ASYMPTOTIC BEHAVIOR OF A DIFFUSED INTERFACE VOLUME-PRESERVING MEAN CURVATURE FLOW

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Abstract. We consider a diffused interface version of the volume-preserving mean curvature flow in the Euclidean space, and prove, in every dimension and under natural assumptions on the initial datum, exponential convergence towards single "diffused balls".

Contents

1. Introduction

1.1. **Overview.** In this paper we introduce a PDE reformulation of the classical volume-preserving mean curvature flow (VPMCF) in $\mathbb{R}^n$ where the role of the perimeter functional is played by the Allen–Cahn energy. From the physical viewpoint this reformulation seems well justified, since it consists in replacing the classical *sharp* interface model for surface tension based on perimeter minimization with the equally interesting and important *diffused* interface model based on the minimization of the Allen–Cahn energy. From the mathematical viewpoint working in the diffused setting clears up the analysis from the ambiguities brought in by the formation of singularities characteristic of geometric flows, which are directly reflected into the abundance of non-equivalent weak formulations of the VPMCF.

Our main result proves, in every dimension and under a variety of natural assumptions on the initial datum, exponential convergence of the diffused VPMCF towards single "diffused balls". This result is indeed stronger than the presently known analogous results for the classical VPMCF, see Remark 1.1 below.

The diffused interface model we adopt is based on the volume-constrained minimization of the classical Allen–Cahn functional

$$\mathcal{AC}_\varepsilon(u) = \varepsilon \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^n} W(u) , \qquad \varepsilon > 0 , \tag{1.1}$$

defined on (dimensionless) density functions $u : \mathbb{R}^n \to [0,1]$, and requiring the choice of (dimensionless) double-well potential $W : [0,1] \to [0,\infty)$. We will always require that $W \in C^{2,1}[0,1]$ and that W satisfies the standard non-degeneracy assumptions

$$W(0) = W(1) = 0 , \quad W > 0 \text{ on } (0,1) , \quad W''(0), W''(1) > 0 , \tag{1.2}$$

as well as the normalization

$$\int_0^1 \sqrt{W} = 1\,. \tag{1.3}$$

As usual, ε has the dimensions of length, so that $\mathcal{AC}_\varepsilon(u)$ has the dimensions of (codimension one) area.

Particular care must be put in the choice of the volume potential $V:[0,1]\to[0,\infty)$ used to impose the volume constraint on u. Indeed, while any choice of V satisfying $V(1)>0$ and $V(0)=0$ will return the correct volume constraint in the sharp interface limit $\varepsilon\to 0^+$ (and will thus be acceptable form the physical viewpoint), not every choice of V will result in a mathematical model that is either well-posed or feasible of in-depth analysis.

A natural choice for V is suggested by the classical isoperimetric lower bound on $\mathcal{AC}_\varepsilon(u)$, and consists in taking

$$V(r)=\Phi(r)^{n/(n-1)}\,,\ \text{where } \Phi(r)=\textstyle\int_0^r\sqrt{W}\ \text{ for } r\in[0,1]\,. \tag{1.4}$$

Indeed, by a classical application of the Cauchy–Schwartz inequality and the chain rule, we have that

$$\varepsilon\,|\nabla u|^2+\frac{W(u)}{\varepsilon}\ge 2\,|\nabla u|\,\sqrt{W(u)}=2\,|\nabla(\Phi\circ u)|\,,$$

from which the isoperimetric lower bound

$$\mathcal{AC}_\varepsilon(u)\ge 2\,|D[\Phi\circ u]|(\mathbb{R}^n)\ge 2\,n\,\omega_n^{1/n}\,\mathcal{V}(u)^{(n-1)/n}\,,\qquad \mathcal{V}(u):=\int_{\mathbb{R}^n}V(u)\,, \tag{1.5}$$

follows, so that[1]

$$n\,\omega_n^{1/n}\,m^{(n-1)/n}=P(B^{(m)})=\inf\{P(E):|E|=m\}\,,\qquad m>0\,,$$

is the optimal value of the **(Euclidean) isoperimetric problem**.

Thanks to (1.5), our choice (1.4) of V is instrumental for obtaining a well-posed **diffused interface (Euclidean) isoperimetric problem**,

$$\Psi(\varepsilon,m)=\inf\Big\{\mathcal{AC}_\varepsilon(u):\mathcal{V}(u)=m\,,u\in L^1_{\mathrm{loc}}(\mathbb{R}^n;[0,1])\Big\}\,,\qquad \varepsilon\,,m>0\,. \tag{1.6}$$

Indeed, by (1.5) and by the (sharp) isoperimetric inequality for functions of bounded variation, we see that

$$\Psi(\varepsilon,m)>2\,n\,\omega_n^{1/n}\,m^{(n-1)/n}\,,\qquad \forall\varepsilon,m>0\,, \tag{1.7}$$

while a simple comparison argument shows that this inequality is saturated in the limit as $\varepsilon\to 0^+$. It is important to keep in mind that simpler choices of V, like $V(t)=t$ or $V(t)=t^2$, would have led[2] to degenerate minimization problems where every competitor has positive energy but where the infimum of the energy is equal to zero.

Problem $\Psi(\varepsilon,m)$ has been studied in great detail in [MR24]. Some of the results obtained therein will play an important role in the present paper, and will therefore be summarized in Section 2. For the moment, with the sole intent of formulating our main result as quickly as possible, we just recall from [MR24] that, in the physical regime where $\varepsilon\in(0,\varepsilon_0\,m^{1/n})$ for some *universal constant*[3] ε_0, there is a unique minimizer $\zeta_{\varepsilon,m}$

[1] Here, $|Dv|$ denotes the total variation measure of $v\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, ω_n is the volume of the unit radius ball in $\mathbb{R}^n$, $|E|$ and $P(E)$ denote the volume and perimeter of $E\subset\mathbb{R}^n$, and $B^{(m)}$ stands for the ball of volume m with center at the origin in $\mathbb{R}^n$.

[2] The size of $V(t)$ for $t\to 0^+$ plays a crucial role here. Our choice of V satisfies $V(t)=\mathrm{O}(t^{2\,n/(n-1)})$ as $t\to 0^+$. This is not the only property of V that plays an important role in our analysis though, and the close relation between V and W will be repeatedly used.

[3] By universal constant we mean a constant depending only on the dimension n and on the double-well potential W. By $C(a,b,\dots)$ we denote a constant depending only on n, W, and the arguments a, b, etc..

in the class of the radially symmetric, strictly decreasing, and everywhere positive functions on $\mathbb{R}^n$ with maximum at the origin; and that u is a minimizer of $\Psi(\varepsilon,m)$ if and only if $u=\tau_{x_0}[\zeta_{\varepsilon,m}]$ for some[4] $x_0\in\mathbb{R}^n$. Each $\zeta_{\varepsilon,m}$ solves the **diffused constant mean curvature equation**

$$2\,\varepsilon^2\,\Delta\zeta_{\varepsilon,m}=W'(\zeta_{\varepsilon,m})-\varepsilon\,\Lambda_{\varepsilon,m}\,V'(\zeta_{\varepsilon,m})\qquad\text{on }\mathbb{R}^n\,,$$

where $\Lambda_{\varepsilon,m}\to 2\,(n-1)\,\omega_n^{1/n}\,m^{-1/n}$ as $\varepsilon\to 0^+$ (and is thus positive in the physical regime $\varepsilon\in(0,\varepsilon_0\,m^{1/n})$).

The **diffused VPMCF** is then defined as the L^2-gradient flow of $\mathcal{AC}_\varepsilon$ with a Lagrange multiplier modification that preserves $\mathcal{V}$ along the flow: that is, we consider the parabolic initial value problem[5]

$$\begin{cases}\varepsilon^2\,\partial_t u=2\,\varepsilon^2\,\Delta u-W'(u)+\varepsilon\,\lambda_\varepsilon[u(t)]\,V'(u)\,, & \text{on }\mathbb{R}^n\times(0,\infty)\,,\\ u(0)=u_0\,,\end{cases}\tag{DF}$$

where we have introduced the Lagrange multiplier functional[6]

$$\lambda_\varepsilon[v]=\frac{\int_{\mathbb{R}^n}2\,\varepsilon^2\,|\nabla v|^2\,V''(v)+W'(v)\,V'(v)}{\varepsilon\,\int_{\mathbb{R}^n}V'(v)^2}\,,\tag{1.8}$$

whose choice guarantees, on smooth solutions of the flow, that

$$\frac{d}{dt}\,\mathcal{V}(u(t))=\int_{\mathbb{R}^n}V'(u(t))\,\partial_t u(t)=0\,,\qquad\text{i.e., }\mathcal{V}(u(t))=\mathcal{V}(u_0)\text{ for all }t>0\,.$$

From the viewpoint of classical parabolic theory, (DF) presents some peculiar features since it is a non-autonomous semilinear PDE, where the non-autonomy is due to the Lagrange multiplier $\lambda_\varepsilon[u(t)]$, which, in turn, is non-local in space (its determination requires knowledge of $u(t)$ over the whole $\mathbb{R}^n$). The following theorem is our main result concerning the long-time behavior of (DF).

Theorem 1.1. *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies (1.2) and (1.3), then there exists a universal constant $\varepsilon_0>0$ with the following property. If $\varepsilon\in(0,\varepsilon_0)$, $u_0\in W^{2,p}(\mathbb{R}^n;[0,1])$ for all $p\ge 2$, $\mathcal{V}(u_0)=1$, and*

$$\text{either }\mathcal{AC}_\varepsilon(u_0)<2\,\Psi(\varepsilon,1/2)\,,\tag{1.9}$$

$$\text{or }\operatorname{spt}u_0\text{ is compact}\,,\tag{1.10}$$

then

$$0\le\mathcal{AC}_\varepsilon(u(t))-\Psi(\varepsilon,1)\le C(\varepsilon,u_0)\,e^{-t/C(\varepsilon)}\,,\qquad\forall t>\frac{1}{C(\varepsilon,u_0)}\,,\tag{1.11}$$

and there exists a unique $x_0\in\mathbb{R}^n$ (depending on ε and u_0) such that for all $p>2$ and $t>1/C(\varepsilon,u_0)$

$$\big\|u(t)-\tau_{x_0}[\zeta_{\varepsilon,1}]\big\|_{(W^{2,p}\cap W^{1,2})(\mathbb{R}^n)}\le C(u_0,\varepsilon,p)\,e^{-t/C(\varepsilon)}\,.\tag{1.12}$$

Remark 1.1 (Asymptotic analysis of the VPMCF)**.** It is convenient to briefly review the state of the art concerning convergence to equilibrium for the classical VPMCF (with the disclaimer that, due to singularities formation, these various results may pertain to different weak formulations of the VPMCF). First, convergence to a single ball has been proved under a variety of suitable geometric restrictions on the initial datum that can be shown to be preserved along the flow, and that *exclude* singularities formation: these are uniform

[4] We set $\tau_{x_0}[v](x)=v(x-x_0)$ for every $x,x_0\in\mathbb{R}^n$ and $v:\mathbb{R}^n\to\mathbb{R}^m$.

[5] Given $t\ge 0$ and $u:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$, we set $u(t):\mathbb{R}^n\to\mathbb{R}$ for the function defined by $u(t)(x):=u(x,t)$ ($x\in\mathbb{R}^n$).

[6] Notice that $\lambda_\varepsilon[v]$ is defined in $[0,\infty]$ on any $v:\mathbb{R}^n\to[0,1]$ with $|\{0<v<1\}|>0$ – this condition guarantees indeed that $\int_{\mathbb{R}^n}V'(v)^2>0$.

convexity [Hui87], C^1-proximity to a sphere [ES98], star-shapedeness [KK20], or integral pinching conditions [Li09]. In general, singularity formation may lead to convergence to multiple balls, a phenomenon called *bubbling* (see [FJM22, Theorem 1.4] for an example) and it is thus unclear for which class of initial data one should expect convergence towards a single ball (of the same volume as the initial datum) or towards multiple balls (all with a same volume equal to a fraction of the initial one). Since perimeter decreases along the VPMCF, a natural **conjecture** is that, if a unit volume initial datum has perimeter *strictly* less than *twice* the perimeter of a ball of volume $1/2$, then convergence to a single ball should follow (with exponential rates of convergence). This conjecture has been proved for "flat flow" solutions of the VPMCF, and in dimensions $n = 2$ and $n = 3$ respectively, in the recent papers [JMPS23, JMOS24]. Moreover, again when $n = 2, 3$, in [JN23] it is proved that flat flow solutions always resolve, as $t \to \infty$, as finite union of balls with possibly moving centers. All these results are based on powerful quantitative bubbling results for sets with L^2-small mean curvature oscillation. The restriction to dimensions $n = 2, 3$ is strongly correlated, on the one hand, to the fact that the L^2-oscillation of the mean curvature is the dissipation of the VPMCF, and is thus the quantity to work with in this setting; and that, on the other hand, the critical L^p-space for the regularity theory of the mean curvature of a boundary in $\mathbb{R}^n$ is $p = n - 1$. For these reasons, the further extension of these methods to dimensions $n \geq 4$ seems delicate.

Remark 1.2 (On assumption (1.9))**.** Assumption (1.9) amounts to asking that the initial datum has strictly less energy than twice that of two diffused balls of volume $1/2$. Hence, by proving Theorem 1.1 under (1.9) we have proved the validity, in every dimension, of the diffused analogue of the VPMCF-conjecture mentioned in Remark 1.1.

Remark 1.3 (On assumption (1.10))**.** Proving Theorem 1.1 under assumption (1.10) is somehow more striking than doing so under assumption (1.9), since (1.10) allows for initial data with arbitrarily large energy as well as for initial data that is arbitrarily close to the characteristic functions of *any* bounded set with finite perimeter and unit volume. Indeed, if $E \subset \mathbb{R}^n$ is a bounded set of finite perimeter with $|E| = 1$, then by a standard construction, we can find a family $\{v_\varepsilon\}_{\varepsilon>0}$ of smooth and compactly supported functions on $\mathbb{R}^n$ such that $v_\varepsilon \to 1_E$ in $L^1(\mathbb{R}^n)$ and $\mathcal{AC}_\varepsilon(v_\varepsilon) \to 2\,P(E)$ as $\varepsilon \to 0^+$, with $\operatorname{spt} v_\varepsilon \subset\subset \{x : \operatorname{dist}(x, E) < 1\}$ for all $\varepsilon > 0$.

1.2. **Open problems and metastability.** Before presenting the proof of Theorem 1.1, and the various intermediate results behind it, we briefly introduce some interesting problems related to Theorem 1.1.

A first natural problem is addressing the existence of non-compactly supported initial data such that the resulting flow does not converge to a single diffused ball, but rather resolves into a superposition of time-drifting diffused balls (compare with conclusion (1.21) in Theorem 1.4). Since the physical or numerical relevance of non-compactly supported initial data is unclear, this is probably a question of very theoretical flavor; still, answering to it may be challenging.

In the $\varepsilon \to 0^+$-limit, and for suitably prepared initial data, the diffused VPMCF should converge to a weak formulation of the VPMCF, and, indeed, this kind of convergence has been established, in absence of volume-preserving Lagrange multipliers, in [Ilm93, BOS06], and for the VPMCF but under spherical symmetry assumptions on Ω, in [BS97]. For this reason, another natural problem related to Theorem 1.1 would be understanding whether the ε-dependency of the decay rates (1.11) and (1.12) can be dropped off or not: the corresponding ε-independent decay rates could then be transferred to the VPMCF. It seems natural to conjecture that, if u_0 is compactly supported *and* is such that $\mathcal{AC}_\varepsilon(u_0) < 2\,\Psi(\varepsilon, 1/2)$ (that is, if both (1.9) and (1.10) hold), then ε-independent decay rates to a single diffused ball hold true, thus providing a strategy to extend the results of [JN23, JMPS23, JMOS24] to arbitrary dimensions.

We do not expect, however, that ε-independent decay rates to a single diffused ball should hold for a generic compactly supported initial datum u_0: such rates should hold, at best, only for $t > T_\varepsilon = T_\varepsilon(u_0)$ with $T_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Indeed, a flow starting from a superposition of two diffused balls truncated so to have compact support should spend a time $T_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ close to its "metastable" initial datum, before eventually converging to a single diffused ball.

Evidence in support of such metastable scenario can be found in a series of "slow-motion" results concerning the initial value problem

$$\begin{cases} \partial_t v = 2\,\varepsilon^2\,\Delta v - W'(v) + \varepsilon\,\mu_\varepsilon[v(t)]\,, & \text{on } \Omega \times (0,\infty)\,, \\ \nabla_{\nu_\Omega} v(t) = 0\,, & \text{on } \partial\Omega\,, \\ v(0) = u_0\,, & \text{on } \Omega\,, \end{cases} \tag{1.13}$$

defined on a bounded open set $\Omega \subset \mathbb{R}^n$ with regular boundary, and involving the Lagrange multiplier $\mu_\varepsilon[v] = \int_\Omega W'(v)/(\varepsilon\,|\Omega|)$, where the choice[7] of μ_ε is such that $\int_\Omega v(t) = \int_\Omega u_0$ for all $t \ge 0$. Problem (1.13) has indeed been the object of study in several papers, as we are now going to informally[8] review. Before doing that, let us stress that in the slow-motion literature it is customary to work with the parabolic operator $\partial_t v - 2\,\varepsilon^2\,\Delta v$ in place of the one used in (DF), namely, $\varepsilon^2\,(\partial_t u - 2\,\Delta u)$. In practice this is a small difference, since one can pass from one setting to the other by just rescaling solutions in time, according to the relation $v(t) = u(\varepsilon^2\,t)$. The choice made in (DF) seems more natural, since the resulting flow is the one that is expected to converge, in the $\varepsilon \to 0^+$-limit, to the VPMCF. It is important to keep this difference in mind when comparing Theorem 1.1 to results from the slow-motion literature, which are typically formulated on (1.13).

In dimension $n = 1$, when Ω is an interval, building up on the pioneering work [BK90, BK91], in [Gra95] (see also [BHNN15]) the following result is proved: if $u_0 : \Omega \to \{0,1\}$ has N-many jumps and $\{u_0^\varepsilon\}_\varepsilon$ is a family of initial data for (1.13) such that $\|u_0^\varepsilon - u_0\|_{L^1(\Omega)} \to 0$ as $\varepsilon \to 0^+$, then solutions $\{v^\varepsilon(t)\}_\varepsilon$ of (1.13) are such that

$$\lim_{\varepsilon\to 0^+}\ \sup_{t < C\,e^{C/\varepsilon}} \int_\Omega |v^\varepsilon(t) - u_0| = 0\,. \tag{1.14}$$

In terms of $u^\varepsilon(t) = v^\varepsilon(t/\varepsilon^2)$, we thus have $\sup_{t<C\,\varepsilon^2\,e^{C/\varepsilon}} \int_\Omega |u^\varepsilon(t) - u_0| = 0$ as $\varepsilon \to 0^+$, which is a non-trivial information since $\varepsilon^2\,e^{C/\varepsilon} \to \infty$ as $\varepsilon \to 0^+$. When $N \ge 2$ we can interpret this statement as a metastability result, since the expectation is that $u^\varepsilon(t)$ will eventually converge to a single transition layer.

Concerning dimensions $n \ge 2$, it is shown in [AF94, ABF98] that for initial data resembling the characteristic function of a sufficiently small ball contained in Ω and located near $\partial\Omega$, solutions $\{v^\varepsilon(t)\}_\varepsilon$ to (1.13) will remain close to the characteristic function of a ball contained in Ω on an interval of times $(0, T_\varepsilon)$ with $T_\varepsilon = \mathrm{O}(e^{C/\varepsilon})$ as $\varepsilon \to 0^+$. This is another result supporting the metastability scenario: indeed, the expected attractors of (1.13) as

[7]The boundedness of Ω allows one to work with the simpler volume potential $V(t) = t$, which also brings some remarkable simplifications in the form of the volume-preserving Lagrange multiplier. Indeed, comparing the definition of λ_ε with that of μ_ε we see how the former choice, crucial in ensuring the well-posedness of $\Psi(\varepsilon, m)$ as a minimization problem on $\mathbb{R}^n$, leads to the presence of the possibly degenerate denominator $\varepsilon \int_{\mathbb{R}^n} V'(v)^2$ (in place of the constant denominator $\varepsilon|\Omega|$), and of the additional term $|\nabla v|^2\,V''(v)(\approx |\nabla v|^2\,v^{2/(n-1)}$ for v small) at numerator.

[8]Our review is informal in the sense that we will gloss over specific assumptions made in the reviewed papers on the double well potential W and the initial data u_0. Moreover, the range of u is often assumed to be $\mathbb{R}$, while we have chosen to consider functions with range in $[0,1]$.

$t \to \infty$ should be close, for ε small, to minimizers of $\inf\{P(E;\Omega) : E \subset \Omega, |E| = m\}$; however, such minimizers, at small volume m, look like half-balls centered at the point of highest mean curvature of $\partial\Omega$ (see [Fal10, MM16]), and thus will *never* be balls contained in Ω.

Another slow-motion result in higher dimension has been obtained in [MR16, LM19]. Its interest lies in the fact that it assumes the proximity of the initial data to generic perimeter minimizing sets (thus, not necessarily to balls/intervals); the drawback is that proximity of solutions $v^\varepsilon(t)$ is shown to be preserved only on a time interval $(0, T_\varepsilon)$ with $T_\varepsilon = C/\varepsilon$ so that, in terms of $u^\varepsilon(t) = v^\varepsilon(t/\varepsilon^2)$, proximity preservation is shown only on a time interval $(0, C\,\varepsilon)$, and no information survives in the limit $\varepsilon \to 0^+$.

1.3. **Analysis of the diffused VPMCF.** We now provide a detailed breakdown of a series of results leading to Theorem 1.1.

The starting point of our analysis is to establish the existence of solutions of the diffused VPMCF and their basic regularity properties. This is a nontrivial task due to the presence of the Lagrange multiplier functional λ_ε. In addition to being non-local and requiring integrating on a non-compact set, this term brings two main technical complications into our analysis: 1) the possible degeneration of the flow because of the smallness of the $\int_{\mathbb{R}^n} V'(u)^2$-term at the denominator, and 2) the lack of regularity because of the $|\nabla u|^2\, V''(u)$-term at numerator, since $V''(u)$ is only Hölder continuous. These difficulties are addressed by first approximating V with regularized volume potentials V_δ such that $\mathrm{Lip}(V_\delta''; [0,1]) < \infty$, and by then discussing the delicate limit as $\delta \to 0^+$. Boundedness, Lipschitz and Hölder continuity estimates on λ_ε, and on its variant $\lambda_{\varepsilon,\delta}$ associated with V_δ, are discussed in Section 3, and are obtained by borrowing the geometric viewpoint of "nucleation lemmas" from the theory of isoperimetric clusters [Alm76]. With these estimates at hand we can implement a fixed point argument to show the existence of mild solutions to the modification of (DF) obtained by replacing V with V_δ. We can then start bootstrapping regularity and monotonicity properties for the δ-approximating flows, up to the point where enough information is obtained and the $\delta \to 0^+$ limit can be addressed, thus establishing the following theorem (proved in Section 4).

Theorem 1.2 (Existence, regularity, and entropies of the flow). *If $\varepsilon > 0$, $n \ge 2$, $W \in C^{2,1}[0,1]$ satisfies* (1.2) *and* (1.3)*, $u_0 \in W^{2,p}(\mathbb{R}^n; [0,1])$ for every $p \ge 2$, and $\mathcal{V}(u_0) = 1$, then there exists a unique $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,\infty))$ that is a classical solution of* (DF)*. Moreover:*

(i): *for every $p \ge 2$ and $t_0 > 0$ we have*

$$\sup_{t>0} \left\{ \|\partial_t u(t)\|_{L^p(\mathbb{R}^n)} , \|u(t)\|_{W^{2,p}(\mathbb{R}^n)} \right\} \le C(\varepsilon, u_0, p)\,, \tag{1.15}$$

$$\sup_{t>t_0} \left\{ \|\partial_{tt} u(t)\|_{L^p(\mathbb{R}^n)} , \|\partial_t u(t)\|_{W^{2,p}(\mathbb{R}^n)} , \|u(t)\|_{W^{3,p}(\mathbb{R}^n)} \right\} \le C(\varepsilon, u_0, p, t_0)\,; \tag{1.16}$$

(ii): *$t \mapsto \lambda_\varepsilon[u(t)]$ is Lispchitz continuous on (t_0, ∞) for every $t_0 > 0$, and is such that*

$$\sup_{t>0} \varepsilon\, |\lambda_\varepsilon[u(t)]| \le C\, \mathcal{AC}_\varepsilon(u_0)^{2n+2}\,;$$

(iii): *$\mathcal{V}(u(t)) = 1$ for every $t \ge 0$;*

(iv)*: $t \mapsto \mathcal{AC}_\varepsilon(u(t))$ is continuous and decreasing on $[0,\infty)$ with*

$$\mathcal{AC}_\varepsilon(u(t_2)) - \mathcal{AC}_\varepsilon(u(t_1)) = -\varepsilon \int_{t_1}^{t_2} dt \int_{\mathbb{R}^n} (\partial_t u(t))^2\,, \qquad \forall t_2 \ge t_1 > 0\,; \tag{1.17}$$

(v): *$0 < u < 1$ on $\mathbb{R}^n \times (0,\infty)$;*

(vi): *the function* $t \mapsto \int_{\mathbb{R}^n} (\partial_t u(t))^2$ *belongs to* $W^{1,1}(t_0, \infty)$ *for every* $t_0 > 0$*, with*

$$\frac{d}{dt} \int_{\mathbb{R}^n} (\partial_t u(t))^2 = - \int_{\mathbb{R}^n} \Big\{ 4 \, |\nabla(\partial_t u)|^2 + \frac{2}{\varepsilon} \Big(\frac{W''(u)}{\varepsilon} - \lambda_\varepsilon[u(t)] \, V''(u) \Big) \, (\partial_t u)^2 \Big\}(t) \, . \quad (1.18)$$

In particular,

$$\lim_{t \to \infty} \int_{\mathbb{R}^n} (\partial_t u(t))^2 = 0 \, .$$

It is well-know that the asymptotic behavior of a semilinear parabolic PDE can be "subsequentially resolved" into a bubbling of mutually drifting stationary states: a good exemplification of this kind of result, whose fundamental idea is rooted into Lions' concentration-compactness principle itself, is found, for example, in [Fei97, Theorem 1.1]. Another powerful idea found in theory of semilinear parabolic PDE is that when the initial datum is compactly supported, then a sort of star-shapedness of the flow (see (5.45)) can be established by means of the parabolic maximum principle, thus excluding bubbling phenomena; for a general exemplification of this idea, see [Fei97, Theorem 1.2]. In Section 5 we adapt these general methods to our specific problem, which, again, does not follow in the range of application of the general theory because of the non-autonomy of (DF) and because of the presence of the non-local Lagrange multiplier λ_ε. Notice that we are not considering (yet) the physical regime when ε is small: in particular, there is no way to attribute any geometric meaning to the stationary states ξ_i appearing in the statement, e.g. by relating them to the minimizers $\zeta_{\varepsilon,m}$ of $\Psi(\varepsilon, m)$.

Theorem 1.3 (Subsequential bubbling, general ε)**.** *Let* $\varepsilon > 0$*,* $n \geq 2$*,* $W \in C^{2,1}[0,1]$ *satisfy* (1.2) *and* (1.3)*,* $u_0 \in W^{2,p}(\mathbb{R}^n; [0,1])$ *for every* $p \geq 2$*,* $\mathcal{V}(u_0) = 1$*, and let* $\{u(t)\}_{t \geq 0}$ *be the diffused VPMCF with* $u(0) = u_0$*.*

Then, for every sequence $t_j \to \infty$ *as* $j \to \infty$*, up to extracting a subsequence, there are* $M \in \mathbb{N}$*,* $\ell_\varepsilon > 0$*, sequences* $(x_j^i)_j$ *(*$i = 1, ..., M$*) satisfying* $|x_j^i - x_j^k| \to \infty$ *as* $j \to \infty$ *(*$i \neq k$*), and radial solutions* $\{\xi_i\}_{i=1}^M$ *of*

$$2 \, \varepsilon^2 \, \Delta \xi_i = W'(\xi_i) - \varepsilon \, \ell_\varepsilon \, V'(\xi_i) \quad \text{on } \mathbb{R}^n \, , \quad (1.19)$$

such that

$$\sum_{i=1}^M \mathcal{V}(\xi_i) = 1 \, , \qquad \sum_{i=1}^M \mathcal{AC}_\varepsilon(\xi_i) \leq \mathcal{AC}_\varepsilon(u_0) \, , \quad (1.20)$$

and, for all $p > 2$*,*

$$\lim_{j \to \infty} \Big\| u(t_j) - \sum_{i=1}^M \tau_{x_i^j}[\xi_i] \Big\|_{(W^{2,p} \cap W^{1,2})(\mathbb{R}^n)} + \big| \lambda_\varepsilon[u(t_j)] - \ell_\varepsilon \big| = 0 \, .$$

Moreover, if u_0 *has compact support in* $\mathbb{R}^n$*, then* $M = 1$ *and* $x_j^1 \to x_*$ *as* $j \to \infty$ *for some* $x_* \in \mathbb{R}^n$*.*

From this point on we work exclusively in the geometric regime when $\varepsilon < \varepsilon_0$ for some small universal constant ε_0 and relate the subsquential bubbling established in Theorem 1.3 to the diffused isoperimetric problem $\Psi(\varepsilon, m)$. A key tool in achieving this result is the diffused Alexandrov theorem proved in [MR24, Theorem 1.1-(iv)], whose statement is recalled in Section 2, and which asserts, roughly speaking, that any solution of $2 \, \varepsilon^2 \, \Delta \xi = W'(\xi) - \varepsilon \, \ell \, V'(\xi)$ on $\mathbb{R}^n$ with $\xi(x) \to 0$ as $|x| \to \infty$ and $\varepsilon \, \ell \in (0, \nu_0)$ for a sufficiently small universal constant ν_0, must satisfy, up to translations, $\xi = \zeta_{\varepsilon,m}$ and $\ell = \Lambda_{\varepsilon,m}$, for some m such that $\varepsilon \in (0, \varepsilon_0 \, m^{1/n})$.

Theorem 1.4 (Subsequential bubbling, small ε)**.** *If* $n \geq 2$*,* $W \in C^{2,1}[0,1]$ *satisfies* (1.2) *and* (1.3)*,* $u_0 \in W^{2,p}(\mathbb{R}^n; [0,1])$ *for all* $p \geq 2$*, and* $\mathcal{V}(u_0) = 1$*, then there is* $\varepsilon_0^* = \varepsilon_0^*(n, W, u_0)$ *with the following property.*

If $\varepsilon \in (0, \varepsilon_0^*)$ and $\{u(t)\}_{t\ge 0}$ is the corresponding solution of (DF) *with $u(0) = u_0$, then, for every $t_j \to \infty$ as $j \to \infty$, up to extracting a subsequence in j, there are $M \in \mathbb{N}$ and sequences $\{(x_j^i)_j\}_{i=1}^M$ with $|x_j^i - x_j^k| \to \infty$ as $j \to \infty$ ($i \ne k$), such that*

$$\lim_{j\to\infty} \Big\| u(t_j) - \sum_{i=1}^M \tau_{x_j^i}[\zeta_{\varepsilon,1/M}] \Big\|_{(W^{2,p}\cap W^{1,2})(\mathbb{R}^n)} + \Big| \lambda_\varepsilon[u(t_j)] - \Lambda_{\varepsilon,1/M} \Big| = 0\,, \tag{1.21}$$

for all $p > 2$. Moreover, M is uniquely characterized by the relation

$$M\,\Psi\Big(\varepsilon, \frac{1}{M}\Big) = \lim_{t\to\infty} \mathcal{AC}_\varepsilon(u(t))\,. \tag{1.22}$$

In particular, M and the limit Lagrange multiplier $\Lambda_{\varepsilon,1/M}$ depend only on u_0 and not on the specific sequence $(t_j)_j$ under consideration.

