RIGIDITY AND COMPACTNESS WITH CONSTANT MEAN CURVATURE IN WARPED PRODUCT MANIFOLDS

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Abstract. We prove the rigidity of *rectifiable* boundaries with constant *distributional* mean curvature in the Brendle class of warped product manifolds (which includes important models in General Relativity, like the deSitter–Schwarzschild and Reissner–Nordstrom manifolds). As a corollary we characterize limits of rectifiable boundaries whose mean curvatures converge, as distributions, to a constant. The latter result is new, and requires the full strength of distributional CMC-rigidity, even when one considers smooth boundaries whose mean curvature oscillations vanish in arbitrarily strong C^k-norms. Our method also establishes that rectifiable boundaries of sets of finite perimeter in the hyperbolic space with constant distributional mean curvature are finite unions of possibly mutually tangent geodesic spheres.

Contents

1. Introduction

1.1. **Overview.** We move from two recent extensions of the classical Alexandrov theorem [Ale62]: *in the Euclidean space, spheres are the only connected, constant mean curvature (CMC) boundaries enclosing finite volumes*:

(i) In [Bre13], Brendle has proved CMC-rigidity in a class of warped product Riemannian manifolds which includes important models in General Relativity, like the deSitter–Schwarzschild and the Reissner–Nordstrom manifolds. In dimension $3 \le n \le 7$, when isoperimetric sets are smooth by local regularity theorems, Brendle's theorem allows one to solve the "horizon homologous" isoperimetric problem in this class of warped product manifolds. In turn, since the works of Eichmair and Metzger [EM12, EM13a, EM13b], the study of horizon-homologous isoperimetric problems in the large volume regime has played a prominent role in the analysis of the Huisken–Yau problem [Hui96]– see, e.g. [Cho16, CCE16, CEV17, CESZ19, CESY21, CE22].

(ii) In [DM19], Delgadino and the first-named author have extended the Alexandrov theorem to the class of (Borel) sets with finite volume, finite *distributional* perimeter, and constant *distributional* mean curvature. CMC-rigidity in the distributional setting in turn implies, by basic varifolds theory, a general *compactness theorem for almost CMC-boundaries*: in the Euclidean space, finite unions of disjoint spheres with equal radii are the only possible limits of sequences of boundaries converging in volume and in perimeter, and whose mean curvatures converge, as distributions, to a constant. With stronger controls on the mean curvature oscillation one can even provide quantitative rates of convergence towards finite unions of balls, as done, for example, in [CM17, DMMN18, JN20, JMS21]. In addition to their geometric interest, these results find applications to the resolution of geometric variational problems (see, e.g., [CM17, DW22] for the characterization of *local* minimizers of "free droplet energies") and to the long time behavior of the volume-preserving mean curvature flow (see, e.g., [JN20, MS22, JMS21]).

The main results of this paper are the distributional version of the Alexandrov theorem in the Brendle class (Theorem 1.1) and a consequent compactness theorem for almost-CMC boundaries (Theorem 1.2).

An interesting aspect of the method of proof of Theorem 1.1 is that we avoid the use of "smoothness intensive" tools, like the Schätzle *strong* maximum principle for integer varifolds [Sch04], which was central in [DM19]; or the Greene–Wu "approximation by convolution" theorem [GW73, GW79], which was crucially used in [Bre13]. We rather put emphasis on the *metric* notion of set of positive reach, and work with one-sided *viscous* mean-curvature/dimensions bounds (as formulated by White in [Whi16]; see also [CC93]); and, as a by-product of these efforts, throughout our analysis, we only need to use the *weak* maximum principle (meant as "one-sided inclusion and contact imply mean curvature ordering"). We take the success of this synthetic approach as a significant indication that *the Alexandrov theorem should hold in some metric version of the Brendle class*.

1.2. **The Brendle class and the main theorem.** Given $n \ge 3$, we denote by $\mathcal{B}_n$ the class of the n-dimensional manifolds (M, g) of the form

$$M = N \times [0, \bar r)\,, \qquad g = dr \otimes dr + h(r)^2\, g_N\,, \tag{1.1}$$

for some $\bar r \in (0, \infty]$, compact $(n-1)$-dimensional Riemannian manifold (N, g_N), and smooth positive function $h : [0, \bar r) \to \mathbb{R}$, such that:

(H0) for some $\rho > 0$, $\mathrm{Ric}_N \ge \rho(n-2)\, g_N$ on N;
(H1) $h'(0) = 0$ and $h''(0) > 0$;
(H2) $h' > 0$ on $(0, \bar r)$;
(H3) $2\,(h''/h) + (n-2)\,[((h')^2 - \rho)/h^2]$ is increasing on $(0, \bar r)$;
(H4) $(h''/h) + [(\rho - (h')^2)/h^2]$ is positive on $(0, \bar r)$.

To obtain geometric interpretations of these conditions we denote by

$$M^\circ = N \times (0, \bar r)\,, \qquad N_0 = N \times \{0\}\,, \qquad N_t = N \times \{t\} \quad (t > 0)\,,$$

the **interior**, the **horizon**, and the **slices** of M, and notice that

$$\mathrm{Ric}_M = \mathrm{Ric}_N - \left\{h\, h'' + (n-2)\,(h')^2\right\} g_N - (n-1)\left(h''/h\right) dr \otimes dr\,, \tag{1.2}$$

$$\mathrm{R}_M = \left(\mathrm{R}_N/h^2\right) - (n-1)\left(2\,(h''/h) + (n-2)\,(h'/h)^2\right). \tag{1.3}$$

The scalar mean curvature of N_t with respect to its g-unit normal vector field $\partial/\partial r$ is $H_{N_t} = \langle \vec{H}_{N_t}, \partial/\partial r\rangle_g = (n-1)\, h'(t)/h(t)$, so that the horizon of M is a minimal surface by (H1), and the slices of M have *positive* CMC (w.r.t. to $\partial/\partial r$) thanks to (H2). We next notice that (H3) implies (in combination with (1.3), (H0), and (H2)) that R_M is *decreasing* along $\partial/\partial r$. Finally, while (1.2) implies that $\partial/\partial r$ is an eigenvector of Ric_M, (H4) (combined with (H0)) adds the information that $\partial/\partial r$ is a *simple* eigenvector. We then have:

Brendle's theorem: [Bre13, Theorem 1.1] *If $n \ge 3$, $(M,g) \in \mathcal{B}_n$, and Σ is a smooth closed, embedded, orientable, CMC hypersurface in M, then Σ is a slice of M.*

Remark 1.1 (Dropping (H4))**.** A simple remark (which seems to have gone uncommented so far) is that Brendle's theorem also holds in the class $\mathcal{B}_n^*$ of those (M,g) satisfying (1.1), (H0), (H1), (H2), and

(H3)* $2\,(h''/h) + (n-2)\,[((h')^2 - \rho)/h^2]$ is *strictly* increasing on $(0,\bar r)$.

In other words, condition (H4) is not needed to conclude rigidity as soon as R_M is *strictly* decreasing along $\partial/\partial r$. (For more details on this point, see the discussions in Section 1.4 and Remark 4.1 below.) In terms of applications to General Relativity, it is interesting to notice that while the Reissner–Nordstrom manifolds belong to the class $\mathcal{B}_n^*$, the deSitter–Schwarzschild manifolds do not; in particular, a stronger stability mechanism for almost-CMC hypersurfaces is at work in the former class than in the latter; see Appendix A for more information.

Remark 1.2 (Formulation with boundaries)**.** Brendle's theorem is, actually, a statement about *boundaries* in M. Indeed, as noticed also in [Bre13, Section 3], under the assumptions of Brendle's theorem on Σ,

$$\begin{aligned}&\text{there is } (a,b) \subset\subset (0,\bar r) \text{ and } \Omega \subset M \text{ open such that}\\ &\Sigma \subset N \times (a,b) \text{ and either } \partial\Omega = \Sigma \text{ or } \partial\Omega = \Sigma \cup N_0\,.\end{aligned} \qquad (\Sigma \leftrightarrow \Omega)$$

Now, as explained in more detail later on, there are two basic geometric problems – the characterization of horizon-homologous isoperimetric regions and the study of sequences of (smooth) boundaries with vanishing mean curvature oscillation – that call for the extension of Brendle's theorem to the class of *sets of finite perimeter*. This extension is the content of our main theorem, Theorem 1.1 below. Referring to [Mag12] for a complete discussion of the subject, we just recall here that a Borel set Ω in (M,g) is a *set of finite perimeter* if $\mathrm{Per}(\Omega) := \sup\{\int_\Omega \mathrm{div}\, X\, d\mathcal{H}^n : X \in \mathcal{X}(M), |X|_g \le 1\}$ is finite (where $\mathcal{X}(M) = \{\text{smooth vector fields on } M\}$). Then one can define the *reduced boundary* $\partial^*\Omega$ (a locally $\mathcal{H}^{n-1}$-rectifiable set with $\mathrm{Per}(\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega)$) and the measure-theoretic outer normal ν_Ω (a Borel g-unit vector field defined on $\partial^*\Omega$) so that the distributional divergence theorem $\int_\Omega \mathrm{div}\, X\, d\mathcal{H}^n = \int_{\partial^*\Omega} \langle X, \nu_\Omega\rangle_g\, d\mathcal{H}^{n-1}$ holds for every $X \in \mathcal{X}(M)$. Finally, one says that H is the *distributional mean curvature* of $\partial^*\Omega$ with respect to ν_Ω, if H is summable in $\mathcal{H}^{n-1} \llcorner \partial^*\Omega$, and

$$\int_{\partial^*\Omega} \mathrm{div}^{\,\partial^*\Omega} X\, d\mathcal{H}^{n-1} = \int_{\partial^*\Omega} H\,\langle X, \nu_\Omega\rangle_g\, d\mathcal{H}^{n-1}\,, \qquad \forall X \in \mathcal{X}(M)\,,$$

where $\mathrm{div}^{\,\partial^*\Omega} X := \mathrm{div}\, X - \langle \nabla_{\nu_\Omega} X, \nu_\Omega\rangle_g$. We thus have the following *distributional* version of Brendle's theorem:

Theorem 1.1 (Rigidity)**.** *If $n \geq 3$, $(M,g) \in \mathcal{B}_n \cup \mathcal{B}_n^*$, Ω is a set of finite perimeter in M such that $M^\circ \cap \overline{\partial^*\Omega}$ is compact in M and $\lambda \in \mathbb{R}$ is such that*

$$\int_{M^\circ \cap \partial^*\Omega} \mathrm{div}^{\,\partial^*\Omega} X \, d\mathcal{H}^{n-1} = \lambda \int_{M^\circ \cap \partial^*\Omega} \langle X, \nu_\Omega \rangle_g \, d\mathcal{H}^{n-1} \,, \tag{1.4}$$

for every $X \in \mathcal{X}(M)$, then, for some $t_0 \in (0, \bar{r})$, either Ω is $\mathcal{H}^n$-equivalent to $N \times (0, t_0)$ (and $\lambda > 0$) or Ω is $\mathcal{H}^n$-equivalent to $N \times (t_0, \bar{r})$ (and $\lambda < 0$).

Remark 1.3. By Allard's regularity theorem [All72, Sim83, DL18], (1.4) implies that $\Sigma = \partial^*\Omega$ is a smooth, CMC hypersurface in M with $\mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0$. Since Σ *is not necessarily closed*, rigidity cannot be deduced by the direct application of Brendle's theorem. The difficulty addressed in Theorem 1.1 is expressing how the distributional CMC condition (1.4) "ties together" the (potentially countably many) connected components of Σ, and forces them to align into a single slice, rather than, say, allowing them to combine through the singular set $\overline{\Sigma} \setminus \Sigma$ into a non-slice CMC hypersurface.

Remark 1.4. Condition (1.4) is equivalent to ask that, if f_t is a smooth volume-preserving flow of Ω ($f_0 = \mathrm{id}$, $(\partial f/\partial t)|_{t=0} = X \in \mathcal{X}(M)$, and $\mathcal{H}^n(f_t(\Omega)) = \mathcal{H}^n(\Omega)$ for every $|t|$ small), then

$$\big(d/dt\big)\big|_{t=0} \mathrm{Per}(f_t(\Omega)) = 0 \,.$$

Theorem 1.1 then says that *among sets of finite perimeter, slices are the only volume-constrained critical points of the area functional in $(M,g) \in \mathcal{B}_n \cup \mathcal{B}_n^*$.*

Remark 1.5 (Necessity of "$M^\circ \cap \overline{\partial^*\Omega}$ compact" in Theorem 1.1)**.** If $v > 0$ is small enough and Ω_v is a minimizer of $\mathcal{H}^{n-1}(M^\circ \cap \partial^*\Omega)$ among sets $\Omega \subset M$ with $\mathcal{H}^n(\Omega) = v$, then $M^\circ \cap \overline{\partial^*\Omega_v}$ is a smooth CMC hypersurface, diffeomorphic to a hemisphere sitting on the horizon (see, e.g., [MM16] for a detailed analysis of this kind of result in the capillarity setting). Alternatively, one can first apply the perturbative construction of Pacard and Xu [PX09] on the horizon of the *doubled* Schwarzschild manifold, as described in [BE13]. Either way, one obtains non-slice, CMC hypersurfaces bounding sets Ω.

1.3. **Compactness for almost-CMC boundaries.** Theorem 1.1 is of course strongly motivated by the following natural **compactness problem for almost-CMC hypersurfaces**, which can be formulated, in very general terms, as follows:

If $(M,g) \in \mathcal{B}_n \cup \mathcal{B}_n^*$, does every sequence $\{\Sigma_j\}_j$ of closed,
embedded, orientable smooth hypersurfaces in M
whose scalar mean curvatures in (M,g) converge
to a constant, have slices as their only possible limits? (CP)

This basic question appears naturally in several contexts. Two important examples are: the analysis of the Huisken–Yau problem [Hui96, QT07, NT09, NT10, Hua10, LMS11, BE14, Ner18, CE20], e.g., an outlying CMC hypersurface in an asymptotically Schwarzschild manifold can be seen as an almost-CMC hypersurface in the Schwarzschild manifold; and the study of the long-time behavior of the volume-preserving mean curvature flow – since the L^2-oscillation of the mean curvature is the dissipation of the flow.

In light of $(\Sigma \leftrightarrow \Omega)$, for each Σ_j in (CP) there are open sets Ω_j in M and intervals $(a_j, b_j) \subset\subset (0, \bar{r})$ such that $\Sigma_j \subset N \times (a_j, b_j)$ and (up to extracting subsequences) either $\Sigma_j = \partial\Omega_j$ (for every j) or $\Sigma_j \cup N_0 = \partial\Omega_j$ (for every j). In both cases, under the natural set of assumptions that

$$a = \inf_j a_j > 0\,, \qquad b = \sup_j b_j < \bar r\,, \qquad \sup_j \mathcal{H}^{n-1}(\Sigma_j) < \infty\,, \tag{1.5}$$

one finds a set Ω with finite perimeter in (M, g) such that, and up to extracting subsequences, it holds

$$\lim_{j\to\infty} \mathcal{H}^n(\Omega_j \Delta \Omega) = 0\,, \qquad \liminf_{j\to\infty} \mathcal{H}^{n-1}(\partial\Omega_j) \geq \mathrm{Per}(\Omega)\,. \tag{1.6}$$

In the above basic setting, Theorem 1.1, combined with standard closure results for integer varifolds, leads to an affirmative answer to (CP). This is the content of our second main result, where the most general case of sequences of sets of finite perimeter is directly addressed:

Theorem 1.2 (Compactness)**.** *If $n \geq 3$, $(M, g) \in \mathcal{B}_n \cup \mathcal{B}_n^*$, and $\{\Omega_j\}_j$ is a sequence of sets of finite perimeter in M such that*

(i) *there are $(a, b) \subset\subset (0, \bar r)$ and a Borel set $\Omega \subset M$ s.t. $M^\circ \cap \overline{\partial^*\Omega_j} \subset N \times (a, b)$ for every j, and $\mathcal{H}^n(\Omega_j \Delta \Omega) \to 0$ as $j \to \infty$;*

(ii) $\mathrm{Per}(\Omega_j) \to \mathrm{Per}(\Omega)$ *as $j \to \infty$;*

(iii) *there is $\lambda \in \mathbb{R}$ such that, for every $X \in \mathcal{X}(M)$, as $j \to \infty$,*

$$\int_{M^\circ \cap \partial^*\Omega_j} \mathrm{div}^{\,\partial^*\Omega_j} X \, d\mathcal{H}^{n-1} - \lambda \int_{M^\circ \cap \partial^*\Omega_j} \langle X, \nu_{\Omega_j} \rangle_g \, d\mathcal{H}^{n-1} \to 0\,; \tag{1.7}$$

then there is $t_0 \in (0, \bar r)$ such that Ω if $\mathcal{H}^n$-equivalent either to $N \times (0, t_0)$ (and then $\lambda > 0$) or to $N \times (t_0, \bar r)$ (and then $\lambda < 0$).

Remark 1.6. The proof of Theorem 1.2 requires the full strength of Theorem 1.1 even if one is only interested in sequences $\{\Sigma_j\}_j$ of closed, embedded, orientable *smooth* hypersurfaces in M whose mean curvature oscillations are assumed to vanish in every C^k-norm.

Remark 1.7. In (1.7) the mean curvature oscillation is required to vanish *only in distributional sense*. This feature points to the possibility of applying Theorem 1.2 to minimizing sequences of horizon-homologous isoperimetric problems that have been suitably *selected* by means of the Ekeland variational principle; see, e.g., [CL12, Theorem 3.2(iv)]. Similarly, Theorem 1.2 will be easily applied to the study of horizon-homologous isoperimetric sets with fixed volume with respect to metrics $\{g_j\}_j$ on M such that, as $j \to \infty$, $g_j \to g$ with $(M, g) \in \mathcal{B}_n \cup \mathcal{B}_n^*$. In both these examples, the perimeter convergence assumption (ii) is trivially checked by energy comparison.

1.4. **Strategy of proof and organization of the paper.** Section 2 and Section 3 are devoted to establishing in the Riemannian setting the several tools from GMT that lie at the core of our analysis. In Section 2.1 we collect several properties of normal bundles to closed sets in complete Riemannian manifolds (Theorem 2.1), extending from the Euclidean case a series of recent results obtained in [Alb15, MS19, San20a, KS23, HS22]. In Section 2.2 we review some theorems of Kleinjohann [Kle81] and Bangert [Ban82] concerning sets of positive reach in complete Riemannian manifolds (Theorem 2.2 and Theorem 2.3). In Section 3.1 we recall the viscous notion of "being m-dimensional with mean curvature vector bounded by λ" introduced on closed subsets of Riemannian manifolds by White in [Whi16], and recall its relation to distributional mean curvature (Theorem 3.1).

Finally, in Section 3.2, we extend to the Riemannian setting a delicate "Lusin-type property" of normal bundles of White's (m, λ)-sets that, in the Euclidean case, was proved in [San20b] (Theorem 3.3).

At the basis of the distributional version of the Alexandrov theorem proved in [DM19] (as well as of the previously cited quantitative versions of it), lies the approach to CMC-rigidity of Ros [Ros87] and Montiel-Ros [MR91]. Their method is based on the analysis of equality cases in the (Euclidean) Heintze-Karcher inequality. Brendle's theorem, in turn, is based on the analysis of the equality cases of *two* different Heintze–Karcher-type inequalities for subsets Ω of $(M, g) \in \mathcal{B}_n \cup \mathcal{B}_n^*$ (corresponding to the cases $\partial\Omega = \Sigma$ and $\partial\Omega = \Sigma \cup N_0$ appearing in $(\Sigma \leftrightarrow \Omega)$), and, specifically, to the fact that, when Σ is such an equality case, then, Σ is umbilic; the umbilicality of Σ is then combined with (H4) to deduce rigidity (i.e., Σ is a slice).

A natural strategy for proving Theorem 1.1 thus consists in: (a) establishing the two Heintze–Karcher-type inequalities of Brendle on sets with finite perimeter; (b) addressing the analysis of their equality cases in the distributional setting; and, (c) deducing rigidity from an established set of necessary conditions for equality.

