SHIFTS MAPS ARE NOT TYPE-PRESERVING

CAROLYN ABBOTT, NICHOLAS MILLER, AND PRIYAM PATEL

Abstract. For a surface S of sufficient complexity, Dehn twists act elliptically on the arc, curve, and relative arc graph of S. We show that composing a Dehn twist with a shift map results in a loxodromic isometry of the relative arc graph $\mathcal{A}(S,p)$ for any surface S with an isolated puncture p admitting a shift map. Therefore, shift maps are not type-preserving.

1. Introduction

A surface S is *finite-type* if its fundamental group is finitely generated, and is otherwise *infinite-type*. The *mapping class group*, $\mathrm{Map}(S)$, of a finite-type surface is well studied, especially through its actions on various hyperbolic graphs including the curve graph, $\mathcal{C}(S)$. The most simple mapping class, a Dehn twist about a simple closed curve, acts elliptically on $\mathcal{C}(S)$.

There have been many developments in the study of infinite-type surfaces and their mapping class groups over the last few years. For an infinite-type surface S with at least one isolated puncture p, the *relative arc graph*, $\mathcal{A}(S,p)$, plays the role of $\mathcal{C}(S)$ and is defined as follows: the vertices correspond to isotopy classes of simple arcs that begin and end at p, and edges connect vertices for arcs admitting disjoint representatives. The subgroup $\mathrm{Map}(S,p)$ of $\mathrm{Map}(S)$ that fixes the isolated puncture p acts on $\mathcal{A}(S,p)$ by isometries. A Dehn twist about a simple closed curve acts elliptically on $\mathcal{A}(S,p)$ as well.

This paper fits into a body of work aimed at constructing and classifying all of the elements of $\mathrm{Map}(S,p)$ acting loxodromically on $\mathcal{A}(S,p)$ and various other hyperbolic graphs associated to infinite-type surfaces (see [2, 3, 1, 5, 4]). Our main result shows that *shift maps* are not type-preserving in the sense that composing a Dehn twist with a shift map results in a mapping class thats acts loxodromically on $\mathcal{A}(S,p)$.

Theorem 1.1. *Let χ be a standard simple closed curve in the biinfinite flute surface S containing ℓ punctures besides p in its interior. Then $g = hT_\chi^j$ is a loxodromic isometry of $\mathcal{A}(S,p)$ for all $j > 0$, unless both ℓ and j are equal to 1, where h is the standard shift on S.*

For simplicity, we prove the theorem for the biinfinite flute surface S and *standard* curves (see Definition 2.6), but the result immediately extends to any surface Σ containing an isolated puncture that admits a shift map since the inclusion of $\mathcal{A}(S,p)$ into $\mathcal{A}(\Sigma,p)$ is a $(2,0)$–quasi-isometric embedding. There are uncountably many such surfaces Σ, which are referred to as surfaces of type $\mathcal{S}$ (see [1, Definition 2.6, Lemma 2.7, and Lemma 2.10] for more details). In addition, Lemma 2.7 shows that we can extend Theorem 1.1 to other simple closed curves as well.

The proof that these mapping classes are loxodromic isometries of $\mathcal{A}(S,p)$ utilizes a "starts like" function, which (roughly) measures how long any arc starting at p fellow travels arcs in a given collection. The function is then used to bound distance in $\mathcal{A}(S,p)$ from below and produce a quasi-axis for g. This method is inspired by Bavard's construction in [2] and that of the authors in [1].

Acknowledgements: Abbott was partially supported by NSF Grant DMS-2106906. Miller was partially supported by NSF Grant DMS-2005438/2300370. Patel was partially supported by NSF CAREER Grant DMS–2046889 and the University of Utah Faculty Fellows Award.

2. Background

2.1. **Coding arcs and standard position.** Let S be the biinfinite flute surface with a distinguished isolated puncture p, and let $\{p_i\}_{i\in\mathbb{Z}}$ be the countably infinite discrete collection of all other punctures on S which exit both ends of the cylinder. We move the distinguished puncture p so that it lies to the right of p_{-1} and to the left of p_0. We also choose one non-isolated end of S to correspond to the left direction (the accumulation point of p_i for $i<0$) and one to correspond to the right direction (the accumulation point of p_i for $i>0$), which gives a well-defined notion of a front and back of the cylinder for S.

Just as in [1], we fix a complete hyperbolic metric on S and let B_0 be a horocycle at a height sufficiently far out the cusp corresponding to p_0. Fix a shift map h on S whose domain contains exactly the collection $\{p_i\}$ for $i\in\mathbb{Z}$ and which shifts p_i to p_{i+1} for all $i\in\mathbb{Z}$.

Definition 2.1. Define the simple closed curves $B_i := h^i(B_0)$ for $i\in\mathbb{Z}$. Then B_i is a simple closed curve bounding the puncture p_i. We identify each B_i with $\mathbb{S}^1$ and fix the north pole of each B_i.

2.2. **Coding arcs.** Suppose γ is an oriented arc on S starting and ending at p. We code γ exactly as in [1]. For the sake of brevity, we give the following examples of arcs and their codes instead of discussing the code in detail.

Example 2.2. Consider the arcs shown in Figure 1. The elements $k\in\mathbb{Z}$ shown under S denote the subscript on the simple closed curves B_k. The code for α is $P_s0_o1_u2_o2_u1_u0_uP_s$, the code for β is $P_sP_uP_o0_o1_o2_o2_u1_o0_oP_s$, the code for γ is given by $P_s(-1)_o(-2)_o(-2)_u(-1)_uP_u0_u1_u1_o0_oP_s$, and the code for δ is given by $P_s(-1)_oC(-2)_o(-2)_uC(-1)_oP_s$. Note that P_s indicates that the arc starts or ends at the puncture p, the subscript o/u corresponds to whether the arc passes over of under that puncture, and the C in the code for δ denotes the fact that δ goes to the back of the surface S.

The appearance of repeated characters in the code of an arc indicates backtracking in the arc so that we have the following.

Definition 2.3. Let γ be an oriented arc on S starting and ending at p. A code for γ is *reduced* if no two adjacent characters are the same and if the character immediately following the initial P_s or preceding the terminal P_s is not $P_{o/u}$.

Note that if a triple appears in the code for an arc, it is reduced to a single character according to our convention, as only *pairs* of repeated characters are removed.

Definition 2.4. The *code length* of an arc γ, denoted $\ell_c(\gamma)$, is the number of characters in a reduced code for γ.

