

Chapter 31

Birational maps and Nori motives

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The monograph by A. Huber and S. Müller-Stach [in: *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, (2017), 207–232] contains a systematic exposition of Nori motives that were developed and studied as the “universal (co)homology theory” of algebraic varieties (or schemes), according to the prophetic vision of A. Grothendieck. Since then, some research was dedicated to applications of Nori motives in various domains of algebraic geometry: geometries in characteristic 1 (see the work of J. F. Lieber, Yu. I. Manin, and M. Marcolli [in: *Facets of Algebraic Geometry. Vol. II*, (2017), 147–227] and Yu. I. Manin and M. Marcolli [in: *Dynamics: topology and numbers* (202), 27–56]), absolute Galois group (see the article of Yu. I. Manin and M. Marcolli [*SIGMA Symmetry Integrability Geom. Methods Appl.* (2020), article no. 038]), persistence formalism (see Yu. I. Manin and M. Marcolli, [*Math. Comput. Sci.* (2020), 77–102]).

In this chapter, we sketch an approach to the problems of equivariant birational geometry developed by M. Kontsevich and Yu. Tschinkel [*Invent. Math.* (2019), 415–432], where Burnside invariants were introduced. We are making explicit the role of Nori constructions in this environment.

1 Introduction and summary

1.1 Birational maps and their symmetries

Our main objects of study here are stable *birational maps*, mostly between algebraic varieties defined over a subfield of $\mathbf{C}$. The general restriction of *stability* is discussed in [2], and the main results about natural categories/“towers” of birational maps we use here are given in [1] and [4].

Symmetries of stable maps and their moduli spaces appear in various contexts. The celebrated Grothendieck approach to the study of the *absolute Galois group* $G_{\mathbf{Q}}$ of the field of all algebraic numbers $\overline{\mathbf{Q}}$ bridged geometry and arithmetic via the tower of stable étale maps to $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. More generally, for any integral scheme X , the exact sequence

$$1 \rightarrow \pi_1(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}) \rightarrow \pi_1(X) \rightarrow G_{\mathbf{Q}} \rightarrow 1$$

merges actions of what we call *arithmetic* and *geometric* symmetries.

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In [10], in place of étale maps $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ we considered moduli spaces of stable curves of genus zero with marked points, and later demonstrated that their geometric symmetries can be approached through the *Nori motivic structures* of these moduli spaces. In this context, the exact sequence of fundamental groups above is replaced by a subextension of the “motivic fundamental groups”

$$1 \rightarrow G_{\text{mot}}(\overline{\mathbf{Q}}, \mathbf{Q}) \rightarrow G_{\text{mot}}(\mathbf{Q}, \mathbf{Q}) \rightarrow G_{\mathbf{Q}} \rightarrow 1.$$

Here we focus on similar constructions, but starting with towers of birational maps replacing towers of stable moduli spaces above.

1.2 Burnside groups and Nori diagrams

In [1, 4], and other papers it was shown that the problem of classification of birational maps can use information encoded in the tower of natural maps structurally similar to the tower of stable genus zero moduli spaces $\overline{M}_{0,n}$. Using these natural maps, in the recent article [18], A. Kresch and Yu. Tschinkel are imposing an additional geometric symmetry group from the start and then showing that this geometric symmetry can be encoded by certain analogues of “modular symbols”.

In this note we demonstrate that the Kresch–Tschinkel modular symbols defined through *Burnside groups*, also have a natural description in terms of Nori motivic structures. Briefly, we prove the following theorem:

Theorem 1.3. *Nori stratifications of appropriate towers of birational maps can be enriched to the homological Nori geometric diagram by Burnside groups of strata.*

For a more precise statement, see Theorem 4.4.

1.4 Bost–Connes structures, Burnside groups, and modular symbols

Finally, in the last section we consider the Bost–Connes formalism, which was originally introduced in [5] to connect arithmetic zeta-functions of fields of algebraic numbers with physicists’ studies of classical and quantum behaviour of physical systems with infinite number of degrees of freedom. More recently, an arithmetic form of the Bost–Connes algebra was shown to encode important aspects of $\mathbf{F}_1$ -geometry, [11].

In [22] we presented an enrichment of (equivariant) Burnside groups through a Bost–Connes algebra structure and associated categorifications. This type of Bost–Connes structures and categorification, and their relevance in the setting of $\mathbf{F}_1$ geometry, were further investigated in [19].

The integral Bost–Connes system, as defined in [11], consists of the ring $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ together with a semigroup homomorphism $\sigma: \mathbf{N} \rightarrow \text{End}(\mathbf{Z}[\mathbf{Q}/\mathbf{Z}])$ and additive maps

$\rho_n: \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ satisfying

$$\sigma_n \circ \rho_n = n \operatorname{id}, \quad \rho_n \circ \sigma_n = n \pi_n,$$

with π_n the idempotent in $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}]$ given by $\pi_n = \frac{1}{n} \sum_{nr=0} e(r)$, for $\{e(r)\}_{r \in \mathbf{Q}/\mathbf{Z}}$ the standard basis of $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$.

It is shown in [11] that the integral Bost–Connes system is a model of $\mathbf{F}_1$ -geometry, encoding the system of extensions $\mathbf{F}_{1^n}$. For that reason, one expects that this kind of structure will appear in several different arithmetic contexts, often together with some form of categorification of the Bost–Connes algebra, as discussed in [19, 22, 23].

In Section 5 of this paper we prove that a similar type of Bost–Connes structure exists at the level of the Kontsevich–Pestun–Tschinkel modular symbols of [16].

Our main result in Section 5 is the following.

Theorem 1.5. *The $\mathbf{Z}$ -modules $\mathcal{M}_n(\mathbf{Z}/N\mathbf{Z})$ of Kontsevich–Pestun–Tschinkel modular symbols, for the tower of $\mathbf{Z}/N\mathbf{Z}$ ordered by divisibility, are endowed with a Bost–Connes structure consisting of endomorphisms σ_n and linear maps ρ_n satisfying the Bost–Connes relations.*

See Lemma 5.7, Lemma 5.8, and Proposition 5.9 for more precise statements and a detailed proof.

1.6 Summary

The paper is organized as follows. Section 2 contains a short dictionary of graphic presentations of categories and functors, including the definition of a category of blowups. Nori diagrams and representations are discussed in Section 3, with focus on diagrams associated to blowups. Burnside groups are discussed in Section 4, containing also the last steps of the proof of Theorem 1.3, which identifies the associated class of Nori diagrams. Section 5 is dedicated to Bost–Connes structures. We first show how $\hat{\mathbf{Z}}$ -equivariant Kontsevich–Tschinkel Burnside rings carry a Bost–Connes structure compatible with the analogous structure on the equivariant Grothendieck ring obtained in [19]. We then focus on the Kontsevich–Pestun–Tschinkel modular symbols and we identify the corresponding Bost–Connes structure.

2 Background

2.1 Categories and their diagrams

A diagram D is family $(V(D), E(D), \partial)$ where ∂ (boundary map) is an embedding of $E(D)$ (edges) into $V(D) \times V(D)$ (ordered pairs of vertices). Each category defines its diagram, whose edges are its morphisms, and vertices are its objects.

