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Measures of finite energy in pluripotential theory: A synthetic approach

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To Bo Berndtsson, on the occasion of his 70ε-th birthday

ABSTRACT. We introduce a synthetic approach to global pluripotential theory, covering in particular the case of a compact Kähler manifold and that of a projective Berkovich space over a non-Archimedean field. We define and study the space of measures of finite energy, introduce twisted energy and free energy functionals thereon, and show that coercivity of these functionals is an open condition with respect to the Kähler class.

CONTENTS

Introduction

Pluripotential theory on compact Kähler spaces is by now a very well developed subject, with key applications to Kähler geometry. Generalizing classical concepts from potential theory, measures and potentials of finite energy play a central role in this theory, see for instance [**GZ07**, **BBGZ13**, **Berm13**, **DN15**, **DGL21**]. In parallel, a non-Archimedean version of pluripotential theory has also emerged, taking place on projective Berkovich spaces [**Berk90**] over a (complete) non-Archimedean field. Initially motivated by Arakelov geometry [**Zha95**, **Cha06**], it has found various other applications, including degenerations of Calabi–Yau manifolds [**Y.Li20**] and the Yau–Tian–Donaldson conjecture [**BBJ21**, **C.Li22**, **BoJ23**].

These two versions of pluripotential theory bear many similarities, and can be formulated in a quite parallel way. The main purpose of the present article is to introduce a synthetic formalism covering in particular these two cases, and use it to extend some of the main results of [**BoJ22**] and [**BoJ23**] (that were taking place

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on projective Berkovich spaces over a trivially valued field, and applied to the study of K-stability). More specifically, we

- define and study the space of measures of finite energy;
- introduce the *twisted energy* and *free energy* functionals on the latter space; the composition of these functionals with the Monge–Ampère operator recover, respectively, the Donaldson J-functional and the Mabuchi K-energy in the Kähler case, and their analogues in the non-Archimedean case;
- show that coercivity of the free energy is an open condition with respect to the Kähler class.

The emphasis in this paper is on *measures* of finite energy, as opposed to *potentials* of finite energy, that we do not seek to investigate here (see for instance [**BFJ15**, **BoJ22**, **Reb21**, **DXZ23**] in the non-Archimedean context).

The general setup. Throughout this paper, we work with a compact topological space X equipped with the following data:

- a dense linear subspace $\mathcal{D} \subset \mathrm{C}^0(X)$ of *test functions*, containing all constants;
- a vector space $\mathcal{Z}$ of *admissible* $(1,1)$*-forms on* X, endowed with a nice[1] partial order, and a linear map $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ vanishing on constants;
- an integer $n \geq 1$ (seen as the 'dimension' of X), and a nonzero n-linear symmetric map taking a tuple $(\theta_1, \dots, \theta_n)$ in $\mathcal{Z}$ to a signed Radon measure $\theta_1 \wedge \dots \wedge \theta_n$ on X, assumed to be positive when all $\theta_i \geq 0$, and such that each bilinear form

$$\mathcal{D} \times \mathcal{D} \to \mathbb{R} \quad (\varphi, \psi) \mapsto \int \varphi \, \mathrm{dd^c}\psi \wedge \theta_1 \wedge \dots \wedge \theta_{n-1} \tag{0.1}$$

  with $\theta_i \in \mathcal{Z}$ is symmetric, and seminegative for $\theta_i \geq 0$.

We then introduce the *Bott–Chern cohomology space*

$$\mathrm{H_{BC}}(X) := \mathcal{Z}/\mathrm{dd^c}\mathcal{D},$$

and define the *positive cone* $\mathrm{Pos}(X) \subset \mathrm{H_{BC}}(X)$ as the interior[2] of the image of the convex cone

$$\mathcal{Z}_+ := \{\theta \in \mathcal{Z} \mid \theta \geq 0\}.$$

This setup is primarily inspired by the case of a compact Kähler manifold X, where $\mathcal{D}$ is the space of smooth functions, and $\mathcal{Z}$ the space of closed $(1,1)$-forms. It also covers the case of a projective Berkovich space X over a complete non-Archimedean field k, where $\mathcal{D}$ is generated by *piecewise linear* (or *model*) functions, and elements of $\mathcal{Z}$ are represented by numerical classes on models over the valuation ring (or test configurations, in the trivially valued case) [**BFJ16a**, **GM16**, **BoJ22**]. At least under reasonable assumptions on X and k, we then have $\mathrm{H_{BC}}(X) = \mathrm{N}^1(X)$, and $\mathrm{Pos}(X)$ coincides with the ample cone, see §1.3.2.

[1] See §1.1 for the terminology.

[2] Here we use the finest vector space topology of $\mathrm{H_{BC}}(X)$, see §1.1.

Measures of finite energy. Fix a form $\omega \in \mathcal{Z}_+$ such that $[\omega] \in \mathrm{Pos}(X)$, with volume $V_\omega = \int \omega^n > 0$. The space of *$\omega$-plurisubharmonic* test functions is defined as

$$\mathcal{D}_\omega := \{\varphi \in \mathcal{D} \mid \omega_\varphi := \omega + dd^c\varphi \geq 0\},$$

and the *Monge–Ampère operator* takes $\varphi \in \mathcal{D}_\omega$ to the probability measure

$$\mathrm{MA}_\omega(\varphi) := V_\omega^{-1}\omega_\varphi^n.$$

It admits a primitive, the *Monge–Ampère energy* $\mathrm{E}_\omega \colon \mathcal{D}_\omega \to \mathbb{R}$, explicitly given by

$$\mathrm{E}_\omega(\varphi) = \frac{1}{n+1}\sum_{j=0}^n V_\omega^{-1}\int \varphi\,\omega_\varphi^j \wedge \omega^{n-j}.$$

Denote by $\mathcal{M}$ the space of (Radon) probability measures on X, and define the *energy* of a measure $\mu \in \mathcal{M}$ as the Legendre transform[3]

$$\mathrm{J}_\omega(\mu) := \sup_{\varphi \in \mathcal{D}_\omega}\{\mathrm{E}_\omega(\varphi) - \int \varphi\,\mu\} \in [0, +\infty].$$

Then $\mathrm{J}_\omega \colon \mathcal{M} \to [0, +\infty]$ is convex, and lsc in the weak topology; the space of *measures of finite energy* is defined as

$$\mathcal{M}_\omega^1 := \{\mu \in \mathcal{M} \mid \mathrm{J}_\omega(\mu) < \infty\},$$

equipped with the *strong topology*, i.e. the coarsest refinement of the weak topology in which J_ω becomes continuous.

As a consequence of the seminegativity of (0.1), the functional E_ω is concave. This is equivalent to the non-negativity of the *Dirichlet functional*

$$\mathrm{J}_\omega(\varphi, \psi) := \mathrm{E}_\omega(\varphi) - \mathrm{E}_\omega(\psi) + \int (\psi - \varphi)\,\mathrm{MA}_\omega(\varphi),$$

which is more explicitly given by the familiar expression

$$\mathrm{J}_\omega(\varphi, \psi) = \frac{1}{2}\int (\varphi - \psi)dd^c(\psi - \varphi)$$

when $n = 1$, and a positive linear combination of integrals of the form

$$\int (\varphi - \psi)dd^c(\psi - \varphi) \wedge \omega_\varphi^j \wedge \omega_\psi^{n-j-1} \quad (0 \leq j < n)$$

in general, see (1.31). Our first main result shows that the Dirichlet functional induces, via the Monge–Ampère operator, a complete quasi-metric[4] space structure on $\mathcal{M}_\omega^1$.

Theorem A. *Assume that ω has the* orthogonality property. *Then:*

(i) *the image of the Monge–Ampère operator $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}$ is a dense subspace of $\mathcal{M}_\omega^1$;*

(ii) *there exists a unique quasi-metric δ_ω on $\mathcal{M}_\omega^1$ that defines the strong topology of $\mathcal{M}_\omega^1$ and such that*

$$\delta_\omega(\mathrm{MA}_\omega(\varphi), \mathrm{MA}_\omega(\psi)) = \mathrm{J}_\omega(\varphi, \psi)$$

for all $\varphi, \psi \in \mathcal{D}_\omega$;

(iii) *the quasi-metric space $(\mathcal{M}_\omega^1, \delta_\omega)$ is complete.*

[3] This corresponds to $\mathrm{E}_\omega^\vee(\mu)$ in the notation of [BBGZ13, BoJ22], and to $\|\mu\|_\omega$ in that of [BoJ23].

[4] See §1.1 for the notion of quasi-metric used in this paper.

The energy can be expressed in terms of the quasi-metric as

$$\mathrm{J}_\omega(\mu) = \delta_\omega(\mu, \mu_\omega) \quad \text{where} \quad \mu_\omega := V_\omega^{-1}\omega^n = \mathrm{MA}_\omega(0).$$

We refer to Definition 2.15 for the precise definition of the orthogonality property. Suffice it to say here that it only depends on $[\omega] \in \mathrm{Pos}(X)$, and holds when X is a compact Kähler manifold, or any projective Berkovich space X over a non-Archimedean field (as a consequence of [**BE21**, **BGM22**]). In the former case, Theorem A can be deduced from [**BBGZ13**, **BBEGZ19**]; in the latter, it was established in the trivially valued case in [**BoJ22**], and is thus extended here to the case of an arbitrary non-Archimedean ground field.

The strategy of proof of Theorem A follows the same lines as in the trivially case treated in [**BoJ22**]. The first key ingredient is a uniform differentiability property for the Legendre transform of the convex functional $\mu \mapsto \mathrm{J}_\omega(\mu)$, which is shown to be equivalent to the orthogonality property. This is used to prove that if (φ_i) is a maximizing sequence for a given $\mu \in \mathcal{M}^1_\omega$ (i.e. a sequence in $\mathcal{D}_\omega$ computing the supremum that defines $\mathrm{J}_\omega(\mu)$), then $\mathrm{MA}_\omega(\varphi_i)$ converges to μ. We emphasize that this 'asymptotic Calabi–Yau theorem' is sufficient for our purposes, and that we do not need to characterize the image of the Monge–Ampère operator (which is a delicate issue in the non-Archimedean context).

The rest of the proof relies on an extensive use of Hölder estimates for mixed Monge–Ampère integrals, obtained from repeated applications of the Cauchy–Schwarz inequality to the seminegative form (0.1). This approach, which goes back to [**Blo03**] and was further exploited in [**BBGZ13**, **BBEGZ19**, **BoJ22**], is put in a simple general setting in Appendix A.

Twisted energy, free energy, and coercivity. Assuming from now on the orthogonality property, we next investigate the dependence on ω of the space $\mathcal{M}^1_\omega$ and the energy functional $\mathrm{J}_\omega\colon \mathcal{M}^1_\omega \to \mathbb{R}_{\geq 0}$. To this end, we require the *submean value property*, i.e. the existence of $C \in \mathbb{R}_{\geq 0}$ such that

$$\sup \varphi \leq \int \varphi\, \mu_\omega + C$$

for all ω-psh test functions $\varphi \in \mathcal{D}_\omega$. We show that this condition is independent of ω, and that it is equivalent to the irreducibility of X when the latter is a compact Kähler or projective Berkovich space (see §1.5).

Theorem B. *Assume that the submean value property holds. Then:*

- *the topological space $\mathcal{M}^1 = \mathcal{M}^1_\omega$ is independent of ω;*
- *for any $\theta \in \mathcal{Z}$, there exists a unique continuous functional $\mathrm{J}^\theta_\omega\colon \mathcal{M}^1 \to \mathbb{R}$ such that*

$$\mathrm{J}^\theta_\omega(\mu) = \frac{d}{dt}\Big|_{t=0} \mathrm{J}_{\omega+t\theta}(\mu)$$

for any $\mu \in \mathcal{M}^1$; furthermore, $\mathrm{J}^\theta_\omega(\mu)$ satisfies Hölder estimates with respect to ω.

The strategy of proof of Theorem B again globally follows the same lines as the trivially valued case treated in [**BoJ22**, **BoJ23**]. However, in the latter case the submean value inequality is actually an equality, i.e. one can take the constant C above to be 0, while an extra layer of complication arises in the general case to handle this constant, which forces us to take a slightly different route.

We call $\mathrm{J}^\theta_\omega(\mu)$ the *θ-twisted energy* of $\mu \in \mathcal{M}^1$. It provides an analogue of the *Donaldson J-functional* on the level of measures, and its relevance comes from its relation to the *Mabuchi K-energy*, when X is a compact Kähler manifold or a smooth projective Berkovich space. In these two cases, the choice of a (smooth or PL) metric ρ on the canonical bundle K_X defines an *entropy* functional $\mathrm{Ent}\colon \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$. In line with [**Berm13**], we then define the *free energy* $\mathrm{F}_\omega\colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\mathrm{F}_\omega(\mu) := \mathrm{Ent}(\mu) - \mathrm{Ent}(\mu_\omega) + \mathrm{J}^\theta_\omega(\mu),$$

where $\theta \in \mathcal{Z}$ denotes the curvature of ρ. The free energy so defined is independent of the choice of ρ, and its composition with the Monge–Ampère operator coincides with the *Mabuchi K-energy* $\mathrm{M}_\omega\colon \mathcal{D}_\omega \to \mathbb{R}$. As a consequence of Theorem B, we then show:

Theorem C. *Assume X is a compact Kähler manifold or a smooth projective Berkovich space over a non-Archimedean field. Then the* coercivity threshold

$$\sigma(X,\omega) := \sup\{\sigma \in \mathbb{R} \mid \mathrm{F}_\omega \geq \sigma \mathrm{J}_\omega + A \text{ for some } A \in \mathbb{R}\}.$$

defines a continuous function of $[\omega] \in \mathrm{Pos}(X)$.

This result actually holds in much greater generality, for the twisted coercivity threshold of an arbitrary given functional on $\mathcal{M}^1$ with no a priori regularity whatsoever (see Theorem 5.5).

In the trivially valued case, the free energy $\mathrm{F}_\omega(\mu)$ coincides with the invariant $\beta_\omega(\mu)$ defined and studied in [**BoJ23**], and $\sigma(X,\omega)$ with the *divisorial stability threshold* of (X,ω), which is positive iff (X,ω) is *divisorially stable* (a strengthening of uniform K-stability, conjecturally equivalent to it, cf. §5.2.2). Specializing to the case of Dirac measures μ recovers the notion of *valuative stability* [**DL23**, **Liu23**], which in the Fano case is equivalent to K-stability [**Fuj19a**, **Li17**].

In the case of a compact Kähler manifold, we have $\sigma(X,\omega) > 0$ iff $[\omega]$ contains a unique constant scalar curvature Kähler (cscK) metric, as a consequence of [**CC21**] and [**DaR17**, **BDL20**]. Theorem C thus recovers the fact, originally due to LeBrun-Simanca [**LS94**], that the existence of a unique cscK metric in a Kähler class is an open condition on that class.

Structure of the paper. The article is organized as follows.

- Section 1 introduces the synthetic pluripotential theoretic formalism, including the energy pairing and the submean value property, and establishes basic properties of the Dirichlet functional.
- Section 2 studies the space of measures of finite energy. It introduces the orthogonality property, and proves Theorem A (cf. §2.6).
- Assuming the submean value property, Section 3 establishes the first part of Theorem B, along with various further estimates for the energy.
- Section 4 is devoted to the twisted energy, which is proved to compute the directional derivatives of the energy, concluding the proof of Theorem B.
- Section 5 studies the (twisted) coercivity threshold of a functional, and proves that it depends continuously on the cohomology classes. This is then applied to the free energy, yielding Theorem C.
- Finally, Appendix A establishes the relevant estimates needed for the Dirichlet functional in a simple general setting, while Appendix B studies the orthogonality property on compact Kähler spaces.

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It is a great pleasure to dedicate this article to Bo Berndtsson. His many fundamental contributions to complex analysis, pluripotential theory and convex analysis have been a constant source of inspiration for us. We would especially like to express our admiration for his fantastic results on positivity of direct images that have played a key role in our own works (among many others!), as well as for his laid-back approach to mathematics in general, which makes Bo such a pleasant person to interact with.

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1. A synthetic pluripotential theoretic formalism

In this section, we introduce the general setup considered in this paper. This is designed to cover in a synthetic manner the case of a compact Kähler space and that of a projective Berkovich space over a non-Archimedean field.

1.1. Notation and terminology.

- For $x, y \in \mathbb{R}$, $x \lesssim y$ or $x = O(y)$ mean in this paper $x \le C_n y$ for a constant $C_n > 0$ only depending on a given integer n fixed in the setup, and $x \approx y$ if $x \lesssim y$ and $y \lesssim x$.
- Recall that any real vector space V admits a *finest vector space topology*, generated by its finite dimensional subspaces, i.e. a subset $A \subset V$ is open (or closed) iff, for each finite dimensional subspace $W \subset V$, $A \cap W$ is open (resp. closed) in the canonical vector space topology of W. This topology is not locally convex as soon as the dimension of V is uncountable.
- Consider a partially ordered $\mathbb{R}$-vector space $(V, \ge)$. We shall say for brevity that the partial order is *nice* if $V_+ := \{x \in V \mid x \ge 0\}$ spans V, and is closed in the finest vector space topology of V.
- In this paper, a *quasi-metric* on a set Z is a function $\delta\colon Z \times Z \to \mathbb{R}_{\ge 0}$ that is
  - *quasi-symmetric*, i.e. there exists $C > 0$ such that
  $$C^{-1}\delta(x, y) \le \delta(y, x) \le C\delta(x, y);$$
  - satisfies the *quasi-triangle inequality*, i.e. there exists $C > 0$ such that
  $$\delta(x, y) \le C(\delta(x, z) + \delta(z, y))$$
  - *separates points*, i.e. $\delta(x, y) = 0 \Leftrightarrow x = y$.

  A quasi-metric space (Z, δ) comes with a Hausdorff topology, and Cauchy sequences and completeness further make sense for (Z, δ).

1.2. Test functions and admissible $(1,1)$-forms. Throughout this paper, we work with a compact Hausdorff topological space X. We denote by $\mathrm{C}^0(X)^\vee$ the space of signed Radon measures on X, and by

$$\mathcal{M} \subset \mathrm{C}^0(X)^\vee$$

the subset of probability measures, which is convex and compact in the weak topology. Recall from the introduction that we assume X equipped with the following data:

- a dense linear subspace $\mathcal{D} \subset \mathrm{C}^0(X)$ of *test functions*, containing all constants;
- a vector space $\mathcal{Z}$ of *admissible* $(1,1)$*-forms on* X, endowed with a nice partial order, and a linear map $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ vanishing on constants;
- an integer $n \geq 1$ (viewed as the 'dimension' of X), and a nonzero n-linear symmetric map taking a tuple $(\theta_1, \ldots, \theta_n)$ in $\mathcal{Z}$ to a signed Radon measure $\theta_1 \wedge \cdots \wedge \theta_n$ on X, assumed to be positive when all $\theta_i \geq 0$, and such that each bilinear form
$$\mathcal{D} \times \mathcal{D} \to \mathbb{R} \quad (\varphi, \psi) \mapsto \int \varphi \, \mathrm{dd^c}\psi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}$$
with $\theta_i \in \mathcal{Z}$ is symmetric, and seminegative when $\theta_i \geq 0$.

Symmetry in the last item amounts to the *integration-by-parts* formula

$$\int \varphi \, \mathrm{dd^c}\psi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} = \int \psi \, \mathrm{dd^c}\varphi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} \tag{1.1}$$

for all $\varphi, \psi \in \mathcal{D}$ and $\theta_i \in \mathcal{Z}$, while seminegativity requires

$$\int \varphi \, \mathrm{dd^c}\varphi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} \leq 0 \tag{1.2}$$

when $\theta_i \geq 0$ for all i.

Remark 1.1. *The above setup induces a similar one by viewing $\theta_1 \wedge \cdots \wedge \theta_n$ as a p-linear symmetric function of $(\theta_1, \ldots, \theta_p)$ for $1 \leq p \leq n$ and $\theta_{p+1}, \ldots, \theta_n \in \mathcal{Z}_+$ fixed, or by replacing $\mathcal{Z}$ with any linear subspace $\mathcal{Z}'$ containing $\mathrm{dd^c}\mathcal{D}$.*

Definition 1.2. *For any $\theta \in \mathcal{Z}$ and $\varphi \in \mathcal{D}$, we set $\theta_\varphi := \theta + \mathrm{dd^c}\varphi$. We say that the test function φ is θ-*plurisubharmonic *(θ-psh for short) if $\theta_\varphi \geq 0$, and denote by*
$$\mathcal{D}_\theta := \{\varphi \in \mathcal{D} \mid \theta_\varphi \geq 0\}$$
the space so defined.

Note that if $\theta \in \mathcal{Z}$ and $\varphi \in \mathcal{D}$, we have

$$\varphi \in \mathcal{D}_{\theta_\tau} \iff \varphi + \tau \in \mathcal{D}_\theta. \tag{1.3}$$

Moreover, for all $\theta, \theta' \in \mathcal{Z}$ and $t \in \mathbb{R}_{>0}$ we have
$$\mathcal{D}_\theta + \mathcal{D}_{\theta'} \subset \mathcal{D}_{\theta+\theta'}, \quad \mathcal{D}_{t\theta} = t\mathcal{D}_\theta.$$
In particular, $\mathcal{D}_\theta$ is a convex subset of $\mathcal{D}$, and

$$\theta \leq \theta' \Longrightarrow \mathcal{D}_\theta \subset \mathcal{D}_{\theta'}. \tag{1.4}$$

Since $\mathrm{dd^c}$ vanishes on constants, we have:

Example 1.3. *Constant functions on X are θ-psh iff $\theta \geq 0$.*

The two main instances of the above setup considered in this paper are as follows.

1.2.1. *The Kähler case.* The above formalism is primarily inspired by the case of a compact Kähler complex analytic space X. Here $n = \dim X$, $\mathcal{D} = \mathcal{C}^\infty(X)$ is the space of smooth functions on X, and $\mathcal{Z}$ is the space of closed $(1,1)$-forms θ on X that are locally $\mathrm{dd^c}$-exact (i.e. global sections of the image $\mathcal{Z}_X^{1,1}$ of the sheaf morphism $\mathrm{dd^c}\colon \mathcal{C}_X^\infty \to \Omega_X^{1,1}$), and $\theta \geq 0$ means that θ is semipositive as a smooth $(1,1)$-form (see for instance [**Dem85**] for the definition of smooth forms in this context).

When X is nonsingular, i.e. a compact Kähler manifold, $\mathcal{Z}$ coincides with the space of closed $(1,1)$-forms on X, but the inclusion might be strict in the singular case.

For all $\varphi, \psi \in \mathcal{D}$ and $\theta_i \in \mathcal{Z}$, the Stokes formula implies

$$\int \varphi \, \mathrm{dd^c}\psi \wedge \Theta = \int \psi \, \mathrm{dd^c}\varphi \wedge \Theta = -\int \mathrm{d}\varphi \wedge \mathrm{d^c}\psi \wedge \Theta$$

with $\Theta := \theta_1 \wedge \cdots \wedge \theta_{n-1}$, which yields (1.1) and (1.2), since the $(1,1)$-form $\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi$ is semipositive.

1.2.2. *The non-Archimedean case.* Assume now that X is a projective Berkovich space over a non-Archimedean field k, i.e. the Berkovich analytification of a projective k-scheme, of dimension $n = \dim X$. We then take $\mathcal{D}$ to be the $\mathbb{R}$-vector space generated by *PL functions*, see [**BE21**, §5.4].

When k is nontrivially valued, $\mathcal{D}$ can be described in terms of *vertical divisors* on (projective, flat) models $\mathcal{X}$ of X over the spectrum S of the valuation ring. More precisely, we have

$$\mathcal{D} \simeq \varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X})_{\mathbb{R}},$$

where $\mathrm{VCar}(\mathcal{X})_{\mathbb{R}}$ denotes the $\mathbb{R}$-vector space generated by Cartier divisors on $\mathcal{X}$ that are vertical, i.e. supported on the special fiber. The same description applies in the trivially valued case as well, if a model is now understood as a *test configuration* $\mathcal{X} \to S := \mathbb{A}^1$ (see [**BHJ17**, §6.1], [**BoJ22**, §2.2]).

In the nontrivially valued case, the space $\mathcal{Z}$ is defined by setting

$$\mathcal{Z} := \varinjlim_{\mathcal{X}} \mathrm{N}^1(\mathcal{X}/S), \tag{1.5}$$

where $\mathrm{N}^1(\mathcal{X}/S)$ denotes the (finite dimensional) vector space of relative numerical classes (see [**BFJ16a**, §4.2], [**GM16**, §4], the definition being inspired by [**BGS95**]). A form $\theta \in \mathcal{Z}$ is thus represented by a numerical class $\theta_{\mathcal{X}} \in \mathrm{N}^1(\mathcal{X}/S)$ for some model $\mathcal{X}$, called a *determination* of θ, two such classes being identified if they coincide after pulling back to some higher model, and we write $\theta \geq 0$ if $\theta_{\mathcal{X}}$ is (relatively) nef for some (hence any) determination $\mathcal{X}$ of θ.

The measure $\theta_1 \wedge \cdots \wedge \theta_n$ associated to a tuple of forms $\theta_i \in \mathcal{Z}$ is a finite linear combination of Dirac masses at divisorial points, whose coefficients can be described in terms of intersection numbers computed on models.

The linear map $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ takes a vertical divisor $D \in \mathrm{VCar}(\mathcal{X})_{\mathbb{R}}$ to its numerical class in $\mathrm{N}^1(\mathcal{X}/S)$, and the seminegativity condition (1.2) follows the local Hodge index theorem of Yuan–Zhang [**YZ17**, Theorem 2.1].

Again, the same discussion applies to the trivially valued case as well, using test configurations instead of models. In that case, pulling back numerical classes on X to the product test configuration further yields an injection

$$\mathrm{N}^1(X) \hookrightarrow \mathcal{Z}. \tag{1.6}$$

Note that only forms lying in $\mathrm{N}^1(X)$ were considered in [**BoJ22**].