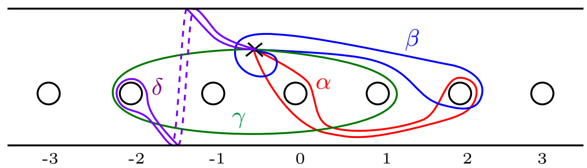


FIGURE 1. Arcs on the front of the surface S, whose codes are given in Example 2.2. The $\times$ denotes the puncture p, and the elements $k \in \mathbb{Z}$ shown under S denote the subscript on the simple closed curves B_k.

Given a string of characters $\alpha = a_1 a_2 \dots a_n$, we denote by $\overline{\alpha}$ the reverse of α, so that $\overline{\alpha} = a_n a_{n-1} \dots a_2 a_1$. If α is an arc, then $\overline{\alpha}$ is the same arc with the opposite orientation.

2.3. **Spokes.** We now introduce spokes, which are special segments on S.

Definition 2.5. A *segment* is a simple path with at least one endpoint which is not a puncture, and no endpoints on a puncture other than p. We code a segment in an analogous way as we do arcs.

Given any essential, separating simple closed curve χ on the front of S such that one connected component of $S \setminus \chi$ is a finite-type surface containing the puncture p, we let $g_{\chi,j} = hT_\chi^j$. We call the connected component of $S \setminus \chi$ containing p the *interior* or *inside of* χ and the other connected component is the *outside of* χ. Below, we prove some of the technical results in the case where $j = 1$ for brevity, which is actually the most difficult case, and refer to $g_{\chi,j}$ as g_χ or g for notational simplicity. All proofs generalize easily to $j > 1$, and in fact simplify a bit.

Definition 2.6. A simple closed curve χ on the front of S is *standard* if it has the form in Figure 2; we assume that χ contains the puncture p in its interior and that the right-most puncture contained in χ is p_0. In addition, there are no punctures "above" χ. The punctures on the interior of χ are called *interior punctures*.

We first consider curves that can be translated to standard curves by powers of h.

Lemma 2.7. *Let χ be a simple closed curve on S such that χ' is homotopic to $h^i(\chi)$ which is standard for some i. Then g_χ and $g_{\chi'}$ are conjugate by a power of h, and thus, g_χ is loxodromic with respect to the action of* $\mathrm{Map}(S,p)$ *on* $\mathcal{A}(S,p)$ *if and only if $g_{\chi'}$ is.*

Proof. Fix any simple closed curve χ on the front of S containing the puncture p. If $h^i(\chi) = \chi'$ is standard, we see that

$$h^i(g_\chi)h^{-i} = h^i(hT_\chi)h^{-i} = h(h^iT_\chi h^{-i}) = hT_{h^i(\chi)} = hT_{\chi'} = g_{\chi'},$$

which concludes the proof. □

We therefore assume in the remainder of the paper that χ is standard. Moreover, we choose the homotopy representative of χ to contain no backtracking. It will be useful to have a code for χ, even though it is not a arc. We define this code in the usual way, by tracking whether χ passes over or under each puncture, always assuming that χ is oriented clockwise. However, since χ does not have a well-defined starting point, such a code is only well-defined up to cyclic permutations. This will

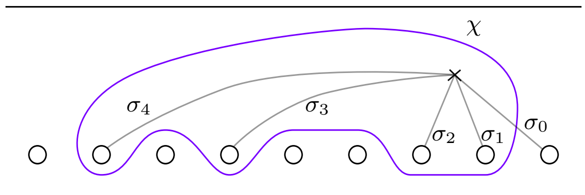


FIGURE 2. The curve χ is a standard simple closed curve on the front of S containing the puncture p. The purple segments are spokes, and the sectors are labeled in green.

cause no problems in this paper. For example, a code for the curve χ shown in Figure 6 is $P_o 0_o 0_u P_u {-1}_u {-2}_u {-2}_o {-1}_o P_o$.

Definition 2.8. For each k such that p_k is contained inside χ, and for p_1, a *spoke* (to p_k) is a segment whose initial point is p and whose terminal point is the north pole of B_k, as in Figure 2. In particular, such a segment passes over all punctures contained in χ between p and p_k. We label the spokes σ_i consecutively, starting from the right; if there are ℓ punctures on the interior of χ, then the spokes are $\sigma_0, \sigma_1, \ldots, \sigma_\ell$. Note that since χ is standard, σ_0 is the only spoke whose terminal point is outside of χ.

Given a spoke σ_i, let $P(i) \in \mathbb{Z}$ be the index so that σ_i is a segment from p to $B_{P(i)}$, i.e., σ_i ends at the simple closed curve corresponding to $P(i)$. If $P(i) \leq -1$, define a code for σ_i to be $P_s(-1)_o(-2)_o \ldots P(i)_s$. Here, $P(i)_s$ indicates that the segment stops at a point on $B_{P(i)}$. If $P(i) = 0$, the code for σ_i is $P_s 0_s$, while if $P(i) = 1$, the code for σ_i is $P_s 0_o 1_s$.

Definition 2.9. The spokes divide the interior of χ into regions which we call *sectors*. We denote the sector bounded by σ_i and σ_{i+1} by S_i for $i = 0, \ldots, \ell - 1$ and S_ℓ is the sector bounded by σ_ℓ and σ_0. See Figure 2.

Definition 2.10. If $i \geq 1$, an arc δ starting at p *initially follows a spoke* σ_i if an initial portion of the reduced code for δ agrees with the code for σ_i with the last character replaced with either $P(i)_u$ or $P(i)_o P(i)_u$. Similarly, we say that an arc δ *initially follows the spoke* σ_0 if the initial portion of its reduced code begins with $P_s 0_o 1_u$.

Example 2.11. Consider the arcs $\beta_1, \beta_2, \beta_3, \beta_4$ in Figure 3. The arc β_1 starts in sector S_1 and initially follows σ_1, the arc β_2 starts in sector S_2 and initially follows σ_3, the arc β_3 starts in sector S_4 and initially follows no spoke, while β_4 starts in S_1 and initially follows no spoke.

2.4. **Standard position.** Every arc γ with reduced code is homotopy equivalent to an arc γ' with the same code that satisfies the following properties.

- There is an initial segment γ'_1 of γ' which is contained in a unique sector.
- If γ'_1 is contained in S_i with $i \geq 1$, then γ'_1 begins with $P_s(-1)_o \ldots P(i)_o$. If $i = 0$, then the arc begins with $P_s 0_o$.
- If γ'_1 is followed by a segment that agrees with the code for χ, then γ' does not intersect χ until after this segment.

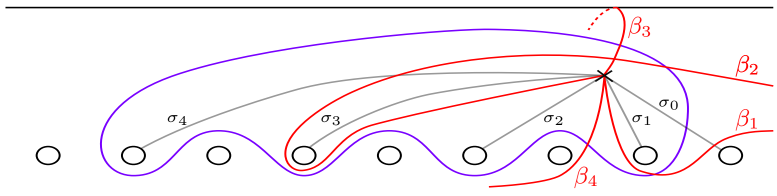


FIGURE 3. The arcs in Example 2.11.

