{R_i}(\nu, \mathcal{F}_i) = \epsilon. \end{cases} \tag{P_i}$$

If $a \in \mathbb{R}^n$ is so that $\mathrm{Tan}_{\Lambda}(\mu, a) \subset \mathcal{M}$ and $\mathrm{Tan}_{\Lambda}(\mu, a) \cap \mathcal{F} \neq \emptyset$ then $\mathrm{Tan}_{\Lambda}(\mu, a) \subset \mathcal{F}$.

Proof. Suppose $\mathrm{Tan}_{\Lambda}(\mu, a) \subset \mathcal{M}$ and $\mathrm{Tan}_{\Lambda}(\mu, a) \cap \mathcal{F} \neq \emptyset$. Let i be the smallest integer so that $\mathrm{Tan}_{\Lambda}(\mu, a) \cap \mathcal{F}_i \neq \emptyset$. Then in particular, $\mathrm{Tan}_{\Lambda}(\mu, a) \cap \cup_{j<i} \mathcal{F}_i = \emptyset$.

We will show that $\mathrm{Tan}_{\Lambda}(\mu, a) \subset \mathcal{F}_i$. Indeed, suppose not. Then by Theorem 3.5(1) applied to $\mathcal{F}_i$ and $\mathrm{Tan}_{\Lambda}(\mu, a)$, for $0 < \epsilon < \min\{1, \epsilon_i\}$, there exists ν_ϵ so that

$$\begin{cases} d_{r_0}(\nu_\epsilon, \mathcal{F}_i) = \epsilon \\ d_r(\nu_\epsilon, \mathcal{F}_i) \leq \epsilon \qquad r > r_0. \end{cases}$$

but this contradicts Property (P_i) □

The next lemma provides information about the measure of balls centered at the origin for Λ-tangents. This is the crucial starting point for Theorem 5.1 as well as for showing the equivalence of studyin the density question with arbitrary weights c_i or in the special case $c_i = cr_i^{-m}$.

Lemma 3.7. *Suppose $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ and $0 < c_0 = \theta^m_{\Lambda,*}(\mu, a) \leq \theta^{m,*}_\Lambda(\mu, a) = C_0 < \infty$. If $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ and $c_i T^\Lambda_{a,r_i}[\mu] \overset{*}{\rightharpoonup} \nu$, then*

$$0 < \liminf_{i\to\infty} c_i r_i^m \leq \limsup_{i\to\infty} c_i r_i^m < \infty. \tag{3.5}$$

In fact,

$$1 \leq \frac{\limsup_i c_i r_i^m}{\liminf_i c_i r_i^m} \leq \frac{\theta^{m,*}_\Lambda(\mu, a)}{\theta^m_{\Lambda,*}(\mu, a)}. \tag{3.6}$$

Moreover, for all $R > 0$,

$$\limsup_i c_i r_i^m \theta^m_{\Lambda,*}(\mu, a) \leq \frac{\nu(B_R)}{R^m} \leq \liminf_i c_i r_i^m \theta^{m,*}_\Lambda(\mu, a). \tag{3.7}$$

The following is a quick corollary of Lemma 3.7.

Corollary 3.8. *If $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ and $0 < \theta^m_{\Lambda,*}(\mu, a) \leq \theta^{m,*}_\Lambda(\mu, a) < \infty$, then for all $C \geq 1$*

$$\frac{\nu(B(0, CR))}{\nu(B(0, R))} \leq \frac{\theta^{m,*}_\Lambda(\mu, a)}{\theta^m_{\Lambda,*}(\mu, a)} C^m.$$

If $\theta^m_\Lambda(\mu, a)$ exists, $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ and $\nu = \lim_i c_i T^\Lambda_{a,r_i}[\mu]$, then

$$\theta^m_\Lambda(\mu, a) \lim_i c_i r_i^m = \nu(B_1).$$

In particular, $\nu = \lim_i \widetilde{c}_i T^\Lambda_{a,r_i}[\mu]$ where $\widetilde{c}_i = \frac{\nu(B_1)}{\theta^m_\Lambda(\mu,a) r_i^m}$.

Proof of Lemma 3.7. Note that for any $R > 0$,

$$\infty > \theta^{m,*}_\Lambda(\mu, a) = \limsup_{r\downarrow 0} \frac{T^\Lambda_{a,r}[\mu](B_R)}{(rR)^m} \geq \limsup_{i\to\infty} \frac{T^\Lambda_{a,r_i}[\mu](B_R)}{(r_i R)^m}.$$

Since ν is a Radon measure, for almost every $R > 0$, $\nu(\partial B(0, R)) = 0$. Choosing such R,

$$\begin{aligned}\limsup_{i\to\infty} \frac{T^\Lambda_{a,r_i}[\mu](B_R)}{(r_i R)^m} &= \limsup_{i\to\infty} \frac{1}{c_i (r_i R)^m} c_i T^\Lambda_{a,r_i}[\mu]\,(B_R) \\ &= \nu(B_R) \limsup_{i\to\infty} \frac{1}{c_i (r_i R)^m}.\end{aligned}$$

Since $0 \in \mathrm{spt}\ \nu$ this implies

$$0 < R^{-m} \nu(B_R) \leq \theta^{m,*}_\Lambda(\mu, a) \liminf_{i\to\infty} c_i r_i^m. \tag{3.8}$$

Similarly, for any such R, it follows

$$\begin{aligned}0 < \theta^m_{\Lambda,*}(\mu, a) &= \liminf_{r\downarrow 0} \frac{T^\Lambda_{a,r}[\mu](B_R)}{(rR)^m} \leq \liminf_{i\to\infty} \frac{c_i T^\Lambda_{a,r_i}[\mu](B_R)}{c_i (r_i R)^m} \\ &= \frac{\nu(B_R)}{\limsup_i c_i (r_i R)^m}.\end{aligned}$$

Since ν is Radon, this implies

$$\theta^m_{\Lambda,*}(\mu, a) \limsup_i c_i r_i^m \leq \frac{\nu(B_R)}{R^m} < \infty. \tag{3.9}$$

Combining (3.8) and (3.9) confirms (3.5) and (3.6). In fact, (3.8) and (3.9) also verifies (3.7) for all R so that $\nu\left(\partial B(0,R)\right) = 0$. To prove (3.7) for general R, note

$$\left(\frac{s}{R}\right)^m \frac{\nu(B_s)}{s^m} \leq \frac{\nu(B_R)}{R^m} \leq \left(\frac{S}{R}\right)^m \frac{\nu(B_S)}{S^m}$$

Choosing any sequence of $s_i \leq R \leq S_i$ so that $\nu(\partial B_{S_i}) = 0 = \nu(\partial B_{s_i})$ and $s_i \uparrow R, S_i \downarrow R$ confirms (3.7) for general R. □

Lemma 3.9. *Let μ be a Radon measure and $A \subset \mathbb{R}^n$. If $\mu(A) > 0$, for μ a.e. $a \in A$*

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B(a,r))}{\mu(B(a,r))} = 1. \tag{3.10}$$

Moreover, for any such a, if $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ and $\nu = \lim_i c_i T^\Lambda_{a,r_i}[\mu]$, then for any $x \in \operatorname{spt} \nu$, there exists $a_i \in A$ with

$$\lim_{i \to \infty} \Lambda(a)^{-1} \left(\frac{a_i - a}{r_i}\right) = x. \tag{3.11}$$

Proof of Lemma 3.9. By [Fed14, Theorem 2.9.11], for any measure μ and $A \subset \mathbb{R}^n$,

$$\mu\left(A \setminus \{x : \liminf_{r \downarrow 0} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))} = 1\}\right) = 0.$$