Remark 1.4. For the quantification of ε_0^* in terms of u_0, see (6.2) below.

Under the assumption (1.9) or (1.10), Theorem 1.4 holds with $M = 1$. Thus to go from Theorem 1.4 to Theorem 1.1 we have to upgrade subsequential convergence to full convergence as $t \to \infty$. The natural approach to this problem consists in proving the differential inequality

$$-\frac{d}{dt}\mathcal{I}_\varepsilon(t) \ge \frac{\mathcal{I}_\varepsilon(t)}{C(\varepsilon)}\,, \qquad \forall t > 1/C(\varepsilon, u_0)\,,$$

for the Fisher information/dissipation $\mathcal{I}_\varepsilon(t) = \varepsilon \int_{\mathbb{R}^n} (\partial_t u(t))^2$ of the flow (compare with Theorem 1.2-(iv,vi)). This is the well-known Bakry-Émery method, which requires establishing the strict stability of the $\mathcal{V}$-constrained second variation

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) = \int_{\mathbb{R}^n} 2\,\varepsilon\,|\nabla\varphi|^2 + \Big\{ \frac{W''(\zeta_\varepsilon)}{\varepsilon} - \Lambda_\varepsilon\, V''(\zeta_\varepsilon) \Big\}\,\varphi^2\,, \qquad \varphi \in W^{1,2}(\mathbb{R}^n)\,, \tag{1.23}$$

of the Allen–Cahn functional at the minimizer $\zeta_\varepsilon = \zeta_{\varepsilon,1}$ of $\Psi(\varepsilon, 1)$ (where we are also setting $\Lambda_\varepsilon = \Lambda_{\varepsilon,1}$ for brevity). This strict stability result is established in the following theorem, proved in Section 7 (whereas the proof of Theorem 1.4 is finally discussed in Section 8).

Theorem 1.5 (Strict stability of $\mathcal{Q}_\varepsilon$ at ζ_ε)**.** *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1.2) *and* (1.3)*, then there exists a positive universal constant ε_0 with the following property.*

For every $\varepsilon \in (0, \varepsilon_0)$, if $\varphi \in W^{1,2}(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \varphi\, V'(\zeta_\varepsilon) = 0\,, \qquad \int_{\mathbb{R}^n} \varphi\, \nabla\zeta_\varepsilon = 0\,, \tag{1.24}$$

then

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) \ge \frac{\varepsilon}{C} \int_{\mathbb{R}^n} \varphi^2\,; \tag{1.25}$$

and, moreover, for a constant $C(\varepsilon) > 0$,

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) \ge \frac{1}{C(\varepsilon)} \int_{\mathbb{R}^n} |\nabla\varphi|^2 + \varphi^2\,. \tag{1.26}$$

It had already been proved in [MR24, Lemma 4.4] that if φ is *radial* (and satisfies (1.24)), then

$$C\,\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) \ge \int_{\mathbb{R}^n} \varepsilon\,|\nabla\varphi|^2 + \frac{\varphi^2}{\varepsilon}\,, \tag{1.27}$$

thus showing that the stability of $\mathcal{Q}_\varepsilon[\zeta_\varepsilon]$ among radial variations is much stronger than among generic variations (the costant ε/C in (1.25) is sharp as seen in the proof of the theorem itself). The proof of Theorem 1.5 combines the decomposition of φ as a Fourier series in angular/radial variables with a geometric change of variables that allows to relate (on special modes of such decomposition) $\mathcal{Q}_\varepsilon[\zeta_\varepsilon]$ with the second variation of the perimeter functional at a ball.

1.4. **Organization of the paper.** Section 2 contains a recap of our basic notation, main results from [MR24], and useful properties of the various potentials involved in our analysis. The other sections of the paper are organized as described in the previous section.

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2. Background material on the diffused isoperimetric problem

In this section we collect some background material that will be repeatedly used in the sequel. In Section 2.1 we set some basic conventions and notation, while in Section 2.2 we recall some key results contained in [MR24] and concerning the diffused isoperimetric problem $\Psi(\varepsilon, m)$. Finally, in Section 2.3 we collect some elementary inequalities concerning the various potentials W, Φ, and V that will be often referred to in later proofs, and in Section 2.4 we introduced some regularized volume potentials V_δ which will be employed as a technical device in the proof of Theorem 1.2.

2.1. **Basic conventions.** Throughout the paper $n \in \mathbb{N}$, $n \geq 2$, denotes the dimension of the Euclidean space we are working in, and $W \in C^{2,1}[0,1]$ is a "double-well potential" satisfying (1.2) and (1.3). By **universal constant** we mean a positive real number depending on W and the value of n under consideration. We denote by C a generic universal constant whose value may increase at each of its subsequent appearances. Universal constants that may also depend on $\varepsilon > 0$ are denoted by $C(\varepsilon)$, with the idea that $C(\varepsilon)$ may diverge to $+\infty$ as $\varepsilon \to 0^+$. Given $k \in \mathbb{N}$, we will write "$f(\varepsilon) = \mathrm{O}(\varepsilon^k)$ as $\varepsilon \to 0^+$" if there exists a universal constant C such that $|f(\varepsilon)| \leq C\,\varepsilon^k$ for every $\varepsilon \in (0, 1/C)$; similar definitions are given for "$\mathrm{O}(t)$ as $t \to \infty$", etc.

2.2. **The diffused isoperimetric problem.** Let us recall that for every $\varepsilon, m > 0$ we set

$$\Psi(\varepsilon, m) = \inf\Big\{\mathcal{AC}_\varepsilon(u) : \mathcal{V}(u) = m\,, u \in W^{1,2}(\mathbb{R}^n; [0,1])\Big\}\,.$$

Among the basic properties of $\Psi(\varepsilon, m)$ we have the scaling law

$$\Psi(\varepsilon, m) = m^{(n-1)/n}\,\Psi\Big(\frac{\varepsilon}{m^{1/n}}, 1\Big)\,, \qquad \forall \varepsilon, m > 0\,, \tag{2.1}$$

and the isoperimetric lower bound and $\varepsilon \to 0^+$ limit

$$\Psi(\varepsilon, m) \;>\; 2\,c_{\mathrm{iso}}(n)\,m^{(n-1)/n}\,, \qquad \forall \varepsilon, m > 0\,, \tag{2.2}$$

$$\lim_{\varepsilon \to 0^+} \Psi(\varepsilon, m) \;=\; 2\,c_{\mathrm{iso}}(n)\,m^{(n-1)/n}\,, \qquad \forall m > 0\,, \tag{2.3}$$

where we have set $c_{\mathrm{iso}}(n) = n\,\omega_n^{1/n}$. A simple concentration compactness argument (see [MNR23, Proof of Theorem A.1, steps one and two]) shows that $\Psi(\varepsilon, m)$ admits radially symmetric decreasing minimizers for each ε and m. Stronger properties are proved in [MR24, Theorem 1.1, Theorem 6.1] under the geometric regime $\varepsilon \ll m^{1/n}$:

Theorem Ψ [MR24] *If $n \geq 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1.2) *and* (1.3)*, then there are positive universal constants ε_0 and ν_0 with the following properties:*

(i) : *we have*

$$\Psi(\cdot, m) \text{ is strictly increasing on } (0, \varepsilon_0\,m^{1/n})\,, \tag{2.4}$$

$$\Psi(\varepsilon, \cdot) \text{ is concave on } (0, \infty) \text{ and strictly concave on } ((\varepsilon/\varepsilon_0)^n, \infty)\,; \tag{2.5}$$

moreover, if $0 < \varepsilon < \varepsilon_0\,m^{1/n}$, then there is a radially symmetric strictly decreasing minimizer $\zeta_{\varepsilon,m}$ of $\Psi(\varepsilon, m)$ with maximum at the origin having the property that u is a minimizer *of $\Psi(\varepsilon,m)$ if and only if $u=\tau_{x_0}[\zeta_{\varepsilon,m}]$ for some $x_0\in\mathbb{R}^n$; and, for some $\Lambda_{\varepsilon,m}>0$, $\zeta_{\varepsilon,m}$ satisfies*

$$2\,\varepsilon^2\,\Delta\zeta_{\varepsilon,m}=W'(\zeta_{\varepsilon,m})-\varepsilon\,\Lambda_{\varepsilon,m}\,V'(\zeta_{\varepsilon,m})\qquad\text{on }\mathbb{R}^n\,,$$

where

$$\lim_{\varepsilon\to 0^+} m^{1/n}\,\Lambda_{\varepsilon,m}=2\,(n-1)\,\omega_n^{1/n}\,.\tag{2.6}$$

(ii): *if $v\in C^2(\mathbb{R}^n;[0,1])$ satisfies $v(x)\to 0^+$ as $|x|\to\infty$ and solves*

$$2\,\varepsilon^2\,\Delta v=W'(v)-\varepsilon\,\lambda\,V'(v)\qquad\text{on }\mathbb{R}^n\,,$$

for some positive λ such that $\varepsilon\,\lambda<\nu_0$, then there exist $x_0\in\mathbb{R}^n$ and $m>0$ such that

$$v=\tau_{x_0}[\zeta_{\varepsilon,m}]\,,\qquad \varepsilon<\varepsilon_0\,m^{1/n}\,,\qquad \lambda=\Lambda_{\varepsilon,m}\,.$$

2.3. **Technical properties of W and related potentials.** In the linearization of (DF) we shall tacitly use the fact that, being $W\in C^{2,1}[0,1]$, we have

$$\Big|W(r)-W(s)-W'(s)(r-s)-W''(s)\frac{(r-s)^2}{2}\Big|\le C\,|r-s|^3\,,\qquad\forall r,s\in[0,1]\,.$$

We also observe that $1/C\le W''\le C$ on $[0,1]$, and that there is a universal constant $\delta_0<1/2$ such that

$$\begin{aligned}&\frac{1}{C}\le\frac{W(r)}{r^2}\,,\frac{W'(r)}{r}\le C &&\forall r\in(0,\delta_0]\,,\\ &\frac{1}{C}\le\frac{W(r)}{(1-r)^2}\,,\frac{-W'(r)}{1-r}\le C &&\forall r\in[1-\delta_0,1)\,.\end{aligned}\tag{2.7}$$

Recalling that $\Phi(r)=\int_0^r\sqrt{W}$ and $V(r)=\Phi(r)^{n/(n-1)}$ for $r\in[0,1]$, we use (2.7) to quantify the behaviors of Φ and V near $r=0$ and $r=1$. By (1.2), $\Phi\in C^3_{\rm loc}(0,1)$, with

$$\Phi'=\sqrt{W}\,,\quad \Phi''=\frac{W'}{2\sqrt{W}}\,,\quad \Phi'''=\frac{W''}{2\sqrt{W}}-\frac{(W')^2}{4\,W^{3/2}}\,,\qquad\text{on }(0,1)\,.$$

By (2.7) and (1.3) we thus see that Φ satisfies

$$\begin{aligned}&\frac{1}{C}\le\frac{\Phi(r)}{r^2}\,,\frac{\Phi'(r)}{r}\,,\Phi''(r)\le C\,, &&\forall r\in(0,\delta_0]\,,\\ &\frac{1}{C}\le\frac{1-\Phi(r)}{(1-r)^2}\,,\frac{\Phi'(r)}{1-r}\,,-\Phi''(r)\le C\,, &&\forall r\in[1-\delta_0,1)\,.\end{aligned}\tag{2.8}$$

By exploiting (2.8) and setting for brevity $a=W''(0)$, we see that, as $r\to 0^+$,

$$\begin{aligned}\Phi'''&=\frac{2\,W''\,W-(W')^2}{4\,W^{3/2}}=\frac{2\,(a+\mathrm{O}(r))\,(a\,(r^2/2)+\mathrm{O}(r^3))-(a\,r+\mathrm{O}(r^2))^2}{4\,(a\,(r^2/2)+\mathrm{O}(r^3))^{3/2}}\\ &=\frac{\mathrm{O}(r^3)}{4\,a^{3/2}\,r^3+\mathrm{o}(r^3)}\,,\end{aligned}$$

and by a similar computation for $r\to 1^-$, we find

$$|\Phi'''|\le C\quad\text{on }(0,\delta_0)\cup(1-\delta_0,1)\,.\tag{2.9}$$

By (2.8) and (2.9) we see that $\Phi\in C^{2,1}[0,1]$ with a universal estimate on its $C^{2,1}[0,1]$-norm: in particular,

$$\Big|\Phi(r)-\Phi(s)-\Phi'(s)(r-s)-\Phi''(s)\frac{(r-s)^2}{2}\Big|\le C\,|r-s|^3\,,\qquad\forall r,s\in(0,1)\,.\tag{2.10}$$

Since $V=\Phi^{1+\alpha}$ for $\alpha=1/(n-1)\in(0,1]$ (recall that $n\ge 2$) and $\Phi(r)=0$ if and only if $r=0$, we easily see that $V\in C^3_{\rm loc}(0,1)$, with

$$
\begin{aligned}
&V'=(1+\alpha)\,\Phi^\alpha\,\Phi'\,,\qquad V''=(1+\alpha)\Big\{\alpha\,\frac{(\Phi')^2}{\Phi^{1-\alpha}}+\Phi^\alpha\,\Phi''\Big\}\,,\\
&|V'''|\le C(\alpha)\,\Big\{\frac{(\Phi')^3}{\Phi^{2-\alpha}}+\frac{\Phi'\,|\Phi''|}{\Phi^{1-\alpha}}+\Phi^\alpha\,|\Phi'''|\Big\}\,.
\end{aligned}
$$

By (2.10), and keeping track of the sign of Φ'' and of the fact that negative powers of $\Phi(r)$ are large only near $r=0$, but are bounded near $r=1$, we find that

$$
\begin{aligned}
&\frac{1}{C}\le\frac{V(r)}{r^{2+2\alpha}}\,,\frac{V'(r)}{r^{1+2\alpha}}\,,\frac{V''(r)}{r^{2\alpha}}\le C\,,\qquad |V'''(r)|\le\frac{C}{r^{1-2\alpha}}\qquad\forall r\in(0,\delta_0]\,,\\
&\frac{1}{C}\le\frac{1-V(r)}{(1-r)^2}\,,\frac{V'(r)}{1-r}\le C\,,\quad |V''(r)|\,,|V'''(r)|\le C\,,\qquad\forall r\in[1-\delta_0,1)\,.
\end{aligned}
\tag{2.11}
$$

In particular, $V''(r)\to\infty$ explodes as $r\to 0^+$. However, $V\in C^{2,\gamma(n)}[0,1]$ (with $\gamma(n)=\min\{1,2/(n-1)\}\in(0,1]$) and we have the second order Taylor expansion

$$
\Big|V(r)-V(s)-V'(s)\,(r-s)-V''(s)\frac{(r-s)^2}{2}\Big|\le C\,|r-s|^{2+\gamma(n)}\,,\qquad\forall r,s\in(0,1)\,.
$$

As much as the analogous expansion for W, this formula will be repeatedly used in linearizing (DF).

2.4. **Regularized volume potentials.** As already discussed at the beginning of Section 1.3, the non-Lipschitzianity of V'' near 0 causes several technical problems, that call for the introduction of regularized volume potentials $V_\delta:[0,1]\to[0,1]$ ($\delta>0$) such that $V_\delta\in C^{2,1}[0,1]$, with

$$
V_\delta(0)=0\,,\qquad V_\delta'(0)=V_\delta'(1)=0\,,\qquad\lim_{\delta\to0^+}\|V_\delta-V\|_{C^2[0,1]}=0\,.
$$

We will of course have

$$
\lim_{\delta\to0^+}{\rm Lip}(V_\delta'';[0,1])=+\infty\,,\qquad\sup_{\delta>0}\,[V_\delta'']_{C^{0,\gamma(n)}}<\infty\,.
$$

Such potentials V_δ can be defined by first considering a family $\{\rho_\delta\}_{\delta>0}$ of smooth mollifiers on $\mathbb{R}$ such that ${\rm spt}\rho_\delta\subset\subset(-\delta^2,\delta^2)$, and then by setting

$$
V_\delta=\rho_\delta\star\Big(L_\delta\circ\Phi\Big)^{n/(n-1)}\,,\tag{2.12}
$$

where

$$
L_\delta(r)=\begin{cases}0\,, & r\in[0,\delta]\,,\\ \dfrac{r-\delta}{1-2\,\delta}\,, & r\in[\delta,1-\delta]\,,\\ 1\,, & r\in[1-\delta,1]\,.\end{cases}
$$

By $W>0$ on $(0,1)$, (2.7) and (2.11), and up to further decreasing the value of δ_0 introduced above, we have

$$
V_\delta(r)\le C\,r^2\le C\,W(r)\,,\qquad V_\delta'(r)\le C\,r\,,\qquad\forall r\in(0,1-\delta_0)\,,
$$

for every $\delta\in\big[0,\delta_0\big]$, as well as (compare with (2.11))

$$
V_\delta(r)\ge\frac{1}{C}\,,\qquad V_\delta'(r)\le C\,(1-r)\qquad\forall r\in(\delta_0,1)\,.\tag{2.13}
$$

3. Estimates for the Lagrange multiplier functional

This section is devoted to the analysis of the Lagrange multiplier functional λ_ε, defined with values in $[0,\infty)$ on any given function $v\in W^{1,2}(\mathbb{R}^n;[0,1])$ with $|\{0<v<1\}|>0$ (and assumption that guarantees $\int_{\mathbb{R}^n} V'(u)^2>0$) by setting

$$\lambda_\varepsilon[v]=\frac{\int_{\mathbb{R}^n} 2\,\varepsilon^2\,|\nabla v|^2\,V''(v)+W'(v)\,V'(v)}{\varepsilon\,\int_{\mathbb{R}^n} V'(v)^2}\,. \tag{3.1}$$

In particular, we address the Lipschitz continuity properties of λ_ε in the Banach space $(X,\|\cdot\|)$ defined by

$$X=(C^0\cap W^{1,2})(\mathbb{R}^n;[0,1])\,,\qquad \|u\|_X=\|u\|_{W^{1,2}(\mathbb{R}^n)}+\|u\|_{C^0(\mathbb{R}^n)}\,,$$

that is the space we shall use to construct mild solutions of (DF). In fact, we shall also need to consider the Lagrange multiplier functionals

$$\lambda_{\varepsilon,\delta}[u]=\frac{\int_{\mathbb{R}^n} 2\,\varepsilon^2\,|\nabla u|^2\,V_\delta''(u)+W'(u)\,V_\delta'(u)}{\varepsilon\,\int_{\mathbb{R}^n} V_\delta'(u)^2}\,, \tag{3.2}$$

obtained by replacing V with the regularized volume potentials V_δ introduced in the previous section. We shall also set $V_0=V$, $\mathcal{V}_0=\mathcal{V}$, and use the notation

$$\mathcal{AC}_\varepsilon(u;\Omega)=\int_\Omega \varepsilon\,|\nabla u|^2+\frac{W(u)}{\varepsilon}\,,\qquad \mathcal{V}_\delta(u;\Omega)=\int_\Omega V_\delta(u)\,,$$

for the localization to a Borel set $\Omega\subset\mathbb{R}^n$ of the functionals $\mathcal{AC}_\varepsilon$ and $\mathcal{V}_\delta$.

Theorem 3.1. *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies (1.2) and (1.3), then there exist positive universal constants ε_0 and δ_0 with the following properties:*

(i): *if $u\in W^{1,2}(\mathbb{R}^n;[0,1])$, $\varepsilon>0$, and $\delta\in[0,\delta_0]$, then*

$$\int_{\mathbb{R}^n} u^2\le C\left\{\varepsilon\,\mathcal{AC}_\varepsilon(u)+\mathcal{V}_\delta(u)\right\}, \tag{3.3}$$

and, with an ε-dependent universal constant $C(\varepsilon)$,

$$C(\varepsilon)\int_{\mathbb{R}^n}|\nabla u|^2\int_{\mathbb{R}^n}V_\delta'(u)^2\ \ge\ \min\Big\{1,\frac{\mathcal{V}_\delta(u)}{\mathcal{AC}_\varepsilon(u)}\Big\}^{2\,n}\,, \tag{3.4}$$

$$|\lambda_{\varepsilon,\delta}[u]|\ \le\ C(\varepsilon)\,\frac{\mathcal{AC}_\varepsilon(u)^{2n+1}}{\mathcal{V}_\delta(u)^{2n}}\int_{\mathbb{R}^n}|\nabla u|^2\,. \tag{3.5}$$

Furthermore, if $\delta=0$ and $\varepsilon\in(0,\varepsilon_0)$, then (3.4) and (3.5) hold with C in place of $C(\varepsilon)$.

(ii): *if $u,v\in X\setminus\{0\}$, $\varepsilon>0$, and $\max\{\|u\|_X\,,\|v\|_X\}\le C(\varepsilon)$, then*

$$|\lambda_{\varepsilon,\delta}[u]-\lambda_{\varepsilon,\delta}[v]|\ \le\ C(\varepsilon,\delta)\,\frac{\|u-v\|_X}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}\,\min\{1,\mathcal{V}_\delta(v)^{2n}\}}\,,\qquad \forall\delta\in(0,\delta_0]\,, \tag{3.6}$$

$$|\lambda_{\varepsilon,\delta}[u]-\lambda_{\varepsilon,\delta}[v]|\ \le\ C(\varepsilon)\,\frac{\|u-v\|_X^{\gamma(n)}}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}\,\min\{1,\mathcal{V}_\delta(v)^{2n}\}}\,,\qquad \forall\delta\in[0,\delta_0]\,, \tag{3.7}$$

$$|\lambda_{\varepsilon,\delta}[u]-\lambda_\varepsilon[u]|\ \le\ C(\varepsilon)\,\frac{\|V_\delta-V\|_{C^2[0,1]}}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}\,\min\{1,\mathcal{V}(u)^{2n}\}}\,,\qquad \forall\delta\in[0,\delta_0]\,, \tag{3.8}$$

where $\gamma(n)=\min\{1,2/(n-1)\}$; and if $u,v\in W^{2,2}(\mathbb{R}^n;[0,1])\setminus\{0\}$, then

$$|\lambda_{\varepsilon,\delta}[u]-\lambda_{\varepsilon,\delta}[v]|\le C(\varepsilon)\frac{\max\{\|u\|_{W^{2,2}},\|v\|_{W^{2,2}}\}\,\|u-v\|_{W^{1,2}(\mathbb{R}^n)}}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}\,\min\{1,\mathcal{V}_\delta(v)^{2n}\}}\,,\qquad \forall\delta\in[0,\delta_0]\,. \tag{3.9}$$

Proof. Step one, diffused relative isoperimetry and nucleation: We prove two relative isoperimetric inequalities in balls in the diffused setting and a consequent nucleation type lemma modeled after [MPS22, Lemma 2.1]. This kind of result is in turn inspired by a tool introduced by Almgren [Alm76] in the study of isoperimetric clusters, see [Mag12, Lemma 29.10]. More precisely, we prove the existence of universal constants η_0, σ_0 and C with the following properties:

(a): if $\delta \in [0,\delta_0]$, $\varepsilon, r > 0$, and if $u \in W^{1,2}(\mathbb{R}^n;[0,1])$ satisfies

$$\fint_{B_r} V_\delta(u) \le \eta_0 \left(\frac{\varepsilon}{r}\right)^{2\,n}, \tag{3.10}$$

then

$$C\,\mathcal{AC}_\varepsilon(u;B_r) \ge \frac{\varepsilon}{r}\,\mathcal{V}_\delta(u;B_r)^{(n-1)/n}\,; \tag{3.11}$$

(b): if $\varepsilon, r > 0$ are such that $\varepsilon/r < \sigma_0$, and if $u \in W^{1,2}(\mathbb{R}^n;[0,1])$ satisfies

$$\fint_{B_r} V(u) \le \frac{1}{2}\,, \tag{3.12}$$

then

$$C\,\mathcal{AC}_\varepsilon(u;B_r) \ge \mathcal{V}(u;B_r)^{(n-1)/n}\,. \tag{3.13}$$

(c): if $\varepsilon > 0$ and

$$\mathcal{B} = \Big\{ B_{\sqrt{n+1}\,R/2}(R\,z) : z \in \mathbb{Z}^n \Big\}\,, \qquad R = \frac{\max\{1,\varepsilon\}}{\sigma_0\,(\sqrt{n+1}/2)}\,,$$

then for every $u \in W^{1,2}(\mathbb{R}^n;[0,1])$ we have

$$C\,\frac{\max\{1,\varepsilon\}}{\min\{1,\varepsilon^{2\,n}\}}\,\sup_{B\in\mathcal{B}} \fint_B V_\delta(u) \;\ge\; \min\Big\{1, \Big(\frac{\mathcal{V}_\delta(u)}{\mathcal{AC}_\varepsilon(u)}\Big)^n\Big\}\,, \qquad \forall \delta \in (0,\delta_0]\,, \tag{3.14}$$

$$C\,\max\{1,\varepsilon\}\,\sup_{B\in\mathcal{B}} \fint_B V(u) \;\ge\; \min\Big\{1, \Big(\frac{\mathcal{V}(u)}{\mathcal{AC}_\varepsilon(u)}\Big)^n\Big\}\,. \tag{3.15}$$

We first derive conclusion (c) from conclusions (a) and (b): Let $u \in W^{1,2}(\mathbb{R}^n;[0,1])$. In proving (3.15) we can assume without loss of generality that

$$\sup_{B\in\mathcal{B}} \fint_B V(u) \le \frac{1}{2}\,.$$

In particular, since the choice of R is such that $\varepsilon/(\sqrt{n+1}R/2) < \sigma_0$ for every $\varepsilon > 0$, we deduce from conclusion (b) that (3.13) holds for every $B \in \mathcal{B}$. The corresponding bounds can be used together with the fact that $\mathcal{B}$ is a covering of $\mathbb{R}^n$ with finite overlapping (depending only on the dimension n) to conclude that

$$\begin{aligned} C(n)\,\mathcal{AC}_\varepsilon(u) \;&\ge\; \sum_{B\in\mathcal{B}} \mathcal{AC}_\varepsilon(u;B) \ge \frac{1}{C}\sum_{B\in\mathcal{B}} \mathcal{V}(u;B)^{(n-1)/n} \\ &\ge\; \frac{1}{C}\,\frac{\sum_{B\in\mathcal{B}} \mathcal{V}(u;B)}{\sup_{B\in\mathcal{B}} \mathcal{V}(u;B)^{1/n}} \ge \frac{1}{C}\,\frac{\mathcal{V}(u)}{\sup_{B\in\mathcal{B}}\Big(\int_B V(u)\Big)^{1/n}}\,. \end{aligned} \tag{3.16}$$

Thanks to $|B|^{1/n} \le C\,\max\{1,\varepsilon\}$ for every $B \in \mathcal{B}$, (3.16) implies (3.15). To prove (3.14), let $\delta \in (0,\delta_0]$, and let us assume without loss of generality that

$$\sup_{B\in\mathcal{B}} \fint_B V_\delta(u) \le \eta_0 \Big(\frac{\varepsilon}{\sqrt{n+1}R/2}\Big)^{2\,n}\,, \tag{3.17}$$

then we can apply (3.11) to each $B\in\mathcal{B}$, and conclude as in (3.16) that

$$C(n)\,\mathcal{AC}_\varepsilon(u)\ge\frac{\varepsilon}{C\,R}\,\frac{\mathcal{V}_\delta(u)}{\sup_{B\in\mathcal{B}}\mathcal{V}_\delta(u;B)^{1/n}}\ge\frac{\min\{1,\varepsilon\}}{C}\,\frac{\mathcal{V}_\delta(u)}{\sup_{B\in\mathcal{B}}\mathcal{V}_\delta(u;B)^{1/n}}\,.$$

In particular, (3.14) follows by taking again into account that $\varepsilon/R=C\,\max\{1,\varepsilon\}$.

We now prove conclusions (a) and (b): Up to a rescaling, we can take $r=1$ in both conclusions. To prove conclusion (a), we notice that, by (3.10) and (2.13),

$$\eta_0\,\varepsilon^{2\,n}\ge\frac{1}{\omega_n}\int_{B_1\cap\{u\ge1/2\}}V_\delta(u)\ge\frac{|B_1\cap\{u\ge1/2\}|}{C}\,. \tag{3.18}$$

Since $W(r)\ge r^2/C$ for $r\in[0,1/2]$, by (2.13), (3.18), and the Hölder inequality, we get

$$\begin{aligned}\int_{B_1}u^2&=\int_{B_1\cap\{u\le1/2\}}u^2+\int_{B_1\cap\{u\ge1/2\}}u^2\\&\le C\int_{B_1}W(u)+C\,\big|B_1\cap\{u\ge1/2\}\big|^{1/n}\Big(\int_{B_1}u^{2n/(n-1)}\Big)^{(n-1)/n}\\&\le C\int_{B_1}W(u)+C\,\eta_0^{1/n}\,\varepsilon^2\Big(\int_{B_1}u^{2n/(n-1)}\Big)^{(n-1)/n}\,.\end{aligned}\tag{3.19}$$

By combining (3.19) with the embedding of $L^{2n/(n-1)}(B_1)$ into $W^{1,2}(B_1)$, we find that

$$\begin{aligned}\int_{B_1}\varepsilon\,|\nabla u|^2+\frac{W(u)}{\varepsilon}&\ge\int_{B_1}\varepsilon\,|\nabla u|^2+\frac{u^2}{C\varepsilon}-C\,\eta_0^{1/n}\,\varepsilon\Big(\int_{B_1}u^{2n/(n-1)}\Big)^{(n-1)/n}\\&\ge\frac{\varepsilon}{C}\Big(\int_{B_1}u^{2n/(n-1)}\Big)^{(n-1)/n}-C\,\eta_0^{1/n}\,\varepsilon\Big(\int_{B_1}u^{2n/(n-1)}\Big)^{(n-1)/n}\\&\ge\frac{\varepsilon}{C}\Big(\int_{B_1}u^{2n/(n-1)}\Big)^{(n-1)/n}\ge\frac{\varepsilon}{C}\Big(\int_{B_1}V_\delta(u)\Big)^{(n-1)/n}\,,\end{aligned}$$

provided η_0 is a sufficiently small universal constant, and where we have used that $V_\delta(r)\le C\,r^{2n/(n-1)}$ for every $r\in[0,1]$ and $\delta\in[0,\delta_0]$. Having proved conclusion (a), we now prove conclusion (b). Arguing by contradiction, we can assume the existence of $\varepsilon_k\to0^+$ and $\{u_k\}_k$ in $W^{1,2}(B_1;[0,1])$ such that, for all $k\in\mathbb{N}$ and setting for brevity

$$M_k=\int_{B_1}V(u_k)=\int_{B_1}\Phi(u_k)^p\,,\qquad p=\frac{n}{n-1}\,,$$

we have $M_k\le\omega_n/2$ and

$$\int_{B_1}\varepsilon_k|\nabla u_k|^2+\frac{W(u_k)}{\varepsilon_k}\le\frac{M_k^{1/p}}{k}\,, \tag{3.20}$$

for every $k\in\mathbb{N}$. Combining $|\nabla(\Phi\circ u)|=|\nabla u|\,\sqrt{W(u)}$ with Young's inequality as in (1.5), we deduce from (3.20) and the BV-Poincaré inequality [AFP00, (3.41)] that

$$\frac{1}{C}\Big(\int_{B_1}|\Phi(u_k)-t_k|^p\Big)^{1/p}\le\int_{B_1}|\nabla(\Phi\circ u_k)|\le\frac{C}{k}\,M_k^{1/p}\,, \tag{3.21}$$

where $t_k=\omega_n^{-1}\int_{B_1}\Phi(u_k)$. In particular, there is $c\in[0,1]$ such that, up to extracting a subsequence, $u_k\to c$ in $L^1(B_1)$ and a.e. in B_1 as $k\to\infty$. Since (3.20) implies $\int_{B_1}W(u_k)\to0$ as $k\to\infty$, by Fatou's lemma we find

$$\int_{B_1}W(c)\le\liminf_{k\to\infty}\int_{B_1}W(u_k)=0\,.$$

In particular, $c \in \{0,1\}$. Since $c = 1$ would contradict $M_k \le \omega_n/2$ for every k, we conclude that $c = 0$, and hence, thanks also to $0 \le u_k \le 1$, that $\Phi(u_k) \to 0$ in $L^p(B_1)$ (i.e., $M_k \to 0$) as $k \to \infty$. On noticing that $0 \le t_k \le C\, M_k^{1/p}$, we deduce by (3.21) that

$$\|\Phi(u_k) - (M_k/\omega_n)^{1/p}\|_{L^p(B_1)} \le \|t_k - (M_k/\omega_n)^{1/p}\|_{L^p(B_1)} + \frac{C}{k}\, M_k^{1/p} \le C\, M_k^{1/p}\,, \tag{3.22}$$

and, in particular, that

$$\lim_{k\to\infty} \Big|B_1 \cap \Big\{\Phi(u_k) \le (M_k/2\omega_n)^{1/p}\Big\}\Big| = 0\,.$$

Since $\Phi(u_k) \to 0$ in $L^p(B_1)$ implies $|B_1 \cap \{\Phi(u_k) > 1/2\}| \to 0$ as $k \to \infty$, we conclude that

$$\Big|B_1 \cap \Big\{\frac{1}{2} \ge \Phi(u_k) \ge (M_k/2\omega_n)^{1/p}\Big\}\Big| \ge \frac{|B_1|}{2}\,, \tag{3.23}$$

for k large enough. In particular, thanks to $\Phi \le C\, W$ on $[0,1/2]$ and to (3.20), for k large enough we have

$$\frac{M_k^{1/p}}{C\,\varepsilon_k} \le \frac{1}{\varepsilon_k}\int_{B_1\cap\{1/2\ge\Phi(u_k)\ge(M_k/2\omega_n)^{1/p}\}} \Phi(u_k) \le \frac{1}{\varepsilon_k}\int_{B_1} W(u_k) \le \frac{M_k^{1/p}}{k}\,, \tag{3.24}$$

that leads to a contradiction as $k \to \infty$.