In Section 4 we implement this strategy. There we work with *not necessarily closed*, smooth, embedded, hypersurfaces Σ satisfying three assumptions: first, $\mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0$ with $\mathcal{H}^{n-1}(\Sigma) < \infty$ ($\overline{\Sigma} \setminus \Sigma$ is understood as the "singular set" of Σ); second, either $\overline{\Sigma} = \partial\Omega$, or $N_0 \cup \overline{\Sigma} = \partial\Omega$, for an open set Ω in M; and, finally, $\overline{\Sigma}$ is compactly contained in M° and has bounded mean curvature in the viscous sense of White, while Σ has positive mean curvature with respect to the outer g-unit normal ν_Ω to Ω.

In Theorem 4.1 we take care of steps (a) and (b). Implementing step (a) is particularly delicate not only because, as expected, several passages of Brendle's argument make a crucial use of smoothness, and thus require considerable effort to be repeated or redesigned in a non-smooth framework; but also because, in anticipation of the non-smooth rigidity discussion of step (c), we need a more detailed list of necessary conditions for equality cases in Brendle's Heintze–Karcher-type inequalities. In this direction, we notice that we shall establish *three* such conditions: (E1) Σ is umbilical in (M, g) (which is the condition already pointed out in [Bre13]); (E2) $M \setminus \Omega$ has positive reach in (M, g^*) (where g^* is a certain complete metric on M, conformal to g); (E3) under (H3)*, $g_N(\nu_\Omega, \nu_\Omega) = 0$ on Σ. The identification of condition (E3) is new even in the smooth case. We also notice that, in the smooth case, condition (E2) is automatically true; and, indeed, the derivation of (E2) requires some careful work on Brendle's argument which is specific to the viscous setting.

In Theorem 4.2, we further assume that $\overline{\Sigma}$ has constant mean curvature in distributional sense, and then address step (c). If (H3)*, and thus condition (E3), holds, then we can infer rigidity directly from it, without using umbilicality: indeed (E3) implies the very strong information that ν_Ω is parallel to $\partial/\partial r$ $\mathcal{H}^{n-1}$-a.e. along Σ – an information that gives $\Sigma = N_{t_0}$ by a simple property of sets of finite perimeter (cf. with [Mag12, Exercise 15.18]). If, otherwise, only (H3) holds, then, as in [Bre13], we need to combine (H4) and umbilicality to deduce that, $\mathcal{H}^{n-1}$-a.e. on Σ, ν_Ω is either parallel *or orthogonal* to $\partial/\partial r$. The dichotomy parallel/orthogonal prevents the use of something as simple as [Mag12, Exercise 15.18]. In the smooth case, one immediately excludes "orthogonality", and thus conclude rigidity, by a sliding argument. However, in the non-smooth setting, sliding arguments are not equally effective (because of multiplicity issues preventing the use of Allard's regularity theorem at contact points). This is the passage where condition (E2) reveals useful, and ultimately allows us to conclude the proof of Theorem 4.2.

Finally, Section 5 contains the proofs of Theorem 1.1 and Theorem 1.2.

1.5. **Rigidity and bubbling in the hyperbolic space.** It is a classical and well known result that Alexandrov rigidity result can be generalized to *smooth* boundaries in the hyperbolic space. In [Bre13, Theorem 1.4] a new proof is obtained based on the Heintze-Karcher inequality. It is a natural and interesting question to understand if finite unions of possibly mutually tangent balls are the only examples of sets of finite perimeter in the hyperbolic space with constant distributional mean curvature. The methods developed in this paper provides a positive answer to this question.

Theorem 1.3. *Suppose Ω is a set of finite perimeter with compact closure in the hyperbolic space $\mathbb{H}^n$ and $\lambda \in \mathbb{R}$ such that*

$$\int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^{n-1} = \lambda \int_{\partial^*\Omega} \langle X, \nu_\Omega \rangle_{\mathbb{H}^n} \, d\mathcal{H}^{n-1}, \tag{1.8}$$

for every $X \in \mathcal{X}(M)$. Then Ω is a finite union of disjoint (possibly mutually tangent) open geodesic balls with equal radii.

Remark 1.8. In [HHW23] the authors develop a beautiful moving plane method for a class of varifolds satisfying a suitable *tameness condition* and they employ it to prove several rigidity results for stationary and CMC varifolds. In particular, cf. [HHW23, Theorem 1.8], they prove that if $\Sigma \subseteq \mathbb{H}^n$ is the support of a $(n-1)$-dimensional *tame* varifold without boundary and with constant mean curvature[1], and if Σ is connected and compact, then it is a geodesic sphere. In this direction we point out that unions of mutually tangent geodesic spheres cannot be tame varifolds in the sense of [HHW23] (since the tangent cone at the singular point between the spheres is a multiplicity two plane). On the other hand, these configurations naturally arise as limits of sequences of connected and compact smooth hypersurfaces with mean curvatures converging to a constant (see [CM17] and references therein). Hence, Theorem 1.3, in addressing rigidity and compactness in the hyperbolic space under assumptions that do not prevent bubbling, provides a useful improvement of [HHW23, Theorem 1.8].

1.6. **Further directions.** A natural question is that of obtaining quantitative estimates for almost-CMC boundaries, both in the Brendle class and on space forms. In this direction we mention the recent results on space forms [CV20, CRV21], where, based on the moving planes method, sharp decays are obtained under a "bubbling-preventing" exterior/interior ball assumption. There are alternative proofs of the Heintze–Karcher type inequalities behind Brendle's theorem, based on integral identities rather than on geodesic flows, that have been developed, for example, in [LX19, FP22]. Correspondingly, non-sharp quantitative estimate have been derived in [Sch21, Theorem 1.3] on space forms, and in [SX22, Theorem 1.4] on

[1] cf. [HHW23, eq. (8)] for the definition of varifold with bounded mean curvature, and [HHW23, Definition 1.6] for the definition of tameness for varifolds

a sub-class of Brendle's class (which still includes the model manifolds from General Relativity). Both these results require bounds on the $C^{2,\beta}$-geometry of the considered boundaries (in addition to interior ball conditions), and the resulting stability constants (together with the non-sharp stability exponents in the case of [SX22, Theorem 1.4]) depend to the particular β under consideration. For this reason the compactness result in Theorem 1.2 is entirely new even on smooth boundaries, as it does not require any uniform control on their geometry.

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2. Sets of positive reach in Riemannian manifolds

In this section (M,g) is a *complete* Riemannian manifold of dimension n, with exponential function exp and Riemannian distance d, and we denote by $\Psi : TM \times [0,\infty) \to M$ the map

$$\Psi(p,\eta,t) = \exp(p,t\,\eta)\,, \qquad \forall (p,\eta) \in TM\,, t \geq 0\,. \tag{2.1}$$

We denote by $\mathrm{d}f$ the differential of $f : M \to \mathbb{R}$, and by ∇f and $\mathrm{D}^2 f$ the gradient and Hessian of f with respect to g. We use ∇ also to denote the metric connection of (M,g). A segment in M is a unit speed geodesic $\gamma : [a,b] \to M$ such that $d(\gamma(a),\gamma(b)) = b-a$.

2.1. **Normal bundles of closed sets.** Given a closed set $\Gamma \subset M$, the projection map on Γ is defined for at $p \in M$ as the subset of Γ given by

$$\xi_\Gamma(p) = \big\{ a \in \Gamma : \mathrm{dist}(p,\Gamma) = d(p,a) \big\}\,.$$

The **unit normal bundle** $\mathcal{N}^1\Gamma$ and the **normal bundle** $\mathcal{N}\Gamma$ of Γ are defined by setting, for $a \in \Gamma$,

$$\begin{aligned}
\mathcal{N}^1_a\Gamma &= \big\{ \eta \in T_aM : |\eta| = 1\,, \exists s > 0 \text{ s.t. } s = \mathrm{dist}\big(\exp(a,s\,\eta),\Gamma\big) \big\}\,,\\
\mathcal{N}_a\Gamma &= \big\{ t\,\eta : t \geq 0\,, \eta \in \mathcal{N}^1_a\Gamma \big\}\,.
\end{aligned}$$

We define $\rho_\Gamma : \mathcal{N}^1\Gamma \to (0,\infty]$, $A_\Gamma \subset TM$, and $\mathrm{Cut}(\Gamma) \subset M$ by setting

$$\rho_\Gamma(x,\eta) = \sup\big\{ s > 0 : s = \mathrm{dist}(\exp(x,s\,\eta),\Gamma) \big\}\,, \tag{2.2}$$

$$A_\Gamma = \big\{ (p,\eta,t) : (p,\eta) \in \mathcal{N}^1\Gamma\,, t \in (0,\rho_\Gamma(p,\eta)) \big\}\,, \tag{2.3}$$

$$\mathrm{Cut}(\Gamma) = \big\{ \exp(x,s\,\eta) : (x,\eta) \in \mathcal{N}^1\Gamma\,, s = \rho_\Gamma(x,\eta) < \infty \big\}\,, \tag{2.4}$$

so that, when Γ is a closed C^2-hypersurface in M, ρ_Γ is continuous on Γ, $\mathrm{Cut}(\Gamma)$ corresponds to the usual notion of **cut-locus** of Γ and satisfies $\mathcal{H}^n(\mathrm{Cut}(\Gamma)) = 0$, and $\Psi|_{A_\Gamma}$ is a diffeomorphism between A_Γ and

$$\mathcal{U}(\Gamma) = \Psi(A_\Gamma) = \big\{ \exp(a,s\,\eta) : (a,\eta) \in \mathcal{N}^1\Gamma\,, s \in \big(0,\rho_\Gamma(a,\eta)\big) \big\}\,.$$

The following theorem relies on a series of recent results [Alb15, MS19, San20a, KS23, HS22] to adapt/extend the above classical facts to the case when Γ is merely a closed set.

Theorem 2.1. *If (M^n,g) is a complete Riemannian manifold and $\Gamma \subset M$ is closed, then:*

(i) $\mathcal{N}\,\Gamma$ *is a countably* n*-rectifiable Borel subset of* TM*;*
(ii) $\mathcal{N}_a\Gamma$ *is a convex cone in* T_aM *for every* $a \in \Gamma$*;*
(iii) *for each* $m = 0, \ldots, n-1$ *the set*

$$\Gamma^{(m)} := \{a \in \Gamma : \dim \mathcal{N}_a\Gamma = n - m\}$$

is countably $(\mathcal{H}^m, m)$*-rectifiable; in particular*[2]

$$\mathcal{H}^{n-1}(\{a \in \Gamma : \mathcal{H}^0(\mathcal{N}^1_a\Gamma) > 2\}) = 0\,; \tag{2.5}$$

(iv) $\rho_\Gamma : \mathcal{N}^1(\Gamma) \to (0, +\infty]$ *is an upper-semicontinuous function;*
(v) $\mathcal{H}^n(\mathrm{Cut}(\Gamma)) = 0$.
(vi) *For each* $p \in \mathcal{U}(\Gamma)$ *there exists a unique* $(a, \eta) \in TM$*,* $|\eta| = 1$*, such that* $d(p, a) = \mathrm{dist}(p, \Gamma)$ *and* $\exp(a, d(p, a)\eta) = p$*;*
(vii) *If* $\tau_0 > 0$ *and* $\rho_\Gamma(a, \eta) \geq \tau_0$ *for* $\mathcal{H}^{n-1}$ *a.e.* $(a, \eta) \in \mathcal{N}^1\Gamma$*, then*

$$\big\{x \in M : 0 < \mathrm{dist}(x, \Gamma) < \tau_0\big\} \subseteq \mathcal{U}(\Gamma)\,.$$

Proof. Let $\phi : U \subset M \to V \subset \mathbb{R}^n$ be a local chart of M. For each $a \in U$ there is a unique symmetric bijective linear map $S_a : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\langle u, v\rangle_g = \mathrm{d}\phi(a)(v) \,\cdot\, S_a\big[\mathrm{d}\phi(a)(u)\big]\,, \qquad \forall u, v \in T_aM\,.$$

A routine argument shows that the map $\Phi : TU \to V \times \mathbb{R}^n$ defined by $\Phi(a, v) = (\phi(a), S_a[\mathrm{d}\phi(a)(v)])$, $(a, v) \in TU$, is a diffeomorphism, with

$$\Phi\big(\mathcal{N}\,\Gamma \llcorner U\big) = \big(\mathcal{N}\,\overline{\phi(\Gamma \cap U)}\big) \llcorner V\,.$$

Similarly, for every $a \in \Gamma$ and $m = 0, ..., n-1$, we have

$$[S_a \circ \mathrm{d}\phi(a)]\big(\mathcal{N}_a\Gamma\big) = \mathcal{N}_{\phi(a)}\,\overline{\phi(\Gamma \cap U)}\,, \quad \phi(\Gamma^{(m)} \cap U) = [\overline{\phi(\Gamma \cap U)}]^{(m)} \cap V\,.$$

Hence, conclusions (i), (ii), and (iii) follow from the analogous statements in the Euclidean case proved in [San20a, Remark 4.3] and [MS19]. Conclusion (iv) follows by an obvious adaptation of the argument for the Euclidean space proved in [KS23, Lemma 2.35]. Conclusion (v) is proved in [Alb15, Theorem 1]. Conclusion (vi) follows from the remark that if $s > 0$ and α is a unit-speed geodesic such that $\alpha(0) \in \Gamma$ and $\mathrm{dist}(\alpha(s), \Gamma) = s$, then, for every $0 \leq t < s$, we have $\mathrm{dist}(\alpha(t), \Gamma) = t$, $\xi_\Gamma(\alpha(t)) = \{\alpha(0)\}$, and $\alpha|_{[0,t]}$ is the unique segment joining $\alpha(0)$ and $\alpha(t)$. We are left to prove conclusion (vii), which requires modifications to the proof of [HS22, Lemma 3.19]. With Ψ as in (2.1), we define

$$Q^* = \{(a, \eta, t) : (a, \eta) \in \mathcal{N}^1\Gamma,\ 0 < t < \inf\{\tau_0, \rho_\Gamma(a, \eta)\}\}$$

and we notice that $\Psi(Q^*) = \{x \in M : 0 < \mathrm{dist}(x, \Gamma) < \tau_0\} \setminus \mathrm{Cut}(\Gamma)$. Let

$$Q = Q^* \cap \{(a, \eta, t) : \rho_\Gamma(a, \eta) \geq \tau_0\}$$

so that $\mathcal{H}^n(Q^* \setminus Q) = 0$ by assumption. Hence $\mathcal{H}^n(\Psi(Q^*) \setminus \Psi(Q)) = 0$, and, by (v), we conclude that $\Psi(Q)$ is dense in $\{x \in M : 0 < \mathrm{dist}(x, \Gamma) < \tau_0\}$. Let $x \in M$ and $t = \mathrm{dist}(x, \Gamma)$ with $0 < t < \tau_0$. There exists a sequence $(a_i, \eta_i, t_i) \in Q$ such that $\Psi(a_i, \eta_i, t_i) \to x$. Since $d(a_i, x) \leq d(a_i, \Psi(a_i, \eta_i, t_i)) + d(\Psi(a_i, \eta_i, t_i), x)$ for $i \geq 1$ and $t_i = \mathrm{dist}(\Psi(a_i, \eta_i, t_i), \Gamma) \to t$, we infer that $\limsup_{i\to\infty} d(a_i, x) \leq t$. By

[2] Indeed, if $\mathcal{H}^0(\mathcal{N}^1_a\Gamma) > 2$, then the convexity of $\mathcal{N}_a\Gamma$ implies that $\dim \mathcal{N}_a\Gamma \geq 2$, i.e. $a \in \Gamma^{(m)}$ for some $m \leq n-2$.

compactness, there are $(a,\eta)\in TM$ with $|\eta|=1$ and a subsequence (a_{i_j},η_{i_j}) such that $(a_{i_j},\eta_{i_j})\to(a,\eta)$ as $j\to\infty$. It follows that

$$t=\lim_{j\to\infty}\mathrm{dist}(\Psi(a_{i_j},\eta_{i_j},t_{i_j}),\Gamma)=\mathrm{dist}(\Psi(a,\eta,t),\Gamma)$$

and $(a,\eta)\in\mathcal{N}^1\Gamma$. By conclusion (iv) and by $\rho_\Gamma(a_i,\eta_i)\ge\tau_0$ we find $\rho_\Gamma(a,\eta)\ge\tau_0$, and thus $x\in\mathcal{U}(\Gamma)$. ☐

2.2. **Sets of positive reach.** Introduced in the Euclidean setting by Federer [Fed59], sets of positive reach have been studied in the Riemannian setting by Kleinjohann [Kle81] and Bangert [Ban82]. Given a closet set $\Gamma\subset M$, the **set of unique projection over** Γ is defined as

$$\mathrm{UP}(\Gamma)=\left\{x\in M:\mathcal{H}^0(\xi_\Gamma(x))=1\right\}.$$

Given $a\in\Gamma$, we denote by $\mathrm{reach}(\Gamma,a)$ the supremum of those $r\ge0$ such that $B(a,r)\subseteq\mathrm{UP}(\Gamma)$, and say that Γ is a **set of locally positive reach** if $\mathrm{reach}(\Gamma,a)>0$ for each $a\in\Gamma$, and is a **set of positive reach** if $\mathrm{reach}(\Gamma,\cdot)$ is uniformly positive on Γ.

Remark 2.1. If Γ is closed and $\rho_\Gamma(x,\eta)\ge\tau_0>0$ for $\mathcal{H}^{n-1}$ a.e. $(x,\eta)\in\mathcal{N}^1\Gamma$, then by Theorem 2.1-(vii),

$$\{x\in M:0<\mathrm{dist}(x,\Gamma)<\tau_0\}\subset\mathcal{U}(\Gamma)\subseteq\mathrm{UP}(\Gamma).$$

In particular, Γ is of positive reach, with $\mathrm{reach}(\Gamma,\cdot)\ge\tau_0$ on Γ.

The following result, contained in [Kle81], is crucial in obtaining (4.50) and (4.51) in the proof of Theorem 4.1.

Theorem 2.2 (Kleinjohann)**.** *If $A\subseteq M$ is a set of positive reach with compact boundary, then there exists $\varepsilon(A)>0$ such that for every $t\in(0,\varepsilon(A))$ the set*

$$A_t=\{x\in M:\mathrm{dist}(A,x)=t\}$$

is a compact $C^{1,1}$-hypersurface contained in $\mathrm{UP}(A)$*, and the geodesic-flow map $\Phi_t:\mathcal{N}^1(A)\to A_t$, defined by $\Phi_t(a,\eta)=\exp(a,t\eta)$ for $(a,\eta)\in\mathcal{N}^1(A)$, is bi-Lipschitz on $\mathcal{N}^1(A)$. In particular, $\mathcal{N}^1(A)$ is an $(n-1)$-dimensional compact Lipschitz submanifold of TM.*

Proof. Let $\mathcal{U}$ be an open neighborhood of the null section of TM and $\mathcal{V}$ an open subset of $M\times M$ such that the map $\mathcal{U}\ni(a,v)\mapsto(a,\exp(a,v))$ is a diffeomorphism of $\mathcal{U}$ onto $\mathcal{V}$. Let Φ be its inverse, so that $\exp(p,\Phi(p,q))=q$ for every $(p,q)\in\mathcal{V}$. As explained in [Kle81, middle of page 336], one can choose for each $a\in A$ a number $\zeta(a)>0$[3] so that

$$B(a,\zeta(a))\times B(a,\zeta(a))\subseteq\mathcal{V}\quad\text{for } a\in A$$

and, defining $W:=\bigcup_{a\in A}B(a,\zeta(a))$, we have that for every $x\in W$ there exists a unique minimizing geodesic joining x and A and ξ_A is locally lipschitz on W (see [Kle81, Satz (2.5)]). For $x\in W\setminus A$ we set

$$\nu(x)=\frac{\Phi(x,\xi_A(x))}{|\Phi(x,\xi_A(x))|}\quad\text{and}\quad\eta(x)=\frac{\Phi(\xi_A(x),x)}{|\Phi(\xi_A(x),x)|}.$$

It follows from [Kle81, Satz (2.1) and Satz (2.3)] that $\mathrm{dist}(A,\cdot)$ is continuously differentiable on $W\setminus A$, with $\nabla\mathrm{dist}(A,x)=-\nu(x)$ for $x\in W\setminus A$. In particular

[3]Denoted with $z'(a,\epsilon)$ in [Kle81, page 336].