Conversely, diagrams themselves form the objects of a category, whose morphisms imitate functors.

For a more detailed discussion of combinatorics of categories based upon diagrams, we refer the reader to [20, Section 0.2] and [8, Section 5]. Here we recall only the definition of *posets in groupoids* ([8, Definition 5.2]).

Definition 2.2. A category $\mathcal{PG}$ is called a poset in groupoids, if

- (a) for any object X of $\mathcal{PG}$, the full subcategory consisting of all objects isomorphic X , is a groupoid, that is, all morphisms in it are isomorphisms;
- (b) whenever X_1 and X_2 are not isomorphic and $\text{Hom}(X_1, X_2) \neq \emptyset$, then $\text{Hom}(X_1, X_2)$ has a single orbit with respect to the left action of the group

$$\text{Hom}(X_1, X_1) \times \text{Hom}(X_2, X_2)^{op}$$

combining precomposition and postcomposition.

A part of posets in groupoids consists of *thin categories* $\mathcal{C}$ such that if any set $\text{Hom}_{\mathcal{C}}(X_1, X_2)$ has cardinality ≤ 1 , and if both $\text{Hom}_{\mathcal{C}}(X_1, X_2)$ and $\text{Hom}_{\mathcal{C}}(X_2, X_1)$ are non-empty, then $X_1 = X_2$.

2.3 Diagrams of effective pairs ([14, Chapter 9, Definition 9.1.1.])

Fix a subfield k of $\mathbf{C}$ and define the diagram $\text{Pairs}^{\text{eff}}$ of effective pairs over k in the following way.

- (a) Vertices of $\text{Pairs}^{\text{eff}}$ are triples (X, Y, i) where X is a variety over k , $Y \subset X$ is a closed subvariety, and $i \in \mathbf{Z}$.
- (b) There are two types of edges of $\text{Pairs}^{\text{eff}}$: *functoriality edges* and *coboundary edges*:
 - (b1) Each morphism $f: X \rightarrow X'$ with $f(Y) \subset Y'$ determines edges denoted by (f^*, i) starting at (X, Y, i) and landing at $(X', Y', i + 1)$ for every $i \in \mathbf{Z}$.
 - (b2) Each ladder $Z \subset Y \subset X$ of closed subvarieties determines edges (∂, i) starting at (Y, Z, i) and ending at $(X, Y, i + 1)$ for every $i \in \mathbf{Z}$.

Whenever one needs to consider tensor structures on the categories of diagrams, one has to consider (super)gradings of such diagrams, as discussed in [14, Chapter 8].

2.4 Categories of blowups

We will now describe some categories of good blowups, following [1, Section 1.3] (with somewhat changed terminology and notation).

We will call a “good” scheme what in the Introduction to [1] is called a “noetherian quasi excellent (qe) regular scheme”.

Consider a morphism of good schemes $\varphi: X_1 \rightarrow X_2$, which is the blowing up of a coherent sheaf of ideals $I \subset \mathcal{O}_{X_2}$ (for the relevant definitions in this context, see [1, Section 2.1.8]). Alternatively, we will call such φ the blowing up of the closed subscheme defined by the equations $f = 0$ for all $f \in I$.

Assume also given normal crossings divisors $D_i \subset X_i$ such that $D_1 = \varphi^{-1}(D_2)$.

Let U be the maximal open subscheme of X_2 upon which the restriction of I is its structure sheaf. It follows that φ induces an isomorphism $\varphi^{-1}(U) \rightarrow U$.

Definition 2.5. A *good morphism* is the structure represented by a set of data (X_i, D_i, I, φ) as above.

In particular, identical morphisms, and generally, automorphisms, are good: for them $D_1 = D_2 = \emptyset$.

Consider a finite connected poset in groupoids $\mathcal{M}$ whose objects are data (X, D, I) and morphisms are good morphisms φ in the sense discussed above.

Consider the diagram (with identities) $D(\mathcal{M})$ whose vertices are objects of $\mathcal{M}$, oriented edges are morphisms of $\mathcal{M}$, and orientation of $X \rightarrow Y$ is from X to Y . Call edges corresponding to isomorphisms (in particular, identities) *horizontal*, and other edges *vertical*.

Form also the following quotient of $D(\mathcal{M})$, which we denote by $T(\mathcal{M})$: vertices of $T(\mathcal{M})$ are isomorphisms classes of $\mathcal{M}$, and oriented edges of $T(\mathcal{M})$ are orbits of non-empty sets $\text{Hom}_{\mathcal{M}}(X_1, X_2)$ with respect to the left action of the group

$$\text{Hom}(X_1, X_1) \times \text{Hom}(X_2, X_2)^{op}$$

combining precomposition and postcomposition, as in Definition 2.2 above. We omit edges corresponding to identities.

Call an edge $X_1 \rightarrow X_2$ in $T(\mathcal{M})$ *indecomposable*, if the respective morphism cannot be expressed as composition of other morphisms.

In an important particular case, the diagram $T(\mathcal{M})$ is in fact a tree oriented downwards. More precisely, starting with any of its vertex X_0 , we may consider the longest sequence of vertices (“a path down”)

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_h.$$

Assume that there is only one vertex from which any longest path down can start.

Definition 2.6. One object of the category Bl_{rs} (the regular surjective category of blowups, cf. [1, Section 1.3]) is a triple (X_2, I, D_2) associated to a good morphism φ as above. We can equivalently write the objects as (X_i, D_i, I, φ) .

A morphism between such objects of $\mathrm{Bl}_{\mathrm{rs}}$

$$(X'_i, D'_i, I', \varphi') \rightarrow (X_i, D_i, I, \varphi)$$

is represented by a regular and surjective morphism

$$g: X'_2 \rightarrow X_2$$

satisfying the following conditions:

$$g^{-1}(D_2) = D'_2, \quad g^*(I) = I'.$$

Remark. From this definition one can deduce that g induces a canonical isomorphism

$$X'_1 \rightarrow X_1 \times_{X_2} X'_2,$$

and moreover, D'_1 is the inverse image of D_1 with respect to the composition of this isomorphism and projection $X_1 \times_{X_2} X'_2 \rightarrow X_1$ (cf. [1, Definition 1.3.1]). This presentation may be helpful for defining and studying compositions of morphisms between objects of $\mathrm{Bl}_{\mathrm{rs}}$.

In the next section, we will be studying posets of groupoids as above from the viewpoint of Nori theory, as presented in [20, Section 1], but before continuing with this part, we need one more definition.

2.7 Simple normal crossings divisors

Let S be a finite set. Consider a family $\{D_s \subset X \mid s \in S\}$ of closed immersions. Following [4, Definition (3.1)], we will call it an S -labelled simple normal crossings divisor on X , if for any finite subset $S' \subseteq S$ the intersection $\bigcap_{s \in S'} D_s$ is smooth of codimension card S' .

3 Nori geometric diagrams of blowups

3.1 Categories $\mathcal{M}$

We will now introduce a class of geometric categories that will be the starting point for our enrichment of birational maps by Nori motives. The notation $\mathcal{M}$ for a generic member of this class should remind the reader that we generalize here the basic example of stable moduli spaces of genus zero and their canonical stratifications studied in [10]. Here are the basic restrictions imposed upon $\mathcal{M}$.