Thus (3.10) holds for μ a.e. $a \in \mathbb{R}^n$. Now suppose $a \in A$ satisfies (3.10) but (3.11) fails. Then there exist $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$, $x \in \operatorname{spt} \nu$, a subsequence $\{i_k\}$, and $\delta > 0$ so that

$$\operatorname{dist}(B(a + r_{i_k}\Lambda(a)x, r_{i_k}\delta), A) > 0. \tag{3.12}$$

Without loss of generality we suppose (3.12) holds for the original sequence. Since μ is Radon, for any sets E, F with $\operatorname{dist}(E, F) > 0$, it follows $\mu(E \cup F) = \mu(E) + \mu(F)$. Therefore, (3.10) and (3.12) imply

$$1 = \lim_{i \to \infty} \frac{\mu\left(B(a, 2r_i|\Lambda(a)x|) \cap A\right)}{\mu(B(a, 2r_i|\Lambda(a)x|))} \leq 1 - \liminf_{i \to \infty} \frac{\mu(B(a + r_i\Lambda(a)x, \delta r_i))}{\mu(B(a, 2r_i|\Lambda(a)x|))} \tag{3.13}$$

We will show (3.13) is a contradiction by producing a non-zero lower bound on the final term. Indeed, following the convention that $B(x,r)$ and $U(x,r)$ are respectively the closed and open balls around x of radius r,

$$\begin{aligned}
\liminf_{i \to \infty} \frac{\mu(B(a + r_i\Lambda(a)x, \delta r_i))}{\mu(B(a, 2r_i|\Lambda(a)x|))} &\geq \liminf_{i \to \infty} \frac{c_i T_{a,r_i}[\mu]\left(U(\Lambda(a)x, \delta)\right)}{c_i T_{a,r_i}[\mu]\left(B(0, 2|\Lambda(a)x|)\right)} \\
&= \liminf_{i \to \infty} \frac{c_i T^\Lambda_{a,r_i}[\mu]\left(\Lambda(a)^{-1}U(\Lambda(a)x, \delta)\right)}{c_i T^\Lambda_{a,r_i}[\mu]\left(\Lambda(a)^{-1}B(0, 2|\Lambda(a)x|)\right)} \\
&\geq \frac{\nu\left(\Lambda(a)^{-1}U(\Lambda(a)x, \delta)\right)}{\nu\left(\Lambda(a)^{-1}B(0, 2|\Lambda(a)x|)\right)} > 0. \qquad (3.14)
\end{aligned}$$

The reason the final term is positive is that $\Lambda(a)^{-1}U(\Lambda(a)x, \delta)$ is an open neighborhood of $x \in \operatorname{spt} \nu$. Now (3.13) and (3.14) yield a contradiction, confirming (3.11). □

4. Rectifiability from existence of principal values

In Section 4.1 we prove Theorem 1.1. Due to Lemma 4.2, the remaining work is in proving Proposition 4.1 which verifies that the lower density assumption and existence of the principal values $T^m_\Lambda \mu(a)$ implies Λ-tangents are symmetric.

In Section 4.2 we prove Corollary 1.2 by verifying the equivalence of the existence of $T_A\mu(a)$ implies symmetry of Λ-tangents in the same way that existence of $T^{n-1}_A \mu(a)$ does, under the assumption that $A \in \widetilde{\mathrm{DMO}}$ is uniformly elliptic.

4.1. **Symmetry and flatness from $T^m_\Lambda\mu$.**

Proposition 4.1 (Symmetry of Λ-tangents)**.** *Suppose that μ is a finite Borel measure over $\mathbb{R}^n$ such that $\theta^m_*(\mu,a)>0$ and $T^m_\Lambda\mu(a)$ exists for almost every a, then for almost every a, every $\nu\in Tan_\Lambda(\mu,a)$ satisfies*

$$\int_{r\le|x-y|\le R}\frac{y-x}{|y-x|^{m+1}}\,d\nu(y)=0\qquad\forall x\in\text{spt }\mu\tag{4.1}$$

for all $0<r<R<\infty$. In particular, $\mathrm{Tan}^m_\Lambda\mu(a)\subset\mathcal{S}_n$ for μ a.e. $a\in\mathbb{R}^n$.

Proof. Consider A to be the set of points $a\in\mathbb{R}^n$ satisfying

A1) $\theta^m_*(\mu,a)>0$
A2) $T^m_\Lambda\mu(a)$ exists and is finite
A3) For all $\nu\in\mathrm{Tan}_\Lambda(\mu,a)$, and all $x\in\text{spt }\nu$, $T_{x,1}[\nu]\in\mathrm{Tan}_\Lambda(\mu,a)$.

By hypothesis, (A1) and (A2) hold almost everywhere. By Theorem 3.3, (A3) also holds almost everywhere, so A is a set of full measure. Suppose $a\in A$ and $\nu=\lim_i c_iT^\Lambda_{a,r_i}[\mu]\in\mathrm{Tan}_\Lambda(\mu,a)$. Then for $0<r<R$, using $\lim a_ib_i\le(\limsup a_i)(\limsup b_i)$

$$\begin{aligned}\left|\int_{r\le|y|\le R}\frac{y}{|y|^{m+1}}d\nu(y)\right|&=\left|\lim_{i\to\infty}c_i\int_{r<|y|<R}\frac{y}{|y|^{m+1}}dT^\Lambda_{a,r_i}[\mu](y)\right|\\&=\left|\lim_{i\to\infty}c_ir_i^m\int_{r<|T^\Lambda_{a,r_i}(y)|<R}\frac{\Lambda(a)^{-1}(y-a)}{|\Lambda(a)^{-1}(y-a)|^{m+1}}d\mu(y)\right|\\&=\lim_{i\to\infty}c_ir_i^m\left|\int_{rr_i\le|\Lambda(a)^{-1}(y-a)|}\frac{\Lambda(a)^{-1}(y-a)}{|\Lambda(a)^{-1}(y-a)|^{m+1}}d\mu(y)-\right.\\&\qquad\left.-\int_{Rr_i\le|\Lambda(a)^{-1}(y-a)|}\frac{\Lambda(a)^{-1}(y-a)}{|\Lambda(a)^{-1}(y-a)|^{m+1}}d\mu(y)\right|\\&\le\limsup_i c_ir_i^m\left|T^m_\Lambda\mu(a)-T^m_\Lambda\mu(a)\right|,\end{aligned}$$

where (A2) implies this final line is well-defined and zero so long as $\limsup_i c_ir_i^m<\infty$. Since $x\mapsto\Lambda(a)x$ is a linear isomorphism from $\mathbb{R}^n\to\mathbb{R}^n$, $\theta^m_*(\mu,a)>0$ if and only if $\theta^m_{\Lambda,*}(\mu,a)>0$. So, (A1) and (3.5) imply $\limsup_i c_ir_i^m<\infty$ verifying (4.1) when $x=0$ for all $\nu\in\mathrm{Tan}_\Lambda(\mu,a)$. Finally, (A3) says $T_{x,1}[\nu]\in\mathrm{Tan}_\Lambda(\mu,a)$ for all $x\in\text{spt }\nu$. Since,

$$\int_{r\le|y|<R}\frac{y}{|y|^{m+1}}dT_{x,1}[\nu](y)=\int_{r\le|y-x|\le R}\frac{y-x}{|y-x|^{m+1}}d\nu(y),$$

(4.1) follows. By Lemma 2.10, this verifies the symmetry of ν. Since $a\in A$ and $\nu\in\mathrm{Tan}^m_\Lambda(\mu,a)$ are arbitrary and A is a set of full measure this completes proof. □

The next Lemma provides the final step to Prove theorem 1.1. As it is interesting in its own right, we state it separately.