$\nabla\mathrm{dist}(\cdot, A)$ is locally Lipschitz on $W \setminus A$. Since $\mathrm{dist}(A, x) = |\Phi(\xi_A(x), x)|$ for $x \in W$, it follows that

$$\Psi_t(\xi_A(x), \eta(x)) = \exp(\xi_A(x), \Phi(\xi_A(x), x)) = x$$

for $x \in W \cap A_t$. By applying the Lebesgue covering lemma to $\{B(a, \zeta(a)) : a \in \partial A\}$, we find a positive number $\varepsilon(A) > 0$ such that $\{x \in M : \mathrm{dist}(A, x) < \varepsilon(A)\} \subseteq W$, and thus conclude the proof. □

Following a standard convention (see, e.g. [Gro93]) we say that $p \in M \setminus \Gamma$ is a **critical point for** $\mathrm{dist}(\cdot, \Gamma)$ if for every $v \in T_p(M)$, $v \neq 0$, there exists $a \in \Gamma$ with $d(p, a) = \mathrm{dist}(p, \Gamma)$ and a segment $\gamma : [0, d(p, a)] \to M$ such that $\gamma(0) = p$, $\gamma(d(p, a)) = a$ and $\mathrm{angle}(v, \gamma'(0)) \leq \pi/2$. Correspondingly, $\tau > 0$ is a **regular value for** $\mathrm{dist}(\Gamma, \cdot)$ if there are no critical points p of $\mathrm{dist}(\cdot, \Gamma)$ with $\mathrm{dist}(p, \Gamma) = \tau$. The following result is a special case of the main result obtained by Bangert in [Ban82], and plays an important role in our analysis (see the proof of (4.48)).

Theorem 2.3 (Bangert)**.** *If $\Gamma \subseteq M$ is compact and $\tau > 0$ is a regular value of $\mathrm{dist}(\cdot, \Gamma)$, then $\{x \in M : \mathrm{dist}(x, \Gamma) \geq \tau\}$ is a set of positive reach.*

Proof. By [Man03, Proposition 3.4], $f = -\mathrm{dist}(\Gamma, \cdot)$ is locally semiconvex on $M \setminus \Gamma$. In particular, f belongs to the class $\mathcal{F}(M \setminus \Gamma)$ introduced by Bangert in [Ban82]. Moreover, by [RZ12, Lemma 5.5], $p \in M \setminus \Gamma$ is a regular point of f if and only if there exists $v \in T_pM$ such that[4] $\partial_p f(v) = \lim_{t\to 0^+}(f(p + t\,v) - f(p))/t$ is negative. It follows that all points of $f^{-1}(-\tau)$ are regular in the sense of [Ban82, Definition (i)], so that $f^{-1}((-\infty, -\tau])$ is a set of locally positive reach by the main theorem of [Ban82]. Since (M, g) is complete, $f^{-1}(-\tau) = \partial[f^{-1}((-\infty, -\tau])]$ is compact, and thus we conclude by the general fact that, if $A \subseteq M$ is a set of locally positive reach and ∂A is compact, then A is a set of positive reach (for example because, by [Kle81, Lemma 1.1], $\mathrm{reach}(A, \cdot)$ is continuous on A). □

3. A Lusin-type property of White's (m, λ)-sets

In Section 3.1 we recall a viscosity formulation of the notion of "being m-dimensional with mean curvature vector bounded by λ" for closed subsets of a Riemannian manifold (M, g), as introduced by White in [Whi16]. Then, in Section 3.2, we extend from the Euclidean to the Riemannian setting a "Lusin condition" for normal bundles proved in [San20b].

3.1. White's (m, λ)-sets.

Given an integer $m \in \{1, ..., n-1\}$ and a constant $\lambda \geq 0$, we say that a closed subset Γ of a Riemannian manifold (M, g) is a **White (m, λ)-set** in (M, g) if, for every $f \in C^2(M)$ such that $f|_\Gamma$ admits a local maximum at $x \in \Gamma$, it holds that

$$\mathrm{trace}_m(\mathrm{D}^2 f(x)) \leq \lambda |\nabla f(x)| \,. \tag{3.1}$$

Here $\mathrm{trace}_m(\mathrm{D}^2 f(x)) = \lambda_1 + \cdots + \lambda_m$ if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $\mathrm{D}^2 f(x)$ listed in increasing order. A fundamental result concerning White (m, λ)-sets relates condition (3.1) to the notion of *distributional* mean curvature for a varifold. This theorem plays a key role in our analysis (specifically, it allows us to use Theorem 4.1 in proving Theorem 4.2, see the next section).

[4] The function f being semiconcave, it may fail to have a differential at p. However, the limit $\partial_p f(v)$ will exist for every p and v. Here are we using the same notation found in [Ban82].

Theorem 3.1 (White)**.** *If $\lambda \geq 0$, V is an m-dimensional varifold in (M, g), and $\vec{H}$ is a Borel vector field in M such that $\|\vec{H}_V\|_{L^\infty(\operatorname{spt}\|V\|)} \leq \lambda$ and*

$$\int (\operatorname{div}^{\tau} X)(x)\, dV(x, \tau) = -\int_M \langle \vec{H}, X \rangle_g \, d\|V\| ,$$

for every $X \in \mathcal{X}(M)$, then $\operatorname{spt}\|V\|$ is a White (m, λ)-set in (M, g).

Proof. This is [Whi16, Corollary 2.8]. □

We shall also need the following simple fact:

Lemma 3.2. *If (M, g) is a Riemannian manifold, Γ is a White (m, λ)-set in (M, g), $\varphi \in C^\infty(M)$, $g^* = e^{2\varphi} g$, and*

$$C(\Gamma, \varphi) = \sup_{\Gamma} |\nabla \varphi| < \infty ,$$

then Γ is a White $(m, \lambda + 3\, m\, C(\Gamma, \varphi))$-set in (M, g^*).

Proof. Denoting with ∇^* and D^2_* the Riemannian connection and the Hessian operator with respect to g^*, it is enough to prove that

$$\mathrm{D}^2_* f(x) \leq \mathrm{D}^2 f(x) + 3\, |\nabla \varphi(x)|\, |\nabla f(x)|\, g_x \tag{3.2}$$

whenever $x \in O$, O is open in M, and $f \in C^2(O)$. To this end, by [Sak96, II, Proposition 3.9], we compute

$$\begin{aligned} \mathrm{D}^2_* f(U, V) = U(V(f)) - (\nabla^*_U V)(f) = \mathrm{D}^2 f(U, V) &- g(\nabla \varphi, U) g(\nabla f, V) \\ -g(\nabla \varphi, V) g(\nabla f, U) &- g(U, V) g(\nabla \varphi, \nabla f) , \end{aligned}$$

whenever $U, V \in \mathcal{X}(M)$. Hence,

$$\mathrm{D}^2_* f(U, U) \leq \mathrm{D}^2 f(U, U) + 3 |\nabla f| |\nabla \varphi| |U|^2 .$$

If now O is a neighborhood of some $x \in \Gamma$ and $f|_\Gamma$ has a local maximum at x, then this last inequality, combined with [Whi16, Lemma 12.3] and the fact that Γ is a White (m, λ)-set in (M, g), implies that

$$\begin{aligned} \operatorname{trace}_m(\mathrm{D}^2_* f(x)) &\leq \operatorname{trace}_m\big(\mathrm{D}^2 f(x) + 3\, |\nabla f(x)|\, |\nabla \varphi(x)|\, g_x\big) \\ &= \operatorname{trace}_m(\mathrm{D}^2 f(x)) + 3\, m\, |\nabla f(x)|\, |\nabla \varphi(x)| \\ &\leq (\lambda + 3\, m\, C(\Gamma, \varphi))\, |\nabla f(x)| , \end{aligned}$$

thus completing the proof. □

3.2. **A Lusin-type property of normal bundles.** The following theorem is needed at a crucial step (4.29) in the proof of Theorem 4.1.

Theorem 3.3 (Lusin-type property of normal bundles)**.** *Suppose $\Gamma \subseteq M$ is a White (m, λ)-set in (M, g) such that Γ is a countable union of sets with finite $\mathcal{H}^m$-measure. Then the following implication holds:*

$$Z \subseteq \Gamma,\ \mathcal{H}^m(Z \cap \Gamma^{(m)}) = 0 \quad \Longrightarrow \quad \begin{cases} \mathcal{H}^n\big(\mathcal{N}(\Gamma) \llcorner Z\big) = 0 , \\ \mathcal{H}^{n-1}(\mathcal{N}^1(\Gamma) \llcorner Z) = 0 . \end{cases} \tag{3.3}$$

Proof. By [San20b, 3.3, 3.8, 3.9], if $W \subseteq \mathbb{R}^k$ is open, Γ is a White (m, λ)-set in W, and Γ is a countable union of sets with finite $\mathcal{H}^m$-measure, then

$$Z \subseteq \Gamma,\ \mathcal{H}^m(Z \cap \Gamma^{(m)}) = 0 \quad \Longrightarrow \quad \begin{cases} \mathcal{H}^k\big(\mathcal{N}(\Gamma) \llcorner Z\big) = 0 , \\ \mathcal{H}^{k-1}\big(\mathcal{N}(\Gamma) \llcorner Z\big) = 0 . \end{cases} \tag{3.4}$$

We now reduce the proof of (3.3) to an application of the Euclidean case (3.4). To this end, by the Nash embedding theorem, we can directly assume that M is an n-dimensional embedded submanifold of the Euclidean space $\mathbb{R}^k$ (for some large k) and g is the Riemannian metric on M induced by the Euclidean metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^k$. If $X \subseteq M$ is relatively closed in M, then we denote by $\widetilde{\mathcal{N}}\overline{X}$ the normal bundle of the closure in $\mathbb{R}^k$ of X, while we keep the symbol $\mathcal{N}X$ for the normal bundle of X as a subset of (M, g) (notice $\mathcal{N}X \subseteq TM$). If $a \in M$, then π_a and $\pi_a^\perp$ denote the orthogonal projections of $\mathbb{R}^k$ onto T_aM and $T_a^\perp M$ respectively. We now divide the rest of the proof into two claims and one final argument.

Claim one: If $X \subseteq M$ is relatively closed in M, then

$$\widetilde{\mathcal{N}}_a\overline{X} = \mathcal{N}_aX + T_a^\perp M\,, \qquad \forall a \in X\,.$$

Indeed, let $u \in \widetilde{\mathcal{N}}_a\overline{X}$ be such that $\pi_a(u) \neq 0$ and $B \cap \overline{X} = \varnothing$, where, for some $r > 0$, B is the open Euclidean ball in $\mathbb{R}^k$ of radius $r|u|$ centered at $a + ru$. Since $u \in T_a^\perp(\partial B)$ and $\langle \pi_a(u), u\rangle > 0$, we conclude that $\pi_a(u) \in T_aM \setminus T_a(\partial B)$. Then we choose an open neighborhood V of a and a continuous function $\eta : V \cap M \to \mathbf{S}^{k-1}$ such that $\eta(a) = \pi_a(u)/|\pi_a(u)|$ and $\eta(b) \in T_bM$ for every $b \in V \cap M$. Since $[b \in V \cap M \cap \partial B] \mapsto \operatorname{dist}(\eta(b), T_b(\partial B))$ is continuous and $\operatorname{dist}(\eta(a), T_a(\partial B)) > 0$, there is an open neighborhood $W \subseteq V$ of a such that $\operatorname{dist}(\eta(b), T_b(\partial B)) > 0$ for every $b \in W \cap \partial B \cap M$. Hence, $\dim\big[T_bM \cap T_b(\partial B)\big] \leq n-1$ and $T_bM + T_b(\partial B) = \mathbb{R}^k$ for every $b \in W \cap M \cap \partial B$. This means that the submanifolds $W \cap \partial B$ and $W \cap M$ are transversal and consequently

$$\partial B \cap M \cap W \quad \text{is an } n-1\text{-dimensional smooth submanifold of } M \tag{3.5}$$

with $T_bM \cap T_b(\partial B) = T_b(M \cap \partial B)$ for $b \in W \cap M \cap \partial B$; see [GP74, pp. 29-30]. Now, observing that if $v \in T_a(\partial B \cap M) = T_a(\partial B) \cap T_a(M)$, then $\langle \pi_a^\perp(u), v\rangle = 0$ and $\langle \pi_a(u), v\rangle = \langle u, v\rangle = 0$, we find that

$$\pi_a(u) \in T_a^\perp(\partial B \cap M)\,. \tag{3.6}$$

For t sufficiently small, since $\langle \pi_a(u), u\rangle > 0$, we notice that $\exp(a, t\pi_a(u)) \in B \cap M$ and we conclude from (3.6) that the open geodesic ball of (M, g) centred at $\exp(a, t\pi_a(u))$ and radius $t|\pi_a(u)|$ is contained in $B \cap M$. This means, always for t sufficiently small, that $d(\exp(a, t\pi_a(u)), X) = t|\pi_a(u)|$ (thanks to $B \cap X = \varnothing$) and $\pi_a(u) \in \mathcal{N}_a(X)$.

We have thus proved $\widetilde{\mathcal{N}}_a\overline{X} \subseteq \mathcal{N}_aX + T_a^\perp M$ for $a \in X$. To prove the opposite inclusion, let now $v \in \mathcal{N}_aX$ with $v \neq 0$. An open geodesic ball G in M with sufficiently small radius is such that ∂G is a smooth $n-1$ dimensional submanifold in $\mathbb{R}^k$, $G \cap X = \varnothing$, $a \in \partial G$, and v is an interior normal of G at a. Then there exists $r > 0$ such that the open Euclidean ball B in $\mathbb{R}^k$ of radius r centered at $a + rv$ satisfies $B \cap \partial G = \varnothing$ and $B \cap G \neq \varnothing$. Choosing r smaller if necessary, we can also ensure that B is contained in the tubular neighbourhood of M where the nearest point projection onto M is single valued. This means that $B \cap M$ is a connected subset of $M \setminus \partial G$; since G is a connected component of $M \setminus \partial G$ and $G \cap B \neq \varnothing$, we infer that $B \cap M \subseteq G$. The latter inclusion implies that $B \cap X = \varnothing$. It follows that $B \cap \overline{X} = \varnothing$ and $\mathcal{N}_aX \subseteq \widetilde{\mathcal{N}}_a\overline{X}$. Now the convexity of $\widetilde{\mathcal{N}}_a\overline{X}$ and the obvious inclusion $T_a^\perp M \subseteq \widetilde{\mathcal{N}}_a\overline{X}$ allow to conclude $\mathcal{N}_aX + T_a^\perp M \subseteq \widetilde{\mathcal{N}}_a\overline{X} + \widetilde{\mathcal{N}}_a\overline{X} = \widetilde{\mathcal{N}}_a\overline{X}$ and to complete the proof of claim one.

Claim two: If $W \subseteq \mathbb{R}^k$ is an open set such that $\overline{W} \cap M$ is compact, then there is $\lambda_W \geq 0$ such that $\Gamma \cap W$ is a White (m, λ_W)-set in W. Firstly, notice that $M \cap W$ and $\Gamma \cap W$ are relatively closed in W. Now, let $b \in \Gamma \cap W$ and $f \in C^2(W)$ such that $f|_\Gamma$ has a local maximum at b. Denoting by $\overline{\nabla}$ both the Euclidean gradient operator and metric connection, and by $\overline{\mathrm{D}}^2$ the Euclidean Hessian operator, we set $\eta(x) = \pi_x^\perp(\overline{\nabla} f(x))$ for $x \in W \cap M$, notice that $\pi_x(\overline{\nabla} f(x)) = \nabla f(x)$, and compute

$$\begin{aligned}\overline{\mathrm{D}}^2 f(x)(v,w) &= \langle \overline{\nabla}_v \overline{\nabla} f(x), w\rangle = \langle \overline{\nabla}_v \nabla f(x), w\rangle + \langle \overline{\nabla}_v \eta(x), w\rangle \\ &= \mathrm{D}^2 f(x)(v,w) + \langle A_{\eta(x)}(v), w\rangle = \mathrm{D}^2 f(x)(v,w) - Q(x)(v,w)\end{aligned}$$

for every $x \in M \cap W$ and $v, w \in T_x(M)$, where $A_{\eta(x)} : T_x M \to T_x$ is the shape operator of M in the direction $\eta(x)$, $Q(x) : T_x M \times T_x M \to \mathbb{R}$ is the symmetric bilinear form defined as $Q(x)(v,w) = \langle S(v,w), \eta(x)\rangle$ and S is the second fundamental form of M. If $Q(b)(v,v) \geq 0$ for all $v \in T_b M$ then we obtain from [JT03, Lemma 2.3] and [Whi16, Lemma 12.3]

$$\begin{aligned}\mathrm{trace}_m(\overline{\mathrm{D}}^2 f(b)) &\leq \mathrm{trace}_m\big[\overline{\mathrm{D}}^2 f(b)|T_b M \times T_b M\big] \\ &= \mathrm{trace}_m[\mathrm{D}^2 f(b) - Q(b)] \\ &\leq \mathrm{trace}_m(\mathrm{D}^2 f(b)) \leq \lambda|\nabla f(b)| \leq \lambda|\overline{\nabla} f(b)|.\end{aligned}$$

Now we assume that $Q(b)(v,v) < 0$ for some $v \in T_b M$. Then we define

$$\mu_W = \inf\{\langle S(u,u), \nu\rangle : x \in \overline{W} \cap M,\ u \in T_x M,\ \nu \in (T_x M)^\perp,\ |u| = |\nu| = 1\}$$

and we notice that $-\infty < \mu_W < 0$ and

$$Q(b)(v,v) \geq \mu_W |\eta(b)| \langle v, v\rangle \qquad \text{for } v \in T_b M.$$

Therefore, by [JT03, Lemma 2.3] and [Whi16, Lemma 12.3]

$$\begin{aligned}\mathrm{trace}_m(\mathrm{D}^2 f(b)) &= \mathrm{trace}_m\big[\overline{\mathrm{D}}^2 f(b)|(T_b M \times T_b M) + Q(b)\big] \\ &\geq \mathrm{trace}_m\big[\big(\overline{\mathrm{D}}^2 f(b) + \mu_W|\eta(b)|\langle\cdot,\cdot\rangle\big)\big|T_b M \times T_b(M)\big] \\ &\geq \mathrm{trace}_m\big[\overline{\mathrm{D}}^2 f(b) + \mu_W|\eta(b)|\langle\cdot,\cdot\rangle\big] \\ &= \mathrm{trace}_m(\overline{\mathrm{D}}^2 f(b)) + m\mu_W|\eta(b)|,\end{aligned}$$

and we deduce that

$$\mathrm{trace}_m(\overline{\mathrm{D}}^2 f(b)) \leq \lambda|\nabla f(b)| - m\mu_W|\eta(b)| \leq (\lambda - m\mu_W)|\overline{\nabla} f(b)|.$$

Conclusion of the proof: Let $Z \subseteq \Gamma$ such that $\mathcal{H}^m(Z \cap \Gamma^{(m)}) = 0$. Since $\mathcal{N}_a(\Gamma) \subseteq T_a M$ it follows from claim one that

$$\dim \widetilde{\mathcal{N}}_a \overline{\Gamma} = \dim \mathcal{N}_a \Gamma + \dim T_a^\perp M \quad \text{for } a \in \Gamma,$$

so that

$$\mathcal{H}^m(Z \cap \{a : \dim \widetilde{\mathcal{N}}_a \overline{\Gamma} = k - m\}) = 0.$$