- (a) Objects of $\mathcal{M}$ are some objects of $\mathrm{Bl}_{\mathrm{rs}}$.

- (b) For any object (X_2, I, D_2) (cf. Definition 2.6 above) of $\mathcal{M}$, the divisor D_2 is a simple normal crossings divisor. Sets of labels S , together with their functorial behaviour, may be included as separate elements of the structure of $\mathcal{M}$.

3.2 Two classes of morphisms in $\mathcal{M}$

Fix a category $\mathcal{M}$ as above. Let $\mathcal{X} := (X_2, I_X, D_1)$ and $\mathcal{Y} := (Y_2, I_Y, D_2)$ be two objects of $\mathcal{M}$.

Assume that we have a locally closed embedding $Y_2 \hookrightarrow X_2$ which extends in a natural way to a morphism between some blowups of X_2 and Y_2 , respectively. The resulting commutative diagrams will be declared some new morphisms in $\mathcal{M}$, “morphisms of closed embeddings”: cf. [10, Section 1.5]

Similarly, “morphisms of complements to locally closed embeddings” are extensions of this definition to $X_2 \setminus \overline{Y}_2 \hookrightarrow X_2$, where $\overline{Y}_2$ denotes the closure of Y_2 in X_2 .

They are presented below as left and right sides of the commutative diagram:

$$\begin{array}{ccccc} Y_1 & \hookrightarrow & X_1 & \longleftarrow & X_1 \setminus \overline{Y}_1 \\ \downarrow \varphi_Y & & \downarrow \varphi_X & & \downarrow \varphi_{X \setminus Y} \\ Y_2 & \hookrightarrow & X_2 & \longleftarrow & X_2 \setminus \overline{Y}_2. \end{array}$$

Subschemes $Y_i \hookrightarrow X_i$ (resp. $X_i \setminus \overline{Y}_i \hookrightarrow X_i$), $i = 1, 2$, will be called *locally closed* (respectively, *open*) strata of X_i .

One can then construct a category of diagrams, as recalled in Section 2 above, using vertices and edges given by objects of $\mathcal{M}$ and the morphisms described above as functoriality edges, together with boundary edges as discussed in Section 2.3.

In the case of moduli spaces considered in [10], this construction of diagrams of blowups is related to some other interesting geometric structures, of which we mention two examples (see [9] for additional discussion).

Example 3.3 (Kapranov’s presentation of $\overline{M}_{0,n}$, $n \geq 3$). This presentation of $\overline{M}_{0,n}$, $n \geq 3$, as a result of successive blowups of projective subspaces in $\mathbf{P}^{n-3}$ was given in [15], and then used in [6] for calculating of regular automorphisms of these stable modular spaces.

Example 3.4 (Connes–Kreimer Hopf algebra of rooted trees). The Connes–Kreimer Hopf algebras of rooted trees and of Feynman graphs were introduced in [12] in the context of renormalization of perturbative quantum field theories.

Later, using the operadic formalism, F. Chapoton and M. Livernet have shown that the Connes–Kreimer Hopf algebra can be equivalently formulated in terms of

a “incidence Hopf algebra” associated to an operad structure, where the coproduct operation is based on an ordering relation on subforests, [7].

In [9, Section 2], we considered such an incidence Hopf algebra in a geometric context, associated to the strata of the moduli spaces $\overline{M}_{0,n}$, labelled by trees. Using the result of [7], we related it to the Connes–Kreimer Hopf algebra of rooted trees, and we showed how to incorporate certain Galois symmetries in the structure.

4 Nori motives and Burnside groups

Below we work over a fixed field k of characteristic 0.

Definition 4.1 ([17, Section 4, Definition 10]). Let $\mathcal{B}$ (“base scheme”) be a separated scheme of finite type over k .

Consider a smooth $\mathcal{B}$ -scheme $f: X \rightarrow \mathcal{B}$. If $U \hookrightarrow X$ is an open embedding with $\overline{U} = X$, then $f|_U: U \rightarrow \mathcal{B}$ is also a smooth $\mathcal{B}$ -scheme.

- (a) Define the set $\text{Burn}_+(\mathcal{B})$ as the set of equivalence classes of smooth $\mathcal{B}$ -schemes modulo equivalence relation generated by $f \sim f|_U$ as above. We may denote the respective equivalence class by $[f: X \rightarrow \mathcal{B}]$, or simply $[f]$.
- (b) Define the monoid structure $+$ upon $\text{Burn}_+(\mathcal{B})$ as generated by disjoint union of smooth $\mathcal{B}$ -schemes.
It generates the respective Grothendieck group $\text{Burn}(\mathcal{B})$.
- (c) Both Burnside group and Burnside monoid a naturally graded: class of X of pure dimension n belongs to $\text{Burn}_{+,n}(\mathcal{B})$ and $\text{Burn}_n(\mathcal{B})$

These constructions are covariant functors of $\mathcal{B}$: a morphism $g: \mathcal{B}' \rightarrow \mathcal{B}$ induces maps $g_*[f] := [g \circ f]$.

4.2 Boundary homomorphisms

Start with a pair $Z \subset X$, in which X is an equidimensional algebraic variety, and Z its closed subvariety of strictly lesser dimension. Moreover, we will assume that X is reduced and separated, but nothing more.

Theorem 4.3 ([17, Section 4, Theorem 11]). *On the set of Burnside groups of members of such pairs $Z \subset X$ one can define graded boundary elements*

$$\partial_Z(X) \in \text{Burn}_{\dim(X)-1}(Z)$$

satisfying two requirements:

- (a) For any proper surjective morphism $g: X' \rightarrow X$ inducing birational equivalence between X and X' and such that $Z' = g^{-1}(Z)$, we have

$$\partial_Z(X) = (g|_Z)_*(\partial_{Z'}(X')).$$

- (b) If X is smooth, and Z is an S -labelled simple normal crossings divisor in the sense explained in Section 2.4 above, then we have an explicit presentation

$$\partial_Z(X) = - \sum_{\emptyset \neq T \subseteq S} (-1)^{\text{card } T} [f_T]$$

where $f_T: D_T \times \mathbb{A}^{\text{card } T-1} \rightarrow Z$ is the composition of projection to D_T and its inclusion into Z .

Moreover, these two requirements uniquely determine boundary elements.

We can now state precisely and prove Theorem 1.3.

Consider a category $\mathcal{M}$ as in Section 3.1 above, with its objects graded by dimension. For each object $\mathcal{B}$ of $\mathcal{M}$, construct its Grothendieck–Burnside group $\text{Burn}(\mathcal{B})$. Denote by GrAb the category of graded abelian groups.

Theorem 4.4. *The natural degree zero map $\text{Ob}(\mathcal{M}) \rightarrow \text{Ob}(\text{GrAb}): \mathcal{B} \mapsto \text{Burn}(\mathcal{B})$ extends to the homological Nori geometric diagram, in which boundary edges correspond to three step towers of closed embeddings $Z \subset Y \subset X$. The morphism*

$$(X, Y, i+1) \rightarrow (Y, Z, i)$$

sends boundary element $\partial_Y(X)$ to $\partial_Z(Y)$. On obtains in this way a category $\mathcal{M}_{\text{Burn}(\mathcal{B})}$ of Nori motivic sheaves over $\mathcal{B}$.