- If the character after γ_1' does not agree with the code for χ, then the arc crosses χ immediately after traversing γ_1'.

If δ is an arc that begins in sector S_i and does not follow σ_{i+1} or σ_i, then we say δ *exits immediately*. Such a δ can exit immediately in two ways. First, it could go to the back directly after crossing χ. Otherwise, there must be an exterior puncture q between $P(i+1)$ and $P(i)$, and δ must contain q_u. For instance, in Figure 3 the former way occurs for β_3 and the latter way occurs for β_4.

3. Blocking cancellation

Suppose δ begins in S_i and initially follows σ_{i+1}. If $i \geq 1$, then by definition a code for δ begins with $P_s(-1)_o \dots (P(i+1)+1)_o P(i+1)_u$. If $i = 0$, then this code begins with $P_s 0_o 0_u$. In particular, because the final character of this code in either case agrees with that of χ, the arc δ must follow χ clockwise for at least one character. We say δ *hooks a puncture* if either: (a) δ follows χ clockwise for less than one full turn and then has a character $k_{o/u}$ that disagrees with the code for χ such that p_k is an interior puncture; or (b) if δ follows σ_{i+1}, loops around $P(i+1)$, and then follows σ_{i+1} backwards to p. In (a), we say δ hooks the puncture p_k, while in (b) we say δ hooks the puncture $P(i+1)$. See Figure 4. Notice that in (a), $k_{o/u}$ is the first character after δ leaves S_i that disagrees with the code for χ, so we can think of this as the point where δ stops fellow traveling χ.

Our strategy will be to understand how the images of arcs begin after iteratively applying T_χ and h. In order to do this, we need to understand what kind of cancellation can occur when we apply T_χ or h to an arc. The following lemma gives specific conditions under which there is no cancellation between a particular initial segment of δ and the remainder of δ. In the following sections, we will focus only on initial segments of arcs that fit into one of these categories.

Lemma 3.1. *If δ is an arc, then the following hold:*

(1) If δ hooks a puncture p_k, then there is no cancellation between the images under T_χ of the portion of δ before p_k and the portion after p_k.

(2) Suppose δ begins in a sector S_i and then δ either
   - *exits immediately, so that the character in a reduced code for δ immediately after crossing χ is either C or q_u for some exterior puncture; or*
   - *follows χ counterclockwise for at least one character $k_{o/u}$.*

   Then there is no cancellation between the images under T_χ of the portion of δ before this character and the portion after this character.

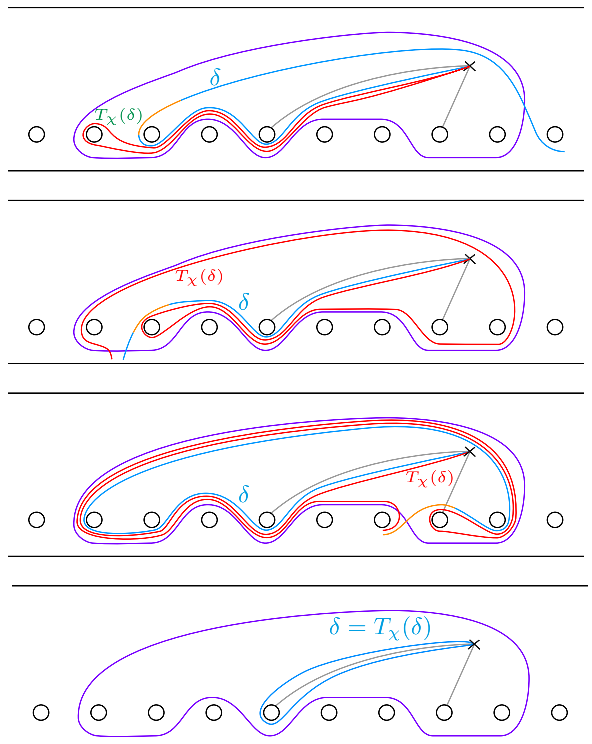


FIGURE 4. Four ways an arc δ that initially follows σ_{i+1} can hook a puncture. The first three pictures fit (a), while the last fits (b). The orange highlighted segments in each of the first three correspond to the character $k_{o/u}$ in the definition. In the last panel, δ is fixed by T_χ. This illustrates part (1) of Lemma 3.1.

(3) *If δ contains the character $k_{o/u}$ for any $k \in [1, \infty)$, then there is no cancellation between the images under g of the portion of δ before $k_{o/u}$ and the portion after $k_{o/u}$.*
(4) *If δ contains the character $k_{o/u}$ for any $k \in [2, \infty)$, then there is no cancellation between the images under g^{-1} of the portion of δ before $k_{o/u}$ and the portion after $k_{o/u}$.*

The proof of this lemma is intuitively straightforward: in each case, the character that blocks cancellation disagrees with the code for χ. When applying T_χ to an arc or segment, a copy of the code for χ is inserted into the code for δ, representing the twisting around χ. Thus, the only cancellation that can occur in $T_\chi(\delta)$ is when the copy of χ is inserted into the code for δ next to a subsegment of $\overline{\chi}$. In (1) and (2), it is clear that the initial segment does not end with a subsegment of $\overline{\chi}$. For (3), the additional intuition is that T_χ fixes the character $k_{o/u}$ because it is not contained in a code for χ, and then h is simply a shift, which does not cause any additional cancellation. For (4), the reasoning is reversed: h^{-1} does not cause any cancellation, and then T_χ^{-1} fixes the character $h^{-1}(k_{o/u}) = (k-1)_{o/u}$, since $k \geq 2$.

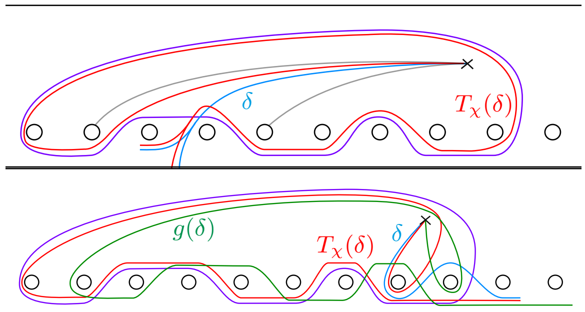


FIGURE 5. Illustration of parts (2) and (3) of Lemma 3.1. Note that in Case (3), the only cancellation occurs in the green highlighted portion.

Rather than giving a lengthy and technical proof of these straightforward facts, we illustrate the proof of each case; see Figure 4 for case (1) and Figure 5 for cases (2) and (3).