It follows from claim two and (3.4) that $\mathcal{H}^k(\widetilde{\mathcal{N}}\,\overline{\Gamma} \llcorner Z) = 0$. Let $P : M \times \mathbb{R}^k \to TM$ be the smooth map defined by

$$P(a,u) = (a, \pi_a(u)) \quad \text{for } (a,u) \in M \times \mathbb{R}^k.$$

Since $P(\widetilde{\mathcal{N}}\,\overline{\Gamma}\llcorner\Gamma) = \mathcal{N}\,\Gamma$ by claim one, noting by Theorem 2.1 that $\widetilde{\mathcal{N}}\,\overline{\Gamma}\llcorner\Gamma$ is a countably k-rectifiable subset of $M \times \mathbb{R}^k$ and $\mathcal{N}\,\Gamma$ is a countably n-rectifiable subset of TM. By the coarea formula [Fed69a, 3.2.22],

$$0 = \int_{\widetilde{\mathcal{N}}\,\overline{\Gamma}\llcorner Z} \mathrm{ap}J_n P \, d\mathcal{H}^k = \int_{\mathcal{N}\Gamma\llcorner Z} \mathcal{H}^{k-n}(T_a^{\perp} M) \, d\mathcal{H}^n_{(a,u)} \, .$$

Since $\mathcal{H}^{k-n}(T_a^{\perp} M) = +\infty$ for every $a \in M$, we have $\mathcal{H}^n(\mathcal{N}\Gamma\llcorner Z) = 0$. □

4. Viscous Heintze–Karcher inequalities

In Theorem 4.1 below, we prove the Heintze–Karcher inequalities of Brendle [Bre13], and address their equality cases, in the viscous setting of White [Whi16]. Starting from this result, in Theorem 4.2, we extend Brendle's rigidity theorem to the distributional setting. Throughout the section, $n \geq 3$ and (M, g) is a Riemannian manifold which satisfies at least (H0)–(H3). We consider $\Sigma \subset M$ such that:

(A1) Σ is a smooth embedded hypersurface in M with $\overline{\Sigma} \subset M^{\circ}$ and

$$\mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0 \, , \qquad \mathcal{H}^{n-1}(\Sigma) < \infty \, .$$

Notice carefully that *we do not assume Σ to be closed*. Thus, $\overline{\Sigma} \setminus \Sigma$ may be non-empty and may contain singular points (i.e., $\overline{\Sigma}$ may fail to be an hypersurface at points in $\overline{\Sigma} \setminus \Sigma$), and Σ may consists of countably many connected components. Our second main assumption is that $\overline{\Sigma}$ is (topologically) a boundary, namely,

(A2) there is $\Omega \subset M$ open such that

$$\text{either } \partial\overline{\Omega} = \overline{\Sigma} \text{ or } \partial\overline{\Omega} = \overline{\Sigma} \cup N_0, \tag{4.1}$$

Now, under (4.1), assumption (A1) implies that Ω is a set of finite perimeter in M thanks to Federer's criterion, see [Fed69b, 4.5.12]. In particular, if we denote by $\partial^*\Omega$ the reduced boundary of Ω, and by ν_Ω its measure theoretic outer g-unit normal, then we observe that $\Sigma \subset \partial^*\Omega$ and ν_Ω is smooth on Σ. It thus makes sense to define

$$H_\Sigma = \vec{H}_\Sigma \cdot \nu_\Omega \quad \text{on } \Sigma,$$

where $\vec{H}_\Sigma$ is the mean curvature vector of Σ in (M, g).

Theorem 4.1. *If $n \geq 3$, (M, g) satisfies* (H0)-(H3)*, and the pair (Σ, Ω) satisfies assumptions* (A1), (A2)*, and*

(A3) *for some $\lambda \geq 0$, $\overline{\Sigma}$ is a White $(n-1, \lambda)$-set in M,*

then, denoting by r the projection of $M = N \times (0, \bar{r})$ over $(0, \bar{r})$, and setting

$$f = h' \circ r \, , \qquad g^* = f^{-2}\, g \, ,$$

the following statements hold:

(a) *if $\partial\Omega = \overline{\Sigma}$, then*

$$(n-1) \int_\Sigma \frac{f}{H_\Sigma} \, d\mathcal{H}^{n-1} \geq n \int_\Omega f \, d\mathcal{H}^n \, ; \tag{4.2}$$

(b) *if $\partial\Omega = \overline{\Sigma} \cup N_0$, then*

$$(n-1) \int_\Sigma \frac{f}{H_\Sigma} \, d\mathcal{H}^{n-1} \geq n \int_\Omega f \, d\mathcal{H}^n + h(0)^n \mathrm{vol}\,(N, g_N) \, ; \tag{4.3}$$

(c) *if either (a) or (b) holds with equality, then Σ is umbilic in (M,g), $M\setminus\Omega$ has positive reach in (M,g^*), and*[5]

$$\Big(2\,\frac{h''}{h}+(n-2)\,\frac{((h')^2-\rho)}{h^2}\Big)'\,g_N(\nu_\Omega,\nu_\Omega)=0\,,\qquad \textit{on }\Sigma\,. \tag{4.4}$$

Theorem 4.2. *If $n\ge 3$, $(M,g)\in\mathcal{B}_n\cup\mathcal{B}_n^*$, and the pair (Σ,Ω) satisfies assumptions* (A1)*,*

(A2)' *there is $\Omega\subset M$ open such that either $\partial\overline{\Omega}=\overline{\Sigma}$ or $\partial\overline{\Omega}=\overline{\Sigma}\cup N_0$;*

(A3)' *there is $H_0\ge 0$ such that, for every $Y\in\mathcal{X}(M)$,*

$$\int_\Sigma \mathrm{div}^{\,\Sigma} Y\,d\mathcal{H}^{n-1}=H_0\int_\Sigma\langle\nu_\Omega,Y\rangle\,d\mathcal{H}^{n-1}\,; \tag{4.5}$$

then, for some $t_0\in(0,\bar r)$, $\Omega=N\times(0,t_0)$ and $H_0=(n-1)\,h'(t_0)/h(t_0)>0$.

Remark 4.1 (On the relation between Theorem 4.1 and Theorem 4.2)**.** As detailed in the proof of Theorem 4.2, by testing the constant mean curvature condition (4.5) with the vector field $h\,(\partial/\partial r)$, we see that H_0 appearing in (4.5) is positive (so that (A2)' implies (A2)), and that either (4.2) or (4.3) (depending on whether $\partial\Omega=\overline{\Sigma}$ or $\partial\Omega=\overline{\Sigma}\cup N_0$) *holds as an identity.* Moreover, by Theorem 3.1, (A3)' implies the validity of (A3). Therefore, under the assumptions of Theorem 4.2, conclusion (c) of Theorem 4.1 holds too. When (H3)* holds, then (4.4) immediately implies that $\nu_\Omega(p)$ is parallel to $(\partial/\partial r)|_p$ at every $p\in\Sigma$: this information, combined with standard facts on sets of finite perimeter and with the positivity of H_Σ, immediately implies that Ω is bounded by a single slice. When, instead, only (H3) is assumed, the information in (4.4) may be trivial. In this second case, arguing as in [Bre13], we deduce from umbilicality and the Codazzi equations that $\nu_\Omega(p)$ is an eigenvector of $(\mathrm{Ric}_M)|_p$ at every $p\in\Sigma$. Since (H4) implies that $(\partial/\partial r)|_p$ is a simple eigenvector of $(\mathrm{Ric}_M)|_p$, we thus find that, at each $p\in\Sigma$, $\nu_\Omega(p)$ is either parallel *or orthogonal* to $(\partial/\partial r)|_p$. Concluding rigidity from this weaker information using only standard facts on sets of finite perimeter does not seem immediate; however, the fact that $M\setminus\Omega$ has positive reach in (M,g^*) can be exploited to quickly reach the desired conclusion.

Proof of Theorem 4.1. Preparation of M: The results of Section 2 and Section 3 require the completeness of the ambient manifold. Notice that (M,g) is not complete. A first problem is that geodesics in (M,g) may arrive in finite time to the horizon N_0: this issue is fixed by passing from g to $g^*=f^{-2}\,g$. A different issue, however, is the behavior of geodesics near the $\bar r$-end of M. To fix this second problem we argue as follows. By assumption $\overline{\Sigma}\subset M^\circ$, there is $(a,b)\subset\subset(0,\bar r)$ such that

$$\overline{\Sigma}\subset N\times\Big[a,\frac{a+b}{2}\Big]\,,\qquad \Omega\subset N\times\Big(0,\frac{a+b}{2}\Big)\,. \tag{4.6}$$

Correspondingly we can consider a smooth positive function $h_b:[0,+\infty)\to\mathbb{R}$ so that $h_b=h$ on $[0,b]$, $h_b'>0$ on $(0,\infty)$, and

$$\sup_{[0,\infty)} h_b'<\infty\,,\qquad \int_b^\infty\frac{dt}{h_b'(t)}=+\infty\,.$$

[5]Here, given $\nu=(\tau,a)\in T_{(x,t)}M\equiv T_xN\times\mathbb{R}$, we have set $(g_N)|_{(x,t)}(\nu,\nu)=(g_N)_x(\tau,\tau)$.

We introduce the metrics $g_b = dr \otimes dr + h_b(r)^2 g_N$ and $g_b^* = f_b^{-2}\, g_b$ on $N \times (0,\infty)$, where $f_b(p) = h_b'(r(p))$. Since (H1) and $h_b = h$ on $[0,b]$ imply

$$\int_0^b \frac{dt}{h_b'(t)} = +\infty\,,$$

we easily see that g_b^*-geodesic balls centered at points $p \in N \times (0,+\infty)$ are contained in compact slabs of the form $N \times [s,t]$ $(0 < s < t < \infty)$. In particular, by the Hopf–Rinow theorem, $(N \times (0,\infty), g_b^*)$ is a *complete* Riemannian manifold. Notice that h_b satisfies assumption (H3) on $(0,b)$, where it coincides with h, but, possibly, not on $(0,\infty)$.

We have thus reduced to the following situation: $M = N \times (0,\infty)$; the metric $g = dr \otimes dr + h(r)^2\, g_N$ is such that (H0) and (H1) hold, (H2) holds on $(0,\infty)$ (i.e., $h' > 0$ on $(0,\infty)$), (H3) holds on $(0,b)$, and

$$\sup_M f < \infty\,; \tag{4.7}$$

the metric $g^* = f^{-2}\, g$ is such that (M, g^*) is a complete Riemannian manifold; and, finally, Σ and Ω satisfy (4.6) in addition to assumptions (A1), (A2), and (A3).

The "vertical" vector field X: The shortest path between points (x,t_1) and (x,t_2) in (M,g^*) with $t_1 \in (0,t_2)$ is given by $s \in [t_1,t_2] \mapsto (x,s)$ (while $\mathrm{dist}_{g^*}((x,0),(x,t)) = +\infty$ for every $x \in N$ and $t > 0$). "Vertical" segments are thus length minimizing geodesics in (M,g^*), and the vector field $\partial/\partial r$ has a special role in the geometry of (M,g^*). It is also convenient to consider, alongside with $\partial/\partial r$, its rescaled version

$$X = h\,\frac{\partial}{\partial r}\,.$$

Simple computations show that

$$\mathrm{div}\,(\partial/\partial r) = (n-1)\,\frac{f}{h \circ r}\,, \tag{4.8}$$

$$\mathrm{div}^{\Gamma}(\partial/\partial r) = \Big\{(n-1) - \sum_{i=1}^{n} \langle \sigma_i, \partial_n \rangle_g^2\Big\}\,\frac{f}{h \circ r}\,, \qquad \text{on } \Gamma\,, \tag{4.9}$$

$$\mathrm{div}\; X = n\,f \quad \text{on } M\,, \qquad \mathrm{div}^{\Gamma} X = (n-1)\,f \quad \text{on } \Gamma\,, \tag{4.10}$$

whenever Γ is a C^1-hypersurface and $\{\sigma_i\}_{i=1}^{n-1}$ denotes a g-orthonormal basis of $T_p\Gamma$ for some $p \in \Gamma$. An immediate consequence of (4.9) is that if Γ is a *closed* $C^{1,1}$-hypersurface in M° with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and with mean curvature vector $\vec{H}_\Gamma$ in (M,g), and if $h'' \circ r \geq 0$ on Γ, then

$$(n-1)\,\mathcal{H}^{n-1}(\Gamma) \geq \int_\Gamma \frac{\langle \vec{H}_\Gamma, X \rangle_g}{f}\, d\mathcal{H}^{n-1}\,. \tag{4.11}$$

Indeed, by (4.10) we find

$$\mathrm{div}^{\Gamma}\Big(\frac{X}{f}\Big) = (n-1) - \frac{\langle \nabla^\Gamma f, X \rangle_g}{f^2}\,,$$

where $\nabla f = [(h''/h) \circ r]\,X$. Denoting by X^Γ the projection of X along $T\Gamma$, we find $\langle X, \nabla^\Gamma f \rangle_g = [(h''/h') \circ r]\,|X^\Gamma|_g^2$. Hence, by applying the divergence theorem to X/f on Γ and by using $h'' \circ r \geq 0$ on Γ, we find (4.11).

"Immersed" geodesic flow from Σ: The vector field

$$\nu_\Omega^* = f\,\nu_\Omega$$

is a smooth g^*-unit normal to Σ, pointing out of Ω. Now, denoting by $\exp^*$ the exponential map in the complete Riemannian manifold (M,g^*), we can define a smooth map $\Phi:\Sigma\times(0,\infty)\to M$ by setting

$$\Phi(x,t)=\Phi_t(x)=\exp^*(x,-t\,\nu_\Omega^*(x))\,,\qquad x\in\Sigma\,,t>0\,.$$

In this way, denoting by $J^\Sigma\Phi_t$ the tangential Jacobian[6] of Φ_t along Σ,

$$\Phi(x,0)=x\,,\quad \frac{\partial\Phi}{\partial t}(x,0)=-f(x)\,\nu_\Omega(x)\,,\quad J^\Sigma\Phi_0(x)=1\,,\qquad \forall x\in\Sigma\,. \tag{4.12}$$

By (4.6) and (4.12), we have that $\Phi_t(x)\in N\times(0,b)$ with $J^\Sigma\Phi_t(x)>0$ for every t small enough. We can thus define a lower semicontinuous, positive function $R_\Sigma:\Sigma\to(0,\infty]$ by setting

$$\begin{aligned}R_\Sigma(x) \;=\; \min\Big\{&\inf\big\{t>0:J^\Sigma\Phi_t(x)=0\big\},\\ &\inf\big\{t>0:\Phi_t(x)\notin N\times(0,b)\big\}\Big\}\,,\qquad x\in\Sigma\,,\end{aligned}$$

so to have

$$\begin{aligned}&\Phi_t(x)\in N\times(0,b)\quad\text{and}\quad J^\Sigma\Phi_t(x)>0\,,\\ &\forall(x,t)\in A_\Sigma:=\Big\{(x,t):x\in\Sigma\,,t\in(0,R_\Sigma(x))\Big\}\,.\end{aligned} \tag{4.13}$$

By the Gauss lemma (see, e.g., [Sak96, pag. 60]), for every $(x,t)\in A_\Sigma$,

$$\frac{\partial\Phi}{\partial t}(x,t)\in\Big(d\Phi_t(x)[T_x\Sigma]\Big)^\perp\,,\qquad \Big|\frac{\partial\Phi}{\partial t}(x,t)\Big|_g=f(\Phi_t(x))\,. \tag{4.14}$$

In particular, the tangential Jacobian $J^{A_\Sigma}\Phi$ of Φ along A_Σ is related to $J^\Sigma\Phi_t$ by the identity

$$(J^{A_\Sigma}\Phi)(x,t)=f(\Phi_t(x))\,J^\Sigma\Phi_t(x)\,,\qquad \forall(x,t)\in A_\Sigma\,. \tag{4.15}$$

We now notice that, for every $t\in(0,\infty)$, $\{R_\Sigma>t\}$ is an open subset of Σ, and

$$\Gamma_t=\Phi_t\big(\{R_\Sigma>t\}\big)\,,\qquad t>0\,,$$

is a smooth *immersed* hypersurface in M. Indeed, by construction, for every $(x,t)\in A_\Sigma$ there is an open neighborhood W of x in Σ such that $(\Phi_t)|_W$ is a smooth embedding. Correspondingly, we denote[7] by $H(x,t)$ and $\mathrm{II}(x,t)$ the scalar mean curvature and the second fundamental form (in the metric g) of the smooth hypersurface $\Phi_t(W)$ at the point $\Phi_t(x)$ and with respect to the normal

$$\nu(x,t)=-\frac{1}{f(\Phi_t(x))}\frac{\partial\Phi}{\partial t}(x,t), \tag{4.16}$$

(see (4.15)). Notice that $\nu(x,0)=\nu_\Omega(x)$ for $x\in\Sigma$. Since, by (4.13), $\Phi_t(x)\in N\times(0,b)$ for every $t\in(0,R_\Sigma(x))$, and since h satisfies (H3) on $(0,b)$, the pointwise

[6]Here and in the following, $\mathcal{H}^k$ and J always denote Hausdorff measures and Jacobians computed with respect to the metric g.

[7]Notice carefully that Φ may not be injective on the whole A_Σ, therefore we will not be able to consider H and II as functions on $\Phi(A_\Sigma)\subset M$.

calculations in [Bre13, Proposition 3.2] (which are based on (H0) and (H3) and the Riccati equation) can be repeated *verbatim* to show that, everywhere on A_Σ,

$$\frac{\partial}{\partial t}\Big(\frac{H}{f\circ\Phi}\Big) \ge |\mathrm{II}|^2 \ge \frac{H^2}{n-1}\,, \tag{4.17}$$

$$\frac{\partial}{\partial t}\Big(\frac{f\circ\Phi}{H}\Big) \le -|\mathrm{II}|^2\,\frac{(f\circ\Phi)^2}{H^2} \ \le -\frac{(f\circ\Phi)^2}{n-1}\,. \tag{4.18}$$

Since $H(\cdot,0)>0$ on Σ by assumption (A2), we see that (4.17) implies

$$H \text{ is positive on } A_\Sigma\,. \tag{4.19}$$

Moreover, we have that

$$\frac{\partial}{\partial t}\,J^\Sigma\Phi_t(x) = -[(f\circ\Phi)\,H](x,t)\,J^\Sigma\Phi_t(x)\,,\qquad \forall (x,t)\in A_\Sigma\,. \tag{4.20}$$

Indeed, given $(x,t)\in A_\Sigma$ and W as above, if W' is an arbitrary open subset of $W\subset\Sigma$ then, by the area formula,

$$\frac{d}{ds}\Big|_{s=t}\mathcal{H}^{n-1}(\Phi_s(W')) = \frac{d}{ds}\Big|_{s=t}\int_{W'} J^\Sigma\Phi_s\,d\mathcal{H}^{n-1} = \int_{W'}\frac{\partial}{\partial t}J^\Sigma\Phi_t\,d\mathcal{H}^{n-1}$$

while, by the formula for the first variation of the area and by (4.14)

$$\frac{d}{ds}\Big|_{s=t}\mathcal{H}^{n-1}(\Phi_s(W')) \ = \ -\int_{W'}[(f\circ\Phi)\,H](y,t)\,J^\Sigma\Phi_t(y)\,d\mathcal{H}^{n-1}_y\,,$$

so that (4.20) follows by arbitrariness of W'. We finally notice that

$$\begin{aligned}&\text{if } x\in\Sigma,\ R_\Sigma(x)<\infty, \text{ and } J^\Sigma\Phi_t(x)\to 0 \text{ as } t\to R_\Sigma(x)^-\,,\\ &\text{then } H(x,t)\to+\infty \text{ as } t\to R_\Sigma(x)^-\,.\end{aligned} \tag{4.21}$$

Indeed, (4.20) gives

$$\log\big(J^\Sigma\Phi_t(x)\big) = -\int_0^t[(f\circ\Phi)\,H](x,s)\,ds\,,\qquad \forall t\in(0,R_\Sigma(x))\,.$$

A refinement of (4.18): We claim that, everywhere on A_Σ,

$$\begin{aligned}-\Big\{\frac{\partial}{\partial t}\Big(\frac{f\circ\Phi}{H}\Big)+\frac{(f\circ\Phi)^2}{n-1}\Big\} \quad = \quad & (f\circ\Phi)^2\Big\{\frac{|\mathrm{II}|^2}{H^2}-\frac{1}{n-1}\Big\}\\ &+\frac{f\circ\Phi}{H^2}\,h'\,\big\{\mathrm{Ric}_N-\rho\,(n-2)\,g_N\big\}(\nu,\nu)\\ &+\frac{h^3}{2}\,\big(\mathcal{M}[h]\big)'\,g_N(\nu,\nu)\,,\end{aligned} \tag{4.22}$$

where, by definition,

$$\mathcal{M}[h] = 2\,\frac{h''}{h}-(n-2)\,\frac{\rho-(h')^2}{h^2}\,. \tag{4.23}$$

Indeed, setting $T=(\Delta f)\,g-\mathrm{D}^2 f+f\,\mathrm{Ric}_M$, by [Bre13, Proposition 2.1] we have that

$$T = h'\,\big\{\mathrm{Ric}_N-\rho\,(n-2)\,g_N\big\}+\frac{h^3}{2}\,\big(\mathcal{M}[h]\big)'\,g_N\,, \tag{4.24}$$

while the computations in [Bre13, Proposition 3.2] give

$$\frac{\partial}{\partial t}\Big(\frac{f\circ\Phi}{H}\Big) = (f\circ\Phi)^2\Big\{\frac{1}{n-1}-\frac{|\mathrm{II}|^2}{H^2}\Big\}-\frac{(f\circ\Phi)}{H^2}\,T(\nu,\nu)\,. \tag{4.25}$$

The combination of (4.24) and (4.25) leads immediately to (4.22).