Proof. Given a base scheme $\mathcal{B}$, we can consider a category $\mathcal{M}_{\mathcal{B}}$ of Nori motivic sheaves in the sense of [3]. This is defined as the abelian category constructed from a Nori diagram $\mathcal{D}(\mathcal{B})$ with the following vertices and edges:

- Vertices are given by elements of the form

$$(f: X \rightarrow \mathcal{B}, Y, i, w)$$

where $f: X \rightarrow \mathcal{B}$ is a smooth $\mathcal{B}$ -scheme, with an embedding $Y \hookrightarrow X$, with $f|_Y$ as structure map $f|_Y: Y \rightarrow \mathcal{B}$, and integers $i \in \mathbb{N}$, $w \in \mathbb{Z}$.

- Edges are of three types:

- (1) edges $h^*: (X' \rightarrow \mathcal{B}, Y', i, w) \rightarrow (X \rightarrow \mathcal{B}, Y, i, w)$ associated to morphisms $h: (X \rightarrow \mathcal{B}, Y) \rightarrow (X' \rightarrow \mathcal{B}, Y')$;
- (2) connecting morphisms $\partial: (Y \rightarrow \mathcal{B}, Z, i, w) \rightarrow (X \rightarrow \mathcal{B}, Y, i+1, w)$ associated to a chain of embeddings $Z \hookrightarrow Y \hookrightarrow X$;

- (3) twisted projections $(X, Y, i, w) \rightarrow (X \times \mathbf{P}^1, Y \times \mathbf{P}^1 \cup X \times \{0\}, i + 2, w + 1)$.

In particular, we can restrict to considering equivalence classes of diagrams in $\mathcal{D}(\mathcal{B})$ where vertices $(f: X \rightarrow \mathcal{B}, Y, i, w)$ and $(f|_U: U \rightarrow \mathcal{B}, Y \cap U, i, w)$ are identified, for $j: U \hookrightarrow X$ an open embedding with $\bar{U} = X$, and edges of type (1) are identified if they fit commutative diagrams of the form

$$\begin{array}{ccc} U & \xrightarrow{h|_U} & U' \\ \downarrow j & & \downarrow j' \\ X & \xrightarrow{h} & X' \end{array}$$

where all the maps are compatible with the structure maps to $\mathcal{B}$, not shown in the diagram. Edges of type (2) descend to edges of type (2) on the quotient diagrams by Theorem 4.3 above, and twisted projections also induce corresponding edges. We can then use this resulting class $\mathcal{D}(\mathcal{B})/\sim$ of Nori diagrams to form a new category $\mathcal{M}_{\text{Burn}(\mathcal{B})}$ of Nori motivic sheaves over $\mathcal{B}$. ■

5 Bost–Connes system and Burnside groups

As we recalled in Section 1.4, the integral Bost–Connes algebra consists of the ring $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ together with a semigroup of endomorphisms σ_n and partial inverses given by linear maps, designed to model the arithmetic Frobenius and Verschiebung maps in characteristic one. Thus, the presence of a Bost–Connes structure (at the level of algebras or of a categorification) can be regarded as a sign of the existence of an underlying arithmetic structure over $\mathbf{F}_1$. In [19] and [22] several such generalizations and categorifications of the Bost–Connes algebra were constructed based on $\hat{\mathbf{Z}}$ -equivariant Grothendieck rings, related to the original Bost–Connes algebra by equivariant Euler characteristics. The categorifications were obtained either through the assembler categories introduced by I. Zakharevich in [24, 25], which categorify the scissor congruence relation of Grothendieck rings, or through categories of Nori motives.

In [22] it was shown that there is a Bost–Connes system of endomorphisms acting on the $\hat{\mathbf{Z}}$ -equivariant version of the Kontsevich–Tschinkel Burnside ring. This is a lift of the integral Bost–Connes algebra of [11], through the map to the graded ring associated to the filtration of the Grothendieck ring by dimension, which in turn maps to the integral Bost–Connes algebra through the equivariant Euler characteristic.

On the other hand, in [19] it was also shown that the lift of the Bost–Connes algebra to the $\hat{\mathbf{Z}}$ -equivariant Grothendieck ring of varieties can be lifted higher, to a

$\hat{\mathbf{Z}}$ -equivariant category of Nori motives, where it maps through the fibre functor to the categorification of the Bost–Connes system constructed in [23].

We review here briefly the setting of [19] and [22], and then we show that a similar categorification of the Bost–Connes system via Nori motives can be constructed in the case of the Kontsevich–Tschinkel Burnside group. As was done in [19] for the Grothendieck ring, we work here with the relative version of the Kontsevich–Tschinkel Burnside ring considered in the previous sections, and with its $\hat{\mathbf{Z}}$ -equivariant version.

5.1 Relative equivariant Kontsevich–Tschinkel Burnside group

In the previous sections we considered a base scheme $\mathcal{B}$ and the Kontsevich–Tschinkel Burnside group $\text{Burn}(\mathcal{B})$.

Here we introduce its equivariant version $\text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B})$ where now $\mathcal{B}$ is endowed with a residually finite action of $\hat{\mathbf{Z}}$. The last condition means that this action factors through some finite $\mathbf{Z}/N\mathbf{Z}$ -quotient of $\hat{\mathbf{Z}}$.

The Burnside group $\text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B})$ then is generated by equivalence classes of smooth $\mathcal{B}$ -schemes $f: X \rightarrow \mathcal{B}$ where X is also endowed with a residually finite action of $\hat{\mathbf{Z}}$, and f is $\hat{\mathbf{Z}}$ -equivariant. Equivalence classes correspond to the equivalence relation $f \sim f|_U$ where $U \hookrightarrow X$ is a $\hat{\mathbf{Z}}$ -equivariant dense open embedding.

As in [19], we will adopt the notation $(\mathcal{B}, \alpha_{\mathcal{B}})$ for the base scheme endowed with action $\alpha_{\mathcal{B}}: \hat{\mathbf{Z}} \times \mathcal{B} \rightarrow \mathcal{B}$, and we similarly write (X, α_X) and $f: (X, \alpha_X) \rightarrow (\mathcal{B}, \alpha_{\mathcal{B}})$ to keep track explicitly of the $\hat{\mathbf{Z}}$ -actions.

5.2 Lifting the Bost–Connes system

We will use notation and conventions of [19], which are briefly repeated below.

Put $Z_n := \text{Spec } K^n$ where K is our ground field.