REMARK 1.4. *In the non-Archimedean case, one could also work with smooth functions and* $(1,1)$*-forms in the sense of* [**CD12**] *(see also* [**GK15**, **GK17**, **GJR21**]*), but this will not be considered in the present paper.*

REMARK 1.5. *More generally, test configurations of any compact Kähler manifold (as in* [**SD18**, **DeR17**]*) can also be approached using the above formalism. This is the topic of recent work of Pietro Mesquita-Piccione* [**MP24**]*.*

REMARK 1.6. *Another setting where the above formalism applies is that of tropical toric pluripotential theory as in* [**BGJK21**]*.*

1.3. Bott–Chern cohomology and positive classes.

DEFINITION 1.7. *We define the* Bott–Chern cohomology space *as*

$$\mathrm{H_{BC}}(X) := \mathcal{Z} / \operatorname{Im} \mathrm{dd^c},$$

and denote by $\theta \mapsto [\theta]$ *the quotient map* $\mathcal{Z} \to \mathrm{H_{BC}}(X)$*. The* positive cone

$$\mathrm{Pos}(X) \subset \mathrm{H_{BC}}(X)$$

is defined as the interior of the image of $\mathcal{Z}_+$*.*

Here the interior is taken with respect to the finest vector space topology (see §1.1). Concretely, a class $\alpha \in \mathrm{H_{BC}}(X)$ belongs to $\mathrm{Pos}(X)$ if, for any $\beta \in \mathrm{H_{BC}}(X)$, $\alpha + t\beta$ lies in the image of $\mathcal{Z}_+$ for all $t \in \mathbb{R}$ small enough.

Since it is assumed that $\mathcal{Z}_+$ spans $\mathcal{Z}$, its image in $\mathrm{H_{BC}}(X)$ is a convex cone that generates $\mathrm{H_{BC}}(X)$. As a consequence, the positive cone $\mathrm{Pos}(X)$ is non-empty as soon as $\mathrm{H_{BC}}(X)$ is finite dimensional.

REMARK 1.8. *The image of* $\mathcal{Z}_+$ *in* $\mathrm{H_{BC}}(X)$ *is not closed in general. Indeed, in the compact Kähler case, this means that a nef* $(1,1)$*-class on* X *does not always admit a smooth semipositive representative (see Example 1.3.1 below, and* [**DPS94**, Example 1.7] *for an explicit example).*

Since $\mathrm{dd^c}$ vanishes on $\mathbb{R} \subset \mathcal{D}$, (1.1) yields

$$\int \mathrm{dd^c}\varphi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} = \int \varphi\, \mathrm{dd^c} 1 \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} = 0$$

for all $\varphi \in \mathcal{D}$ and $\theta_i \in \mathcal{Z}$. As a result, $(\theta_1, \dots, \theta_n) \mapsto \int \theta_1 \wedge \cdots \wedge \theta_n$ descends to a symmetric n-linear pairing

$$\mathrm{H_{BC}}(X)^n \to \mathbb{R} \quad (\alpha_1, \dots, \alpha_n) \mapsto \alpha_1 \cdot \ldots \cdot \alpha_n,$$

which we call the *intersection pairing.*

LEMMA 1.9. *For all classes* $\alpha_1, \dots, \alpha_n \in \mathrm{Pos}(X)$ *we have* $\alpha_1 \cdot \ldots \cdot \alpha_n > 0$*.*

PROOF. By assumption, the measure $\theta_1 \wedge \cdots \wedge \theta_n$ is nonzero for some tuple $\theta_1, \dots, \theta_n \in \mathcal{Z}$. Since $\mathcal{Z}_+$ generates $\mathcal{Z}$, we can further assume $\theta_i \in \mathcal{Z}_+$ for all i. Then $[\theta_1] \cdot \ldots \cdot [\theta_n] = \int \theta_1 \wedge \cdots \wedge \theta_n > 0$. Now we can find $0 < \varepsilon \ll 1$ such that $\alpha_i - \varepsilon[\theta_i] \in \mathrm{Pos}(X)$ for all i, and hence $\alpha_1 \cdot \ldots \cdot \alpha_n \geq \varepsilon^n [\theta_1] \cdot \ldots \cdot [\theta_n] > 0$. □

DEFINITION 1.10. *For each* $\omega \in \mathcal{Z}_+$ *and* $\theta \in \mathcal{Z}$ *we set*

$$\|\theta\|_\omega := \inf\{C \geq 0 \mid \pm\theta \leq C\omega\} \in [0, +\infty],$$

and we say that θ *is* ω-bounded *if* $\|\theta\|_\omega < \infty$*.*

The set

$$\mathcal{Z}_\omega := \{\theta \in \mathcal{Z} \mid \|\theta\|_\omega < \infty\}$$

of ω-bounded forms is a linear subspace of $\mathcal{Z}$, on which $\|\cdot\|_\omega$ defines a norm.

REMARK 1.11. *In general, $\mathcal{Z}_\omega$ is a strict subspace of $\mathcal{Z}$. More precisely, $\mathcal{Z}_\omega = \mathcal{Z}$ iff ω lies in the interior of $\mathcal{Z}_+$ in the finest vector space topology of $\mathcal{Z}$, and this interior is empty in the non-Archimedean case (see §1.3.2 below).*

We can now characterize positive classes as follows:

PROPOSITION 1.12. *A class $\alpha \in \mathrm{H}_{\mathrm{BC}}(X)$ lies in $\mathrm{Pos}(X)$ iff, for each finite dimensional subspace $V \subset \mathcal{Z}$, α admits a representative $\omega \in \mathcal{Z}_+$ such that $V \subset \mathcal{Z}_\omega$.*

PROOF. Assume $\alpha \in \mathrm{Pos}(X)$, and pick a basis $(\theta_i)_{1\le i\le r}$ of a finite dimensional vector space $V \subset \mathcal{Z}$. Since $\mathcal{Z}_+$ spans $\mathcal{Z}$, for each i we can write $\theta_i = \theta_i^+ - \theta_i^-$ with $\theta_i^\pm \ge 0$. Since $\alpha \in \mathrm{Pos}(X)$ we have $\alpha - \varepsilon[\theta_i^\pm] \in \mathrm{Pos}(X)$ for all $0 < \varepsilon \ll 1$, and we can thus find $\varepsilon > 0$ and $\omega_i^\pm \in \alpha$ such that $\omega_i^\pm - \varepsilon\theta_i^\pm \ge 0$ for all i. Now set

$$\omega := \tfrac{1}{2r}\sum_{i=1}^r (\omega_i^+ + \omega_i^-) \in \mathcal{Z}_+.$$

Then $[\omega] = \alpha$, and for each i we have $\omega \ge \frac{\varepsilon}{2r}\theta_i^\pm \ge \pm\frac{\varepsilon}{2r}\theta_i$, and hence $\theta_i \in \mathcal{Z}_\omega$, i.e. $V \subset \mathcal{Z}_\omega$. This proves the 'only if' part, and the converse is clear. □

COROLLARY 1.13. *Pick $\theta \in \mathcal{Z}$ such that $[\theta] \in \mathrm{Pos}(X)$. Then any $f \in \mathcal{D}$ can be written as $f = f^+ - f^-$ with $f^\pm \in \mathcal{D}_{t\theta} = t\mathcal{D}_\theta$ for some $t > 0$. In particular, $\mathcal{D}_\theta$ spans $\mathcal{D}$.*

PROOF. By Proposition 1.12, we can find $\psi \in \mathcal{D}_\theta$ and $t > 0$ such that $-\mathrm{dd^c} f \le t(\theta + \mathrm{dd^c}\psi)$. Thus $f^+ := f + t\psi$ lies in $\mathcal{D}_{t\theta}$, and the result follows with $f^- := t\psi \in \mathcal{D}_{t\theta}$. □

Following [**Tho63**], we define the *Thompson distance* between $\omega, \omega' \in \mathcal{Z}_+$ as

$$\mathrm{d_T}(\omega,\omega') := \inf\{\delta \ge 0 \mid e^{-\delta}\omega \le \omega' \le e^\delta\omega\} \in [0,+\infty]. \tag{1.7}$$

We say that ω and ω' are *commensurable* if $\mathrm{d_T}(\omega,\omega') < \infty$. Note that this holds iff $\omega' \in \mathcal{Z}_\omega$ and $\omega \in \mathcal{Z}_{\omega'}$. Commensurability is an equivalence relation on $\mathcal{Z}_+$. The linear subspace $\mathcal{Z}_\omega$ only depends on the commensurability class of $\omega \in \mathcal{Z}_+$, and so does the equivalence class of the norm $\|\cdot\|_\omega$.

The next result is readily checked, and left to the reader.

LEMMA 1.14. *The commensurability class of any $\omega \in \mathcal{Z}_+$ forms an open convex cone in the normed vector space $(\mathcal{Z}_\omega, \|\cdot\|_\omega)$, whose topology is defined by the restriction of the Thompson metric.*

We conclude this section with the following fact, that we will put to use in §3.

PROPOSITION 1.15. *Each finite subset of $\mathrm{Pos}(X)$ can be represented by commensurable forms in $\mathcal{Z}_+$.*

PROOF. Consider $\alpha_1,\dots,\alpha_r \in \mathrm{Pos}(X)$ and set $\beta := \frac{1}{r}\sum_1^r \alpha_i$. If $0 < t \ll 1$, then the classes α_i' defined by $\alpha_i = (1-t)\alpha_i' + t\beta$ lie in $\mathrm{Pos}(X)$. Pick any representatives $\theta_i' \in \mathcal{Z}_+$ of α_i', $1 \le i \le r$. By Proposition 1.12, we can find a representative $\omega \in \mathcal{Z}_+$ of β such that $\theta_i' \in \mathcal{Z}_\omega$ for all i. Let $\omega_i := (1-t)\theta_i' + t\omega$ for $1 \le i \le r$. Then $\omega_i \in \mathcal{Z}_+$ is a representative of α_i and ω, ω_i are commensurable for all i. □

1.3.1. *The Kähler case.* As in §1.2.1, assume that X is a compact Kähler space. When X is nonsingular, $\mathrm{H}_{\mathrm{BC}}(X)$ coincides with the $(1,1)$-part of the Hodge decomposition of $\mathrm{H}^2(X,\mathbb{R})$. In general, denote by $\mathcal{PH}_X$ the sheaf of germs of *pluriharmonic functions*, i.e. the kernel of the sheaf morphism $\mathrm{dd^c}\colon \mathcal{C}^\infty_X \to \Omega^{1,1}_X$. The existence of partitions of unity implies that the sheaf $\mathcal{C}^\infty_X$ is soft, and the cohomology long exact sequence associated to the short exact sequence of sheaves

$$0 \to \mathcal{PH}_X \to \mathcal{C}^\infty_X \to \mathcal{Z}^{1,1}_X \to 0$$

thus yields

$$\mathrm{H}_{\mathrm{BC}}(X) \simeq \mathrm{H}^1(X, \mathcal{PH}_X).$$

We do not know whether the right-hand side is always finite dimensional, but this holds at least when X is normal, as a consequence of the fact that $\mathcal{PH}_X$ then coincides with the sheaf $\Re\mathcal{O}_X$ of real parts of holomorphic functions, see [**BG13**, §4.6.1].

The positive cone $\mathrm{Pos}(X)$ is compatible with the usual definition, i.e. $\alpha \in \mathrm{H}_{\mathrm{BC}}(X)$ lies in $\mathrm{Pos}(X)$ iff α can be represented by a Kähler form. Indeed, given any Kähler form ω, a class $\alpha \in \mathrm{Pos}(X)$ can be represented by a form $\theta \in \mathcal{Z}$ such that $\theta - \varepsilon\omega \in \mathcal{Z}_+$ for some $0 < \varepsilon \ll 1$, so that θ is a Kähler form. In particular, Kähler forms constitute a single commensurability class that maps onto $\mathrm{Pos}(X)$.

1.3.2. *The non-Archimedean case.* Assume, as in §1.2.2, that X is a projective Berkovich space over a non-Archimedean field. By definition, any $\theta \in \mathcal{Z}$ is represented by a numerical class on some model $\mathcal{X}$ of X, and the restriction of θ to the generic fiber defines a numerical class $\{\theta\} \in \mathrm{N}^1(X)$ on (the projective variety underlying) X. This induces a surjective map

$$\mathrm{H}_{\mathrm{BC}}(X) \twoheadrightarrow \mathrm{N}^1(X), \tag{1.8}$$

which is an isomorphism when X is smooth and k is discretely valued of residue characteristic 0 [**BFJ16a**, Theorem 4.3], or when X is normal and k is algebraically closed [**Jell16**, Theorem 4.2.7]). It is also an isomorphism when k is trivially valued, with inverse provided by pulling back classes in $\mathrm{N}^1(X)$ to the trivial test configuration $\mathcal{X}_{\mathrm{triv}} = X \times \mathbb{A}^1$.

We claim that $\mathrm{Pos}(X)$ coincides with the preimage of the ample cone of X under (1.8), and that $\mathcal{Z}_+$ has empty interior in the finest vector space topology of $\mathcal{Z}$ (so that Kähler forms do not admit an analogue in the non-Archimedean case).

To see this, we say as in [**BFJ16a**, §5.1] that a form $\omega \in \mathcal{Z}_+$ is *$\mathcal{X}$-positive* for a given model/test configuration $\mathcal{X}$ if it is represented by a (relatively) ample class in $\mathrm{N}^1(\mathcal{X}/S)$. Note that all $\mathcal{X}$-positive forms are commensurable. As observed in [**BFJ16a**, Proposition 5.2] (see also [**GM16**, Proposition 4.14], [**BoJ22**, Lemma 3.11]), any $\alpha \in \mathrm{H}_{\mathrm{BC}}(X)$ whose image in $\mathrm{N}^1(X)$ under (1.8) is ample can be represented by an $\mathcal{X}$-positive form ω for any sufficiently high model $\mathcal{X}$. Now $\mathcal{Z}_\omega$ contains all forms determined on $\mathcal{X}$, but does not contain any $\mathcal{X}'$-positive form for a model $\mathcal{X}'$ strictly dominating $\mathcal{X}$. The claim easily follows.

1.4. The energy pairing. Mimicking the properties of induced metrics on Deligne pairings (see for instance [**BE21**, Theorem 8.16]), and generalizing [**SD18**, §2.2] (for a compact Kähler manifold) and [**BoJ22**, §3.2] (for a projective Berkovich space over a trivially valued field), we introduce:

Proposition-Definition 1.16. *The* energy pairing

$$(\mathcal{Z}\times\mathcal{D})^{n+1}\to\mathbb{R}\quad((\theta_0,\varphi_0),\dots,(\theta_n,\varphi_n))\mapsto(\theta_0,\varphi_0)\cdot\ldots\cdot(\theta_n,\varphi_n)$$

is defined as the unique $(n+1)$-linear symmetric map such that

$$(0,\varphi_0)\cdot(\theta_1,\varphi_1)\cdot\ldots\cdot(\theta_n,\varphi_n)=\int\varphi_0\,(\theta_1+\mathrm{dd^c}\varphi_1)\wedge\cdots\wedge(\theta_n+\mathrm{dd^c}\varphi_n)\tag{1.9}$$

and

$$(\theta_0,0)\cdot\ldots\cdot(\theta_n,0)=0.\tag{1.10}$$

It further satisfies

$$(\theta_0,\varphi_0+c_0)\cdot\ldots\cdot(\theta_n,\varphi_n+c_n)=(\theta_0,\varphi_0)\cdot\ldots\cdot(\theta_n,\varphi_n)+\sum_{i=0}^n c_i[\theta_0]\cdot\ldots\cdot\widehat{[\theta_i]}\cdot\ldots\cdot[\theta_n]\tag{1.11}$$

for all $c_i\in\mathbb{R}$, and

$$\begin{aligned}(\theta_0+\mathrm{dd^c}\tau_0,\varphi_0)\cdot\ldots\cdot(\theta_n+\mathrm{dd^c}\tau_n,\varphi_n)\\=(\theta_0,\tau_0+\varphi_0)\cdot\ldots\cdot(\theta_n,\tau_n+\varphi_n)-(\theta_0,\tau_0)\cdot\ldots\cdot(\theta_n,\tau_n).\end{aligned}\tag{1.12}$$

for all $\tau_i\in\mathcal{D}$.

Proof. Using multilinearity, symmetry and (1.9), (1.10), we necessarily have

$$(\theta_0,\varphi_0)\cdot\ldots\cdot(\theta_n,\varphi_n)=\sum_{i=0}^n\int\varphi_i\,\theta_0\wedge\cdots\wedge\theta_{i-1}\wedge(\theta_{i+1}+\mathrm{dd^c}\varphi_{i+1})\wedge\cdots\wedge(\theta_n+\mathrm{dd^c}\varphi_n).\tag{1.13}$$

This proves uniqueness. To show existence, the only nontrivial part is to check that the right-hand side of (1.13) is a symmetric function of the (θ_i,φ_i). It suffices to see invariance under transpositions, which is an easy consequence of the integration-by-parts formula (1.1) (compare [**SD18**, Proposition 2.3]).

To see (1.11), we may assume $c_i=0$ for $i>0$, and the result is then a direct consequence of (1.9).

Finally, pick $\tau_i\in\mathcal{D}$, and set

$$F(\varphi_0,\dots,\varphi_n):=(\theta_0,\tau_0+\varphi_0)\cdot\ldots\cdot(\theta_n,\tau_n+\varphi_n)-(\theta_0+\mathrm{dd^c}\tau_0,\varphi_0)\cdot\ldots\cdot(\theta_n+\mathrm{dd^c}\tau_n,\varphi_n).$$

By (1.9), we have

$$(0,\varphi_0)\cdot(\theta_1,\tau_1+\varphi_1)\cdot\ldots\cdot(\theta_n,\tau_n+\varphi_n)-(0,\varphi_0)\cdot(\theta_1+\mathrm{dd^c}\tau_1,\varphi_1)\cdot\ldots\cdot(\theta_n+\mathrm{dd^c}\tau_n,\varphi_n)=0.$$

This implies that $F(\varphi_0,\dots,\varphi_n)$ is independent of φ_0, and hence equal to $F(0,\varphi_1,\dots,\varphi_n)$. Applying the same argument successively to $\varphi_1,\dots,\varphi_n$, we end up with $F(\varphi_0,\dots,\varphi_0)=F(0,\dots,0)=(\theta_0,\tau_0)\cdot\ldots\cdot(\theta_n,\tau_n)$, which proves (1.12). □

By (1.9), the seminegativity property (1.2) translates into

$$(0,\varphi)^2\cdot(\theta_1,\varphi_1)\cdot\ldots\cdot(\theta_{n-1},\varphi_{n-1})\le 0\tag{1.14}$$

for all $\varphi\in\mathcal{D}$ and $\varphi_i\in\mathcal{D}_{\theta_i}$.

As in [**BoJ22**, §3.2], we further note the following straightforward monotonicity properties:

PROPOSITION 1.17. *For all* $\theta_i, \theta'_i \in \mathcal{Z}$ *and* $\varphi_i, \varphi'_i \in \mathcal{D}_{\theta_i}$, *we have*

$$\varphi_i \leq \varphi'_i \text{ for all } i \Longrightarrow (\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n) \leq (\theta_0, \varphi'_0) \cdot \ldots \cdot (\theta_n, \varphi'_n); \tag{1.15}$$

$$\varphi_i \leq 0 \text{ and } 0 \leq \theta_i \leq \theta'_i \text{ for all } i \Longrightarrow (\theta'_0, \varphi_0) \cdot \ldots \cdot (\theta'_n, \varphi_n) \leq (\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n). \tag{1.16}$$

Combining (1.15) and (1.11), we infer the Lipschitz property

$$|(\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n) - (\theta_0, \varphi'_0) \cdot \ldots \cdot (\theta_n, \varphi'_n)| \leq \sum_{i=0}^{n} \sup |\varphi_i - \varphi'_i| [\theta_0] \cdot \ldots \cdot \widehat{[\theta_i]} \cdot \ldots \cdot [\theta_n] \tag{1.17}$$

for all $\varphi_i, \varphi'_i \in \mathcal{D}_{\theta_i}$.

Following [**BoJ22**, §7.2], we next establish a lower bound for the energy pairing, which will play a crucial role in §3.2 below.

THEOREM 1.18. *Assume* $\omega_0, \ldots, \omega_n \in \mathcal{Z}_+$ *are commensurable, and set* $\delta := \max_{i,j} \mathrm{d}_{\mathrm{T}}(\omega_i, \omega_j)$. *If* $0 \geq \varphi_i \in \mathcal{D}_{\omega_i}$ *for* $i = 1, \ldots, n$, *then*

$$0 \geq (\omega_0, \varphi_0) \cdot \ldots \cdot (\omega_n, \varphi_n) \gtrsim e^{O(\delta)} \min_i (\omega_i, \varphi_i)^{n+1}.$$

Here d_{T} denotes the Thompson metric (1.7), and the implicit constant in $O(\delta)$ only depends on n (see §1.1).

LEMMA 1.19. *For any* $\theta \in \mathcal{Z}$, $\varphi \mapsto (\theta, \varphi)^{n+1}$ *is concave on* $\mathcal{D}_\theta$.

PROOF. This is a formal consequence of (1.14), see Appendix A. □

LEMMA 1.20. *Pick* $\omega, \omega' \in \mathcal{Z}_+$ *and* $t \geq 1$ *such that* $\omega \leq \omega' \leq t\omega$. *For any* $\varphi \in \mathcal{D}_\omega \subset \mathcal{D}_{\omega'}$ *such that* $\varphi \leq 0$, *we then have*

$$0 \geq (\omega', \varphi)^{n+1} \geq t^n (\omega, \varphi)^{n+1}.$$

PROOF. By (1.16) we have

$$0 \geq (\omega', \varphi)^{n+1} \geq (t\omega, \varphi)^{n+1} = t^{n+1} (\omega, t^{-1}\varphi)^{n+1}.$$

Since $t^{-1} \in [0, 1]$, concavity of the energy (Lemma 1.19) yields $(\omega, t^{-1}\varphi)^{n+1} \geq t^{-1}(\omega, \varphi)^{n+1}$, and the result follows. □

LEMMA 1.21. *Pick* $\omega_0, \ldots, \omega_r \in \mathcal{Z}_+$ *and* $0 \geq \varphi_i \in \mathcal{D}_{\omega_i}$ *for* $i = 0, \ldots, r$. *Assume also given* $t \geq 1$ *such that* $\omega_i \leq t\omega_j$ *for all* i, j. *Then*

$$\left(\sum_i \omega_i, \sum_i \varphi_i \right)^{n+1} \geq C_{r,n} t^n \sum_i (\omega_i, \varphi_i)^{n+1} \tag{1.18}$$

with $C_{r,n} := (r(r+1))^n$.

PROOF. Set $\overline{\omega} := \frac{1}{r+1}\sum_0^r \omega_i$ and $\overline{\omega}_t := \frac{t}{r+t}\sum_0^r \omega_i$. Then $\overline{\omega} \le \overline{\omega}_t \le \frac{t(1+rt)}{t+r}\omega_i$ for $0 \le i \le r$. If we set $\overline{\varphi} := \frac{1}{r+1}\sum_i \varphi_i$, then

$$\begin{aligned}\left(\sum_i \omega_i, \sum_i \varphi_i\right)^{n+1} &= (r+1)^{n+1}(\overline{\omega}, \overline{\varphi})^{n+1} \\ &\ge (r+1)^{n+1}(\overline{\omega}_t, \overline{\varphi})^{n+1} \\ &\ge (r+1)^n \sum_i (\overline{\omega}_t, \varphi_i)^{n+1} \\ &\ge (r+1)^n \left(\frac{t(1+rt)}{t+r}\right)^n \sum_i (\omega_i, \varphi_i)^{n+1} \\ &\ge C_{r,n} t^n \sum_i (\omega_i, \varphi_i)^{n+1},\end{aligned}$$

where the first inequality holds by (1.16), the second one by Lemma 1.19, and the last two inequalities from Lemma 1.20 and the estimate $\frac{1+rt}{t+r} \le r$. □

PROOF OF THEOREM 1.18. Expanding out $(\omega_0 + \cdots + \omega_n, \varphi_0 + \cdots + \varphi_n)^{n+1}$ yields

$$(n+1)!(\omega_0, \varphi_0) \cdot \ldots \cdot (\omega_n, \varphi_n) \ge (\omega_0 + \cdots + \omega_n, \varphi_0 + \cdots + \varphi_n)^{n+1},$$

and we conclude by Lemma 1.21 with $r = n$. □

1.5. The Monge–Ampère operator and the submean value property. Pick $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$. We define its *volume* as

$$V_\omega := \int \omega^n = [\omega]^n,$$

which is positive by Lemma 1.9, and introduce the probability measure

$$\mu_\omega := V_\omega^{-1}\omega^n.$$

DEFINITION 1.22. *The* Monge–Ampère operator $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}$ *is defined by setting*

$$\mathrm{MA}_\omega(\varphi) := V_\omega^{-1}\omega_\varphi^n.$$

Equivalently,

$$\mathrm{MA}_\omega(\varphi) = \mu_{\omega_\varphi}. \tag{1.19}$$

As an illustration of the energy pairing formalism, we recover the following version of the classical *Chern–Levine–Nirenberg inequality.*

LEMMA 1.23. *For all* $\varphi, \psi, \tau \in \mathcal{D}_\omega$ *we have*

$$\left|\int \tau\left(\mathrm{MA}_\omega(\varphi) - \mathrm{MA}_\omega(\psi)\right)\right| \le 2n \sup|\varphi - \psi|.$$

PROOF. By (1.9) we have

$$V_\omega \int \tau\left(\mathrm{MA}_\omega(\varphi) - \mathrm{MA}_\omega(\psi)\right) = (0,\tau)\cdot(\omega,\varphi)^n - (0,\tau)\cdot(\omega,\psi)^n$$

$$= (\omega,\tau)\cdot(\omega,\varphi)^n - (\omega,0)\cdot(\omega,\varphi)^n - (\omega,\tau)\cdot(\omega,\psi)^n + (\omega,0)\cdot(\omega,\psi)^n.$$

Now (1.17) yields

$$|(\omega,\tau)\cdot(\omega,\varphi)^n-(\omega,\tau)\cdot(\omega,\psi)^n|\le n\sup|\varphi-\psi|V_\omega,$$
$$|(\omega,0)\cdot(\omega,\varphi)^n-(\omega,0)\cdot(\omega,\psi)^n|\le n\sup|\varphi-\psi|V_\omega,$$

and the result follows. □

We next consider the quantity

$$T_\omega:=\sup\left\{\sup\varphi-\int\varphi\,\mu_\omega\mid\varphi\in\mathcal{D}_\omega\right\}\in[0,+\infty]. \tag{1.20}$$

Thus $T_\omega<\infty$ iff there exists $C>0$ such that the submean value inequality

$$\sup\varphi\le\int\varphi\,\mu_\omega+C \tag{1.21}$$

holds for all $\varphi\in\mathcal{D}_\omega$.