Geodesic flow from Σ: We now consider the (embedded) geodesic flow of Σ (which is the main structure used in Brendle's argument), i.e. we relate Φ to the distance function from Σ in (M,g^*). Let us define, for the sake of brevity, $u^*_\Sigma:\overline{\Omega}\to[0,\infty)$ and $R^*_\Sigma:\Sigma\to(0,\infty)$ by setting

$$\begin{aligned} u^*_\Sigma(p) &= \mathrm{dist}_{g^*}(p,\Sigma)\,, && p\in\overline{\Omega}\,,\\ R^*_\Sigma(x) &= \rho^*_{\overline{\Sigma}}(x,-\nu^*_\Omega(x))\\ &= \sup\big\{s>0: s=u^*_\Sigma(\Phi_s(x))\big\}\,, && x\in\Sigma\,, \end{aligned}$$

(where, for Γ closed in M, ρ^*_Γ is defined as in (2.2) with respect to the metric g^*) and then consider the sets

$$\begin{aligned} \Omega_t &= \big\{p\in\Omega: u^*_\Sigma(p)>t\big\}\,,\\ \Sigma_t &= \big\{p\in\Omega: u^*_\Sigma(p)=t\big\} = M^\circ\cap\partial\Omega_t\,,\\ \Sigma^*_t &= \Phi_t(\{R^*_\Sigma>t\})\subset\Sigma_t\,,\\ A^*_\Sigma &= \big\{(x,t): x\in\Sigma,\, t\in(0,R^*_\Sigma(x))\big\}\,. \end{aligned}$$

(Notice that if $\partial\Omega=\overline{\Sigma}$, then $\partial\Omega_t=\Sigma_t$; if, otherwise, $\partial\Omega=\overline{\Sigma}\cup N_0$, then $\partial\Omega_t=\Sigma_t\cup N_0$.) It is easily seen that R^*_Σ is continuous on Σ, so that $\{R^*_\Sigma>t\}$ is an open subset of Σ for every $t>0$, and A^*_Σ is open. The fact that Φ is a diffeomorphism on A^*_Σ with values in Ω is standard (since Σ is smooth), so that

$$\Phi(A^*_\Sigma)\subset\Omega\,,\qquad R_\Sigma(x)\ge R^*_\Sigma(x)\quad\forall x\in\Sigma\,,\qquad A^*_\Sigma\subset A_\Sigma\,,\tag{4.26}$$

and Φ_t is a smooth embedding of $\{R^*_\Sigma>t\}$ into M. In particular, for each $t>0$, Σ^*_t is a (possibly empty, embedded) hypersurface in M. (Notice that Γ_t is, in general, *larger* than Σ^*_t, immersed but not embedded, and *unrelated* to u^*_Σ.) The vector field (see (4.16))

$$\nu_t(y)=\nu(\Phi_t^{-1}(y),t)\,,\qquad y\in\Sigma^*_t\,,\tag{4.27}$$

is a unit normal vector field to Σ^*_t in (M,g) with the property that

$$H(\Phi_t^{-1}(y),t)=H_{\Sigma^*_t}(y)\qquad y\in\Sigma^*_t\,,\tag{4.28}$$

where $H_{\Sigma^*_t}$ is the scalar mean curvature of Σ^*_t with respect to ν_t. We now prove three important geometric properties of the family $\{\Sigma^*_t\}_t$, namely, we show that

$$\mathcal{H}^n(\Omega\setminus\Phi(A^*_\Sigma))=0\,,\tag{4.29}$$

$$\mathcal{H}^{n-1}(\Sigma_t\setminus\Sigma^*_t)=0\,,\qquad\text{for }\mathcal{L}^1\text{-a.e. }t>0\,,\tag{4.30}$$

and that, when $\partial\Omega=\Sigma\cup N_0$,

$$\mathcal{H}^{n-1}(\Sigma^*_t)\ge h(0)^{n-1}\,\mathrm{vol}\,(N)\,,\qquad\text{for }\mathcal{L}^1\text{-a.e. }t>0\,.\tag{4.31}$$

We begin noticing that (4.29) is immediate to prove when Σ is a *closed* smooth hypersurface, since, in that case, we trivially see that $\Omega\setminus\Phi(A^*_\Sigma)\subseteq\mathrm{Cut}^*(\Sigma)$, where Cut^* denotes the cut-locus in (M,g^*), and $\mathcal{H}^n(\mathrm{Cut}^*(\Sigma))=0$ by Theorem 2.1-(v). In our non-smooth setting, we begin noticing that, by assumption (A3) and Lemma 3.2, there is $\lambda^*>0$ such that $\overline{\Sigma}$ is a White $(n-1,\lambda^*)$-subset of (M,g^*). Now, by construction,

$$\Omega\setminus\Phi(A^*_\Sigma)\subseteq\mathrm{Cut}^*(\overline{\Sigma})\cup\exp\big[\mathcal{N}(\overline{\Sigma})\llcorner(\overline{\Sigma}\setminus\Sigma)\big]\,,$$

Since $\overline{\Sigma}$ is a White $(n-1,\lambda^*)$-set in (M,g^*) and $\mathcal{H}^{n-1}(\overline{\Sigma}\setminus\Sigma)=0$ we conclude from Theorem 3.3 that

$$\mathcal{H}^n(\mathcal{N}(\overline{\Sigma})\llcorner(\overline{\Sigma}\setminus\Sigma))=0 \quad\text{and}\quad \mathcal{H}^n\big(\exp^*\big[\mathcal{N}(\overline{\Sigma})\llcorner(\overline{\Sigma}\setminus\Sigma)\big]\big)=0\,.$$

Since $\mathcal{H}^n(\mathrm{Cut}^*(\overline{\Sigma}))=0$, we conclude the proof of (4.29). By (4.29) and the coarea formula we have

$$0=\int_{\Omega\setminus\Phi(A^*_\Sigma)}|\nabla u^*_\Sigma|_g\,d\mathcal{H}^n=\int_0^\infty\mathcal{H}^{n-1}(\Sigma_s\setminus\Phi(A^*_\Sigma))\,ds\,,$$

which immediately implies (4.30) by $\Sigma^*_s=\Sigma_s\cap\Phi(A^*_\Sigma)$. Finally, to prove (4.31), we notice that, since u^*_Σ is a Lipschitz function, $\Omega_t=\{u^*_\Sigma>t\}$ is a set of finite perimeter in M for $\mathcal{L}^1$-a.e. $t>0$. Since $\partial\Omega=\Sigma\cup N_0$ implies $\partial\Omega_t=N_0\cup\Sigma_t$, by (4.30) the reduced boundary $\partial^*\Omega_t$ of Ω_t is $\mathcal{H}^{n-1}$-equivalent to the union of N_0 and Σ^*_t, with measure theoretic outer g-unit normal ν_{Ω_t} such that $\nu_{\Omega_t}=-\partial/\partial r$ on N_0 and $\nu_{\Omega_t}=\nu_t$ on Σ^*_t; in particular,

$$\int_{\Omega_t}\mathrm{div}\,(\partial/\partial r)=\int_{\Sigma^*_t}\langle\nu_t,\partial/\partial r\rangle_g\,d\mathcal{H}^{n-1}-\mathcal{H}^{n-1}(N_0)\,.$$

By (4.8), and since both $\partial/\partial r$ and ν_t have unit length in g, we deduce $\mathcal{H}^{n-1}(\Sigma^*_t)\ge\mathcal{H}^{n-1}(N_0)$, which is (4.31).

A general Heintze–Karcher inequality: We now prove a general Heintze–Karcher inequality, see (4.33) below, which implies both (4.2) and (4.3), and which allows one to deduce the crucial positive reach information contained in conclusion (c) when equality holds in either (4.2) or (4.3). We start noticing that, by (4.18) and (4.20),

$$\begin{aligned}\int_\Sigma d\mathcal{H}^{n-1}_x&\int_0^{R_\Sigma(x)}(f\circ\Phi)^2(x,t)\,J^\Sigma\Phi_t(x)\,dt \qquad (4.32)\\ &\le-(n-1)\int_\Sigma d\mathcal{H}^{n-1}_x\int_0^{R_\Sigma(x)}\frac{\partial}{\partial t}\Big(\frac{f\circ\Phi}{H}\Big)(x,t)\,J^\Sigma\Phi_t(x)\,dt\\ &=-(n-1)\int_\Sigma d\mathcal{H}^{n-1}_x\int_0^{R_\Sigma(x)}(f\circ\Phi)^2\,(x,t)\,J^\Sigma\Phi_t(x)\,dt\\ &\qquad-(n-1)\int_\Sigma\Big[\Big(\frac{f\circ\Phi}{H}\Big)(x,t)\,J^\Sigma\Phi_t(x)\Big]\Big|_{t=0}^{t=R_\Sigma(x)}d\mathcal{H}^{n-1}_x\,.\end{aligned}$$

By (4.15), the area formula, and (4.29) we obtain

$$\begin{aligned}&\int_\Sigma d\mathcal{H}^{n-1}_x\int_0^{R^*_\Sigma(x)}(f\circ\Phi)^2\,(x,t)\,J^\Sigma\Phi_t(x)\,dt\\ &=\int_{A^*_\Sigma}(f\circ\Phi)\,J^{A_\Sigma}\Phi\,d\mathcal{H}^n=\int_{\Phi(A^*_\Sigma)}f\,d\mathcal{H}^n=\int_\Omega f\,d\mathcal{H}^n\,,\end{aligned}$$

while by (4.12),

$$\Big[\int_\Sigma\Big(\frac{f\circ\Phi}{H}\Big)(x,t)\,J^\Sigma\Phi_t(x)d\mathcal{H}^{n-1}_x\Big]\Big|_{t=0}=\int_\Sigma\frac{f}{H_\Sigma}\,d\mathcal{H}^{n-1}\,,$$

so that (4.32) gives

$$n\int_\Omega f\,d\mathcal{H}^n+\int_\Sigma d\mathcal{H}^{n-1}_x\int_{R^*_\Sigma(x)}^{R_\Sigma(x)}(f\circ\Phi)^2\,(x,t)\,J^\Sigma\Phi_t(x)\,dt+\mathcal{L}(\Sigma)$$
$$\le(n-1)\int_\Sigma\frac{f}{H_\Sigma}\,d\mathcal{H}^{n-1}\,,\tag{4.33}$$

where

$$\mathcal{L}(\Sigma)=(n-1)\int_\Sigma\Big[\lim_{t\to R_\Sigma(x)^-}\Big(\frac{f\circ\Phi}{H}\Big)(x,t)\,J^\Sigma\Phi_t(x)\Big]\,d\mathcal{H}^{n-1}_x\,.$$

Notice that, for every $x\in\Sigma$, $t\mapsto J^\Sigma\Phi_t(x)\,[(f\circ\Phi)/H](x,t)$ is decreasing on $(0,R_\Sigma(x))$ thanks to (4.18) and (4.20): in particular, the integrand in the definition of $\mathcal{L}(\Sigma)$ is a well-defined non-negative function, (4.2) follows immediately from (4.33), and conclusion (a) is proved.

Conditional proof of conclusions (b) and (c): We now prove conclusions (b) and (c) assuming the validity of the following inequality:

$$\mathcal{L}(\Sigma)\ge h(0)^n\,\mathrm{vol}(N)\,,\qquad\text{when }\partial\Omega=\Sigma\cup N_0\,.\tag{4.34}$$

Indeed, if (4.34) holds, then (4.33) definitely implies (4.3), that is is conclusion (b). Moreover, if equality holds in either (4.2) or (4.3), then inequality (4.32) (appearing in the derivation of (4.33)) must hold as an identity. Therefore, since (4.18) was used in proving (4.32), we find that if equality holds in either (4.2) or (4.3), then

$$\frac{\partial}{\partial t}\Big(\frac{f\circ\Phi}{H}\Big)=-\frac{(f\circ\Phi)^2}{n-1}\,,\qquad\text{on }A_\Sigma\,,\tag{4.35}$$
$$\int_\Sigma d\mathcal{H}^{n-1}_x\int_{R^*_\Sigma(x)}^{R_\Sigma(x)}(f\circ\Phi)^2\,(x,t)\,J^\Sigma\Phi_t(x)\,dt=0\,.\tag{4.36}$$

By (4.22), we see that (4.35) gives

$$|\mathrm{II}|^2=\frac{H^2}{n-1}\,,\qquad\text{on }A_\Sigma\,,\tag{4.37}$$

which, tested with $t=0$, implies that Σ is umbilical in (M,g) (the first part of conclusion (c)), as well as

$$\frac{h^3}{2}\,\big(\mathcal{M}[h]\big)'\,g_N(\nu,\nu)=0\,,\qquad\text{on }A_\Sigma\,,$$

which, tested with $t=0$, implies the validity of (4.4). A more delicate argument is needed to deduce from (4.36) that $M\setminus\Omega$ has positive reach in (M,g^*) (the second part of conclusion (c)), and it goes as follows: Since $f>0$ on M (by assumption (H2)) and $J^\Sigma\Phi_t(x)>0$ for every $t\in(0,R_\Sigma(x))$ (by definition of $R_\Sigma(x)$), (4.36) implies that

$$R_\Sigma(x)=R^*_\Sigma(x)\,,\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in\Sigma\,,$$

whence we infer from the lower semicontinuity of R_Σ and the continuity of R^*_Σ that

$$R_\Sigma(x)=R^*_\Sigma(x)\,,\qquad\text{for every }x\in\Sigma\,.\tag{4.38}$$

Let $x\in\Sigma$ be such that $R_\Sigma(x)<\infty$. Should it be that

$$\mathrm{dist}_{g^*}(\Phi_t(x),N_0\cup N_b)\to0^+\qquad\text{as }t\to R_\Sigma(x)^-\,,\tag{4.39}$$

the facts that $R_\Sigma(x) < \infty$ and $\mathrm{dist}_{g^*}(N_0, \Sigma) \ge \mathrm{dist}_{g^*}(N_0, N_a) = +\infty$ would imply

$$\mathrm{dist}_{g^*}(\Phi_t(x), N_b) \to 0^+ \qquad \text{as } t \to R_\Sigma(x)^- \,,$$

and thus, by (4.38), $\Phi(A^*_\Sigma) \subset \Omega$, and (4.6), that

$$\lim_{t\to R_\Sigma(x)^-} \Phi_t(x) \in \overline{\Omega} \cap N_b = \varnothing \,,$$

a contradiction. Since (4.39) cannot occur, $R_\Sigma(x) < \infty$ must then imply $J^\Sigma \Phi_t(x) \to 0^+$ as $t \to R_\Sigma(x)^-$: in this case, (4.21) holds, and we can integrate (4.35) over $t \in (0, R_\Sigma(x))$ and take advantage of (4.12) so to find

$$-\frac{f(x)}{H_\Sigma(x)} = -\int_0^{R_\Sigma(x)} \frac{f(\Phi_t(x))^2}{n-1}\, dt \ge -\frac{R_\Sigma(x)}{n-1} \sup_M f^2 \,,$$

where $\sup_M f^2 < \infty$ thanks to (4.7). Since $0 < H_\Sigma \le \lambda$ on Σ by assumption (A3), we conclude (using again (4.6)) that

$$R_\Sigma(x) \ge \frac{(n-1)}{\lambda \sup_M f^2} \inf_\Sigma f \ge \frac{(n-1)}{\lambda \sup_M f^2} \inf_{[a,b]} h' \,.$$

We have thus proved the existence of a positive constant $c(\Sigma)$ such that[8] By assumption (A2), $\partial(M \setminus \Omega) = \overline{\Sigma}$, therefore

$$\mathcal{N}^1(M \setminus \Omega) \ \subset \ \mathcal{N}^1\overline{\Sigma} \,, \tag{4.41}$$

$$\mathcal{N}^1(M \setminus \Omega) \cap \big(\mathcal{N}^1\overline{\Sigma} \llcorner \Sigma\big) \ = \ \big\{(x, -\nu^*_\Omega(x)) : x \in \Sigma\big\} \,. \tag{4.42}$$

At the same time, by applying Theorem 3.3 with $\Gamma = \overline{\Sigma}$ (which is admissible by assumption (A3)), $m = n-1$, and $Z = \overline{\Sigma} \setminus \Sigma$ (which is admissible since, by assumption (A1), $\mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0$), we find that

$$\mathcal{H}^{n-1}\big(\mathcal{N}^1\overline{\Sigma} \llcorner \big(\overline{\Sigma} \setminus \Sigma\big)\big) = 0 \,. \tag{4.43}$$

By combining (4.41), (4.42) and (4.43) we thus find

$$(x, \eta) = (x, -\nu^*_\Omega(x)) \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } (x,\eta) \in \mathcal{N}^1(M \setminus \Omega) \,,$$

so that the function, for $\mathcal{H}^{n-1}$-a.e. $(x, \eta) \in \mathcal{N}^1(M \setminus \Omega)$ we have

$$\begin{aligned} \rho^*_{M\setminus\Omega}(x,\eta) &= \rho^*_{M\setminus\Omega}(x, -\nu^*_\Omega(x)) \\ &= \sup\big\{s > 0 : s = \mathrm{dist}_{g^*}\big(\exp^*\big(x, -s\,\nu^*_\Omega(x)\big), M \setminus \Omega\big)\big\} \,. \end{aligned}$$

Since $\mathrm{dist}_{g^*}(p, M \setminus \Omega) = \mathrm{dist}_{g^*}(p, \overline{\Sigma})$ for $p \in \Omega$ and since

$$\exp^*(x, -s\,\nu^*_\Omega(x)) \in \Omega \,, \qquad \forall s \in (0, \rho^*_{M\setminus\Omega}(x, -\nu^*_\Omega(x))) \,,$$

by (4.40) we conclude that, for $\mathcal{H}^{n-1}$-a.e. $(x,\eta) \in \mathcal{N}^1(M \setminus \Omega)$,

$$\rho^*_{M\setminus\Omega}(x,\eta) = \rho^*_{\overline{\Sigma}}(x, -\nu^*_\Omega(x)) = R^*_\Sigma(x) \ge c(\Sigma) > 0 \,.$$

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[8]Since it is *false* that every $(x,\eta) \in \mathcal{N}^1\overline{\Sigma}$ is an accumulation point of $\{(x, -\nu^*_\Omega(x)) : x \in \Sigma\}$, we cannot deduce a lower bound for $\rho_{\overline{\Sigma}}$ on $\mathcal{N}^1\overline{\Sigma}$ by simply combining (4.40), $R^*_\Sigma(x) = \rho^*_{\overline{\Sigma}}(x, -\nu^*_\Omega(x))$, and the upper semicontinuity of $\rho^*_{\overline{\Sigma}}$ on $\mathcal{N}^1\overline{\Sigma}$. In general, $\mathcal{N}^1\overline{\Sigma}$ minus the closure of $\{(x, -\nu^*_\Omega(x)) : x \in \Sigma\})$ may even be of positive $\mathcal{H}^{n-1}$-measure.

$$R^*_\Sigma(x) = R_\Sigma(x) \ge c(\Sigma) \,, \qquad \forall x \in \Sigma \,. \tag{4.40}$$

We can thus apply Theorem 2.1-(vii) and Remark 2.1 to finally conclude that $M \setminus \Omega$ is a set of positive reach in (M, g^*). We are thus left to prove (4.34) to complete the proof of conclusions (b) and (c).