Lemma 4.2. *Fix $\Lambda:\mathbb{R}^n\to GL(n,\mathbb{R})$. Suppose μ is a Radon measure so that at almost every a, $0<\theta^m_*(\mu,a)$ and $\mathrm{Tan}_\Lambda(\mu,a)\subset\mathcal{S}_n$. Then for almost every a, $\mathrm{Tan}_\Lambda(\mu,a)\subset\mathcal{M}_n$. In particular, if $\theta^m_*(\mu,a)<\infty$ almost everywhere, μ is m-rectifiable.*

Proof. Lemma 2.11(1) implies that $\mathcal{F}_i=\mathcal{M}_{n,i}$ and $\mathcal{M}=\mathcal{S}_n$ satisfy (P_i). Since $\mathrm{Tan}_\Lambda(\mu,a)\subset\mathcal{S}_n$, Lemma 2.11(2) and Theorem 3.3 imply $\mathrm{Tan}_\Lambda(\mu,a)\cap\mathcal{M}_n\neq\emptyset$ for μ a.e. a. So, Corollary 3.6 verifies $\mathrm{Tan}_\Lambda(\mu,a)\subset\mathcal{M}_n$ for almost every a. If additionally $\theta^m_*(\mu,a)<\infty$, Theorem 1.5(iii) implies rectifiability. □

Proof of Theorem 1.1. By Proposition 4.1, $\mathrm{Tan}_\Lambda(\mu, a) \subset \mathcal{S}_n$ for almost every a. Theorem 1.1 now follows from Lemma 4.2. $\square$

4.2. **Symmetry and flatness from $T_A\mu$.** To prove Corollary 1.2 we show that for suitable matrix-valued functions A, and suitable Radon measures μ, (1.5) implies that μ a.e., if $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ then (4.1) holds. This implies $\mathrm{Tan}_\Lambda(\mu, a) \subset \mathcal{S}_n$, which implies flat tangents and rectifiability by Lemma 4.2. We achieve this first step by adapting the estimates in [MMPT23, Lemma 3.12, 3.13] to prove integrals of $\nabla_1\Gamma_A(x, y)$ and $\nabla_1\Theta(x, y; A(x))$ are sufficiently close at small scales, see Lemma 4.7.

We first introduce some terminology and notation from [MMPT23] which we will adhere to. We warn the reader that in [MMPT23] is working in $\mathbb{R}^{n+1}$, while here we adapt to the setting of $\mathbb{R}^n$. A Lebesgue measurable function $\theta : [0, \infty] \to [0, \infty]$ is called κ-doubling if $\theta(t) \leq \kappa\theta(s)$ for all $s \in [t/2, t]$. A κ-doubling function θ is in $DS(\kappa)$ (resp. $DL_d(\kappa)$) if $\int_0^1 \theta(t)\frac{dt}{t} < \infty$ (resp. $\int_1^\infty \theta(t)\frac{dt}{t^{d+1}} < \infty$). These spaces are the Dini spaces for κ-doubling functions at small (resp. large) scales. Note that if $d_1 < d_2$ then $DL_{d_1}(\kappa) \subset DL_{d_2}(\kappa)$. Given a matrix-valued function $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, for any $x \in \mathbb{R}^n$ and $r > 0$ define $\overline{A}_{x,r} = \fint_{B(x,r)} A(z)dz$ and

$$\omega_A(r) = \sup_{x\in\mathbb{R}^n} \fint_{B(x,r)} |A(z) - \overline{A}_{x,r}|dz.$$

Further, denote

$$\mathfrak{L}^d_\theta(r) = r^d \int_r^\infty \theta(t)\frac{dt}{t^{d+1}} \quad \text{and} \quad \mathfrak{I}_\theta(r) = \int_0^r \theta(t)\frac{dt}{t}.$$

The matrix-valued function A is said to be in $\mathrm{DMO_s}$ (resp. DMO_ℓ) if $\omega_A \in DS(\kappa)$ (resp. $\omega_A \in DL_{n-2}(\kappa)$) some $\kappa < \infty$. It is said $A \in \mathrm{DDMO_s}$ if $A \in \mathrm{DMO_s}$ and

$$\int_0^1 \int_0^r \omega_A(t)\frac{dt}{t}\frac{dr}{r} < \infty.$$

The spaces $\mathrm{DMO_s}$ (resp. DMO_ℓ) stand for Dini mean oscillation at small scales (resp. at large scales) and $\mathrm{DDMO_s}$ stands for double Dini mean oscillation at small scales. We write $A \in \widetilde{\mathrm{DMO}}$ if $A \in \mathrm{DDMO_s} \cap \mathrm{DMO}_\ell$.

Remark 4.3. It is known that $A \in \widetilde{\mathrm{DMO}}$ if and only if the precise representative of A is uniformly continuous with modulus of continuity $\mathfrak{I}_{\omega_A}$, [HK20, Appendix A]. If we consider $A, \widetilde{A} \in \widetilde{\mathrm{DMO}}$ so that $A = \widetilde{A}$ Lebesgue a.e., then for all $y \in \mathbb{R}^n$, $\nabla\Gamma_A(\cdot, y) = \nabla\Gamma_{\widetilde{A}}(\cdot, y)$ on $\mathbb{R}^n \setminus \{y\}$ and $\omega_A = \omega_{\widetilde{A}}$ on $[0, \infty]$. In particular, there is no loss in generality in assuming that $A \in \widetilde{\mathrm{DMO}}$ is uniformly continuous, even when studying measures μ which are mutually singular with respect to the Lebesgue measure.

Finally, we define

$$\tau_A(r) = \mathfrak{I}_{\omega_A}(r) + \mathfrak{L}^{n-1}_{\omega_A}(r) = \int_0^r \omega_A(t)\frac{dt}{t} + r^{n-1}\int_r^\infty \omega_A(t)\frac{dt}{t^n}$$

and

$$\widehat{\tau}_A(R) = \mathfrak{I}_{\omega_A}(R) + \mathfrak{L}^{n-2}_{\omega_A}(R) = \int_0^R \omega_A(t)\frac{dt}{t} + R^{n-2}\int_R^\infty \omega_A(t)\frac{dt}{t^{n-1}}.$$

Remark 4.4. In [MMPT23, p. 7] it is observed that $A \in \widetilde{\mathrm{DMO}}$ implies both $\mathfrak{I}_{\tau_A}(1) < \infty$ and $\widehat{\tau}_A(R) < \infty$ for all $R > 0$ and whenever $A \in C^\alpha$, $\tau_A(r) \lesssim r^\alpha$. In particular, this means that when $A \in \widetilde{\mathrm{DMO}}$, $\lim_{r\to 0}\tau_A(r) = 0$. In fact, if $A \in \widetilde{DMO}$ then $\lim_{r\to 0}\tau_{\omega_A}(r) + \widehat{\tau}_{\omega_A}(r) = 0$. Indeed, [MMPT23, Remark 2.2] says that if $\omega_A \in DS(\kappa) \cap DL_{n-2}(\kappa)$, then $\mathcal{L}^{n-2}_{\omega_A}(R) \to 0$ as

$R \to 0$. The fact that $\omega_A \in DS(\kappa) \cap DL_{n-2}(\kappa)$ is precisely the statement $A \in \mathrm{DMO}_\ell \cap \mathrm{DMO_s}$. Thus we also know $\lim_{r\to 0} \widehat{\tau}_A(r) = 0$ when $A \in \widetilde{\mathrm{DMO}}$.