PROPOSITION 1.24. *If $T_\omega<\infty$, then $T_{\omega'}<\infty$ for any $\omega'\in\mathcal{Z}_+$ with $[\omega']\in\operatorname{Pos}(X)$.*

DEFINITION 1.25. *We say that the* submean value property *holds if $T_\omega<\infty$ for some (hence any) $\omega\in\mathcal{Z}_+$ with $[\omega]\in\operatorname{Pos}(X)$.*

When X is a compact Kähler or projective Berkovich space, this property holds iff X is irreducible (see Theorem 1.27 below).

LEMMA 1.26. *Pick $\tau\in\mathcal{D}_\omega$, and $\omega'\in\mathcal{Z}_+$ commensurable to ω. Then*

$$T_{\omega_\tau}\le T_\omega+(2n+2)\sup|\tau| \tag{1.22}$$

and

$$T_{\omega'}\le e^{O(\delta)}T_\omega \tag{1.23}$$

with $\delta:=\mathrm{d_T}(\omega,\omega')$.

Recall that, in this paper, the implicit constant in O only depends on n (see §1.1).

PROOF. Pick $\varphi\in\mathcal{D}_{\omega_\tau}$. Then $\varphi+\tau\in\mathcal{D}_\omega$, and the Chern–Levine–Nirenberg inequality (see Lemma 1.23) yields

$$\varphi+\tau\le\int(\varphi+\tau)\operatorname{MA}_\omega(0)+T_\omega\le\int(\varphi+\tau)\operatorname{MA}_\omega(\tau)+2n\sup|\tau|+T_\omega.$$

This proves (1.22), in view of (1.19).

For each $t>0$ we have $\mathcal{D}_{t\omega}=t\mathcal{D}_\omega$, and hence $T_{t\omega}=tT_\omega$. Further, $\omega'\le\omega$ implies $\mathcal{D}_{\omega'}\subset\mathcal{D}_\omega$ and

$$V_{\omega'}\mu_{\omega'}=(\omega')^n\le\omega^n=V_\omega\mu_\omega,$$

and hence $V_{\omega'}T_{\omega'}\le V_\omega T_\omega$. These properties imply (1.23). □

PROOF OF PROPOSITION 1.24. The condition $T_\omega<\infty$ only depends on the commensurability class of ω, by (1.23), and it only depends on $[\omega]\in\operatorname{Pos}(X)$, by (1.22). This condition is thus independent of ω, since any two classes in $\operatorname{Pos}(X)$ admit commensurable representatives (see Proposition 1.15). □

THEOREM 1.27. *Assume X is either a compact Kähler space or a projective Berkovich space over a non-Archimedean field k. Then the submean value property holds iff X is irreducible.*

LEMMA 1.28. *Let $\mathfrak{a} \subset \mathcal{O}_X$ be a coherent ideal sheaf. Then we can find $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$, and a decreasing sequence $(\varphi_j)_j$ in $\mathcal{D}_\omega$ such that, for any given choice of local generators (f_ν) of $\mathfrak{a}$ near a point of X, $\varphi_j - \max\{\log \max_\nu |f_\nu|, -j\}$ is locally bounded uniformly with respect to j.*

In the non-Archimedean case, $\mathfrak{a}$ can be viewed as a coherent ideal sheaf on the underlying algebraic variety, by the GAGA principle.

PROOF. Assume first X is compact Kähler. Fix a Kähler form ω, and pick a finite open cover (U_α) of X such that $\mathfrak{a}$ is generated on each U_α by a finite subset $(f_{\alpha\nu})_\nu$ of $\mathcal{O}(U_\alpha)$. Introduce the psh function on U_α

$$\varphi_{\alpha j} := \tfrac{1}{2}\log\left(\sum_\nu |f_{\alpha\nu}|^2 + e^{-2j}\right) = \max\left\{\log\max_\nu |f_{\alpha\nu}|, -j\right\} + O(1),$$

and observe that, for all α, β, $|\varphi_{\alpha j} - \varphi_{\beta j}|$ is uniformly bounded with respect to j on each compact subset of $U_\alpha \cap U_\beta$. Now choose relatively compact open subsets $U''_\alpha \Subset U'_\alpha \Subset U_\alpha$ such that the U''_α still cover X. Since $\overline{U'_\alpha} \cap \overline{U'_\beta}$ is compact in $U_\alpha \cap U_\beta$, we have

$$C := \sup_{\alpha,\beta,j} \sup_{U'_\alpha \cap U'_\beta} |\varphi_{\alpha j} - \varphi_{\beta j}| < \infty.$$

Pick also cut-off functions $\chi_\alpha \in C_c^\infty(U'_\alpha)$ such that $\chi_\alpha \geq 0$ and $\chi_\alpha \equiv 1$ on U''_α. We claim that the smooth functions

$$\varphi_j := \log \sum_\alpha \chi_\alpha^2 e^{\varphi_{\alpha j}}$$

satisfy $\mathrm{dd^c}\varphi_j \geq -A\omega$ for a uniform constant $A > 0$, which will yield the desired conclusion. The result follows indeed from [**Dem92**, Lemma 3.5], whose proof we now briefly recall. A direct computation using $\mathrm{dd^c}\varphi_{\alpha j} \geq 0$ yields

$$\mathrm{dd^c}\varphi_j \geq -\frac{\sum_\alpha \theta_\alpha e^{\varphi_{\alpha j}}}{\sum_\alpha \chi_\alpha^2 e^{\varphi_{\alpha j}}}$$

with $\theta_\alpha := 2\mathrm{d}\chi_\alpha \wedge \mathrm{d^c}\chi_\alpha - 2\chi_\alpha \mathrm{dd^c}\chi_\alpha$. Since the latter are smooth $(1,1)$-forms, we have $\theta_\alpha \leq A\omega$ for some constant $A > 0$. Now each $x \in X$ in the support of some θ_α lies in $U'_\alpha \setminus U''_\alpha$, and also in U''_β for some β, and hence $\sum_\beta \chi_\beta^2 e^{\varphi_{\beta j}} \geq e^C e^{\varphi_{\alpha j}}$ at x. This shows

$$\frac{\sum_\alpha \theta_\alpha e^{\varphi_{\alpha j}}}{\sum_\alpha \chi_\alpha^2 e^{\varphi_{\alpha j}}} \leq e^C A\omega,$$

which yields the result.

Assume now X is a projective Berkovich space. Pick an ample line bundle L such that $L \otimes \mathfrak{a}$ is generated by global sections (s_ν), and choose a PL metric on L with curvature form $\omega \in \mathcal{Z}_+$. Then setting $\varphi_j := \max\{\log\max_\nu |s_\nu|, -j\}$ yields the result. □

PROOF OF THEOREM 1.27. Assume first that the submean value property holds, and pick an n-dimensional irreducible component Y of X. Applying Lemma 1.28 to the ideal sheaf of Y in X yields $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$ and a decreasing sequence (φ_j) in $\mathcal{D}_\omega$ such that $\sup_Y \varphi_j \to -\infty$ while (φ_j) is uniformly bounded on compact subsets of $X \setminus Y$. Since $\int_Y \omega^n = [\omega]^n \cdot [Y] > 0$, we get $\int \varphi_j \, \mu_\omega \to -\infty$, and hence $\sup \varphi_j \to -\infty$, by the submean value property. It follows that $X = Y$ is irreducible.

Conversely, assume X is irreducible. In the Kähler case, pick a Kähler form ω on X, and choose a resolution of singularities $\pi\colon X'\to X$ with X' a connected compact Kähler manifold, and pick a Kähler form ω' on X' such that $\omega'\ge\pi^\star\omega$. Since $\mu:=V_\omega^{-1}\pi^\star\omega^n=f\omega'^n$ satisfies $\mathrm{PSH}(X',\omega')\subset L^1(\mu)$, [GZ05, Proposition 2.7] yields $C>0$ such that $\sup_{X'}\psi\le\int\psi\,\mu+C$ for all $\psi\in\mathrm{PSH}(X',\omega')$. Applying this to $\psi=\pi^\star\varphi$ with $\varphi\in\mathcal{D}_\omega$ yields the submean value property.

In the non-Archimedean case, pick an ample line bundle L and a PL metric on L determined by an ample model/test configuration $(\mathcal{X},\mathcal{L})$ of (X,L), with curvature form $\omega\in\mathcal{Z}_+$. Each $\varphi\in\mathcal{D}_\omega$ satisfies $\sup\varphi=\max_\Gamma\varphi$, where $\Gamma\subset X$ is the (finite) set of Shilov points attached to $\mathcal{X}$ (see [GM16, Proposition 4.22] or [BE21, Lemma 6.3]). On the other hand, Γ is also the support of μ_ω, by definition of the measure ω^n in terms of intersection numbers. Now [BoJ18a, Theorem 2.21] yields $C>0$ such that $|\varphi(x)-\varphi(y)|\le C$ for all $x,y\in\Gamma$ and all $\varphi\in\mathcal{D}_\omega$, and we infer, as desired, $\sup\varphi\le\int\varphi\,\mu_\omega+C$ for all $\varphi\in\mathcal{D}_\omega$. □

1.6. Monge–Ampère energy and the Dirichlet functional. From now on we fix $\omega\in\mathcal{Z}_+$ with $[\omega]\in\mathrm{Pos}(X)$.

Definition 1.29. *The* Monge–Ampère energy $\mathrm{E}_\omega\colon\mathcal{D}\to\mathbb{R}$ *is defined by*

$$\mathrm{E}_\omega(\varphi):=\frac{(\omega,\varphi)^{n+1}}{(n+1)V_\omega}. \tag{1.24}$$

For all $\varphi,\psi\in\mathcal{D}$ we have

$$\mathrm{E}_\omega(\varphi)-\mathrm{E}_\omega(\psi)=\frac{1}{n+1}\sum_{j=0}^n V_\omega^{-1}\int(\varphi-\psi)\,\omega_\varphi^j\wedge\omega_\psi^{n-j}. \tag{1.25}$$

Indeed, by (1.9), this amounts to the basic identity

$$(\omega,\varphi)^{n+1}-(\omega,\psi)^{n+1}=(0,\varphi-\psi)\cdot\sum_{j=0}^n(\omega,\varphi)^j\cdot(\omega,\psi)^{n-j}.$$

Assume now $\varphi,\psi\in\mathcal{D}_\omega$. By (1.25), we then have

$$\varphi\le\psi\Longrightarrow\mathrm{E}_\omega(\varphi)\le\mathrm{E}_\omega(\psi). \tag{1.26}$$

Further, $\mathrm{E}_\omega((1-t)\varphi+t\psi)$ is a polynomial function of $t\in[0,1]$, with

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{E}_\omega((1-t)\varphi+t\psi)=\int(\psi-\varphi)\,\mathrm{MA}_\omega(\varphi). \tag{1.27}$$

This characterizes E_ω as the unique primitive of the Monge–Ampère operator that vanishes at $0\in\mathcal{D}_\omega$. By Lemma 1.19, E_ω is concave on $\mathcal{D}_\omega$, which translates into

$$\mathrm{E}_\omega(\psi)\le\mathrm{E}_\omega(\varphi)+\int(\psi-\varphi)\,\mathrm{MA}_\omega(\varphi) \tag{1.28}$$

for all $\varphi,\psi\in\mathcal{D}_\omega$.

As a direct consequence of (1.12), we have

$$\mathrm{E}_{\omega_\tau}(\varphi)=\mathrm{E}_\omega(\varphi+\tau)-\mathrm{E}_\omega(\tau). \tag{1.29}$$

for any $\varphi,\tau\in\mathcal{D}$.

Definition 1.30. *We define the* Dirichlet functional $\mathrm{J}_\omega\colon\mathcal{D}_\omega\times\mathcal{D}_\omega\to\mathbb{R}_{\ge0}$ *by setting*

$$\mathrm{J}_\omega(\varphi,\psi):=\mathrm{E}_\omega(\varphi)-\mathrm{E}_\omega(\psi)+\int(\psi-\varphi)\,\mathrm{MA}_\omega(\varphi).$$

By concavity of E_ω, $\mathrm{J}_\omega(\varphi,\psi)$ is a convex function of $\psi\in\mathcal{D}_\omega$. Note also that

$$\mathrm{J}_\omega(\varphi,\psi)+\mathrm{J}_\omega(\psi,\varphi)=\int(\varphi-\psi)(\mathrm{MA}_\omega(\psi)-\mathrm{MA}_\omega(\varphi)). \tag{1.30}$$

By (A.4), we have the formula

$$\mathrm{J}_\omega(\varphi,\psi)=V_\omega^{-1}\sum_{j=0}^{n-1}\frac{j+1}{n+1}\int(\varphi-\psi)\,\mathrm{dd^c}(\psi-\varphi)\wedge\omega_\varphi^j\wedge\omega_\psi^{n-1-j}. \tag{1.31}$$

We simply write

$$\mathrm{J}_\omega(\varphi):=\mathrm{J}_\omega(0,\varphi)=\int\varphi\,\mu_\omega-\mathrm{E}_\omega(\varphi). \tag{1.32}$$

EXAMPLE 1.31. *When $n=1$, we have*

$$\mathrm{J}_\omega(\varphi,\psi)=\mathrm{J}_\omega(\psi,\varphi)=\tfrac{1}{2}\int(\varphi-\psi)\mathrm{dd^c}(\psi-\varphi),$$

which recovers the usual expression for the Dirichlet functional on a Riemann surface.

For each $R>0$ we set

$$\mathcal{D}_{\omega,R}:=\{\varphi\in\mathcal{D}_\omega\mid \mathrm{J}_\omega(\varphi)\le R\}. \tag{1.33}$$

LEMMA 1.32. *For each $R>0$, $\mathcal{D}_{\omega,R}$ is a convex subset of $\mathcal{D}$ that generates it.*

PROOF. By convexity of $\mathrm{J}_\omega=\mathrm{J}_\omega(0,\cdot)$, $\mathcal{D}_{\omega,R}$ is convex. Since $\mathcal{D}_\omega$ spans $\mathcal{D}$ (see Corollary 1.13), to see that $\mathcal{D}_{\omega,R}$ spans it suffices to show that any $\varphi\in\mathcal{D}_\omega$ satisfies $\mathrm{J}_\omega(t\varphi)\le R$ for $0<t\ll 1$, which holds since $\mathrm{J}_\omega(t\varphi)$ is a polynomial function of t with $\mathrm{J}_\omega(0)=0$. □

We may now collect the fundamental properties of the Dirichlet functional in the next result.

THEOREM 1.33. *For all $\varphi,\varphi',\psi,\psi',\tau\in\mathcal{D}_\omega$ and $t\in[0,1]$, the following holds:*

- quasi-symmetry:

$$\mathrm{J}_\omega(\varphi,\psi)\approx\mathrm{J}_\omega(\psi,\varphi); \tag{1.34}$$

- quasi-triangle inequality:

$$\mathrm{J}_\omega(\varphi,\psi)\lesssim\mathrm{J}_\omega(\varphi,\tau)+\mathrm{J}_\omega(\tau,\psi); \tag{1.35}$$

- quadratic estimate:

$$\mathrm{J}_\omega(\varphi,(1-t)\varphi+t\psi)\lesssim t^2\mathrm{J}_\omega(\varphi,\psi); \tag{1.36}$$

- uniform concavity:

$$\mathrm{E}_\omega((1-t)\varphi+t\psi)-[(1-t)\,\mathrm{E}_\omega(\varphi)+t\,\mathrm{E}_\omega(\psi)]\gtrsim t(1-t)\mathrm{J}_\omega(\varphi,\psi).$$

For all $\varphi,\varphi',\psi,\psi'\in\mathcal{D}_{\omega,R}$, we further have the following Hölder estimates:

$$\left|\int(\varphi-\varphi')\,(\mathrm{MA}_\omega(\psi)-\mathrm{MA}_\omega(\psi'))\right|\lesssim\mathrm{J}_\omega(\varphi,\varphi')^\alpha\mathrm{J}_\omega(\psi,\psi')^{1/2}R^{1/2-\alpha}; \tag{1.37}$$

and

$$|\mathrm{J}_\omega(\varphi,\psi)-\mathrm{J}_\omega(\varphi',\psi')|\lesssim\max\{\mathrm{J}_\omega(\varphi,\varphi'),\mathrm{J}_\omega(\psi,\psi')\}^\alpha R^{1-\alpha}, \tag{1.38}$$

where $\alpha:=2^{-n}$.

PROOF. In view of (1.14), this is a direct consequence of Theorem A.3 applied to

- the vector space $V := \mathcal{Z} \times \mathcal{D}$ with projection $\pi \colon V \to \mathcal{Z}$ onto the first factor;
- the convex cone $P := \{(\theta, \varphi) \in V \mid \varphi \in \mathcal{D}_\theta\}$;
- the homogeneous polynomial $F \colon V \to \mathbb{R}$ defined by $F(\theta, \varphi) := -\frac{(\theta,\varphi)^{n+1}}{(n+1)V_\omega}$. □

As a simple consequence of (1.29), we finally note:

LEMMA 1.34. *For each $\tau \in \mathcal{D}_\omega$ and $\varphi, \psi \in \mathcal{D}_{\omega_\tau}$ we have $\varphi + \tau, \psi + \tau \in \mathcal{D}_\omega$ and*

$$\mathrm{J}_{\omega_\tau}(\varphi, \psi) = \mathrm{J}_\omega(\varphi + \tau, \psi + \tau). \tag{1.39}$$

REMARK 1.35. *The above formalism recovers that of* [**BBGZ13**, **BBEGZ19**, **BFJ15**, **BoJ22**] *in the Kähler and non-Archimedean settings. However, in contrast to those works, we do not explicitly introduce the functional $\mathrm{I}_\omega(\varphi, \psi)$, which corresponds to the right-hand side of* (1.30)*.*

2. Measures of finite energy

In what follows, we pick $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$. We define the space $\mathcal{M}^1_\omega$ of measures of finite energy with respect to ω, and show, assuming a certain orthogonality property, that it is complete with respect to a quasi-metric δ_ω induced by the Dirichlet functional.

2.1. The energy of a measure.

DEFINITION 2.1. *We define the* energy of $\mu \in \mathcal{M}$ relative to $\psi \in \mathcal{D}_\omega$ *as*

$$\mathrm{J}_\omega(\mu, \psi) := \sup_{\varphi \in \mathcal{D}_\omega} \left\{ \mathrm{E}_\omega(\varphi) - \mathrm{E}_\omega(\psi) + \int (\psi - \varphi)\mu \right\} \in [0, +\infty]. \tag{2.1}$$

The choice of notation is justified by (2.5) below. When $\psi = 0$, we simply write

$$\mathrm{J}_\omega(\mu) := \mathrm{J}_\omega(\mu, 0) = \sup_{\varphi \in \mathcal{D}_\omega} \left\{ \mathrm{E}_\omega(\varphi) - \int \varphi\, \mu \right\}, \tag{2.2}$$

and call it the *energy*[5] *of* μ (with respect to ω). Note that

$$\mathrm{J}_\omega(\mu) = \sup_{\varphi \in \mathcal{D}_\omega} \left\{ \int \varphi(\mu_\omega - \mu) - \mathrm{J}_\omega(\varphi) \right\}, \tag{2.3}$$

by (1.32).

PROPOSITION 2.2. *For each $\psi \in \mathcal{D}_\omega$, the functional $\mathrm{J}_\omega(\cdot, \psi) \colon \mathcal{M} \to [0, +\infty]$ is convex and weakly lsc, and satisfies, for all $\mu \in \mathcal{M}$ and $\varphi \in \mathcal{D}_\omega$,*

$$\mathrm{J}_\omega(\mu, \psi) = \mathrm{J}_\omega(\mu) + \int \psi\, \mu - \mathrm{E}_\omega(\psi); \tag{2.4}$$

$$\mathrm{J}_\omega(\mathrm{MA}_\omega(\varphi), \psi) = \mathrm{J}_\omega(\varphi, \psi); \tag{2.5}$$

$$\mathrm{J}_\omega(\mathrm{MA}_\omega(\varphi)) \approx \mathrm{J}_\omega(\varphi); \tag{2.6}$$

$$\mathrm{J}_\omega(\varphi, \psi) \lesssim \mathrm{J}_\omega(\mu, \varphi) + \mathrm{J}_\omega(\mu, \psi). \tag{2.7}$$

[5] This corresponds to $\mathrm{E}^\vee_\omega(\mu)$ in the notation of [**BBGZ13**, **BoJ22**], and to $\|\mu\|_\omega$ in that of [**BoJ23**].

PROOF. Convexity and lower semicontinuity are clear from (2.1), which also directly yields (2.4). By (1.28), any $\varphi \in \mathcal{D}_\omega$ further computes the supremum defining $\mathrm{J}_\omega(\mathrm{MA}_\omega(\varphi), \psi)$, which is thus equal to

$$\mathrm{E}_\omega(\varphi) - \mathrm{E}_\omega(\psi) + \int (\psi - \varphi)\,\mathrm{MA}_\omega(\varphi) = \mathrm{J}_\omega(\varphi, \psi).$$

This proves (2.5), which implies

$$\mathrm{J}_\omega(\mathrm{MA}_\omega(\varphi)) = \mathrm{J}_\omega(\mathrm{MA}_\omega(\varphi), 0) = \mathrm{J}_\omega(\varphi, 0) \approx \mathrm{J}_\omega(0, \varphi) = \mathrm{J}_\omega(\varphi),$$

see (1.34). Finally, pick $\mu \in \mathcal{M}$, and set $\tau = \frac{1}{2}(\varphi + \psi) \in \mathcal{D}_\omega$. By (2.1), we have

$$\mathrm{J}_\omega(\mu, \varphi) \ge \mathrm{E}_\omega(\tau) - \mathrm{E}_\omega(\varphi) + \int (\varphi - \tau)\,\mu, \quad \mathrm{J}_\omega(\mu, \psi) \ge \mathrm{E}_\omega(\tau) - \mathrm{E}_\omega(\psi) + \int (\psi - \tau)\,\mu,$$

and hence

$$\mathrm{J}_\omega(\mu, \varphi) + \mathrm{J}_\omega(\mu, \psi) \ge 2\,\mathrm{E}_\omega(\tau) - (\mathrm{E}_\omega(\varphi) + \mathrm{E}_\omega(\psi)).$$

On the other hand,

$$2\,\mathrm{E}_\omega(\tau) - (\mathrm{E}_\omega(\varphi) + \mathrm{E}_\omega(\psi)) \gtrsim \mathrm{J}_\omega(\varphi, \psi)$$

by uniform concavity of E_ω (see Theorem 1.33), and (2.7) follows. □

Generalizing Lemma 1.34, we note:

LEMMA 2.3. *For all $\tau \in \mathcal{D}_\omega$, $\psi \in \mathcal{D}_{\omega_\tau}$ and $\mu \in \mathcal{M}$ we have*

$$\mathrm{J}_{\omega_\tau}(\mu, \psi) = \mathrm{J}_\omega(\mu, \psi + \tau). \tag{2.8}$$

In particular,

$$\mathrm{J}_{\omega_\tau}(\mu) = \mathrm{J}_\omega(\mu) + \int \tau\,\mu - \mathrm{E}_\omega(\tau). \tag{2.9}$$

PROOF. By (1.3) and (1.29) we have $\varphi \in \mathcal{D}_{\omega_\tau} \Leftrightarrow \varphi + \tau \in \mathcal{D}_\omega$, and

$$\mathrm{E}_{\omega_\tau}(\varphi) - \mathrm{E}_{\omega_\tau}(\psi) + \int (\varphi - \psi)\mu = \mathrm{E}_\omega(\varphi + \tau) - \mathrm{E}_\omega(\psi + \tau) + \int ((\varphi + \tau) - (\psi + \tau))\mu.$$

Taking the sup over φ yields (2.8), and (2.9) follows, by (2.4). □

REMARK 2.4. *If we drop the assumption that $\omega \ge 0$, but still require $[\omega] \in \mathrm{Pos}(X)$, then $\mathrm{E}_\omega(\varphi)$ and $\mathrm{J}_\omega(\mu)$ can still be defined by (1.24) and (2.2), respectively. Then (1.29), and hence (2.9), remain valid for any $\tau \in \mathcal{D}$. This will only get used in the context of Theorem 4.8 below.*

2.2. Measures of finite energy.

DEFINITION 2.5. *The space of* measures of finite energy (with respect to ω) *is defined as*

$$\mathcal{M}^1_\omega := \{\mu \in \mathcal{M} \mid \mathrm{J}_\omega(\mu) < \infty\}.$$

It is endowed with the strong topology*, defined as the coarsest refinement of the weak topology in which $\mathrm{J}_\omega \colon \mathcal{M}^1_\omega \to \mathbb{R}_{\ge 0}$ becomes continuous.*

In other words, a net (μ_i) converges strongly to μ in $\mathcal{M}^1_\omega$ iff $\mu_i \to \mu$ weakly in $\mathcal{M}$ and $\mathrm{J}_\omega(\mu_i) \to \mathrm{J}_\omega(\mu)$. For any $R > 0$ we also set

$$\mathcal{M}^1_{\omega,R} := \{\mu \in \mathcal{M}^1_\omega \mid \mathrm{J}_\omega(\mu) \le R\}. \tag{2.10}$$

By Proposition 2.2, this set is convex and weakly compact. By (2.4),

$$\mathrm{J}_\omega(\cdot, \psi) \colon \mathcal{M}^1_\omega \to \mathbb{R}_{\ge 0}$$

is continuous in the strong topology for any $\psi \in \mathcal{D}_\omega$. By Lemma 2.3, this yields:

PROPOSITION 2.6. *The topological space $\mathcal{M}^1_\omega$ only depends on the positive class $[\omega] \in \mathrm{Pos}(X)$.*

One should be wary of the fact that, in the present generality, even a 'nice' probability measure of the form $\mu = \theta_1 \wedge \cdots \wedge \theta_n$ with $\theta_i \in \mathcal{Z}_+$ need not be of finite energy with respect to ω in general (see however Theorem 3.4 below):

EXAMPLE 2.7. *Let X be either a compact Kähler or projective Berkovich space. For each irreducible component Y of X and each $\mu \in \mathcal{M}^1_\omega$, we then have*

$$\mu(Y) = \frac{[\omega|_Y]^n}{[\omega]^n}. \tag{2.11}$$

Indeed, this is proved in [**BoJ22**, Corollary 9.13] *in the trivially valued case, and the proof can be adapted to the general case. Now* (2.11) *fails in general for $\mu = \mu_{\omega'}$ with $\omega' \in \mathcal{Z}_+$ such that $[\omega] \neq [\omega'] \in \mathrm{Pos}(X)$, and hence $\mathcal{M}^1_{\omega'} \neq \mathcal{M}^1_\omega$.*

By (2.3), we have, for all $\varphi \in \mathcal{D}_\omega$ and $\mu \in \mathcal{M}$,

$$\int \varphi\,(\mu_\omega - \mu) \le \mathrm{J}_\omega(\varphi) + \mathrm{J}_\omega(\mu).$$

The following converse will come in handy.