Proof of (4.34): We start by setting, for every $(x,t) \in \Sigma \times [0,\infty)$,

$$g(x,t) = \begin{cases} \Big(\dfrac{f\circ\Phi}{H}\Big)(x,t)\, J^\Sigma\Phi_t(x)\,, & x\in\Sigma\,, t\in(0,R_\Sigma(x))\,,\\ 0\,, & x\in\Sigma\,, t\ge R_\Sigma(x)\,. \end{cases}$$

By (4.18), (4.20), and the non-negativity of f, H and $J^\Sigma\Phi_t$, we see that $t\in[0,\infty)\mapsto g(x,t)$ is decreasing, and thus provides a non-negative extension of $J^\Sigma\Phi\,(f\circ\Phi)/H$ from A_Σ to the whole $\Sigma\times[0,\infty)$ such that

$$\mathcal{L}(\Sigma) = (n-1)\lim_{t\to\infty}\int_\Sigma g(x,t)\,d\mathcal{H}^{n-1}_x\,.$$

Now, since $\{R^*_\Sigma > t\} \subset \Sigma$ and since $t < R^*_\Sigma(x)$ implies $t < R_\Sigma(x)$ we have

$$\int_\Sigma g(x,t)\,d\mathcal{H}^{n-1}_x \ge \int_{\{R^*_\Sigma>t\}}\Big(\frac{f\circ\Phi}{H}\Big)J^\Sigma\Phi_t\,d\mathcal{H}^{n-1}_x = \int_{\Sigma^*_t}\frac{f}{H_{\Sigma^*_t}}\,d\mathcal{H}^{n-1}\,,$$

where we have used (4.28). Now, by the Cauchy–Schwartz inequality,

$$\int_{\Sigma^*_t}\frac{f}{H_{\Sigma^*_t}}\,d\mathcal{H}^{n-1} \ge \mathcal{H}^{n-1}(\Sigma^*_t)^2\Big(\int_{\Sigma^*_t}\frac{H_{\Sigma^*_t}}{f}\,d\mathcal{H}^{n-1}\Big)^{-1}$$

so that, in summary,

$$\mathcal{L}(\Sigma) \ge (n-1)\limsup_{t\to\infty}\mathcal{H}^{n-1}(\Sigma^*_t)^2\Big(\int_{\Sigma^*_t}\frac{H_{\Sigma^*_t}}{f}\,d\mathcal{H}^{n-1}\Big)^{-1}\,. \tag{4.44}$$

Claim: for every $\lambda\in(0,1)$ there is $t_0 = t_0(\lambda)$ so that, if $t > t_0$, then

$$\inf_{\Sigma^*_t}\langle X,\nu_t\rangle_g \;\ge\; \lambda\,h(0)\,, \tag{4.45}$$

$$(n-1)\,\mathcal{H}^{n-1}(\Sigma^*_t) \;\ge\; \int_{\Sigma^*_t} H_{\Sigma^*_t}\,\frac{\langle X,\nu_t\rangle_g}{f}\,d\mathcal{H}^{n-1}\,, \tag{4.46}$$

with ν_t as in (4.27). Notice that by combining (4.44), (4.45) and (4.46) we obtain indeed that

$$\begin{aligned}\mathcal{L}(\Sigma) &\ge (n-1)\limsup_{t\to\infty}\inf_{\Sigma^*_t}\langle X,\nu_t\rangle_g\,\mathcal{H}^{n-1}(\Sigma^*_t)^2\Big(\int_{\Sigma^*_t}\frac{H_{\Sigma^*_t}}{f}\,\langle X,\nu_t\rangle_g\Big)^{-1}\\ &\ge \lambda\,h(0)\limsup_{t\to\infty}\mathcal{H}^{n-1}(\Sigma^*_t) \ge \lambda h(0)^n\,\mathrm{vol}\,(N)\,,\end{aligned}$$

where in the last step we have used (4.31). By letting $\lambda\to 1^-$ we deduce (4.34). We are thus left to prove (4.45) and (4.46) to complete the proof of conclusions (b) and (c).

Proof of (4.45): Recalling that $\Omega_t = \{x\in\Omega : \mathrm{dist}_{g^*}(x,\overline{\Sigma}) > t\}$, we now consider

$$u^*_{\Omega_t} = \mathrm{dist}_{g^*}(\cdot,\Omega_t)\,,\qquad t>0\,,$$

and notice that, with the same argument used in the proof of [Bre13, Lemma 3.6], for every $\lambda\in(0,1)$ there is $t_0 = t_0(\lambda)$ such that, if $p\in\Omega_{t_0}$ and α is a g^*-unit speed geodesic with $\alpha(0) = p$ and $\alpha(u^*_\Sigma(p))\in\overline{\Sigma}$, then $|\alpha'(0)|_g = f(p)$ and

$$\langle\alpha'(0),\partial/\partial r\rangle_g \ge \lambda\,f(p)\,. \tag{4.47}$$

(In more geometric terms, every g^*-unit speed geodesic that ends up in $\overline{\Sigma}$ after originating in Ω from a point at a sufficiently large distance from $\overline{\Sigma}$, must have an initial velocity with "almost vertical" direction). If $t > t_0$ we can apply this statement to any $p \in \Sigma_t^* \subset \Sigma_t \subset \Omega_{t_0}$ and with $\alpha'(0) = f(p)\,\nu_t(p)$, so to find

$$\langle \nu_t(p), \partial/\partial r\rangle_g \ge \lambda \qquad \forall p \in \Sigma_t^*\,,$$

from which (4.45) follows since $X = h\,\partial/\partial r$ and $h \ge h(0)$ on M.

As an additional consequence of (4.47), setting from now on $t_0 = t_0(\lambda)|_{\lambda=1/2}$, we also notice that, for every $t \ge t_0$,

$$\Omega_t \text{ has positive reach in } (M, g^*)\,. \tag{4.48}$$

To prove this, thanks to Theorem 2.3, we only need to show that u^*_Σ has no critical points in Ω_{t_0} ($t_0 = t_0(\lambda)|_{\lambda=1/2}$), that is, that there cannot be $p \in \Omega_{t_0}$ such that for every $v \in T_pM$ with $|v|_{g^*} = 1$ one can find a g^*-unit speed geodesic α with $\alpha(0) = p$ and $\alpha(u^*_\Sigma(p)) \in \overline{\Sigma}$ such that $\langle v, \alpha'(0)\rangle_g \ge 0$; and, indeed, any such α would satisfy $\langle \alpha'(0), \partial/\partial r\rangle_{g^*} \ge f(p)/2$ by (4.47), so that, taking $v = -f(p)\,(\partial/\partial r)$, we would obtain a contradiction.

Proof of (4.46): The proof is based on an approximation argument. Precisely, for $t > 0$ and $\varepsilon < \min\{1, t\}$, we consider the sets

$$W_{t,\varepsilon} = \big\{x \in M : u^*_{\Omega_t}(x) = \varepsilon\big\}\,,$$

so that

$$\Sigma^*_{t-\varepsilon} \subset W_{t,\varepsilon} \subset \Omega_{t-1} \setminus \overline{\Omega}_t\,, \qquad \forall t > 0\,, \varepsilon < \min\{1, t\}\,, \tag{4.49}$$

and reduce the proof of (4.46) to showing that

$$\lim_{\varepsilon\to 0^+} \mathcal{H}^{n-1}(W_{t,\varepsilon}) = \mathcal{H}^{n-1}(\Sigma_t)\,, \tag{4.50}$$

$$\lim_{\varepsilon\to 0^+} \int_{\Sigma^*_{t-\varepsilon}} H_{\Sigma^*_{t-\varepsilon}} \frac{\langle X, \nu_{t-\varepsilon}\rangle_g}{f}\, d\mathcal{H}^{n-1} = \int_{\Sigma^*_t} H_{\Sigma^*_t} \frac{\langle X, \nu_t\rangle_g}{f}\, d\mathcal{H}^{n-1}\,, \tag{4.51}$$

$$\liminf_{\varepsilon\to 0^+} \int_{W_{t,\varepsilon}\setminus\Sigma^*_{t-\varepsilon}} \frac{\langle \vec{H}_{W_{t,\varepsilon}}, X\rangle_g}{f}\, d\mathcal{H}^{n-1} \ge 0\,. \tag{4.52}$$

Indeed, thanks to (H1), there is $r_1 > 0$ such that $h'' > 0$ on $(0, 2\,r_1)$. Up to further increase the value of t_0, we can ensure $\Omega_{t-1} \subseteq N \times [0, r_1]$ for every $t \ge t_0$. In particular, by $W_{t,\varepsilon} \subset \Omega_{t-1}$, we conclude that $h'' > 0$ on $W_{t,\varepsilon}$. Since $W_{t,\varepsilon}$ is a closed $C^{1,1}$-hypersurface in M° with $\mathcal{H}^{n-1}(W_{t,\varepsilon}) < \infty$, by (4.11) we find

$$\begin{aligned}(n-1)\,\mathcal{H}^{n-1}(W_{t,\varepsilon}) &\ge \int_{W_{t,\varepsilon}} \frac{\langle \vec{H}_{W_{t,\varepsilon}}, X\rangle_g}{f}\, d\mathcal{H}^{n-1} \\ &= \int_{W_{t,\varepsilon}\setminus\Sigma^*_{t-\varepsilon}} \frac{\langle \vec{H}_{W_{t,\varepsilon}}, X\rangle_g}{f}\, d\mathcal{H}^{n-1} + \int_{\Sigma^*_{t-\varepsilon}} H_{\Sigma^*_{t-\varepsilon}} \frac{\langle \nu_{t-\varepsilon}, X\rangle_g}{f}\, d\mathcal{H}^{n-1}\end{aligned} \tag{4.53}$$

where we have used $\Sigma^*_{t-\varepsilon} \subset W_{t,\varepsilon}$ to deduce

$$\vec{H}_{W_{t,\varepsilon}} = \vec{H}_{\Sigma^*_{t-\varepsilon}} = H_{\Sigma^*_{t-\varepsilon}}\,\nu_{t-\varepsilon} \quad \text{on } \Sigma^*_{t-\varepsilon}\,.$$

By using (4.50), (4.51), (4.52), and (4.53) we deduce immediately (4.46). We now turn to the proof of (4.50), (4.51), and (4.52). We shall use the following preliminary step:

Proof of (4.50) *and* (4.51): Thanks to (4.48) we can apply Theorem 2.2 with $A=\overline{\Omega}_t$, $t\ge t_0$ and find $\varepsilon_t\in(0,\min\{1,t\})$ such that, if $\varepsilon\in(0,\varepsilon_t)$, then $W_{t,\varepsilon}=\{u^*_{\Omega_t}=\varepsilon\}$ is a compact $C^{1,1}$-hypersurface contained in $\mathrm{UP}(\Omega_t)$ and the g^*-geodesic flow $\Psi^t:\mathcal{N}^1(\overline{\Omega}_t)\times[0,\varepsilon_t)\to\{0\le u^*_{\Omega_t}<\varepsilon_t\}$ defined by

$$\Psi^t(p,\eta,\varepsilon)=\Psi^t_\varepsilon(p,\eta)=\exp^*(p,\varepsilon\,\eta)\,,\qquad (p,\eta,\varepsilon)\in\mathcal{N}^1(\overline{\Omega}_t)\times[0,\varepsilon_t)\,,$$

is such that Ψ^t_ε is a locally bi-Lipschitz map $\mathcal{N}^1(\overline{\Omega}_t)$ to $W_{t,\varepsilon}$ when $\varepsilon>0$, with $\Psi^t_0(p,\eta)=p$. In particular,

$$\Theta^t:=\mathcal{N}^1(\overline{\Omega}_t)$$

is a $(n-1)$-dimensional compact Lipschitz submanifold of TM. On noticing that $\mathcal{N}^1_p(\overline{\Omega}_t)=\varnothing$ for every $p\in\Omega_t\cup N_0$, since $\partial\Omega_t=\Sigma_t\cup N_0$ we find that

$$\mathcal{N}^1(\overline{\Omega}_t)=\mathcal{N}^1(\overline{\Omega}_t)\llcorner\Sigma_t\,,\qquad\text{i.e. }(p,\eta)\in\Theta^t\text{ implies }p\in\Sigma_t\,.\tag{4.54}$$

Moreover, by Theorem 2.1-(iii) (applied to $\overline{\Omega}_t$, see, in particular, (2.5)),

$$\mathcal{H}^0(\mathcal{N}^1_p(\overline{\Omega}_t))=1\,,\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }p\in\Sigma_t\,.\tag{4.55}$$

Finally, the smoothness of $(p,\eta)\in TM\mapsto\exp^*(p,\varepsilon\,\eta)\in M$ ensures that

$$\lim_{\varepsilon\to0^+}J^{\Theta^t}\Psi^t_\varepsilon(p,\eta)=J^{\Theta^t}\Psi^t_0(p,\eta)\,,\tag{4.56}$$

for $\mathcal{H}^{n-1}$-a.e. $(p,\eta)\in\Theta^t$ (i.e., at every (p,η) such that $T_{(p,\eta)}\Theta^t$ exists). Since Ψ^t_ε and its differential are locally bounded in (M,g), we can apply the area formula (to Ψ^t_ε), the dominated convergence theorem (in combination with (4.56)), and the area formula again (to Ψ^t_0) to find

$$\begin{aligned}\lim_{\varepsilon\to0^+}\mathcal{H}^{n-1}(W_{t,\varepsilon})&=\lim_{\varepsilon\to0^+}\int_{\Theta^t}J^{\Theta^t}\Psi^t_\varepsilon\,d\mathcal{H}^{n-1}=\int_{\Theta^t}J^{\Theta^t}\Psi^t_0\,d\mathcal{H}^{n-1}\\&=\int_{\Psi^t_0(\Theta^t)}\mathcal{H}^0\big((\Psi^t_0)^{-1}(x)\big)\,d\mathcal{H}^{n-1}_x=\int_{\Psi^t_0(\Theta^t)}\mathcal{H}^0\big(\mathcal{N}^1_x(\overline{\Omega}_t)\big)\,d\mathcal{H}^{n-1}_x\,,\end{aligned}$$

which, combined with (4.55) and $\Psi^t_0(\Theta^t)\subset\Sigma_t$ (i.e. (4.54)), gives

$$\lim_{\varepsilon\to0^+}\mathcal{H}^{n-1}(W_{t,\varepsilon})=\int_{\Psi^t_0(\Theta^t)}\mathcal{H}^0\big(\mathcal{N}^1_x(\overline{\Omega}_t)\big)\,d\mathcal{H}^{n-1}_x\le\mathcal{H}^{n-1}(\Sigma_t)\,.\tag{4.57}$$

Now, to prove (4.50), let us consider the diffeomorphisms $\phi_\varepsilon:\Sigma^*_t\to\Sigma^*_{t-\varepsilon}$ defined by

$$\phi_\varepsilon(x)=\Phi_{t-\varepsilon}\big(\Phi_t^{-1}(x)\big)=\exp^*(x,\varepsilon\,f(x)\,\nu_t(x))\,,\qquad\forall x\in\Sigma^*_t\,.$$

By the area formula,

$$\mathcal{H}^{n-1}(\phi_\varepsilon(\Sigma^*_t))=\int_{\Sigma^*_t}J^{\Sigma^*_t}\phi_\varepsilon\,d\mathcal{H}^{n-1}\,.$$

Since $\phi_\varepsilon\to\mathrm{Id}$ and $J^{\Sigma^*_t}\phi_\varepsilon\to1$ on Σ^*_t as $\varepsilon\to0$, we conclude by dominated convergence that

$$\lim_{\varepsilon\to0^+}\mathcal{H}^{n-1}(\Sigma^*_{t-\varepsilon})=\mathcal{H}^{n-1}(\Sigma^*_t).\tag{4.58}$$

Thus, by combining (4.57) and (4.58) with the facts that $\Sigma^*_{t-\varepsilon}\subset W_{t,\varepsilon}$ and Σ^*_t is $\mathcal{H}^{n-1}$-equivalent to Σ_t (recall (4.30)), we deduce (4.50) and

$$\lim_{\varepsilon\to0^+}\mathcal{H}^{n-1}(W_{t,\varepsilon}\setminus\Sigma^*_{t-\varepsilon})=0\,.\tag{4.59}$$

To prove (4.51) we notice that, since $(\Phi^{-1})|_{\Sigma_s^*} = (\Phi_s)^{-1}$ on Σ_s^*, we have

$$(\Phi^{-1})|_{\Sigma^*_{t-\varepsilon}} \circ \phi_\varepsilon = (\Phi_t)^{-1} = (\Phi^{-1})|_{\Sigma_t^*} \qquad \text{on } \Sigma_t^* \,,$$

so that, the area formula gives

$$\begin{aligned}\int_{\Sigma^*_{t-\varepsilon}} H_{\Sigma^*_{t-\varepsilon}} \frac{\langle X, \nu_{t-\varepsilon}\rangle_g}{f} \, d\mathcal{H}^{n-1} &= \int_{\Sigma^*_{t-\varepsilon}} (H \circ (\Phi_{t-\varepsilon})^{-1}) \, \frac{\langle X, \nu_\Omega \circ (\Phi_{t-\varepsilon})^{-1}\rangle_g}{f} \\ &= \int_{\Sigma_t^*} H_{\Sigma_t^*} \, \frac{\langle X \circ \phi_\varepsilon, \nu_t\rangle_g}{f \circ \phi_\varepsilon} \, J\phi_\varepsilon \, d\mathcal{H}^{n-1} \,,\end{aligned}$$

for every $\varepsilon > 0$. Then (4.51) follows by dominated convergence.

Proof of (4.52): We finally prove (4.52), that is,

$$\liminf_{\varepsilon \to 0^+} \int_{W_{t,\varepsilon} \setminus \Sigma^*_{t-\varepsilon}} \frac{\langle \vec{H}_{W_{t,\varepsilon}}, X\rangle_g}{f} \, d\mathcal{H}^{n-1} \geq 0 \,. \tag{4.60}$$

Setting, for the sake of brevity,

$$\nu_{t,\varepsilon} = \frac{\nabla u^*_{\Omega_t}}{|\nabla u^*_{\Omega_t}|_g} \,, \qquad H_{t,\varepsilon} = \vec{H}_{W_{t,\varepsilon}} \cdot \nu_{t,\varepsilon} \,, \qquad \text{on } W_{t,\varepsilon} \,, \tag{4.61}$$

we notice that $\nu_{t,\varepsilon}$ defines a Lipschitz continuous g-unit normal to $W_{t,\varepsilon}$, and that $H_{t,\varepsilon}$ is the scalar mean curvature (as usual, with respect to g) of $W_{t,\varepsilon}$ relative to ν_t. With this notation, and thanks to (4.59), (4.60) follows by showing that

$$\inf_{W_{t,\varepsilon}} \langle \nu_{t,\varepsilon}, \partial/\partial r\rangle_g \geq 0 \,, \tag{4.62}$$

$$H_{t,\varepsilon}(x) \geq -\Lambda(t) \,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in W_{t,\varepsilon} \,, \tag{4.63}$$

for a (positive) constant $\Lambda(t)$ independent of ε.