Throughout this section, we will always let A denote a uniformly elliptic matrix valued function from $\mathbb{R}^n \to \mathbb{R}^{n\times n}$ with uniform ellipticity constant Λ_0. That is, $|\xi|^2\Lambda_0^{-1} \le \langle A(x)\xi, \xi\rangle$ and $\langle A(x)\xi, \eta\rangle \le \Lambda_0|\xi||\eta|$ for all $x, \xi, \eta \in \mathbb{R}^n$. We also fix $\kappa < \infty$ so that ω_A is κ-doubling.
Both the formulation and presentation of the next two lemmas come from [MMPT23]. Some of the ideas behind this "frozen coefficient method" are already present in [KS11, CAMT19].

Lemma 4.5 (Lemma 3.12 from [MMPT23])**.** *Suppose $A \in \mathrm{DMO_s} \cap \mathrm{DMO}_\ell$ and $n \ge 3$. For $R_0 > 0$, there exists $C_0 = C(n, \Lambda_0, R_0) > 0$ such that for $x, y \in \mathbb{R}^n$ and $0 < |x-y| < R < R_0$,*

$$\left|\nabla_1\Gamma_A(x,y) - \nabla_1\Theta\left(x,y;\overline{A}_{x,\frac{|x-y|}{2}}\right)\right| \le C\frac{\tau_A(\frac{|y-x|}{2})}{|y-x|^{n-1}} + C\frac{\widehat{\tau}_A(R)}{R^{n-1}}$$

Lemma 4.6 (Lemma 3.13 from [MMPT23])**.** *Let $A \in \mathrm{DMO_s}$, $n \ge 3$, $0 < \delta < r < 1$, and $x \in \mathbb{R}^n$. Assume $\Omega_{x,\delta} \subset \mathbb{R}^n$ is a Borel set such that for some $C_1 \ge 1$,*

$$B(x,\delta) \subset \Omega_{x,\delta} \subset B(x, C_1\delta).$$

Moreover, for all $x \neq y \in \mathbb{R}^n$,

$$\left|\nabla_1\Theta(x-y,0;\overline{A}_{x,r/2}) - \nabla_1\Theta(x-y,0;\overline{A}_{x,\delta/2})\right| \lesssim_{n,\Lambda_0,C_1} \frac{1}{|x-y|^{n-1}}\int_\delta^r \omega_A(t)\frac{dt}{t}.$$

The next Lemma is a consequence of integrating the previous two and using the κ-doubling property of ω_A.

Lemma 4.7. *Let $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a uniformly elliptic matrix-valued function, $n \ge 3$, satisfying $A \in \mathrm{DMO_s} \cap \mathrm{DMO}_\ell$. Let μ be a finite Borel measure and $0 < c_2 < C_2$. For each $R > 0$, define a Borel set E_R such that*

$$E_R \subset B(0, C_2R) \setminus B(0, c_2R).$$

Then for every $x \in \mathbb{R}^n$ and $R \le 1$,

$$\int_{x-y\in E_R}\left|\nabla_1\Gamma_A(x,y) - \nabla_1\Theta\left(x,y;\overline{A}_{x,\frac{3}{2}R}\right)\right| d\mu(y) \lesssim \frac{\mu(B(x,C_2R))}{R^{n-1}}(\tau_A(R) + \widehat{\tau}_A(R)).$$

Proof. We write,

$$\left|\nabla_1\Gamma_A(x,y) - \nabla_1\Theta\left(x,y;\overline{A}_{x,\frac{3}{2}R}\right)\right| \le \left|\nabla_1\Gamma_A(x,y) - \nabla_1\Theta\left(x,y;\overline{A}_{x,\frac{|x-y|}{2}}\right)\right|$$
$$+\left|\nabla_1\Theta\left(x,y;\overline{A}_{x,\frac{|x-y|}{2}}\right) - \nabla_1\Theta\left(x,y;\overline{A}_{x,\frac{3}{2}R}\right)\right| =: I + II$$

If $x - y \in E_R$ then

$$\frac{c_2}{2}R \le \frac{|x-y|}{2} \le \frac{C_2}{2}R. \tag{4.2}$$

We claim that because $\tau_A(\cdot)$ is monotone increasing and $\omega_A(t)$ is κ-doubling, it follows from Lemma 4.5 that for all $x - y \in E_R$

$$I \lesssim_{C_0} \frac{\tau_A\left(\frac{C_2R}{2}\right)}{\left(\frac{c_2R}{2}\right)^{n-1}} + \frac{\widehat{\tau}_A\left(\frac{C_2R}{2}\right)}{\left(\frac{c_2R}{2}\right)^{n-1}} \lesssim_{c_2,C_2,\kappa,n} \frac{\tau_A(R)}{R^{n-1}} + \frac{\widehat{\tau}_A(R)}{R^{n-1}}. \tag{4.3}$$

Indeed, if $C_2 \geq 2$

$$\mathfrak{I}_{\omega_A}\left(\frac{C_2 R}{2}\right) = \int_0^{C_2R/2} \omega_A(t)\frac{dt}{t} = \int_0^R \omega_A\left(\frac{C_2 s}{2}\right)\frac{ds}{s} \leq \kappa^{\lfloor \log_2(C_2)\rfloor}\int_0^R \omega_A(s)\frac{ds}{s} = \mathfrak{I}_{\omega_A}(R).$$

On the other hand,

$$\mathfrak{L}^d_{\omega_A}\left(\frac{C_2R}{2}\right) = \left(\frac{C_2R}{2}\right)^d \int_{\frac{C_2R}{2}}^\infty \omega_A(t)\frac{dt}{t} \leq \left(\frac{C_2}{2}\right)^d R^d \int_R^\infty \omega_A(t)\frac{dt}{t} = \left(\frac{C_2}{2}\right)^d \mathfrak{L}^d_{\omega_A}(R)$$

verifying (4.3) when $C_2/2 \geq 1$. The case $C_2/2 < 1$ is verified by interchanging the arguments used to estimate $\mathfrak{I}_{\omega_A}$ and $\mathfrak{L}^d_{\omega_A}$. Since $\tau_A, \widehat{\tau}_A$ are sums of $\mathfrak{I}_{\omega_A}$, $\mathfrak{L}^{n-1}_{\omega_A}$, and $\mathfrak{L}^{n-2}_{\omega_A}$ this verifies (4.3). Recalling $E_R \subset B(0, C_2R)$ it follows

$$\int_{E_R} I \, d\mu(y) \lesssim_{C_0, \frac{C_2}{c_2}, \kappa} \frac{\mu(B(x, C_2R))}{R^{n-1}}\left(\tau_A(R) + \widehat{\tau}_A(R)\right).$$

Analogous reasoning allows one to estimate the integral of II. Indeed, when $x - y \in E_R$ it follows from Lemma 4.6 and (4.2) that

$$II \leq \frac{1}{|x-y|^{n-1}}\left|\int_{|x-y|}^{3R} \omega_A(t)\frac{dt}{t}\right| \lesssim_{c_2,n} \frac{1}{R^{n-1}}\int_{\frac{c_2R}{2}}^{\max\{3R, \frac{C_2R}{2}\}} \omega_A(t)\frac{dt}{t} \lesssim_{\kappa, C_2} \frac{\mathfrak{I}_{\omega_A}(R)}{R^{n-1}}.$$

In particular,

$$\int_{x-y\in E_R} II d\mu(y) \lesssim_{C_2, c_2, \kappa, n} \frac{\mu(B(x, C_2R))}{R^{n-1}}\mathfrak{I}_{\omega_{A}(r))} \leq \frac{\mu(B(x, C_2R))}{R^{n-1}}(\tau_A(R) + \widehat{\tau}_A(R))$$

as desired. □

We are now ready to prove Corollary 1.2.