LEMMA 2.8. *Assume that $\mu \in \mathcal{M}$ satisfies*

$$S := \sup_{\varphi \in \mathcal{D}_{\omega,R}} \int \varphi(\mu_\omega - \mu) < \infty$$

for some $R > 0$. Then μ has finite energy, and

$$\mathrm{J}_\omega(\mu) \lesssim S(1 + R^{-1}S). \tag{2.12}$$

PROOF. Pick $\varphi \in \mathcal{D}_\omega$ and set $J := \mathrm{J}_\omega(\varphi)$. By (1.36), we have $\mathrm{J}_\omega(t\varphi) \lesssim t^2 J$ for any $t \in [0,1]$, and we can thus choose

$$1 \le a \lesssim 1 + (R^{-1}J)^{1/2}$$

such that $\mathrm{J}_\omega(a^{-1}\varphi) \le R$. By assumption, we then have $\int a^{-1}\varphi\,(\mu_\omega - \mu) \le S$, and hence

$$\begin{aligned} \mathrm{E}_\omega(\varphi) - \int \varphi\,\mu &= \int \varphi\,(\mu_\omega - \mu) - \mathrm{J}_\omega(\varphi) \le aS - J \\ &\lesssim S + SR^{-1/2}J^{1/2} - J \le S + \tfrac{1}{4}S^2R^{-1}, \end{aligned}$$

where the last inequality follows from the elementary estimate $\sup_{y\ge 0}(xy^{1/2} - y) = x^2/4$ for any $x \ge 0$. Taking the supremum over φ yields (2.12). □

2.3. Legendre transform of the energy. Here we compute the Legendre transform of the convex functional $\mathrm{J}_\omega = \mathrm{J}_\omega(\cdot, 0)\colon \mathcal{M} \to [0, +\infty]$.

DEFINITION 2.9. *For any $f \in \mathrm{C}^0(X)$ we set*

$$\widetilde{\mathrm{E}}_\omega(f) := \sup_{f \ge \varphi \in \mathcal{D}_\omega} \mathrm{E}_\omega(\varphi).$$

By monotonicity of E_ω on $\mathcal{D}_\omega$ (see (1.26)), the functional $\widetilde{\mathrm{E}}_\omega\colon \mathrm{C}^0(X) \to \mathbb{R}$ so defined restricts to E_ω on $\mathcal{D}_\omega$. Like the latter, $\widetilde{\mathrm{E}}_\omega$ is further concave, monotone increasing, and equivariant with respect to translation, i.e.

$$\widetilde{\mathrm{E}}_\omega(f + c) = \widetilde{\mathrm{E}}_\omega(f) + c \text{ for } c \in \mathbb{R}.$$

Proposition 2.10. *For all* $f \in \mathrm{C}^0(X)$ *and* $\mu \in \mathcal{M}$ *we have*

$$\widetilde{\mathrm{E}}_\omega(f) = \inf_{\nu \in \mathcal{M}} \left\{ \mathrm{J}_\omega(\nu) + \int f\,\nu \right\}; \quad \mathrm{J}_\omega(\mu) = \sup_{g \in \mathrm{C}^0(X)} \left\{ \widetilde{\mathrm{E}}_\omega(g) - \int g\,\mu \right\}. \tag{2.13}$$

Proof. Define the (convex) Legendre transform $\widetilde{\mathrm{E}}_\omega^\vee\colon \mathrm{C}^0(X)^\vee \to \mathbb{R} \cup \{+\infty\}$ as the right-hand side of (2.13), i.e.

$$\widetilde{\mathrm{E}}_\omega^\vee(\mu) := \sup_{g \in \mathrm{C}^0(X)} \left\{ \widetilde{\mathrm{E}}_\omega(g) - \int g\,\mu \right\}.$$

Since $\widetilde{\mathrm{E}}_\omega$ is increasing and equivariant, it is straightforward to see that $\widetilde{\mathrm{E}}_\omega^\vee(\mu) < \infty$ implies $\mu \geq 0$ and $\int \mu = 1$, i.e. $\mu \in \mathcal{M}$ (compare [**BoJ22**, Proposition 9.8]). By Legendre duality, the result is thus equivalent to $\widetilde{\mathrm{E}}_\omega^\vee(\mu) = \mathrm{J}_\omega(\mu)$ for $\mu \in \mathcal{M}$. Since $\widetilde{\mathrm{E}}_\omega$ restricts to E_ω on $\mathcal{D}_\omega$, we trivially have $\widetilde{\mathrm{E}}_\omega^\vee(\mu) \geq \mathrm{J}_\omega(\mu)$. Conversely, pick $f \in \mathrm{C}^0(X)$ and $\varphi \in \mathcal{D}_\omega$ with $\varphi \leq f$. Then

$$\mathrm{J}_\omega(\mu) \geq \mathrm{E}_\omega(\varphi) - \int \varphi\,\mu \geq \mathrm{E}_\omega(\varphi) - \int f\,\mu,$$

where the first and second inequality respectively follow from (2.2) and $\varphi \leq f$. Taking the supremum over φ and then over f yields $\mathrm{J}_\omega(\mu) \geq \widetilde{\mathrm{E}}_\omega(f) - \int f\,\mu$ and $\mathrm{J}_\omega(\mu) \geq \widetilde{\mathrm{E}}_\omega^\vee(\mu)$. □

2.4. Orthogonality and differentiability.

Definition 2.11. *We say that* $\mathcal{D}_\omega$ admits maxima *if, for all* $\varphi, \psi \in \mathcal{D}_\omega$ *and* $f \in \mathcal{D}$ *such that* $\max\{\varphi, \psi\} < f$ *pointwise on* X, *there exists* $\tau \in \mathcal{D}_\omega$ *with* $\max\{\varphi, \psi\} \leq \tau < f$.

This equivalently means that, for any $f \in \mathcal{D}$, the poset

$$\mathcal{D}_{\omega,<f} := \{\varphi \in \mathcal{D}_\omega \mid \varphi < f\}$$

is inductive. We can then consider limits of nets indexed by $\mathcal{D}_{\omega,<f}$. For instance, note that

$$\widetilde{\mathrm{E}}_\omega(f) = \lim_{\varphi \in \mathcal{D}_{\omega,<f}} \mathrm{E}_\omega(\varphi). \tag{2.14}$$

Example 2.12. *If* X *is a compact Kähler space, then* $\mathcal{D}_\omega$ *admits maxima: take* $\tau := \widetilde{\max}(\varphi, \psi)$ *for an appropriate regularized max function* $\widetilde{\max}$.

Example 2.13. *If* X *is a projective Berkovich space, then* $\mathcal{D}_\omega$ *also admits maxima, since* $\mathbb{Q}$*-PL functions in* $\mathcal{D}_\omega$ *are dense in* $\mathcal{D}_\omega$, *and stable under max.*

Remark 2.14. *While we will not pursue this direction here, one can also introduce as in* [**BFJ16a**, **BE21**, **BoJ22**] *the space* $\mathrm{PSH}(\omega)$ *of* ω*-psh functions* $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$, *defined as usc functions that can be obtained as pointwise limits of decreasing nets in* $\mathcal{D}_\omega$, *and such that* $\int \varphi\,\mu_\omega > -\infty$. *Then* $\mathcal{D}_\omega$ *admits maxima iff* $\mathrm{PSH}(\omega)$ *(or, equivalently, the subspace* $\mathrm{CPSH}(\omega) := \mathrm{PSH}(\omega) \cap \mathrm{C}^0(X)$ *of continuous* ω*-psh functions) is stable under max.*

Definition 2.15. *We say that* ω *has the* orthogonality property *if* $\mathcal{D}_\omega$ *admits maxima and*

$$\lim_{\varphi \in \mathcal{D}_{\omega,<f}} \int (f - \varphi)\,\mathrm{MA}_\omega(\varphi) = 0 \tag{2.15}$$

for all $f \in \mathcal{D}$.

Explicitly, this means for that for any $\varepsilon > 0$ there exists $\varphi_0 \in \mathcal{D}_\omega$ such that $\varphi_0 < f$ and $\int (f - \varphi) \operatorname{MA}_\omega(\varphi) \le \varepsilon$ for all $\varphi \in \mathcal{D}_\omega$ with $\varphi_0 \le \varphi < f$.

Remark 2.16. *The orthogonality property for ω only depends on $[\omega] \in \operatorname{Pos}(X)$. Indeed, for any $\tau \in \mathcal{D}_\omega$ and $f \in \mathcal{D}$ we have*

$$\varphi \in \mathcal{D}_{\omega_\tau, <f} \Longleftrightarrow \varphi + \tau \in \mathcal{D}_{\omega, <f+\tau} \tag{2.16}$$

This implies that $\mathcal{D}_\omega$ admits finite maxima iff $\mathcal{D}_{\omega_\tau}$ does, and similarly for the orthogonality property, using

$$\operatorname{MA}_{\omega_\tau}(\varphi) = \operatorname{MA}_\omega(\varphi + \tau) \quad \textit{and}$$
$$\int (f - \varphi) \operatorname{MA}_{\omega_\tau}(\varphi) = \int ((f + \tau) - (\varphi + \tau)) \operatorname{MA}_\omega(\varphi + \tau).$$

Example 2.17. *Assume X is a compact Kähler space. Conjecturally, the orthogonality property always holds. This is known when X is normal, or X is projective and $[\omega] \in \operatorname{Amp}(X)$, see Appendix B for a more detailed discussion.*

Recall from Corollary 1.13 that any test function $f \in \mathcal{D}$ can be written as

$$f = f^+ - f^-, \quad f^\pm \in \mathcal{D}_{C\omega} \tag{2.17}$$

for some $C = C(f) > 0$. In line with [**BFJ15**, §7], we show:

Proposition 2.18. *Assume $\mathcal{D}_\omega$ admits maxima. The following properties are then equivalent:*

(i) *ω has the orthogonality property;*
(ii) *for any $f \in \mathcal{D}$ written as (2.17) for a given $C > 0$, we have*

$$\left| \widetilde{\operatorname{E}}_\omega(\varphi + f) - \operatorname{E}_\omega(\varphi) - \int f \operatorname{MA}_\omega(\varphi) \right| \lesssim C \sup |f| \tag{2.18}$$

for all $\varphi \in \mathcal{D}_\omega$;

(iii) *in the setting of (ii), we have*

$$\left| \widetilde{\operatorname{E}}_\omega(\varphi + tf) - \operatorname{E}_\omega(\varphi) - t \int f \operatorname{MA}_\omega(\varphi) \right| \lesssim C t^2 \sup |f| \tag{2.19}$$

for all $\varphi \in \mathcal{D}_\omega$ and $t \in \mathbb{R}$.

Note that the uniform differentiability property (2.19) implies in particular

$$\left. \frac{d}{dt} \right|_{t=0} \widetilde{\operatorname{E}}_\omega(\varphi + tf) = \int f \operatorname{MA}_\omega(\varphi).$$

Proof. Assume (i). Write $f \in \mathcal{D}$ as in (2.17), and pick $\varphi \in \mathcal{D}_\omega$. For any $\psi \in \mathcal{D}_{\omega, <\varphi + \varepsilon f}$, (1.28) yields

$$\int (\psi - \varphi) \operatorname{MA}_\omega(\psi) \le \operatorname{E}_\omega(\psi) - \operatorname{E}_\omega(\varphi) \le \int (\psi - \varphi) \operatorname{MA}_\omega(\varphi) \le \int f \operatorname{MA}_\omega(\varphi).$$

By Lemma 1.23, we also have $\left| \int f (\operatorname{MA}_\omega(\varphi) - \operatorname{MA}_\omega(\psi)) \right| \le 2nC \sup |\varphi - \psi|$, and we infer

$$\left| \operatorname{E}_\omega(\psi) - \operatorname{E}_\omega(\varphi) - \int f \operatorname{MA}_\omega(\varphi) \right| \le \int ((\varphi + f) - \psi) \operatorname{MA}_\omega(\psi) + 2nC \sup |\varphi - \psi|. \tag{2.20}$$

Now $\lim_{\psi\in\mathcal{D}_{\omega,<\varphi+f}} \mathrm{E}_\omega(\psi) = \widetilde{\mathrm{E}}_\omega(\varphi+f)$ (see (2.14)), while orthogonality yields

$$\lim_{\psi\in\mathcal{D}_{\omega,<\varphi+f}} \int ((\varphi+f)-\psi)\,\mathrm{MA}_\omega(\psi) = 0.$$

Further, any $\psi\in\mathcal{D}_{\omega,<\varphi+f}$ large enough is greater than $\varphi-\sup|f|\in\mathcal{D}_{\omega,<\varphi+f}$, and hence satisfies $\sup|\varphi-\psi|\le\sup|f|$. As a result, (2.20) implies

$$\left|\widetilde{\mathrm{E}}_\omega(\varphi+f)-\mathrm{E}_\omega(\varphi)-\int f\,\mathrm{MA}_\omega(\varphi)\right|\le 2nC\sup|f|,$$

which shows (i)⇒(ii). Next, (ii)⇒(iii), since $f=f^+-f^-$ with $f\in\mathcal{D}_{C\omega}$ implies $tf=\mathrm{sgn}(t)(|t|f^+-|t|f^-)$ with $|t|f^\pm\in\mathcal{D}_{C|t|\omega}$.

Finally, assume (iii), and pick $f\in\mathcal{D}$. We need to show that

$$L:=\limsup_{\varphi\in\mathcal{D}_{\omega,<f}}\int (f-\varphi)\,\mathrm{MA}_\omega(\varphi)\ge 0$$

vanishes. Write f as in (2.17) for some $C>0$. Pick also $\varphi\in\mathcal{D}_{\omega,<f}$, and set $g:=f-\varphi\in\mathcal{D}$. For any $t\in[0,1]$ we have $\varphi+tg=(1-t)\varphi+tf\le f$, and hence $\widetilde{\mathrm{E}}_\omega(\varphi+tg)\le\widetilde{\mathrm{E}}_\omega(f)$. On the other hand, since $f-\varphi=f^+-(f^-+\varphi)$ with $f^+, f^-+\varphi\in\mathcal{D}_{(C+1)\omega}$, (iii) yields a constant $A>0$ only depending on φ such that

$$\begin{aligned}0&\le t\int (f-\varphi)\,\mathrm{MA}_\omega(\varphi)\\&\le\widetilde{\mathrm{E}}_\omega(\varphi+tg)-\mathrm{E}_\omega(\varphi)+t^2A\sup|g|\le\widetilde{\mathrm{E}}_\omega(f)-\mathrm{E}_\omega(\varphi)+t^2A\sup|g|.\end{aligned}$$

Since any $\varphi\in\mathcal{D}_{\omega,<f}$ large enough is greater then $f^+-\sup f^-$, it satisfies

$$0\le g=f-\varphi\le\sup f^- - f^-\le B$$

with B only depending on f. We infer $0\le tL\le t^2AB$. Dividing by $t>0$ and letting $t\to 0_+$ yields, as desired, $L=0$. This proves (iii)⇒(i). □

EXAMPLE 2.19. *Assume X is a projective Berkovich space over a non-Archimedean field. Then* [**BE21**, Theorem A] *combined with the uniform differentiability estimate of* [**BGM22**, Lemma 3.2] *shows that* (2.18) *is satisfied (compare* [**BoJ22**, Lemma 8.7]*). By Proposition* 2.18*, it follows that the orthogonality property always holds in this setting.*

2.5. Maximizing sequences.

DEFINITION 2.20. *We say that a sequence (ψ_i) in $\mathcal{D}_\omega$ is* maximizing for $\mu\in\mathcal{M}^1_\omega$ *if it computes the energy of μ* (2.2)*, i.e. $\mathrm{E}_\omega(\psi_i)-\int\psi_i\,\mu\to\mathrm{J}_\omega(\mu)$.*

Equivalently, (ψ_i) is maximizing for μ iff $\mathrm{J}_\omega(\mu,\psi_i)\to 0$, see (2.4).

EXAMPLE 2.21. *For any $\varphi\in\mathcal{D}_\omega$ the constant sequence $\psi_i=\varphi$ is maximizing for $\mu=\mathrm{MA}_\omega(\varphi)$ (see* (2.5)*).*

As a key consequence of Proposition 2.18, we show:

THEOREM 2.22. *Assume ω has the orthogonality property. Pick $\mu\in\mathcal{M}^1_\omega$ and a maximizing sequence $\psi_i\in\mathcal{D}_\omega$. Then the measures $\mu_i:=\mathrm{MA}_\omega(\psi_i)$ converge strongly to μ in $\mathcal{M}^1_\omega$, i.e. $\mu_i\to\mu$ weakly and $\mathrm{J}_\omega(\mu_i)\to\mathrm{J}_\omega(\mu)$. In particular, the image of the Monge–Ampère operator*

$$\mathrm{MA}_\omega\colon\mathcal{D}_\omega\to\mathcal{M}^1_\omega$$

is dense in the strong topology.

Proof. Pick $f \in \mathcal{D}$, and choose $C > 0$ such that $f = f^+ - f^-$ with $f^\pm \in \mathcal{D}_{C\omega}$. Since we assume orthogonality, Proposition 2.18 yields $A > 0$ such that

$$\left|\widetilde{\mathrm{E}}_\omega(\psi_i + tf) - \mathrm{E}_\omega(\psi_i) - t\int f\,\mu_i\right| \le At^2$$

for all i and $t > 0$. By Proposition 2.10 and (2.4), we have, on the other hand,

$$\widetilde{\mathrm{E}}_\omega(\psi_i + tf) \le \mathrm{J}_\omega(\mu) + \int(\psi_i + tf)\mu = \mathrm{J}_\omega(\mu, \psi_i) + \mathrm{E}_\omega(\psi_i) + t\int f\,\mu.$$

Combining these estimates, we get

$$t\int f\,\mu_i \le t\int f\,\mu + \mathrm{J}_\omega(\mu, \psi_i) + At^2.$$

Since $\mathrm{J}_\omega(\mu, \psi_i) \to 0$, we infer

$$t\limsup_i \int f\,\mu_i \le t\int f\,\mu + At^2.$$

Dividing by t and letting $t \to 0_+$ yields $\limsup_i \int f\,\mu_i \le \int f\,\mu$. Replacing f with $-f$, we get $\lim_i \int f\,\mu_i = \int f\,\mu$. By density of $\mathcal{D}$ in $\mathrm{C}^0(X)$, this shows $\mu_i \to \mu$ weakly.

For each i we have $\mathrm{J}_\omega(\mu_i) = \mathrm{E}_\omega(\psi_i) - \int \psi_i\,\mu_i$ (see (2.5)), and $\mathrm{E}_\omega(\psi_i) - \int \psi_i\,\mu \to \mathrm{J}_\omega(\mu)$, since (ψ_i) is maximizing for μ. It only remains to prove $\int \psi_i\,(\mu_i - \mu) \to 0$. Since $\mathrm{J}_\omega(\psi_i)$ is bounded (see (2.7)), (1.37) yields $C > 0$ such that

$$\left|\int \psi_i\,(\mu_i - \mu_j)\right| \lesssim C\mathrm{J}_\omega(\psi_i, \psi_j)^\alpha$$

for all i, j, and hence

$$\left|\int \psi_i\,(\mu_i - \mu_j)\right| \lesssim C\max\{\mathrm{J}_\omega(\mu, \psi_i), \mathrm{J}_\omega(\mu, \psi_j)\}^\alpha,$$

by (2.7). Since $\mu_j \to \mu$ weakly and $\mathrm{J}_\omega(\mu, \psi_j) \to 0$ as $j \to \infty$, we infer

$$\left|\int \psi_i\,(\mu_i - \mu)\right| \lesssim C\mathrm{J}_\omega(\mu, \psi_i)^\alpha,$$

and we conclude, as desired, that the left-hand side tends to 0 as $i \to \infty$. □

2.6. The Dirichlet quasi-metric. From now on, **we assume that the orthogonality property holds** for ω. Recall from Examples 2.17 and 2.19, that this is the case if X is a normal compact Kähler space, or X is any projective Berkovich space.

Theorem 2.23. *There exists a unique continuous functional*

$$\delta_\omega \colon \mathcal{M}^1_\omega \times \mathcal{M}^1_\omega \to \mathbb{R}_{\ge 0},$$

such that

$$\delta_\omega(\mathrm{MA}_\omega(\varphi), \mathrm{MA}_\omega(\psi)) = \mathrm{J}_\omega(\varphi, \psi) \tag{2.21}$$

for all $\varphi, \psi \in \mathcal{D}_\omega$. Furthermore:

(i) *for all $\mu \in \mathcal{M}^1_\omega$ and $\psi \in \mathcal{D}_\omega$ we have*

$$\delta_\omega(\mu, \mathrm{MA}_\omega(\psi)) = \mathrm{J}_\omega(\mu, \psi), \quad \delta_\omega(\mu, \mu_\omega) = \mathrm{J}_\omega(\mu); \tag{2.22}$$

(ii) *δ_ω is a quasi-metric: for all $\mu, \nu, \rho \in \mathcal{M}^1_\omega$ we have*

$$\delta_\omega(\mu,\nu) = 0 \Leftrightarrow \mu = \nu, \quad \delta_\omega(\mu,\nu) \approx \delta_\omega(\nu,\mu), \quad \delta_\omega(\mu,\nu) \lesssim \delta_\omega(\mu,\rho) + \delta_\omega(\rho,\nu); \tag{2.23}$$

(iii) *the quasi-metric δ_ω satisfies the Hölder continuity property*

$$|\delta_\omega(\mu,\nu) - \delta_\omega(\mu',\nu')| \lesssim \max\{\delta_\omega(\mu,\mu'), \delta_\omega(\nu,\nu')\}^\alpha R^{1-\alpha} \tag{2.24}$$

for all $R > 0$ and $\mu, \mu', \nu, \nu' \in \mathcal{M}^1_{\omega,R}$, with $\alpha := 2^{-n}$;

(iv) *for all $R > 0$ and $\varphi, \psi \in \mathcal{D}_{\omega,R}$, $\mu, \nu \in \mathcal{M}^1_{\omega,R}$, we have the Hölder estimate*

$$\left| \int (\varphi - \psi)(\mu - \nu) \right| \lesssim \mathrm{J}_\omega(\varphi,\psi)^\alpha \delta_\omega(\mu,\nu)^{1/2} R^{1/2-\alpha}. \tag{2.25}$$

We call δ_ω the *Dirichlet quasi-metric* on $\mathcal{M}^1_\omega$.

Lemma 2.24. *For all $\mu \in \mathcal{M}^1_{\omega,R}$ and $\varphi, \psi, \tau \in \mathcal{D}_{\omega,R}$, we have*

$$|\mathrm{J}_\omega(\mu,\varphi) - \mathrm{J}_\omega(\mathrm{MA}_\omega(\tau),\psi)| \lesssim \max\{\mathrm{J}_\omega(\mu,\tau), \mathrm{J}_\omega(\varphi,\psi)\}^\alpha R^{1-\alpha}; \tag{2.26}$$

$$\left| \int (\varphi - \psi)(\mu - \mathrm{MA}_\omega(\tau)) \right| \lesssim \mathrm{J}_\omega(\varphi,\psi)^\alpha \mathrm{J}_\omega(\mu,\tau)^{1/2} R^{1/2-\alpha}. \tag{2.27}$$

Proof. When μ lies in the image of $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}^1_\omega$, this is equivalent to (1.38) and (1.37), in view of (2.5) and (2.6). By Theorem 2.22 the image of MA_ω is dense in $\mathcal{M}^1_\omega$, and the general case thus follows by continuity in the strong topology of all functions of μ involved. □

Proof of Theorem 2.23. Uniqueness is clear, since (2.21) determines δ_ω on the image of $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}^1_\omega$, which is dense in the strong topology, by Theorem 2.22. To show existence, pick $\mu, \nu \in \mathcal{M}^1_\omega$, and choose a maximizing sequences (ψ_i) for ν. We can then find $R > 0$ such that $\mu \in \mathcal{M}^1_{\omega,R}$ and $\psi_i \in \mathcal{D}_{\omega,R}$ for all i, and (2.26) and (2.7) yield

$$\begin{aligned} |\mathrm{J}_\omega(\mu,\psi_i) - \mathrm{J}_\omega(\mu,\psi_j)| &\lesssim \mathrm{J}_\omega(\psi_i,\psi_j)^\alpha R^{1-\alpha} \\ &\lesssim \max\{\mathrm{J}_\omega(\nu,\psi_i), \mathrm{J}_\omega(\nu,\psi_j)\}^\alpha R^{1-\alpha}. \end{aligned}$$

This estimate implies that $(\mathrm{J}_\omega(\mu,\psi_i))$ is a Cauchy sequence, which thus admits a limit

$$\delta_\omega(\mu,\nu) := \lim_i \mathrm{J}_\omega(\mu,\psi_i). \tag{2.28}$$

The same estimate also shows that the limit is independent of the choice of maximizing sequence (ψ_i), and that the convergence in (2.28) is uniform with respect to $\mu \in \mathcal{M}^1_{\omega,R}$. As a consequence, $\mu \mapsto \delta_\omega(\mu,\nu)$ so defined is continuous on $\mathcal{M}^1_\omega$ for each $\nu \in \mathcal{M}^1_\omega$.