Step two: We prove statement (a). To prove (3.3) it suffices to recall that $W(r) \ge r^2/C$ for $r \in [0,1/2]$ and $V_\delta(r) \ge r^2/C$ for $r \in [1/2,1]$, so that

$$\int_{\mathbb{R}^n} u^2 = \int_{\{u\le 1/2\}} u^2 + \int_{\{u>1/2\}} u^2 \le C \int_{\mathbb{R}^n} W(u) + V_\delta(u)\,.$$

To prove (3.5) we first notice that $|V_\delta''(r)| \le C$ and $|W'(r)\,V_\delta'(r)| \le C\,W(r)$ for every $r \in [0,1]$ (and every $\delta \in [0,\delta_0]$), so that

$$\Big|\int_{\mathbb{R}^n} 2\varepsilon\,|\nabla u|^2\, V_\delta''(u) + \frac{1}{\varepsilon}\, W'(u)\, V_\delta'(u)\Big| \le C\,\mathcal{AC}_\varepsilon(u)\,. \tag{3.25}$$

Since $V_\delta(r) \le C\, r^2$ for $r \in [0,1]$ and $u \in L^2(\mathbb{R}^n)$ we can find a sequence $R_j \to \infty$ such that $\int_{\partial B_{R_j}} V_\delta(u) \to 0$ as $j \to \infty$, and thus apply the divergence theorem to deduce that, for every $x_0 \in \mathbb{R}^n$,

$$\begin{aligned}(n-1)\int_{\mathbb{R}^n} \frac{V_\delta(u)}{|x-x_0|}\,dx &= \int_{\mathbb{R}^n} V_\delta(u)\,\mathrm{div}\Big(\frac{x-x_0}{|x-x_0|}\Big)\,dx \\ &= -\int_{\mathbb{R}^n} V_\delta'(u)\,\nabla u\cdot\frac{x-x_0}{|x-x_0|}\,dx \le \|V_\delta'(u)\|_{L^2(\mathbb{R}^n)}\,\|\nabla u\|_{L^2(\mathbb{R}^n)}\,.\end{aligned} \tag{3.26}$$

Setting $R = 2\,\max\{1,\varepsilon\}/\sigma_0\,\sqrt{n+1}$, we can apply (3.14) to find $x_0 \in \mathbb{R}^n$ such that

$$C(\varepsilon)\, ⨍_{B_R(x_0)} V_\delta(u) \ge \min\Big\{1, \frac{\mathcal{V}_\delta(u)}{\mathcal{AC}_\varepsilon(u)}\Big\}^n$$

which combined with (3.26) gives

$$\|V_\delta'(u)\|_{L^2(\mathbb{R}^n)}\,\|\nabla u\|_{L^2(\mathbb{R}^n)} \ge \frac{(n-1)}{R}\int_{B_R(x_0)} V_\delta(u) \ge \frac{1}{C(\varepsilon)}\,\min\Big\{1, \frac{\mathcal{V}_\delta(u)}{\mathcal{AC}_\varepsilon(u)}\Big\}^n\,, \tag{3.27}$$

that is (3.4). In summary, by combining (3.25) and (3.27) we find

$$|\lambda_{\varepsilon,\delta}[u]| \;\le\; C\,\frac{\mathcal{AC}_\varepsilon(u)}{\int_{\mathbb{R}^n} V_\delta'(u)^2} \le C(\varepsilon)\,\frac{\mathcal{AC}_\varepsilon(u)}{\min\Big\{1, \frac{\mathcal{V}_\delta(u)}{\mathcal{AC}_\varepsilon(u)}\Big\}^{2n}}\int_{\mathbb{R}^n} |\nabla u|^2 \tag{3.28}$$

that is (3.5). When $\varepsilon \in (0,\varepsilon_0]$ and $\delta = 0$ we can replace the constant $C(\varepsilon)$ in (3.5) with a plain universal constant C by exploiting the fact that (3.15) can be used in place of (3.14) (notice indeed that $\max\{1,\varepsilon\} \le C$ when $\varepsilon \in (0,\varepsilon_0]$).

Step three: We prove statement (b). Setting for brevity,

$$\mathcal{N}_{\varepsilon,\delta}[u]=\int_{\mathbb{R}^n}2\,\varepsilon\,|\nabla u|^2\,V_\delta''(u)+\frac{W'(u)\,V_\delta'(u)}{\varepsilon}\,,\qquad \delta\in[0,\delta_0]\,,$$

we notice that for every $\delta,\delta_*\in[0,\delta_0]$ and $u,v\in X\setminus\{0\}$,

$$\lambda_{\varepsilon,\delta}[u]-\lambda_{\varepsilon,\delta_*}[v]=\frac{\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta_*}[v]}{\int_{\mathbb{R}^n}V_\delta'(u)^2}+\frac{\mathcal{N}_{\varepsilon,\delta_*}[v]\left\{\int_{\mathbb{R}^n}V_{\delta_*}'(v)^2-\int_{\mathbb{R}^n}V_\delta'(u)^2\right\}}{\int_{\mathbb{R}^n}V_\delta'(u)^2\int_{\mathbb{R}^n}V_{\delta_*}'(v)^2}\,. \tag{3.29}$$

We first work on (3.29) *with* $\delta=\delta_*$, with the goal of proving (3.6), (3.7), and (3.9). Since $\mathcal{AC}_\varepsilon(u)\le\|u\|_{W^{1,2}}^2/\varepsilon$, we deduce from (3.4) that, if $\|u\|_{W^{1,2}}\le C(\varepsilon)$, then

$$\frac{1}{\int_{\mathbb{R}^n}V_\delta'(u)^2}\le\frac{C(\varepsilon)}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}}\,. \tag{3.30}$$

Recalling from (3.25) that $|\mathcal{N}_{\varepsilon,\delta}[u]|\le C\,\mathcal{AC}_\varepsilon(u)\le C(\varepsilon)\,\|u\|_{W^{1,2}}$, and using again first $\mathrm{Lip}(V_\delta',[0,1])\le C$, and then $|V_\delta'(r)|\le C\,t$ for $r\in[0,1]$, we find that, if $\|u\|_{W^{1,2}}\le C(\varepsilon)$, then for every $v\in W^{1,2}(\mathbb{R}^n;[0,1])$

$$\begin{aligned}&\Big|\mathcal{N}_{\varepsilon,\delta}[v]\int_{\mathbb{R}^n}\big(V_\delta'(v)^2-V_\delta'(u)^2\big)\Big|\le C(\varepsilon)\int_{\mathbb{R}^n}|V_\delta'(u)-V_\delta'(v)|\,\big(|V_\delta'(u)|+|V_\delta'(v)|\big)\\&\le C(\varepsilon)\,\|u-v\|_{L^2}\,\Big(\int_{\mathbb{R}^n}|V_\delta'(u)|^2+|V_\delta'(v)|^2\Big)^{1/2}\\&\le C(\varepsilon)\,\max\{\|u\|_{L^2},\|v\|_{L^2}\}\,\|u-v\|_{L^2}\,.\end{aligned} \tag{3.31}$$

In summary, by (3.29), (3.30) and (3.31) for every $\varepsilon>0$ and $u,v\in W^{1,2}(\mathbb{R}^n;[0,1])\setminus\{0\}$ with $\max\{\|u\|_{W^{1,2}(\mathbb{R}^n)},\|v\|_{W^{1,2}(\mathbb{R}^n)}\}\le C(\varepsilon)$ we have proved that

$$\big|\lambda_{\varepsilon,\delta}[u]-\lambda_{\varepsilon,\delta}[v]\big|\le C(\varepsilon)\,\frac{|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}}+\frac{C(\varepsilon)\,\|u-v\|_{L^2}}{\min\{1,\mathcal{V}_\delta(u)^{2n}\}\,\min\{1,\mathcal{V}_\delta(v)^{2n}\}}\,. \tag{3.32}$$

We first estimate that

$$\begin{aligned}|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|\;\le\;&2\,\varepsilon\int_{\mathbb{R}^n}|\nabla(u-v)|\,\big(|\nabla u|+|\nabla v|\big)\,|V_\delta''(u)|\\&+2\,\varepsilon\int_{\mathbb{R}^n}|\nabla v|^2\,|V_\delta''(u)-V_\delta''(v)|\\&+\frac{1}{\varepsilon}\int_{\mathbb{R}^n}|W'(u)-W'(v)|\,|V_\delta'(u)|+\frac{1}{\varepsilon}\int_{\mathbb{R}^n}|W'(v)|\,|V_\delta'(u)-V_\delta'(v)|\,.\end{aligned} \tag{3.33}$$

By recalling that $\mathrm{Lip}(W';[0,1])\le C$, $\mathrm{Lip}(V_\delta';[0,1])\le C$, $\mathrm{Lip}(V_\delta'';[0,1])\le C(\delta)$ (when $\delta>0$), and $\max\{|V_\delta'(r)|,|W'(r)|\}\le C\,r$ for $r\in[0,1]$ we find

$$\begin{aligned}&|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|\le C\,\max\big\{\|\nabla u\|_{L^2},\|\nabla v\|_{L^2}\big\}\,\|\nabla u-\nabla v\|_{L^2}\\&+C(\varepsilon,\delta)\,\|u-v\|_{C^0}\int_{\mathbb{R}^n}|\nabla v|^2+\frac{C}{\varepsilon}\,\|u-v\|_{L^2}\,\max\{\|u\|_{L^2},\|v\|_{L^2}\}\\\le\;&C(\varepsilon)\,\max\big\{\|u\|_{W^{1,2}},\|v\|_{W^{1,2}}\big\}\,\|u-v\|_{W^{1,2}}+C(\delta)\,\|\nabla v\|_{L^2}^2\,\|u-v\|_{C^0}\,;\end{aligned} \tag{3.34}$$

while, using $[V_\delta'']_{C^{0,\gamma(n)}[0,1]}\le C$ (which also holds when $\delta=0$) in place of $\mathrm{Lip}(V_\delta'';[0,1])\le C(\delta)$, we find instead

$$\begin{aligned}&|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|\\\le\;&C(\varepsilon)\,\max\big\{\|u\|_{W^{1,2}},\|v\|_{W^{1,2}}\big\}\,\|u-v\|_{W^{1,2}}+C\,\|\nabla v\|_{L^2}^2\,\|u-v\|_{C^0}^{\gamma(n)}\,;\end{aligned} \tag{3.35}$$

thanks to $\max\{\|u\|_X,\|v\|_X\}\le C(\varepsilon)$; with the convention that $C(\varepsilon,\delta)=+\infty$ if $\delta=0$, we thus conclude from (3.33), (3.34) and (3.35) that

$$|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|\le\min\Big\{C(\varepsilon,\delta)\,\|u-v\|_X,C(\varepsilon)\|u-v\|_X^{\gamma(n)}\Big\}\,. \tag{3.36}$$

By means of (3.32) and (3.36) we find that, if $\max\{\|u\|_X,\|v\|_X\}\le C(\varepsilon)$, then (3.6) and (3.7) hold. We now assume that $u,v\in W^{2,2}(\mathbb{R}^n;[0,1])\setminus\{0\}$ and prove (3.9). To this end let us first notice that an integration by parts gives

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla u|^2\,V_\delta''(u)-\int_{\mathbb{R}^n}|\nabla v|^2\,V_\delta''(v)\\
=\;&\int_{\mathbb{R}^n}(\nabla u-\nabla v)\cdot\nabla v\;V_\delta''(v)+\int_{\mathbb{R}^n}\nabla u\cdot\nabla u\;V_\delta''(u)-\nabla u\cdot\nabla v\;V_\delta''(v)\\
=\;&\int_{\mathbb{R}^n}(\nabla u-\nabla v)\cdot\nabla v\,V_\delta''(v)-\int_{\mathbb{R}^n}(\Delta u)\big(V_\delta'(u)-V_\delta'(v)\big)\,,
\end{aligned}
$$

which can be used to replace (3.33) with

$$
\begin{aligned}
|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|\;\le\;&2\,\varepsilon\int_{\mathbb{R}^n}|\Delta u|\,|V_\delta'(u)-V_\delta'(v)|+|\nabla(u-v)|\,|\nabla v|\,|V_\delta''(v)|\\
&+\frac{1}{\varepsilon}\int_{\mathbb{R}^n}|W'(u)-W'(v)|\,|V_\delta'(u)|+\frac{1}{\varepsilon}\int_{\mathbb{R}^n}|W'(v)|\,|V_\delta'(u)-V_\delta'(v)|\,.
\end{aligned}
\tag{3.37}
$$

By $\mathrm{Lip}(W';[0,1])\le C$, $\mathrm{Lip}(V_\delta';[0,1])\le C$ and $\max\{|V_\delta'(r)|,|W'(r)|\}\le C\,r$ for $r\in[0,1]$ we thus find

$$
|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_{\varepsilon,\delta}[v]|\le C(\varepsilon)\,\max\{\|u\|_{W^{2,2}},\|v\|_{W^{1,2}}\}\,\|u-v\|_{W^{1,2}}
$$

which, combined with (3.32) and with the assumption $\max\{\|u\|_{W^{2,2}},\|v\|_{W^{2,2}}\}\le C(\varepsilon)$, gives (3.9).

We now work on (3.29) *with* $u=v$ *and* $\delta_*=0$ *to prove* (3.8). To this end, we first notice that

$$
\begin{aligned}
\big|\mathcal{N}_{\varepsilon,\delta}[u]-\mathcal{N}_\varepsilon[u]\big|\;\le\;&2\,\varepsilon\int_{\mathbb{R}^n}|\nabla u|^2\,|V_\delta''(u)-V''(u)|+\frac{1}{\varepsilon}\int_{\mathbb{R}^n}|W'(u)|\,|V_\delta'(u)-V'(u)|\\
\le\;&2\,\mathcal{AC}_\varepsilon(u)\,\|V_\delta-V\|_{C^2[0,1]}\,,
\end{aligned}
$$

while $V_\delta'(0)=V'(0)=0$ gives $|V_\delta'(u)-V'(u)|\le\|V_\delta''-V''\|_{C^0[0,1]}\,|u|$ on $\mathbb{R}^n$, and thus, arguing as in (3.31), that

$$
\Big|\int_{\mathbb{R}^n}V_\delta'(u)^2-\int_{\mathbb{R}^n}V'(u)^2\Big|\le C\,\|V_\delta-V\|_{C^2[0,1]}\,\|u\|_{L^2(\mathbb{R}^n)}^2\,.
$$

Combining this last two estimates with (3.29), (3.30) and $|\mathcal{N}_{\varepsilon,\delta}[u]|\le C(\varepsilon)\,\|u\|_X$ we immediately prove (3.8). ☐

4. Existence, regularity and entropies of the flow (Proof of Theorem 1.2)

Proof of Theorem 1.2. Step one, existence of mild solutions: For every $\varepsilon,\delta>0$ we introduce the regularized flows

$$
\begin{cases}
\varepsilon^2\,\partial_t u=2\,\varepsilon^2\,\Delta u-W'(u)+\varepsilon\,\lambda_{\varepsilon,\delta}[u(t)]\,V_\delta'(u)\,, & \text{on }\mathbb{R}^n\times(0,\infty)\,,\\
u(0)=u_0\,,
\end{cases}
\tag{DF$_\delta$}
$$

that are obtained by replacing V with V_δ in (DF). If we set

$$
G(x,t)=\frac{e^{-|x|^2/8\,t}}{(8\,\pi\,t)^{n/2}}\,,\qquad (x,t)\in\mathbb{R}^n\times(0,\infty)\,,
\tag{4.1}
$$

and $S_t v=v\star G(t)$ for $v:\mathbb{R}^n\to\mathbb{R}$ and $t>0$, then a solution u of

$$
u_t-2\,\Delta u=f\text{ on }\mathbb{R}^n\times(0,\infty)\,,\qquad u(0)=u_0\text{ on }\mathbb{R}^n\,,
\tag{4.2}
$$

with data $f:\mathbb{R}^n\times(0,\infty)\to\mathbb{R}$ and $u_0:\mathbb{R}^n\to\mathbb{R}$, is formally given by the *Duhamel formula* (see, for example, [Eva98, Section 2.3.1]),

$$
\begin{aligned}
u(x,t) &= S_t u_0(x) + \int_0^t S_{t-s}[f(s)](x)\,ds \qquad (4.3)\\
&= \int_{\mathbb{R}^n} G(x-y,t)\,u_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}^n} G(x-y,t-s)\,f(y,s)\,dy\,.
\end{aligned}
$$

Solutions to the integral equation (4.3) are usually called *mild solutions* of the parabolic PDE (4.2), and can be constructed by fixed points arguments. We now set up the stage to prove the short-time existence of a unique mild solution to (DF_δ).

From now on we fix u_0 with

$$
u_0 \in \bigcap_{p\ge 2} W^{2,p}(\mathbb{R}^n;[0,1])\,, \qquad \mathcal{V}(u_0)=1\,. \tag{4.4}
$$

By the properties of V_δ and W it is easily seen that $\mathcal{V}(u_0)=1$ implies $\mathcal{V}_\delta(u_0)\in(0,\infty)$ for every $\delta>0$. Moreover, we also have

$$
\lim_{\delta\to 0^+} \mathcal{V}_\delta(u_0)=\mathcal{V}(u_0)=1\,. \tag{4.5}
$$

Given parameters $\tau>0$ and $\sigma\in(0,1)$ to be chosen in a moment, we then introduce the vector space

$$
Y=\big\{u\in C^0\big([0,\tau);B_\sigma^X(u_0)\big) : u(0)=u_0\big\}\,, \tag{4.6}
$$

where $B_\sigma^X(u_0)$ denotes the ball in X of radius σ and center u_0. If we pick σ small enough depending on u_0, and since $\mathcal{V}_\delta(u_0)>0$ for every $\delta>0$, we find that (see (4.10) below) $\mathcal{V}_\delta(u(t))>0$ for every $t\in[0,\tau)$: in particular, $\lambda_{\varepsilon,\delta}[u(t)]$ is well defined for every $t\in[0,\tau)$. Hence, for each $\varepsilon>0$, $\delta\in(0,\delta_0]$, and $u\in Y$ we can define $F_\delta[u]:\mathbb{R}^n\times[0,\tau)\to\mathbb{R}$ and $T_\delta[u]:\mathbb{R}^n\times[0,\tau)\to\mathbb{R}$ by setting

$$
\begin{aligned}
F_\delta[u] &= -\frac{W'(u)}{\varepsilon^2}+\lambda_{\varepsilon,\delta}[u(t)]\,\frac{V_\delta'(u)}{\varepsilon}\,, \qquad (4.7)\\
T_\delta[u] &= S_t u_0 + \int_0^t S_{t-s}[F_\delta[u](s)]\,ds\,. \qquad (4.8)
\end{aligned}
$$

We claim that, if τ and σ (introduced in the definition of Y) are small enough with respect to ε, δ, and u_0, then $u\mapsto T_\delta[u]$ defines a contraction of the Banach space Y. By the Banach fixed point theorem we will then deduce the existence of a unique $u\in C^0([0,\tau);B_\sigma^X(u_0))$ such that $u(0)=u_0$ and

$$
u(t)=S_t u_0+\int_0^t S_{t-s}[F_\delta[u](s)]\,ds\,, \tag{4.9}
$$

for every $t\in(0,\tau)$. In particular, this claim will prove the existence of a unique, short-time, mild solution u of (DF_δ).

To prove our claim, we begin by showing that if σ is small enough in terms of u_0 and δ, then for every $u\in Y$ we have

$$
\mathcal{V}_\delta(u(t))\ge\frac{\mathcal{V}_\delta(u_0)}{2}\,, \qquad \forall t\in[0,\tau)\,. \tag{4.10}
$$

Indeed, by $|V_\delta'(r)|\le C\,|r|$ for every $r\in(0,1)$ and by $\|u_0-u(t)\|_X<\sigma<1$ we find that

$$
\begin{aligned}
\mathcal{V}_\delta(u(t))-\mathcal{V}_\delta(u_0) &= \int_{\mathbb{R}^n}\int_0^1 V_\delta'\big(s\,u_0+(1-s)\,u(t)\big)\,(u_0-u(t))\,ds\\
&\ge -C\int_{\mathbb{R}^n}|u_0-u(t)|\,\big(|u_0|+|u(t)|\big)\\
&\ge -C\,\|u(t)-u_0\|_{L^2}\,\big(\|u_0\|_{L^2}+\|u(t)\|_{L^2}\big)\\
&\ge -C\,\sigma\,\big(2\,\|u_0\|_{L^2}+\sigma\big)\ge-\frac{\mathcal{V}_\delta(u_0)}{2}\,,
\end{aligned}
$$

provided σ is small enough in terms of δ and u_0 (recall that $\mathcal{V}_\delta(u_0)>0$). Having proved (4.10) we notice that, combined with (3.5), (3.6), and $\mathcal{AC}_\varepsilon(u)\le C(\varepsilon)\,\|u\|_{W^{1,2}}$, it implies

$$
\begin{aligned}
\big|\lambda_{\varepsilon,\delta}[u(t)]\big| &\le C(\varepsilon,\delta,u_0) && (4.11)\\
\big|\lambda_{\varepsilon,\delta}[u(t)]-\lambda_{\varepsilon,\delta}[v(t)]\big| &\le C(\varepsilon,\delta,u_0)\,\|u(t)-v(t)\|_X\,, && (4.12)
\end{aligned}
$$

for every $u,v\in Y$ and every $t\in[0,\tau)$. Next we notice that if $u\in Y$, then for every $t\in(0,\tau)$ we have

$$
|F_\delta[u(t)]|\le C(\varepsilon)\,\max\big\{1,|\lambda_{\varepsilon,\delta}[u(t)]\big\}\,|u(t)|\,,\qquad\text{on }\mathbb{R}^n\,; \tag{4.13}
$$

moroever, since $|W'(r)|\le C\,|r|$ and $|V_\delta'|\le C\,|r|$ for $r\in(0,1)$, W'' and V_δ'' are bounded on $[0,1]$, and

$$
\nabla\big(F_\delta[u](t)\big)=\Big\{-\frac{W''(u)}{\varepsilon^2}+\lambda_{\varepsilon,\delta}[u(t)]\,\frac{V_\delta''(u)}{\varepsilon}\Big\}\,\nabla u(t)\,, \tag{4.14}
$$

then

$$
|\nabla F_\delta[u(t)]|\le C(\varepsilon)\,\max\big\{1,|\lambda_{\varepsilon,\delta}[u(t)]\big\}\,|\nabla u(t)|\,,\qquad\text{on }\mathbb{R}^n\,. \tag{4.15}
$$

By combining (4.13) and (4.15) with (4.11), we see that if $u\in Y$, then $F_\delta[u]\in C^0([0,\tau);X)$ and

$$
\begin{aligned}
\|F_\delta[u](t)\|_{C^0} &\le C(\varepsilon)\,\max\big\{1,|\lambda_{\varepsilon,\delta}[u(t)]\big\}\,\|u(t)\|_{C^0}\,, && (4.16)\\
\|F_\delta[u](t)\|_{L^2} &\le C(\varepsilon)\,\max\big\{1,|\lambda_{\varepsilon,\delta}[u(t)]\big\}\,\|u(t)\|_{L^2}\,, && (4.17)\\
\|\nabla(F_\delta[u](t))\|_{L^2} &\le C(\varepsilon)\,\max\big\{1,|\lambda_{\varepsilon,\delta}[u(t)]\big\}\,\|\nabla u(t)\|_{L^2},\qquad\forall t\in[0,\tau)\,. && (4.18)
\end{aligned}
$$

By combining these estimates with (4.11) we thus find

$$
\|F_\delta[u(t)]\|_X\le C(\varepsilon,\delta,u_0)\,\|u\|_X\,,\qquad\forall t\in[0,\tau)\,. \tag{4.19}
$$

Since $\|S_tv\|_{C^0}\le\|v\|_{C^0}$, $\|S_tv\|_{L^2}\le\|v\|_{L^2}$, and $\|\nabla(S_tv)\|_{L^2}\le\|\nabla v\|_{L^2}$ for every $v\in X$ and every $t>0$, we deduce that if $u\in Y$, then $T_\delta[u]\in C^0([0,\tau);X)$. Moreover, since $\|S_tu_0-u_0\|_X\to0$ as $t\to0^+$, if we pick τ small enough in terms of ε, δ, u_0 and σ (where σ has already been chosen small enough in terms of u_0 and δ), then we find

$$
\begin{aligned}
\|T_\delta[u](t)-u_0\|_X &\le \|S_tu_0-u_0\|_X+t\sup_{0<s<t}\|F_\delta[u](t)\|_X\\
&\le \|S_tu_0-u_0\|_X+C(\varepsilon,\delta,u_0)\,t\,\|u(t)\|_X<\sigma\,,
\end{aligned}
$$

for every $t\in[0,\tau_0)$, so that $T_\delta[u]\in Y$ for every $u\in Y$. Having proved that T_δ is a self-map of Y, we now pick $u,v\in Y$ and notice that for every $t\in[0,\tau)$ we have

$$
\|T_\delta[u]-T_\delta[v]\|_Y=\sup_{0<t<\tau}\|T_\delta[u](t)-T_\delta[v](t)\|\le\tau\sup_{0<t<\tau}\|F_\delta[u](t)-F_\delta[v](t)\|\,. \tag{4.20}
$$

Now for every $u, v \in Y$ and $t \in [0, \tau)$ we have, pointwise on $\mathbb{R}^n$,

$$
\begin{aligned}
|F_\delta[u](t) - F_\delta[v](t)| &\le \Big\{ \frac{\mathrm{Lip}(W')}{\varepsilon^2} + \big|\lambda_{\varepsilon,\delta}[u(t)]\big| \, \frac{\mathrm{Lip}(V_\delta')}{\varepsilon} \Big\} \, |u(t) - v(t)| \\
&\qquad + \big|\lambda_{\varepsilon,\delta}[u(t)] - \lambda_{\varepsilon,\delta}[v(t)]\big| \, \frac{|V_\delta'(v(t))|}{\varepsilon} \qquad (4.21)\\
&\le C(\varepsilon) \Big\{ \max\big\{1, \big|\lambda_{\varepsilon,\delta}[u(t)]\big|\big\} \, |u(t) - v(t)| + \big|\lambda_{\varepsilon,\delta}[u(t)] - \lambda_{\varepsilon,\delta}[v(t)]\big| \, |v(t)| \Big\} ,
\end{aligned}
$$

where we have used $|V_\delta'(r)| \le C\,|r|$ for $r \in [0,1]$; then, by (4.11), (4.12), and (4.21), we find that

$$
\|F_\delta[u](t) - F_\delta[v](t)\|_X \le C(\varepsilon, \delta, u_0) \, \|u(t) - v(t)\|_X \,, \qquad \forall t \in [0, \tau) \,. \tag{4.22}
$$

Similarly, starting from (4.14), we find that, pointwise on $\mathbb{R}^n$,

$$
\begin{aligned}
|\nabla(F_\delta[u](t)) - \nabla(F_\delta[v](t))| &\le \Big\{ \frac{\mathrm{Lip}(W')}{\varepsilon^2} + \big|\lambda_{\varepsilon,\delta}[u(t)]\big| \, \frac{\mathrm{Lip}(V_\delta')}{\varepsilon} \Big\} \, |\nabla u(t) - \nabla v(t)| \\
&\qquad + \Big\{ \frac{\mathrm{Lip}(W'')}{\varepsilon^2} + \frac{\mathrm{Lip}(V_\delta'')}{\varepsilon} \, |\lambda_{\varepsilon,\delta}[v(t)]| \Big\} \, |\nabla v(t)| \, |u(t) - v(t)| \\
&\qquad + \Big|\lambda_{\varepsilon,\delta}[u(t)] - \lambda_{\varepsilon,\delta}[v(t)]\Big| \, |\nabla v(t)| \, \frac{|V_\delta''(u(t))|}{\varepsilon} \\
\le C(\varepsilon, \delta) \Big\{ \max\big\{1, \big|\lambda_{\varepsilon,\delta}[u(t)]\big|\big\} & \big\{ |\nabla v(t)| \, |u(t) - v(t)| + |\nabla u(t) - \nabla v(t)| \big\} \qquad (4.23) \\
&\qquad + \Big|\lambda_{\varepsilon,\delta}[u(t)] - \lambda_{\varepsilon,\delta}[v(t)]\Big| \, |\nabla v(t)| \Big\} ,
\end{aligned}
$$

where we have made crucial use of the regularization V_δ of V to assert that $\mathrm{Lip}(V_\delta'') \le C(\delta)$. By (4.11), (4.12), (4.23), and $\sigma < 1$,

$$
\|\nabla(F_\delta[u](t)) - \nabla(F_\delta[v](t))\|_{L^2} \le C(\varepsilon, \delta, u_0) \, \|u(t) - v(t)\|_X \,, \qquad \forall t \in [0, \tau) \,. \tag{4.24}
$$

By combining (4.22) and (4.24) with (4.20) we conclude that

$$
\|T_\delta[u] - T_\delta[v]\|_Y \le C(\varepsilon, \delta, u_0) \, \tau \, \|u - v\|_Y \,.
$$

In particular, up to further decrease τ depending on ε, δ and u_0, we can ensure that $\mathrm{Lip}(T_\delta; Y) < 1$, and that T_δ is a contraction of Y.