Proof. Without loss of generality suppose A is continuous, see Remark 4.3. Let G be the collection of $x \in \mathbb{R}^n$ such that $\theta^{n-1}_*(\mu, x) > 0$, $\theta^{n-1,*}(\mu, x) < \infty$, (1.5) holds, and the conclusion of Theorem 3.3 holds. Then by assumption, and Theorem 3.3, $\mu(\mathbb{R}^n \setminus G) = 0$. Fix $a \in G$ and $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$. Then there exists there exists $c_i > 0$ and a sequence of r_i converging to zero so that $\nu = \lim_i c_i T^\Lambda_{a,r_i}[\nu]$. Fix $0 < r < R < \infty$. Adopting the notation that $\Lambda(a)^{-2} = (\Lambda(a)^{-1})^2$, it follows as in the proof of Proposition 4.1 that

$$\begin{aligned}\left|\Lambda(a)^{-1}\int_{r\leq|y|\leq R}\frac{y}{|y|^n}d\nu(y)\right| &= \lim_{i\to\infty} c_i r_i^{n-1}\left|\int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i))}\frac{\Lambda(a)^{-2}(y-a)}{|\Lambda(a)^{-1}(y-a)|^n}d\mu(y)\right| \\ &= \frac{1}{c_{\Lambda(a)}}\lim_{i\to\infty} c_i r_i^{n-1}\left|\int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i)} \nabla_1\Theta(a,y;A(a))d\mu(y)\right|\end{aligned}$$

where $c_{\Lambda(a)}$ is a positive and finite constant depending on n and $\det(\Lambda(a))$, c.f., (1.2). By the definition of G and Lemma 3.7, $\limsup_i c_i r_i^{n-1} < \infty$. We claim that (1.5) implies that for any $0 < r < R < \infty$

$$\lim_{i\to\infty}\int_{B_\Lambda(a,r_iR)\setminus B_\Lambda(a,r_ir)} \nabla_1\Theta(a,y;\Lambda(a))d\mu(y) = 0. \tag{4.4}$$

Indeed,

$$\begin{aligned}\left|\int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i)}\nabla_1\Theta(a,y;A(a))d\mu(y)\right| &\leq \left|\int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i)}\nabla_1\Gamma_A(a,y)d\mu(y)\right| \\ &+ \int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i)}\left|\nabla_1\Theta(a,y;A_{a,\frac{3}{2}r_i}) - \nabla_1\Gamma_A(x,y)\right|d\mu(y) \\ &+ \int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i)}\left|\nabla_1\Theta(a,y;A(a)) - \nabla_1\Theta(a,y;A_{a,\frac{3}{2}r_i})\right|d\mu(y).\end{aligned}$$

We label the terms on the right hand side respectively as I, II and III. By (1.5) and $a \in G$, the term I tends to zero as $i \to \infty$. Define $E_i = B_\Lambda(0, Rr_i) \setminus B_\Lambda(0, rr_i)$. Note, since A is Λ_0 uniformly elliptic, there exists $0 < c_2 < C_2 < \infty$ independent of i so that

$$E_i \subset B(0, C_2 r_i) \setminus B(0, c_2 r_i).$$

Thus, by Lemma 4.7,

$$\int_{B_\Lambda(a,Rr_i)\setminus B_\Lambda(a,rr_i)} |\nabla_1 \Gamma_A(a,y) - \nabla_1 \Theta(a, y; \overline{A}_{a,\frac{3}{2}r_i})| d\mu(y) \le \frac{\mu(B(a, C_2 r_i))}{r_i^{n-1}} \left(\tau_A(r_i) + \widehat{\tau}_A(r_i)\right)$$

Since $a \in G$ implies $\theta^{n-1,*}(\mu, a) < \infty$, it now follows from Remark 4.4 that the term II vanishes as $i \to \infty$. We now focus on the term III. For notational simplicity, let $A_i = \overline{A}_{a,\frac{3}{2}r_i}$. Applying (1.2) we further decompose the integrand of III:

$$\begin{aligned}
&|\nabla_1 \Theta(a, y; \overline{A}_{a,\frac{3}{2}r_i}) - \nabla_1 \Theta(a, y; A(a))| \\
&= c_n \left| \frac{A_i^{-1}(y-a)}{\det(A_i)^{1/2} \langle A_i^{-1}(y-a), y-a\rangle^{n/2}} - \frac{A(a)^{-1}(y-a)}{\det(A(a))^{1/2}\langle A(a)^{-1}(y-a), y-a\rangle^{n/2}} \right| \\
&\le c_n \left| \frac{A_i^{-1}(y-a)}{\det(A_i)^{1/2} \langle A_i^{-1}(y-a), y-a\rangle^{n/2}} - \frac{A(a)^{-1}(y-a)}{\det(A_i)^{1/2}\langle A_i^{-1}(y-a), y-a\rangle^{n/2}} \right| \\
&\quad + c_n \left| \frac{A(a)^{-1}(y-a)}{\det(A_i)^{1/2} \langle A_i^{-1}(y-a), y-a\rangle^{n/2}} - \frac{A(a)^{-1}(y-a)}{\det(A(a))^{1/2}\langle A(a)^{-1}(y-a), y-a\rangle^{n/2}} \right| \\
&= c_n \left| \frac{(A_i^{-1} - A(a)^{-1})(y-a)}{\det(A_i)^{1/2} \langle A_i^{-1}(y-a), y-a\rangle^{n/2}} \right| \\
&\quad + c_n \left| A(a)^{-1}(y-a) \frac{\left\langle L(y-a), y-a\right\rangle^{n/2} - \left\langle L_i(y-a), y-a\right\rangle^{n/2}}{\left\langle L(y-a), y-a\right\rangle^{n/2} \left\langle L_i(y-a), y-a)\right\rangle^{n/2}} \right|
\end{aligned}$$

where $L_i = \det(A_i)^{1/n} A_i^{-1}$ and $L = \det(A(a))^{1/n} A(a)^{-1}$. Call these terms III_1 and III_2. We recall that A is assumed to be Λ_0 elliptic. Consequently A^{-1}, A_i, and A_i^{-1} are all Λ_0^2 elliptic. So,

$$III_1 \quad \le c_n \frac{\|A_i^{-1} - A(a)^{-1}\|}{\det(A_i)^{1/2}} \frac{|y-a|}{(\Lambda_0^{-1})^{n/2}|y-a|^n} \lesssim_{\Lambda_0,n} \frac{\|A_i^{-1} - A(a)^{-1}\|}{|y-a|^{n-1}}.$$

Note that $y - a \in E_i$ implies $|y-a|^{1-n} \gtrsim_{\frac{C_2}{c_2},n} r_i^{1-n}$. On the other hand, since $A_i = \overline{A}_{a,\frac{3}{2}r_i}$ and A is uniformly continuous, $\lim_{i\to\infty} A_i = A(a)$. Because matrix inversion is a continuous function on the space of invertible matrices, it follows $\lim_{i\to\infty} A_i^{-1} = A(a)^{-1}$. Thus, for any $\epsilon > 0$ and all i large enough,

$$III_1 \le \frac{\epsilon}{r_i^{n-1}} \qquad \forall y - a \in E_i.$$

We now turn our attention to III_2. By uniform ellipticity, $\det(A_i)^{1/n}, \det(A)^{1/n} \ge \Lambda_0^{-1}$. So L, L_i are Λ_0^3 uniformly elliptic. Therefore,

$$\begin{aligned}
III_2 &\lesssim_{\Lambda_0,n} \frac{|A(a)^{-1}(y-a)|}{|y-a|^{2n}} \left| \left\langle L(y-a), y-a\right\rangle^{n/2} - \left\langle L_i(y-a), y-a\right\rangle^{n/2} \right| \\
&\lesssim_{\Lambda_0,n} \frac{1}{|y-a|^{2n-1}} \left| \left\langle (L-L_i)(y-a), y-a\right\rangle \right| \left\langle (L+L_i)(y-a), y-a\right\rangle^{n/2-1}
\end{aligned}$$

since $a, b > 0$ implies $|a^{\frac{n}{2}} - b^{\frac{n}{2}}| \lesssim_n |a-b|(a+b)^{\frac{n}{2}-1}$. Furthermore, since L, L_i are Λ_0^3 uniformly elliptic, $\|L + L_i\|$ is bounded by a dimensional multiple of Λ_0^3. Hence we conclude

$$III_2 \lesssim_{\Lambda_0, n} \frac{|y-a|^2|y-a|^{n-2}}{|y-a|^{2n-1}}\|L - L_i\| = \frac{\|L-L_i\|}{|y-a|^{n-1}}.$$