By construction, (2.22) holds, and hence also (2.21), by (2.5). This proves (i).

Next, (2.24) holds when ν, ν' lie in the image of $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}^1_\omega$, by applying (2.26) to a maximizing sequence for μ', and the general case follows by using maximizing sequences for ν, ν'. This shows (iii), which also yields the continuity of δ_ω on $\mathcal{M}^1_\omega \times \mathcal{M}^1_\omega$ (and hence concludes the proof of existence), since $\mu_i \to \mu$ strongly implies $\delta_\omega(\mu_i,\mu) \to \delta_\omega(\mu,\mu) = 0$, by continuity of $\delta_\omega(\cdot,\mu)$.

Similarly, (iv) follows follows by applying (2.27) to a maximizing sequence (τ_i) for ν.

Finally, the first point in (ii) follows from (2.25), since $\mathcal{D}_\omega$ spans the dense subspace $\mathcal{D}$ of $\mathrm{C}^0(X)$ (see Corollary 1.13). By (1.34) and (1.35), the last two properties in (ii) hold when the measures lie in the image of MA_ω, and hence in general, by continuity of δ_ω. □

We next show:

THEOREM 2.25. *The quasi-metric space* $(\mathcal{M}^1_\omega, \delta_\omega)$ *only depends on the class* $[\omega]$. *It is complete, and its topology coincides with the strong topology.*

LEMMA 2.26. *For any* $\nu \in \mathcal{M}^1$ *and* $R > 0$, $\delta_\omega(\cdot, \nu)$ *is weakly lsc on* $\mathcal{M}^1_{\omega,R}$.

PROOF. When $\nu = \mathrm{MA}_\omega(\psi)$ with $\psi \in \mathcal{D}_\omega$, (2.22) yields $\delta_\omega(\cdot, \nu) = \mathrm{J}_\omega(\cdot, \psi)$, which is weakly lsc on $\mathcal{M}^1_\omega$ (see (2.1)). In the general case, pick a maximizing sequence (ψ_i) for ν, and set $\nu_i := \mathrm{MA}_\omega(\psi_i)$. By (2.24), we have $\delta_\omega(\mu, \nu_i) \to \delta_\omega(\mu, \nu)$ uniformly for $\mu \in \mathcal{M}^1_{\omega,R}$, and the result follows. □

PROOF OF THEOREM 2.25. We already know that $\mathcal{M}^1_\omega$ only depends on $[\omega]$ (see Proposition 2.6). Pick $\tau \in \mathcal{D}_\omega$, $\mu, \nu \in \mathcal{M}^1_\omega = \mathcal{M}^1_{\omega_\tau}$, and choose maximizing sequences (φ_i), (ψ_i) in $\mathcal{D}_{\omega_\tau}$ for μ, ν, so that $\delta_{\omega_\tau}(\mu, \nu) = \lim_i \mathrm{J}_{\omega_\tau}(\varphi_i, \psi_i)$. By (2.8), $(\varphi_i + \tau)$ and $(\psi_i + \tau)$ are maximizing sequences in $\mathcal{D}_\omega$ for μ, ν, and hence $\delta_\omega(\mu, \nu) = \lim_i \mathrm{J}_\omega(\varphi_i, \psi_i)$. Now (1.39) yields $\mathrm{J}_{\omega_\tau}(\varphi_i, \psi_i) = \mathrm{J}_\omega(\varphi_i + \tau, \psi_i + \tau)$, which proves $\delta_{\omega_\tau}(\mu, \nu) = \delta_\omega(\mu, \nu)$. Thus δ_ω only depends on $[\omega]$.

We next show that the topology of $(\mathcal{M}^1_\omega, \delta_\omega)$ is the strong topology, i.e. a net (μ_i) converges strongly to $\mu \in \mathcal{M}^1_\omega$ iff $\delta_\omega(\mu_i, \mu) \to 0$. When the latter holds, (2.25) implies $\mu_i \to \mu$ weakly (since $\mathcal{D}_\omega$ spans the dense subspace $\mathcal{D}$ of $\mathrm{C}^0(X)$), while (2.24) yields $\mathrm{J}_\omega(\mu_i) = \mathrm{J}_\omega(\mu_i, 0) \to \mathrm{J}_\omega(\mu)$. Thus $\mu_i \to \mu$ strongly, and the converse holds by strong continuity of δ_ω.

Finally, consider a Cauchy net (μ_i) in $(\mathcal{M}^1, \delta_\omega)$. Then $\mathrm{J}_\omega(\mu_i) = \delta_\omega(\mu_i, \mu_\omega)$ is eventually bounded. By weak compactness of $\mathcal{M}$, we may assume, after passing to a subnet, that (μ_i) admits a weak limit $\mu \in \mathcal{M}$. Since J_ω is weakly lsc on $\mathcal{M}$, we get $\mathrm{J}_\omega(\mu) \le \liminf_i \mathrm{J}_\omega(\mu_i) < +\infty$, i.e. $\mu \in \mathcal{M}^1$. It remains to show $\delta_\omega(\mu_i, \mu) \to 0$. To see this, pick $\varepsilon > 0$ and i_0 such that $\delta_\omega(\mu_i, \mu_j) \le \varepsilon$ for all $i, j \ge i_0$. Since $\mathrm{J}_\omega(\mu_j)$ is bounded and $\mu_j \to \mu$ weakly, Lemma 2.26 yields $\delta_\omega(\mu_i, \mu) \le \liminf_j \delta_\omega(\mu_i, \mu_j) \le \varepsilon$, and we are done. □

To conclude this section, we show:

PROPOSITION 2.27. *For each* $\nu \in \mathcal{M}^1_\omega$, $\delta_\omega(\cdot, \nu)\colon \mathcal{M}^1_\omega \to \mathbb{R}_{\ge 0}$ *is strictly convex, and we further have the uniform convexity estimate*

$$(1-t)\delta_\omega(\mu_0, \nu) + t\delta_\omega(\mu_1, \nu) - \delta_\omega((1-t)\mu_0 + t\mu_1, \nu) \gtrsim t(1-t)\delta_\omega(\mu_0, \mu_1).$$

for all $\mu_0, \mu_1 \in \mathcal{M}^1_\omega$ *and* $t \in [0, 1]$.

PROOF. By density of the image of $\mathrm{MA}_\omega\colon \mathcal{D}_\omega \to \mathcal{M}^1_\omega$ and continuity of δ_ω, we may assume without loss $\nu = \mathrm{MA}_\omega(\psi)$ with $\psi \in \mathcal{D}_\omega$, and hence $\delta_\omega(\cdot, \nu) = \mathrm{J}_\omega(\cdot, \psi)$. Set

$$J_t := (1-t)\mathrm{J}_\omega(\mu_0, \psi) + t\mathrm{J}_\omega(\mu_1, \psi), \quad \mu_t := (1-t)\mu_0 + t\mu_1,$$

and pick $\varphi \in \mathcal{D}_\omega$. Applying (2.4) to μ_0 and μ_1 yields

$$J_t = \mathrm{E}_\omega(\varphi) - \mathrm{E}_\omega(\psi) + \int (\psi - \varphi)\mu_t + (1-t)\mathrm{J}_\omega(\mu_0, \varphi) + t\mathrm{J}_\omega(\mu_1, \varphi),$$

and hence

$$J_t - \mathrm{E}_\omega(\varphi) + \mathrm{E}_\omega(\psi) + \int (\varphi - \psi)\mu_t \geq t(1-t)(\mathrm{J}_\omega(\mu_0, \psi) + \mathrm{J}_\omega(\mu_1, \psi)),$$

using the elementary estimate $(1-t)a + tb \geq t(1-t)(a+b)$ for $a, b \geq 0$ (see for instance [**BoJ22**, Lemma 7.29]). By (2.23), this implies

$$J_t - \mathrm{E}_\omega(\varphi) + \mathrm{E}_\omega(\psi) + \int (\varphi - \psi)\mu_t \gtrsim t(1-t)\delta_\omega(\mu_0, \mu_1),$$

and taking the infimum over φ shows $J_t - \mathrm{J}_\omega(\mu_t, \psi) \gtrsim t(1-t)\, \mathrm{d}_\omega(\mu_0, \mu_1)$, which concludes the proof. □

2.7. An equivalent metric on $\mathcal{M}^1$. One can show that the quasi-metric space $(\mathcal{M}^1_\omega, \delta_\omega)$ is metrizable, by general theory. Here we introduce a concrete metric that defines the strong topology of $\mathcal{M}^1_\omega$. Recall from (1.33) that

$$\mathcal{D}_{\omega,1} = \{\varphi \in \mathcal{D}_\omega \mid \mathrm{J}_\omega(\varphi) \leq 1\}.$$

PROPOSITION 2.28. *Setting*

$$\mathrm{d}_\omega(\mu, \nu) := \sup_{\varphi \in \mathcal{D}_{\omega,1}} \left| \int \varphi(\mu - \nu) \right| \tag{2.29}$$

yields a complete metric on $\mathcal{M}^1_\omega$ that defines the strong topology. Furthermore:

(i) *the metric d_ω and the Dirichlet quasi-metric δ_ω share the same bounded sets;*

(ii) *they are Hölder equivalent on bounded sets; more precisely:*

$$\mathrm{d}_\omega(\mu, \nu) \lesssim \delta_\omega(\mu, \nu)^{1/2} R^{1/2} \quad \textit{and} \quad \delta_\omega(\mu, \nu) \lesssim \mathrm{d}_\omega(\mu, \nu) R^{1/2} \tag{2.30}$$

for all $\mu, \nu \in \mathcal{M}^1_{\omega,R}$ with $R \geq 1$;

(iii) *for all $\varphi \in \mathcal{D}_\omega$ and $\mu, \nu \in \mathcal{M}^1_\omega$ we have*

$$\left| \int \varphi\,(\mu - \nu) \right| \lesssim (\mathrm{J}_\omega(\varphi)^{1/2} + 1)\, \mathrm{d}_\omega(\mu, \mu). \tag{2.31}$$

REMARK 2.29. *In the compact Kähler and non-Archimedean cases, the metric d_ω just constructed is* a priori *unrelated to the usual Darvas-type metric d_1* [**Dar15**, **Reb22**].

PROOF OF PROPOSITION 2.28. Pick $\varphi \in \mathcal{D}_\omega$. Since $\mathrm{J}_\omega(a^{-1}\varphi) \lesssim a^{-2}\mathrm{J}_\omega(\varphi)$ for $a \geq 1$ (see (1.36)), we can choose $1 \leq a \lesssim \mathrm{J}_\omega(\varphi)^{1/2} + 1$ such that $\mathrm{J}_\omega(a^{-1}\varphi) \leq 1$. Then $\left|\int a^{-1}\varphi\,(\mu - \nu)\right| \leq \mathrm{d}_\omega(\mu, \nu)$, which proves (2.31).

The first part of (2.30) is a direct consequence of (2.25). It shows, in particular, that d_ω is finite valued. It is also clear that d_ω is symmetric, vanishes on the diagonal, and satisfies the triangle inequality. Since $\mathcal{D}_{\omega,1}$ spans the dense subspace $\mathcal{D}$ of $\mathrm{C}^0(X)$ (see Lemma 1.32), d_ω further separates points, and hence defines a metric on $\mathcal{M}^1_\omega$.

The first part of (2.30) also shows that $\mu_i \to \mu$ in $\mathcal{M}^1_\omega$ implies $\mathrm{d}_\omega(\mu_i, \mu) \to 0$, by continuity of δ_ω, and it follows that the metric d_ω is continuous. By density of the image of MA_ω, it is thus enough to show the second half of (2.30) when $\mu = \mathrm{MA}_\omega(\varphi)$ and $\nu = \mathrm{MA}_\omega(\psi)$ with $\varphi, \psi \in \mathcal{D}_\omega$. Then $\mathrm{J}_\omega(\varphi) \approx \mathrm{J}_\omega(\mu) \leq R$ and $\mathrm{J}_\omega(\psi) \approx \mathrm{J}_\omega(\nu) \leq R$ (see (2.6)), while

$$\delta_\omega(\mu, \nu) = \mathrm{J}_\omega(\varphi, \psi) \leq \int (\varphi - \psi)(\mu - \nu),$$

by (2.21) and (1.30). Using (2.31), we get the second half of (2.30). Next pick $\mu \in \mathcal{M}^1_\omega$ and set $R := \max\{1, \delta_\omega(\mu, \mu_\omega)\}$ and $S := \max\{1, \mathrm{d}_\omega(\mu, \mu_\omega)\}$. Applying (2.30) to $\nu = \mu_\omega$ yields $S \lesssim R$, and also $R \lesssim SR^{1/2}$, i.e. $R \lesssim S^2$. This proves (i) and (ii). Since δ_ω defines the strong topology of $\mathcal{M}^1_\omega$ and is complete (see Theorem 2.25), the same therefore holds, as desired, for d_ω. □

By Theorem 2.25, the quasi-metric space $(\mathcal{M}^1_\omega, \delta_\omega)$ only depends on $[\omega] \in \mathrm{Pos}(X)$. Here we show:

LEMMA 2.30. *For each $\tau \in \mathcal{D}_\omega$ we have $\mathrm{d}_{\omega_\tau} \lesssim (\mathrm{J}_\omega(\tau)^{1/2} + 1)\,\mathrm{d}_\omega$. In particular, the Lipschitz equivalence class of the metric space $(\mathcal{M}^1_\omega, \mathrm{d}_\omega)$ only depends on $[\omega]$.*

PROOF. For any $\varphi \in \mathcal{D}_{\omega_\tau}$, (1.39) yields $\mathrm{J}_{\omega_\tau}(\varphi) = \mathrm{J}_{\omega_\tau}(0, \varphi) = \mathrm{J}_\omega(\tau, \varphi + \tau)$. When $\mathrm{J}_{\omega_\tau}(\varphi) \le 1$, the quasi-triangle inequality (1.35) thus yields $\mathrm{J}_\omega(\varphi + \tau) = \mathrm{J}_\omega(0, \varphi + \tau) \lesssim 1 + J$ with $J := \mathrm{J}_\omega(\tau) = \mathrm{J}_\omega(0, \tau)$. By (2.31) we infer

$$\left| \int (\varphi + \tau)(\mu - \nu) \right| \lesssim (1 + J^{1/2})\,\mathrm{d}_\omega(\mu, \nu), \qquad \left| \int \tau(\mu - \nu) \right| \lesssim (1 + J^{1/2})\,\mathrm{d}_\omega(\mu, \nu).$$

Thus $\left|\int \varphi(\mu - \nu)\right| \lesssim (1 + J^{1/2})\,\mathrm{d}_\omega(\mu, \nu)$. Taking the supremum over $\varphi \in \mathcal{D}_{\omega_\tau}$ such that $\mathrm{J}_{\omega_\tau}(\varphi) \le 1$ yields the result. □

3. Lipschitz and Hölder estimates for the energy

In what follows we consider $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$. As above, we assume that the orthogonality property holds. From now on, we further assume the submean value property (see Definition 1.25), and use it to investigate the dependence of $\mathcal{M}^1_\omega$ on ω and establish a Hölder continuity estimate for the energy pairing.

Recall that the standing assumptions hold when X is a normal irreducible compact Kähler space, or any irreducible projective Berkovich space (see Theorem 1.27 and Examples 2.17 and 2.19).

3.1. Lipschitz estimates for the energy. Recall from §2.7 the metric d_ω, which defines the strong topology of $\mathcal{M}^1_\omega$. As a first key consequence of the submean value property, we show:

THEOREM 3.1. *The Lipschitz equivalence class of the metric space $(\mathcal{M}^1_\omega, \mathrm{d}_\omega)$ is independent of ω.*

In particular, the topological space $\mathcal{M}^1_\omega$ is independent of ω (see Proposition 2.28), and will henceforth simply by denoted by $\mathcal{M}^1$.

LEMMA 3.2. *Assume $\omega' \in \mathcal{Z}_+$ is commensurable to ω, with $[\omega'] \in \mathrm{Pos}(X)$. For all $\mu, \nu \in \mathcal{M}$ we then have*

$$\mathrm{d}_{\omega'}(\mu, \nu) \le e^{O(\delta)}(1 + T_\omega)^{1/2}\,\mathrm{d}_\omega(\mu, \nu) \tag{3.1}$$

with $\delta := \mathrm{d}_{\mathrm{T}}(\omega, \omega') \in \mathbb{R}_{\ge 0}$.

PROOF. Pick any $\varphi \in \mathcal{D}_{\omega'}$ such that $\mathrm{J}_{\omega'}(\varphi) = \int \varphi\,\mu_{\omega'} - \mathrm{E}_{\omega'}(\varphi) \le 1$. We need to show

$$\left| \int \varphi(\mu - \nu) \right| \le e^{O(\delta)}(1 + T_\omega)^{1/2}\,\mathrm{d}_\omega(\mu, \nu). \tag{3.2}$$

By translation invariance of J_ω on $\mathcal{D}_\omega$, we may assume without loss $\sup \varphi = 0$. Then

$$-\mathrm{E}_{\omega'}(\varphi) \le 1 + T_{\omega'} \le 1 + e^{O(\delta)} T_\omega,$$

by (1.23). On the other hand,

$$0 \geq e^{(n+1)\delta}(\omega, e^{-\delta}\varphi)^{n+1} = (e^{\delta}\omega, \varphi)^{n+1} \geq e^{O(\delta)}(\omega', \varphi)^{n+1},$$

where the last inequality follows from Lemma 1.20, since $\omega' \leq e^{\delta}\omega \leq e^{2\delta}\omega'$. Dividing by $(n+1)V_\omega = e^{O(\delta)}(n+1)V_{\omega'}$, we get

$$0 \leq \mathrm{J}_\omega(e^{-\delta}\varphi) \leq -\mathrm{E}_\omega(e^{-\delta}\varphi) \leq -e^{O(\delta)}\,\mathrm{E}_{\omega'}(\varphi) \leq e^{O(\delta)}(1+T_\omega).$$

By (2.31) this implies

$$\left|\int e^{-\delta}\varphi(\mu-\nu)\right| \leq e^{O(\delta)}(1+T_\omega)^{1/2}\,\mathrm{d}_\omega(\mu,\nu),$$

and hence (3.2). □

Proof of Theorem 3.1. We argue as in the proof of Proposition 1.24. On the one hand, the Lipschitz equivalence class only depends on the positive class $[\omega]$, by Lemma 2.30. On the other hand, it only depends on the commensurability class of ω, by Lemma 3.2, and we conclude since any two positive classes admit commensurable representatives, see Proposition 1.15. □

3.2. Mixed Monge–Ampère measures. It will be convenient to introduce, for $\varphi \in \mathcal{D}_\omega$ and $\mu \in \mathcal{M}$, the quantities

$$\mathrm{J}^+_\omega(\varphi) := \mathrm{J}_\omega(\varphi) + T_\omega, \quad \text{and} \quad \mathrm{J}^+_\omega(\mu) := \mathrm{J}_\omega(\mu) + T_\omega. \tag{3.3}$$

Lemma 3.3. *For each $\varphi \in \mathcal{D}_\omega$ we have*

$$0 \leq \sup\varphi - \mathrm{E}_\omega(\varphi) \leq \mathrm{J}^+_\omega(\varphi); \tag{3.4}$$

$$\mathrm{J}^+_\omega(\varphi) \approx \mathrm{J}^+_\omega(\mathrm{MA}_\omega(\varphi)). \tag{3.5}$$

Proof. By (1.32), we have

$$\sup\varphi - \mathrm{E}_\omega(\varphi) = \mathrm{J}_\omega(\varphi) + (\sup\varphi - \int \varphi\,\mu_\omega) \leq \mathrm{J}_\omega(\varphi) + T_\omega.$$

This yields (3.4), while (3.5) is a direct consequence of (2.6). □

We next establish a key energy estimate for mixed Monge–Ampère measures.

Theorem 3.4. *For $i = 1, \dots, n$, pick $\omega_i \in \mathcal{Z}_+$ with $[\omega_i] \in \mathrm{Pos}(X)$ and $\varphi_i \in \mathcal{D}_{\omega_i}$, and set*

$$\mu := ([\omega_1]\cdot \dots \cdot [\omega_n])^{-1}(\omega_1 + dd^c\varphi_1)\wedge \dots \wedge (\omega_n + dd^c\varphi_n) \in \mathcal{M}. \tag{3.6}$$

Then μ lies in $\mathcal{M}^1$, and satisfies:

(i) *if each ω_i is commensurable to ω, then*

$$\mathrm{J}^+_\omega(\mu) \lesssim e^{O(\delta)} \max_i \mathrm{J}^+_{\omega_i}(\varphi_i)$$

with $\delta := \max_i \mathrm{d_T}(\omega_i, \omega)$;

(ii) *in the general case,*

$$\mathrm{J}_\omega(\mu) \lesssim C(\max_i \mathrm{J}_{\omega_i}(\varphi_i) + 1),$$

where $C > 0$ only depends on ω and the ω_i.

PROOF. Assume first that each ω_i is commensurable to ω. Set

$$J := \max_i \mathrm{J}^+_{\omega_i}(\varphi_i), \quad V := V_\omega, \quad V_i := V_{\omega_i},$$

and observe that

$$e^{-n\delta} V \le V_i \le e^{n\delta} V, \quad e^{-n\delta} V \le [\omega_1] \cdot \ldots \cdot [\omega_n] \le e^{n\delta} V. \tag{3.7}$$

Since μ is unchanged when the φ_i's are translated by constants, we may assume without loss that $\sup \varphi_i = 0$. Then

$$0 \ge (\omega_i, \varphi_i)^{n+1} = (n+1) V_i \, \mathrm{E}_{\omega_i}(\varphi_i) \ge -(n+1) V_i \mathrm{J}^+_{\omega_i}(\varphi_i),$$

by (3.4), and hence

$$0 \ge (\omega_i, \varphi_i)^{n+1} \gtrsim -e^{O(\delta)} V J, \tag{3.8}$$

using (3.7). Now pick $\psi \in \mathcal{D}_\omega$ such that $\sup \psi = 0$, and set $R := \mathrm{J}_\omega(\psi)$. On the one hand,

$$0 \ge (\omega, \psi)^{n+1} = (n+1) V \, \mathrm{E}_\omega(\psi) \ge -(n+1) V (R + T_\omega), \tag{3.9}$$

using (3.4) again. On the other hand, (3.7) yields

$$\begin{aligned}
0 &\ge e^{-n\delta} V \int \psi \, \mu \\
&\ge \int \psi \, (\omega_1 + dd^c \varphi_1) \wedge \cdots \wedge (\omega_n + dd^c \varphi_n) \\
&= (\omega, \psi) \cdot (\omega_1, \varphi_1) \cdot \ldots \cdot (\omega_n, \varphi_n) - (\omega, 0) \cdot (\omega_1, \varphi_1) \cdot \ldots \cdot (\omega_n, \varphi_n) \\
&\ge (\omega, \psi) \cdot (\omega_1, \varphi_1) \cdot \ldots \cdot (\omega_n, \varphi_n) \\
&\gtrsim e^{O(\delta)} \min\{(\omega, \psi)^{n+1}, \min_i (\omega_i, \varphi_i)^{n+1}\},
\end{aligned}$$

by (1.16) and Theorem 1.18. Combined with (3.8) and (3.9), this yields

$$\int \psi (\mu_\omega - \mu) \le - \int \psi \, \mu \lesssim e^{O(\delta)} (R + J + T_\omega) \lesssim e^{O(\delta)} (R + J),$$

since $T_\omega \le e^{O(\delta)} T_{\omega_i} \le e^{O(\delta)} J$ by (1.23). By Lemma 2.8, we infer,

$$\mathrm{J}_\omega(\mu) \lesssim e^{O(\delta)} \inf_{R > 0} (R + J) \left(1 + R^{-1}(R + J)\right) \le e^{O(\delta)} J,$$

which concludes the proof of (i) (using (1.23)).

We now consider the general case. By Proposition 1.15, we can choose $\tau \in \mathcal{D}_\omega$ and $\tau_i \in \mathcal{D}_{\omega_i}$ such that $\omega' := \omega_\tau$ and $\omega'_i := \omega_{i,\tau_i}$ are commensurable for all i. Then

$$\mu = ([\omega'_1] \cdot \ldots \cdot [\omega'_n])^{-1} (\omega'_1 + dd^c \varphi'_1) \wedge \cdots \wedge (\omega'_n + dd^c \varphi_n)$$

with $\varphi'_i := \varphi_i - \tau_i$, and hence

$$\mathrm{J}_{\omega'}(\mu) \lesssim \max_i (\mathrm{J}_{\omega'_i}(\varphi'_i) + T_{\omega'_i}),$$

by Theorem 3.4. By (2.9), we have $\mathrm{J}_\omega(\mu) \le \mathrm{J}_{\omega'}(\mu) + C$ and $\mathrm{J}_{\omega'_i}(\varphi'_i) \le \mathrm{J}_\omega(\varphi_i) + C$, with $C > 0$ independent of μ, and (ii) follows. □

As a consequence, we get the following estimate for the energy:

COROLLARY 3.5. *Pick* $\omega' \in \mathcal{Z}_+$ *such that* $[\omega'] \in \mathrm{Pos}(X)$, *and* $\mu \in \mathcal{M}^1$.