4. A "STARTS LIKE" FUNCTION

In the remainder of the paper, we will construct an explicit quasi-geodesic axis in $\mathcal{A}(S,p)$ on which g acts by translations and therefore show that g acts loxodromically on $\mathcal{A}(S,p)$. To accomplish this, we will follow a strategy similar to that in [2, 1], in which a "starts like" function is constructed and used to bound distance in $\mathcal{A}(S,p)$ from below.

We begin by defining the vertices in the quasi-axis for g via the following formula

$$\alpha_k = \begin{cases} P_s 0_o 1_u 2_o 3_o 3_u 2_o 1_u 0_o P_s, & k=0 \\ g^k(\alpha_0), & k \neq 0 \end{cases} \tag{1}$$

where $k \in \mathbb{Z}$.

4.1. **Defining a "starts like" function.** The following two definitions were first introduced in [1] and inspired by [2]; here they are modified to fit our current situation.

Definition 4.1. Given any arc γ on S, we define the *beginning of* γ, denoted by $\mathring{\gamma}$, to be the first $\lfloor \frac{1}{2}\ell_c(\gamma)\rfloor - 2$ characters in a reduced code for γ. Recall that code length was defined in Definition 2.4.

Definition 4.2. Fix any $k \in \mathbb{Z}_{\geq 0}$. An arc δ *starts like* α_k if the maximal initial or the maximal terminal segment of δ which agrees with an initial or terminal segment of α_k (not necessarily respectively) has code length at least $\lfloor \frac{1}{2}\ell_c(\alpha_k)\rfloor - 2$.

The following lemma is a straightforward application of Definition 4.1 and the definition of α_0 in Equation (1).

Lemma 4.3. *If δ is disjoint from $\mathring{\alpha}_0$, then δ does not contain an instance of χ or $\overline{\chi}$ in its reduced code.*

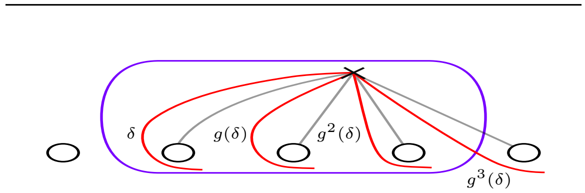


FIGURE 6. The most straightforward situation: after applying g 3 times, we have an arc that begins in S_0 and follows σ_0.

Consider the map

$$\varphi\colon \mathcal{A}(S,p) \to \mathbb{Z}_{\geq 0}, \tag{2}$$

defined by setting $\varphi(\delta)$ equal to the largest $i \geq 0$ such that δ starts like α_i. If no such i exists, then set $\varphi(\delta) = 0$. Using this function, we have the following simple lemma, where we do not require that j is non-negative.

Lemma 4.4. *Suppose an arc δ starts like α_i for some $i \geq 1$. Then for any j such that $i + j \geq 0$, the arc $g^j(\delta)$ starts like α_{i+j}.*

Proof. First, notice that the arc α_0 is symmetric in the sense that

$$\alpha_0 = P_s 0_o 1_u 2_o 3_o 3_u 2_o 1_u 0_o P_s = \tau 3_o 3_u \overline{\tau}.$$

By Lemma 3.1(3), the characters 3_o and 3_u block cancellation. In particular, there is no cancellation between the first half of α_0 and the second half. Moreover, any cancellation that occurs in $\tau 3_o$, the first half of α_0, must also occur in $3_u\overline{\tau}$, the second half of α_0. Thus the image of the first half of α_0 is the first half of α_1. Since the characters 1_u and 2_o also block cancellation by Lemma 3.1(3) and the last character of $\mathring{\alpha}_0$ is 1_u, it follows that $g(\mathring{\alpha}_0) = \mathring{\alpha}_1$ and that the last character of $\mathring{\alpha}_1$ is 2_u. Inductively, we conclude that $g(\mathring{\alpha}_i) = \mathring{\alpha}_{i+1}$ and the last character of $\mathring{\alpha}_{i+1}$ is $(i+1)_u$.

To show the lemma, it suffices to prove the result in the special case that $j = \pm 1$ and $i + j \geq 0$. An inductive argument then exhibits the general case. To this end, if $j = 1$ and δ starts like α_i, then the initial subsegment of δ is given by $\mathring{\alpha}_i$. Moreover, the terminal character $(i+1)_u$ of $\mathring{\alpha}_i$, which must appear in δ, blocks cancellation by Lemma 3.1(3). In particular $g(\mathring{\alpha}_i) = \mathring{\alpha}_{i+1}$ is the initial subsegment of $g(\delta)$, as required. Similarly, if $j = -1$, then since $i + j \geq 0$, we have $i + 1 \geq 2$, and therefore $(i+1)_u$ blocks cancellation by Lemma 3.1(4). In particular, $g^{-1}(\delta)$ will contain $\mathring{\alpha}_{i-1}$, as required. □

4.2. **A lower bound on the "starts like" function.** The goal of this section is to prove the existence of a uniform M so that given any arc δ disjoint from $\mathring{\alpha}_0$, there exists $0 \leq k \leq M$ such that $g^k(\delta)$ starts like α_1. We begin with an example that motivates the method of proof; see Figure 6. The arc δ shown in the figure begins in sector S_3 and follows σ_3. An initial subpath is invariant under T_χ, so $g(\delta)$ begins in S_2 and follows σ_2. Applying g two more times yields an arc $g^3(\delta)$ that begins in S_0 and follows σ_0. The following lemma shows that applying g one final time ensures that $g^4(\delta)$ begins like α_1.

Lemma 4.5. *Suppose δ is an arc which begins in sector S_0 and initially follows σ_0. Then $g(\delta)$ starts like α_1.*

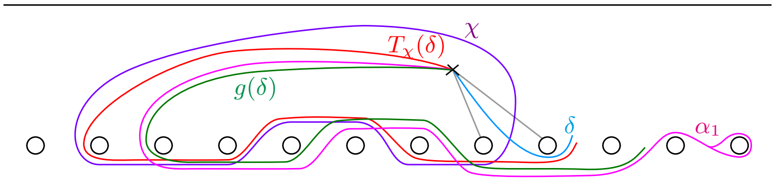


FIGURE 7. An example of an arc δ as in Lemma 4.5. The arc $g(\delta)$ begins by following $\mathring{\alpha}_1$, and so $\varphi(g(\delta)) \geq 1$.