Given that $\lim_{i\to\infty} A_i = A(a)$ and both the determinant and matrix inverse are continuous functions on the space of invertible matrices, it follows analogously to before that $\|L - L_i\|$ is arbitrarily small for large enough i. Thus for all i large enough and all $y - a \in E_i$,

$$III_2 \leq \frac{\epsilon}{r_i^{n-1}}.$$

Combining the estimates on III_1 and III_2 we have shown for i large enough it holds,

$$III \leq \int_{y-a\in E_i} III_1 + III_2 d\mu(y) \leq 2\epsilon \frac{\mu(B(a, C_2 r_i)}{r_i^{n-1}}.$$

Since $\epsilon > 0$ is arbitrary, this confirms (4.4). It now follows as in the proof of Proposition 4.1 that (4.4) implies $\nu \in \mathcal{S}_n$. Since $a \in G$ and $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$ are arbitary, Lemma 4.2 in turn implies μ is $(n-1)$-rectifiable since $\theta^{m,*}(\mu, a) < \infty$ almost everywhere. □

5. Rectifiability from existence of densities

In this section we prove Theorem 1.6. In Theorem 5.1, we show that almost everywhere existence of Λ-densities implies Λ-tangents are uniform almost everywhere and theorem 5.2 shows that existence of Λ-densities implies rectifiability. In Theorem 5.3 we switch gears and instead of using Λ-tangents, decompose the measure μ into countably many pieces to show that a small Λ-density gap also implies the measure is rectifiable. We then put together all the pieces to prove the equivalences in Theorem 1.6.

Theorem 5.1. *Suppose $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$ and for μ almost every a that $\theta^m_\Lambda(\mu, a)$ exists. Then for μ almost every a, and every $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$,*

$$\nu(B(x, r)) = \nu(B(0, 1))r^m \qquad \forall x \in \mathrm{spt}\ \nu.$$

Before beginning the proof, we note that one can identify $GL(n, \mathbb{R})$ with a subset of $\mathbb{R}^{n\times n}$, and we recall that the eigenvalues of a matrix depend continuously upon the coefficients. Therefore, by considering only elements of $GL(n, \mathbb{R})$ with rational coefficients, given any $\epsilon > 0$ we can cover $GL(n, \mathbb{R})$ with countably many sets $\{U_i^\epsilon\}_{i\in\mathbb{N}}$ so that for all $i \in \mathbb{N}$,

$$B_M(0, (1-\epsilon)r) \subset B_{\widetilde{M}}(0, r) \subset B_M(0, (1+\epsilon)r) \quad \forall M, \widetilde{M} \in U_i^\epsilon \quad \forall i \in \mathbb{N}. \tag{5.1}$$

Proof. By Corollary 3.8 if $\theta^m_\Lambda(\mu, a)$ exists, then,

$$\nu(B(0, r)) = \nu(B(0, 1))r^m \qquad \forall \nu \in \mathrm{Tan}_\Lambda(\mu, a). \tag{5.2}$$

In fact, by Theorem 3.3 and another application of Corollary 3.8, we know that for almost every a, and all $\nu \in \mathrm{Tan}_\Lambda(\mu, a)$

$$\nu\left(B(x, r)\right) = T_{x,1}[\nu](B(0, r)) = T_{x,1}[\nu](B(0, 1))r^m \quad \forall x \in \mathrm{spt}\ \nu. \tag{5.3}$$

So the theorem follows from showing that for almost every a,

$$T_{x,1}[\nu](B(0, 1)) = \nu(B(0, 1)) \qquad \forall \nu \in \mathrm{Tan}_\Lambda(\mu, a) \quad \forall x \in \mathrm{spt}\ \nu. \tag{5.4}$$

Indeed, briefly assuming (5.4), Theorem 5.1 follows from (5.2) and (5.3) that

$$\nu\left(B(x, r)\right) = \nu(B(0, 1))r^m = \nu(B(0, r)) \quad \forall \nu \in \mathrm{Tan}_\Lambda(\mu, a) \quad \forall x \in \mathrm{spt}\ \nu.$$

Define $E \subset \mathbb{R}^n$ as the set of points a so that,

$$\exists \nu_a \in \mathrm{Tan}_\Lambda(\mu, a) \text{ and } \exists x_a \in \mathrm{spt}\ \nu_a \text{ so that } \nu_a(B(x_a, 1)) \neq \nu_a(B(0,1)).$$

Assume that $\mu(E) > 0$. Consequently, for some k large enough,

$$E(k) = \Big\{ a \in B(0,k) : \exists \nu_a \in \mathrm{Tan}_\Lambda(\mu, a)\ \exists x_a \in \mathrm{spt}\ \nu_a, \tag{5.5}$$
$$\text{so that } \frac{\nu_a(B(x_a,1))}{\nu_a(B(0,1))} \notin ((1+k^{-1})^{-1}, 1+k^{-1}) \Big\}$$

has positive measure. Fix such a k_0. We will reach a contradiction by showing that in fact

$$\frac{\nu_a(B(x_a,1))}{\nu_a(B(0,1))} < 1 + k_0^{-1}. \tag{5.6}$$

The proof that

$$\frac{\nu_a(B(x_a,1))}{\nu_a(B(0,1))} > (1 + k_0^{-1})^{-1}$$

follows by applying (5.6) to $\widetilde{\nu_a} = T_{x_a,1}[\nu]$ with the point $-x_a \in \mathrm{spt}\ \widetilde{\nu_a}$.

Let A be the set of all $a \in \mathbb{R}^n$ such that $\theta^m_\Lambda(\mu, a) \in (0,\infty)$ and on A define the function

$$F_i(a) = \sup_{r < 2^{-i}} \frac{\mu(B_\Lambda(a,r))}{\theta^m_\Lambda(\mu,a) r^m}.$$

Note that for μ almost every a, $1 \leq F_{i+1}(a) \leq F_i(a)$ and $\lim_{i\to\infty} F_i(a) = 1$. In particular, since $\mu(E(k)) > 0$, for $\epsilon_0 > 0$ there exists i_0 large enough so that

$$E_{i_0} = \{a \in E(k_0) : 0 \leq F_{i_0}(a) - 1 < \epsilon_0\}$$

has $\mu(E_{i_0}) > 0$. It follows that

$$\frac{\frac{\mu(B_\Lambda(a,r))}{r^m}}{\theta^m_\Lambda(\mu,a)} < 1 + \epsilon_0 \qquad \forall r < 2^{-i_0} \quad \forall a \in E_{i_0} \tag{5.7}$$

For $\epsilon_1 > 0$, let $\{U^{\epsilon_1}_j\}_{j\in\mathbb{N}}$ be a countable cover of $GL(n,\mathbb{R})$ as in (5.1). Define $A_j = \{a \in E_{i_0} : \Lambda(a) \in U^{\epsilon_1}_i\}$. Since $\cup_j A_j$ covers E_{i_0} there exists some j_0 so that $\mu(A_{j_0}) > 0$. For $\epsilon_2 > 0$, define for $k \in \mathbb{Z}$

$$A^k_{j_0} = \left\{ a \in A_{j_0} : \theta^m_\Lambda(\mu,a) \in [(1+\epsilon_2)^k, (1+\epsilon_2)^{k+1}) \right\}.$$

If $a_1, a_2 \in A^k_{j_0}$ for some k, it follows

$$(1+\epsilon_2)^{-1} \leq \frac{\theta^m_\Lambda(\mu,a_1)}{\theta^m_\Lambda(\mu,a_2)} \leq (1+\epsilon_2). \tag{5.8}$$

Since $(0,\infty) = \cup_{k\in\mathbb{Z}}[(1+\epsilon_2)^k, (1+\epsilon_2)^{k+1})$, there exists some k with $\mu(A^k_{j_0}) > 0$ and we denote this set by A.