(i) *If* ω' *is commensurable to* ω, *then*

$$\mathrm{J}^+_{\omega'}(\mu) \approx e^{O(\delta)} \mathrm{J}^+_\omega(\mu)$$

with $\delta := \mathrm{d_T}(\omega, \omega')$.

(ii) *In the general case, there exists* $C > 0$ *only depending on* ω, ω' *such that*

$$\mathrm{J}_{\omega'}(\mu) \le C(\mathrm{J}_\omega(\mu) + 1).$$

We refer to (4.21) below for a more precise estimate when δ is small.

PROOF. Assume first ω' commensurable. Pick a maximizing sequence (φ_j) for μ in $\mathcal{D}_\omega$, and set $\mu_j := \mathrm{MA}_\omega(\varphi_j)$. Then $\mu_j \to \mu$ strongly in $\mathcal{M}^1$ (see Theorem 2.22), and hence $\mathrm{J}_\omega(\mu_j) \to \mathrm{J}_\omega(\mu)$ and $\mathrm{J}_{\omega'}(\mu_j) \to \mathrm{J}_{\omega'}(\mu)$. For each j we have $\mathrm{J}_\omega(\varphi_j) \approx \mathrm{J}_\omega(\mu_j)$ (see (2.6)). Theorem 3.4 thus yields $\mathrm{J}^+_{\omega'}(\mu_j) \lesssim e^{O(\delta)} \mathrm{J}^+_\omega(\mu_j)$, and (i) follows.

In the general case, we can choose $\tau \in \mathcal{D}_\omega$ and $\tau' \in \mathcal{D}_{\omega'}$ such that ω_τ and $\omega_{\tau'}$ are commensurable (see Proposition 1.15). By (2.9), we then have $\mathrm{J}_{\omega'}(\mu) \le \mathrm{J}_{\omega_\tau}(\mu) + C$ and $\mathrm{J}_{\omega_{\tau'}}(\mu) \le \mathrm{J}_{\omega'}(\mu) + C$ with $C > 0$ independent of μ, and (ii) now follows from (i). □

3.3. Hölder continuity of the energy pairing. Recall from §1.3 that

$$\mathcal{Z}_\omega = \{\theta \in \mathcal{Z} \mid \|\theta\|_\omega < \infty\}.$$

Using the above results, we establish a general Hölder continuity property for the energy pairing.

THEOREM 3.6. *For* $i = 0, \dots, n$, *pick* $\theta_i, \theta'_i \in \mathcal{Z}_\omega$ *and* $\varphi_i, \varphi'_i \in \mathcal{D}_\omega$, *normalized by* $\int \varphi_i \, \mu_\omega = \int \varphi'_i \, \mu_\omega = 0$. *Then*

$$|(\theta_0, \varphi_0) \cdot \ldots \cdot (\theta_n, \varphi_n) - (\theta'_0, \varphi'_0) \cdot \ldots \cdot (\theta'_n, \varphi'_n)| \lesssim A \left(\max_i \|\theta_i - \theta'_i\|_\omega J + \max_i \mathrm{J}_\omega(\varphi_i, \varphi'_i)^\alpha J^{1-\alpha} \right).$$

with $\alpha := 2^{-n}$ *and*

$$A := V_\omega \prod_i (1 + \|\theta_i\|_\omega + \|\theta'_i\|_\omega), \quad J := \max_i \mathrm{J}^+_\omega(\varphi_i).$$

In particular,

$$|(\theta_1, \varphi_1) \cdot \ldots \cdot (\theta_n, \varphi_n)| \lesssim V_\omega \prod_i (1 + \|\theta_i\|_\omega) \max_i \mathrm{J}^+_\omega(\varphi_i). \tag{3.10}$$

PROOF. Assume first $\theta_i = \theta'_i$ for all i. By symmetry of the energy pairing, we may assume $\varphi_i = \varphi'_i$ for $i \ge 1$. For $i = 1, \dots, n$ set $t_i := 1 + \|\theta_i\|_\omega$. Then $\|t_i^{-1}\theta_i\|_\omega \le 1$, and

$$\mathrm{J}_\omega(t_i^{-1}\varphi_i) \le t_i^{-1} \mathrm{J}_\omega(\varphi_i) \le \mathrm{J}_\omega(\varphi_i),$$

by convexity of J_ω on $\mathcal{D}_\omega$. By homogeneity, we may thus assume $\|\theta_i\|_\omega \le 1$ for all $i = 1, \dots, n$. Thus $-\omega \le \theta_i \le \omega$, and hence $\theta_i = \theta_i^+ - \theta_i^-$ where $\theta_i^+ := \theta_i + 2\omega$ and $\theta_i^- := 2\omega$ both satisfy $\omega \le \theta_i^\pm \le 3\omega$. By multilinearity, we may finally assume $\omega \le \theta_i \le 3\omega$. By Proposition 1.17 (i), it then suffices to show

$$\left| \int (\varphi_0 - \varphi'_0)(\theta_1 + dd^c\varphi_1) \wedge \cdots \wedge (\theta_n + dd^c\varphi_n) \right| \le V_\omega \mathrm{J}_\omega(\varphi_0, \varphi'_0)^\alpha J^{1-\alpha}. \tag{3.11}$$

Since $\omega \le \theta_i \le 3\omega$, Theorem 3.4 shows that

$$\mu := ([\theta_0] \cdot \ldots \cdot [\theta_n])^{-1}(\theta_1 + \mathrm{dd^c}\varphi_1) \wedge \cdots \wedge (\theta_n + \mathrm{dd^c}\varphi_n) \in \mathcal{M}$$

satisfies $\mathrm{J}_\omega(\mu) \lesssim J$. By (2.25), we infer

$$\left| \int (\varphi_0 - \varphi_0)(\mu - \mu_\omega) \right| \lesssim \mathrm{J}_\omega(\varphi_0, \varphi_0)^\alpha J^{1-\alpha},$$

which yields (3.11) since $\int (\varphi_0 - \varphi_0)\mu_\omega = 0$ and $[\theta_0] \cdot \ldots \cdot [\theta_n] \lesssim V_\omega$.

In the general case, we may again assume $(\theta_i, \varphi_i) = (\theta_i', \varphi_i')$ for $i \ge 1$, by symmetry and multilinearity of the energy pairing. Then

$$(\theta_0, \varphi_0) \cdot (\theta_1, \varphi_1) \cdot \ldots \cdot (\theta_n, \varphi_n) - (\theta_0', \varphi_0') \cdot (\theta_1, \varphi_1) \cdot \ldots \cdot (\theta_n, \varphi_n)$$
$$= (\theta_0 - \theta_0', 0) \cdot (\theta_1, \varphi_1) \cdot \ldots \cdot (\theta_n, \varphi_n) + (0, \varphi_0 - \varphi_0') \cdot (\theta_1, \varphi_1) \cdot \ldots \cdot (\theta_n, \varphi_n),$$

where the last term has already been estimated by (3.11). We are thus reduced to showing

$$|(\theta_0, 0) \cdot (\theta_1, \varphi_1) \cdot \ldots \cdot (\theta_n, \varphi_n)| \lesssim AJ\|\theta_0\|_\omega.$$

By homogeneity we may further assume $\|\theta_0\|_\omega = 1$, and the desired estimate now follows from the first step of the proof applied to $\varphi_i' = 0$, using $(\theta_0, 0) \cdot \ldots \cdot (\theta_n, 0) = 0$. □

4. Twisted energy and differentiability

As in §3, we assume that the orthogonality and submean value properties hold, and recall that this is satisfied when X is a normal irreducible compact Kähler space or any irreducible projective Berkovich space. We fix $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$. In this section we introduce and study the twisted energy of a measure, and show that it computes the directional derivatives of the energy.

4.1. The twisted energy of a measure. For any $\theta \in \mathcal{Z}$ and $\varphi \in \mathcal{D}$,

$$\mathrm{E}_{\omega+t\theta}(\varphi) := V_{\omega+t\theta}^{-1}(\omega + t\theta, \varphi)^{n+1}$$

make sense for all $t \in \mathbb{R}$ small enough, and

$$\nabla_\theta \mathrm{E}_\omega(\varphi) := \frac{d}{dt}\bigg|_{t=0} \mathrm{E}_{\omega+t\theta}(\varphi) \tag{4.1}$$

satisfies

$$\nabla_\theta \mathrm{E}_\omega(\varphi) = \mathrm{E}_\omega^\theta(\varphi) - V_\omega^\theta \, \mathrm{E}_\omega(\varphi), \tag{4.2}$$

where

$$\mathrm{E}_\omega^\theta(\varphi) := V_\omega^{-1}(\theta, 0) \cdot (\omega, \varphi)^n \tag{4.3}$$

and

$$V_\omega^\theta := nV_\omega^{-1}[\theta] \cdot [\omega]^{n-1}.$$

Note that $\mathrm{E}_\omega^\theta(\varphi)$ is a linear function of $\theta \in \mathcal{Z}$, and

$$\mathrm{E}_\omega^\theta(\varphi + c) = \mathrm{E}_\omega^\theta(\varphi) + cV_\omega^\theta \text{ for } c \in \mathbb{R}, \tag{4.4}$$

while $\nabla_\theta \mathrm{E}_\omega$ is translation invariant. For all $\varphi, \psi \in \mathcal{D}$, we further have

$$\mathrm{E}_\omega^\theta(\varphi) - \mathrm{E}_\omega^\theta(\psi) = \sum_{j=0}^{n-1} V_\omega^{-1} \int (\varphi - \psi)\theta \wedge \omega_\varphi^j \wedge \omega_\psi^{n-1-j}. \tag{4.5}$$

EXAMPLE 4.1. *By* (1.24), *we have* $\mathrm{E}_{(1+t)\omega}((1+t)\varphi) = (1+t)\,\mathrm{E}_\omega(\varphi)$. *By* (1.27), *this implies* $\nabla_\omega \mathrm{E}_\omega(\varphi) + \int \varphi\, \mathrm{MA}_\omega(\varphi) = \mathrm{E}_\omega(\varphi)$, *and hence*

$$\nabla_\omega \mathrm{E}_\omega(\varphi) = \mathrm{E}_\omega(\varphi) - \int \varphi\, \mathrm{MA}_\omega(\varphi) = \mathrm{J}_\omega(\varphi, 0) \approx \mathrm{J}_\omega(0,\varphi) = \mathrm{J}_\omega(\varphi). \tag{4.6}$$

EXAMPLE 4.2. *For each* $\psi \in \mathcal{D}$ *and* $t \in \mathbb{R}$, (1.29) *yields* $\mathrm{E}_{\omega+t\mathrm{dd}^c\psi}(\varphi) = \mathrm{E}_\omega(\varphi + t\psi) - \mathrm{E}_\omega(t\psi)$, *and using again* (1.27) *we get*

$$\nabla_{\mathrm{dd}^c\psi} \mathrm{E}_\omega(\varphi) = \int \psi(\mathrm{MA}_\omega(\varphi) - \mu_\omega). \tag{4.7}$$

EXAMPLE 4.3. *For any* $\tau \in \mathcal{D}_\omega$, *we similarly have* $\mathrm{E}_{\omega_\tau + t\theta}(\varphi) = \mathrm{E}_{\omega+t\theta}(\varphi+\tau) - \mathrm{E}_{\omega+t\theta}(\tau)$ *and*

$$\nabla_\theta \mathrm{E}_{\omega_\tau}(\varphi) = \nabla_\theta \mathrm{E}_\omega(\varphi+\tau) - \nabla_\theta \mathrm{E}_\omega(\tau). \tag{4.8}$$

LEMMA 4.4. *For all* $\theta \in \mathcal{Z}$ *and* $\varphi, \psi \in \mathcal{D}_\omega$, *the following holds:*

(i) *if* $\theta \in \mathcal{Z}_\omega$, *then*

$$|\nabla_\theta \mathrm{E}_\omega(\varphi) - \nabla_\theta \mathrm{E}_\omega(\psi)| \lesssim \mathrm{J}_\omega(\varphi,\psi)^\alpha \max\{\mathrm{J}^+_\omega(\varphi), \mathrm{J}^+_\omega(\psi)\}^{1-\alpha} \|\theta\|_\omega \tag{4.9}$$

with $\alpha := 2^{-n}$;

(ii) *in the general case, there exist* $C > 0$ *only depending on* ω *and* θ *such that*

$$|\nabla_\theta \mathrm{E}_\omega(\varphi) - \nabla_\theta \mathrm{E}_\omega(\psi)| \le C \mathrm{J}_\omega(\varphi,\psi)^\alpha (\max\{\mathrm{J}_\omega(\varphi), \mathrm{J}_\omega(\psi)\} + 1)^{1-\alpha}. \tag{4.10}$$

PROOF. By translation invariance of $\nabla_\theta \mathrm{E}_\omega$, we may assume $\int \varphi\, \mu_\omega = \int \psi\, \mu_\omega = 0$. When θ lies in $\mathcal{Z}_\omega$, Theorem 3.6 applied to (1.24) and (4.3) yields

$$|\mathrm{E}_\omega(\varphi) - \mathrm{E}_\omega(\psi)| \le C \quad \text{and} \quad \left|\mathrm{E}^\theta_\omega(\varphi) - \mathrm{E}^\theta_\omega(\psi)\right| \le C(1 + \|\theta\|_\omega)$$

with $C := \mathrm{J}_\omega(\varphi,\psi)^\alpha \max\{\mathrm{J}^+_\omega(\varphi), \mathrm{J}^+_\omega(\psi)\}^{1-\alpha}$. By homogeneity of E^θ_ω with respect to θ, we may replace $1 + \|\theta\|_\omega$ with $\|\theta\|_\omega$ in the last estimate, and (i) now follows from (4.2) together with $|V^\theta_\omega| \le n\|\theta\|_\omega$.

In the general case, pick $\tau \in \mathcal{D}_\omega$ such that $\theta \in \mathcal{Z}_{\omega_\tau}$ (see Proposition 1.12). Given $\varphi, \psi \in \mathcal{D}_\omega$, we then have $\varphi - \tau, \psi - \tau \in \mathcal{D}_{\omega_\tau}$, and (4.8), (1.39) yield

$$\nabla_\theta \mathrm{E}_{\omega_\tau}(\varphi-\tau) - \nabla_\theta \mathrm{E}_{\omega_\tau}(\psi-\tau) = \nabla_\theta \mathrm{E}_\omega(\varphi) - \nabla_\theta \mathrm{E}_\omega(\psi), \quad \mathrm{J}_{\omega_\tau}(\varphi-\tau, \psi-\tau) = \mathrm{J}_\omega(\varphi,\psi),$$

$$\mathrm{J}_{\omega_\tau}(\varphi-\tau) = \mathrm{J}_{\omega_\tau}(0, \varphi-\tau) = \mathrm{J}_\omega(\tau,\varphi) \lesssim \mathrm{J}_\omega(\varphi) + \mathrm{J}_\omega(\tau), \quad \mathrm{J}_{\omega_\tau}(\psi-\tau) \lesssim \mathrm{J}_\omega(\psi) + \mathrm{J}_\omega(\tau).$$

By (i) we thus get

$$|\nabla_\theta \mathrm{E}_\omega(\varphi) - \nabla_\theta \mathrm{E}_\omega(\psi)| \lesssim \mathrm{J}_\omega(\varphi,\psi)^\alpha \left(\max\{\mathrm{J}^+_\omega(\varphi), \mathrm{J}^+_\omega(\psi)\} + \mathrm{J}^+_\omega(\tau)\right)^{1-\alpha} \|\theta\|_{\omega_\tau},$$

which proves (ii). □

PROPOSITION 4.5. *For any* $\theta \in \mathcal{Z}$, *there exists a unique strongly continuous functional* $\mathrm{J}^\theta_\omega \colon \mathcal{M}^1 \to \mathbb{R}$, *the* θ-twisted energy, *such that*

$$\mathrm{J}^\theta_\omega(\mathrm{MA}_\omega(\varphi)) = \nabla_\theta \mathrm{E}_\omega(\varphi) \tag{4.11}$$

for all $\varphi \in \mathcal{D}_\omega$. *For all* $\mu, \nu \in \mathcal{M}^1$, *we also have:*

(i) $\mathrm{J}^\theta_\omega(\mu)$ *is a linear function of* θ;

(ii) *if* $\theta \in \mathcal{Z}_\omega$, *then*

$$|\mathrm{J}^\theta_\omega(\mu) - \mathrm{J}^\theta_\omega(\nu)| \lesssim \delta_\omega(\mu,\nu)^\alpha \max\{\mathrm{J}^+_\omega(\mu), \mathrm{J}^+_\omega(\nu)\}^{1-\alpha} \|\theta\|_\omega; \tag{4.12}$$

with $\alpha := 2^{-n}$, *and hence*

$$|\mathrm{J}^\theta_\omega(\mu)| \lesssim \mathrm{J}^+_\omega(\mu) \|\theta\|_\omega; \tag{4.13}$$

(iii) *in the general case, there exist $C > 0$ only depending on ω and θ such that*

$$|\mathrm{J}^\theta_\omega(\mu) - \mathrm{J}^\theta_\omega(\nu)| \le C\delta_\omega(\mu,\nu)^\alpha(\max\{\mathrm{J}_\omega(\mu),\mathrm{J}_\omega(\nu)\}+1)^{1-\alpha}, \tag{4.14}$$

and hence

$$|\mathrm{J}^\theta_\omega(\mu)| \le C(\mathrm{J}_\omega(\mu)+1). \tag{4.15}$$

PROOF. Assume $\varphi, \psi \in \mathcal{D}_\omega$ satisfy $\mathrm{MA}_\omega(\varphi) = \mathrm{MA}_\omega(\psi)$. Then $\mathrm{J}_\omega(\varphi,\psi) = 0$, and hence $\nabla_\theta \mathrm{E}_\omega(\varphi) = \nabla_\theta \mathrm{E}_\omega(\psi)$, by (4.10). As a result, there exists a unique function J^θ_ω on the image of $\mathrm{MA}_\omega\colon \mathcal{D}_\omega \to \mathcal{M}^1$ such that (4.11) is satisfied. For all $\mu,\nu \in \mathcal{M}^1$ in the image of MA_ω, it further follows from (4.10) that (4.14) holds. This shows that J^θ_ω is uniformly continuous on a dense subspace of the quasi-metric space $(\mathcal{M}^1, \delta_\omega)$, and hence admits a unique continuous extension $\mathrm{J}^\theta_\omega\colon \mathcal{M}^1 \to \mathbb{R}$. Finally, (i) holds by linearity of $\nabla_\theta \mathrm{E}_\omega$ with respect to θ,(ii) and (iii) follow, by continuity, from Lemma 4.4. □

Using (4.11) (with $\varphi = 0$), (4.6), (4.7) and (4.8), we further get, for all $\mu \in \mathcal{M}^1$, $\psi \in \mathcal{D}$ and $\tau \in \mathcal{D}_\omega$,

$$\mathrm{J}^\theta_\omega(\mu_\omega) = 0; \tag{4.16}$$

$$\mathrm{J}^\omega_\omega(\mu) = \mathrm{J}_\omega(\mu); \tag{4.17}$$

$$\mathrm{J}^{\mathrm{dd^c}\psi}_\omega(\mu) = \int \psi(\mu-\mu_\omega); \tag{4.18}$$

$$\mathrm{J}^\theta_{\omega_\tau}(\mu) = \mathrm{J}^\theta_\omega(\mu) + c \tag{4.19}$$

with $c \in \mathbb{R}$ uniquely determined by (4.16), i.e. $c = \mathrm{J}^\theta_{\omega_\tau}(\mu_\omega) = -\mathrm{J}^\theta_\omega(\mu_{\omega_\tau})$.

4.2. Hölder continuity of the twisted energy. The following estimates will be the key ingredients for the continuity of coercivity thresholds (see Theorem 5.5 below).

THEOREM 4.6. *Pick $\omega, \omega' \in \mathcal{Z}_+$ with $\delta := \mathrm{d_T}(\omega,\omega') \le 1$ and $[\omega'] \in \mathrm{Pos}(X)$. For all $\theta, \theta' \in \mathcal{Z}_\omega$ and $\mu \in \mathcal{M}^1$, we then have*

$$\left|\mathrm{J}^\theta_\omega(\mu) - \mathrm{J}^{\theta'}_{\omega'}(\mu)\right| \lesssim (\delta^a\|\theta\|_\omega + \|\theta-\theta'\|_\omega)\,\mathrm{J}^+_\omega(\mu) \tag{4.20}$$

and

$$\mathrm{J}^+_\omega(\mu) = (1+O(\delta^\alpha))\mathrm{J}^+_{\omega'}(\mu). \tag{4.21}$$

LEMMA 4.7. *Assume $\omega \le \omega' \le e^\delta\omega$ with $\delta \in [0,2]$. Pick $\varphi \in \mathcal{D}_\omega \subset \mathcal{D}_{\omega'}$, and set $\mu := \mathrm{MA}_\omega(\varphi)$, $\mu' := \mathrm{MA}_{\omega'}(\varphi)$. Then:*

$$|\nabla_\theta \mathrm{E}_\omega(\varphi) - \nabla_\theta \mathrm{E}_{\omega'}(\varphi)| \lesssim \delta \mathrm{J}^+_\omega(\mu)\|\theta\|_\omega; \tag{4.22}$$

$$\max\{\mathrm{J}^+_{\omega'}(\mu), \mathrm{J}^+_{\omega'}(\mu')\} \lesssim \mathrm{J}^+_\omega(\mu); \tag{4.23}$$

$$\mathrm{J}_{\omega'}(\mu,\mu') \lesssim \delta \mathrm{J}^+_\omega(\mu). \tag{4.24}$$

Proof. All three estimates are invariant under translation of φ by a constant, and we shall rely on a different normalization for each of them. We first normalize φ by $\int \varphi\, \mu_\omega = 0$. By homogeneity, we may further assume $\|\theta\|_\omega = 1$. Since

$$\mathrm{E}^\theta_\omega(\varphi) - \mathrm{E}^\theta_{\omega'}(\varphi) = V_\omega^{-1}(\theta, 0) \cdot (\omega, \varphi)^n - V_{\omega'}^{-1}(\theta, 0) \cdot (\omega', \varphi)^n$$

with $V_{\omega'}/V_\omega = 1 + O(\delta)$, Theorem 3.6 yields

$$\left|\mathrm{E}^\theta_\omega(\varphi) - \mathrm{E}^\theta_{\omega'}(\varphi)\right| \lesssim \|\omega' - \omega\|_\omega \mathrm{J}^+_\omega(\mu) \lesssim \delta \mathrm{J}^+_\omega(\mu).$$

We similarly get $|\mathrm{E}_\omega(\varphi) - \mathrm{E}_{\omega'}(\varphi)| \lesssim \delta \mathrm{J}^+_\omega(\mu)$, and (4.22) follows, using the trivial estimate

$$V^\theta_{\omega'} = V^\theta_\omega + O(\delta).$$

Next, we normalize φ by $\sup \varphi = 0$. By Lemma 1.20, we then have

$$\mathrm{J}_{\omega'}(\mu') \approx \mathrm{J}_{\omega'}(\varphi) = \int \varphi\, \mu_{\omega'} - \mathrm{E}_{\omega'}(\varphi) \le -\mathrm{E}_{\omega'}(\varphi) \lesssim -\mathrm{E}_\omega(\varphi) \le \mathrm{J}^+_\omega(\mu).$$

On the other hand, Corollary 3.5 yields $\mathrm{J}^+_{\omega'}(\mu) \lesssim \mathrm{J}^+_\omega(\mu)$, and (4.23) follows.