Proof. Since δ begins in sector S_0 and initially follows σ_0, a reduced code for δ is $P_s 0_o 1_u \delta'$ for some δ'. We may compute the image of $P_s 0_o 1_u$ directly and see that it agrees with the code for $\mathring{\alpha}_1$; see Figure 7. Thus it remains to show that there is no cancellation between $g(P_s 0_0 1_u)$ and $g(\delta')$ in $g(\delta)$. The initial character of δ' must be either 1_o, $2_{o/u}$, or C. All of these block cancellation by Lemma 3.1. □

The remainder of this section shows that if δ is *any* arc that is disjoint from $\mathring{\alpha}_0$, applying g sufficiently many times results in an arc which begins in S_0 and follows σ_0 so that applying g one more time results in an arc that starts like α_1. Moreover, we will show that we only need to apply g at most $3\ell + 2$ times for this behavior to occur, where ℓ is the number of punctures in the interior of χ that are not p.

From our standard position, we will assume for the time being that every arc begins in a sector S_i for some $i \in \{1, \ldots, \ell\}$; the case $i = 0$ will be handled later. Given such an arc δ, there are three possibilities: δ either exits immediately, follows σ_i, or follows σ_{i+1}. We first show that either $g(\delta)$ or $g^2(\delta)$ must begin in sector S_{i-1}, that is, after applying g at most 2 times, the image of δ has moved into the sector to the right. The first two cases are dealt with in the following lemma, while the third case is dealt with in Lemma 4.7.

Lemma 4.6. *Let δ be an arc which begins in sector S_i for some $i \in \{1, \ldots, \ell\}$. Suppose that either:*

(a) δ exits immediately, or
(b) δ follows σ_i.

Then $g(\delta)$ begins in sector S_{i-1} and does not follow σ_i. In particular, $g^i(\delta)$ begins in sector S_0 and does not follow σ_1.

Proof. In case (a), since T_χ corresponds to a counterclockwise twist about χ, we see that $T_\chi(\delta)$ initially follows σ_i by Lemma 3.1(2); see Figure 8. In case (b), the arc δ follows σ_i, and by Lemma 3.1(2), the character $P(i)_u$ in δ blocks cancellation when applying T_χ; see Figure 8. This implies that $T_\chi(\delta)$ follows σ_i as well. Applying h, we have that $g(\delta)$ begins in sector S_{i-1}. Moreover, since $T_\chi(\delta)$ initially follows σ_i, there are two possibilities for the behavior of $g(\delta)$: either $g(\delta)$ follows σ_{i-1} when there are no punctures between $P(i)$ and $P(i-1)$, or $g(\delta)$ exits the sector S_{i-1} immediately. Either way, $g(\delta)$ does not follow σ_i.

Applying this argument i times yields the final statement of the lemma. □

We now analyze the behavior of the image of δ when neither (a) nor (b) of Lemma 4.6 holds, that is, when δ begins in sector S_i and follows σ_{i+1}.

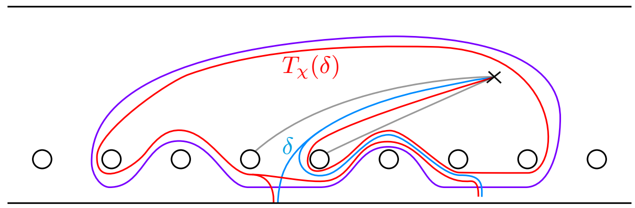


FIGURE 8. In both cases of Lemma 4.6, $T_\chi(\delta)$ follows σ_i so that $g(\delta)$ begins in S_{i-1} and does not follow σ_i. Both examples of δ are shown in blue and $T_\chi(\delta)$ are shown in red.

Lemma 4.7. *Let δ be an arc which begins in sector S_i for some $i \in \{1, \dots, \ell - 1\}$. If δ initially follows σ_{i+1}, then there exists $s \in \{1, 2\}$ such that $g^s(\delta)$ begins in sector S_{i-1}. Moreover, if $g^s(\delta)$ follows σ_i, then $s = 1$ and either*

(i) δ follows σ_{i+1} and then follows χ clockwise long enough to intersect the sector S_i again; or

(ii) δ hooks a puncture.

Proof. Since the sector S_i is invariant under T_χ and applying T_χ cannot cause cancellation with the initial subarc of δ contained in S_i, it must be the case that $T_\chi(\delta)$ also begins in S_i. Therefore, $T_\chi(\delta)$ must either initially follow σ_i, initially follow σ_{i+1}, or exit immediately. In particular, since $i > 0$, this implies that $g(\delta) = h(T_\chi(\delta))$ either begins in sector S_{i-1} or exits S_i immediately. In the latter case, $g(\delta)$ fits the hypothesis of Lemma 4.6(a) and therefore $g^2(\delta)$ is contained in sector S_{i-1}. This shows the first statement of the lemma.

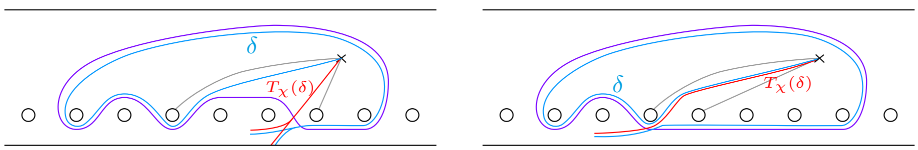
We now prove the second statement of the lemma. Suppose $g^s(\delta)$ follows σ_i. If $s = 2$, then we applied Lemma 4.6(a) to conclude that $g^2(\delta)$ begins in sector S_{i-1}. However, the moreover statement of that lemma shows that $g^2(\delta)$ does not follow σ_i. Therefore $s = 1$.

It remains to show that δ satisfies (i) or (ii). Recall that we assume that δ follows σ_{i+1}. There are three cases to consider depending on the behavior of $T_\chi(\delta)$. In the second two cases, we determine the initial behavior of δ by taking the preimage of an initial segment of $T_\chi(\delta)$, i.e., the image of the segment under T_χ^{-1}. In general, taking preimages is only well-defined if the initial and terminal characters of the segment are fixed by the mapping class (see [1, Section 3.4] for details).

Case 1: If $T_\chi(\delta)$ follows σ_i, then $g(\delta) = h(T_\chi(\delta))$ doesn't follow σ_i. Indeed, this was the case analyzed in the first paragraph of the proof of Lemma 4.6

Case 2: Suppose $T_\chi(\delta)$ exits immediately. Since $g(\delta)$ follows σ_i, $T_\chi(\delta)$ must exit between $P(i)$ and $P(i) - 1$. The character immediately following where $T_\chi(\delta)$ exits is fixed by T_χ and blocks cancellation by Lemma 3.1. Thus by taking the preimage of this initial segment under T_χ^{-1}, we obtain an initial segment of δ. As seen in Figure 9(A), (i) occurs.