Proof of (4.62): We start by proving the existence of a positive constant c such that

$$\big\langle \eta, X(p) \big\rangle_{g^*} \geq c \,, \quad \forall (p, \eta) \in \Theta_t \,, t > t_0 \,. \tag{4.64}$$

Since $\langle \eta, v\rangle_{g^*} \leq 0$ whenever $\eta \in \mathcal{N}_p(\overline{\Omega}_t)$ and $v \in T_p(\Omega_t)$ (where $T_p(\Omega_t)$ is tangent cone to Ω_t at p), and recalling (4.55), the validity of (4.64) (for an explicitly computable constant σ_0) can be easily deduced by showing that, for every $t > t_0$ and $p \in \Sigma_t \subset \Omega_{t_0}$,

$$\begin{cases} v \in T_p M \,, \quad |v|_g = 1 \,, \\ \big\langle v, -(\partial/\partial r)|_p \big\rangle_g > \dfrac{15}{16} \,, \end{cases} \qquad \Rightarrow \qquad v \in T_p(\Omega_t) \,. \tag{4.65}$$

(In geometric terms: leaving $p \in \Sigma_t = (\partial \Omega_t) \cap M^\circ$ along a sufficiently "vertical and downward" direction, we stay inside Ω_t.) The proof of (4.65) follows closely that of [Bre13, Lemma 3.7], but since the two statements are not immediate to compare, we include the details. We need to consider an arbitrary g-unit speed curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, and prove the existence of $\sigma \in (0,1)$ such that $u^*_\Sigma(\gamma(s)) > t$ for every $s \in (0, \sigma)$. To begin with, we can definitely chose σ so that

$$\gamma(s) \in \Omega_{t_0} \,, \qquad \big\langle \gamma'(s), -(\partial/\partial r)|_{\gamma(s)} \big\rangle_g \geq \frac{15}{16} \,, \quad \forall s \in [0, \sigma] \,.$$

For every $s \in (0,\sigma)$, the fact that $\gamma(s) \in \Omega_{t_0}$ implies $u^*_\Sigma(\gamma(s)) > 0$, and thus the existence of a g^*-unit speed geodesic α_s with $\alpha_s(0) = \gamma(s)$ and $\alpha_s(u^*_\Sigma(\gamma(s))) \in \overline{\Sigma}$. By (4.47) (with $\lambda = 1/2$), since $w(s) = \alpha'_s(0)/f(\gamma(s))$ is a g-unit vector, we find

$$\begin{aligned}
\frac{1}{2} &\le \big\langle w(s), (\partial/\partial r)|_{\gamma(s)} \big\rangle_g = \big\langle w(s), \gamma'(s) + (\partial/\partial r)|_{\gamma(s)} \big\rangle_g - \langle w(s), \gamma'(s)\rangle_g \\
&\le \big|\gamma'(s) + (\partial/\partial r)|_{\gamma(s)}\big|_g - \langle w(s), \gamma'(s)\rangle_g \\
&= \sqrt{2 - 2\,\langle \gamma'(s), -(\partial/\partial r)|_{\gamma(s)}\rangle_g} - \langle w(s), \gamma'(s)\rangle_g \\
&\le \frac{1}{4} - \langle w(s), \gamma'(s)\rangle_g\,, \qquad \text{i.e.} \quad \langle -w(s), \gamma'(s)\rangle_g \ge \frac{1}{4} \quad \forall s \in (0,\sigma)\,.
\end{aligned}$$

Using the facts that $u^*_\Sigma(p) = t$ and that, for a.e. $s \in (0,\sigma)$, $u^*_\Sigma \circ \gamma$ is differentiable at $\gamma(s)$ with

$$(u^*_\Sigma \circ \gamma)'(s) = \langle \nabla^* u^*_\Sigma(\gamma(s)), \gamma'(s)\rangle_{g^*} = \langle -\alpha'_s(0), \gamma'(s)\rangle_{g^*} = \frac{\langle \gamma'(s), -w(s)\rangle_g}{f(\gamma(s))}\,,$$

we thus find that, for every $s' \in (0,\sigma)$,

$$u^*_\Sigma(\gamma(s')) = t + \int_0^{s'} \frac{ds}{4\,f(\gamma(s))} > t$$

as desired. This proves (4.65), and thus, as explained, (4.64).

We are now ready to deduce (4.62) from (4.64). First, by (4.64) there are positive constants c and δ such that

$$\langle \eta', \partial/\partial r\rangle_g \ge c\,, \tag{4.66}$$

whenever (q,η') lies in the δ-neighborhood $\mathcal{A}_\delta(\Theta^t)$ of $\Theta^t = \mathcal{N}^1(\overline{\Omega}_t)$ in TM. Second, by smoothness of $(p,\eta,\varepsilon) \mapsto \exp^*(p,\varepsilon\,\eta)$, we can find $\varepsilon' < \varepsilon_t$ depending on δ such that

$$\Big\{\Big(\Psi^t(p,\eta,\varepsilon), \frac{\partial \Psi^t}{\partial\varepsilon}(p,\eta,\varepsilon)\Big) : (p,\eta) \in \Theta^t\,, 0 < \varepsilon < \varepsilon'\Big\} \subset \mathcal{A}_\delta(\Theta^t)\,.$$

Third, since Ψ^t_ε is a bijection from Θ^t to $W_{t,\varepsilon}$, we see that for each $x \in W_{t,\varepsilon}$ there is a unique pair (p,η) with $p \in \Sigma_t$ and $\eta \in \mathcal{N}^1_p(\overline{\Omega}_t)$ such that $x = \Psi^t_\varepsilon(p,\eta)$. Thus, taking also into account that (in general) $|\nabla^* v|_{g^*} = f\,|\nabla v|_g$ (by $\nabla^* v = f^2\,\nabla v$), that $|\nabla^* u^*_{\Omega_t}|_{g^*} = 1$ (wherever $u^*_{\Omega_t}$ is differentiable), and that $u^*_{\Omega_t}$ is differentiable along $\varepsilon \mapsto \Psi^t_\varepsilon(p,\eta)$ with g^*-gradient given by $\partial\Psi^t_\varepsilon/\partial\varepsilon$, we conclude that

$$\begin{aligned}
\langle \nu_{t,\varepsilon}(x), \partial/\partial r\rangle_g &= \Big\langle \frac{\nabla u^*_{\Omega_t}(\Psi^t_\varepsilon(p,\eta))}{|\nabla u^*_{\Omega_t}(\Psi^t_\varepsilon(p,\eta))|_g}, \partial/\partial r\Big\rangle_g \\
&= f\big(\Psi^t_\varepsilon(p,\eta)\big)\,\langle \nabla u^*_{\Omega_t}(\Psi^t_\varepsilon(p,\eta)), \partial/\partial r\rangle_g \\
&= f\big(\Psi^t_\varepsilon(p,\eta)\big)\,\Big\langle \frac{\partial\Psi^t}{\partial\varepsilon}(p,\eta,\varepsilon), \partial/\partial r\Big\rangle_g \ge c\,\inf_{[a,b]} h' > 0\,,
\end{aligned}$$

provided $\varepsilon < \varepsilon'$. This proves (4.62).

Proof of (4.63): Let us consider the open set

$$A_{t,\varepsilon} = \Omega_t \cup \big\{p \in \Omega : u^*_{\Omega_t}(p) < \varepsilon\big\}\,,$$

that has $C^{1,1}$-boundary and g-unit outer unit normal given by $\nu_{A_{t,\varepsilon}} = \nu_{t,\varepsilon}$ on $\partial A_{t,\varepsilon} = W_{t,\varepsilon}$, with $\nu_{t,\varepsilon}$ as in (4.61). Given $x \in W_{t,\varepsilon}$ such that $\nu_{t,\varepsilon}$ is differentiable at x (this holds at $\mathcal{H}^{n-1}$-a.e. $x \in W_{t,\varepsilon}$), let B^*_x denote the g^*-geodesic ball centered at $\exp^*(x, (\mu_t/2)\,\nabla^* u^*_{\Omega_t}(x))$, where $\mu_t < \min\{1,\varepsilon_t\}$ is smaller than the injectivity radius of M in $\overline{\Omega}_{t-2} \setminus \Omega_{t+1}$, so to entail that B_x^* has smooth boundary. Since, by construction,

$$x \in \partial A_{t,\varepsilon} \cap \partial B_x^* , \qquad A_{t,\varepsilon} \subset M \setminus B_x^* ,$$

by the weak maximum principle for $C^{1,1}$-vs-smooth hypersurfaces we find

$$H_{W_{t,\varepsilon}}(x) = \vec{H}_{A_{t,\varepsilon}}(x) \cdot \nu_{t,\varepsilon}(x) \geq \vec{H}_{M \setminus B_x^*} \cdot \nu_{t,\varepsilon}(x) = -H_{B_x^*}(x) ,$$

where $H_{B_x^*}(x)$ denotes the scalar mean curvature of B_x^* in (M, g), computed with respect to the outer g-unit normal to B_x^* at x (i.e., with respect to $-\nu_{t,\varepsilon}(x)$). Now, if $H^*_{B_x^*}$ denotes the scalar mean curvature of B_x^* in (M, g^*) computed with respect to the outer g^*-unit normal to B_x^* at x, then by (3.2) we find

$$\begin{aligned} H_{B_x^*}(x) &\leq H^*_{B_x^*}(x) + 3\,(n-1)\,|\nabla \log(f)(x)|_g \\ &\leq (n-1)\,\sqrt{\kappa_t}\coth(\sqrt{\kappa_t}\,\mu_t/2) + 3\,(n-1)\,|\nabla \log(f)(x)|_g , \end{aligned}$$

where in the last inequality we have denoted by $-\kappa_t$ a negative lower bound for the sectional curvatures of (M, g^*) in $\overline{\Omega}_{t-2} \setminus \Omega_{t+1}$, and have used [Kar89, pag. 184] (comparison with the mean curvature of geodesic balls in an hyperbolic model space). Since the right-hand side can be bounded by a positive constant $\Lambda(t)$, we have concluded the proof of (4.63), and thus of conclusions (b) and (c) of the theorem. □

Proof of Theorem 4.2. Preparation: As in the proof of Theorem 4.1, we reduce the case when $M = N \times (0, \infty)$, (H0) and (H1) hold, (H2) holds on $(0, \infty)$ (i.e., $h' > 0$ on $(0, \infty)$), (H3) and (H4) hold on $(0, b)$ (if $(M, g) \in \mathcal{B}_n$) **or** (H3)* holds on $(0, b)$ (if $(M, g) \in \mathcal{B}_n^*$); $f = h' \circ r$ is bounded on M and the metric $g^* = f^{-2}\, g$ is such that (M, g^*) is a complete Riemannian manifold; and, finally, Σ and Ω satisfy (4.6) in addition to assumptions (A1), (A2)', and (A3)'. Next, by testing (4.5) with vector fields compactly supported in $M \setminus (\overline{\Sigma} \setminus \Sigma)$ we see that

$$H_\Sigma \equiv H_0 \qquad \text{on } \Sigma ,$$

while testing (4.5) with $X = h\,\partial/\partial r$ and taking into account that $\operatorname{div}^\Sigma X = (n-1)\, f$ on Σ and $\operatorname{div}(X) = n\, f$ on M by (4.10), we find, in the case $\partial\Omega = \overline{\Sigma}$,

$$\begin{aligned} (n-1) \int_\Sigma f \, d\mathcal{H}^{n-1} &= \int_\Sigma \operatorname{div}^\Sigma X \, d\mathcal{H}^{n-1} = H_0 \int_\Sigma \langle X, \nu_\Omega \rangle_g \, d\mathcal{H}^{n-1} \\ &= H_0 \int_\Omega \operatorname{div} X = H_0\, n \int_\Omega f , \end{aligned}$$

i.e., $H_\Sigma \equiv H_0 > 0$ (so that (A2) holds) and (4.2) holds as an equality; and, in the case $\partial\Omega = \overline{\Sigma} \cup N_0$,

$$\begin{aligned} (n-1) \int_\Sigma f \, d\mathcal{H}^{n-1} &= \int_\Sigma \operatorname{div}^\Sigma X \, d\mathcal{H}^{n-1} = H_0 \int_\Sigma \langle X, \nu_\Omega \rangle_g \, d\mathcal{H}^{n-1} \\ &= H_0 \int_\Omega \operatorname{div} X \, d\mathcal{H}^n - \int_{N_0} \langle X, \partial/\partial r \rangle_g \, d\mathcal{H}^{n-1} \\ &= H_0\, n \int_\Omega f - h(0)^n \operatorname{vol}(N) , \end{aligned}$$

i.e, $H_\Sigma \equiv H_0 > 0$ (so that (A2) holds) and (4.3) holds as an equality; finally, Theorem 3.1 and (4.5) imply the validity of assumption (A3). We can thus apply conclusion (c) of Theorem 4.1 to conclude that

$$\Sigma \text{ is umbilical and has constant mean curvature in } (M,g)\,, \tag{4.67}$$

$$M\setminus\Omega \text{ has positive reach in } (M,g^*)\,, \tag{4.68}$$

$$(\mathcal{M}[h])'\,g_N(\nu_\Omega,\nu_\Omega)=0 \text{ on } \Sigma\,, \tag{4.69}$$

with $\mathcal{M}[h]$ defined as in (4.23).

Conclusion of the proof if $(M,g)\in\mathcal{B}_n^*$: In this case, (4.6) and the validity of (H3)* on $(0,b)$ implies that $(\mathcal{M}[h])'>0$ on Σ, so that (4.69) gives

$$\text{if } p\in\Sigma, \text{ then } \nu_\Omega(p) \text{ is parallel to } (\partial/\partial r)|_p\,. \tag{4.70}$$

Now, if ϕ is a local chart of N, defined on a ball B in $\mathbb{R}^{n-1}$, then $\psi(x,t)=(\phi(x),t)$ defines a local chart of M defined on the open set $V=B\times(0,\bar r)$; clearly, $\psi^{-1}(\Omega)$ is a set of finite perimeter in $V\subset\mathbb{R}^n$, with $\nu_{\psi^{-1}(\Omega)}$ parallel to e_n $\mathcal{H}^{n-1}$-a.e. on $\partial^*[\psi^{-1}(\Omega)]$; by [Mag12, Exercise 15.18], $\psi^{-1}(\Omega)\cap V$ is $\mathcal{H}^n$-equivalent to $B\times J$, where J is a finite union of open intervals compactly contained in $(0,\bar r)$; covering N by such charts ϕ, and going back to M, we conclude that Ω is $\mathcal{H}^n$-equivalent to a $N_0\times J$. Then, the fact that H_Σ is constant implies that J is either equal to $(0,t_0)$ or to $(t_0,\bar r)$ for some $t_0\in(0,\bar r)$, and the theorem is proved.

Conclusion of the proof if $(M,g)\in\mathcal{B}_n$: Condition (4.67) combined with the Codazzi equations implies that

$$(\mathrm{Ric}_M)_p(\nu_\Omega(p),\sigma_i(p))=0\,,\qquad \forall p\in\Sigma\,, i=1,...,n-1\,, \tag{4.71}$$

provided $\{\sigma_i(p)\}_{i=1}^{n-1}$ is an g-orthonormal basis of $T_p\Sigma$; in particular,

$$\text{if } p\in\Sigma, \text{ then } \nu_\Omega(p) \text{ is an eigenvector of } \mathrm{Ric}_p\,. \tag{4.72}$$

Since (H4) implies that $(\partial/\partial r)|_p$ is a *simple* eigenvector of (Ric_M) (with eigenvalue $-(n-1)\,(h''/h)(r(p)))$, it follows from (4.72) that

$$\text{if } p\in\Sigma, \text{ then } \nu_\Omega(p) \text{ is either orthogonal or parallel to } (\partial/\partial r)|_p\,. \tag{4.73}$$

Now, by (4.6), there is $t_0>0$ and $p_0\in\overline{\Sigma}$ such that

$$N\times(0,t_0)\subset\Omega\,,\qquad p_0\in\overline{\Sigma}\cap N_{t_0}\subset\partial\Omega\,. \tag{4.74}$$

From here, *in the smooth case when* $\overline{\Sigma}=\Sigma$ (i.e., in the case considered in [Bre13]), (4.73) and (4.74) immediately imply, first, that $\nu_\Omega(p_0)=(\partial/\partial r)|_{p_0}$, and, second, that $N_{t_0}\subset\Sigma$; from which a sliding argument (also required and detailed below in the non-smooth case) proves the theorem.

However, *in the non-smooth case*, we cannot immediately conclude the containment of N_{t_0} into Σ, and, actually, it is not even clear if $\overline{\Sigma}$ is regular at the contact point p_0 defined in (4.74): indeed, the blow-up of (the multiplicity one varifold associated to) $\overline{\Sigma}$ at p_0 is an hyperplane with multiplicity possibly higher than one – thus preventing the use of Allard's regularity theorem to infer $p_0\in\Sigma$. To exit this impasse we make crucial use of the positive reach property (4.68), which we use to prove the following *approximation property*: for every $(p,\eta)\in\mathcal{N}^1(M\setminus\Omega)$ there are a connected component Σ' of Σ and a sequence $\{p_j\}_j\subset\Sigma'$ such that

$$\big(p_j,-f(p_j)\,\nu_\Omega(p_j)\big)\to(p,\eta)\quad \text{in } \mathcal{N}^1(M\setminus\Omega)\,. \tag{4.75}$$

Indeed, by (4.68) and Theorem 2.2, there is $s_0 > 0$ such that, for every $s \in (0, s_0)$, the sets

$$Z_s = \left\{x \in M : \mathrm{dist}^*(x, M \setminus \Omega) = s\right\},$$

are $C^{1,1}$-hypersurfaces, and the map $\Psi_s(p,\eta) = \exp^*(p, s\,\eta)$ is bi-Lipschitz from $\mathcal{N}^1(M \setminus \Omega)$ to Z_s. We notice that

$$\mathcal{N}^1(M \setminus \Omega)\lfloor\Sigma = \left\{(p, -f(p)\,\nu_\Omega(p)) : p \in \Sigma\right\},$$

and set

$$Z_s^* = \Phi_s\big(\mathcal{N}^1(M \setminus \Omega)\lfloor\Sigma\big) \subset Z_s\,.$$

By arguing as in the proof of (4.30), with the aid of Theorem 3.3 we find that, for $\mathcal{L}^1$-a.e. $s \in (0, s_0)$, Z_s^* is $\mathcal{H}^{n-1}$-equivalent to Z_s. For any such s, Z_s^* is an open dense subset of Z_s, and we can find a sequence $\{q_j\}_j$, contained in same connected component Z_s^{**} of Z_s^*, such that $q_j \to \Phi_s(p,\eta) \in Z_s$. Setting $\pi(q,\tau) = q$ for every $(q,\tau) \in TM$, we see that $p_j = \pi[(\Phi_s)^{-1}(q_j)]$ defines a sequence contained in a same connected subset $\pi[(\Phi_s)^{-1}(Z_s^{**})]$ of Σ, and such that

$$\big(p_j, -f(p_j)\,\nu_\Omega(p_j)\big) = (\Phi_s)^{-1}(q_j) \to (p,\eta)\,,$$

in TM, thus proving (4.75). We now combine (4.75) with the fact that, by definition of p_0 (recall (4.74)) it holds

$$-f(p_0)\,\frac{\partial}{\partial r}\Big|_{p_0} \in \mathcal{N}^1_{p_0}(M \setminus \Omega)\,, \tag{4.76}$$

to find a sequence $\{p_j\}_j$, contained in a connected component Σ' of Σ, such that $(p_j, -f(p_j)\,\nu_\Omega(p_j)) \to (p_0, -f(p_0)(\partial/\partial r)|_{p_0})$ as $j \to \infty$. By (4.73), up to extracting subsequences, there are two alternatives: either

$$-f(p_j)\,\nu_\Omega(p_j) \text{ is parallel to } (\partial/\partial r)|_{p_j} \text{ for every } j\,, \tag{4.77}$$

or $g_{p_j}(\nu_\Omega(p_j), (\partial/\partial r)|_{p_j}) = 0$ for every j, where the latter is clearly contradictory, since $|\partial/\partial r|_g = 1$. By smoothness and connectedness of Σ', by (4.73), and since Σ' contains points p_j as in (4.77) with $p_j \to p_0$ as $j \to \infty$, we conclude that Σ' is an open connected subset of N_{t_0}. In fact, it must be $\Sigma' = N_{t_0}$, because the above argument, with p_0 replaced by a possible point p_0' in the boundary of Σ' relative to N_{t_0}, would lead to the contradiction that an open neighborhood of p_0' in N_{t_0} would be contained in Σ' itself. We have thus proved that

$$N_{t_0} \subset \Sigma\,. \tag{4.78}$$

The same argument also shows that

$$N_{t_0} \cap \overline{\Sigma \cap [N \times (t_0, \infty)]} = \varnothing\,. \tag{4.79}$$