Step two, regularity and global-in-time existence for (DF_δ): Given $\varepsilon > 0$, $\delta \in (0, \delta_0]$ and u_0 as in (4.4), by step five we can define $\tau_* \in (0, \infty]$ as the supremum of those $\tau > 0$ such that there exists $u \in C^0([0, \tau); X)$ such that $u(0) = u_0$ and (4.9) holds for every $t \in (0, \tau)$ and

$$
\sup_{0<t<\tau} \big\{ \mathcal{AC}_\varepsilon(u(t)), \mathcal{V}_\delta(u(t)) \big\} < \infty \,, \qquad \inf_{0<t<\tau} \mathcal{V}_\delta(u(t)) > 0 \,. \tag{4.25}
$$

In this step we prove that $\tau_* = +\infty$, that $\mathcal{AC}_\varepsilon(u(t))$ is Lipschitz continuous and decreasing on $[0, \infty)$, that $\mathcal{V}_\delta(u(t)) = \mathcal{V}_\delta(u_0)$ for every $t \in [0, \infty)$, and that $u(t) \in W^{3,p}(\mathbb{R}^n)$ and $\partial_t u(t) \in W^{1,p}(\mathbb{R}^n)$ for every $t > 0$ and $p \ge 2$ with

$$
\begin{aligned}
&\max\big\{ \|u(t)\|_{W^{3,p}(\mathbb{R}^n)}, \|\partial_t u(t)\|_{W^{1,p}} \big\} \qquad (4.26)\\
&\le C(\varepsilon, p) \, M\big(\mathcal{AC}_\varepsilon(u_0), \mathcal{V}_\delta(u_0), 1/\mathcal{V}_\delta(u_0), \|\nabla u_0\|_{W^{2,p}}, t\big) \,,
\end{aligned}
$$

where M denotes a generic constant which is *increasing and continuous* in its arguments.

We first notice that by combining $0 \le u \le 1$, (3.3), (3.5) and (4.25) we find that

$$
\sup_{0<s<t} \|u(s)\|_{L^p} < \infty \,, \qquad \sup_{0<s<t} \big|\lambda_{\varepsilon,\delta}[u(s)]\big| < \infty \tag{4.27}
$$

for every $t \in (0, \tau_*)$ and $p \ge 2$. Next, setting $G(t) = G(\cdot, t)$ (recall (4.1)), we notice that for every $t > 0$ we have

$$
\|G(t)\|_{L^1} = 1 \,, \qquad \max\big\{ t^{1/2} \|\nabla G(t)\|_{L^1}, \, t^{3/2} \|\nabla^2 G(t)\|_{L^1}, \, t \|\partial_t G(t)\|_{L^1} \big\} \le C(n) \,. \tag{4.28}
$$

Combining $u(t) = T_\delta[u(t)]$ with (4.28), (4.13) and standard applications of Fubini's theorem and Hölder's inequality, we find that, if $p \ge 2$, then, for every $t \in (0, \tau_*)$,

$$\begin{aligned} \|\nabla u(t)\|_{L^p} &\le \|\nabla u_0\|_{L^p} \|G(t)\|_{L^1} \\ &\quad + C(\varepsilon) \sup_{0<s<t} \max\left\{1, \left|\lambda_{\varepsilon,\delta}[u(s)]\right|\right\} \|u(s)\|_{L^p} \int_0^t \|\nabla G(t-s)\|_{L^1}\, ds \\ &\le \|\nabla u_0\|_{L^p} + C(\varepsilon)\, t^{1/2} \sup_{0<s<t} \max\left\{1, \left|\lambda_{\varepsilon,\delta}[u(s)]\right|\right\} \|u(s)\|_{L^p}\,, \end{aligned} \tag{4.29}$$

so that (4.27) gives

$$\sup_{0<s<t} \|\nabla u(s)\|_{L^p} < \infty\,, \qquad \forall t \in (0, \tau_*)\,, p \ge 2\,. \tag{4.30}$$

Similarly, again from $u(t) = T_\delta[u(t)]$ and (4.28), we obtain that

$$\begin{aligned} \|\nabla^2 u(t)\|_{L^p} &\le \|\nabla^2 u_0\|_{L^p} \|G(t)\|_{L^1} \\ &\quad + C(\varepsilon) \sup_{0<s<t} \max\left\{1, \left|\lambda_{\varepsilon,\delta}[u(s)]\right|\right\} \|\nabla u(s)\|_{L^p} \int_0^t \|\nabla G(t-s)\|_{L^1}\, ds \\ &\le \|\nabla^2 u_0\|_{L^p} + C(\varepsilon)\, t^{1/2} \sup_{0<s<t} \max\left\{1, \left|\lambda_{\varepsilon,\delta}[u(s)]\right|\right\} \|\nabla u(s)\|_{L^p} \end{aligned} \tag{4.31}$$

which, combined with (4.11), (4.27), and (4.30), gives

$$\sup_{0<s<t} \|u(s)\|_{W^{2,p}} < \infty\,, \qquad \forall t \in (0, \tau_*)\,, p \ge 2\,. \tag{4.32}$$

In particular, $u \in L^\infty_{\rm loc}((0,\tau); W^{2,2}(\mathbb{R}^n))$, so that, by standard properties of mild solutions (see e.g. [CH98, Proposition 4.1.9]) (4.9) implies that

$$u \in W^{1,1}_{\rm loc}((0,\tau_*); X)\,, \tag{4.33}$$

with

$$\partial_t u = 2\,\Delta u + f_\delta \qquad \text{a.e. on } \mathbb{R}^n \times (0, \tau_*)\,, \tag{4.34}$$

where we have set $f_\delta(x,t) = F_\delta[u](x,t)$, and where, by (4.13) and (4.15),

$$\|f_\delta(t)\|_{L^p} \le C(\varepsilon) \max\left\{1, |\lambda_{\varepsilon,\delta}[u(t)]\right\} \|u(t)\|_{L^p}\,, \tag{4.35}$$

$$\|\nabla f_\delta(t)\|_{L^p} \le C(\varepsilon) \max\left\{1, |\lambda_{\varepsilon,\delta}[u(t)]\right\} \|\nabla u(t)\|_{L^p}\,, \qquad \forall p \ge 2\,. \tag{4.36}$$

Moreover, by (4.34), we see that for every $t \in (0, \tau_*)$ and $p \ge 2$,

$$\|\partial_t u(t)\|_{L^p} \le C(\varepsilon) \left\{ \|\nabla^2 u(t)\|_{L^p} + \|f_\delta(t)\|_{L^p} \right\}. \tag{4.37}$$

We now differentiate the flow in space to obtain L^p-estimates on $\nabla^3 u$ and on $\nabla(\partial_t u)$. Given $e \in \mathbb{R}^n$ with $|e| = 1$ and $v = v(x,t)$ we set $e_h v(x,t) = (v(x + h\,e, t) - v(x,t))/h$ for the (spatial) incremental ratio of v in the direction v of step h. In this way (4.34) implies that $e_h u$ solves

$$\partial_t(e_h u) - 2\,\Delta(e_h u) = e_h f_\delta \qquad \text{on } \mathbb{R}^n \times (0,\infty), \qquad e_h u(0) = e_h u_0 \text{ on } \mathbb{R}^n\,. \tag{4.38}$$

Next we consider the decomposition $e_h u = u_{1,h} + u_{2,h}$ where $u_{1,h}(t) = S_t[e_h u_0]$ and thus

$$\partial_t u_{2,h} - 2\,\Delta u_{2,h} = e_h f_\delta \qquad \text{on } \mathbb{R}^n \times (0,\infty), \qquad u_{2,h}(0) = 0 \text{ on } \mathbb{R}^n\,. \tag{4.39}$$

Since $u_{2,h}(0) = 0$ on $\mathbb{R}^n$ we can apply [Lie96, Corollary 7.31] to deduce that for every $(a,b) \subset\subset (0,\tau_*)$ and $p \ge 2$

$$\int_a^b dt \int_{\mathbb{R}^n} |\partial_t u_{2,h}(t)|^p + |\nabla^2 u_{2,h}(t)|^p \le C(p) \int_a^b dt \int_{\mathbb{R}^n} |e_h f_\delta(t)|^p\,. \tag{4.40}$$

Since $\|e_h f_\delta(t)\|_{L^p} \le \|\nabla f_\delta(t)\|_{L^p}$ and, thanks to (4.36), (4.27), and (4.30),

$$\sup_{0<s<t} \|\nabla f_\delta(s)\|_{L^p} < \infty\,, \qquad \forall t \in (0,\tau_*)\,,$$

we conclude that, for every $(a,b)\subset(0,t)$, $t<\tau_*$, and $p\ge 2$,

$$\int_a^b dt\int_{\mathbb{R}^n}|\partial_t u_{2,h}(t)|^p+|\nabla^2 u_{2,h}(t)|^p\le C(p)(b-a)\sup_{a<s<b}\|\nabla f_\delta(s)\|_{L^p}^p\,,\tag{4.41}$$

so that, by arbitrariness of (a,b), for every $(a,b)\subset\subset(0,\tau_*)$ and $p\ge 2$,

$$\sup_{a<s<b}\big\{\|\partial_t u_{2,h}(s)\|_{L^p},\|\nabla^2 u_{2,h}(s)\|_{L^p}\big\}\le C(p)\sup_{a<s<b}\|\nabla f_\delta(s)\|_{L^p}\,.\tag{4.42}$$

At the same time, recalling that $u_{1,h}(s)=S_s(e_h u_0)$, by (4.28) we find that for every $s>0$

$$\begin{aligned}\|\partial_t u_{1,h}(s)\|_{L^p}&\le\|\partial_t G(s)\|_{L^1}\,\|e_h u_0\|_{L^p}\le\frac{C}{s}\,\|\nabla u_0\|_{L^p}\,,\\ \|\nabla^2 u_{1,h}(s)\|_{L^p}&\le\|\nabla^2 G(s)\|_{L^1}\,\|e_h u_0\|_{L^p}\le\frac{C}{s^{3/2}}\,\|\nabla u_0\|_{L^p}\,.\end{aligned}$$

By combining these last two estimates with (4.42) and $e_h u=u_{1,h}+u_{2,h}$, by the uniformity in $h>0$ and e with $|e|=1$, we conclude that for every $p\ge 2$ and $(a,b)\subset\subset(0,\tau_*)$ we have

$$\sup_{a<s<b}\big\{\|\nabla(\partial_t u)(s)\|_{L^p},\|\nabla^3 u(s)\|_{L^p}\big\}\le C(p)\sup_{a<s<b}\Big\{\frac{\|\nabla u_0\|_{L^p}}{a^{3/2}},\|\nabla f_\delta(s)\|_{L^p}\Big\}\,.\tag{4.43}$$

In combination with (4.32), (4.37), (4.35), and (4.27) this implies that, for every $p\ge 2$ and $(a,b)\subset\subset(0,\tau_*)$ we have

$$\sup_{a<t<b}\big\{\|\partial_t u(t)\|_{W^{1,p}},\|u(t)\|_{W^{3,p}}\big\}<\infty\,.\tag{4.44}$$

In particular, $u(t)\in C^{2,\alpha}(\mathbb{R}^n)$ and $\partial_t u(t)\in C^{0,\alpha}(\mathbb{R}^n)$ for every $\alpha\in(0,1)$ and $t\in(0,\tau_*)$.

The regularity of $u(t)$ established in (4.44) is sufficient to prove that

$$\mathcal{V}_\delta(u(t))=\mathcal{V}_\delta(u_0)\,,\qquad\forall t\in[0,\tau_*)\,,\tag{4.45}$$

$$\begin{aligned}&t\mapsto\mathcal{AC}_\varepsilon(u(t))\text{ is decreasing, continuous on }[0,\tau_*),\\ &\qquad\qquad\text{and locally Lipschitz on }(0,\tau_*)\,.\end{aligned}\tag{4.46}$$

Indeed, let us set $u_\eta=(\psi_\eta\,u)\star\rho_\eta$, where, for each $\eta>0$, ρ_η is a mollifying kernel on $\mathbb{R}^n\times\mathbb{R}$, and where $\psi_\eta\in C^\infty_c(\mathbb{R}^n\times(0,\infty))$ is monotonically converging to 1 as $\eta\to 0^+$. If we set $D(t)=\int_{\mathbb{R}^n}|\nabla u(t)|^2$ and $D_\eta(t)=\int_{\mathbb{R}^n}|\nabla u_\eta(t)|^2$, then by (4.30) we find that, for every $\varphi\in C^\infty_c(0,\tau)$,

$$\begin{aligned}&\int_0^\tau D(t)\,\varphi'(t)\,dt=\lim_{\eta\to0^+}\int_0^\tau D_\eta(t)\,\varphi'(t)\,dt\\ &=-2\lim_{\eta\to0^+}\int_0^\tau\varphi(t)\,dt\int_{\mathbb{R}^n}\nabla u_\eta(t)\cdot\nabla(\partial_t u_\eta(t))\\ &=-2\int_0^\tau\varphi(t)\,dt\int_{\mathbb{R}^n}\nabla u(t)\cdot\nabla(\partial_t u(t))=2\int_0^\tau\varphi(t)\,dt\int_{\mathbb{R}^n}\Delta u(t)\,\partial_t u(t)\,,\end{aligned}$$

where, in computing the second limit, we have used (4.44). We have thus proven that D is locally Lipschitz continuous on $(0,\tau_*)$, with

$$D'(t)=\frac{d}{dt}\int_{\mathbb{R}^n}|\nabla u(t)|^2=-\int_{\mathbb{R}^n}2\,\Delta u(t)\,\partial_t u(t)\,.\tag{4.47}$$

for a.e. $t\in(0,\tau_*)$. By an analogous approximation argument we see that, for a.e. $t\in(0,\tau_*)$,

$$\frac{d}{dt}\int_{\mathbb{R}^n}W(u(t))=\int_{\mathbb{R}^n}W'(u)\,\partial_t u\,,\qquad\frac{d}{dt}\int_{\mathbb{R}^n}V_\delta(u(t))=\int_{\mathbb{R}^n}V_\delta'(u)\,\partial_t u\,.\tag{4.48}$$

In particular, by (4.34), for a.e. $t \in (0,\tau_*)$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} V_\delta'(u)\,\partial_t u &= \int_{\mathbb{R}^n} \Big(2\,\Delta u - \frac{W'(u)}{\varepsilon^2} + \lambda_{\varepsilon,\delta}[u(t)]\,\frac{V_\delta'(u)}{\varepsilon}\Big)\,V_\delta'(u)\\
&= -\int_{\mathbb{R}^n} 2\,\nabla u\cdot\nabla(V_\delta'(u)) - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^n} W'(u)\,V_\delta'(u) + \frac{\lambda_{\varepsilon,\delta}[u(t)]}{\varepsilon}\int_{\mathbb{R}^n} V_\delta'(u)^2 = 0\,,
\end{aligned}
$$

where the last identity follows from the definition (3.2) of $\lambda_{\varepsilon,\delta}[u(t)]$. This proves that $\mathcal{V}_\delta(u(t))$ is constant on $(0,\tau_*)$, and since $t\mapsto\mathcal{V}_\delta(u(t))$ is continuous on $[0,\tau_*)$, we deduce (4.45). Finally, by (4.47) and (4.48) we have that $t\mapsto\mathcal{AC}_\varepsilon(u(t))$ is locally Lipschitz continuous on $(0,\tau_*)$ with

$$
\begin{aligned}
\frac{d}{dt}\mathcal{AC}_\varepsilon(u(t)) &= \varepsilon\int_{\mathbb{R}^n} \partial_t u\Big\{-2\,\Delta u + \frac{W'(u)}{\varepsilon^2}\Big\}\\
&= \varepsilon\int_{\mathbb{R}^n} \partial_t u\Big\{-\partial_t u + \lambda_{\varepsilon,\delta}[u(t)]\,\frac{V_\delta'(u)}{\varepsilon}\Big\} = -\varepsilon\int_{\mathbb{R}^n}(\partial_t u)^2\,.
\end{aligned}
$$

where we have taken into account $\int_{\mathbb{R}^n} V_\delta'(u)\,\partial_t u = 0$. The continuity of $t\mapsto\mathcal{AC}_\varepsilon(u(t))$ on $[0,\tau_*)$ is of course immediate from $u\in C^0([0,\tau_*);X)$. This proves (4.46).

We now prove that $\tau_* = +\infty$. We argue by contradiction, and assume that $\tau_* < \infty$. By combining (4.45) and (4.46) (which implies $\mathcal{AC}_\varepsilon(u(t))\le\mathcal{AC}_\varepsilon(u_0)$ for every $t\in(0,\tau_*)$) with (3.3) and (3.5) we deduce that, for every $p\ge 2$,

$$
\sup_{0<t<\tau_*}\big\{\|u(t)\|_{L^p},|\lambda_{\varepsilon,\delta}[u(t)]|\big\} \le C(\varepsilon)\,M\big(\varepsilon,\mathcal{AC}_\varepsilon(u_0),\mathcal{V}_\delta(u_0),1/\mathcal{V}_\delta(u_0)\big)\,. \tag{4.49}
$$

Then, by (4.49) and (4.29) we find that, for every $p\ge 2$,

$$
\sup_{0<t<\tau_*}\|\nabla u(t)\|_{L^p} \le C(\varepsilon)\,M\big(\mathcal{AC}_\varepsilon(u_0),\mathcal{V}_\delta(u_0),1/\mathcal{V}_\delta(u_0),\|\nabla u_0\|_{L^p},\tau_*\big)\,, \tag{4.50}
$$

which, combined with (4.31) gives that, for every $p\ge 2$,

$$
\sup_{0<t<\tau_*}\|\nabla^2 u(t)\|_{L^p} \le C(\varepsilon,p)\,M\big(\varepsilon,\mathcal{AC}_\varepsilon(u_0),\mathcal{V}_\delta(u_0),1/\mathcal{V}_\delta(u_0),\|\nabla u_0\|_{W^{1,p}},\tau_*\big)\,. \tag{4.51}
$$

By combining (4.35) and (4.36) with (4.49) and (4.50) we find that, for every $p\ge 2$,

$$
\sup_{0<t<\tau_*}\|f_\delta(t)\|_{W^{1,p}} \le C(\varepsilon)\,M\big(\varepsilon,\mathcal{AC}_\varepsilon(u_0),\mathcal{V}_\delta(u_0),1/\mathcal{V}_\delta(u_0),\|\nabla u_0\|_{L^p},\tau_*\big)\,, \tag{4.52}
$$

so that by (4.37) and (4.43) with (4.51) and (4.52) we find

$$
\begin{aligned}
&\sup_{\tau_*/2<t<\tau_*}\big\{\|\nabla^3 u(t)\|_{L^p},\|\partial_t u(t)\|_{W^{1,p}}\big\}\\
&\qquad\le C(\varepsilon,p)\,M\big(\mathcal{AC}_\varepsilon(u_0),\mathcal{V}_\delta(u_0),1/\mathcal{V}_\delta(u_0),\|\nabla u_0\|_{W^{2,p}},\tau_*,1/\tau_*\big)\,,
\end{aligned}
$$

which combined with (4.49), (4.50) and (4.51) finally gives, for every $p\ge 2$,

$$
\begin{aligned}
&\sup_{\tau_*/2<t<\tau_*}\big\{\|u(t)\|_{W^{3,p}},\|\partial_t u(t)\|_{W^{1,p}}\big\}\\
&\qquad\le C(\varepsilon,p)\,M\big(\mathcal{AC}_\varepsilon(u_0),\mathcal{V}_\delta(u_0),1/\mathcal{V}_\delta(u_0),\|\nabla u_0\|_{W^{2,p}},\tau_*,1/\tau_*\big)\,.
\end{aligned} \tag{4.53}
$$

By the $W^{1,2}$-estimate on $\partial_t u$ contained in (4.53), we can deduce that for every $t,s\in(\tau_*/2,\tau_*)$ it holds

$$
\|u(t)-u(s)\|_X \le C(\varepsilon,\delta,u_0,\tau_*)\,|t-s|\,. \tag{4.54}
$$

Since X is a Banach space, this means that there is $u_*\in X = C^0\cap W^{1,2}(\mathbb{R}^n;[0,1])$ such that

$$
\lim_{t\to\tau_*^-}\|u(t)-u_*\|_X = 0\,.
$$

By combining this last fact with (4.53) we see that $u_* \in W^{3,p}(\mathbb{R}^n)$ for every $p \geq 2$, with

$$\mathcal{V}_\delta(u_*) = \lim_{t \to \tau_*^-} \mathcal{V}_\delta(u(t)) = \mathcal{V}_\delta(u_0) > 0\,.$$

Therefore, for some $\eta > 0$, we can repeat the argument of step five to extend u as an element of $C^0([0, \tau_* + \eta); X)$ such that (4.9) holds for every $t \in (0, \tau_* + \eta)$. We can also repeat the proof of (4.45) and (4.46) to show that $\mathcal{AC}_\varepsilon(u(t))$ is decreasing on $[\tau_*, \tau_* + \eta)$ and $\mathcal{V}_\delta(u(t))$ is constant on $t \in [\tau_*, \tau_* + \eta)$. Since τ_* was introduced as the supremum of those $\tau > 0$ such that there exists $u \in C^0([0, \tau); X)$ with $u(0) = u_0$, solving (4.9) for every $t \in (0, \tau)$, and such that

$$\sup_{0<t<\tau} \big\{\mathcal{AC}_\varepsilon(u(t)), \mathcal{V}_\delta(u(t))\big\} < \infty\,, \qquad \inf_{0<t<\tau} \mathcal{V}_\delta(u(t)) > 0\,. \tag{4.55}$$

and since we have just proved that the bounds stated in (4.55) holds with $\tau = \tau_* + \eta$, thus reaching a contradiction with the maximality of τ_*. This proves that $\tau_* = +\infty$. The monotonicity of $\mathcal{AC}_\varepsilon(u(t))$ and constancy of $\mathcal{V}_\delta(u(t))$ then hold on $[0, \infty)$, and the validity of (4.26) is deduced by arguing as in the proof of (4.53).

Step three: We prove, for an initial datum u_0 as in (4.4), the existence of $u \in C^0(\mathbb{R}^n \times [0, \infty))$ with $u(0) = u_0$ which is a classical solution of (DF) on $\mathbb{R}^n \times (0, \infty)$, with $\mathcal{V}(u(t)) = \mathcal{V}(u_0) = 1$ for every $t \geq 0$ and such that $t \mapsto \mathcal{AC}_\varepsilon(u(t))$ is continuous and decreasing on $[0, \infty)$.

Given $\varepsilon > 0$ and $\delta \in (0, \delta_0]$, let us denote by u^δ the unique global solution of (DF$_\delta$) with $u^\delta(0) = u_0$ constructed in step six. Recalling that u^δ satisfies (4.26), and keeping in mind that $\mathcal{V}_\delta(u_0) \to \mathcal{V}(u_0) = 1$ by (4.5), up to make δ_0 depend on u_0 too, we can ensure that $1/2 \leq \mathcal{V}_\delta(u_0) \leq 2$ for every $\delta \in (0, \delta_0]$, and thus deduce from (4.26) that for every $(a, b) \subset (0, \infty)$ and $p \geq 2$, uniformly on $\delta \in (0, \delta_0]$,

$$\sup_{a<s<b} \big\{\|u^\delta(s)\|_{W^{3,p}}, \|\partial_t u^\delta(s)\|_{W^{1,p}}\big\} \leq C\big(\varepsilon, p, \mathcal{AC}_\varepsilon(u_0), \|\nabla u_0\|_{W^{2,p}}, a, b\big)\,. \tag{4.56}$$

By Morrey's embedding theorem we can obtain a $C^{0,1/2}$-estimate on $\partial_t u^\delta$ from (4.56), which combined with the $W^{1,2}$-estimate on $\partial_t u^\delta$ contained in (4.56) leads to prove that, for every $(a, b) \subset (0, \infty)$,

$$\|u^\delta(r) - u^\delta(s)\|_X \leq C(\varepsilon, u_0, a, b)\,|r - s|, \qquad \forall r, s \in (a, b)\,, \tag{4.57}$$

uniformly on $\delta \in (0, \delta_0]$.

If we now consider a sequence $\delta_j \to 0^+$, then, up to extracting a subsequence, we deduce from (4.57) and (4.56) that there is $u \in \mathrm{Lip}_{\mathrm{loc}}((0, \infty); X)$ with

$$\sup_{r,s\in(a,b)} \frac{\|u(r) - u(s)\|_X}{|r - s|} \;\leq\; C(\varepsilon, u_0, a, b)\,, \tag{4.58}$$

$$\sup_{a<s<b} \big\{\|u(s)\|_{W^{3,p}}, \|\partial_t u(s)\|_{W^{1,p}}\big\} \;\leq\; C\big(\varepsilon, p, \mathcal{AC}_\varepsilon(u_0), \|\nabla u_0\|_{W^{2,p}}, a, b\big)\,, \tag{4.59}$$

for every $p \geq 2$ and $(a, b) \subset (0, \infty)$, and such that

$$\lim_{j\to\infty} \sup_{a<s<b} \big\{\|u^{\delta_j}(s) - u(s)\|_{W^{2,p}}, \|\partial_t u^{\delta_j}(s) - \partial_t u(s)\|_{L^p \cap C^0}\big\} = 0\,, \tag{4.60}$$

for every $p \geq 2$ and $(a, b) \subset (0, \infty)$. By (4.59), $u(t) \in C^{2,\alpha}(\mathbb{R}^n)$, $\partial_t u(t) \in C^{0,\alpha}(\mathbb{R}^n)$ for every $\alpha \in (0, 1)$ and $t > 0$. By (4.60), and since

$$\varepsilon^2\, \partial_t u^{\delta_j} = 2\varepsilon^2\, \Delta u^{\delta_j} - W'(u^{\delta_j}) + \varepsilon\, \lambda_{\varepsilon,\delta_j}[u^{\delta_j}(t)]\, V'_{\delta_j}(u^{\delta_j})\,,$$

holds in classical sense on $\mathbb{R}^n \times (0, \infty)$, we can deduce that

$$\varepsilon^2\, \partial_t u = 2\,\varepsilon^2\, \Delta u - W'(u) + \varepsilon\, \lambda_\varepsilon[u(t)]\, V'(u)\,, \tag{4.61}$$

also holds in classical sense on on $\mathbb{R}^n \times (0,\infty)$, provided we show that

$$\lim_{j\to\infty} \lambda_{\varepsilon,\delta_j}[u^{\delta_j}(t)] = \lambda_\varepsilon[u(t)]\,, \qquad \forall t>0\,. \tag{4.62}$$

To prove (4.62) we first notice that, by $0 \le V_\delta(r) \le C\,r^2$ for $r \in [0,1]$ any by dominated convergence, for every $t>0$ it holds that $\mathcal{V}_{\delta_j}(u^{\delta_j}(t)) \to \mathcal{V}(u(t))$ as $j\to\infty$. At the same time, for every $t>0$, $\mathcal{V}_{\delta_j}(u^{\delta_j}(t)) = \mathcal{V}_{\delta_j}(u_0) \to \mathcal{V}(u_0)$ as $j\to\infty$ so that, in summary,

$$\mathcal{V}(u(t)) = \mathcal{V}(u_0) = 1\,, \qquad \forall t>0\,. \tag{4.63}$$

Now, by (3.7), (3.8), $\mathcal{V}_\delta(u^\delta(s)) = \mathcal{V}_\delta(u_0)$, (4.63), and (4.59), we have that

$$\begin{aligned}
|\lambda_{\varepsilon,\delta}[u^\delta(t)] - \lambda_{\varepsilon,\delta}[u(t)]| &\le C(\varepsilon,u_0,b)\,\frac{\|u^\delta(t)-u(t)\|_X^{\gamma(n)}}{\min\{1,\mathcal{V}_\delta(u_0)^{2n}\}\,\min\{1,\mathcal{V}_\delta(u(t))\}^{2n}}\,,\\
|\lambda_{\varepsilon,\delta}[u(t)] - \lambda_\varepsilon[u(t)]| &\le C(\varepsilon,u_0,a,b)\,\frac{\|V_\delta - V\|_{C^2[0,1]}}{\min\{1,\mathcal{V}_\delta(u(t))^{2n}\}}\,,
\end{aligned}$$

for all $t\in(a,b)\subset(0,\infty)$. Since $\mathcal{V}_\delta(u(t))\in(0,\infty)$ for every $t>0$ and $\delta\in(0,\delta_0]$, by letting $\delta=\delta_j$ and $j\to\infty$ in the above two estimates we obtain (4.62), and thus (4.61). The constancy of $\mathcal{V}(u(t))$ and the monotonicity of $\mathcal{AC}_\varepsilon(u(t))$ are then immediate to prove.

We are left to prove that $u(0)=u_0$. To begin with, we notice that, by (3.5), (4.45) and (4.46), for all $t>0$ we have

$$|\lambda_{\varepsilon,\delta}[u^\delta(t)]| \le C(\varepsilon)\,\frac{\mathcal{AC}_\varepsilon(u^\delta(t))^{2n+2}}{\mathcal{V}_\delta(u^\delta(t))^{2n}} \le C(\varepsilon)\,\frac{\mathcal{AC}_\varepsilon(u_0)^{2n+2}}{\mathcal{V}_\delta(u_0)^{2n}}\,,$$

so that (4.62) and (4.5) imply

$$|\lambda_\varepsilon[u(t)]| \le C(\varepsilon,u_0)\,, \qquad \forall t>0\,. \tag{4.64}$$

Next, if we set

$$\begin{aligned}
F[u](t) &= -\frac{W'(u(t))}{\varepsilon^2} + \lambda_\varepsilon[u(t)]\,\frac{V'(u(t))}{\varepsilon}\,, && (4.65)\\
T[u](t) &= S_t u_0 + \int_0^t S_{t-s}[F[u](s)]\,ds\,, && (4.66)
\end{aligned}$$

then (4.61) implies $u(t) = T[u(t)]S_t u_0 + \int_0^t S_{t-s}[F[u](s)]\,ds$ for every $t>0$, and by the contraction properties of the heat flow, (4.16), (4.17), (4.18), and (4.64), we find that

$$\begin{aligned}
\|u(t)-u_0\|_X &\le \|S_t u_0 - u_0\|_X + C(\varepsilon)\int_0^t \big(1+|\lambda_\varepsilon[u(s)]|\big)\,\|u(s)\|_X\,ds\\
&\le \|S_t u_0 - u_0\|_X + C(\varepsilon,u_0)\,t\,\sup_{0<s<t}\|u(s)\|_X\,.
\end{aligned}$$

By $u_0\in X$ and (4.59) we find that $\|u(t)-u_0\|_X \to 0$ as $t\to 0^+$, and thus that $u\in C^0(\mathbb{R}^n\times[0,\infty))$ with $u(0)=u_0$, as claimed. This completes the proof of step seven.