Given $(\mathcal{B}, \alpha_{\mathcal{B}})$ as above, denote by $\Phi(\alpha_{\mathcal{B}})$ the $\hat{\mathbf{Z}}$ -action upon $\mathcal{B} \times Z_n$ through composition with geometric *Verschiebung* map: see [19, (2.12) and (2.13)]. Namely, for ζ_N a primitive N th root of unity, we write

$$V_n(\zeta_{Nn}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha(\zeta_N) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

so that we can write the *Verschiebung* map as $V_n(\zeta_{Nn}) \cdot \underline{x} = (x, a_{i+1})$ for $i = 1, \dots, n-1$ and $V_n(\zeta_{Nn}) \cdot \underline{x} = (\alpha(\zeta_N) \cdot x, a_1)$ for $i = n$, so that it satisfies

$V_n(\zeta_{Nn})^n = \alpha(\zeta_N) \times Id_{Z_n}$. The resulting action $\Phi_n(\alpha)$ of $\hat{\mathbf{Z}}$ on $\mathcal{B} \times Z_n$ that factors through $\mathbf{Z}/Nn\mathbf{Z}$ is specified by setting

$$\Phi_n(\alpha)(\zeta_{Nn}) \cdot (x, a) = (V_n(\alpha(\zeta_N)) \cdot \underline{x})_a,$$

where for $x \in \mathcal{B}$ we write $\underline{x} = (x, a_i)_{a_i \in Z_n} = (x_i)_{i=1}^n$ for the subset $\{x\} \times Z_n \subset \mathcal{B} \times Z_n$.

This action is residually finite as well: if the action $\alpha_{\mathcal{B}}$ of $\hat{\mathbf{Z}}$ factors through $\mathbf{Z}/N\mathbf{Z}$, then the action $\Phi(\alpha_{\mathcal{B}})$ factors through $\mathbf{Z}/Nn\mathbf{Z}$. Denote by σ_n the endomorphism of multiplication by n .

Now, by lifting σ_n (and avoiding extra notation), define the maps σ_n and $\tilde{\rho}_n$ as follows:

$$\begin{aligned} \sigma_n: (f: (X, \alpha_X) \rightarrow (\mathcal{B}, \alpha_{\mathcal{B}})) &\mapsto (f: (X, \alpha_X \circ \sigma_n) \rightarrow (\mathcal{B}, \alpha_{\mathcal{B}} \circ \sigma_n)), \\ \tilde{\rho}_n: (f: (X, \alpha_X) \rightarrow (\mathcal{B}, \alpha_{\mathcal{B}})) &\mapsto (f \times \text{id}: (X \times Z_n, \Phi_n(\alpha_X)) \rightarrow (\mathcal{B} \times Z_n, \Phi_n(\alpha_{\mathcal{B}}))). \end{aligned}$$

Proposition 5.3. *The maps above induce ring homomorphisms*

$$\sigma_n: \text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}}) \rightarrow \text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}} \circ \sigma_n)$$

and group homomorphisms

$$\tilde{\rho}_n: \text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}}) \rightarrow \text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B} \times Z_n, \Phi_n(\alpha_{\mathcal{B}})).$$

For proofs, see [19, Theorem 3.15].

The induced morphisms of the Kontsevich–Tschinkel Burnside group lift the maps σ_n and $\tilde{\rho}_n$ of the integral Bost–Connes algebra of [11] in the following way.

Proposition 5.4. *The action of the σ_n and $\tilde{\rho}_n$ on the $\text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$ is compatible with the σ_n and $\tilde{\rho}_n$ constructed in [19] on the Grothendieck rings $K_0^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$, through the morphism $\text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}}) \rightarrow \text{gr } K_0^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$ to the associated graded object with respect to filtration by dimension.*

Proof. The maps σ_n and $\tilde{\rho}_n$ on $K_0^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$ are constructed in [19] using the same formulas as above. These preserve the filtration by dimension, since the schemes Z_n are zero dimensional. Thus, the map $\text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}}) \rightarrow \text{gr } K_0^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$ intertwines the action of the σ_n and $\tilde{\rho}_n$ on the Burnside ring and on the Grothendieck ring.

It is also shown in [19, Theorem 2.11] that the actions of σ_n and $\tilde{\rho}_n$ on $K_0^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$ are compatible with the σ_n and $\tilde{\rho}_n$ acting on $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$, through an equivariant Euler characteristic map. ■

5.5 Equivariant Nori motives

Given a base scheme $\mathcal{B}$ with a residually finite $\hat{\mathbf{Z}}$ -action $\alpha_{\mathcal{B}}$, we can consider, as in [19], Section 7, a category $\mathcal{M}_{(\mathcal{B}, \alpha_{\mathcal{B}})}$ of $\hat{\mathbf{Z}}$ -equivariant Nori motivic sheaves. This is defined as the abelian category constructed from a Nori diagram $\mathcal{D}(\mathcal{B}, \alpha_{\mathcal{B}})$ with the following vertices and edges:

- Vertices are given by elements of the form

$$(f: (X, \alpha_X) \rightarrow (\mathcal{B}, \alpha_{\mathcal{B}}), (Y, \alpha_X|_Y), i, w)$$

where $f: (X, \alpha_X) \rightarrow (\mathcal{B}, \alpha_{\mathcal{B}})$ is a smooth $\hat{\mathbf{Z}}$ -equivariant $\mathcal{B}$ -scheme, with a $\hat{\mathbf{Z}}$ -equivariant embedding $(Y, \alpha_X|_Y) \hookrightarrow (X, \alpha_X)$, and $i \in \mathbf{N}$, $w \in \mathbf{Z}$.

- Edges are of three types:

- (1) edges $h^*: (X' \rightarrow \mathcal{B}, Y', i, w) \rightarrow (X \rightarrow \mathcal{B}, Y, i, w)$ associated to morphisms $h: (X \rightarrow \mathcal{B}, Y) \rightarrow (X' \rightarrow \mathcal{B}, Y')$;
- (2) connecting morphisms $\partial: (Y \rightarrow \mathcal{B}, Z, i, w) \rightarrow (X \rightarrow \mathcal{B}, Y, i + 1, w)$ associated to a chain of embeddings $Z \hookrightarrow Y \hookrightarrow X$;
- (3) twisted projections $(X, Y, i, w) \rightarrow (X \times \mathbf{P}^1, Y \times \mathbf{P}^1 \cup X \times \{0\}, i + 2, w + 1)$.

As was shown in [19], the maps σ_n and $\tilde{\rho}_n$ from above induce functors of the categories of Nori motivic sheaves considered above, with

$$\begin{aligned} \sigma_n: \mathcal{M}_{(\mathcal{B}, \alpha_{\mathcal{B}})} &\rightarrow \mathcal{M}_{(\mathcal{B}, \alpha_{\mathcal{B}} \circ \sigma_n)} \\ \tilde{\rho}_n: \mathcal{M}_{(\mathcal{B}, \alpha_{\mathcal{B}})} &\rightarrow \mathcal{M}_{(\mathcal{B} \times \mathbf{Z}_n, \Phi_n(\alpha_{\mathcal{B}}))}. \end{aligned}$$

The functors σ_n are compatible with the monoidal structure, but $\tilde{\rho}_n$ are not, as discussed in [19].