By (4.79) we could then start sliding N_t upwards to prove that either $\Omega = N \times (0, t_0)$ with $M^\circ \cap \partial\Omega = \overline{\Sigma} = \Sigma = N_{t_0}$, thus concluding the proof of the theorem, or we could find $t_1 > t_0$ such that

$$(t_0, t_1) \setminus M \setminus \overline{\Omega}\,, \qquad p_1 \in N_{t_1} \cap \overline{\Sigma} \subset M \setminus \Omega\,. \tag{4.80}$$

By construction, $-f(p_1)\,(\partial/\partial r)|_{p_1} \in \mathcal{N}^1_{p_1}(M \setminus \Omega)$, and, by arguing as in the proof of (4.78), we would find $N_{t_1} \subset \Sigma$, with $\nu_\Omega = -(\partial/\partial r)|_{p_1}$ along N_{t_1}. In turn, this would give that H_Σ is negative along N_{t_1}, a contradiction. This finally proves the theorem. □

5. Rigidity and compactness theorem

Proof of Theorem 1.1. Up to change Ω with $M \setminus \Omega$, since $\nu_\Omega = -\nu_{M\setminus\Omega}$ on $\partial^*\Omega = \partial^*(M \setminus \Omega)$, we can assume that $\lambda \geq 0$. Let V_Ω be the multiplicity one rectifiable varifold associated to $M^\circ \cap \partial^*\Omega$. The distributional constant mean curvature condition (1.4) imply lower density bounds on $\|V_\Omega\|$, which in turn imply

$$\mathcal{H}^{n-1}(\overline{\partial^*\Omega} \setminus \partial^*\Omega) = 0.$$

Therefore, it is not restrictive to assume that Ω is an open set such that $M^\circ \cap \partial\Omega$ is compact, $\overline{\partial^*\Omega} = \partial\Omega$ and

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0. \tag{5.1}$$

(see for instance the construction in the proof of [DRKS20, Lemma 6.2]). Notice that $\partial^*\Omega = \partial^*\overline{\Omega} \subseteq \partial\overline{\Omega} \subseteq \partial\Omega$, hence taking the closure we find that $\partial\Omega = \partial\overline{\Omega}$. By Allard's regularity theorem [All72], if we set

$$\Sigma = \left\{ x \in \operatorname{spt}\|V_\Omega\| : \lim_{\rho\to 0^+} \frac{\|V_\Omega\|(B_\rho(x))}{\omega_{n-1}\,\rho^{n-1}} = 1 \right\},$$

then Σ is a smooth, embedded hypersurface and

$$\Sigma = M^\circ \cap \partial^*\Omega\,. \tag{5.2}$$

We now check that the pair (Σ, Ω) satisfies the assumptions (A1), (A2)' and (A3)' of Theorem 4.2, thus concluding the proof of the theorem. Clearly, (A3)' is equivalent to (1.4). Since $M^\circ \cap \partial\Omega$ is compact, we infer that $\overline{\Sigma} \subseteq M^\circ$; moreover (5.1) means that $\mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0$. Henceforth (A1) holds. Concerning (A2)', since $M^\circ \cap \partial\Omega$ is compact, we notice that 1_Ω is constant in a neighborhood A of N_0 in M; if $1_\Omega = 0$ on A then $\partial\Omega = \overline{\Sigma}$; if, otherwise $1_\Omega = 1$ on A, then $N_0 \subset \partial\Omega$, and thus $\partial\Omega = N_0 \cup \overline{\Sigma}$. □

Proof of Theorem 1.2. From $\mathcal{H}^n(\Omega_j \Delta \Omega) \to 0$ as $j \to \infty$ we easily deduce that for every $x_0 \in M^\circ \cap \overline{\partial^*\Omega}$ there is $x_j \in \partial^*\Omega_j$ such that $x_j \to x_0$ in M; for, otherwise, there would be $\rho > 0$, with $\overline{B}_\rho(x_0) \cap \overline{\partial^*\Omega_j} = \varnothing$ for every j, and $X \in C^\infty_c(B_\rho(x_0))$ such that

$$\begin{aligned} 1 &= \int_{B_\rho(x_0)\cap\partial^*\Omega} \langle X, \nu_\Omega\rangle_g \, d\mathcal{H}^{n-1} = \int_\Omega \operatorname{div} X \, d\mathcal{H}^n \\ &= \lim_{j\to\infty} \int_{\Omega_j} \operatorname{div} X \, d\mathcal{H}^n = \lim_{j\to\infty} \int_{B_\rho(x_0)\cap\partial^*\Omega_j} \langle X, \nu_{\Omega_j}\rangle_g \, d\mathcal{H}^{n-1} = 0\,. \end{aligned}$$

We thus conclude that

$$M^\circ \cap \overline{\partial^*\Omega} \subset N \times [a,b] \subset\subset M^\circ\,. \tag{5.3}$$

By $\mathcal{H}^n(\Omega_j \Delta \Omega) \to 0$ and, crucially, by $\mathrm{Per}(\Omega_j) \to \mathrm{Per}(\Omega)$, as $j \to \infty$, we see that the multiplicity one rectifiable varifolds V_j associated to $\partial^*\Omega_j$ converge, in the sense of varifolds on M, to the multiplicity one rectifiable varifold V associated to $\partial^*\Omega$: in particular, for every $X \in \mathcal{X}(M)$,

$$\lim_{j\to\infty} \int_{M^\circ\cap\partial^*\Omega_j} \operatorname{div}^{\partial^*\Omega_j} X \, d\mathcal{H}^{n-1} = \int_{M^\circ\cap\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^{n-1}\,.$$

Again by $\mathcal{H}^n(\Omega_j \Delta \Omega) \to 0$ as $j \to \infty$ and thanks to the divergence theorem $\int_{M^\circ \cap \partial^* \Omega_j} \langle X, \nu_{\Omega_j} \rangle_g \, d\mathcal{H}^{n-1} \to \int_{M^\circ \cap \partial^* \Omega} \langle X, \nu_\Omega \rangle_g \, d\mathcal{H}^{n-1}$ as $j \to \infty$. Therefore, (1.7) implies

$$\int_{M^\circ \cap \partial^* \Omega} \operatorname{div}^{\partial^* \Omega} X \, d\mathcal{H}^{n-1} = \lambda \int_{M^\circ \cap \partial^* \Omega} \langle X, \nu_\Omega \rangle_g \, d\mathcal{H}^{n-1} . \tag{5.4}$$

By (5.3) and (5.4) we can apply Theorem 1.1 to Ω and conclude the proof of the theorem. □

6. Proof of Theorem 1.3

The key observation to prove Theorem 1.3 is contained in the following result, that can be proved employing the same method of Theorem 4.1.

Theorem 6.1. *Suppose (M, g) is a n-dimensional Riemannian manifold (notice carefully that we do not assume this space to be geodesically complete) and f is a smooth positive function on M such that*

$$f\mathrm{Ric} - \mathrm{D}^2 f + (\Delta f) g \geq 0 \quad \text{on } M. \tag{6.1}$$

Suppose $\Omega \subseteq M$ is an open set with finite perimeter, with exterior unit-normal ν_Ω, such that $\overline{\Omega}$ is compact, and suppose $\Sigma \subseteq \partial\overline{\Omega}$ is a smooth embedded hypersurface such that $\overline{\Sigma} = \partial\overline{\Omega}$ is a White $(n-1, \lambda)$-set of (M, g), $\mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0$, and $H_\Sigma = \overrightarrow{H}_\Sigma \cdot \nu_\Omega$ is positive on Σ.

Then

$$n \int_\Omega f \, d\mathcal{H}^n \leq (n-1) \int_\Sigma \frac{f}{H_\Sigma} \, d\mathcal{H}^{n-1}. \tag{6.2}$$

If the equality holds, then there exists $t_0 > 0$ such that the sets

$$\Sigma_t = \{p \in \Omega : \mathrm{dist}_{g^*}(p, \overline{\Sigma}) = t\}, \quad \text{for } 0 < t < t_0,$$

where $g^* = \frac{g}{f^2}$, are closed embedded $\mathcal{C}^{1,1}$-hypersurfaces, and for $\mathcal{L}^1$ a.e. $t \in (0, t_0)$ there exists a smooth embedded umbilical hypersurface $\Sigma_t^* \subseteq \Sigma_t$ such that

$$\mathcal{H}^{n-1}(\Sigma_t \setminus \Sigma_t^*) = 0.$$

Proof. Let $N \subseteq M$ be a compact set with smooth boundary such that $\overline{\Omega} \subseteq \mathrm{int}(N)$. By [PV20, Corollary B], there exists a geodesically complete Riemannian extension (M^*, g^*) of $(N, g/f^2)$ with $\partial M^* = \varnothing$. We denote by $\exp^*$ the exponential map of (M^*, g^*) and define

$$\Phi : \Sigma \times [0, +\infty) \to M^*, \quad \Phi(x, t) = \Phi_t(x) = \exp^*(x, -tf(x)\nu_\Omega(x))$$

and, for $x \in \Sigma$,

$$R_\Sigma(x) = \min\{\inf\{t > 0 : J^\Sigma \Phi_t(x) = 0\}, \inf\{t > 0 : \Phi_t(x) \notin \mathrm{int}(N)\}\}.$$

The conclusion now can be obtained by tracing the argument of Theorem 4.1 that gives the Heintze-Karcher inequality (4.2). We omit to repeat these details here, and we point out a couple of remarks. First, one needs to employ (6.1) in order to obtain (4.17) and (4.18) in the present setting: in fact, one can repeat verbatim the pointwise computations of [Bre13, Proposition 3.2], where only (6.1) is used. Second, to analyze the equality case, firstly we observe, exactly in the same way as in the proof of Theorem 4.1, that $M^* \setminus \Omega$ is a set of positive reach; henceforth, by Theorem 2.2, there exists $t_0 > 0$ such that Σ_t is a compact embedded $\mathcal{C}^{1,1}$-hypersurface for every $0 < t < t_0$; then, combining (4.30) and (4.37) we infer that

Σ_t^* is a smooth umbilical embedded hypersurface relatively open in Σ_t for every $t > 0$, and $\mathcal{H}^{n-1}(\Sigma_t \setminus \Sigma_t^*) = 0$ for $\mathcal{L}^1$ a.e. $t > 0$. □

We consider the upper-half space model of the hyperbolic space $\mathbb{H}^n$: namely $\mathbb{H}^n = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ and

$$g_{\mathbb{H}^n} = \frac{1}{x_n^2}(dx_1^2 + \ldots + dx_n^2).$$

Proof of Theorem 1.3. Choose $p \in \mathbb{H}^n \setminus \overline{\Omega}$ and define

$$r(x) = \mathrm{dist}_{\mathbb{H}^n}(x,p), \quad f(x) = \cosh(r(x)),$$

for $x \in \mathbb{H}^n$. Recalling that the geodesic spheres in $\mathbb{H}^n$ of radius ρ are smooth embedded umbilical hypersurface with all principal curvatures equal to $\coth(\rho)$, a straightforward computation gives that

$$\mathrm{D}^2 f = \sinh(r)\,\mathrm{D}^2 r + \cosh(r)\, dr \otimes dr,$$

$$\mathrm{D}^2 f(x)(v,v) = \cosh(r(x)), \tag{6.3}$$

$$\Delta f(x) = n\cosh(r(x)), \tag{6.4}$$

$$\begin{aligned} f(x)\,\mathrm{Ric}_{\mathbb{H}^n}(v,v) &- \mathrm{D}^2 f(x)(v,v) + \Delta f(x) \\ &= -(n-1)\cosh(r(x)) - \cosh(r(x)) + n\cosh(r(x)) = 0, \end{aligned}$$

for every $v \in T_x(\mathbb{H}^n)$ with $|v| = 1$. Henceforth, the Riemannian manifold $(\mathbb{H}^n \setminus \{p\}, g_{\mathbb{H}^n})$ endowed with the function f satisfies the hypothesis of Theorem 6.1. Arguing as in the proof of Theorem 1.1, we notice that it is not restrictive to assume that Ω is an open set such that $\partial\overline{\Omega} = \partial\Omega = \overline{\partial^*\Omega}$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$; by Allard regularity theorem we also have that $\partial^*\Omega$ is a smooth embedded hypersurface. We set $\Sigma = \partial^*\Omega$. By Theorem 3.1, $\overline{\Sigma}$ is a White $(n-1,\lambda)$-set of $\mathbb{H}^n$. Finally, we need to check that H_Σ is positive and the couple (Ω,Σ) fulfills the equality in (6.2). The equality (1.8) clearly implies that $H_\Sigma(x) = \lambda$ for $x \in \Sigma$. Moreover, since by (6.3) and (6.4) we have that

$$\mathrm{div}^\Sigma(\nabla f)(x) = \Delta f(x) - \mathrm{D}^2 f(\nu_\Omega(x), \nu_\Omega(x)) = (n-1)f(x) \quad \text{for } x \in \Sigma,$$

we infer from (1.8) that

$$\begin{aligned} (n-1)\int_\Sigma f\, d\mathcal{H}^{n-1} = \int_\Sigma \mathrm{div}^\Sigma(\nabla f)\, d\mathcal{H}^{n-1} &= \lambda \int_\Sigma \langle \nu_\Omega, \nabla f\rangle_{\mathbb{H}^n}\, d\mathcal{H}^{n-1} \\ &= \lambda \int_\Omega \Delta f\, d\mathcal{H}^n = n\lambda \int_\Omega f\, d\mathcal{H}^n. \end{aligned}$$

This implies that $H_\Sigma \equiv \lambda > 0$ and (Ω,Σ) fulfills the equality case in (6.2). We conclude that there exists $t_0 > 0$ such that the sets

$$\Sigma_t = \{p \in \Omega : \mathrm{dist}_{g^*}(p, \overline{\Sigma}) = t\}, \quad \text{for } \mathcal{L}^1 \text{ a.e. } 0 < t < t_0,$$

where $g^* = \frac{g_{\mathbb{H}^n}}{f^2}$, are closed embedded $\mathcal{C}^{1,1}$-hypersurfaces with respect to the hyperbolic metric. Since the hyperbolic metric is conformally equivalent to the Euclidean metric, and since changing conformally the metric of the ambient space preserves umbilicity, we infer that Σ_t is also umbilical with respect to the Euclidean metric. Henceforth, by [DRKS20, Lemma 3.2], we conclude that each Σ_t is a finite disjoint union of Euclidean spheres. Now the conclusion follows letting $t \to 0$.

Appendix A. Assumption (H3)* and models in General Relativity

In this appendix we check that the Reissner–Nordstrom manifolds satisfy assumption (H3)*, while the deSitter–Schwarzschild manifolds do not. This simple fact, combined with the analysis of equality cases in Brendle's Heintze–Karcher type inequalities, shows that a stronger stability mechanism for almost-CMC hypersurfaces is at a play in the R–N manifolds. Indeed, when (H3)* holds, Brendle's argument also provides, in addition to almost-umbilicality, a direct control on the oscillation of the normals with respect to the radial directions as measured by $g_N(\nu_\Omega,\nu_\Omega)$; see, for example, condition (4.4).

Let us set $G = S^{n-1}\times(s_1,s_2)$ for some $0<s_1<s_2\le+\infty$. The dS–S manifold is then $(G, g_{\rm dSS})$, where

$$g_{\rm dSS} = \frac{ds\otimes ds}{1-m\,s^{2-n}-\kappa s^2} + s^2\,g_{S^{n-1}}\,,$$

with

$$m>0\,,\qquad -\infty<\kappa<(n-2)\Big(\frac{4}{n^n\,m^2}\Big)^{1/(n-2)}\,.$$

When $\kappa>0$ the upper bound on κ guarantees that $1-m\,s^{2-n}-\kappa s^2$ has exactly two zeros $s_1<s_2$ on $(0,\infty)$, while if $\kappa\le 0$ we set $s_2=+\infty$, while s_1 is the unique zero of $1-m\,s^{2-n}-\kappa s^2$ on $(0,\infty)$. The R–N manifold is defined instead as $(G,g_{\rm RN})$, where

$$g_{\rm RN} = \frac{ds\otimes ds}{1-m\,s^{2-n}+q^2\,s^{4-2n}} + s^2\,g_{S^{n-1}}\,,\qquad m>2\,q>0\,.$$

In this case s_1 is the largest of the two solutions of $1-m\,s^{2-n}+q^2\,s^{4-2n}=0$ on $(0,\infty)$, while we set $s_2=+\infty$. Both examples can be modeled as

$$g_\omega = (1/\omega(s))\,ds\otimes ds + s^2\,g_{S^{n-1}}$$

for a smooth function $\omega:(s_1,s_2)\to(0,\infty)$. We then define

$$F(s)=\int_{s_1}^s\frac{1}{\sqrt{\omega}}\quad\forall s\in(s_1,s_2)\,,\qquad h(t)=F^{-1}(t)\quad\forall t\in(0,\bar r)\,,\bar r:=F(s_2)\,,$$

so that $h(F(s))=s$ for every $s\in(s_1,s_2)$, and

$$h'(F(s))=\sqrt{\omega(s)}\,,\qquad h''(F(s))=\omega'(s)/2\,,\qquad\forall t\in(0,\bar r)\,.\tag{A.1}$$

Setting $M=S^{n-1}\times(0,\bar r)$, the map $\phi:M\to G$ defined by $\phi(\tau,t)=(\tau,h(t))$ is such that

$$(d\phi)^*g_\omega = dr\otimes dr + h(r)^2\,g_{S^{n-1}} =: g_h\,,$$

so that (G,g_ω) is isometric to (M,g), and rigidity of CMC-hypersurfaces in dS–S and R–N manifolds can be studied in their (M,g) representations. Since (H0) holds with $\rho=1$ we have

$$2\,\frac{h''}{h}+(n-2)\,\frac{(h')^2-1}{h^2}\Big|_{F(s)} = \frac{\omega'(s)}{s}+(n-2)\,\frac{\omega(s)-1}{s^2}=\ell(s)\,,$$

for every $s\in(s_1,s_2)$. We thus find, in the case of $g_{\rm dSS}$, where $\omega(s)=1-m\,s^{2-n}-\kappa\,s^2$,

$$\ell(s)=\frac{1}{s}\Big((n-2)\,m\,s^{1-n}-2\kappa\,s\Big)+\frac{n-2}{s^2}\Big(-m\,s^{2-n}-\kappa\,s^2\Big)=-n\,\kappa\,,$$

so that (H3) holds (but (H3)* does not); in the case of g_{RN}, where $\omega(s) = 1 - m\, s^{2-n} + q^2\, s^{4-2n}$, we have an identical cancellation of the mass term, but thanks to the q^2-term we rather find

$$\ell(s) = -\frac{(n-2)\, q^2}{s^{2n-2}}\,,$$

so that $\ell(s)$ is strictly increasing on (s_1, s_2), and (H3)* holds.

□

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