Step four: We now prove that

$$|\lambda_\varepsilon[u(t)]| \le C\,\mathcal{AC}_\varepsilon(u_0)^{2n+2}\,, \qquad \forall t>0\,, \tag{4.67}$$

and that, for every $t_0>0$ and $p\ge 2$ we have

$$\sup_{t\ge 0}\big\{\|u(t)\|_{W^{2,p}}, \|\partial_t u(t)\|_{L^p}\big\} \le C(\varepsilon,p,u_0)\,, \tag{4.68}$$

$$\sup_{t\ge t_0}\max\big\{\|u(t)\|_{W^{3,p}}, \|\partial_t u(t)\|_{W^{2,p}}, \|\partial_{tt} u(t)\|_{L^p}\big\} \le C(\varepsilon,p,u_0,t_0)\,. \tag{4.69}$$

Indeed, recalling that in step seven we have proved $\mathcal{V}(u(t))=1$ and $\mathcal{AC}_\varepsilon(u(t))\le\mathcal{AC}_\varepsilon(u_0)$ for every $t\ge0$, and recalling that the constant $C(\varepsilon)$ in (3.5) can be taken independent from ε when (3.5) is applied with $\delta=0$, we find that, for all $t>0$,

$$|\lambda_\varepsilon[u(t)]|\le C\,\frac{\mathcal{AC}_\varepsilon(u(t))^{2n+1}}{\mathcal{V}(u(t))^{2n}}\int_{\mathbb{R}^n}|\nabla u|^2\le\frac{C}{\varepsilon}\,\mathcal{AC}_\varepsilon(u_0)^{2n+2}\,,$$

that is (4.67). Similarly, we deduce from (3.3) (with $\delta=0$) and $0\le u\le1$ that

$$\sup_{t\ge0}\|u(t)\|_{L^p}\le C(\varepsilon,u_0)\,,\qquad\forall p\ge2\,,\tag{4.70}$$

while by combining (4.29) with (4.67) and (4.70) we get

$$\sup_{0\le t\le1}\|\nabla u(t)\|_{L^p}\le C(\varepsilon,p,u_0)\,,\qquad\forall p\ge2\,.\tag{4.71}$$

Now, using the semigroup property of the heat flow we see that for every $t>s\ge0$ we have

$$u(t)=S_{t-s}u(s)+\int_s^t S_{t-r}\big[F[u](r)\big]\,dr\,.\tag{4.72}$$

By differentiating (4.72), and by using (4.28), (4.67) and (4.70), we find that

$$\begin{aligned}\|\nabla u(t)\|_{L^p}&\le\|\nabla G(t-s)\|_{L^1}\,\|u(s)\|_{L^p}+\sup_{s<r<t}\|F[u](r)\|_{L^p}\int_s^t\|\nabla G(t-r)\|_{L^1}\,dr\\&\le\frac{C(\varepsilon,u_0)}{(t-s)^{1/2}}+C(\varepsilon,u_0)\,(t-s)^{1/2}\,,\end{aligned}\tag{4.73}$$

where we have used the analog to (4.13) with $\delta=0$ in estimating $\|F[u](r)\|_{L^p}$. If $t\ge1$, then we can apply (4.73) with $s=t-1\ge0$ to deduce that $\|\nabla u(t)\|_{L^p}\le C(\varepsilon,u_0)$, which, combined with (4.71), gives

$$\sup_{t\ge0}\|\nabla u(t)\|_{L^p}\le C(\varepsilon,p,u_0)\,,\qquad\forall p\ge2\,.\tag{4.74}$$

Similarly, combining (4.31) with (4.67) and (4.74) we see that, on the one hand

$$\sup_{0\le t\le1}\|\nabla^2u(t)\|_{L^p}\le C(\varepsilon,p,u_0)\,,\qquad\forall p\ge2\,;\tag{4.75}$$

on other hand, using again (4.72), (4.28), and (4.67), this time in combination with (4.70), (4.15), and (4.74), we find that, if $t>s\ge0$, then

$$\begin{aligned}\|\nabla^2u(t)\|_{L^p}&\le\|\nabla^2G(t-s)\|_{L^1}\,\|u(s)\|_{L^p}+\sup_{s<r<t}\|\nabla F[u](r)\|_{L^p}\int_s^t\|\nabla G(t-r)\|_{L^1}\,dr\\&\le\frac{C(\varepsilon,u_0)}{(t-s)^{3/2}}+C(\varepsilon,u_0)\,(t-s)^{1/2}\,,\end{aligned}$$

and using this last estimate for $t\ge1$ (and with $s=t-1\ge0$), we find $\|\nabla^2u(t)\|_{L^p}\le C(\varepsilon,u_0)$ for every $t\ge1$ and $p\ge2$. By combining this last fact with (4.75) we have thus proved

$$\sup_{t>0}\big\{\|u(t)\|_{W^{2,p}},\|\partial_tu(t)\|_{L^p}\big\}\le C(\varepsilon,p,u_0)\,,\qquad\forall p\ge2\,.\tag{4.76}$$

Here the L^p-estimate on $\partial_tu(t)$ has been obtained by combining the $W^{2,p}$-estimate for $u(t)$ with $\partial_tu=2\,\Delta u+f$, where $f(t)=F[u(t)]$, and thus $\|f(t)\|_{L^p}\le C(\varepsilon,p,u_0)$ thanks to (4.13) and (4.70). In fact, thanks also to (4.15), (4.67), and (4.74) for every $p\ge2$, we have

$$\sup_{t\ge0}\|f(t)\|_{W^{1,p}}\le C(\varepsilon,p,u_0)\,.$$

We can now repeat the argument based on the incremental ratios method and on the parabolic Calderon–Zygmund theorem used in the proof of (4.43) to deduce that

$$\sup_{t>t_0}\big\{\|\nabla(\partial_t u)(t)\|_{L^p},\|\nabla^3 u(t)\|_{L^p}\big\}\le C(p)\,\max\Big\{\frac{\|\nabla u_0\|_{L^p}}{t_0^{3/2}},\sup_{t>t_0}\|\nabla f(t)\|_{L^p}\Big\}\,,$$

and conclude, in summary, that, for every $p\ge 2$ and $t_0>0$,

$$\sup_{t>t_0}\big\{\|u(t)\|_{W^{3,p}},\|\partial_t u(t)\|_{W^{1,p}}\big\}\le C(\varepsilon,p,u_0,t_0)\,. \tag{4.77}$$

The $W^{1,2}$-estimate on $\partial_t u(t)$ allows one to deduce by elementary means that

$$\|u(t)-u(s)\|_{W^{1,2}(\mathbb{R}^n)}\le C(\varepsilon,u_0,t_0)\,|t-s|\,,\qquad\forall t,s>t_0\,. \tag{4.78}$$

Combining (4.78) with (3.9) (with $\delta=0$) and (4.76) with $p=2$ we conclude that

$$|\lambda_\varepsilon[u(t)]-\lambda_\varepsilon[u(s)]|\le C(\varepsilon,u_0,t_0)\,|t-s|\,,\qquad\forall t,s>t_0\,, \tag{4.79}$$

that is $t\mapsto\lambda_\varepsilon[u(t)]$ is Lipschitz continuous on (t_0,∞) for every $t_0>0$.

To obtain L^p-estimates for $\partial_{tt}u(t)$ and $\nabla^2(\partial_t u)$ we need to differentiate $\partial_t u=2\,\Delta u+f$ in time. To this end, given $t_0>0$, we introduce that incremental ratio operator T_h that acts on $v=v(x,t)$ by taking $T_h v(t)=(v(t_0+t+h)-v(t_0+t))/h$ for every $t\ge 0$ and $h\in(-t_0,t_0)\setminus\{0\}$. With this notation, $\partial_t u=2\,\Delta u+f$ on $\mathbb{R}^n\times(0,\infty)$ implies that

$$\begin{aligned}\partial_t(T_h u)-2\,\Delta(T_h u)&=T_h f\,,\qquad\text{on }\mathbb{R}^n\times(0,\infty)\,,\\ T_h u(0)&=\frac{u(t_0+h)-u(t_0)}{h}\,,\qquad\text{on }\mathbb{R}^n\,.\end{aligned} \tag{4.80}$$

Setting

$$u_{1,h}=S_t\Big[\frac{u(t_0+h)-u(t_0)}{h}\Big]\,,\qquad u_{2,h}=T_h u-u_{1,h}\,,$$

we find that $u_{2,h}$ satisfies

$$\partial_t u_{2,h}-2\,\Delta u_{2,h}=T_h f\,,\qquad\text{on }\mathbb{R}^n\times(0,\infty)\,,\qquad u_{2,h}(0)=0\text{ on }\mathbb{R}^n\,. \tag{4.81}$$

By [Lie96, Corollary 7.31], for every $(a,b)\subset(0,\infty)$ we have

$$\int_a^b dt\int_{\mathbb{R}^n}|\nabla^2 u_{2,h}(t)|^p+|\partial_t u_{2,h}(t)|^p\le C(p)\int_a^b dt\int_{\mathbb{R}^n}|T_h f(t)|^p\,.$$

Setting

$$g(t)=\Big\{-\frac{W''(u)}{\varepsilon^2}+\lambda_\varepsilon[u(t)]\,\frac{V''(u)}{\varepsilon}\Big\}\,v+\frac{V'(u)}{\varepsilon}\,\frac{d}{dt}\,\lambda_\varepsilon[u(t)]\,, \tag{4.82}$$

by (4.67) and (4.79) we find that

$$\int_a^b dt\int_{\mathbb{R}^n}|T_h f(t)|^p\le\int_{a-|h|}^{b+|h|}dt\int_{\mathbb{R}^n}|g(t)|^p\le C(\varepsilon,u_0,a)\,(b-a+2\,|h|)\,,$$

so that, in summary,

$$\int_a^b dt\int_{\mathbb{R}^n}|\nabla^2 u_{2,h}(t)|^p+|\partial_t u_{2,h}(t)|^p\le C(\varepsilon,p,u_0,a)\,(b-a+2\,|h|)\,. \tag{4.83}$$

At the same time, thanks to (4.76), we have, for $|h|<t_0/2$ and $t>t_0$,

$$\begin{aligned}\|\nabla^2 u_{1,h}(t)\|_{L^p}&\le\|\nabla^2 G(t)\|_{L^1}\Big\|\frac{u(t_0+h)-u(t_0)}{h}\Big\|_{L^p}\\&\le\frac{C}{t^{3/2}}\sup_{t_0-|h|<s<t_0+|h|}\|\partial_t u(s)\|_{L^p}\le\frac{C(\varepsilon,p,u_0)}{t_0^{3/2}}\,,\\ \|\partial_t u_{1,h}(t)\|_{L^p}&\le\|\partial_t G(t)\|_{L^1}\Big\|\frac{u(t_0+h)-u(t_0)}{h}\Big\|_{L^p}\le\frac{C(\varepsilon,p,u_0)}{t_0}\,,\end{aligned}$$

and thus, by $T_h u = u_{1,h} + u_{2,h}$ and (4.83)

$$\int_a^b dt \int_{\mathbb{R}^n} |\nabla^2(T_h u)(t)|^p + |\partial_t(T_h u)(t)|^p \le C(\varepsilon, p, u_0, a)\,(b - a + 2\,|h|)\,. \tag{4.84}$$

Letting $h \to 0^+$ in (4.84) we obtain

$$\int_a^b dt \int_{\mathbb{R}^n} |\nabla^2(\partial_t u)(t)|^p + |\partial_{tt} u(t)|^p \le C(\varepsilon, p, u_0, a)\,(b - a)\,,$$

from which we easily conclude that, for every $t_0 > 0$ and $p \ge 2$,

$$\sup_{t \ge t_0} \left\{ \|\nabla^2(\partial_t u)(t)\|_{L^p}, \|\partial_{tt} u(t)\|_{L^p} \right\} \le C(\varepsilon, p, u_0, t_0)\,. \tag{4.85}$$

This last estimate, combined with (4.77) concludes the proof of (4.69).

Step five: We prove the uniqueness of the solution u of (DF) constructed in step seven. Indeed, let v be another solution of (DF) with $v(0) = u_0$. By combining (3.9) with $\mathcal{V}(u(t)) = \mathcal{V}(v(t)) = \mathcal{V}(u_0) = 1$ and with (4.76) with $p = 2$ for both u and v, we find that, for all $t > 0$,

$$|\lambda_\varepsilon[u(t)] - \lambda_\varepsilon[v(t)]| \le C(\varepsilon, u_0)\, \|u(t) - v(t)\|_{W^{1,2}}\,. \tag{4.86}$$

Since $w = u - v$ satisfies $\partial_t w - 2\,\Delta w = F[u] - F[v]$ with $w(0) = 0$, by the Duhamel formula (4.3), for all $t \ge 0$ we can represent $w(t)$ as

$$w(t) = \int_0^t S_{t-s}\big[F[u](s) - F[v](s)\big]\, ds\,. \tag{4.87}$$

Notice that by using, in the order, (4.67), $0 \le V'(r) \le C\,r$ for $r \in [0, 1]$, and (4.86), we find

$$\begin{aligned} |F[u](s) - F[v](s)| &\le C(\varepsilon) \Big\{ \mathrm{Lip}(W') + |\lambda_\varepsilon[u(s)]|\, \mathrm{Lip}(V') \Big\}\, |w(s)| \\ &\quad + V'(v(s))\, |\lambda_\varepsilon[u(s)] - \lambda_\varepsilon[v(s)]| \\ &\le C(\varepsilon, u_0) \left\{ |w(s)| + \|w(s)\|_{W^{1,2}}\, |v(s)| \right\}, \end{aligned}$$

so that, by (4.74) with $p = 2$ applied to v,

$$\|F[u](s) - F[v](s)\|_{L^2} \le C(\varepsilon, u_0)\, \|w(s)\|_{W^{1,2}}\,, \qquad \forall s > 0\,. \tag{4.88}$$

Setting

$$a(t) = \sup_{0<s<t} \|w(s)\|_{W^{1,2}}$$

and combining (4.88) with (4.28) we deduce from (4.87) that, if $0 < s < t$, then

$$\begin{aligned} \|w(s)\|_{L^2} &\le \int_0^s \|F[u](r) - F[v](r)\|_{L^2}\, dr \le C(\varepsilon, u_0) \int_0^s \|w(r)\|_{W^{1,2}}\, dr \\ &\le C(\varepsilon, u_0)\, t\, a(t)\,, \\ \|\nabla w(s)\|_{L^2} &\le \sup_{0<r<s} \|F[u](r) - F[v](r)\|_{L^2} \int_0^s \|\nabla G(s - r)\|_{L^1}\, ds \\ &\le C(\varepsilon, u_0)\, \sqrt{s}\, \sup_{0<r<s} \|w(r)\|_{W^{1,2}} \le C(\varepsilon, u_0)\, \sqrt{t}\, a(t)\,. \end{aligned}$$

Combining this last two estimates, we find that $a(t) \le C(\varepsilon, u_0)\,\sqrt{t}\, a(t)$ for every $t > 0$. In particular, setting $t_0 = 1/4\, C(\varepsilon, u_0)^2$, if $a(t) > 0$ for some $t \in (0, t_0)$, then a contradiction follows. We have thus proved that $u(t) = v(t)$ on $\mathbb{R}^n$ for every $t \in (0, t_0)$. The argument can of course be iterated (using t_0 and then integer multiples of t_0 as initial times) to prove that $u(t) = v(t)$ on $\mathbb{R}^n$ for every $t > 0$.

Conclusion: Statement (i) was proved in (1.15) and (1.16). Statement (ii) was proved in (4.67) and (4.79). Statement (iii) was proved in (4.63). Moreover, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{AC}_\varepsilon(u(t)) &= \int_{\mathbb{R}^n}\Big\{2\,\varepsilon\,\nabla u\cdot\partial_t(\nabla u)+\frac{W'(u)}{\varepsilon}\,\partial_t u\Big\}(t)\\ &= -\int_{\mathbb{R}^n}\partial_t u(t)\Big\{2\,\varepsilon\,\Delta u-\frac{W'(u)}{\varepsilon}\Big\}(t)\\ &= -\int_{\mathbb{R}^n}\partial_t u(t)\Big\{\varepsilon\,\partial_t u+\lambda_\varepsilon[u(t)]\,V'(u)\Big\}(t)\\ &= -\varepsilon\int_{\mathbb{R}^n}(\partial_t u(t))^2+\lambda_\varepsilon[u(t)]\int_{\mathbb{R}^n}\partial_t u(t)\,V'(u(t)) = -\varepsilon\int_{\mathbb{R}^n}(\partial_t u(t))^2\,,\end{aligned}$$

where we have used the regularity properties of u to differentiate in time and to apply the divergence theorem, (DF), and, in the last identity, statement (iii), to deduce that $0=(d/dt)\mathcal{V}(u(t))=\int_{\mathbb{R}^n}\partial_t u(t)\,V'(u(t))$. This proves statement (iv).

We prove that $0<u<1$ on $\mathbb{R}^n\times(0,\infty)$, i.e. statement (v). Since $V,W\in C^2[0,1]$ with $V'(1)=W'(1)=0$ and $|\lambda_\varepsilon[u(t)]|\le C(\varepsilon,u_0)$ for all $t>0$, we can find a positive constant $K=K(\varepsilon,u_0)$ such that $r\mapsto K\,r-W'(r)+\varepsilon\lambda_\varepsilon[u(t)]\,V'(r)$ is strictly increasing on $[0,1]$. Correspondingly,

$$K\,u-W'(u)+\varepsilon\lambda_\varepsilon[u(t)]\,V'(u)\le K\,,\qquad\text{on }\mathbb{R}^n\times(0,\infty)\,,$$

which, combined with (DF), implies

$$\varepsilon^2\partial_t u=2\,\varepsilon^2\,\Delta u-W'(u)+\varepsilon\,\lambda_\varepsilon[u(t)]\,V'(u)\le 2\,\varepsilon^2\,\Delta u-K\,u+K$$

that is, $v=1-u$ is a non-negative solution of $\varepsilon^2(\partial_t-2\,\Delta)v+K\,v\ge 0$ on $\mathbb{R}^n\times(0,\infty)$. By the strong maximum principle, either $v\equiv 0$ or $v>0$ on $\mathbb{R}^n\times(0,\infty)$, where the first option is excluded a priori since $V(1)=1$ and $\mathcal{V}(u(t))$ is finite for every $t>0$. We conclude that $u<1$ on $\mathbb{R}^n\times(0,\infty)$, and argue analogously for proving $u>0$ on $\mathbb{R}^n\times(0,\infty)$.

We finally prove statement (vi). The argument of step eight shows that $v=\partial_t u$ satisfies $\partial_t v-2\,\Delta v=g$ for g as in (4.82). By testing this equation with $\partial_t u$ and integrating by parts,

$$\begin{aligned}\varepsilon^2\int_{\mathbb{R}^n}\big(\partial_t u\,\partial_{tt}u\big)(t) &= -2\,\varepsilon^2\int_{\mathbb{R}^n}|\nabla(\partial_t u)|^2-\int_{\mathbb{R}^n}\Big\{W''(u)-\varepsilon\,\lambda_\varepsilon[u(t)]\,V''(u)\Big\}(t)\,(\partial_t u(t))^2\\ &\quad+\varepsilon\Big(\frac{d}{dt}\,\lambda_\varepsilon[u(t)]\Big)\int_{\mathbb{R}^n}V'(u(t))\,\partial_t u(t)\,,\end{aligned}\tag{4.89}$$

where the last integral is equal to zero since $\mathcal{V}(u(t))=0$ for every $t>0$. This implies the validity of (1.18). Setting $b(t)=\int_{\mathbb{R}^n}(\partial_t u(t))^2$, (1.18) combined with (1.16) implies that $b'\in L^1(t_0,\infty)$ for every $t_0>0$. Since (1.17) and the monotonicity of $\mathcal{AC}_\varepsilon(u(t))$ imply $b\in L^1(0,\infty)$, we have proved that $b\in W^{1,1}(t_0,\infty)$ for every $t_0>0$. This completes the proof of the theorem. □

5. Subsequential bubbling resolution (Proof of Theorem 1.3)

Proof of Theorem 1.3. Let $u_0\in W^{2,p}(\mathbb{R}^n;[0,1])$ for all $p\ge 2$, with $\mathcal{V}(u_0)=1$. By Theorem 1.2, there is a unique solution u to the diffused VPMCF (DF) with initial datum u_0, satisfying the various statements (i)–(vi) listed therein.

Given $\{t_j\}_{j\in\mathbb{N}}$ with $t_j\to\infty$ as $j\to\infty$, we now want to prove that, up to extracting a subsequence, there are $M\in\mathbb{N}$, $\ell_\varepsilon>0$ such that

$$\ell_\varepsilon=\lim_{j\to\infty}\lambda_\varepsilon[u(t_j)]\,,\tag{5.1}$$

sequences $\{x_j^i\}_{j\in\mathbb{N}}$ $(i=1,...,M)$ satisfying $|x_j^i - x_j^k| \to \infty$ as $j\to\infty$ $(i\neq k)$, and strictly radially decreasing solutions $\{\xi_i\}_{i=1}^M$ of

$$2\,\varepsilon^2\,\Delta\xi_i = W'(\xi_i) - \varepsilon\,\ell_\varepsilon\,V'(\xi_i) \quad \text{on } \mathbb{R}^n\,, \tag{5.2}$$

with

$$\sum_{i=1}^M \mathcal{V}(\xi_i) = 1\,, \qquad \sum_{i=1}^M \mathcal{AC}_\varepsilon(\xi_i) \le \mathcal{AC}_\varepsilon(u_0)\,. \tag{5.3}$$

such that, for all $p>2$,

$$\lim_{j\to\infty}\Big\|u(t_j) - \sum_{i=1}^M \tau_{x_i^j}[\xi_i]\Big\|_{(W^{2,p}\cap W^{1,2})(\mathbb{R}^n)} = 0\,. \tag{5.4}$$

This will be proved in step one through five. Finally, in step six, we shall prove that, if $\operatorname{spt} u_0 \subset\subset \mathbb{R}^n$, then $M=1$ and $x_j^1 \to x_*$ as $j\to\infty$ for some $x_*\in\mathbb{R}^n$.

Step one: We start by proving that, if $\xi\in W^{1,2}(\mathbb{R}^n;[0,1])\setminus\{0\}$ and $\ell\in\mathbb{R}$ satisfy

$$2\,\varepsilon^2\,\Delta\xi = W'(\xi) - \varepsilon\,\ell\,V'(\xi) \quad \text{on } \mathbb{R}^n, \tag{5.5}$$

then $\ell>0$, ξ is strictly radially decreasing with respect to some $x_0\in\mathbb{R}^n$, and

$$\mathcal{V}(\xi) \ge \frac{1}{C\,\ell^n}\,. \tag{5.6}$$

Indeed, $\xi\in W^{2,p}(\mathbb{R}^n)$ for every $p\ge 2$ thanks to the Calderon–Zygmund theorem, $\|\xi\|_{L^p} \le \|\xi\|_{L^2} < \infty$ (as $0\le\xi\le 1$ on $\mathbb{R}^n$), and $|W'(r)|, |V'(r)| \le C\,t$ for all $r\in[0,1]$. Since $\xi\in W^{2,p}(\mathbb{R}^n)$ we have enough regularity to test (5.5) with $\varphi = X\cdot\nabla\xi$ for $X(x) = \eta(x/R)\,x$, η a cut-off function between B_1 and B_2, and $R>0$, and deduce that (see [LM89], or [MR24, Step five, Proof of Theorem 2.1], for the details)

$$n\,\ell\,\mathcal{V}(\xi) = n\,\mathcal{AC}_\varepsilon(\xi) - 2\,\varepsilon\int_{\mathbb{R}^n}|\nabla\xi|^2\,. \tag{5.7}$$

Since $\xi\neq 0$ and $n\ge 2$, the right-hand side of (5.7) is strictly positive, thus proving that $\ell>0$. Moreover, since $\xi\in W^{2,p}(\mathbb{R}^n)$, we have that $\xi(x)\to 0$ as $|x|\to\infty$, so that the moving planes method of [GNN81] can be applied (see for example [MR24, Theorem 6.2-(i)]) to deduce that ξ is strictly radially decreasing with respect to some $x_0\in\mathbb{R}^n$.

We finally prove (5.6). If $n\ge 3$, then we can combine (5.7) with (1.5) to find

$$n\,\ell\,\mathcal{V}(\xi) \ge (n-2)\,\mathcal{AC}_\varepsilon(\xi) > (n-2)\,n\,\omega_n^{1/n}\,\mathcal{V}(\xi)^{(n-1)/n}\,,$$

which immediately gives (5.6). In the case $n=2$, we argue as follows. We test (5.5) with $\varphi_k\,\xi$ for $\varphi_k\in C_c^\infty(B_{k+2};[0,1])$ with $\varphi_k = 1$ on B_k and $|\nabla\varphi_k| \le 1_{B_{k+2}\setminus B_k}$ for each $k\in\mathbb{N}$. In this way we find

$$2\,\varepsilon^2\int_{\mathbb{R}^n}\varphi_k\,|\nabla\xi|^2 \le 2\,\varepsilon^2\int_{\mathbb{R}^n}\xi\,|\nabla\xi|\,|\nabla\varphi_k| + \int_{\mathbb{R}^n}\varphi_k\,\xi\,W'(\xi) + \varepsilon\,\ell\,\varphi_k\,\xi\,V'(\xi)\,.$$

Letting $k\to\infty$ (and using monotone convergence for all the integrals but the one involving W', which may be negative, and is dealt with by dominated convergence), we find that

$$2\,\varepsilon^2\int_{\mathbb{R}^n}|\nabla\xi|^2 \le \int_{\mathbb{R}^n}\xi\,W'(\xi) + \varepsilon\,\ell\,\xi\,V'(\xi)\,.$$

Thanks to (1.2) there is $\delta_0>0$ such that $W'(r)\le 0$ for all $r\in[1-\delta_0,1]$ and $r\,|W'(r)| \le C\,r^2 \le C\,W(r)$ for all $r\in[0,1]$. Therefore, $r\,W'(r)\le C\,W(r)$ for all $r\in[0,1]$, and, taking also into account that $r\,V'(r)\le C\,V(r)$ for all $r\in[0,1]$, we deduce that

$$2\,\varepsilon^2\int_{\mathbb{R}^n}|\nabla\xi|^2 \le C\Big\{\varepsilon\,\ell\,\mathcal{V}(\xi) + \int_{\mathbb{R}^n}W(\xi)\Big\}\,. \tag{5.8}$$

Since, when $n = 2$, (5.7) boils down to $\varepsilon\,\ell\,\mathcal{V}(\xi) = \int_{\mathbb{R}^n} W(\xi)$, we have finally proved $C\,\ell\,\mathcal{V}(\xi) \ge \mathcal{AC}_\varepsilon(\xi)$. Again by (1.5), as in the case $n \ge 3$, we deduce (5.6).

Step two: We now begin the proof of (5.1), (5.2), (5.3), and (5.4) by discussing a compactness argument aimed at "extracting one bubble" from $u(t_j)$ as $t_j \to \infty$. In later steps we will of course discuss the iteration of this argument.

We begin by recalling that, by Theorem 1.2-(i,ii), for all $p \ge 2$ and $\alpha \in (0,1)$ we have

$$\sup_{t\ge 0}\Big\{\|u(t)\|_{W^{2,p}(\mathbb{R}^n)}, \|u(t)\|_{C^{1,\alpha}(\mathbb{R}^n)}, |\lambda_\varepsilon[u(t)]|\Big\} < \infty\,, \tag{5.9}$$

$$\sup_{t\ge 1}\Big\{\|u(t)\|_{W^{3,p}(\mathbb{R}^n)}, \|u(t)\|_{C^{2,\alpha}(\mathbb{R}^n)}\Big\} < \infty\,. \tag{5.10}$$

By (5.9), $\sup_{\mathbb{R}^n} u(t)$ is achieved for every $t > 0$, and we actually claim that

$$\max_{\mathbb{R}^n} u(t) \ge \min\Big\{\frac{1}{2}, \Big(\frac{1}{C\,\varepsilon\,\mathcal{AC}_\varepsilon(u_0)}\Big)^{(n-1)/2}\Big\}\,, \qquad \forall t > 0\,. \tag{5.11}$$

Indeed, setting for brevity $\beta(t) = \max_{\mathbb{R}^n} u(t)$, if $\beta(t) \le 1/2$, then we can use the elementary estimate

$$V(r) \le C\,r^{2n/(n-1)} \le C\,r^{2/(n-1)}\,r^2 \le C\,r^{2/(n-1)}\,W(r)\,, \qquad \forall r \in [0,1/2]\,,$$

to deduce from Theorem 1.2-(iii,iv) that, for all $t > 0$,

$$\begin{aligned} 1 &= \mathcal{V}(u(t)) \le C \int_{\mathbb{R}^n} u(t)^{2/(n-1)}\,W(u(t)) \le C\,\beta(t)^{2/(n-1)}\,\varepsilon\,\mathcal{AC}_\varepsilon(u(t)) \\ &\le C\,\beta(t)^{2/(n-1)}\,\varepsilon\,\mathcal{AC}_\varepsilon(u_0)\,. \end{aligned}$$

This proves (5.11). Combining (5.11) with (5.9) we find $\beta_0 = \beta_0(\varepsilon, u_0) \in (0, 1/2]$ and $r_0 = r_0(\varepsilon, u_0) > 0$ such that for each $t > 0$ there is $x_t \in \mathbb{R}^n$ with the property that

$$u(t) \ge \beta_0 \qquad \text{on } B_{r_0}(x_t)\,. \tag{5.12}$$

In particular, given $t_j \to \infty$ as $j \to \infty$, then, thanks to (5.9), (5.10) and (5.12), and up to extracting subsequences, we can find $\ell_\varepsilon \in \mathbb{R}$, $\xi_1 \in \cap_{p\ge 2} W^{3,p}(\mathbb{R}^n;[0,1]) \setminus \{0\}$, and a sequence $(x_j^1)_j$ in $\mathbb{R}^n$ such that, as $j \to \infty$,

$$\lim_{j\to\infty} \lambda_\varepsilon[u(t_j)] = \ell_\varepsilon\,, \tag{5.13}$$

$$\tau_{(-x_j^1)}[u(t_j)] \rightharpoonup \xi_1 \text{ weakly in } W^{3,p}(\mathbb{R}^n) \text{ as } j \to \infty\,, \tag{5.14}$$

$$\lim_{j\to\infty} \big\|\tau_{(-x_j^1)}[u(t_j)] - \xi_1\big\|_{W^{2,p}(B_R)} = 0\,, \qquad \forall p \ge 2\,, \forall R > 0\,. \tag{5.15}$$

By Theorem 1.2-(vi), if we first take $t = t_j$ in

$$\partial_t u(t) = 2\,\varepsilon^2\,\Delta u(t) - W'(u(t)) + \varepsilon\,\lambda_\varepsilon[u(t)]\,V'(u(t)) \quad \text{on } \mathbb{R}^n\,, \tag{5.16}$$

and then let $j \to \infty$, we deduce that

$$2\,\varepsilon^2\,\Delta\xi_1 = W'(\xi_1) - \varepsilon\,\ell_\varepsilon\,V'(\xi_1) \quad \text{on } \mathbb{R}^n\,.$$

By step one, we conclude that $\ell_\varepsilon > 0$, that ξ_1 is strictly radially decreasing, and that

$$\mathcal{V}(\xi_1) \ge \frac{1}{C\,\ell_\varepsilon^n}\,. \tag{5.17}$$

Step three: We now iterate the construction of step two. Let us consider the following statement, depending on $k \in \mathbb{N}$:

$(S)_k$: There are $\{(x_j^i)_j\}_{i=1}^k$ sequences in $\mathbb{R}^n$ with $|x_j^i - x_j^\ell| \to \infty$ as $j \to \infty$ $(i \neq \ell)$, and there are $\{\xi_i\}_{i=1}^k$ radially symmetric decreasing functions solving (5.2), such that

$$\sum_{i=1}^k \mathcal{V}(\xi_i) \le 1\,, \tag{5.18}$$

and, for each $i = 1, ..., k$,

$$\begin{aligned} &\tau_{(-x_j^i)}[u(t_j)] \rightharpoonup \xi_i \text{ weakly in } W^{3,p}(\mathbb{R}^n) \text{ as } j \to \infty\,,\\ &\lim_{j\to\infty} \big\|\tau_{(-x_j^i)}[u(t_j)] - \xi_i\big\|_{W^{2,p}(B_R)} = 0\,, \qquad \forall R > 0\,, \forall p \ge 2\,. \end{aligned} \tag{5.19}$$

Notice that if $(S)_k$ holds with equality in (5.18) for some $k \in \mathbb{N}$, then the theorem is proved (up to showing the validity of (5.4)) with $M = k$. We also notice that, in step two, we have proved that $(S)_1$ holds. In this step we prove that if $(S)_k$ holds for some $k \in \mathbb{N}$ with strict sign in (5.18), that is, with

$$\sum_{i=1}^k \mathcal{V}(\xi_i) < 1\,,$$

then, up to extracting a subsequence in j, we can find $(x_j^{k+1})_j$ and ξ_{k+1} such that $(S)_{k+1}$ holds. Since, by step one, $\mathcal{V}(\xi_i) \ge (1/C\,\ell_\varepsilon)^n$ for each $i = 1, ..., k+1$, we conclude that, for some $M \in \mathbb{N}$, $(S)_M$ must hold with equality in (5.18).