Case 3: Suppose $T_\chi(\delta)$ follows σ_{i+1}. Then after following σ_{i+1}, $T_\chi(\delta)$ follows χ in the clockwise direction for at least the character $P(i+1)_u$. Let $k_{o/u}$ be the first character that $T_\chi(\delta)$ stops following χ. Then p_k is either an interior or exterior puncture. If p_k is an exterior puncture, then $k_{o/u}$ is fixed by T_χ and blocks



(A) Case 2 (B) Case 3

Figure 9. Cases 2 and 3 of Lemma 4.7. Note that the exterior punctures between $P(i+1)$ and $P(i)$ that appear in (A) do not appear in (B). This is because, under the assumption of the lemma, $g(\delta)$ must follow σ_i.

cancellation by Lemma 3.1. By taking the image of this initial segment of $T_\chi(\delta)$ under T_χ^{-1}, we see that (i) occurs. On the other hand, if p_k is an interior puncture, then $T_\chi(\delta)$ hooks a puncture, and so $k_{o/u}$ blocks cancellation, again by Lemma 3.1. This initial segment is fixed by T_χ^{-1}, and so δ also hooks a puncture, ensuring that (ii) holds. See Figure 9(B) for the first possibility. □

Remark 4.8. The reason that S_ℓ is excluded from the previous lemma is because S_ℓ is bounded by σ_ℓ and σ_0, so that arcs in this sector either follow σ_ℓ or exit immediately.

The previous two lemmas show that arcs that begin in S_i then begin in S_{i-1} after applying either g or g^2. Iteratively applying these lemmas shows that, after applying g a uniform number of times, the image of any arc δ begins in S_0. Our goal is to show that the image of δ follows σ_0 under iterates of g; however this may not be the case the first time this image begins in sector S_0. The following lemma gives a precise description of when the image of δ follows σ_1.

Corollary 4.9. *Let δ be an arc which begins in sector S_i for any $i \in \{1, \ldots, \ell\}$. There exists $0 \le k \le 2\ell - 1$ such that the following two properties hold:*

- *the arc $g^k(\delta)$ begins in sector S_0; and*
- *if the arc $g^k(\delta)$ follows σ_1, then $i = k$ and for all $0 \le s \le k$, $g^s(\delta)$ begins in S_{i-s}, follows σ_{i+1-s}, and either*
  - *(i) follows χ clockwise long enough to return to sector S_{i-s}; or*
  - *(ii) hooks a puncture.*

In the special case where the original arc δ hooks a puncture, then either $g^k(\delta)$ also hooks a puncture or $g^k(\delta)$ does not follow σ_1.

Proof. This follows from an inductive application of Lemmas 4.6 and 4.7, keeping track of which of the different cases is occurring at each step. □

In the remainder of the section, we use the expanded notation T_χ^j instead of T_χ so we can easily record the power of the Dehn twist about χ that we are applying. This power did not affect the proofs before this point. Let φ be the starts like function from (2). Then the key result for showing that $(\alpha_i)_{i\in\mathbb{N}}$ is a quasi-geodesic axis is the following:

Proposition 4.10. *Suppose that either $\ell \ge 2$ or that $\ell = 1$ and $g = hT_\chi^j$, where $j > 1$. If δ is an arc disjoint from $\mathring{\alpha}_0$, then there is some $0 \le k \le 3\ell + 1$ such that $g^k(\delta)$ starts like α_1.*

Before beginning the proof, we prove one additional lemma which will be used frequently throughout the proof of the proposition.

Lemma 4.11. *Under the assumptions of Proposition 4.10, if δ begins in sector S_i for any $i \in \{0, \dots, \ell\}$ and exits immediately or follows σ_i, then $g^k(\delta)$ starts like α_1 for some $0 \le k \le i + \ell + 1$.*

Proof. For $i \neq 0$, if δ exits immediately or follows σ_i, then Lemma 4.6(1) shows that $g^i(\delta)$ begins in sector S_0 and does not follow σ_1. Of course, when $i = 0$ this fact about δ holds by assumption, so we allow for the possibility that $i = 0$ as well. If $g^i(\delta)$ follows σ_0, then it follows from Lemma 4.5 that $g^{i+1}(\delta)$ starts like α_1.

On the other hand, suppose $g^i(\delta)$ exits immediately. Then $T^j_\chi(g^i(\delta))$ begins in S_ℓ and follows σ_ℓ. Thus $g^{i+1}(\delta) = h(T^j_\chi(g^i(\delta)))$ begins in $S_{\ell-1}$ and either follows $\sigma_{\ell-1}$ or exits immediately. We now repeat the reasoning of this lemma for the element $g^{i+1}(\delta)$, and we obtain that $g^{i+\ell}(\delta)$ starts in S_0 and either follows σ_0 or exits immediately. However, notice that since $P(1) = P(0) + 1$, the only way an arc that begins in S_0 can exit immediately is if it goes to the back of the surface right after exiting. By construction, $g^{i+\ell}(\delta)$ will not immediately go to the back; to see this, note that applying T^j_χ to δ introduces a copy of χ before the arc goes to the back of the surface. Since we are assuming that $\ell > 1$ or $j \ge 2$, the image of some portion of this copy of χ will remain as we apply Lemma 4.6, and it will always occur before the image of δ goes to the back. Therefore, $g^{i+\ell}(\delta)$ follows σ_0, and $g^{i+\ell+1}(\delta)$ starts like α_1 by Lemma 4.5. $\square$

We now turn to the proof of the proposition.

Proof of Propositin 4.10. Suppose that δ begins in sector S_i for any $i \in \{0, \dots, \ell\}$. Then by Corollary 4.9, there exists $0 \le k_0 \le 2\ell - 1$ such that $g^{k_0}(\delta)$ starts in sector S_0. If $g^{k_0}(\delta)$ starts like σ_0 or exits immediately, Lemma 4.11 shows that $g^{k_0+\ell+1}(\delta)$ starts like α_1. Let $j_0 = k_0 + \ell + 1 \le 3\ell$.

By Corollary 4.9, the remaining case to analyze is when $k_0 = i$ and $g^i(\delta)$ begins in sector S_0 and follows σ_1.

If $i = 0$, then δ itself begins in sector S_0 and follows σ_1. In this case, since δ is disjoint from $\mathring{\alpha}_0$ by assumption, either δ hooks a puncture or it follows χ clockwise for at least the character 0_u but not long enough to intersect the sector S_0 again, and then exits.

On the other hand, if $i \neq 0$, then we conclude from Corollary 4.9 that $g^s(\delta)$ follows σ_{i+1-s} for all $0 \le s \le i$ and that the initial segment of $g^s(\delta)$ satisfies either (i) or (ii) of that corollary. Again using that δ is disjoint from $\mathring{\alpha}_0$, we see that δ cannot follow χ long enough to intersect the sector S_i again. Thus, when $i \neq 0$, we must be in the case of Corollary 4.9(ii).