Finally, we normalize φ by $\int \varphi\, \mu_{\omega'} = 0$. Pick a maximizing sequence (ψ_i) in $\mathcal{D}_{\omega'}$ for μ, also normalized by $\int \psi_i\, \mu_{\omega'} = 0$, and set $\mu_i := \mathrm{MA}_{\omega'}(\psi_i)$. By Theorem 2.22, $\mathrm{J}_{\omega'}(\mu, \mu')$ is the limit of

$$\mathrm{J}_{\omega'}(\mu_i, \mu') = \mathrm{J}_{\omega'}(\psi_i, \varphi) \approx \int (\psi_i - \varphi)(\mu' - \mu_i) = \int (\psi_i - \varphi)(\mu' - \mu) + o(1),$$

where the last two points hold by (1.30) and (2.25), respectively. Now

$$\begin{aligned}\int (\psi_i - \varphi)(\mu' - \mu) = &\left[V_{\omega'}^{-1}(0, \psi_i) \cdot (\omega', \varphi)^n - V_\omega^{-1}(0, \psi_i) \cdot (\omega, \varphi)^n\right] \\ &- \left[V_{\omega'}^{-1}(0, \varphi) \cdot (\omega', \varphi)^n - V_\omega^{-1}(0, \varphi) \cdot (\omega, \varphi)^n\right],\end{aligned}$$

where

$$\left|V_{\omega'}^{-1}(0, \psi_i) \cdot (\omega', \varphi)^n - V_\omega^{-1}(0, \psi_i) \cdot (\omega, \varphi)^n\right| \lesssim \|\omega' - \omega\|_{\omega'} \max\{\mathrm{J}^+_{\omega'}(\varphi), \mathrm{J}^+_{\omega'}(\psi_i)\}$$

and

$$\left|V_{\omega'}^{-1}(0, \varphi) \cdot (\omega', \varphi)^n - V_\omega^{-1}(0, \varphi) \cdot (\omega, \varphi)^n\right| \lesssim \|\omega' - \omega\|_{\omega'} \mathrm{J}^+_{\omega'}(\varphi),$$

by Theorem 3.6. Since $\|\omega' - \omega\|_{\omega'} \lesssim \delta$, $\mathrm{J}_{\omega'}(\varphi) \approx \mathrm{J}_{\omega'}(\mu')$ and $\mathrm{J}_{\omega'}(\psi_i) \approx \mathrm{J}_{\omega'}(\mu_i) \to \mathrm{J}_{\omega'}(\mu)$, this proves (4.24), thanks to (4.23). □

Proof of Theorem 4.6. Note first that $\omega'' := e^{-\delta}\omega \le \omega' \le e^\delta \omega$, and hence

$$\omega'' \le \omega' \le e^{2\delta}\omega'', \quad \omega'' \le \omega \le e^\delta \omega''.$$

Arguing successively with ω'', ω', and with ω'', ω, and relying on Corollary 3.5, it is thus enough to prove the result when $\omega \le \omega' \le e^\delta \omega$, which we henceforth assume. By density of the image of $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}^1$ and strong continuity of J^θ_ω and $\mathrm{J}^\theta_{\omega'}$ (recall that the strong topology of $\mathcal{M}^1$ is independent of ω), we may assume $\mu = \mathrm{MA}_\omega(\varphi)$ with $\varphi \in \mathcal{D}_\omega \subset \mathcal{D}_{\omega'}$. As in Lemma 4.7, set $\mu' := \mathrm{MA}_{\omega'}(\varphi)$. By (4.11), we have

$$\mathrm{J}^\theta_\omega(\mu) = \nabla_\theta \mathrm{E}_\omega(\varphi), \quad \mathrm{J}^\theta_{\omega'}(\mu') = \nabla_\theta \mathrm{E}_{\omega'}(\varphi),$$

and (4.22) thus yields

$$\left|\mathrm{J}^\theta_\omega(\mu) - \mathrm{J}^\theta_{\omega'}(\mu')\right| \lesssim \delta \mathrm{J}^+_\omega(\mu) \|\theta\|_\omega. \tag{4.25}$$

On the other hand, (4.12), implies

$$\left|\mathrm{J}^{\theta}_{\omega'}(\mu) - \mathrm{J}^{\theta}_{\omega'}(\mu')\right| \lesssim \delta^\alpha \mathrm{J}^+_\omega(\mu)\|\theta\|_\omega, \tag{4.26}$$

thanks to (4.23) and (4.24). Finally, (4.13) yields

$$\left|\mathrm{J}^{\theta}_{\omega'}(\mu) - \mathrm{J}^{\theta'}_{\omega'}(\mu)\right| = \left|\mathrm{J}^{\theta-\theta'}_{\omega'}(\mu)\right| \lesssim \mathrm{J}^+_{\omega'}(\mu)\|\theta-\theta'\|_{\omega'}, \tag{4.27}$$

and summing up (4.25), (4.26) and (4.27) yields (4.20). Applying the latter estimate with $\theta := \omega$, $\theta' := \omega'$, we get

$$|\mathrm{J}_\omega(\mu) - \mathrm{J}_{\omega'}(\mu)| \lesssim \delta^\alpha \mathrm{J}^+_\omega(\mu), \tag{4.28}$$

in view of (4.17). Since we also have $T_{\omega'} = (1+O(\delta))T_\omega$ (see (1.23)), (4.21) follows. □

4.3. Differentiability of the energy.

THEOREM 4.8. *For any $\theta \in \mathcal{Z}$ and $\mu \in \mathcal{M}^1$, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{J}_{\omega+t\theta}(\mu) = \mathrm{J}^\theta_\omega(\mu).$$

If we do not assume $\theta \in \mathcal{Z}_\omega$, the condition $\omega + t\theta \geq 0$ will fail in general, but one can still make sense of $\mathrm{J}_{\omega+t\theta}(\mu)$, see Remark 2.4.

LEMMA 4.9. *Pick $\theta \in \mathcal{Z}_\omega$ and $\varphi \in \mathcal{D}_\omega$ normalized by $\int \varphi\, \mu_\omega = 0$. Then*

$$|\mathrm{E}_\omega(\varphi)| \lesssim \mathrm{J}^+_\omega(\varphi), \quad \left|\int \varphi\, \mathrm{MA}_\omega(\varphi)\right| \lesssim \mathrm{J}^+_\omega(\varphi), \quad \left|\mathrm{E}^\theta_\omega(\varphi)\right| \lesssim \|\theta\|_\omega \mathrm{J}^+_\omega(\varphi).$$

PROOF. We have

$$\mathrm{E}_\omega(\varphi) = \frac{(\omega,\varphi)^{n+1}}{(n+1)V_\omega}, \quad \int \varphi\, \mathrm{MA}_\omega(\varphi) = \frac{(0,\varphi)\cdot(\omega,\varphi)^n}{V_\omega}, \quad \mathrm{E}^\theta_\omega(\varphi) = \frac{(\theta,0)\cdot(\omega,\varphi)^n}{V_\omega},$$

and the estimates thus follow from (3.10) (and homogeneity in θ). □

LEMMA 4.10. *Pick $\theta \in \mathcal{Z}_\omega$ such that $\|\theta\|_\omega < 1$ and $[\omega+\theta] \in \mathrm{Pos}(X)$. For all $\mu \in \mathcal{M}^1$ we then have*

$$\mathrm{J}_{\omega+\theta}(\mu) \geq \mathrm{J}_\omega(\mu) + \mathrm{J}^\theta_\omega(\mu) - O(\mathrm{J}^+_\omega(\mu)\|\theta\|^2_\omega).$$

Recall that, in this paper, the implicit constant in O only depends on n.

PROOF. Set $J := \mathrm{J}^+_\omega(\mu)$ and $\varepsilon := \|\theta\|_\omega$. If $\varepsilon = 0$, then $\theta = 0$ and the result is clear. We may thus assume $\varepsilon > 0$ and write $\theta = \varepsilon\tilde{\theta}$ with $\|\tilde{\theta}\|_\omega = 1$ and $\varepsilon \in (0,1)$.

By density of the image of $\mathrm{MA}_\omega \colon \mathcal{D}_\omega \to \mathcal{M}^1$, it is enough to prove the result when $\mu = \mathrm{MA}_\omega(\varphi)$ with $\varphi \in \mathcal{D}_\omega$, which we normalize by $\int \varphi\, \mu_\omega = 0$. By (2.5), (4.11) and (4.2), we then have

$$\mathrm{J}_\omega(\mu) = \mathrm{E}_\omega(\varphi) - \int \varphi\, \mu, \quad \mathrm{J}^\theta_\omega(\mu) = \mathrm{E}^\theta_\omega(\varphi) - V^\theta_\omega\, \mathrm{E}_\omega(\varphi). \tag{4.29}$$

Note that

$$(1-\varepsilon)\omega \leq \omega + \theta \leq (1+\varepsilon)\omega \leq (1-\varepsilon)^{-1}\omega, \tag{4.30}$$

and hence

$$(1-\varepsilon)\varphi \in \mathcal{D}_{(1-\varepsilon)\omega} \subset \mathcal{D}_{\omega+\theta}.$$

By (2.2), this yields

$$\mathrm{J}_{\omega+\theta}(\mu) \geq \mathrm{E}_{\omega+\theta}\left((1-\varepsilon)\varphi\right) - (1-\varepsilon)\int \varphi\,\mu. \tag{4.31}$$

Further,

$$\begin{aligned}(n+1)V_{\omega+\theta}\,\mathrm{E}_{\omega+\theta}\left((1-\varepsilon)\varphi)\right) &= (\omega+\theta,(1-\varepsilon)\varphi)^{n+1} = ((\omega,\varphi)+(\theta,-\varepsilon\varphi))^{n+1} \\ &= (\omega,\varphi)^{n+1} + (n+1)\left[(\theta,0)\cdot(\omega,\varphi)^n - \varepsilon(0,\varphi)\cdot(\omega,\varphi)^n\right] + \varepsilon^2 a(\varepsilon) \\ &= (n+1)V_\omega\left[\mathrm{E}_\omega(\varphi) + \mathrm{E}^\theta_\omega(\varphi) - \varepsilon\int\varphi\,\mu\right] + \varepsilon^2 a(\varepsilon)\end{aligned}$$

with

$$a(\varepsilon) := \sum_{j=2}^{n+1}\binom{n+1}{j}\varepsilon^{j-2}(\tilde\theta,-\varphi)^j\cdot(\omega,\varphi)^{n+1-j}.$$

Since $\|\tilde\theta\|_\omega = 1$, (3.10) yields $|a(\varepsilon)| \lesssim V_\omega J$. Combining this with

$$V_{\omega+\theta} = [\omega+\theta]^n = [\omega]^n + n[\theta]\cdot[\omega]^{n-1} + O(\varepsilon^2[\omega]^n) = V_\omega\left(1 + V^\theta_\omega + O(\varepsilon^2)\right),$$

and

$$\left|V^\theta_\omega\right| \lesssim \varepsilon, \quad |\mathrm{E}_\omega(\varphi)| \lesssim J, \quad \left|\int\varphi\,\mu\right| \leq J, \quad \left|\mathrm{E}^\theta_\omega(\varphi)\right| \lesssim \varepsilon J,$$

(see Lemma 4.9), we infer

$$\begin{aligned}\mathrm{E}_{\omega+\theta}((1-\varepsilon)\varphi) &\geq \left(1 - V^\theta_\omega + O(\varepsilon^2)\right)\left(\mathrm{E}_\omega(\varphi) + \mathrm{E}^\theta(\varphi) - \varepsilon\int\varphi\,\mu - O(\varepsilon^2 J)\right) \\ &= \mathrm{E}_\omega(\varphi) + \mathrm{E}^\theta_\omega(\varphi) - V^\theta_\omega\,\mathrm{E}_\omega(\varphi) - \varepsilon\int\varphi\,\mu - O(\varepsilon^2 J).\end{aligned}$$

Injecting this into (4.31) and using (4.29), we get

$$\mathrm{J}_{\omega+\theta}(\mu) \geq \mathrm{J}_\omega(\mu) + \mathrm{J}^\theta_\omega(\mu) - O(\varepsilon^2 J),$$

which completes the proof. □

Proof of Theorem 4.8. Assume first $\theta \in \mathcal{Z}_\omega$. Set $J := \mathrm{J}^+_\omega(\mu)$. By Lemma 4.10 we then have

$$\mathrm{J}_{\omega+t\theta}(\mu) \geq \mathrm{J}_\omega(\mu) + t\mathrm{J}^\theta_\omega(\mu) - O(t^2\|\theta\|_\omega J).$$

For $|t| \ll 1$, we also have $\delta(\omega+t\theta,\omega) \leq 1$, and hence $\mathrm{J}^+_{\omega+t\theta}(\mu) \lesssim J$, by Corollary 3.5. Reversing the roles of ω and $\omega+t\theta$, Lemma 4.10 thus yields

$$\mathrm{J}_\omega(\mu) \geq \mathrm{J}_{\omega+t\theta}(\mu) - t\mathrm{J}^\theta_{\omega+t\theta}(\mu) - O(t^2\|\theta\|_\omega J).$$

By Theorem 4.6, $\mathrm{J}^\theta_{\omega+t\theta}(\mu) \to \mathrm{J}^\theta_\omega(\mu)$ as $t \to 0$, and we conclude, as desired,

$$\mathrm{J}_{\omega+t\theta}(\mu) = \mathrm{J}_\omega(\mu + t\mathrm{J}^\theta_\omega(\mu) + o(t).$$

as $t \to 0$.

In the general case, pick $\tau \in \mathcal{D}_\omega$ such that $\theta \in \mathcal{Z}_{\omega_\tau}$ (see Proposition 1.12). Then (2.9) yields

$$\mathrm{J}_{\omega+t\theta}(\mu) = \mathrm{J}_{\omega_\tau+t\theta}(\mu) - \int\tau\,\mu + \mathrm{E}_{\omega+t\theta}(\tau),$$

see Remark 2.4. We infer

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{J}_{\omega+t\theta}(\mu) = \mathrm{J}^\theta_{\omega_\tau}(\mu) + \nabla_\theta\,\mathrm{E}_\omega(\tau) = \mathrm{J}^\theta_{\omega_\tau}(\mu) + \mathrm{J}^\theta_\omega(\mu_{\omega_\tau}) = \mathrm{J}^\theta_\omega(\mu),$$

where the first equality follows from the first step of the proof and (4.2), the second one from (4.11), and the third from (4.19). □

5. Coercivity thresholds and free energy

As in §4, we assume that the orthogonality and submean value properties hold. We will establish a general continuity result for coercivity thresholds, and apply this to the free energy, which induces the Mabuchi K-energy on potentials.

5.1. Continuity of coercivity thresholds. Fix for the moment $\omega \in \mathcal{Z}_+$ with $[\omega] \in \mathrm{Pos}(X)$, and consider an arbitrary functional $F \colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$.

DEFINITION 5.1. *We define the* coercivity threshold *of F as*

$$\sigma_\omega(F) := \sup\{\sigma \in \mathbb{R} \mid F \geq \sigma \mathrm{J}_\omega + A \text{ for some } A \in \mathbb{R}\} \in [-\infty, +\infty]. \tag{5.1}$$

We say that F is coercive *if $\sigma_\omega(F) > 0$, i.e. $F \geq \sigma \mathrm{J}_\omega + A$ for some $\sigma > 0$ and $A \in \mathbb{R}$.*

By Corollary 3.5 (ii), the condition that F is coercive (resp. $\sigma_\omega(F) = \pm\infty$) is independent of ω.

Recall from (2.22) that the energy of any $\mu \in \mathcal{M}^1$ coincides with the quasi-distance to the base point μ_ω, i.e. $\mathrm{J}_\omega(\mu) = \delta_\omega(\mu, \mu_\omega)$. As a result, the coercivity threshold measures the linear growth of F with respect to quasi-metric δ_ω. As we next show, any other base point can be used in place of μ_ω (something that could fail for a general quasi-metric).

LEMMA 5.2. *For any $F \colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$ and $\nu \in \mathcal{M}^1$ we have*

$$\sigma_\omega(F) = \sup\{\sigma \in \mathbb{R} \mid F \geq \sigma \delta_\omega(\cdot, \nu) + A \text{ for some } A \in \mathbb{R}\}. \tag{5.2}$$

In particular, $\sigma_\omega(F)$ only depends on $[\omega] \in \mathrm{Pos}(X)$.

PROOF. For any $\mu \in \mathcal{M}^1$, (2.24) yields

$$|\delta_\omega(\mu, \nu) - \mathrm{J}_\omega(\mu)| \lesssim \mathrm{J}_\omega(\nu)^\alpha \max\{\mathrm{J}_\omega(\mu), \mathrm{J}_\omega(\nu)\}^{1-\alpha}.$$

For any $\varepsilon > 0$, we can thus find $C_\varepsilon > 0$ (depending on ν) such that

$$(1-\varepsilon)\mathrm{J}_\omega(\mu) - C_\varepsilon \leq \delta_\omega(\mu, \nu) \leq (1+\varepsilon)\mathrm{J}_\omega(\mu) + C_\varepsilon$$

for all $\mu \in \mathcal{M}^1$. This implies (5.2), and the last point follows, since the Dirichlet quasi-metric δ_ω only depends on $[\omega]$ (see Theorem 2.25). □

DEFINITION 5.3. *For each $\theta \in \mathcal{Z}$, we introduce the* twisted coercivity threshold

$$\sigma_\omega^\theta(F) := \sigma_\omega(F + \mathrm{J}_\omega^\theta). \tag{5.3}$$

We refer to §5.2 below for a discussion of the concrete cases we have in mind.

LEMMA 5.4. *The following holds:*

(i) *$\sigma_\omega^0(F) = \sigma_\omega(F)$, and $\sigma_\omega^{\theta + t\omega}(F) = \sigma_\omega^\theta(F) + t$ for all $\theta \in \mathcal{Z}$ and $t \in \mathbb{R}$;*
(ii) *we have $\sigma_\omega(F) \in \mathbb{R}$ (resp. $\sigma_\omega(F) = \pm\infty$) iff $\sigma_\omega^\theta(F) \in \mathbb{R}$ (resp. $\sigma_\omega^\theta(F) = \pm\infty$) for all $\theta \in \mathcal{Z}$.*

PROOF. The first point is a direct consequence of (4.17). For any $\theta \in \mathcal{Z}$, (4.15) yields a constant $C > 0$ such that $|\mathrm{J}_\omega^\theta| \leq C(\mathrm{J}_\omega + 1)$. This implies

$$\sigma_\omega(F) - C \leq \sigma_\omega^\theta(F) \leq \sigma_\omega(F) + C,$$

and (ii) follows. □

We can now state the main result of this section.

THEOREM 5.5. *For any functional* $F\colon \mathcal{M}^1 \to \mathbb{R}\cup\{+\infty\}$, *the twisted coercivity threshold* $\sigma_\omega^\theta(F)$ *is a continuous function of* $([\omega],[\theta]) \in \mathrm{Pos}(X)\times \mathrm{H}_{\mathrm{BC}}(X)$.

We emphasize that F here is a fixed functional, independent of ω and θ. Continuity is understood with respect to the finest vector space topology of $\mathrm{H}_{\mathrm{BC}}(X)$, i.e. for $[\omega]$ and $[\theta]$ constrained to any given finite dimensional subspace.

LEMMA 5.6. *There exists* $\delta_n > 0$ *only depending on* n *such that, for all* $\omega,\omega' \in \mathcal{Z}_+$ *with* $\delta := \mathrm{d_T}(\omega,\omega') \le \delta_n$ *and all* $\theta,\theta' \in \mathcal{Z}_\omega$, *we have*

$$\sigma_{\omega'}^{\theta'}(F) \ge (1+O(\delta^\alpha))\left[\sigma_\omega^\theta(F) + O(\delta^\alpha\|\theta\|_\omega + \|\theta-\theta'\|_\omega)\right]. \tag{5.4}$$

PROOF. Since $\mathrm{J}_\omega^+ = \mathrm{J}_\omega + T_\omega$, we can replace J_ω with J_ω^+ in (5.1), and hence

$$\sigma_\omega^\theta(F) = \sup\left\{\sigma\in\mathbb{R} \mid F + \mathrm{J}_\omega^\theta \ge \sigma \mathrm{J}_\omega^+ + A \text{ for some } A\in\mathbb{R}\right\}. \tag{5.5}$$

Pick $\sigma, A \in \mathbb{R}$ such that $F + \mathrm{J}_\omega^\theta \ge \sigma\mathrm{J}_\omega^+ + A$ on $\mathcal{M}^1$. By (4.20) we get

$$\begin{aligned} F + \mathrm{J}_{\omega'}^{\theta'} &\ge F + \mathrm{J}_\omega^\theta + O(\delta^\alpha\|\theta\|_\omega + \|\theta-\theta'\|_\omega)\mathrm{J}_\omega^+ \\ &\ge \sigma\mathrm{J}_\omega^+ + A + O(\delta^\alpha\|\theta\|_\omega + \|\theta-\theta'\|_\omega)\mathrm{J}_\omega^+ \\ &\ge (1+O(\delta^\alpha))\left[\sigma + O(\delta^\alpha\|\theta\|_\omega + \|\theta-\theta'\|_\omega\right]\mathrm{J}_{\omega'}^+ + A, \end{aligned}$$

using (4.21), and hence $\sigma_{\omega'}^{\theta'}(F) \ge (1+O(\delta^\alpha))\left[\sigma + O(\delta^\alpha\|\theta\|_\omega + \|\theta-\theta'\|_\omega\right]$. Choosing $\delta_n > 0$ such that $1+O(\delta^\alpha) \ge 0$ for $\delta \le \delta_n$ and taking the supremum over σ yields (5.4). □

PROOF OF THEOREM 5.5. We first show that $\sigma_\omega^\theta(F)$ only depends on the classes $[\omega]\in\mathrm{Pos}(X)$ and $[\theta]\in\mathrm{H}_{\mathrm{BC}}(X)$. For each $\tau\in\mathcal{D}_\omega$, we have

$$\sigma_{\omega_\tau}^\theta(F) = \sigma_{\omega_\tau}(F + \mathrm{J}_{\omega_\tau}^\theta) = \sigma_{\omega_\tau}(F+\mathrm{J}_\omega^\theta) = \sigma_\omega(F+\mathrm{J}_\omega^\theta) = \sigma_\omega^\theta(F),$$

where the second equality follows from (4.19), and the third from Lemma 5.2. This proves that $\sigma_\omega^\theta(F)$ only depends on $[\omega]$. On the other hand, for any $\psi\in\mathcal{D}$, (4.18) yields

$$F + \mathrm{J}_\omega^\theta - C \le F + \mathrm{J}_\omega^{\theta+\mathrm{dd^c}\psi} \le F + \mathrm{J}_\omega^\theta + C$$

with $C := 2\sup|\psi|$. This implies $\sigma_\omega^\theta(F) = \sigma_\omega^{\theta+\mathrm{dd^c}\psi}(F)$, which thus only depends on $[\theta]$.

As noted above, Corollary 3.5 (ii) shows that the condition $\sigma_\omega(F) = \pm\infty$ is independent of ω, and in that case the result trivially holds, since all twisted thresholds are then equal to $\pm\infty$, by Lemma 5.4 (ii).

Assume now that the twisted thresholds are finite valued. Pick $\omega_0\in\mathcal{Z}_+$ with $[\omega_0]\in\mathrm{Pos}(X)$, $\theta_0\in\mathcal{Z}$. Choose a finite dimensional subspace $W\subset\mathrm{H}_{\mathrm{BC}}(X)$ containing $[\omega_0]$, $[\theta_0]$, and a finite dimensional subspace $V\subset\mathcal{Z}$ containing θ_0, and whose image in $\mathrm{H}_{\mathrm{BC}}(X)$ contains W. Since $\sigma_{\omega_0}^{\theta_0}(F)$ only depends on $[\omega_0]$, we can assume without loss $V\subset\mathcal{Z}_{\omega_0}$ (see Proposition 1.12). After enlarging V, we may further assume that it contains ω_0. For each $\omega,\theta\in V\subset\mathcal{Z}_{\omega_0}$ with ω close enough to ω_0, (5.5) yields

$$|\sigma_\omega^\theta(F) - \sigma_{\omega_0}^{\theta_0}(F)| \lesssim \delta^\alpha\left[\sigma_{\omega_0}^{\theta_0}(F) + O(\delta^\alpha\|\theta_0\|_{\omega_0} + \|\theta-\theta_0\|_{\omega_0})\right]$$

with $\delta = \mathrm{d_T}(\omega,\omega_0)$. By Lemma 1.14, this shows that the restriction of $(\omega,\theta)\mapsto\sigma_\omega^\theta(F)$ to V is continuous at (ω_0,θ_0), and the result follows, since the image of V contains W. □

5.2. Free energy vs. Mabuchi K-energy.

5.2.1. *The Kähler case.* Consider first a compact connected Kähler manifold X. Any smooth metric ρ on K_X induces a volume form μ_ρ (normalized to mass one), and hence a *(relative) entropy* functional

$$\mathrm{Ent}_\rho \colon \mathcal{M} \to \mathbb{R} \cup \{+\infty\}, \tag{5.6}$$

such that

$$\mathrm{Ent}_\rho(\mu) := \int \log\left(\frac{\mu}{\mu_\rho}\right)\mu$$

if μ is absolutely continuous with respect to μ_ρ, and $\mathrm{Ent}_\rho(\mu) = +\infty$ otherwise. Note that

$$\mathrm{Ent}_{\rho'}(\mu) = \mathrm{Ent}_\rho(\mu) + \int (\rho - \rho')\,\mu \tag{5.7}$$

for any other metric ρ' on K_X. As is well-known, the relative entropy can also be written as the Legendre transform

$$\mathrm{Ent}_\rho(\mu) := \sup_{f \in \mathcal{D}} \left(\int f\,\mu - \log \int e^f\,\mu_\rho\right)$$

which shows that (5.6) is convex and lsc. In particular, the restriction $\mathrm{Ent}_\rho \colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$ is lsc in the strong topology. While it is not continuous, we nevertheless have (see [**BDL17**, Theorem 4.7]):

LEMMA 5.7. *For each $\mu \in \mathcal{M}^1$, there exists a sequence (φ_i) in $\mathcal{D}_\omega$ such that $\mu_i := \mathrm{MA}_\omega(\varphi_i)$ satisfies $\mu_i \to \mu$ strongly in $\mathcal{M}^1$ and $\mathrm{Ent}_\rho(\mu_i) \to \mathrm{Ent}_\rho(\mu)$.*

Extending the terminology of [**Berm13**], we introduce:

DEFINITION 5.8. *The* free energy $\mathrm{F}_\omega \colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$ *is defined by setting*

$$\mathrm{F}_\omega(\mu) := \mathrm{Ent}_\rho(\mu) - \mathrm{Ent}_\rho(\mu_\omega) + \mathrm{J}_\omega^{\theta_\rho}(\mu), \tag{5.8}$$

where $\theta_\rho \in \mathcal{Z}$ denotes the curvature of ρ.

As the notation suggests, F_ω is independent of the choice of ρ; this follows from (5.7) combined with (4.18). Furthermore, (4.16) and (4.19) show that F_ω only depends on the Kähler class $[\omega]$, up to an additive constant uniquely determined by the normalization $\mathrm{F}_\omega(\mu_\omega) = 0$.

The *raison d'être* of the free energy is that its composition with the Monge–Ampère operator coincides with the *Mabuchi K-energy* $\mathrm{M}_\omega \colon \mathcal{D}_\omega \to \mathbb{R}$, i.e. we have

$$\mathrm{F}_\omega(\mathrm{MA}_\omega(\varphi)) = \mathrm{M}_\omega(\varphi) \tag{5.9}$$

for all $\varphi \in \mathcal{D}_\omega$. Indeed, in view of (4.11), (5.9) is equivalent to the well-known *Chen–Tian formula* for the K-energy, which can be written as

$$\mathrm{M}_\omega(\varphi) = \mathrm{Ent}_\rho(\mathrm{MA}_\omega(\varphi)) + \nabla_{\theta_\rho}\mathrm{E}_\omega(\varphi) + c \tag{5.10}$$

in the present formalism, for a constant $c \in \mathbb{R}$ determined by the normalization $\mathrm{M}_\omega^\theta(0) = 0$, i.e. $c = -\mathrm{Ent}_\rho(\mu_\omega)$.