So let us consider $k \in \mathbb{N}$ such that $(S)_k$ holds with strict sign in (5.18), and let $m = \sum_{i=1}^k \mathcal{V}(\xi_i)$. Given $h \in \mathbb{N}$, let

$$A_j^h = \bigcup_{i=1}^k B_h(x_j^i)\,.$$

By (5.19), for every $h \in \mathbb{N}$ there is $j(h) \in \mathbb{N}$ such that

$$\mathcal{V}(u(t_j); B_h(x_j^i)) \ge \mathcal{V}(\xi_i; B_h) - \frac{1-m}{2k}\,, \qquad \forall i = 1, ..., k, \forall j \ge j(h)\,,$$

so that if, $j \ge j(h)$,

$$\begin{aligned} 1 - m &\le 1 - \sum_{i=1}^k \mathcal{V}(\xi_i; B_h) \le \frac{1-m}{2} + 1 - \sum_{i=1}^k \mathcal{V}\big(u(t_j); B_h(x_j^i)\big)\\ &= \frac{1-m}{2} + \mathcal{V}(u(t_j); \mathbb{R}^n \setminus A_j^h)\,, \end{aligned}$$

that is

$$\frac{1-m}{2} \le \mathcal{V}(u(t_j); \mathbb{R}^n \setminus A_j^h)\,, \qquad \forall h \in \mathbb{N}\,, \forall j \ge j(h)\,,$$

If we now set

$$\beta_j^h = \max_{\mathbb{R}^n \setminus A_j^h} u(t_j)\,,$$

then, either $\beta_j^h \ge 1/2$ or, by arguing as in step two, for every $h \in \mathbb{N}$ and $j \ge j(h)$,

$$\frac{1-m}{2} \le \mathcal{V}(u(t_j); \mathbb{R}^n \setminus A_j^h) \le C\,(\beta_j^h)^{2/(n-1)} \int_{\mathbb{R}^n \setminus A_j^h} W(u(t_j)) \le C\,(\beta_j^h)^{2/(n-1)}\,\varepsilon\,\mathcal{AC}_\varepsilon(u_0)\,,$$

that is, for every $h \in \mathbb{N}$ and $j \ge j(h)$,

$$\beta_j^h \ge \min\Big\{\frac{1}{2}, \Big(\frac{1-m}{C\,\varepsilon\,\mathcal{AC}_\varepsilon(u_0)}\Big)^{(n-1)/2}\Big\} =: \beta_* > 0\,.$$

In particular, up to extracting a subsequence in j, for each j we can find $x_j^{k+1}\in\mathbb{R}^n$ such that

$$\inf_{i=1,\dots,k}|x_j^{k+1}-x_j^i|\ge j\,,\qquad u_j(x_j^{k+1},t_j)\ge\beta_*\,,\qquad\forall j\,;$$

and then, thanks to (5.9), we can find $r_*>0$ such that

$$u(t_j)\ge\frac{\beta_*}{2}\quad\text{on } B_{r_*}(x_j^{k+1})\,,\qquad\forall j\,.$$

Hence, by (5.10), there exists $\xi_{k+1}\in\cap_{p\ge2}W^{3,p}(\mathbb{R}^n;[0,1])\setminus\{0\}$ such that

$$\tau_{(-x_j^{k+1})}[u(t_j)]\rightharpoonup\xi_{k+1}\text{ weakly in } W^{3,p}(\mathbb{R}^n)\text{ as } j\to\infty\,,\tag{5.20}$$

$$\lim_{j\to\infty}\big\|\tau_{(-x_j^{k+1})}[u(t_j)]-\xi_{k+1}\big\|_{W^{2,p}(B_R)}=0\,,\qquad\forall R>0\,,\forall p\ge2\,,\tag{5.21}$$

and by the same argument of step two we see that ξ_{k+1} satisfies

$$2\,\varepsilon^2\,\Delta\xi_{k+1}=W'(\xi_{k+1})-\varepsilon\,\ell_\varepsilon\,V'(\xi_{k+1})\quad\text{on }\mathbb{R}^n\,,$$

and thus

$$\mathcal{V}(\xi_{k+1})\ge\frac{1}{C\,\ell_\varepsilon^n}\,.\tag{5.22}$$

We have thus proved the desired induction step.

Step four: We have so far proved the existence of $M\in\mathbb{N}$, $\ell_\varepsilon>0$, $\{(x_j^i)_j\}_{i=1}^M$ sequences in $\mathbb{R}^n$, and $\{\xi_i\}_{i=1}^M$ radially symmetric decreasing functions on $\mathbb{R}^n$ such that, up to extracting a subsequence, (5.1), and (5.2) hold, together with part of (5.3) (i.e., $1=\sum_{i=1}^M\mathcal{V}(\xi_i)$), and

$$\tau_{(-x_j^i)}[u(t_j)]\rightharpoonup\xi_i\text{ weakly in } W^{3,p}(\mathbb{R}^n)\text{ as } j\to\infty\,,$$

$$\lim_{j\to\infty}\big\|\tau_{(-x_j^i)}[u(t_j)]-\xi_i\big\|_{W^{2,p}(B_R)}=0\,,\qquad\forall R>0\,,p\ge2\,,i=1,\dots,M\,.\tag{5.23}$$

To complete the proof of (5.3) we just need to notice that, by (5.23), for every $R>0$

$$\begin{aligned}\sum_{i=1}^M\mathcal{AC}_\varepsilon(\xi_i;B_R)&\le\liminf_{j\to\infty}\sum_{i=1}^M\mathcal{AC}_\varepsilon\big(\tau_{-x_j^i}[u(t_j)];B_R\big)=\liminf_{j\to\infty}\sum_{i=1}^M\mathcal{AC}_\varepsilon\big(u(t_j);B_R(x_j^i)\big)\\&=\liminf_{j\to\infty}\mathcal{AC}_\varepsilon\Big(u(t_j);\bigcup_{i=1}^M B_R(x_j^i)\Big)\le\mathcal{AC}_\varepsilon(u_0)\,,\end{aligned}$$

where in the last equality we have used $|x_j^i-x_j^\ell|\to\infty$ as $j\to\infty$ if $i\ne\ell$, and in the last inequality we have used the monotonicity of $t\mapsto\mathcal{AC}_\varepsilon(u(t))$. By arbitrariness of R, we have completed the proof of (5.3). We are thus left to prove (5.4). In this step, we shall prove a slightly weaker statement, namely

$$\lim_{j\to\infty}\Big\|u(t_j)-\sum_{i=1}^M\tau_{x_i^j}[\xi_i]\Big\|_{W^{2,p}(\mathbb{R}^n)}=0\,,\qquad\forall p>2\,.\tag{5.24}$$

We will then improve (5.24) to (5.4) in step five. (This improvement is of crucial importance in discussing convergence to equilibrium of the flow.) We begin the proof of (5.24) by showing that, with $n'=n/(n-1)$,

$$\lim_{j\to\infty}\|u(t_j)-v_j\|_{L^{2n'}(\mathbb{R}^n)}=0\,,\qquad\text{where } v_j=\sum_{i=1}^M\tau_{x_j^i}[\xi_i]\,.\tag{5.25}$$

Indeed, for every $\sigma>0$ we can find $R>0$ such that

$$\mathcal{V}(\xi_i;B_R)\ge(1-\sigma)\,\mathcal{V}(\xi_i)\,,\qquad\forall i=1,\dots,M\,.\tag{5.26}$$

Setting $A_j^R = \mathbb{R}^n \setminus \cup_{i=1}^M B_R(x_j^i)$, for j large enough to have $B_R(x_j^i) \cap B_R(x_j^\ell) = \emptyset$ $(i \neq \ell)$, we have

$$\begin{aligned}\frac{1}{C(M)} \int_{\mathbb{R}^n} |u(t_j) - v_j|^{2n'} &\le \int_{A_j^R} |u(t_j)|^{2n'} + |v_j|^{2n'} + \sum_{i=1}^M \int_{B_R(x_j^i)} |u(t_j) - \tau_{x_j^i}[\xi_i]|^{2n'} \\ &\quad + \sum_{i=1}^M \int_{B_R(x_j^i)} \Big| \sum_{\ell \neq i} \tau_{x_j^\ell}[\xi_\ell] \Big|^{2n'} . \end{aligned} \tag{5.27}$$

We first notice that, since $r^{2n'} \le C\,V(r)$ for all $r \in [0,1]$,,

$$\begin{aligned}\int_{A_j^R} |v_j|^{2n'} &\le C(M) \sum_{i=1}^M \int_{A_j^R} |\tau_{x_j^i}[\xi_i]|^{2n'} \le C(M) \sum_{i=1}^M \int_{\mathbb{R}^n \setminus B_R(x_j^i)} |\tau_{x_j^i}[\xi_i]|^{2n'} \\ &\le C(M) \sum_{i=1}^M \int_{\mathbb{R}^n \setminus B_R} V(\xi_i) \le C(M)\,\sigma \sum_{i=1}^M \mathcal{V}(\xi_i) = C(M)\,\sigma; \end{aligned} \tag{5.28}$$

similarly,

$$\begin{aligned}\frac{1}{C} \int_{A_j^R} |u(t_j)|^{2n'} &\le \int_{A_j^R} V(u(t_j)) = 1 - \sum_{i=1}^M \mathcal{V}(u(t_j); B_R(x_j^i)) \\ &= \sum_{i=1}^M \mathcal{V}(\xi_i) - \mathcal{V}(u(t_j); B_R(x_j^i)) = \sum_{i=1}^M \mathcal{V}(\xi_i) - \mathcal{V}(\tau_{-x_j^i}[u(t_j)]; B_R)\end{aligned}$$

so that, by (5.23),

$$\limsup_{j\to\infty} \int_{A_j^R} |u(t_j)|^{2n'} \le C \sum_{i=1}^M \mathcal{V}(\xi_i; \mathbb{R}^n \setminus B_R) \le C\,M\,\sigma\,. \tag{5.29}$$

Coming to the last term in (5.27) we see that

$$\begin{aligned}\sum_{i=1}^M \int_{B_R(x_j^i)} \Big| \sum_{\ell\neq i} \tau_{x_j^\ell}[\xi_\ell] \Big|^{2n'} &\le C(M) \sum_{i=1}^M \sum_{\ell \neq i} \int_{B_R(x_j^i)} \big| \tau_{x_j^\ell}[\xi_\ell] \big|^{2n'} \\ &\le C(M) \sum_{i=1}^M \sum_{\ell\neq i} \int_{B_R(x_j^i - x_j^\ell)} \big|\xi_\ell\big|^{2n'} ,\end{aligned} \tag{5.30}$$

where the last integral converges to zero since $|x_j^i - x_j^\ell| \to 0$ as $j \to \infty$ by $i \neq \ell$, and since $\xi_\ell \in L^{2n'}(\mathbb{R}^n)$. Finally, since $2n' \ge 2$ we can use (5.23) to address the second integral in (5.27), and conclude from (5.27), (5.28), (5.29) and (5.30) that

$$\limsup_{j\to\infty} \|u(t_j) - v_j\|_{L^{2n'}(\mathbb{R}^n)} \le C(M)\,\sigma\,, \qquad \forall \sigma > 0\,,$$

that is (5.25). Now, by (3.3) we have that

$$\|u(t_j) - v_j\|_{L^2(\mathbb{R}^n)} \le C(\varepsilon) \Big\{ \mathcal{AC}_\varepsilon(u(t_j)) + \mathcal{V}(u(t_j)) + \sum_{i=1} M\,\mathcal{AC}_\varepsilon(\xi_i) + \mathcal{V}(\xi_i) \Big\} \le C(\varepsilon, u_0)\,,$$

so that (5.25) implies

$$\lim_{j\to\infty} \|u(t_j) - v_j\|_{L^p(\mathbb{R}^n)} = 0\,, \qquad \forall p > 2\,. \tag{5.31}$$

We finally recall that given $k, k_1, k_2 \in \mathbb{N}$ and $p, p_1, p_2 \in [1,\infty]$ related by $k = \theta\,k_1 + (1-\theta)\,k_2$ and $(1/p) = (\theta/p_1) + (1-\theta)/p_2$ for some $\theta \in (0,1)$, then we have

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} \le C\,\|f\|^\theta_{W^{k_1,p_1}(\mathbb{R}^n)}\,\|f\|^{1-\theta}_{W^{k_2,p_2}(\mathbb{R}^n)}$$

for every $f \in W^{k_1,p_1}(\mathbb{R}^n) \cap W^{k_2,p_2}(\mathbb{R}^n)$ and with a constant C depending only on n, p_1, p_2, k_1 and k_2. By combining this interpolation inequality (with $k = 1, 2$, $k_1 = 0$, $k_2 = 3$, any $p_1 > 2$ and $p_2 \ge 2$) with (5.31) and the uniform $W^{3,p}$-bounds satisfied by $u(t_j)$ (recall (5.10)) and each of the ξ_i's, we conclude the proof of (5.24).

Step five: To complete the proof of (5.4) we need to prove the $W^{1,2}$-convergence stated in (5.24), that is

$$\lim_{j\to\infty} \Big\| u(t_j) - \sum_{i=1}^M \tau_{x_j^i}[\xi_i] \Big\|_{W^{1,2}(\mathbb{R}^n)} = 0 \,. \tag{5.32}$$

To this end we set, for the sake of brevity,

$$u_j = u(t_j)\,, \qquad v_j = \sum_{i=1}^M \tau_{x_j^i}[\xi_i]\,, \qquad w_j = u_j - v_j = u(t_j) - \sum_{i=1}^M \tau_{x_j^i}[\xi_i]\,,$$

as well as

$$Z(r) = W'(r) - \varepsilon\,\ell_\varepsilon\,V'(r)\,, \qquad \forall r \in [0,1]\,.$$

If we multiply by w_j in $\varepsilon^2(\partial_t u_j - 2\,\Delta u_j) = -Z(u_j) + \varepsilon\,(\ell_\varepsilon - \lambda_\varepsilon[u_j])\,V'(u_j)$ and in $2\,\varepsilon^2\,\Delta v_j = \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i])$, and we then add the resulting identities, we obtain

$$\begin{aligned} -2\,\varepsilon^2\,w_j\,\Delta w_j \;&=\; -\varepsilon^2\,w_j\,\partial_t u_j + \varepsilon\,(\ell_\varepsilon - \lambda_\varepsilon[u_j])\,w_j\,V'(u_j) \\ &\quad + w_j\Big\{ - Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i])\Big\} \end{aligned} \tag{5.33}$$

By (3.3), $\mathcal{V}(u_j) = 1$, $\mathcal{AC}_\varepsilon(u_j) \le \mathcal{AC}_\varepsilon(u_0)$, $\mathcal{V}(\xi_i) \le 1$, and $\mathcal{AC}_\varepsilon(\xi_i) \le \mathcal{AC}_\varepsilon(u_0)$ we have $\|w_j\|_{L^2(\mathbb{R}^n)} \le C(\varepsilon, u_0)$, so that $|w_j| \le M + 1$ on $\mathbb{R}^n$ and (5.31) imply

$$\|w_j\|_{L^2(\mathbb{R}^n)} \le C(\varepsilon, u_0, M)\,, \qquad \lim_{j\to\infty} \|w_j\|_{L^p(\mathbb{R}^n)} = 0\,, \qquad \forall p > 2\,. \tag{5.34}$$

Combining (5.34) and $0 \le V'(u_j) \le C\,u_j^{2n'-1}$ with (5.33) and the Hölder inequality, we thus find

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla w_j|^2 \;&\le\; C(\varepsilon, u_0, M)\,\|\partial_t u_j\|_{L^2(\mathbb{R}^n)} + C(\varepsilon, u_0)\,\big|\ell_\varepsilon - \lambda_\varepsilon[u_j]\big|\,\|w_j\|_{L^{2n'}(\mathbb{R}^n)} \\ &\quad + \int_{\mathbb{R}^n} w_j\,\Big\{ - Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i])\Big\}\,. \end{aligned} \tag{5.35}$$

where we have also used $\|u(t)\|_{L^{2n'}(\mathbb{R}^n)} \le C(\varepsilon, u_0)$ for all $t > 0$. The first two terms on the right-hand side of (5.35) converge to zero as $j \to \infty$, respectively, thanks to Theorem 1.2-(vi) and to (5.1) and (5.34). To deal with the third term on the right hand side of (5.35), we consider $\sigma > 0$ and pick $R > 0$ as in (5.26), so that, as proved in (5.28) and (5.29), we have

$$\int_{A_j^R} \sum_{i=1}^M \tau_{x_j^i}[\xi_i]^{2n'} \le C(M)\,\sigma\,, \qquad \limsup_{j\to\infty} \int_{A_j^R} u_j^{2n'} \le C(M)\,\sigma\,, \tag{5.36}$$

where $A_j^R = \mathbb{R}^n \setminus \cup_{i=1}^M B_R(x_j^i)$. Moreover, up to further increase the value of R we can ensure that

$$\max_{i=1,\dots,M} \|\xi_i\|_{L^p(\mathbb{R}^n)} \le C(p)\,\sigma\,, \qquad \forall p \ge 2\,, \tag{5.37}$$

as well as

$$|\xi_i| \le \kappa(\varepsilon) \text{ on } \mathbb{R}^n \setminus B_R, \text{ where } \kappa(\varepsilon) > 0 \text{ is s.t. } Z'(r) \ge \kappa(\varepsilon) \text{ for all } r \in [0,\kappa]\,. \tag{5.38}$$

(Notice that the existence of $\kappa(\varepsilon)$ is guaranteed by $W''(0) > 0$, $V''(0) = 0$, and $|\ell_\varepsilon| \le C(u_0)$, which in turn follows from $\sup_j |\lambda_\varepsilon[u_j]| \le C(u_0)$ (recall Theorem 1.2-(ii)) and (5.1).

Now, provided j is large enough depending on R (thus on σ), $\{A_j^R, B_R(x_i^j)\}_{i=1}^M$ is a partition of $\mathbb{R}^n$, that we can use to decompose the third term on the right hand side of (5.35). We begin by working with the integration over A_j^R, that we decompose by assigning a privileged role to ξ_1, and writing

$$
\begin{aligned}
&\int_{A_j^R} w_j \Big\{ -Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i]) \Big\} = \int_{A_j^R} \big(u_j - \tau_{x_j^1}[\xi_1]\big)\big(Z(\tau_{x_j^1}[\xi_1]) - Z(u_j)\big) \\
&- \int_{A_j^R} \sum_{\ell=2}^M \tau_{x_j^\ell}[\xi_\ell] \Big\{ -Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i]) \Big\} - \int_{A_j^R} \big(u_j - \tau_{x_j^1}[\xi_1]\big) \sum_{\ell=2}^M Z(\tau_{x_j^\ell}[\xi_\ell]) \\
&\le \int_{A_j^R} \big(u_j - \tau_{x_j^1}[\xi_1]\big)\big(Z(\tau_{x_j^1}[\xi_1]) - Z(u_j)\big) \qquad (5.39) \\
&+ C(M) \max_{1\le i\le M} \|\xi_i\|_{L^2(\mathbb{R}^n\setminus B_R)} \max_{1\le i\le M} \big\{ \|u_j\|_{L^2(\mathbb{R}^n)}, \|\xi_i\|_{L^2(\mathbb{R}^n)} \big\},
\end{aligned}
$$

where we have used $|Z(r)| \le C\, r$ for all $r \in [0,1]$. Now by the properties of W and V we have

$$
Z(r) - Z(s) = Z'(s)\,(r-s) + \mathrm{O}(r-s)^{2n'-1}, \qquad \forall r, s \in [0,1], \tag{5.40}
$$

so that by plugging $s = \xi_1$ and $r = u_j$ in (5.40), and by recalling that, thanks to (5.38), $Z'(\tau_{x_j^1}[\xi_1]) \ge \kappa(\varepsilon)$ on A_j^R,

$$
\begin{aligned}
&\int_{A_j^R} \big(u_j - \tau_{x_j^1}[\xi_1]\big)\big(Z(\tau_{x_j^1}[\xi_1]) - Z(u_j)\big) \qquad (5.41) \\
&\le - \int_{A_j^R} \big(u_j - \tau_{x_j^1}[\xi_1]\big)^2 Z'(\tau_{x_j^1}[\xi_1]) + C \int_{A_j^R} |u_j - \tau_{x_j^1}[\xi_1]|^{2n'} \\
&\le -\kappa(\varepsilon) \int_{A_j^R} \big(u_j - \tau_{x_j^1}[\xi_1]\big)^2 + C \int_{A_j^R} |u_j - \tau_{x_j^1}[\xi_1]|^{2n'} .
\end{aligned}
$$

Now, for every $p \ge 2$,

$$
\Big| \int_{A_j^R} |u_j - \tau_{x_j^1}[\xi_1]|^p - \int_{A_j^R} |w_j|^p \Big| \le C(p) \sum_{i=2}^M \int_{A_j^R} \tau_{x_j^i}[\xi_i]^p \le C(p) \sum_{i=2}^M \|\xi_i\|_{L^p(\mathbb{R}^n\setminus B_R)}, \tag{5.42}
$$

so that, combining (5.41) and (5.42) with (5.39), we finally conclude that

$$
\int_{A_j^R} w_j \Big\{ -Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i]) \Big\} \le -\kappa(\varepsilon) \int_{A_j^R} w_j^2 + C(p,M)\,\sigma .
$$

Going back to (5.35) we have proved that

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\nabla w_j|^2 + \kappa(\varepsilon) \int_{A_j^R} w_j^2 \ \le\ & \sum_{i=1}^M \int_{B_R(x_j^i)} w_j \Big\{ -Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i]) \Big\} \qquad (5.43) \\
& + \mathrm{o}_j + C(M)\,\sigma
\end{aligned}
$$

where $\mathrm{o}_j \to 0$ as $j \to \infty$. Next we observe that

$$
\begin{aligned}
&\int_{B_R(x_j^1)} w_j \Big\{ -Z(u_j) + \sum_{i=1}^M Z(\tau_{x_j^i}[\xi_i]) \Big\} \\
&\le C \int_{B_R(x_j^1)} \Big\{ |u_j - \tau_{x_j^1}[\xi_1]| + \sum_{i=2}^M \tau_{x_j^1}[\xi_1] \Big\} \Big\{ u_j + \sum_{k=1}^M |\tau_{x_j^k}[\xi_k] \Big\} \\
&\le C \max_{1\le i\le M} \big\{ \|u_j\|_{L^2}, \|\xi_i\|_{L^2}^2 \big\} \Big\{ \int_{B_R(x_j^1)} |u_j - \tau_{x_j^1}[\xi_1]|^2 + \sum_{k=2}^M \int_{B_R(x_j^1)} \tau_{x_j^k}[\xi_k]^2 \Big\} \\
&\le C(\varepsilon, u_0, M) \Big\{ \int_{B_R} |\tau_{-x_j^1}[u_j] - \xi_1|^2 + \sum_{k=2}^M \int_{B_R(x_j^1 - x_j^k)} \xi_k^2 \Big\} ,
\end{aligned}
$$

where the last two integrals converge to zero as $j \to \infty$ thanks to (5.23), $\xi_k \in L^2$ and $|x_j^1 - x_j^k| \to \infty$ as $j \to \infty$ if $k \ne 1$. Of course we can repeat this same argument with any other $B_R(x_j^i)$ with $i = 2, ..., M$, and go back to (5.43) to conclude that

$$
\int_{\mathbb{R}^n} |\nabla w_j|^2 + \kappa(\varepsilon) \int_{A_j^R} w_j^2 \le \mathrm{o}_j + C(M)\, \sigma \, . \tag{5.44}
$$

We finally conclude the proof of (5.32) since, for each $i = 1, ..., M$, we have

$$
\begin{aligned}
\int_{B_R(x_j^i)} w_j^2 &\le 2 \int_{B_R(x_j^i)} |u_j - \tau_{x_j^i}[\xi_i]|^2 + 2 \sum_{k \ne i} \int_{B_R(x_j^i)} \tau_{x_j^k}[\xi_k]^2 \\
&= 2\, \|\tau_{-x_j^i}[u_j] - \xi_i\|_{L^2(B_R)}^2 + 2 \sum_{k\ne i} \|\xi_k\|_{L^2(B_R(x_j^i - x_j^k))}^2 ,
\end{aligned}
$$

where the right hand side converges to zero as $j \to \infty$ by the same arguments used in the proof of (5.44). This concludes the proof of (5.32).

Step six: We finally prove that if $\mathrm{spt}\, u_0 \subset\subset \mathbb{R}^n$, then $M = 1$. We use the moving plane method, following an argument in [Fei97, Theorem 1.2]. Let $s_0 > 0$ be such that $\mathrm{spt}\, u_0 \subset B_{s_0}(0)$. For $s \ge s_0$, let $H_s = \{x \in \mathbb{R}^{n+1} : x_1 > s\}$, and define

$$
\rho_s[v](x) = v\big(2s - x_1, x_2, ..., x_{n+1}\big) \, , \qquad \forall x \in \mathbb{R}^{n+1}\, , v : \mathbb{R}^{n+1} \to \mathbb{R} \, .
$$

In this way, $u_s(t) = \rho_s[u(t)]$ solves

$$
\varepsilon^2 \big(\partial_t u_s - 2\,\Delta u_s\big) = -W'(u_s) + \varepsilon\, \lambda_\varepsilon[u_s(t)]\, V'(u_s) \, , \qquad \text{on } \mathbb{R}^n \times (0,\infty) \, ,
$$

with initial datum $u_s(0) = \rho_s[u(0)] = \rho_s[u_0]$. Since, trivially, $\lambda_\varepsilon[u_s(t)] = \lambda_\varepsilon[u(t)]$ for all $t > 0$, we have that $v_s = u_s - u$ solves

$$
\varepsilon^2 \big(\partial_t v_s - 2\,\Delta v_s\big) = h(x,t) \, , \qquad \text{on } \mathbb{R}^n \times (0,\infty) \, ,
$$

with $v_s(0) = \rho_s[u_0] - u_0$ and

$$
h(x,t) = W'(u) - W'(u_s) + \varepsilon\, \lambda_\varepsilon[u(t)]\, (V'(u_s) - V'(u)) = c(x,t)\, v_s
$$

for some $c \in L^\infty(\mathbb{R}^n \times (0,\infty))$. Since $\mathrm{spt}\, u_0 \subset B_{s_0}(0)$ and $s > s_0$ imply that $v_s(0) \ge 0$ on H_s, by the parabolic maximum principle [Lie96, Lemma 2.3] we deduce that $v_s(t) \ge 0$ on H_s for all $t > 0$. In particular, if $h > s$ and $(x_2, ..., x_{n+1}) \in \mathbb{R}^n$, then the non-negativity of $v_s(t)$ at $(h, x_2, ..., x_{n+1}) \in H_s$ implies that

$$
\begin{aligned}
0 &\le \lim_{h\to s^+} \frac{v_s(h, x_2, ..., x_{n+1}, t)}{h - s} \\
&= \lim_{h\to s^+} \frac{u(2s - h, x_2, ..., x_{n+1}, t) - u(h, x_2, ..., x_{n+1}, t)}{h - s} = -e_1 \cdot \nabla u(s, x_2, ..., x_{n+1}, t) \, .
\end{aligned}
$$

We have thus proved that $e_1 \cdot \nabla u(t) \le 0$ on H_s for every $s > s_0$. By arbitrariness of $s > s_0$ and of the choice of the direction with respect to which we reflect $u(t)$, we conclude that

$$\frac{x}{|x|} \cdot \nabla u(x,t) \le 0\,, \qquad \forall x \in \mathbb{R}^{n+1} \setminus B_{s_0}(0)\,, t > 0\,. \tag{5.45}$$

By (5.10), for every $t > 0$, $u(x,t) \to 0$ as $|x| \to \infty$. Now pick a sequence $t_j \to \infty$ such that (5.24) holds. Since $m = \max_{1 \le i \le M} \sup_{\mathbb{R}^n} \xi_i > 0$, if $M \ge 2$, then the fact that for $i \ne k$ we have $|x_j^i - x_j^k| \to \infty$ as $j \to \infty$, combined with (5.24), implies that, for every $R > 0$, there is $j(R) \in \mathbb{N}$ such that if $j \ge j(R)$ then

$$\sup_{\mathbb{R}^{n+1} \setminus B_R} u(t_j) \ge \frac{m}{2}\,.$$

This leads to a contradiction with $u(x,t_j) \to 0$ as $|x| \to \infty$. Therefore $M = 1$ and, by a similar argument, (5.45) is also seen to imply the boundedness of $(x_j^1)_j$, and thus its convergence modulo a further subsequence extraction. □

6. Subsequential bubbling into diffused balls (Proof of Theorem 1.4)

Proof of Theorem 1.4. We start by noticing that, thanks to (1.7), we have

$$N\,\Psi\Big(\varepsilon, \frac{1}{N}\Big) \ge 2\,n\,\omega_n^{1/n}\,N^{1/n}\,, \qquad \forall N > 0\,.$$

In particular, if M_0 is the least integer such that $n\,\omega_n^{1/n}\,M_0^{1/n} > \int_{\mathbb{R}^n} |\nabla u_0|$, then by $\mathcal{AC}_\varepsilon(u_0) \to 2\int_{\mathbb{R}^n} |\nabla u_0|$ we can find $\varepsilon_* > 0$ (depending on u_0) such that

$$M_0\,\Psi\Big(\varepsilon, \frac{1}{M_0}\Big) \ge \mathcal{AC}_\varepsilon(u_0)\,, \qquad \forall \varepsilon \in (0, \varepsilon_*)\,. \tag{6.1}$$

We shall prove the theorem for every solution $u(t)$ of (DF) corresponding to $\varepsilon < \varepsilon_0^*$, where

$$\varepsilon_0^* := \min\Big\{\varepsilon_*, \frac{\sigma_0}{M_0^{1/n}}\Big\}\,, \tag{6.2}$$

for some suitably small universal constant σ_0 to be the determined below.