To summarize, we are further reduced to the following two cases:

(1) $i \ge 0$ and δ begins in sector S_i, follows σ_{i+1}, and hooks a puncture.
(2) $i = 0$ and δ begins in sector S_0, follows σ_1, follows χ clockwise for at least one character but not long enough to intersect S_0 again, and then exits. Note that in this case δ does not hook a puncture.

Case 1: Assume first that (1) holds. Since $g^i(\delta)$ follows σ_1, the moreover statement from Corollary 4.9 implies that $g^i(\delta)$ must also hook a puncture. This hooking blocks cancellation in $T^j_\chi(g^i(\delta))$ by Lemma 3.1. Therefore, a reduced code for $T^j_\chi(g^i(\delta))$ begins with $P_s 0_o 0_u$, and so a reduced code for $g^{i+1}(\delta)$ begins with

$P_s 0_o 1_o 1_u$. In particular, $g^{i+1}(\delta)$ begins in sector S_ℓ and exits immediately. By Lemma 4.11 applied to $g^{i+1}(\delta)$, there exists $0 \leq k_1 \leq 2\ell + 1$ such that $g^{i+1+k_1}(\delta)$ starts like α_1. Let $j_1 = i + 1 + k_1 \leq 3\ell + 2$.

Case 2: Next, assume that (2) holds, so that $i = 0$ and δ begins in sector S_0, follows σ_1, then turns left and follows χ clockwise for at least one character but not long enough to return to sector S_0, and then exits. There are now two possibilities to consider, depending on whether δ exits in S_ℓ or not. First, if δ exits in S_r for some $0 < r < \ell$, then $T_\chi^j(\delta)$ begins in S_ℓ and follows σ_ℓ (Figure 10). In particular, $g(\delta)$ begins in $S_{\ell-1}$ and does not follow σ_ℓ. Therefore, by applying Lemma 4.11 to $g(\delta)$, there is some $0 \leq k_2 \leq 2\ell$ such that $g^{1+k_2}(\delta)$ starts like α_1. Let $j_2 = 1 + k_2 \leq 2\ell + 1$.

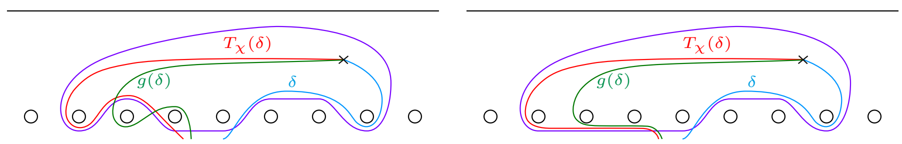


FIGURE 10. Case (2) of the proof of Proposition 4.10 if δ exits in S_r for some $0 < r < l$.

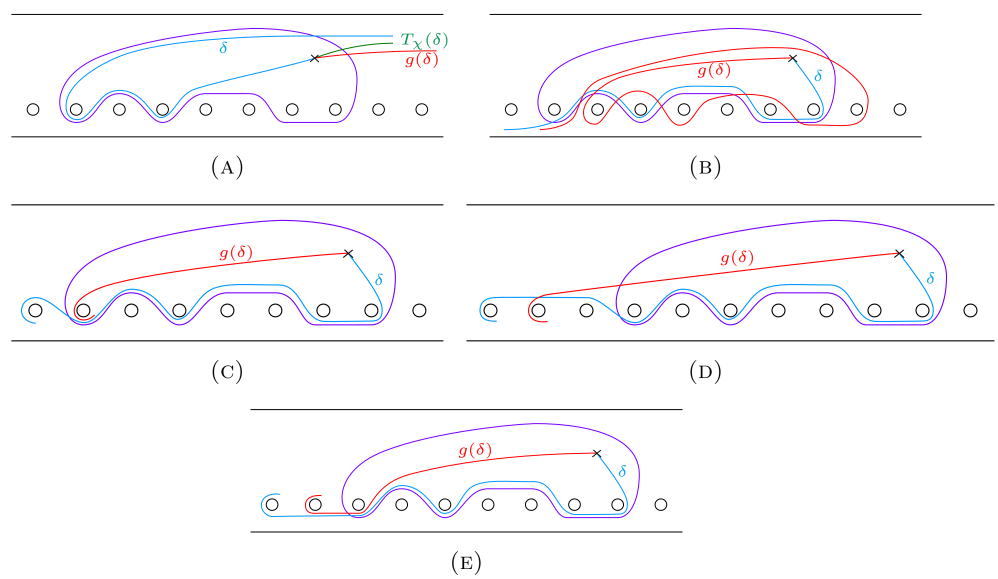


FIGURE 11. Case (2) of the proof of Proposition 4.10 if δ begins in sector S_0, follows σ_1 and then exits in S_ℓ.

On the other hand, suppose δ follows σ_1 and then exits in S_ℓ. Let $q_{o/u}$ be the first character of a reduced code for δ after δ exits; we allow for the possibility that $q = C$. If $q > P(\ell)$ or $q = C$ (Figure 11(A)), then $T_\chi^j(\delta)$ begins in S_ℓ and exits

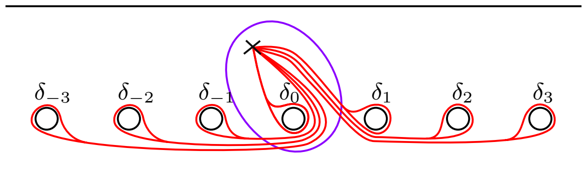


FIGURE 12. A collection of pairwise disjoint arcs $(\delta_i)_{i\in\mathbb{Z}}$, represented as train tracks, such that $\cup_i \delta_i$ is fixed by the composition of a shift and single twist about χ, when $\ell = 1$.

immediately. Therefore, so does $g(\delta)$. By Lemma 4.11 applied to $g(\delta)$, there exists $0 \leq k_3 \leq 2\ell + 1$ such that $g^{k_3+1}(\delta)$ starts like α_1. Let $j_3 = k_3 + 1 \leq 2\ell + 2$. If $q < P(\ell)$, then we must have $q = P(\ell) - 1$. If $j \geq 2$, then $T_\chi^j(\delta)$ begins in S_ℓ and follows σ_ℓ (see Figure 11(B)). Therefore, $g(\delta)$ begins in $S_{\ell-1}$ and does not follow σ_ℓ. The argument again concludes by applying Lemma 4.11: there exists $0 \leq k_2 \leq 2\ell$ such that $g^{k_2+1}(\delta)$ starts like α_1.