By Lemma 3.9 almost every $a \in A$ satisfies the density condition (3.10). Fix such an a. Let $\nu_a \in \mathrm{Tan}(\mu,a)$ and $x_a \in \mathrm{spt}\ \nu_a$ be as in (5.5). By Corollary 3.8, suppose without loss of generality that

$$\nu_a = \frac{\nu_a(B_1)}{\theta^m_\Lambda(\mu,a)} \lim_i r_i^{-m} T^\Lambda_{a,r_i}[\mu]. \tag{5.9}$$

By (3.11) of Lemma 3.9, there exists $\{a_i\} \subset A$ so that

$$\lim_{i\to\infty} \Lambda(a)^{-1}\left(\frac{a_i - a}{r_i}\right) = x_a. \tag{5.10}$$

Since ν_a is a uniform Radon measure, $\nu_a(\partial B(0,1)) = \nu_a(\partial B(x_a,1)) = 0$. Applying Proposition 2.2 twice, once with the choices $\mu = \nu_a$, $\mu_i = T^\Lambda_{a,r_i}[\mu]$, $T = T_{x_a,1}$,, and $T_i = T_{\Lambda(a)^{-1}\left(\frac{a_i-a}{r_i}\right),1}$ and again with the choices $\mu = \nu_a$, $\mu_i = T^\Lambda_{a,r_i}[\mu]$, and $T = T_{x_a,1} = T_i$ then using (5.9), (5.10), and Corollary 3.8, it follows

$$\lim_{i\to\infty} \frac{T_{x_a,1} \circ T^\Lambda_{a,r_i}[\mu](B(0,1))}{r_i^m} = \theta^m_\Lambda(\mu, a) = \lim_{i\to\infty} \frac{T_{\Lambda(a)^{-1}\left(\frac{a_i-a}{r_i}\right),1} \circ T^\Lambda_{a,r_i}[\mu](B(0,1))}{r_i^m}. \tag{5.11}$$

Let Λ_a denote the constant matrix-valued function from $\mathbb{R}^n \to GL(n,\mathbb{R})$ given by $y \mapsto \Lambda(a)$. A computation shows

$$T^{\Lambda_a}_{a_i,r_i}(y) = T_{\Lambda(a)^{-1}\left(\frac{a_i-a}{r_i}\right),1} \circ T^\Lambda_{a,r_i}(y).$$

Therefore (5.11) implies

$$\frac{\nu_a(B(x_a,1))}{\nu_a(B_1)} = \frac{1}{\theta^m_\Lambda(\mu,a)} \lim_{i\to\infty} \frac{\mu(B_{\Lambda_a}(a_i,r_i))}{r_i^m}.$$

Now (5.1) and $a \in A_{j_0}$ ensures

$$\frac{\nu_a(B(x_a,1))}{\nu_a(B(0,1))} \le \frac{(1+\epsilon_1)^m}{\theta^m_\Lambda(\mu,a)} \limsup_{i\to\infty} \frac{\mu\left(B_\Lambda(a_i,(1+\epsilon_1)r_i)\right)}{(1+\epsilon_1)^m r_i^m}.$$

When i is large enough that $(1+\epsilon_1)r_i < 2^{-i_0}$, (5.7) and (5.8) imply

$$\begin{aligned}\frac{\nu_a(B(x_a,1))}{\nu_a(B(0,1))} &< \frac{(1+\epsilon_1)^m}{\theta^m_\Lambda(\mu,a)}(1+\epsilon_0) \limsup_{i\to\infty} \theta^m_\Lambda(\mu,a_i) \\ &\le (1+\epsilon_1)^m(1+\epsilon_0)(1+\epsilon_2).\end{aligned}$$

For $\epsilon_0, \epsilon_1, \epsilon_2$ small enough, this is less than $1+k_0^{-1}$, verifying (5.6) and reaching a contradiction. □

Theorem 5.2. *If μ is a Radon measure on $\mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$ are such that $0 < \theta^m_\Lambda(\mu,a) < \infty$ for μ almost every a, then $\operatorname{Tan}(\mu,a) \subset \mathcal{M}_{n,m}$ for almost every a. In particular, μ is countably m-rectifiable.*

Proof. By Theorems 3.3 and 5.1 for almost every a, and all $\nu \in \operatorname{Tan}_\Lambda(\mu,a)$, $\operatorname{Tan}[\nu] \subset \operatorname{Tan}_\Lambda(\mu,a) \subset \mathcal{U}^m(\mathbb{R}^n)$. Lemma 2.12 implies $\operatorname{Tan}[\nu] \cap \mathcal{M}_n \neq \emptyset$.
Moreover, since $\nu \in \mathcal{U}^m(\mathbb{R}^n)$ it follows in fact that whenever $\nu_a \in \operatorname{Tan}[\nu] \cap \mathcal{M}_n$ then $\nu_a \in \mathcal{M}_{n,m}$. By Lemma 2.8 and 2.16, we can apply Lemma 3.5 to $\mathcal{F} = \mathcal{M}_{n,m}$ and $\mathcal{M} = \mathcal{U}^m(\mathbb{R}^n)$ to conclude that for almost every a, $\operatorname{Tan}_\Lambda(\mu,a) \subset \mathcal{M}_{n,m}$. By [Pre87, Theorem 5.6] this implies μ is countably m-rectifiable. □

Theorem 5.3. *Let δ_n be the dimensional constant in Theorem 1.5. Suppose μ is a Radon measure on $\mathbb{R}^m$, with the following properties at μ almost every a: $\theta^m_*(\mu,a) > 0$ and there exists a $\Lambda(a) \in GL(n,\mathbb{R})$ so that*

$$\frac{\theta^{m,*}_\Lambda(\mu,a)}{\theta^m_{\Lambda,*}(\mu,a)} - 1 < \delta_n. \tag{5.12}$$

Then μ is countably m-rectifiable.

The idea of the proof relies on the equivalence of (1) and (3) in Lemma 3.4. We use the Lebesgue-Besicovitch differentiation theorem to decompose the measure μ into countably many pieces μ_i, so that each μ_i has the following two properties: (a) μ_i almost everywhere $\theta^m_\Lambda(\mu_i,a)$ exists, and (b) Λ has small oscillation on μ_i. Together these two properties will imply that a linear transformation of μ_i, denoted by ν_i, has small density gap, i.e., $\frac{\theta^{m,*}(\nu_i,a)}{\theta^m_*(\nu_i,a)} - 1$ is small μ_i almost everywhere. Then Theorem 1.5(iv) will imply ν_i, and consequently μ_i, is rectifiable. This type of proof cannot be used to prove Theorem 1.1 because the cancellation present in the definition of the principal value does not behave well when decomposing a measure into small pieces.