Step one: We prove the existence of a universal constant σ_0 such that

$$\Psi(\sigma, 1) \ge \Psi(\varepsilon, 1)\,, \qquad \forall \varepsilon < \min\{\sigma_0, \sigma\}\,.$$

We shall take $\sigma_0 \le \varepsilon_0$, so that, thanks to (2.4), we will be able to focus directly on the case when $\varepsilon < \sigma_0 < \sigma$. Should the the claim fail in this case, we could then find sequences $(\varepsilon_j)_j$ and $(\sigma_j)_j$ with $\varepsilon_j \to 0^+$ as $j \to \infty$ and

$$\sigma_j > \varepsilon_j\,, \qquad \Psi(\sigma_j, 1) < \Psi(\varepsilon_j, 1)\,, \qquad \forall j \in \mathbb{N}\,. \tag{6.3}$$

For each j we denote by u_j a minimizer of $\Psi(\sigma_j, 1)$, and notice that, by (2.3) and up to extracting a subsequence in j, there is $\sigma_* \in [0, \infty]$ such that

$$\lim_{j\to\infty} \sigma_j = \sigma_* \in [0,\infty]\,, \qquad \lim_{j\to\infty} \Psi(\sigma_j, 1) = \lim_{j\to\infty} \mathcal{AC}_{\sigma_j}(u_j) = 2\,c_{\rm iso}(n)\,. \tag{6.4}$$

We can immediately exclude that $\sigma_* = 0$ thanks to (2.4) and (6.3). If $\sigma_* \in (0,\infty)$, then a minor modification of the elementary compactness argument of [MNR23, Proof of Theorem A.1, steps one and two] shows that, up to extracting subsequences, there is a minimizer u_* in $\Psi(\sigma_*, 1)$ such that $u_j \to u_*$ in $L^1_{\rm loc}(\mathbb{R}^n)$. In particular, by combining (2.2) with (6.4) we find that

$$2\,c_{\rm iso}(n) < \Psi(\sigma_*, 1) = \mathcal{AC}_{\sigma_*}(u_*) \le \liminf_{j\to\infty} \mathcal{AC}_{\sigma_j}(u_j) = \lim_{j\to\infty} \Psi(\sigma_j, 1) = 2\,c_{\rm iso}(n)\,,$$

a contradiction. We are thus left with the possibility that $\sigma_* = +\infty$. To obtain a contradiction, we notice that, by $1 = \mathcal{V}(u_j) = \|\Phi(u_j)\|_{L^{n/(n-1)}(\mathbb{R}^n)}$, we have

$$c_{\rm iso}(n) \le |D[\Phi(u_j)]|(\mathbb{R}^n) < \frac{\mathcal{AC}_{\sigma_j}(u_j)}{2} \to c_{\rm iso}(n)\,, \qquad \text{as } j\to\infty.$$

In particular, by a standard compactness argument (see, e.g., [FMP07, Theorem A.1])), for some $a, r > 0$ and up to extracting a subsequence, $\Phi(u_j) \to a\,1_{B_r}$ in $L^{n/(n-1)}(\mathbb{R}^n)$ as $j \to \infty$. Setting $b = \Phi^{-1}(a)$ we have $u_j \to b\,1_{B_r}$ in $L^1_{\rm loc}(\mathbb{R}^n)$, so that, for every $\varphi \in C^\infty_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} b\,1_{B_r}\,\nabla\varphi = \lim_{j\to\infty}\int_{\mathbb{R}^n} u_j\,\nabla\varphi \le \|\varphi\|_{L^2(\mathbb{R}^n)}\,\|\nabla u_j\|_{L^2(\mathbb{R}^n)}$$

where $\|\nabla u_j\|_{L^2(\mathbb{R}^n)} \le \mathcal{AC}_{\sigma_j}(u_j)/\sigma_j \to 0$ as $j\to\infty$ thanks to $\sigma_j \to +\infty$. In summary, $|D(b\,1_{B_r})|(\mathbb{R}^n) = 0$, so that $0 = b = \Phi^{-1}(a)$, and thus $a = \Phi(0) = 0$, against $a > 0$.

Step two: Given a sequence $t_j \to \infty$ as $j\to\infty$, we want to show that, up to extracting a subsequence, there is $M \in \mathbb{N}$ (with $M \le M_0$) and sequences $\{(x^i_j)_j\}_{i=1}^M$ with $|x^i_j - x^k_j| \to \infty$ as $j\to\infty$ $(i \ne k)$, such that, for all $p > 2$,

$$\lim_{j\to\infty}\Big\|u(t_j) - \sum_{i=1}^M \tau_{x^i_j}\big[\zeta_{\varepsilon,1/M}\big]\Big\|_{(W^{2,p}\cap W^{1,2})(\mathbb{R}^n)} = 0\,. \tag{6.5}$$

By Theorem 1.3, we know that (6.5) holds if in place of $\zeta_{\varepsilon,1/M}$ we have some ξ_i, $i = 1, ..., M$, solving, for a same $\ell_\varepsilon > 0$,

$$2\,\varepsilon^2\,\Delta\xi_i = W'(\xi_i) - \varepsilon\,\ell_\varepsilon\,V'(\xi_i) \quad \text{on } \mathbb{R}^n, \tag{6.6}$$

with $\mathcal{AC}_\varepsilon(u_0) \ge \sum_{i=1}^M \mathcal{AC}_\varepsilon(\xi_i)$ and $1 = \sum_{i=1}^M \mathcal{V}(\xi_i)$. We set for brevity $m_i = \mathcal{V}(\xi_i)$ and assume without loss of generality that $m_i \ge m_{i+1}$. Our goal is thus proving the existence of $z_i \in \mathbb{R}^n$ such that $\xi_i = \tau_{z_i}[\zeta_{\varepsilon,1/M}]$ for each i.

We first notice that it must be $m_1 \ge 1/M_0$. Indeed, should this not be the case, then using also step one and the fact that $\varepsilon < \varepsilon_0^*$ implies $\varepsilon\,M_0^{1/n} < \sigma_0$, we would have

$$\frac{1}{m_i^{1/n}} > M_0^{1/n}\,, \qquad \Psi\Big(\varepsilon/m_i^{1/n}, 1\Big) \ge \Psi\Big(\varepsilon\,M_0^{1/n}, 1\Big)\,, \qquad \forall i\,.$$

We could then combine these inequalities with $\mathcal{AC}_\varepsilon(u_0) \ge \sum_{i=1}^M \mathcal{AC}_\varepsilon(\xi_i)$ and (2.1) to find that

$$\begin{aligned}
\mathcal{AC}_\varepsilon(u_0) &\ge \sum_{i=1}^M \Psi(\varepsilon, m_i) = \sum_{i=1}^M m_i^{(n-1)/n}\,\Psi\Big(\varepsilon/m_i^{1/n}, 1\Big) \\
&> M_0^{1/n}\sum_{i=1}^M m_i\,\Psi\Big(\varepsilon/m_i^{1/n}, 1\Big) \ge M_0^{1/n}\,\Psi(\varepsilon\,M_0^{1/n}, 1)\sum_{i=1}^M m_i \\
&= M_0^{1/n}\,\Psi(\varepsilon\,M_0^{1/n}, 1) = M_0^{1/n}\,M_0^{(n-1)/n}\,\Psi(\varepsilon, 1/M_0) = M_0\,\Psi(\varepsilon, 1/M_0)\,,
\end{aligned}$$

thus leading to a contradiction with (6.1).

We can thus assume that $m_1 \ge 1/M_0$. By arguing as in step one in the proof of Theorem 1.3 (see, in particular, the proof of (5.7)), we see that (6.6) implies

$$n\,\ell_\varepsilon\,m_i = n\,\ell_\varepsilon\,\mathcal{V}(\xi_i) = n\,\mathcal{AC}_\varepsilon(\xi_i) - 2\,\varepsilon\int_{\mathbb{R}^n}|\nabla\xi_i|^2\,.$$

By (6.1), we find

$$\varepsilon\,\ell_\varepsilon \le \varepsilon \sum_{i=1}^M \mathcal{AC}_\varepsilon(\xi_i) \le \varepsilon\,\mathcal{AC}_\varepsilon(u_0) \le \varepsilon\,M_0^{1/n}\,\Psi(\varepsilon\,M_0^{1/n},1)\,.$$

By $\varepsilon\,M_0^{1/n} < \sigma_0$ and by step one we also have $\Psi(\varepsilon\,M_0^{1/n},1) \le \Psi(\sigma_0,1)$, so that

$$\varepsilon\,\ell_\varepsilon \le \sigma_0\,\Psi(\sigma_0,1)\,.$$

Since $\sigma\,\Psi(\sigma,1)\to 0$ as $\sigma\to 0^+$, we conclude that, up to further decreasing the value of σ_0, we can guarantee $\varepsilon\,\ell_\varepsilon < \nu_0$ with ν_0 as in Theorem Ψ-(ii). In particular, for each i there exists $z_i\in\mathbb{R}^n$ such that

$$\xi_i = \tau_{z_i}[\zeta_{\varepsilon,m_i}]\,,\qquad \ell_\varepsilon = \Lambda_{\varepsilon,m_i}\,.$$

Since ℓ_ε is independent of i, $\ell_\varepsilon = \Lambda_{\varepsilon,m_i}$ implies that $m_i = m_1$ for all i. Therefore $1 = \sum_{i=1}^M m_i$ gives $m_1 = 1/M$, and the proof that $\xi_i = \tau_{z_i}[\zeta_{\varepsilon,1/M}]$ for each i is complete.

Conclusion: We are left to prove that M is uniquely determined by the relation

$$M\,\Psi(\varepsilon,1/M) = \lim_{t\to\infty}\mathcal{AC}_\varepsilon(u(t))\,. \tag{6.7}$$

Since $M\le M_0$ and $\varepsilon\,M_0^{1/n} < \sigma_0 \le \varepsilon_0$ we know that $M\in(0,(\varepsilon_0/\varepsilon)^n)$. To conclude that (6.7) uniquely characterizes M we are going to prove that

$$x\mapsto x\,\Psi\Big(\varepsilon,\frac{1}{x}\Big)\ \text{is strictly increasing on}\ \Big(0,\Big(\frac{\varepsilon_0}{\varepsilon}\Big)^n\Big)\,,$$

which, in turn, is equivalent to showing that $x\mapsto\Psi(\varepsilon,x)/x$ is strictly decreasing on $((\varepsilon/\varepsilon_0)^n,\infty)$.

Let us set $f(x)=\Psi(\varepsilon,x)$ so that, by (2.5), f is concave on $(0,\infty)$ and strictly concave on (a,∞) (with $a=(\varepsilon/\varepsilon_0)^n$). Since $f(0^+)\ge 0$ we deduce from $f(0^+)\le f(x)+f'(x)(0-x)$ for all $x>0$ that $x\,f'(x)\le f(x)$ for all $x>0$. In particular,

$$\frac{f(a)}{a}\ge f'(a) > f'(x)\qquad \forall x>a\,. \tag{6.8}$$

If we now use first $f(a) < f(x)+f'(x)(a-x)$ for all $x>a$, and then (6.8), we find that

$$x\,f'(x)-f(x) < a\,f'(x)-f(a) < 0\,,\qquad \forall x>a\,.$$

This shows that $x\mapsto f(x)/x$ is strictly decreasing on (a,∞), as claimed. $\square$

7. Strict stability of the diffused isoperimetric problem (Proof of Theorem 1.5)

Proof of Theorem 1.5. We aim at proving the theorem by combining two results. The first one is the radial case of Theorem 1.5, which was proved in [MR24, Lemma 4.4]: if $\varepsilon\in(0,\varepsilon_0)$, then

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) \ge \frac{1}{C}\int_{\mathbb{R}^n}\varepsilon\,|\nabla\varphi|^2+\frac{\varphi^2}{\varepsilon}\,, \tag{7.1}$$

for every $\varphi\in W^{1,2}(\mathbb{R}^n)$ which is radial with respect to the origin and such that $\int_{\mathbb{R}^n}V'(\zeta_\varepsilon)\,\varphi = 0$. The second one is the strict stability of the second variation of the (volume-constrained) area functional on the unit sphere. By the latter, we mean the following stability result: Let $\{A_i\}_{i\in\mathbb{N}}$ denote the normalized eigenfunctions of the Laplacian on $\mathbb{S}^{n-1}$ and let $\{\mu_i\}_{i\in\mathbb{N}}$ denote the corresponding eigenvalues listed in increasing order, so that $-\Delta^{\mathbb{S}^{n-1}}A_i = \mu_i\,A_i$ on $\mathbb{S}^{n-1}$ for each i, A_0 is constant with $\mu_0=0$, $A_i(\theta)=\theta_i$ with $\mu_i=(n-1)$ for $i=1,...,n$,

$\mu_{n+1} > \mu_n$ and $\int_{\mathbb{S}^{n-1}} A_i^2 = 1$. It is well known (see, e.g. [CL12, Lemma 4.2]) that, for every $\alpha \in W^{1,2}(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} |\nabla^{\mathbb{S}^{n-1}}\alpha|^2 - (n-1)\,\alpha^2 \geq 0\,, \qquad \text{if} \int_{\mathbb{S}^{n-1}} \alpha = 0\,, \tag{7.2}$$

(stability of the sphere with respect to volume-preserving variations) and that

$$\int_{\mathbb{S}^{n-1}} |\nabla^{\mathbb{S}^{n-1}}\alpha|^2 - (n-1)\,\alpha^2 \geq c(n) \int_{\mathbb{S}^{n-1}} |\nabla^{\mathbb{S}^{n-1}}\alpha|^2 + \alpha^2\,, \tag{7.3}$$

if

$$\int_{\mathbb{S}^{n-1}} \alpha = 0\,, \qquad \int_{\mathbb{S}^{n-1}} \theta\,\alpha(\theta) = 0\,, \tag{7.4}$$

for some positive constant $c(n)$ (strict stability of the sphere with respect to volume-preserving variations orthogonal to translations). Combining these two stability results will require some a careful decompositions of functions φ satisfying (1.24).

Step one: In this first step we introduce a convenient way to rewrite

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) = \int_{\mathbb{R}^n} 2\,\varepsilon\,|\nabla\varphi|^2 + \Big\{\frac{W''(\zeta_\varepsilon)}{\varepsilon} - \Lambda_\varepsilon\,V''(\zeta_\varepsilon)\Big\}\,\varphi^2\,, \tag{7.5}$$

by means of the transformation $\psi = \varphi/\zeta_\varepsilon'$, which allows one to relate $\mathcal{Q}_\varepsilon[\zeta_\varepsilon]$ to the second variation of the volume-constrained area functional on the sphere. (This kind of transformation is inspired by similar computations found in [Ton05, Le11, Gas20].) More precisely, setting $\hat{x} = x/|x|$ and denoting by ζ_ε', ζ_ε'', etc. the derivatives of the radial profile of ζ_ε, we show that if $\varphi \in W^{1,2}(\mathbb{R}^n)$ and, correspondingly, we define $\psi \in W^{1,2}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ by

$$\varphi = \psi\,\zeta_\varepsilon'\,, \tag{7.6}$$

then

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) = 2\,\varepsilon \int_{\mathbb{R}^n} (\zeta_\varepsilon'^2)\,\Big\{|\nabla\psi|^2 - \frac{(n-1)}{|x|^2}\,\psi^2\Big\}\,. \tag{7.7}$$

Indeed, under (7.6), we have $\nabla\varphi = \zeta_\varepsilon'\,\nabla\psi + \zeta_\varepsilon''\,\psi\,\hat{x}$, and integrating by parts[9] the mixed term in

$$|\nabla\varphi|^2 = (\zeta_\varepsilon')^2\,|\nabla\psi|^2 + 2\,\zeta_\varepsilon'\,\zeta_\varepsilon''\,\psi\,(\hat{x}\cdot\nabla\psi) + (\zeta_\varepsilon'')^2\,\psi^2\,, \tag{7.8}$$

we find

$$\begin{aligned} 2\int_{\mathbb{R}^n} \zeta_\varepsilon'\,\zeta_\varepsilon''\,\psi\,(\hat{x}\cdot\nabla\psi) &= -\int_{\mathbb{R}^n} \psi^2\,\mathrm{div}\,\big(\zeta_\varepsilon'\,\zeta_\varepsilon''\hat{x}\big) \\ &= -\int_{\mathbb{R}^n} \psi^2\,(\zeta_\varepsilon'')^2 - \int_{\mathbb{R}^n} \psi^2\,\zeta_\varepsilon'\,\Big\{\zeta_\varepsilon''' + \frac{n-1}{|x|}\,\zeta_\varepsilon''\Big\}\,. \end{aligned} \tag{7.9}$$

To rewrite the term with ζ_ε''' we make use of $2\,\varepsilon^2\,\Delta\zeta_\varepsilon = W'(\zeta_\varepsilon) - \varepsilon\,\Lambda_\varepsilon\,V'(\zeta_\varepsilon)$ on $\mathbb{R}^n$, which, writing $\Delta\zeta_\varepsilon$ in radial coordinates, takes the form

$$2\,\varepsilon^2\,\Big\{\zeta_\varepsilon'' + \frac{(n-1)}{|x|}\,\zeta_\varepsilon'\Big\} = W'(\zeta_\varepsilon) - \varepsilon\,\Lambda_\varepsilon\,V'(\zeta_\varepsilon)\,. \tag{7.10}$$

Differentiating (7.10) in the radial direction we thus find

$$2\,\varepsilon^2\,\Big\{\zeta_\varepsilon''' + \frac{n-1}{|x|}\,\zeta_\varepsilon'' - \frac{n-1}{|x|^2}\,\zeta_\varepsilon'\Big\} = \zeta_\varepsilon'\,\Big\{W''(\zeta_\varepsilon) - \varepsilon\,\Lambda_\varepsilon\,V''(\zeta_\varepsilon)\Big\}\,,$$

which can be combined into (7.9) to obtain

$$2\int_{\mathbb{R}^n} \zeta_\varepsilon'\,\zeta_\varepsilon''\,\psi\,(\hat{x}\cdot\nabla\psi) \;=\; -\int_{\mathbb{R}^n} \psi^2\,(\zeta_\varepsilon'')^2 - \int_{\mathbb{R}^n} \psi^2\,(\zeta_\varepsilon')^2\,\Big\{\frac{W''(\zeta_\varepsilon) - \varepsilon\,\Lambda_\varepsilon\,V''(\zeta_\varepsilon)}{2\,\varepsilon^2} + \frac{n-1}{|x|^2}\Big\}\,.$$

[9] This is easily justified since ζ_ε and its derivatives all decay exponentially at infinity, see [MR24, Theorem 3.1].

On combining this last identity with (7.8) and the definition of $\mathcal{Q}_\varepsilon[\zeta_\varepsilon]$ we thus find

$$\begin{aligned}
\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) &= 2\,\varepsilon\int_{\mathbb{R}^n}|\nabla\varphi|^2+\int_{\mathbb{R}^n}\Big\{\frac{W''(\zeta_\varepsilon)}{\varepsilon}-\Lambda_\varepsilon\,V''(\zeta_\varepsilon)\Big\}\,\varphi^2\\
&= 2\,\varepsilon\int_{\mathbb{R}^n}(\zeta_\varepsilon')^2\,|\nabla\psi|^2+(\zeta_\varepsilon'')^2\,\psi^2\\
&\quad -2\,\varepsilon\int_{\mathbb{R}^n}\psi^2\,(\zeta_\varepsilon'')^2-\int_{\mathbb{R}^n}\psi^2\,(\zeta_\varepsilon')^2\Big\{\frac{W''(\zeta_\varepsilon)}{\varepsilon}-\Lambda_\varepsilon\,V''(\zeta_\varepsilon)+2\,\varepsilon\,\frac{n-1}{|x|^2}\Big\}\\
&\quad +\int_{\mathbb{R}^n}\Big\{\frac{W''(\zeta_\varepsilon)}{\varepsilon}-\Lambda_\varepsilon\,V''(\zeta_\varepsilon)\Big\}\,\varphi^2
\end{aligned}$$

which boils down to (7.7) thanks to $\varphi=\zeta_\varepsilon'\,\psi$.

Step two: We prove that conclusion (1.26) follows from conclusion (1.25). Indeed, arguing by contradiction, should (1.24) imply (1.25) but not (1.26), then we could find a sequence $(\varphi_j)_j$ in $W^{1,2}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\varphi_j\,V'(\zeta_\varepsilon)=0\,,\qquad\int_{\mathbb{R}^n}\varphi_j\,\nabla\zeta_\varepsilon=0\,,\qquad\forall j\,,\tag{7.11}$$

$$\int_{\mathbb{R}^n}|\nabla\varphi_j|^2+\varphi_j^2=1\,,\qquad\forall j\,,\tag{7.12}$$

$$\lim_{j\to\infty}\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_j)=0\,.\tag{7.13}$$

Having assumed that (1.24) implies (1.25), we could deduce from (7.11) that

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_j)\ge\frac{\varepsilon}{C}\int_{\mathbb{R}^n}\varphi_j^2\,,\qquad\forall j\,.\tag{7.14}$$

By combining (7.14) with (7.13) we would then find

$$\lim_{j\to\infty}\int_{\mathbb{R}^n}\varphi_j^2=0\,,\quad\text{and, hence, by (7.12),}\ \lim_{j\to\infty}\int_{\mathbb{R}^n}|\nabla\varphi_j|^2=1\,.\tag{7.15}$$

But then, taking into account that $(W''(\zeta_\varepsilon)/\varepsilon-\Lambda_\varepsilon\,V''(\zeta_\varepsilon))\in L^\infty(\mathbb{R}^n)$, and combining (7.5) and (7.15), we could conclude that $\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_j)\to 2\,\varepsilon$ as $j\to\infty$, in contradiction with (7.13).

Step three: By step two, we are left to prove that (1.24) implies (1.25). In this step, we present an additional reduction. Let $L^2_{\rm rad}(\mathbb{R}^n)$ denote the set of radial functions with respect to the origin belonging to $L^2(\mathbb{R}^n)$, and let

$$\begin{aligned}
W^{1,2}_{\rm rad}(\mathbb{R}^n) &= L^2_{\rm rad}(\mathbb{R}^n)\cap W^{1,2}(\mathbb{R}^n)\,,\\
Z &= L^2_{\rm rad}(\mathbb{R}^n)^\perp\cap W^{1,2}(\mathbb{R}^n)\,,\\
Z^* &= \Big\{\varphi\in Z:\int_{\mathbb{R}^n}\varphi\,\nabla\zeta_\varepsilon=0\Big\}\,.
\end{aligned}$$

We prove that if

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi)\ge\frac{\varepsilon^2}{C}\int_{\mathbb{R}^n}\varphi^2\,,\qquad\forall\varphi\in Z^*\,,\tag{7.16}$$

then (1.24) implies (1.25) (and the theorem is proved).

To begin with, denoting by $\varphi_{\rm rad}$ and $\varphi_{\rm rad}^\perp$ the L^2-projections of $\varphi\in W^{1,2}(\mathbb{R}^n)$ on, respectively, $W^{1,2}_{\rm rad}(\mathbb{R}^n)$ and Z, so that $\varphi=\varphi_{\rm rad}+\varphi_{\rm rad}^\perp$, we notice that

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi)=\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_{\rm rad})+\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_{\rm rad}^\perp)\,,\qquad\forall\varphi\in W^{1,2}(\mathbb{R}^n)\,.\tag{7.17}$$

Indeed,

$$
\begin{aligned}
\mathcal{Q}_\varepsilon[\zeta_\varepsilon]\big(\varphi_{\rm rad},\varphi_{\rm rad}^\perp\big) &= \int_{\mathbb{R}^n} 2\,\varepsilon\,\nabla\varphi_{\rm rad}\cdot\nabla\varphi_{\rm rad}^\perp+\Big(\frac{W''(\zeta_\varepsilon)}{\varepsilon}-\Lambda_\varepsilon\,V''(\zeta_\varepsilon)\Big)\varphi_{\rm rad}\,\varphi_{\rm rad}^\perp\\
&= \int_{\mathbb{R}^n}\Big\{-2\,\varepsilon\,\Delta\varphi_{\rm rad}+\Big(\frac{W''(\zeta_\varepsilon)}{\varepsilon}-\Lambda_\varepsilon\,V''(\zeta_\varepsilon)\Big)\varphi_{\rm rad}\Big\}\,\varphi_{\rm rad}^\perp=0\,,
\end{aligned}
$$

where in the last identity we have used that $\varphi_{\rm rad}^\perp\in Z$ and the fact that the function in the curly bracket is radial.

Next, we observe that the orthogonality relations in (1.24) can be equivalently reformulated as follows for any $\varphi\in W^{1,2}(\mathbb{R}^n)$:

$$
\int_{\mathbb{R}^n} V'(\zeta_\varepsilon)\,\varphi=0 \qquad\text{if and only if}\qquad \int_{\mathbb{R}^n} V'(\zeta_\varepsilon)\,\varphi_{\rm rad}=0\,, \tag{7.18}
$$

$$
\int_{\mathbb{R}^n} \varphi\,\nabla\zeta_\varepsilon=0 \qquad\text{if and only if}\qquad \int_{\mathbb{R}^n} \varphi_{\rm rad}^\perp\,\nabla\zeta_\varepsilon=0\,, \tag{7.19}
$$

since $V'(\zeta_\varepsilon)\in L^2_{\rm rad}(\mathbb{R}^n)$, $\nabla\zeta_\varepsilon=\zeta_\varepsilon'\,\hat x$, and $\int_{\mathbb{R}^n}\hat x\,\psi=0$ for all $\psi\in L^2_{\rm rad}(\mathbb{R}^n)$. In summary, by combining in the order (7.17), (7.1) applied to $\varphi_{\rm rad}$ (as we can do thanks to (7.18)) and (7.16) to $\varphi_{\rm rad}^\perp$ (as we can do since $\varphi_{\rm rad}^\perp\in Z^*$ by (8.11)) we deduce that

$$
\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi)=\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_{\rm rad})+\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_{\rm rad}^\perp)\ge\frac{1}{C\,\varepsilon}\int_{\mathbb{R}^n}(\varphi_{\rm rad})^2+\frac{\varepsilon}{C}\int_{\mathbb{R}^n}(\varphi_{\rm rad}^\perp)^2\ge\frac{\varepsilon}{C}\int_{\mathbb{R}^n}\varphi^2\,,
$$

that is (1.25).

Step four: We now begin the proof of (7.16) by introducing a Fourier decomposition of $\varphi\in W^{1,2}(\mathbb{R}^n)$ that is particularly convenient when $\varphi\in Z^*$. Since the radial Laplacian on $\mathbb{R}^n$, i.e. the map $B(r)\mapsto B''(r)+(n-1)\,B'(r)/r$, defines a self-adjoint operator on $L^2((0,\infty),r^{n-1}\,dr)$, we can consider an orthonormal basis $\{B_j\}_{j=0}^\infty$ of $L^2((0,\infty),r^{n-1}\,dr)$ made up of its eigenfunctions. Since $L^2(\mathbb{S}^{n-1})$ and $L^2((0,\infty),r^{n-1}\,dr)$ are separable Hilbert spaces, it follows that $\{A_i\otimes B_j\}_{i,j=0}^\infty$ is an orthonormal basis of $L^2(K;\kappa)$, where $K=\mathbb{S}^{n-1}\times(0,\infty)$ and $\kappa=(\mathcal{H}^{n-1}\llcorner\mathbb{S}^{n-1})\times(r^{n-1}\,dr)$. In particular, since $\Phi:K\to\mathbb{R}^n$, $\Phi(\theta,r)=\theta\,r$, naturally induces an isometry between $L^2(\mathbb{R}^n)$ and $L^2(K,\kappa)$, we conclude that $\{(A_i\otimes B_j)\circ\Phi\}_{i,j=0}^\infty$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. Moreover, each $a_{ij}=(A_i\otimes B_j)\circ\Phi$ is an eigenfunction of the Laplacian on $\mathbb{R}^n$, so that the orthogonality of a_{ij} and a_{hk} in $L^2(\mathbb{R}^n)$ (which holds true for $i\ne h$ or $j\ne k$) implies the orthogonality of a_{ij} and a_{hk} in $W^{1,2}_0(\mathbb{R}^n)$. In summary, whenever $\varphi\in W^{1,2}(\mathbb{R}^n)$ we have

$$
\varphi=\sum_{i,j=0}^\infty\varphi_{ij}\,a_{ij}\qquad\text{in }W^{1,2}(\mathbb{R}^n)\,,
$$

$$
\text{where }\varphi_{ij}=\int_{\mathbb{R}^n}a_{ij}\,\varphi=\int_{\mathbb{R}^n}A_i(\hat x)\,B_j(|x|)\,\varphi(x)\,dx\,.
$$

If we define $\varphi_i^*\in W^{1,2}_{\rm rad}(\mathbb{R}^n)$ and $\varphi_i\in W^{1,2}(\mathbb{R}^n)$ by

$$
\varphi_i^*(x)=\sum_{j=0}^\infty\varphi_{ij}\,B_j(|x|)\,,\qquad \varphi_i(x)=A_i(\hat x)\,\varphi_i^*(x)\,,\qquad x\in\mathbb{R}^n\,,
$$

then we have

$$
\varphi=\sum_{i=0}^\infty\varphi_i\qquad\text{in }W^{1,2}(\mathbb{R}^n)\,, \tag{7.20}
$$

as well as

$$
\int_{\mathbb{R}^n}|\nabla\varphi|^2=\sum_{i=0}^\infty\int_{\mathbb{R}^n}|\nabla\varphi_i|^2\,,\qquad \int_{\mathbb{R}^n}f\,\varphi^2=\sum_{i=0}^\infty\int_{\mathbb{R}^n}f\,\varphi_i^2\,, \tag{7.21}
$$

for every $\varphi \in W^{1,2}(\mathbb{R}^n)$ and $f \in L^\infty(\mathbb{R}^n)$ that is either radial or angular (i.e., $f \circ \Phi$ depends either on r or on θ only). We can also notice that

$$\varphi_0 = \varphi_0^* = 0 \qquad \forall \varphi \in Z\,, \tag{7.22}$$

and then, since A_0 is a constant, if $\varphi \in Z$, then for every $j \in \mathbb{N}$

$$\varphi_{0j} = \int_{\mathbb{R}^n} A_0(|\hat{x}|)\, B_j(|x|)\, \varphi(x)\, dx = A_0 \int_{\mathbb{R}^n} B_j(|x|)\, \varphi(x)\, dx = 0\,.$$

In particular, by applying (7.21) with $f = (W''(\zeta_\varepsilon)/\varepsilon) - \Lambda_\varepsilon V''(\zeta_\varepsilon)$ and (7.22) we find

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) = \sum_{i=1}^\infty \mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_i)\,, \qquad \forall \varphi \in Z\,. \tag{7.23}$$

We make two claims:

Claim one: if $\varphi \in Z^*$ and $i \ge n+1$, then

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_i) \ge \frac{\varepsilon}{C} \int_{\mathbb{R}^n} \frac{\varphi_i^2}{|x|^2}\,. \tag{7.24}$$

Claim two: if $\varphi \in Z^*$ and $i = 1, ..., n$, then

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_i) \ge \frac{1}{C\,\varepsilon} \int_{\mathbb{R}^n} \varphi_i^2\,. \tag{7.25}$$

We first show how to complete the proof of the theorem starting from these two claims, and then we prove the claims themselves.

Conclusion of the theorem from the claims: Since $W''(0) > 0$, $\zeta_\varepsilon \to 0$ as $|x| \to \infty$ (uniformly on $\varepsilon \in (0, \varepsilon_0)$), and $\varepsilon\, \Lambda_\varepsilon \to 0$ as $\varepsilon \to 0^+$, we see that there are universal constants κ and R_1 such that

$$W''(\zeta_\varepsilon) - \varepsilon\, \Lambda_\varepsilon\, V''(\zeta_\varepsilon) \ge \kappa\,, \qquad \text{on } \mathbb{R}^n \setminus B_{R_1}\,, \tag{7.26}$$

for every $\varepsilon \in (0, \varepsilon_0)$. By (7.24),

$$\int_{B_{R_1}} \varphi_i^2 \le \frac{C\, R_1^2}{\varepsilon}\, \mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_i)\,, \qquad \forall i \ge n+1\,, \tag{7.27}$$

so that, combining (7.22), (7.27) and (7.25) we conclude that

$$\int_{B_{R_1}} \varphi^2 \le \frac{C}{\varepsilon}\, \mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi)\,, \qquad \forall \varphi \in Z^*\,. \tag{7.28}$$

Now let $L = \max_{B_{R_1}} |W''(\zeta_\varepsilon) - \varepsilon\, \Lambda_\varepsilon\, V''(\zeta_\varepsilon)|$. If

$$\frac{\kappa}{2\,L} \int_{\mathbb{R}^n \setminus B_{R_1}} \varphi^2 \le \int_{B_{R_1}} \varphi^2\,,$$

then we deduce $\int_{\mathbb{R}^n} \varphi^2 \le (C/\varepsilon)\, \mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi)$ by (7.28); if, instead,

$$\frac{\kappa}{2\,L} \int_{\mathbb{R}^n \setminus B_{R_1}} \varphi^2 > \int_{B_{R_1}} \varphi^2\,, \tag{7.29}$$

then by (7.26) and (7.29) we see that

$$\begin{aligned} \mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi) &\ge \int_{\mathbb{R}^n} \Big\{ \frac{W''(\zeta_\varepsilon)}{\varepsilon} - \Lambda_\varepsilon\, V''(\zeta_\varepsilon) \Big\} \varphi^2 \ge \frac{\kappa}{\varepsilon} \int_{\mathbb{R}^n \setminus B_{R_1}} \varphi^2 - \frac{L}{\varepsilon} \int_{B_{R_1}} \varphi^2 \\ &\ge \frac{\kappa}{2\,\varepsilon} \int_{\mathbb{R}^n \setminus B_{R_1}} \varphi^2\,, \end{aligned}$$

which combined with (7.28) gives again $\int_{\mathbb{R}^n}\varphi^2\le (C/\varepsilon)\,\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi)$, as desired. We are thus left to prove the two claims.