By construction, the Bost–Connes structure on the categories $\mathcal{M}_{(\mathcal{B}, \alpha_{\mathcal{B}})}$ of Nori motives is compatible with those discussed above on $K_0^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$ and on $\text{Burn}^{\hat{\mathbf{Z}}}(\mathcal{B}, \alpha_{\mathcal{B}})$. In particular, the same construction can also be applied to the $\hat{\mathbf{Z}}$ -equivariant version of the Nori motives considered in Theorem 4.4, by working with the $\hat{\mathbf{Z}}$ -equivariant version $\mathcal{M}_{\text{Burn}(\mathcal{B}, \alpha_{\mathcal{B}})}$ of the category $\mathcal{M}_{\text{Burn}(\mathcal{B})}$.

5.6 The Bost–Connes structure of the Kontsevich–Pestun–Tschinkel modular symbols

As summarised above, in [19] and [22] we considered various lifts of the Bost–Connes algebra to Grothendieck rings, assemblers, spectra, and Nori motives, based on varieties with good actions of $\hat{\mathbf{Z}}$. This setting includes the case of equivariant Kontsevich–Tschinkel Burnside ring and associated assembler and Nori motives

described here in the previous subsections. There is another setting in birational geometry where the Bost–Connes structure naturally appears, which we discuss in this subsection, namely the Kontsevich–Pestun–Tschinkel modular symbols of [16].

As in [16] we consider, for $n \in \mathbb{N}$ and G a finite abelian group, the $\mathbb{Z}$ -modules $\mathcal{M}_n(G)$, generated by symbols $\langle a_1, \dots, a_n \rangle$ with $a_i \in A = G^\vee := \text{Hom}(G, \mathbb{C}^*)$, such that $a_1, \dots, a_n$ generate A , with relations:

- (1) $\langle a_{\sigma(1)}, \dots, a_{\sigma(n)} \rangle = \langle a_1, \dots, a_n \rangle$, for all permutations $\sigma \in S_n$;
- (2) for all $2 \leq k \leq n$ and all $a_1, \dots, a_k$ and $b_1, \dots, b_{n-k}$ in A satisfying

$$\sum_i \mathbb{Z}a_i + \sum_j \mathbb{Z}b_j = A$$

one has the relation

$$\langle a_1, \dots, a_k, b_1, \dots, b_{n-k} \rangle = \sum_{1 \leq i \leq k} \langle a_1 - a_i, \dots, a_i, \dots, a_k - a_i, b_1, \dots, b_{n-k} \rangle,$$

with a_i in the i th place.

As shown in [16], these relations reflect certain scissor-congruence relations on convex cones in lattices. We also consider as in [16] the quotient $\mathcal{M}_n^-(G)$ of $\mathcal{M}_n(G)$ by the further relation $\langle -a_1, \dots, a_n \rangle = -\langle a_1, \dots, a_n \rangle$.

There is a $\mathbb{Z}$ -bilinear multiplication, for $n = n' + n''$,

$$\nabla: \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \rightarrow \mathcal{M}_n(G)$$

associated to an exact sequence of finite abelian groups

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0,$$

with both G' and G'' nontrivial, given by

$$\nabla: \langle a_1, \dots, a_{n'} \rangle \otimes \langle b_1, \dots, b_{n''} \rangle \mapsto \sum \langle \tilde{a}_1, \dots, \tilde{a}_{n'}, \tilde{b}_1, \dots, \tilde{b}_{n''} \rangle,$$

where the sum is over all the lifts $\tilde{a}_i$ in A , in the dual exact sequence

$$0 \rightarrow A'' \rightarrow A \rightarrow A' \rightarrow 0,$$

while the $\tilde{b}_j$ are the images in A of the b_j under the embedding $A'' \rightarrow A$. The multiplication map descends to the $\mathcal{M}_n^-(G)$ in the same form.

There is similarly a $\mathbf{Z}$ -bilinear comultiplication, for $n = n' + n''$, determined by

$$\Delta: \mathcal{M}_n(G) \rightarrow \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}^-(G'')$$

for a sequence as above with G'' nontrivial, given by

$$\Delta: \langle a_1, \dots, a_n \rangle \mapsto \sum \langle a_{I'} \bmod A'' \rangle \otimes \langle a_{I''} \rangle^-,$$

for $\{1, \dots, n\} = I' \sqcup I''$ with $\#I' = n'$, $\#I'' = n''$, $a_{I'} = a_{i_1} \dots a_{i_{n'}}$, and $a_{I''} = a_{j_1} \dots a_{j_{n''}}$ for $I' = \{i_1, \dots, i_{n'}\}$ and $I'' = \{j_1, \dots, j_{n''}\}$, such that all the a_{j_k} are in $A'' \hookrightarrow A$. Here taking the quotient $\mathcal{M}_{n''}^-(G'')$ instead of $\mathcal{M}_{n''}(G'')$ is necessary because of the second type of relations in $\mathcal{M}_n(G)$, see Proposition 9 of [16]. The comultiplication also descends to $\mathcal{M}_n^-(G)$.

In particular, we will focus here on the case of $\mathcal{M}_{n,N} := \mathcal{M}_n(\mathbf{Z}/N\mathbf{Z})$. We write $G_N = \mathbf{Z}/N\mathbf{Z}$, and A_N for the characters and we consider the projective system of the A_N ordered by divisibility, with the maps $\sigma_k: A_N \rightarrow A_M$ for $M|N$ with $N = Mk$, given by $\sigma_k: \zeta \mapsto \zeta^k$ when we identify A_N with the group of N th roots of unity (multiplication by k if written additively). Dually we have the injective system of the G_N with the corresponding inclusions $j_k: G_M \rightarrow G_N$, so that

$$\varinjlim G_N = \mathbf{Q}/\mathbf{Z}, \quad \varprojlim A_N = \hat{\mathbf{Z}}.$$

We define $\mathcal{M}_n(\mathbf{Q}/\mathbf{Z})$ as the $\mathbf{Z}$ -module spanned by the $\mathcal{M}_n(G_N)$ for all N . We write elements of $\mathcal{M}_n(\mathbf{Q}/\mathbf{Z})$ in the form $\sum_i c_i \langle a_i \rangle$ where we use the shorthand notation $\langle a_i \rangle := \langle a_{i,1}, \dots, a_{i,n} \rangle$ with $a_{i,k} \in A_{N_i}$. We also write $\mathcal{M}_{\mathbf{Q}}(\mathbf{Q}/\mathbf{Z}) := \mathcal{M}(\mathbf{Q}/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$, and denote by $\mathcal{M}_n(\mathbf{Q}/\mathbf{Z})^-$ the span of the $\mathcal{M}_n(G_N)^-$.