Proof. Fix some $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$ so that for μ a.e. a,

$$A_k = \left\{ a \;\middle|\; \frac{\theta^{m,*}_\Lambda(\mu, a)}{\theta^m_{\Lambda,*}(\mu, a)} - 1 < (1 - 2^{-k})\delta_n \right\}. \tag{5.13}$$

For $k > 2$ and $\epsilon_k > 0$ to be chosen later, decompose $GL(n, \mathbb{R})$ into countably many neighborhoods $\{U_i^{\epsilon_k}\}_{i\in\mathbb{N}}$ as in (5.1), so that

$$M'B(0, (1 - \epsilon_k)) \subset MB(0, 1) \subset M'B(0, (1 + \epsilon_k)) \qquad \forall M, M' \in U^i_{\epsilon_k}.$$

Define $E_{i,k} = \{a \in A_k : \Lambda(a) \in U_i^{\epsilon_k}\}$. Since

$$\mu\left(\mathbb{R}^n \setminus \cup_{i,k} E_{i,k}\right) = 0,$$

rectifiability of μ follows from confirming $\mu \llcorner E_{i,k}$ is rectifiable for each i, k.

Fix some $i, k \in \mathbb{N}$. Suppose $M \in U_i^{\epsilon_k}$ and define

$$\mu_M = (M^{-1})_\sharp(\mu \llcorner E_{i,k}).$$

Since $M \in U_i^{\epsilon_k}$,

$$B_\Lambda(a, (1 - \epsilon_k)r) \subset B_M(a, r) \subset B_\Lambda(a, (1 + \epsilon_k)r). \tag{5.14}$$

Since M is bilipschitz, μ_M is rectifiable if and only if $\mu \llcorner E_{i,k}$ is rectifiable. The Lebesgue Besicovitch differentiation theorem ensures that for μ a.e. $a \in E_{i,k}$,

$$\begin{cases} \theta^{m,*}_\Lambda(\mu \llcorner E_{i,k}, a) = \theta^{m,*}_\Lambda(\mu, a) \\ \theta^m_{\Lambda,*}(\mu \llcorner E_{i,k}, a) = \theta^m_{\Lambda,*}(\mu, a). \end{cases} \tag{5.15}$$

In particular, (5.14) implies

$$\frac{\mu(B_\Lambda(a, (1 - \epsilon_k)r))}{r^m} \leq \frac{\mu_M(B(M^{-1}a, r))}{r^m} \leq \frac{\mu(B_\Lambda(a, (1 + \epsilon_k)r))}{r^m}$$

so that (5.15) and (5.13) respectively guarantee

$$\begin{aligned} \frac{\theta^{m,*}(\mu_M, M^{-1}a)}{\theta^m_*(\mu_M, M^{-1}a)} - 1 &\leq \frac{(1 + \epsilon_k)^m}{(1 - \epsilon_k)^m} \frac{\theta^{m,*}_\Lambda(\mu, a)}{\theta^m_{\Lambda,*}(\mu, a)} - 1 \\ &< \frac{(1 + \epsilon_k)^m}{(1 - \epsilon_k)^m}(1 + (1 - 2^{-k})\delta_n) - 1. \end{aligned}$$

Therefore, if ϵ_k is chosen small enough so that

$$\frac{(1 + \epsilon_k)^m}{(1 - \epsilon_k)^m}(1 + (1 - 2^{-k})\delta_n) - 1 < \delta_n \tag{5.16}$$

then the measure μ_M satisfies Theorem 1.5(iv) and consequently is countably m-rectifiable. Thus $\mu \llcorner E_{i,k}$ is rectifiable and since $\mu\left(\mathbb{R}^n \setminus \cup_{i,k} E_{i,k}\right) = 0$, this implies μ is countably m-rectifiable. □

We now put together all the pieces to prove Theorem 1.6.

Proof. For (1) $\iff$ (ii), note that by Lemma 3.4, $(\Lambda(a)^{-1})_\sharp \mathrm{Tan}(\mu, a) \subset \mathcal{M}_{n,m}$ if and only if $\mathrm{Tan}_\Lambda(\mu, a) \subset \mathcal{M}_{n,m}$. But the prior condition is equivalent to $\mathrm{Tan}(\mu, a) \subset \mathcal{M}_{n,m}$, so Theorem 1.5 now verifies (1) $\iff$ (ii). The equivalence (1) $\iff$ (iii) follows similarly. That (i) $\implies$ (1) and (iv) $\implies$ (1) are respectively Theorem 5.2 and Theorem 5.3.

Clearly (i) $\implies$ (iv), so it suffices to show (1) $\implies$ (i). Since by m-rectifiable, we in particular mean $\mu \ll \mathcal{H}^m$, it follows that there exist countably many m-dimensional C^1 embedded manifolds Σ_i so that $\mu\left(\mathbb{R}^n \setminus \cup_i \Sigma_i\right) = 0$. Without loss of generality, each Σ_i has a global chart $\varphi_i : \Sigma_i \to \mathbb{R}^m$. Fix (Σ_i, φ_i) and define ν on $\mathbb{R}^m$ as the image measure $(\varphi_i)\sharp\mu$. Since φ_i is a C^1 diffeomorphism onto its image, $\nu \ll \mathcal{L}^m \llcorner \varphi_i(\Sigma_i)$. Hence, by Radon-Nikodym and additionally the Lebesgue differentiation theorem in the form of [Fol99, Theorem 3.21], for ν almost every x,

$$\frac{d\nu}{d\mathcal{L}^m}(x) = \lim_{r\to 0} \frac{\nu(E_r(x))}{\mathcal{L}^m(E_r(x))} \in (0,\infty)$$

for ν almost every x and for any family of sets $E_r(x)$ shrinking nicely to $\{x\}$. In particular when $E_r(x) = \varphi(B_\Lambda(\varphi^{-1}(x), r))$. By the Lebesgue differentiation theorem in the form of [Fed14, Theorem 2.9.11],

$$\lim_{r\to 0} \frac{\mu(B_\Lambda(a,r)\cap \Sigma_i)}{\mu(B_\Lambda(a,r))} = 1 \tag{5.17}$$

for μ almost every $a \in \Sigma_i$. Consequently, for almost every $a \in \Sigma_i$,

$$\begin{aligned}
\lim_{r\to 0} \frac{\mu(B_\Lambda(a,r))}{r^m} &= \lim_{r\to 0} \frac{\mu(B_\Lambda(a,r)\cap \Sigma_i)}{r^m} \\
&= \left(\lim_{r\to 0} \frac{\mathcal{H}^m(E_r(\varphi(a)))}{r^m}\right)\left(\lim_{r\to 0} \frac{\mu \llcorner \Sigma_i(B_\Lambda(a,r))}{r^m} \frac{r^m}{\mathcal{H}^m(E_r(\varphi(a)))}\right) \\
&= \omega_m (J\varphi)(a) \lim_{r\to 0} \frac{\mu \llcorner \Sigma_i\left(\varphi^{-1}(E_r(a))\right)}{\mathcal{L}^m(E_r(\varphi(a))} \\
&= \omega_m (J\varphi)(a) \lim_{r\to 0} \frac{\nu(E_r(x))}{\mathcal{L}^m(E_r(x))} \in (0,\infty),
\end{aligned}$$

where the final conclusion of positive and finite is justified for almost every a since φ is a C^1 diffeomorphism and (5.17). Thus for any Σ_i and μ almost every $a \in \Sigma_i$ we have shown $\theta^m_\Lambda(\mu, a) \in (0,\infty)$. Since $\mu(\mathbb{R}^n \setminus \cup_i \Sigma_i) = 0$ this proves (i). □

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