Proof of claim one: Setting $\psi_i=\varphi_i/\zeta_\varepsilon'$ on $\mathbb{R}^n$, we see that $\psi_i(r\,\theta)=A_i(\theta)\,\xi_i(r)$ (where $\xi_i=\varphi_i^*/\zeta_\varepsilon'$), with

$$|\nabla\psi_i|^2(r\,\theta)=|\nabla^{\mathbb{S}^{n-1}}A_i(\theta)|^2\,\frac{\xi_i(r)^2}{r^2}+|A_i(\theta)|^2\,\xi_i'(r)^2\,.$$

Since $i\ge n+1$ we can exploit (7.3) which, combined with (7.7), gives

$$\begin{aligned}
&\frac{\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_i)}{2\,\varepsilon}=\int_{\mathbb{R}^n}(\zeta_\varepsilon')^2\Big\{|\nabla\psi_i|^2-\frac{(n-1)}{|x|^2}\psi_i^2\Big\}\\
\ge\;&\int_0^\infty\zeta_\varepsilon'(r)^2\Big\{\Big(\frac{\xi_i(r)^2}{r^2}\int_{\mathbb{S}^{n-1}}|\nabla^{\mathbb{S}^{n-1}}A_i|^2-(n-1)\,A_i^2\Big)+\xi_i'(r)^2\int_{\mathbb{S}^{n-1}}A_i^2\Big\}\,r^{n-1}\,dr\\
\ge\;&c(n)\int_0^\infty\zeta_\varepsilon'(r)^2\Big\{\Big(\frac{\xi_i(r)^2}{r^2}\int_{\mathbb{S}^{n-1}}|\nabla^{\mathbb{S}^{n-1}}A_i|^2+\,A_i^2\Big)+\xi_i'(r)^2\int_{\mathbb{S}^{n-1}}A_i^2\Big\}\,r^{n-1}\,dr\\
=\;&c(n)\int_{\mathbb{R}^n}(\zeta_\varepsilon')^2\Big\{|\nabla\psi_i|^2+\frac{\psi_i^2}{|x|^2}\Big\}\ge c(n)\int_{\mathbb{R}^n}(\zeta_\varepsilon')^2\,\frac{\psi_i^2}{|x|^2}=c(n)\int_{\mathbb{R}^n}\frac{\varphi_i^2}{|x|^2}\,,
\end{aligned}$$

that is (7.24).

Proof of claim two: Since $\varphi\in Z^*$ we have, for each $i=1,...,n$,

$$0=\int_{\mathbb{R}^n}\hat{x}_i\,\zeta_\varepsilon'\,\varphi=\sum_{k=1}^\infty\int_{\mathbb{R}^n}\hat{x}_i\,\zeta_\varepsilon'\,\varphi_k\,,$$

which, combined with $\varphi_k(x)=A_k(\hat{x})\,\varphi_k^*(|x|)=c(n)\,\hat{x}_k\,\varphi_k^*(|x|)$, gives

$$\int_0^\infty\varphi_i^*(r)\,\zeta_\varepsilon'(r)\,r^{n-1}\,dr=0\,,\qquad\forall i=1,...,n\,.\tag{7.30}$$

We now prove, if $\varphi\in Z^*$, then

$$\mathcal{Q}_\varepsilon[\zeta_\varepsilon](\varphi_i)\ge\frac{1}{C\,\varepsilon}\int_{\mathbb{R}^n}(\varphi_i^*)^2\,,\qquad\forall i=1,...,n\,.\tag{7.31}$$

Notice that (7.31) implies (7.25) since $\varphi_i\le c(n)\,\varphi_i^*$ for $i=1,...,n$.

We prove (7.31) by contradiction, following closely the proof of [MR24, Lemma 4.4]. Indeed, should (7.31) fail, then there would be sequences $\varepsilon_j\to 0^+$ as $j\to\infty$ and $(\varphi_j)_j$ in Z^* such that (up to rotations taking $i_j\in\{1,...,n\}$ to $i_j=1$ for all j)

$$\frac{1}{\varepsilon_j}\int_{\mathbb{R}^n}((\varphi_j)_1^*)^2=1\,,\qquad\forall j\,,\tag{7.32}$$

$$\int_0^\infty(\varphi_j)_1^*(r)\,\zeta_{\varepsilon_j}'(r)\,r^{n-1}\,dr=0\,,\qquad\forall j\,,\tag{7.33}$$

$$\lim_{j\to\infty}\mathcal{Q}_{\varepsilon_j}[\zeta_{\varepsilon_j}]\big((\varphi_j)_1\big)=0\,.\tag{7.34}$$

Setting $R_0=1/\omega_n^{1/n}$ and

$$\beta_j(s)=(\varphi_j^*)_1(R_0+\varepsilon_j\,s)\,,\qquad\eta_j(s)=\zeta_{\varepsilon_j}(R_0+\varepsilon_j\,s)\,,\qquad s\in\mathbb{R}\,,$$

we can rewrite (7.32) and (7.33) as

$$\int_{I_j}\beta_j(s)^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds=1\,,\qquad\forall j\,,\tag{7.35}$$

$$\int_{I_j}\beta_j(s)\,\eta_j'(s)\,(R_0+\varepsilon_j\,s)^{n-1}\,ds=0\,,\qquad\forall j\,,\tag{7.36}$$

where $I_j = (-R_0/\varepsilon_j, \infty)$. Concerning (7.34) we notice that by $(\varphi_j)_1(x) = A_1(\hat{x})\,(\varphi_j)_1^*(|x|)$ it follows that

$$|\nabla(\varphi_j)_1|^2 = |\nabla^{\mathbb{S}^{n-1}} A_1|^2 \left((\varphi_j)_1^*\right)^2 + A_1^2\,|\nabla(\varphi_j)_1^*|^2$$

so that, as $\|A_1\|_{L^2(\mathbb{S}^{n-1})} = 1$,

$$\varepsilon_j \int_{\mathbb{R}^n} |\nabla(\varphi_j)_1|^2 \geq \varepsilon_j \int_{\mathbb{S}^n} A_1^2 \int_0^\infty |\nabla(\varphi_j)_1^*|^2(r)\,r^{n-1}\,dr = \int_{I_j} \beta_j'(s)^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\,.$$

Again by Fubini's theorem and thanks to $\|A_1\|_{L^2(\mathbb{S}^{n-1})} = 1$ we find

$$\int_{\mathbb{R}^n} \Big\{\frac{W''(\zeta_{\varepsilon_j})}{\varepsilon_j} - \Lambda_{\varepsilon_j}\,V''(\zeta_{\varepsilon_j})\Big\}\,(\varphi_j)_1^2 = \int_{I_j} \big\{W''(\eta_j) - \varepsilon_j\,\Lambda_{\varepsilon_j}\,V''(\eta_j)\big\}\,\beta_j^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds$$

We can thus deduce from (7.34) that

$$\lim_{j\to\infty}\int_{I_j} \Big((\beta_j')^2 + \big\{W''(\eta_j) - \varepsilon_j\,\Lambda_{\varepsilon_j}\,V''(\eta_j)\big\}\,\beta_j^2\Big)\,(R_0+\varepsilon_j\,s)^{n-1}\,ds = 0\,; \tag{7.37}$$

and, in fact, by taking into account that $\varepsilon_j\,\Lambda_{\varepsilon_j} \to 0^+$ as $j\to\infty$ and that $|V''| \leq C$ on $[0,1]$, we see from (7.35) that (7.37) is equivalent to

$$\lim_{j\to\infty}\int_{I_j} \Big((\beta_j')^2 + W''(\eta_j)\,\beta_j^2\Big)\,(R_0+\varepsilon_j\,s)^{n-1}\,ds = 0\,. \tag{7.38}$$

In turn, since $|W''| \leq C$ on $[0,1]$, by combining (7.35) with (7.38) we see that $(\beta_j)_j$ is bounded in $W^{1,2}_{\rm loc}(\mathbb{R})$. Hence, up to extracting a subsequence, we can find $\beta \in W^{1,2}_{\rm loc}(\mathbb{R})$ such that $\beta_j \rightharpoonup \beta$ weakly in $W^{1,2}(\mathbb{R})$.

In this position, we can repeat *verbatim* two arguments contained in the proof of [MR24, Lemma 4.4]. The first argument shows that the sequence of probability measures $(\mu_j)_j$ defined by $\mu_j = \beta_j^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds$ is in the compactness case of the concentration-compactness principle, and thus satisfies

$$\lim_{s\to\infty}\sup_j\,\mu_j\big(\mathbb{R}\setminus(-s,s)\big) = 0\,. \tag{7.39}$$

The second argument shows that

$$\lim_{j\to\infty}\int_{I_j} W''(\eta_j)\,\beta_j^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds = R_0^{n-1}\int_{\mathbb{R}} W''(\eta_0)\,\beta^2\,, \tag{7.40}$$

where $\eta_0(s) = \eta(s-\tau_0)$, η is the unique solution of $-\eta' = \sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = 1/2$, and $\tau_0 = \int_{\mathbb{R}} V'(\eta(s))\,\eta(s)\,s\,ds$.

By (7.39), (7.35), (7.40) and (7.38) we thus find

$$R_0^{n-1}\int_{\mathbb{R}} \beta^2 = 1\,, \tag{7.41}$$

$$\int_{\mathbb{R}} 2\,(\beta')^2 + W''(\eta_0)\,\beta^2 \leq 0\,. \tag{7.42}$$

By [MR24, Lemma 4.3], (7.42) implies that $\beta(s+\tau_0) = t\,\eta'(s)$ for some $t \neq 0$ (the case $t=0$ is ruled out by (7.41)). In other words, $\beta = t\,\eta_0'$.

We now claim that

$$\lim_{j\to\infty}\int_{I_j} \beta_j\,\eta_j'\,(R_0+\varepsilon_j\,s)^{n-1}\,ds = R_0^{n-1}\int_{\mathbb{R}} \beta\,\eta_0'\,. \tag{7.43}$$

Indeed, by $|\eta_j'(s)|\le C\,e^{-|s|/C}$ for $s\in\mathbb{R}$ and by (7.39) we see that

$$
\begin{aligned}
&\Big|\int_{I_j\setminus(-s_0,s_0)}\beta_j\,\eta_j'\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\Big|\\
&\le\Big(\int_{I_j}(\eta_j')^2\,(R_0+\varepsilon_j\,s)^{n-1}\,ds\Big)^{1/2}\,\mu_j\big(I_j\setminus(-s_0,s_0)\big)^{1/2}\le\omega(s_0)\,,
\end{aligned}
$$

for some function ω, independent of j, such that $\omega(s)\to0^+$ as $s\to\infty$. Similarly

$$
\Big|\int_{\mathbb{R}\setminus(-s_0,s_0)}\beta\,\eta_0'\Big|\le\omega(s_0)\,,
$$

and therefore (7.43) follows since, as $j\to\infty$, $\beta_j\to\beta$ in $L^2_{\rm loc}(\mathbb{R})$ and by $\eta_j'\to\eta_0'$ locally uniformly on $\mathbb{R}$. On combining $\beta=t\,\eta_0'$ with (7.36) and (7.43) we conclude that

$$
0=t\,R_0^{n-1}\int_{\mathbb{R}}(\eta_0')^2\,,
$$

and thus, that $\eta\equiv$ constant, reaching a contradiction. □

8. Exponential convergence to a single diffused bubble (Proof of Theorem 1.1)

Proof of Theorem 1.1. We are proving the theorem by showing the existence if $\varepsilon\in(0,\varepsilon_0)$ and u_0 is as in the statement, then

$$
\begin{aligned}
\mathcal{AC}_\varepsilon(u(t))-\Psi(\varepsilon,1)&\le C(\varepsilon,u_0)\,e^{-t/C_*(\varepsilon)}\,, && (8.1)\\
\big\|u(t)-\tau_{x_0}[\zeta_\varepsilon]\big\|_{L^2(\mathbb{R}^n)}&\le C(\varepsilon,u_0)\,e^{-t/C(\varepsilon)}\,,\qquad\forall t>1/C(\varepsilon,u_0)\,. && (8.2)
\end{aligned}
$$

By the assumptions on u_0, Theorem 1.4 holds with $M=1$. In particular, the only accumulation points for the sequences $(\mathcal{AC}_\varepsilon(u(t_j)))_j$ and $(\lambda_\varepsilon[u(t_j)])_j$ corresponding to any $t_j\to\infty$ as $j\to\infty$ are, respectively, $\Psi(\varepsilon,1)$ and Λ_ε, so that we have

$$
\lim_{t\to\infty}\mathcal{AC}_\varepsilon(u(t))=\Psi(\varepsilon,1)\,,\qquad\lim_{t\to\infty}\lambda_\varepsilon[u(t)]=\Lambda_\varepsilon\,. \tag{8.3}
$$

Step one: We prove that

$$
\lim_{t\to\infty}\big\|u(t)-\tau_{x(t)}[\zeta_\varepsilon]\big\|_{(W^{1,2}\cap C^0)(\mathbb{R}^n)}=0\,. \tag{8.4}
$$

where, for each $t>0$, we have defined $x(t)\in\mathbb{R}^n$ so that

$$
\big\|u(t)-\tau_{x(t)}[\zeta_\varepsilon]\big\|_{L^2(\mathbb{R}^n)}\le\big\|u(t)-\tau_x[\zeta_\varepsilon]\big\|_{L^2(\mathbb{R}^n)}\,,\qquad\forall x\in\mathbb{R}^n\,.
$$

Indeed, by (8.3), if $(t_j)_j$ is an arbitrary sequence with $t_j\to\infty$ as $j\to\infty$, then $((u(t_j))_j$ is a minimizing sequence of $\Psi(\varepsilon,1)$. Now, in [MR24, step two, proof of Theorem 2.1] it is proved that if u_j is a minimizing sequence of $\Psi(\varepsilon,1)$, then, up to extracting subsequences, there is $x\in\mathbb{R}^n$ such that

$$
\lim_{j\to\infty}\big\|u_j-\tau_x[\zeta_\varepsilon]\big\|_{(W^{1,2}\cap C^0)(\mathbb{R}^n)}=0\,.
$$

By combining this fact with the definition of $x(t)$ we conclude the proof of (8.4).

Step two: We introduce the Fisher information of the flow

$$
\mathcal{I}_\varepsilon(t)=\varepsilon\int_{\mathbb{R}^n}\big(\partial_t u(t)\big)^2\,,\qquad\forall t>0\,, \tag{8.5}
$$

and prove that

$$
\mathcal{I}_\varepsilon(t)\le C(\varepsilon,u_0)\,e^{-t/C(\varepsilon)}\,,\qquad\forall t>1/C(\varepsilon,u_0)\,. \tag{8.6}
$$

Indeed, by Theorem 1.2-(vi) (see, in particular, (1.18)) we have that $\mathcal{I}_\varepsilon \in W^{1,1}(a,\infty)$ for every $a > 0$, with

$$\frac{d}{dt}\frac{\mathcal{I}_\varepsilon(t)}{2} = -\int_{\mathbb{R}^n}\Big\{2\,\varepsilon\,|\nabla(\partial_t u)|^2 + \Big(\frac{W''(u)}{\varepsilon} - \lambda_\varepsilon[u(t)]\,V''(u)\Big)\,(\partial_t u)^2\Big\}\,. \tag{8.7}$$

By $W'' \in \mathrm{Lip}[0,1]$, $V'' \in C^{0,\alpha(n)}[0,1]$, and $0 < \Lambda_\varepsilon < C$ for all $\varepsilon \in (0,\varepsilon_0)$ (recall (2.6)) we find that, pointwise on $\mathbb{R}^n$,

$$\begin{aligned}
&\Big|\Big(\frac{W''(u(t))}{\varepsilon} - \lambda_\varepsilon[u(t)]\,V''(u(t))\Big) - \Big(\frac{W''(\tau_{x(t)}[\zeta_\varepsilon])}{\varepsilon} - \Lambda_\varepsilon\,V''(\tau_{x(t)}[\zeta_\varepsilon])\Big)\Big|\\
&\le C\,\Big\{\frac{|u(t)-\tau_{x(t)}[\zeta_\varepsilon]|}{\varepsilon} + \big|\lambda_\varepsilon[u(t)] - \Lambda_\varepsilon\big| + \big|u(t)-\tau_{x(t)}[\zeta_\varepsilon]\big|^{\alpha(n)}\Big\}\,.
\end{aligned}$$

Combining this inequality with (8.7) we find that

$$\begin{aligned}
-\frac{d}{dt}\frac{\mathcal{I}_\varepsilon(t)}{2} &\ge \mathcal{Q}_\varepsilon[\tau_{x(t)}[\zeta_\varepsilon]]\big(\partial_t u(t)\big)\\
&-C\,\Big\{\frac{\|u-\tau_{x(t)}[\zeta_\varepsilon]\|_{C^0(\mathbb{R}^n)}}{\varepsilon} + \big|\lambda_\varepsilon[u(t)] - \Lambda_\varepsilon\big| + \big\|u-\tau_{x(t)}[\zeta_\varepsilon]\big\|^{\alpha(n)}_{C^0(\mathbb{R}^n)}\Big\}\int_{\mathbb{R}^n}(\partial_t u(t))^2\,,
\end{aligned}$$

so that (8.3) and (8.4) imply the existence, for every $\eta > 0$, of a positive constant $t_* = t_*(\varepsilon, u_0, \eta)$, such that

$$-\frac{d}{dt}\frac{\mathcal{I}_\varepsilon(t)}{2} \ge \mathcal{Q}_\varepsilon[\tau_{x(t)}[\zeta_\varepsilon]]\big(\partial_t u(t)\big) - \eta\,\mathcal{I}_\varepsilon(t)\,, \qquad \forall t > t_*\,. \tag{8.8}$$

Now let P_t denote the projection operator of $L^2(\mathbb{R}^n)$ onto its closed subspace

$$Y_t = \Big\{\varphi \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} V'\big(\tau_{x(t)}[\zeta_\varepsilon]\big)\,\varphi = \int_{\mathbb{R}^n}\partial_i\big(\tau_{x(t)}[\zeta_\varepsilon]\big)\,\varphi = 0\Big\}\,.$$

By Theorem 1.5 we have

$$\mathcal{Q}_\varepsilon[\tau_{x(t)}[\zeta_\varepsilon]]\big(\partial_t u(t)\big) \ge \frac{1}{C(\varepsilon)}\int_{\mathbb{R}^n}\big(P_t[\partial_t u(t)]\big)^2\,. \tag{8.9}$$

To get a control on $\mathcal{I}_\varepsilon(t)$ we thus need to estimate $\|\partial_t u(t) - P_t[\partial_t u(t)]\|_{L^2(\mathbb{R}^n)}$.

To this end, let us begin by noticing that, since $\int_{\mathbb{R}^n} V'(u(t))\,\partial_t u(t) = 0$ for all $t > 0$, for every $\delta > 0$ there is $t_1 = t_1(\delta, u_0) > 0$ such that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} V'(\tau_{x(t)}[\zeta_\varepsilon])\,\partial_t u(t)\Big| &\le C\,\|u(t)-\tau_{x(t)}[\zeta_\varepsilon]\|_{L^2(\mathbb{R}^n)}\,\|\partial_t u(t)\|_{L^2(\mathbb{R}^n)}\\
&\le \delta\,\|\partial_t u(t)\|_{L^2(\mathbb{R}^n)}\,, \qquad \forall t > t_1\,.
\end{aligned} \tag{8.10}$$

On the other hand, testing (DF) with $\partial_i u(t) \in W^{2,2}(\mathbb{R}^n)$ we find that

$$\varepsilon^2\int_{\mathbb{R}^n}\partial_t u(t)\,\nabla_{e_i} u(t) = 2\,\varepsilon^2\int_{\mathbb{R}^n}\Delta u(t)\,\partial_i u(t) - \int_{\mathbb{R}^n}\partial_i\big(W(u(t)) - \lambda_\varepsilon[u(t)]\,V(u(t))\big)\,,$$

(notice that $\lambda_\varepsilon[u(t)]$ is a function of t alone, and is not affected by differentiation along e_i here), and since $W(u(t)), V(u(t)) \in W^{1,2}(\mathbb{R}^n)$ for all $t > 0$, we conclude that

$$\int_{\mathbb{R}^n}\partial_t u(t)\,\partial_i u(t) = 2\int_{\mathbb{R}^n}\Delta u(t)\,\partial_i u(t) = -2\int_{\mathbb{R}^n}\nabla u(t)\cdot\partial_i(\nabla u(t)) = -\int_{\mathbb{R}^n}\partial_i|\nabla u(t)|^2 = 0\,,$$

for all $t > 0$. Combining this identity with (8.4) we thus conclude that, up to further increase the value of $t_1 = t_1(\delta, u_0)$, we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n}\partial_i\big(\tau_{x(t)}[\zeta_\varepsilon]\big)\,\partial_t u(t)\Big| &\le C\,\|\partial_i u(t) - \partial_i\tau_{x(t)}[\zeta_\varepsilon]\|_{L^2(\mathbb{R}^n)}\,\|\partial_t u(t)\|_{L^2(\mathbb{R}^n)}\\
&\le \delta\,\|\partial_t u(t)\|_{L^2(\mathbb{R}^n)}\,, \qquad \forall t > t_1\,, i = 1,\dots,n\,.
\end{aligned} \tag{8.11}$$

Therefore, by choosing $\delta=\delta(\varepsilon)>0$ small enough in terms of the constant $C(\varepsilon)$ appearing in (8.9), we conclude from (8.10) and (8.11) that, if $t>t_1=t_1(\delta(\varepsilon),u_0)=t_1(\varepsilon,u_0)$, then, for a positive constant $C_*(\varepsilon)$ depending only on ε,

$$\mathcal{Q}_\varepsilon\big[\tau_{x(t)}\zeta_\varepsilon\big](\partial_t u(t))\geq\frac{\mathcal{I}_\varepsilon(\partial_t u(t))}{C_*(\varepsilon)}\,,\qquad\forall t>t_1\,.\tag{8.12}$$

If we now choose $\eta=1/(2\,C_*(\varepsilon))$ in (8.8), and correspondingly set $t_0=\min\{t_1,t_*\}$, then, we deduce from (8.12) that

$$-\frac{d}{dt}\,\frac{\mathcal{I}_\varepsilon(t)}{2}\geq\frac{\mathcal{I}_\varepsilon(t)}{2\,C_*(\varepsilon)}\,,\qquad\forall t>t_0\,,$$

from which (8.6) immediately follows.

Conclusion: By Theorem 1.2-(iv),

$$\mathcal{AC}_\varepsilon(u(T))-\mathcal{AC}_\varepsilon(u(t))=-\varepsilon\int_t^T ds\int_{\mathbb{R}^n}(\partial_t u(s))^2\,,\qquad\forall T>t>0\,.$$

Combining this identity with (8.3) and (8.6) we find that, if $t>1/C(\varepsilon,u_0)$, then

$$\mathcal{AC}_\varepsilon(u(t))-\Psi(\varepsilon,1)=\int_t^\infty\mathcal{I}_\varepsilon(s)\,ds\leq C(\varepsilon,u_0)\int_t^\infty e^{-s/C(\varepsilon)}\,ds\leq C(\varepsilon,u_0)\,e^{-t/C(\varepsilon)}\,,$$

thus proving (8.1). Next we notice that if $1/C(\varepsilon,u_0)<t<T$, then by combining the fundamental theorem of Calculus with the Minkowski inequality, the Hölder inequality and then with (8.6), we obtain

$$\begin{aligned}\|u(T)-u(t)\|_{L^2(\mathbb{R}^n)}&\leq\Big(\int_{\mathbb{R}^n}\Big|\int_t^T\partial_t u(s)ds\Big|^2\Big)^{1/2}\leq\int_t^T\Big(\int_{\mathbb{R}^n}|\partial_t u(s)|^2\Big)^{1/2}ds\\ \leq&\sum_{k=0}^\infty\int_{t+k}^{t+k+1}\Big(\int_{\mathbb{R}^n}|\partial_t u(s)|^2\Big)^{1/2}ds\leq\sum_{k=0}^\infty\Big(\int_{t+k}^{t+k+1}ds\int_{\mathbb{R}^n}|\partial_t u(s)|^2\Big)^{1/2}\\ \leq&\sum_{k=0}^\infty\Big(\int_{t+k}^\infty ds\int_{\mathbb{R}^n}|\partial_t u(s)|^2\Big)^{1/2}\leq C(\varepsilon,u_0)\sum_{k=0}^\infty e^{-(t+k)/C(\varepsilon)}\leq C(\varepsilon,u_0)\,e^{-t/C(\varepsilon)}\,,\end{aligned}$$

that is

$$\|u(T)-u(t)\|_{L^2(\mathbb{R}^n)}\leq C(\varepsilon,u_0)\,e^{-t/C(\varepsilon)}\,,\qquad\forall T>t>1/C(\varepsilon,u_0)\,.\tag{8.13}$$

Now let $t_j\to\infty$ as $j\to\infty$: since $(u(t_j))_j$ is a minimizing sequence of $\Psi(\varepsilon,1)$, then, by the argument in step one and up to extracting a subsequence, there is $x_0\in\mathbb{R}^n$ such that $\|u(t_j)-\tau_{x_0}[\zeta_\varepsilon]\|_{L^2(\mathbb{R}^n)}\to 0$ as $j\to\infty$. By taking $T=t_j$ in (8.13) and letting $j\to\infty$ in the corresponding inequality we thus complete the proof of (8.2), and thus, of the theorem. □

References

[ABF98] N. D. Alikakos, L. Bronsard, and G. Fusco. Slow motion in the gradient theory of phase transitions via energy and spectrum. *Calc. Var. Partial Differential Equations*, 6(1):39–66, 1998.

[AF94] N. D. Alikakos and G. Fusco. Slow dynamics for the Cahn-Hilliard equation in higher space dimensions. I. Spectral estimates. *Comm. Partial Differential Equations*, 19(9-10), 1994.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[Alm76] F. J. Jr. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 4(165):viii+199 pp, 1976.

[BHNN15] G. Bellettini, A. Hassem Nayam, and M. Novaga. Γ-type estimates for the one-dimensional Allen-Cahn's action. *Asymptotic Analysis*, 94(1-2):161–185, 2015.

[BK90] L. Bronsard and R. V. Kohn. On the slowness of phase boundary motion in one space dimension. *Comm. Pure Appl. Math.*, 43(8):983–997, 1990.

[BK91] L. Bronsard and R. V. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Differential Equations*, 90(2):211–237, 1991.

[BOS06] F. Bethuel, G. Orlandi, and D. Smets. Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature. *Ann. of Math. (2)*, 163(1):37–163, 2006.

[BS97] L. Bronsard and B. Stoth. Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation. *SIAM J. Math. Anal.*, 28(4):769–807, 1997.

[CH98] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[CL12] M. Cicalese and G. P. Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Rat. Mech. Anal.*, 206(2):617–643, 2012.

[ES98] J. Escher and G. Simonett. The volume preserving mean curvature flow near spheres. *Proc. Amer. Math. Soc.*, 126(9):2789–2796, 1998.

[Eva98] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[Fal10] M. M. Fall. Area-minimizing regions with small volume in Riemannian manifolds with boundary. *Pacific J. Math.*, 244(2):235–260, 2010.

[Fei97] E. Feireisl. On the long time behaviour of solutions to nonlinear diffusion equations on $\mathbb{R}^n$. *Nonlinear Differential Equations and Applications NoDEA*, 4(1):43–60, 1997.

[FJM22] N. Fusco, V. Julin, and M. Morini. Stationary sets and asymptotic behavior of the mean curvature flow with forcing in the plane. *J. Geom. Anal.*, 32(2):Paper No. 53, 29, 2022.

[FMP07] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative Sobolev inequality for functions of bounded variation. *J. Funct. Anal.*, 244(1):315–341, 2007.

[Gas20] P. Gaspar. The second inner variation of energy and the morse index of limit interfaces. *The Journal of Geometric Analysis*, 30(1):69–85, 2020.

[GNN81] B. Gidas, W.-M. Ni, and L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. *Adv. Math. Suppl. Stud., A*, 7:369–402, 1981.

[Gra95] C. P. Grant. Slow motion in one-dimensional Cahn-Morral systems. *SIAM J. Math. Anal.*, 26(1):21–34, 1995.

[Hui87] G. Huisken. The volume preserving mean curvature flow. *J. Reine Angew. Math.*, 382:35–48, 1987.

[Ilm93] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, 1993.

[JMOS24] V. Julin, M. Morini, F. Oronzio, and E. Spadaro. A sharp quantitative Alexandrov inequality and applications to volume preserving geometric flows in 3D. 2024. arxiv 2406.17691.

[JMPS23] V. Julin, M. Morini, M. Ponsiglione, and E. Spadaro. The asymptotics of the area-preserving mean curvature and the Mullins-Sekerka flow in two dimensions. *Math. Ann.*, 387(3-4):1969–1999, 2023.

[JN23] V. Julin and J. Niinikoski. Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow. *Anal. PDE*, 16(3):679–710, 2023.

[KK20] I. Kim and D. Kwon. Volume preserving mean curvature flow for star-shaped sets. *Calc. Var. Partial Differential Equations*, 59(2):Paper No. 81, 40, 2020.

[Le11] N. Q. Le. On the second inner variation of the Allen–Cahn functional and its applications. *Indiana University Mathematics Journal*, pages 1843–1856, 2011.

[Li09] H. Li. The volume-preserving mean curvature flow in Euclidean space. *Pacific J. Math.*, 243(2):331–355, 2009.

[Lie96] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[LM89] S. Luckhaus and L. Modica. The Gibbs-Thompson relation within the gradient theory of phase transitions. *Arch. Rational Mech. Anal.*, 107(1):71–83, 1989.

[LM19] G. Leoni and R. Murray. Local minimizers and slow motion for the mass preserving Allen-Cahn equation in higher dimensions. *Proc. Amer. Math. Soc.*, 147(12):5167–5182, 2019.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2012.

[MM16] F. Maggi and C. Mihaila. On the shape of capillarity droplets in a container. *Calc. Var. Partial Differential Equations*, 55(5):Art. 122, 42, 2016.

[MNR23] F. Maggi, M. Novack, and D. Restrepo. A hierarchy of Plateau problems and the approximation of Plateau's laws via the Allen–Cahn equation. 2023. arXiv 2312.11139.

[MPS22] M. Morini, M. Ponsiglione, and E. Spadaro. Long time behavior of discrete volume preserving mean curvature flows. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2022(784):27–51, 2022.

[MR16] R. Murray and M. Rinaldi. Slow motion for the nonlocal Allen-Cahn equation in n dimensions. *Calc. Var. Partial Differential Equations*, 55(6):Art. 147, 33, 2016.

[MR24] F. Maggi and D. Restrepo. Uniform stability in the Euclidean isoperimetric problem for the Allen–Cahn energy. *Anal. PDE*, 17(5):1761–1830, 2024.

[Ton05] Y. Tonegawa. On stable critical points for a singular perturbation problem. *Communications in Analysis and Geometry*, 13(2):439–459, 2005.

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