Let $\sigma_k: \mathcal{M}_n(\mathbf{Q}/\mathbf{Z}) \rightarrow \mathcal{M}_n(\mathbf{Q}/\mathbf{Z})$ be the map of $\mathbf{Z}$ -modules determined by

$$\sigma_k: \langle a \rangle \mapsto \langle \sigma_k(a) \rangle,$$

for $\langle a \rangle = \langle a_1, \dots, a_n \rangle$ in $\mathcal{M}_n(\mathbf{Z}/N\mathbf{Z})$ and $\langle \sigma_k(a) \rangle = \langle \sigma_k(a_1), \dots, \sigma_k(a_n) \rangle$. We set $\sigma_k \langle a \rangle = 0$ whenever $\langle \sigma_k(a) \rangle$ would not be an acceptable symbol: for instance this happens for $\langle a \rangle \in \mathcal{M}_n(\mathbf{Z}/k\mathbf{Z})$ when all $\sigma_k(a_i) = 0 \in \mathbf{Z}/k\mathbf{Z}$. We also consider the maps of $\mathbf{Z}$ -modules

$$\rho_k: \mathcal{M}_n(G_M) \rightarrow \mathcal{M}_n(G_N)$$

that maps a symbol to the sum over preimages

$$\rho_k: \langle a_1, \dots, a_n \rangle \mapsto \sum_{\sigma_k(b_i)=a_i} \langle b_1, \dots, b_n \rangle.$$

With a slight abuse of notation we will write for convenience

$$\langle \rho_k(a) \rangle := \sum_{\sigma_k(b_i)=a_i} \langle b_1, \dots, b_n \rangle,$$

so that we can write $\tilde{\rho}_k \langle a \rangle = \langle \tilde{\rho}_k(a) \rangle$. On $\mathcal{M}_{\mathbf{Q}}(\mathbf{Q}/\mathbf{Z})$ we also consider the $\mathbf{Q}$ -linear maps

$$\hat{\rho}_k = \frac{1}{k^n} \tilde{\rho}_k: \mathcal{M}_{\mathbf{Q},n}(\mathbf{Q}/\mathbf{Z}) \rightarrow \mathcal{M}_{\mathbf{Q},n}(\mathbf{Q}/\mathbf{Z}),$$

$$\hat{\rho}_k: \langle a_1, \dots, a_n \rangle \mapsto \frac{1}{k^n} \sum_{\sigma_k(b_i)=a_i} \langle b_1, \dots, b_n \rangle.$$

Notice that the notation we are using here differs from [11, 19, 22] where we used $\tilde{\rho}_k$ for what is here called ρ_k and ρ_k for what is here called $\hat{\rho}_k$. The reason for this change of notation is that, as maps on the group ring $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$, the $\tilde{\rho}_k$ of [11] are not ring homomorphisms, while the ρ_k of [11] are algebra homomorphisms of $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}]$ and are the original Bost–Connes maps. Here however, since we will be considering the multiplication structure on the $\mathcal{M}_n(G)$ as in [16] rather than the group ring $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$, it will turn out that the $\tilde{\rho}_k$ of [11] are the fundamental maps, which we call simply ρ_k .

Lemma 5.7. *The maps σ_k and ρ_k on $\mathcal{M}_n(\mathbf{Q}/\mathbf{Z})$ defined as above satisfy the relations $\sigma_k \sigma_\ell = \sigma_{k\ell}$, $\rho_k \rho_\ell = \rho_{k\ell}$. Moreover, if $(k, \ell) = \gcd\{k, \ell\} = 1$, then*

$$\sigma_k \circ \rho_\ell = \rho_\ell \circ \sigma_k,$$

while

$$\sigma_k \circ \rho_k = k \cdot \text{id}, \quad \rho_k \circ \sigma_k = e_k,$$

with the map $e_k: \mathcal{M}_n(\mathbf{Q}/\mathbf{Z}) \rightarrow \mathcal{M}_n(\mathbf{Q}/\mathbf{Z})$ given by

$$e_k \langle a_1, \dots, a_n \rangle = \sum_{ks_i=0} \langle a_1 + s_1, \dots, a_n + s_n \rangle.$$

Proof. The compositions $\sigma_k \sigma_\ell = \sigma_{k\ell}$ and $\rho_k \rho_\ell = \rho_{k\ell}$ follow directly from the definition. For $(k, \ell) = 1$ we have

$$\begin{aligned} \sigma_k \sum_{\ell b_i=a_i} \langle b_1, \dots, b_n \rangle &= \sum_{\ell b_i=a_i} \langle kb_1, \dots, kb_n \rangle \\ &= \sum_{\ell kb_i=ka_i} \langle kb_1, \dots, kb_n \rangle = \rho_\ell \langle ka_1, \dots, ka_n \rangle. \end{aligned}$$

On the other hand $\sigma_k \sum_{kb_i=a_i} \langle b_1, \dots, b_n \rangle = \sum_{kb_i=a_i} \langle a_1, \dots, a_n \rangle = k \cdot \langle a_1, \dots, a_n \rangle$ and $\rho_k \langle ka_1, \dots, ka_n \rangle = \sum_{kb_i=ka_i} \langle b_1, \dots, b_n \rangle = \sum_{ks_i=0} \langle a_1 + s_1, \dots, a_n + s_n \rangle$. The more general case with $(k, \ell) \neq 1$ is also obtained by combining the relations above. ■

Note that, unlike in the case of the maps σ_n and ρ_n acting on $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}]$ in the Bost–Connes system, the e_k here are not idempotents with respect to ∇ .

The following lemma is a direct consequence of the explicit expression for the multiplication and comultiplication and of the maps σ_k and ρ_k defined above.

Lemma 5.8. *The multiplication ∇ induces a multiplication*

$$\nabla: \mathcal{M}_{n'}(\mathbf{Q}/\mathbf{Z}) \otimes \mathcal{M}_{n''}(\mathbf{Q}/\mathbf{Z}) \rightarrow \mathcal{M}_{n'+n''}(\mathbf{Q}/\mathbf{Z}),$$

which is given by

$$\nabla(\langle a \rangle, \langle b \rangle) = \langle \rho_\ell(a), b \rangle,$$

for $\langle b \rangle \in \mathcal{M}_n(\mathbf{Z}/\ell\mathbf{Z})$. The comultiplication Δ induces a comultiplication

$$\Delta: \mathcal{M}_n(\mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{n'+n''=n} \mathcal{M}_{n'}(\mathbf{Q}/\mathbf{Z}) \otimes \mathcal{M}_{n''}(\mathbf{Q}/\mathbf{Z})$$

given by

$$\Delta\langle a \rangle = \bigoplus \langle \sigma_\ell(a_{I'}) \rangle \otimes \langle a_{I''} \rangle^-,$$

where for $\langle a \rangle \in \mathcal{M}_n(G_N)$ the sum is over $\ell|N$ and over all partitions

$$I' \sqcup I'' = \{1, \dots, n\},$$

with $\langle a_{I''} \rangle$ in $\mathcal{M}_n(\text{Ker}(\sigma_\ell))$.

Thus, we can consider the graded ring $\mathcal{M}(\mathbf{Q}/\mathbf{Z}) = \bigoplus_n \mathcal{M}_n(\mathbf{Q}/\mathbf{Z})$ endowed with this multiplication, and the $\mathbf{Q}$ -algebra $\mathcal{M}_{\mathbf{Q}}(\mathbf{Q}/\mathbf{Z})$.