If $j = 1$, then either: $g(\delta)$ begins in $S_{\ell-1}$ and follows σ_ℓ; $g(\delta)$ begins in S_ℓ and follows σ_ℓ; or $g(\delta)$ begins in S_ℓ and exits immediately. Which of these occurs depends on the behavior of δ after exiting (see Figure 11(C)-(E)). In the latter two cases, Lemma 4.11 implies that there exists $0 \leq k_4 \leq 2\ell + 1$ such that $g^{k_4+1}(\delta)$ starts like α_1; let $j_4 = k_4 + 1 \leq 2\ell + 2$. In the former case, note that $\ell \geq 2$ implies that $\ell - 1 \geq 1$, and hence $g(\delta)$ does not begin in sector S_0. Moreover, the initial form of δ ensures that $g(\delta)$ neither hooks a puncture nor follows χ long enough to intersect the sector $S_{\ell-1}$ again; see Figure 11(E). In particular, applying Corollary 4.9, we conclude that $g^{k_0+1}(\delta)$ begins in sector S_0 and does not follow σ_1, for the constant $k_0 \leq 2\ell+1$ at the beginning of the proof. Applying Lemma 4.11 as in the beginning of the proof, we conclude that there exists $j_5 = k_0 + 1 + \ell + 1$ for which $g^{j_5}(\delta)$ starts like α_1, where $j_5 \leq 3\ell + 1$.

Taking $j = \max\{j_0, j_1, j_2, j_3, j_4, j_5\}$, we have that $j \leq 3\ell + 1$ and that for any arc δ disjoint from $\mathring{\alpha}_0$, there is some $k \leq j$ such that $g^k(\delta)$ starts like α_1. $\square$

Corollary 4.12. *Suppose that either $\ell \geq 2$ or that $\ell = 1$ and $g = hT_\chi^j$, where $j > 1$. If $\varphi(\gamma) = i$ for some $i > 3\ell$ and δ is disjoint from γ, then $|\varphi(\delta) - \varphi(\gamma)| \leq 3\ell$.*

Proof. If γ starts like α_i and δ is disjoint from γ, then $g^{-i}(\gamma)$ starts like α_0 and $g^{-i}(\delta)$ is disjoint from $g^{-i}(\gamma)$ by Lemma 4.4. Thus, Proposition 4.10 applied to $g^{-i}(\delta)$ shows that there exists some $k \leq 3\ell + 1$ such that $g^{-i+k}(\delta)$ starts like α_1. Applying Lemma 4.4 again shows that δ starts like α_{i-k+1} and where $i - k + 1 \geq i - 3\ell$, and so $\varphi(\delta) \geq i - 3\ell$. Asymmetric argument swapping the roles of δ and γ in the proof above shows that $\varphi(\delta) \leq i + 3\ell$. Therefore, if δ is disjoint from γ, then $|\varphi(\delta) - \varphi(\gamma)| \leq 3\ell$. $\square$

Remark 4.13. If $g = hT_\chi$ (a single Dehn twist composed with h) and $\ell = 1$, then Theorem 1.1 is actually false. Indeed, g will act elliptically with respect to the action on $\mathcal{A}(S, p)$. To see this, consider the infinite sequence of pairwise disjoint arcs $(\delta_i)_{i\in\mathbb{Z}}$ for which $\delta_i = g^i(\delta_0)$, as in Figure 12. Such arcs form an infinite 1-simplex in $\mathcal{A}(S, p)$ preserved by g.

5. Proof of main theorem

Let φ be the starts like function defined in (2), and let $M = 3\ell$.

Lemma 5.1. *For any $\gamma, \delta \in \mathcal{A}(S,p)$, we have $d_{\mathcal{A}(S,p)}(\gamma,\delta) \geq \frac{1}{M}|\varphi(\gamma) - \varphi(\delta)|$.*

Proof. Suppose $d_{\mathcal{A}(S,p)}(\gamma,\delta) = 1$, and assume without loss of generality that $\varphi(\delta) \leq \varphi(\gamma)$. If $\varphi(\gamma) = j$, then $g^k(\gamma)$ begins like $g^k(\alpha_j) = \alpha_{j+k}$ for any $k \geq 0$, by Lemma 4.4. Since $d_{\mathcal{A}(S,p)}(-,-)$ is $\mathrm{Map}(S,p)$–invariant, by replacing γ and δ with $g^k(\gamma)$ and $g^k(\delta)$, respectively, we may assume that $j \geq M$. By Corollary 4.12, it follows that $|\varphi(\delta) - \varphi(\gamma)| \leq M$. We conclude using the subadditivity of the absolute value. □

Proof of Theorem 1.1. It suffices to show that $(\alpha_j)_{j \geq M}$ is a quasi-geodesic half-axis for g. To see this, note that this is an unbounded orbit of g, so g is not elliptic, and since g acts as translation along this half-axis, it cannot be parabolic.

Let $f\colon \mathbb{Z}_{\geq 0} \to \mathcal{A}(S,p)$ be the map $i \mapsto \alpha_{M+i} = g^{M+i}(\alpha_0)$. Since $\varphi(\alpha_j) = j$ for any $j \geq 0$, it follows from Lemma 5.1 that for all $i \geq 0$,

$$d_{\mathcal{A}(S,p)}(\alpha_M, \alpha_{M+i}) \geq \frac{i}{M}.$$

Moreover, the arc $\beta = P_s 0_o 1_o 1_u 0_o P_s$ is disjoint from α_0 and α_1, and so $g^j(\beta)$ is disjoint from α_j and α_{j+1} for all $j \in \mathbb{Z}$. Thus $d_{\mathcal{A}(S,p)}(\alpha_j, \alpha_{j+1}) \leq 2$, and so for all $i \geq 0$,

$$d_{\mathcal{A}(S,p)}(\alpha_M, \alpha_{M+i}) \leq 2i.$$

Therefore, the map f is a $(2M, 0)$–quasi-isometric embedding. □

References

[1] C. Abbott, N. Miller, and P. Patel. Infinite-type loxodromic isometries of the relative arc graph. Preprint: arXiv:2109.06106.

[2] J. Bavard. Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire. *Geom. Topol.*, 20(1):491–535, 2016.

[3] J. Bavard and A. Walker. The Gromov boundary of the ray graph. *Trans. Amer. Math. Soc.*, 370(11):7647–7678, 2018.

[4] I. Morales and F. Valdez. Loxodromic elements in big mapping class groups via the Hooper–Thurston–Veech construction. Preprint: arXiv:2003.00102.

[5] P. Patel and S. Taylor. Constructing endperiodic loxodromics of infinite-type arc graphs. Preprint: arXiv:2211.00678.

Department of Mathematics, Brandeis University, Waltham, MA 02453
Email address: `carolynabbott@brandeis.edu`

Department of Mathematics, University of Oklahoma, Norman, OK 73019
Email address: `nickmbmiller@ou.edu`

Department of Mathematics, University of Utah, Salt Lake City, UT 84112
Email address: `patelp@math.utah.edu`