DEFINITION 5.9. *The* coercivity threshold *of (X,ω) is defined as $\sigma(X,\omega) := \sigma_\omega(\mathrm{F}_\omega)$.*

THEOREM 5.10. *The coercivity threshold of (X,ω) is a continuous function of the Kähler class $[\omega]$, and it satisfies*

$$\sigma(X,\omega) = \sup\left\{\sigma \in \mathbb{R} \mid \mathrm{M}_\omega \geq \sigma \mathrm{J}_\omega + A \text{ on } \mathcal{D}_\omega \text{ for some } A \in \mathbb{R}\right\}.$$

By [**CC21**], $\sigma(X,\omega)>0$ iff there exists a unique constant scalar curvature Kähler (cscK) metric in $[\omega]$. In particular, we recover the fact, originally proved in [**LS94**], that the set of Kähler classes of X that contain a unique cscK metric is open.

Proof. Pick a smooth metric ρ on K_X. By (5.8), we have $\sigma(X,\omega)=\sigma_\omega^{\theta_\rho}(\mathrm{Ent}_\rho)$, and the first point thus follows from Theorem 5.5. As to the second point, it is a simple consequence of (5.9) and Lemma 5.7. □

Remark 5.11. *Given $\theta\in\mathcal{Z}$, one can more generally consider the θ-*twisted free energy $\mathrm{F}_\omega^\theta := \mathrm{F}_\omega + \mathrm{J}_\omega^\theta$, *whose composition with the Monge–Ampère operator coincides with the θ-twisted K-energy $\mathrm{M}_\omega^\theta = \mathrm{M}_\omega + \nabla_\theta \mathrm{E}_\omega$ (see [**BDL17**, **CC21**]). Again, Theorem 5.5 shows that the coercivity threshold*

$$\sigma^\theta(X,\omega) := \sigma_\omega(\mathrm{F}_\omega^\theta) = \sigma_\omega^\theta(\mathrm{Ent}_\rho)$$

is a continuous function of $[\omega]$ and $[\theta]$, while Lemma 5.7 shows that

$$\sigma^\theta(X,\omega) := \sup\left\{\sigma\in\mathbb{R} \mid \mathrm{M}_\omega^\theta \geq \sigma \mathrm{J}_\omega + A \text{ on } \mathcal{D}_\omega \text{ for some } A\in\mathbb{R}\right\}$$

*as considered for instance in [**SD20**]. Combining this with [**CC21**], and assuming $\theta>0$, this implies that the set of Kähler classes of a compact Kähler manifold that contain a θ-twisted cscK metric is open—something that can also be directly obtained along the lines of [**LS94**].*

5.2.2. *The non-Archimedean case.* Next we consider a smooth, irreducible projective Berkovich space X over a non-Archimedean field k of characteristic 0. Pick a PL metric ρ on K_X, and define the associated *non-Archimedean entropy* $\mathrm{Ent}_\rho\colon \mathcal{M}\to\mathbb{R}\cup\{+\infty\}$ by setting

$$\mathrm{Ent}_\rho(\mu) := \int (\mathrm{A}_X - \rho)\,\mu.$$

Here A_X denotes the Temkin metric (see [**Tem16**] and also [**Ste19**]) on K_X, and $\mathrm{A}_X-\rho\colon X\to\mathbb{R}\cup\{+\infty\}$ is the corresponding lsc function, using additive notation for metrics (see [**BoJ17**, §5.7]). Again, Ent_ρ is convex and lsc, and (5.7) holds for any other choice of PL metric ρ' on K_X. See also [**Ino22**] for a related notion in the trivially valued case.

As above, one can then define the (non-Archimedean) *free energy* $\mathrm{F}_\omega\colon\mathcal{M}^1\to\mathbb{R}\cup\{+\infty\}$ by (5.8), with $\theta_\rho\in\mathcal{Z}$ the curvature of ρ. Its composition with the Monge–Ampère operator coincides with the (non-Archimedean) *Mabuchi K-energy* $\mathrm{M}_\omega\colon\mathcal{D}_\omega\to\mathbb{R}$, defined by (5.10).

However, the major difference in the non-Archimedean case is that the analogue of Lemma 5.7 is only a conjecture (compare [**BoJ18a**, Conjecture 2.5] and [**C.Li22**, Conjecture 4.4]). Explicitly:

Conjecture 5.12. *For each $\mu\in\mathcal{M}^1$, there exists a sequence (φ_i) of ω-psh PL functions such that $\mu_i := \mathrm{MA}_\omega(\varphi_i)$ satisfies $\mu_i\to\mu$ in $\mathcal{M}^1$ and $\mathrm{Ent}_\rho(\mu_i)\to\mathrm{Ent}_\rho(\mu)$.*

As a consequence, the coercivity threshold $\sigma(X,\omega):=\sigma_\omega(\mathrm{F}_\omega)$ only satisfies

$$\sigma(X,\omega)\leq \sup\left\{\sigma\in\mathbb{R}\mid \mathrm{M}_\omega\geq\sigma\mathrm{J}_\omega + A \text{ on } \mathcal{D}_\omega \text{ for some } A\in\mathbb{R}\right\}, \tag{5.11}$$

and equality holds if Conjecture 5.12 is valid.

These definitions are especially relevant when k is trivially valued and ω lies in $\mathrm{Amp}(X) \hookrightarrow \mathcal{Z}_+$ (see (1.6)). Indeed, the free energy $\mathrm{F}_\omega(\mu)$ then coincides with the invariant $\beta_\omega(\mu)$ introduced and studied in [**BoJ23**]; see also [**DL23**, **Liu23**]. By homogeneity with respect to the action of $\mathbb{R}_{>0}$, $\sigma(X,\omega) = \sigma_\omega(\mathrm{F}_\omega)$ is further equal to the *divisorial stability threshold* $\sigma_{\mathrm{div}}(X,\omega)$, which is positive iff (X,ω) is *divisorially stable*.

On the other hand, the right-hand side of (5.11) coincides, by definition, with the *K-stability threshold* $\sigma_K(X,\omega)$, which is positive iff (X,ω) is *uniformly K-stable* in the sense of [**Der16**, **BHJ17**]). Thus divisorial stability implies uniform K-stability, and the converse holds if Conjecture 5.12 is satisfied.

Remark 5.13. *Assume that (X,L) is a polarized smooth projective variety over $\mathbb{C}$, and pick a Kähler form $\omega \in c_1(L)$. Using* [**SD18**] *and* [**C.Li22**]*, one can then show that the above thresholds satisfy*

$$\sigma_{\mathrm{div}}(X,L) \le \sigma(X,\omega) \le \sigma_K(X,L).$$

Conjecture 5.12 would further yield $\sigma_K(X,L) = \sigma_{\mathrm{div}}(X,L)$, and hence conclude the proof of the 'uniform' version of the Yau–Tian–Donaldson conjecture, as noted in [**C.Li22**, **BoJ23**].

Appendix A. Convexity estimates

We consider the following data:

- a surjective map $\pi\colon V \to \Theta$ of $\mathbb{R}$-vector spaces, with fibers $V_\theta := \pi^{-1}(\theta)$;
- a homogeneous polynomial $F\colon V \to \mathbb{R}$ of degree $n+1$, $n \ge 1$, with associated symmetric multilinear map
$$V^{n+1} \to \mathbb{R} \quad (x_0,\dots,x_n) \mapsto x_0 \cdot \ldots \cdot x_n,$$
i.e. $F(x) = x^{n+1}$;
- a convex cone $P \subset V$ such that

$$x^2 \cdot x_2 \cdot \ldots \cdot x_n \ge 0 \text{ for all } x \in V_0 \text{ and } x_i \in P. \tag{A.1}$$

Example A.1. *The main example that we have in mind is the negative of the energy pairing in §1.4, where $V = \mathcal{Z} \times \mathcal{D}$ with its projection to $\Theta = \mathcal{Z}$, and $P = \{(\theta,\varphi) \in V \mid \theta_\varphi \ge 0\}$. Another example is given by the negative of intersection pairing on a flat projective scheme over $\mathbb{Z}$.*

Our goal is to use (A.1) and the resulting *Cauchy-Schwarz* inequality

$$(x \cdot y \cdot x_2 \cdot \ldots \cdot x_n)^2 \le (x^2 \cdot x_2 \cdot \ldots \cdot x_n)(y^2 \cdot x_2 \cdot \ldots \cdot x_n) \tag{A.2}$$

for all $x, y \in V_0$ and $x_i \in P$ to derive various inequalities and convexity statements.

For each $x \in V$ we define the linear form $F'(x) \in V^\vee$ by

$$\langle F'(x), y\rangle := \frac{d}{dt}\Big|_{t=0} F(x+ty) = (n+1)x^n \cdot y,$$

and we set for all $x, y \in V$

$$\delta(x,y) := F(x) - F(y) - \langle F'(y), x-y\rangle. \tag{A.3}$$

A simple computation yields

$$\delta(x,y) = \sum_{j=0}^{n-1} (j+1)(x-y)^2 \cdot y^j \cdot x^{n-1-j}. \tag{A.4}$$

In what follows, we fix $\theta \in \Theta$ and set $P_\theta := V_\theta \cap P$.

LEMMA A.2. *We have $\delta(x,y) \geq 0$ for $x, y \in P_\theta$. Moreover, F is convex on P_θ, and for every $y \in P_\theta$ we have that $x \mapsto \delta(x,y)$ is convex.*

PROOF. By (A.1) and (A.4), we have $\delta(x,y) \geq 0$ for $x, y \in P_\theta$, and this implies that F is convex on P_θ; it then follows from (A.3) that $x \mapsto \delta(x,y)$ is convex. □

THEOREM A.3. *For all $x, y, z \in P_\theta$ and $t \in [0,1]$, the following holds:*

- quasi-symmetry*:*
$$\delta(x,y) \approx \delta(y,x);$$
- quasi-triangle inequality*:*
$$\delta(x,z) \lesssim \delta(x,y) + \delta(y,z);$$
- quadratic estimate*:*
$$\delta(x,(1-t)x+ty) \lesssim t^2 \delta(x,y);$$
- uniform convexity*:*
$$[(1-t)F(x) + tF(y)] - F((1-t)x+ty) \gtrsim t(1-t)\delta(x,y).$$

For any base point $x_* \in P_\theta$ and $x_i, y_i, z_j \in P_\theta$, the following Hölder estimates *further hold:*

$$|(x_0 - y_0) \cdot (x_1 - y_1) \cdot z_2 \cdot \ldots \cdot z_n| \lesssim \delta(x_0,y_0)^\alpha \delta(x_1,y_1)^\alpha M^{1-2\alpha}; \tag{A.5}$$

$$|\langle F'(x_0) - F'(y_0), x_1 - y_1 \rangle| \lesssim \delta(x_0,y_0)^{1/2} \delta(x_1,y_1)^\alpha M^{1/2-\alpha}; \tag{A.6}$$

$$|\delta(x_0,x_1) - \delta(y_0,y_1)| \lesssim \max\{\delta(x_0,y_0), \delta(x_1,y_1)\}^\alpha M^{1-\alpha}; \tag{A.7}$$

with $\alpha := 2^{-n} \in (0,1/2]$ and $M = \max_\xi \delta(\xi, x_*)$, where in each case ξ ranges over the elements of P_θ appearing in the left-hand side of the inequality.

The strategy to get these types of Hölder estimates goes back to [**BBGZ13**, **BBEGZ19**, **BoJ22**], building upon an original idea of [**Blo03**]. In the rest of this section we prove Theorem A.3, largely following [**BoJ22**, §3.3].

Given $x, y \in P_\theta$, we set
$$d(x,y) := \max_{0 \leq j \leq n-1} (x-y)^2 \cdot y^j \cdot x^{n-1-j}.$$

Using (A.4) it is then clear that
$$\delta(x,y) \approx \delta(y,x) \approx d(x,y) \approx (x-y)^2 \cdot (\tfrac{1}{2}(x+y))^{n-1}. \tag{A.8}$$

To prove the inequality $\delta(x,(1-t)x+ty) \lesssim t^2 \delta(x,y)$ it suffices to prove the corresponding inequality $d(x,(1-t)x+ty) \lesssim t^2 d(x,y)$. But
$$d(x,(1-t)x+ty) = t^2 \max_{0 \leq j \leq n-1} (x-y)^2 \cdot ((1-t)x+ty)^j \cdot x^{n-1-j} \lesssim t^2 d(x,y)$$
using multilinearity and the binomial theorem.

Next we prove what is essentially a special case of (A.5).

LEMMA A.4. *If $x, y, z \in P_\theta$, then*
$$(x-y)^2 \cdot z^{n-1} \lesssim d(x,y)^{2\alpha} \max\{d(x,z), d(y,z)\}^{1-2\alpha}.$$

Proof. Set $w := \frac{1}{2}(x+y)$, $A := d(x,y)$, $B := \max\{d(x,z), d(y,z)\}$, and

$$b_j := (x-y)^2 \cdot z^j \cdot w^{n-1-j}$$

for $0 \le j \le n-1$. Then $b_0 \approx A$, and our goal is to show $b_{n-1} \lesssim A^{2\alpha} B^{1-2\alpha}$. By the triangle inequality for the seminorm $v \mapsto \sqrt{v^2 \cdot z^{n-1}}$ on V_0, we have

$$b_{n-1} \le (\sqrt{(x-z)^2 \cdot z^{n-1}} + \sqrt{(y-z)^2 \cdot z^{n-1}})^2 \le 4B.$$

If $A \ge B$, then $b_{n-1} \le 4B \le 4A^{2\alpha} B^{1-2\alpha}$ and we are done, so we may assume $A \le B$. In this case, we show by induction that

$$b_j \lesssim A^{2^{-j}} B^{1-2^{-j}}$$

for $0 \le j \le n-1$. The case $j = 0$ is clear, so suppose $0 \le j \le n-2$, and note that

$$\begin{aligned} b_{j+1} - b_j &= (x-y)^2 \cdot (z-w) \cdot z^j \cdot w^{n-2-j} \\ &= (x-y) \cdot (z-w) \cdot x \cdot z^j \cdot w^{n-2-j} - (x-y) \cdot (z-w) \cdot y \cdot z^j \cdot w^{n-2-j}. \end{aligned}$$

Here we can use the Cauchy-Schwartz inequality to estimate the last two terms. For example:

$$\begin{aligned} &|(x-y) \cdot (z-w) \cdot x \cdot z^j \cdot w^{n-2-j}|^2 \\ &\qquad \le ((x-y)^2 \cdot x \cdot z^j \cdot w^{n-2-j})((z-w)^2 \cdot x \cdot z^j \cdot w^{n-2-j}). \end{aligned}$$

Using that $2w - x = y \in P$, we can bound the first factor by $2b_j$, and the second factor by $2(z-w)^2 \cdot z^j \cdot w^{n-1-j} \le 2d(z,w)$. Adding the two terms, we get $b_{j+1} - b_j \le 4\sqrt{b_j}\sqrt{d(z,w)}$. Now $d(z,w) \approx \delta(w,z) \le \max\{\delta(x,z), d(y,z)\} \approx B$ using the convexity of $\delta(\cdot, z)$, see Lemma A.2. All in all, this yields

$$b_{j+1} - b_j \lesssim \sqrt{b_j B}.$$

for $0 \le j \le n-2$. Using the induction hypothesis $b_j \lesssim A^{2^{-j}} B^{1-2^{-j}}$, we get

$$b_{j+1} \lesssim b_j + \sqrt{B b_j} \lesssim A^{2^{-j}} B^{1-2^{-j}} + A^{2^{-j-1}} B^{1-2^{-j-1}} \le 2A^{2^{-j-1}} B^{1-2^{-j-1}},$$

where the last inequality follows from our assumption that $A \le B$. We are done. □

Using Lemma A.4, we can now prove the quasi-triangle inequality for δ, or equivalently for d. Fix $x, y, z \in P_\theta$, and set $w := \frac{1}{2}(x+y)$. Then

$$\begin{aligned} d(x,y) &\approx (x-y)^2 \cdot w^{n-1} \lesssim (x-z)^2 \cdot w^{n-1} + (y-z)^2 \cdot w^{n-1} \\ &\lesssim \max\{d(x,z), d(y,z)\})^{2\alpha} \max\{d(x,w), d(y,w), d(z,w)\}^{1-2\alpha}, \end{aligned}$$

by the triangle inequality for the norm $v \mapsto \sqrt{v^2 \cdot w^{n-1}}$ and by Lemma A.4. As noted above, the convexity of $\delta(\cdot, z) \approx d(\cdot, z)$ gives $d(z,w) \lesssim \max\{d(x,z), d(y,z)\}$, as well as $d(x,w), d(y,w) \lesssim d(x,y)$. Thus

$$d(x,y) \lesssim \max\{d(x,z), d(y,z)\})^{2\alpha} \max\{d(x,y), d(x,z), d(y,z)\}^{1-2\alpha},$$

which easily implies $d(x,y) \le \max\{d(x,z), d(y,z)\}$, as desired.

Next we prove (A.5) in general. By the Cauchy–Schwartz inequality we may assume $x_0 = x_1$ and $y_0 = y_1$. We may also assume $n \ge 2$, or else we are done by (A.8). Set $z := \frac{1}{n-1}(z_2 + \cdots + z_n)$. Then

$$\begin{aligned} (x_1 - y_1)^2 \cdot z_2 \cdot \ldots \cdot z_n \lesssim (x_1 - y_1)^2 \cdot z^{n-1} &\lesssim d(x_1, y_1)^{2\alpha} \max\{d(x_1, z), d(y_1, z)\}^{1-2\alpha} \\ &\approx \delta(x_1, y_1)^{2\alpha} \max\{\delta(x_1, z), \delta(y_1, z)\}^{1-2\alpha}. \end{aligned}$$

By the quasi-triangle inequality we have $\delta(x_1, z) \lesssim \max\{\delta(x_1, x_*), \delta(z, x_*)\}$, and by quasi-symmetry and Lemma A.2 we have $\delta(z, x_*) \lesssim \max_{i\geq 2} \delta(z_i, x_*)$. A similar estimate for $\delta(y_1, z)$ completes the proof of (A.5).

Next we prove (A.6), which is equivalent to

$$|(x_0^n - y_0^n) \cdot (x_1 - y_1)| \leq \delta(x_0, y_0)^{1/2} \delta(x_1, y_1)^\alpha M^{1/2-\alpha}.$$

By the Cauchy–Schwartz inequality, we have

$$\begin{aligned} |(x_0^n - y_0^n) \cdot (x_1 - y_1)|^2 &= |(x_0 - y_0)(x_1 - y_1) \sum_{j=0}^{n-1} x_0^j y_0^{n-1-j}|^2 \\ &\leq \left((x_0 - y_0)^2 \sum_{j=0}^{n-1} x_0^j y_0^{n-1-j} \right) \left((x_1 - y_1)^2 \sum_{j=0}^{n-1} x_0^j y_0^{n-1-j} \right). \end{aligned}$$

Here the first factor on the right is $\approx \delta(x_0, y_0)$, whereas the second factor can be bounded above using (A.5).

It only remains to prove (A.7). By the quasi-triangle inequality for δ, it suffices to consider the case when $x_0 = y_0$ or $x_1 = y_1$. Now

$$\begin{aligned} \delta(x_0, x_1) - \delta(x_0, y_1) &= n(x_1^{n+1} - y_1^{n+1}) - (n+1)x_0(x_1^n - y_1^n) \\ &= (x_1 - y_1) \sum_{j,k} (x_1^j y_1^{n-j} - x_0 x_1^k y_1^{n-k-1}). \end{aligned}$$

If $j \leq k$, then

$$x_1^j y_1^{n-j} - x_0 x_1^k y_1^{n-k-1} = x_1^j y_1^{n-k-1} (y_1^{k-j+1} - x_0^{k-j+1}) + x_1^j y_1^{n-k-1} x_0 (x_0^{k-j} - x_1^{k-j}),$$

and by factoring each term of the right-hand side we see from (A.5) that

$$|(x_1 - y_1)(x_1^j y_1^{n-k} - x_0 x_1^j y_1^{n-k-1})| \lesssim \delta(x_1, y_1)^\alpha M^{1-\alpha}.$$

The case when $j > k$ is handled in a similar way, and adding all the terms yields $|\delta(x_0, x_1) - \delta(x_0, y_1)| \lesssim \delta(x_1, y_1)^\alpha M^{1-\alpha}$.

A similar argument shows that $|\delta(x_0, x_1) - \delta(y_0, x_1)| \lesssim \delta(x_0, y_0)^\alpha M^{1-\alpha}$, and completes the proof.

Appendix B. Regularization and orthogonality on Kähler spaces

By relying on a variant of the classical Richberg regularization technique, it was proved in [**BK07**] that any ω-psh function on a compact Kähler manifold (X, ω) can be written as the limit of a decreasing sequence of smooth ω-psh functions. It is natural to hope that this holds in the singular case as well:

CONJECTURE B.1. *Let (X, ω) be a compact Kähler space. Then any ω-psh function φ on X can be written as the pointwise limit of a decreasing sequence (φ_i) of smooth ω-psh functions.*

Note that the conclusion only depends on the Kähler class $[\omega]$. Besides the case when X is nonsingular,[6] mentioned above, we have:

[6] Conjecture B.1 is now known to hold for any normal space X as a consequence of the recent work [**CC24**] combined with [**Ric68**].

Example B.2. *Conjecture B.1 holds if X is projective and $[\omega]$ lies in the ample cone, i.e. the open convex cone generated by classes of ample line bundles on X. This is a consequence of* [**CGZ22**, Theorem 1.1].

Lemma B.3. *Let (X,ω) be a compact Kähler space for which Conjecture B.1 holds. Then $[\omega]$ has the orthogonality property (cf. Definition 2.15).*

Proof. Pick a resolution of singularities $\pi\colon Y \to X$, and set $\theta := \pi^\star\omega \geq 0$. Since $\int \theta^n = \int \omega^n > 0$, the $(1,1)$-class $[\theta]$ is semipositive and big. For any $g \in \mathrm{C}^0(Y)$, consider the θ-psh envelope

$$\mathrm{P}_\theta(g) := \sup\{\psi \in \mathrm{PSH}(Y,\theta) \mid \psi \leq g\}.$$

As is well-known, $\mathrm{P}_\theta(g)$ is θ-psh, and $\mathrm{MA}_\theta(\mathrm{P}_\theta(g))$ is supported in $\{\mathrm{P}_\theta(g) = g\}$, i.e.

$$\int_Y (g - \mathrm{P}_\theta(g))\,\mathrm{MA}_\theta(\mathrm{P}_\theta(g)) = 0, \tag{B.1}$$

see for instance [**BB10**, Proposition 2.10]. For any $f \in \mathrm{C}^0(X)$, we next claim that $\mathrm{P}_\theta(\pi^\star f)$ is the limit in $\mathcal{E}^1(Y,\theta)$ (the space of θ-psh functions of finite energy) of the increasing net $\{\pi^\star\varphi\}_{\varphi\in\mathcal{D}_{\omega,<f}}$. Assume this for the moment. By continuity of Monge–Ampère integrals along increasing nets in $\mathcal{E}^1(Y,\theta)$, and using

$$\int_X (f-\varphi)\,\mathrm{MA}_\omega(\varphi) = \int_Y (\pi^\star f - \pi^\star\varphi)\,\mathrm{MA}_\theta(\pi^\star\varphi),$$

for any $\varphi \in \mathcal{D}_\omega$, we infer

$$\lim_{\varphi\in\mathcal{D}_{\omega,<f}} \int_X (f-\varphi)\,\mathrm{MA}_\omega(\varphi) = \int_Y (\pi^\star f - \mathrm{P}_\theta(\pi^\star f))\,\mathrm{MA}_\theta(\mathrm{P}_\theta(\pi^\star f)) = 0, \tag{B.2}$$

by (B.1).

To prove the claim, note first that Conjecture B.1 implies that the increasing net $\{\varphi\}_{\varphi\in\mathcal{D}_{\omega,<f}}$ converges pointwise to

$$\mathrm{P}_\omega(f) := \sup\{\psi \in \mathrm{PSH}(X,\omega) \mid \psi \leq f\}.$$

Indeed, given $\delta > 0$ and a function $\psi \in \mathrm{PSH}(X,\omega)$ with $\psi \leq f$, Conjecture B.1 and a Dini-type argument guarantees the existence of a function $\psi' \in \mathcal{D}_{\omega,<f}$ with $\psi \leq \psi' < f + \delta$. The claim is thus equivalent to the statement that $\pi^\star\mathrm{P}_\omega(f)$ coincides a.e. with $\varphi := \mathrm{P}_\theta(\pi^\star f)$. To prove this, pick $\tau \in \mathrm{PSH}(X,\omega)$ with $\tau \leq f$ and $\{\tau = -\infty\} = X_{\mathrm{sing}}$ (compare the proof of Lemma 1.28). Since $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\pi^\star\tau$ is $\pi^\star\omega$-psh outside $\pi^{-1}(X_{\mathrm{sing}})$, and $\psi_\varepsilon \equiv -\infty$ on the latter, we have $\psi_\varepsilon = \pi^\star\varphi_\varepsilon$ for a unique $\varphi_\varepsilon \in \mathrm{PSH}(X,\omega)$ (see [**Dem85**, Théorème 1.10]). Since $\pi^\star\varphi_\varepsilon \leq \pi^\star f$, and hence $\varphi_\varepsilon \leq f$, we have $\varphi_\varepsilon \leq \mathrm{P}_\omega(f)$. Thus

$$(1-\varepsilon)\varphi + \varepsilon\pi^\star\tau = \pi^\star\varphi_\varepsilon \leq \pi^\star\mathrm{P}_\omega(f) \leq \varphi,$$

and hence $\pi^\star\mathrm{P}_\omega(f) = \mathrm{P}_\theta(\pi^\star f)$ on $Y \setminus \pi^{-1}(X_{\mathrm{sing}})$. □

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