Proposition 5.9. *The maps $\rho_k: \mathcal{M}(\mathbf{Q}/\mathbf{Z}) \rightarrow \mathcal{M}(\mathbf{Q}/\mathbf{Z})$ are ring homomorphisms, while the maps $\sigma_k: \mathcal{M}(\mathbf{Q}/\mathbf{Z}) \rightarrow \mathcal{M}(\mathbf{Q}/\mathbf{Z})$ are coalgebra homomorphisms.*

Proof. As mentioned above, the multiplication in $\mathcal{M}(\mathbf{Q}/\mathbf{Z})$ is given by $\nabla(\langle a \rangle, \langle b \rangle) = \langle \rho_\ell(a), b \rangle$, hence we have

$$\rho_k \circ \nabla(\langle a \rangle, \langle b \rangle) = \langle \rho_k \rho_\ell(a), \rho_k(b) \rangle = \langle \rho_\ell \rho_k(a), \rho_k(b) \rangle = \nabla \circ \rho_k(\langle a \rangle, \langle b \rangle),$$

while the comultiplication is given by $\Delta\langle a \rangle = \bigoplus \langle \sigma_\ell(a_{I'}) \rangle \otimes \langle a_{I''} \rangle^-$ so that

$$\sigma_k \circ \Delta\langle a \rangle = \bigoplus \langle \sigma_k \sigma_\ell(a_{I'}) \rangle \otimes \langle \sigma_k(a_{I''}) \rangle^-.$$

The sum is over all $\ell|N$, for $\langle a \rangle \in \mathcal{M}_n(\mathbf{Z}/N\mathbf{Z})$ and over all decompositions I', I'' where $\langle a_{I''} \rangle$ is in $\mathcal{M}_n(\mathbf{Z}/\ell\mathbf{Z})$. If $(k, N) = 1$, then also $(k, \ell) = 1$ and we have $\sigma_\ell: \mathbf{Z}/N\mathbf{Z} \rightarrow \mathbf{Z}/N\mathbf{Z}$ and $\langle a_{I''} \rangle$ is in $\mathcal{M}_n(\mathbf{Z}/\ell\mathbf{Z})$ if and only if the same holds for $\langle \sigma_k(a_{I''}) \rangle$, hence the sum on the right hand side is the same as $\Delta \circ \sigma_k\langle a \rangle$. It suffices then to check the case where $k|N$, with $N = kM$. Then $\langle \sigma_k(a) \rangle \in \mathcal{M}_n(\mathbf{Z}/M\mathbf{Z})$ and $\Delta\langle \sigma_k(a) \rangle$ is the sum over all the $\ell'|M$ and the partitions I', I'' with $\langle b_{I''} \rangle \in$

$\mathcal{M}_n(\mathbf{Z}/\ell'\mathbf{Z})$, up to terms that are mapped to zero by σ_k when $a_{I''} \in \mathbf{Z}/k\mathbf{Z}$. Thus, the maps ρ_k are compatible with the multiplication ∇ and the maps σ_k are compatible with the comultiplication Δ .

Note that the maps σ_k are not compatible with multiplication since

$$\sigma_k \nabla(\langle a \rangle, \langle b \rangle) = \langle \sigma_k \rho_\ell(a), \sigma_k(b) \rangle.$$

For $(k, \ell) = 1$ one has $\sigma_k \rho_\ell = \rho_\ell \sigma_k$, which gives the right compatibility, but in the case where $(k, \ell) \neq 1$ this is not the case. For simplicity, consider the case where $\ell = k$, where $\langle \sigma_k \rho_k(a), \sigma_k(b) \rangle = \langle a, \sigma_k(b) \rangle$ while

$$\langle \rho_k \sigma_k(a), \sigma_k(b) \rangle = \sum_{ks_i=0} \langle a + s, \sigma_k(b) \rangle,$$

with $a + s = (a_1 + s_1, \dots, a_n + s_n)$. Similarly, the maps ρ_k are not compatible with comultiplication for exactly the same reason.

The data $(\mathcal{M}(\mathbf{Q}/\mathbf{Z}), \nabla, \Delta, \rho_k, \sigma_k)$ provide the Bost–Connes structure associated to the Kontsevich–Pestun–Tschinkel modular symbols.

In the original Bost–Connes quantum statistical mechanical system, the maps ρ_k have a geometric interpretation in terms of lattices and commensurability, see [13, Chapter 3]. One defines $\mathbf{Q}$ -lattices as pairs (L, ϕ) of a rank n -lattice L (with $n = 1$ in the original Bost–Connes case) and a group homomorphism $\phi: \mathbf{Q}^n/\mathbf{Z}^n \rightarrow \mathbf{Q}L/L$. Two $\mathbf{Q}$ -lattices are commensurable if $\mathbf{Q}L_1 = \mathbf{Q}L_2$ and $\phi_1 = \phi_2$ modulo $L_1 + L_2$. In the rank one case, the commensurability relation is precisely captured by the maps ρ_k of the Bost–Connes system.

In the case of the Kontsevich–Pestun–Tschinkel modular symbols, it is shown in [16] that the $\mathcal{M}_n(G)$ can also be described in terms of lattices and cones. For $\mathcal{M}_n(G)$, one considers isomorphism classes of data (L, χ, Λ) with $L \simeq \mathbf{Z}^n$ a lattice of rank n , with an element $\chi \in L \otimes A$ for which the induced homomorphism $L^\vee \rightarrow A$ is surjective, and a strictly convex cone $\Lambda \subset L_{\mathbf{R}}$ spanned by a basis of L . To a datum (L, χ, Λ) one associates a symbol $\psi(L, \chi, \Lambda) = \langle a_1, \dots, a_n \rangle$ obtained by a choice of a basis $\{e_i\}$ of L with respect to which $\chi = \sum_i e_i \otimes a_i$. The relations of $\mathcal{M}_n(G)$ are then described geometrically in terms of scissor congruence relations $\psi(L, \chi, \Lambda) = \sum_i \psi(L, \chi, \Lambda_i)$ determined by possible decompositions of the cone.

To see the compatibility between these two settings, note that a lattice L with the choice of a basis $\{e_i\}$ determines an invertible $\mathbf{Q}$ -lattice of rank n , by taking $\phi: \mathbf{Q}^n/\mathbf{Z}^n \rightarrow \mathbf{Q}L/L$ that maps $\phi: (a_1, \dots, a_n) \mapsto \sum_i a_i e_i$. To a (not necessarily invertible) $\mathbf{Q}$ -lattice with $\phi: \mathbf{Q}^n/\mathbf{Z}^n \rightarrow \mathbf{Q}L/L$, we can assign an element $\chi \in L \otimes \mathbf{Q}/\mathbf{Z}$ by $\chi = \sum_i \phi(e_i) \otimes a_i$ with e_i the tuple in $(\mathbf{Q}/\mathbf{Z})^n$ with 1 in the i th position and zeroes elsewhere. For a fixed $\mathbf{Z}/N\mathbf{Z}$, the restriction of ϕ to $(\mathbf{Z}/N\mathbf{Z})^n$ induces on those n -tuples that determine corresponding symbols $\langle a_1, \dots, a_n \rangle$ map

$\langle a_1, \dots, a_n \rangle \mapsto \sum_i e_i \otimes a_i$ that gives rise to an associated χ as above. The maps ρ_k are then implementing the commensurability relation as in the Bost–Connes case on the rank one $\mathbf{Q}$ -lattices obtained by projecting along the e_i directions. ■

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