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Almgren-type monotonicity formulas

Mariana Smit Vega Garcia

Western Washington University, Department of Mathematics, 516 High Street, Bellingham, WA, USA

Abstract. The first main goal of this survey is to showcase the celebrated Almgren monotonicity formula. Having provided different examples of its far-reaching consequences, we apply the techniques developed in [16] and [17] to show how one can prove a parabolic Almgren monotonicity formula as a high-dimensional limit of elliptic ones.

Keywords: Almgren monotonicity problem, parabolic, elliptic, free boundary problem, variable coefficients.

2020 Mathematics Subject Classification: 35R35.

1 Introduction

In this survey, we showcase the celebrated Almgren monotonicity formula, [1], [2]. This groundbreaking result is a cornerstone in the study of harmonic functions; it also plays a crucial role in studying unique continuation, and has been used extensively in free boundary problems. It is usually stated as follows: let $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function and let B_r denote a ball of radius r. Then

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 dx}{\int_{\partial B_r} u(x)^2 d\sigma} \qquad \text{is non-decreasing.}$$

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E-mail: Mariana.SmitVegaGarcia@wwu.edu

One might let $r \to 0$ to deduce information from the limit. Conversely, the presence of oscillation implies the existence of larger oscillations at a larger scale.

Firstly, we will discuss the importance of this formula in the proof of the regularity of energy minimizers and almost minimizers (and their free boundaries, in the case of free boundary problems). Secondly, we will discuss the techniques from [16] and [17], where the authors prove parabolic results as high-dimensional limits of elliptic ones. We exemplify the ideas from [16] and [17] by showing how to prove parabolic Almgren monotonicity formulas, both in the constant and variable coefficient cases.

In Section 2, we discuss the Dirichlet problem for the Laplacian, showcasing how Almgren's monotonicity formula can be used to address the regularity of local minimizers of the Dirichlet energy. An overview of the classical obstacle problem is given in Section 3.1. In Section 3.2, we move towards the thin obstacle problem, where an Almgren-type monotonicity formula is again crucial. We discuss the variable coefficient case in more detail, addressing both the regularity of solutions, and of the free boundary. In Section 4, we introduce almost minimizers, first in the context of the Laplacian, as considered by Anzellotti in [3], and in the context of the Signorini problem. We discuss the most important results of [33] and [34], which address the regularity of almost minimizers and their free boundaries for the Signorini problem for constant and variable coefficients, once again using an Almgren-type monotonicity formula.

It is worth mentioning that Almgren monotonicity formulas further appear in discrete settings. The recent paper [49], for example, proves a discrete analogue of the Almgren monotonicity formula for harmonic functions on infinite combinatorial graphs $G = (V, E)$.

Finally, in Section 5, we discuss the techniques from [16] and [17], as mentioned above. In Section 5.1, we address the constant coefficient case (see [16]). In Section 5.2, we provide a sketch of the proof for the variable coefficient case from [17].

2 Dirichlet problem for the Laplacian

As a first motivation for the importance of Almgren-type monotonicity formulas, we give a heuristic idea of how to prove that solutions of the Dirichlet problem for the Laplacian are regular, see for example [50] for another friendly introduction to these ideas. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain. Given $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$, the Dirichlet problem consists of finding $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

One can show (for example, using the divergence theorem), that if the Dirichlet problem has a solution, it must be unique. This leads us to focus on the question of existence. One way to address the question of existence of a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is to notice that u solves (2.1) if and only if u minimizes the energy functional

$$D(v, \Omega) = \int_\Omega |\nabla v|^2 dx$$

in the space $K_\psi(\Omega) = \{v \in C^2(\Omega) \cap C(\overline{\Omega}) \ : \ v|_{\partial\Omega} = \psi\}$.

To prove that this energy attains a minimum, one enlarges the class of candidate functions to the Sobolev space $W^{1,2}_\psi(\Omega) = \{v \in W^{1,2}(\Omega) \ : \ v - \psi \in W^{1,2}_0(\Omega)\}$ and equips it with an appropriate topology. This ensures that the space is compact and $D(\cdot, \Omega)$ is lower semi-continuous. Under these conditions, if $v_n \in W^{1,2}_\psi(\Omega)$ is such that

$$D(v_n, \Omega) \to \inf_{w \in W^{1,2}_\psi(\Omega)} D(w, \Omega),$$

then up to a subsequence, $v_n \to v \in W^{1,2}_\psi(\Omega)$. By the lower semi-continuity of $D(\cdot, \Omega)$,

$$v = \min_{w \in W^{1,2}_\psi(\Omega)} D(w, \Omega).$$

The last step is then to show $v \in C^2(\Omega) \cap C(\overline{\Omega})$, that is, one needs to prove the regularity of minimizers. Instead of doing only that, one can work with a more general class of functions:

Definition 2.1. We say that u is a local minimizer of D in Ω if for all $B \Subset \Omega$,

$$D(u, B) \leq D(u + v, B), \quad \forall v \in W_0^{1,2}(B).$$

Notice that minimizers of $D(\cdot, \Omega)$ in $W_\psi^{1,2}(\Omega)$ are also local minimizers, therefore it suffices to prove the regularity of local minimizers. One way of doing that is through the use of Almgren's monotonicity formula (see [1]):

Theorem 2.2 (Almgren, 1979)**.** *If $\Delta u = 0$ in B_1, then the frequency of u, given by*

$$r \to N(u, r) = \frac{r \int_{B_r} |\nabla u|^2 dx}{\int_{\partial B_r} u^2 d\sigma}$$

is non-decreasing in $(0, 1)$. Furthermore, $N(r) \equiv \kappa$ if and only if u is homogeneous of degree κ, i.e., $u(rx) = r^\kappa u(x)$.

This theorem can be proved in multiple plays, one of which relies solely on the PDE, integration by parts, and the Cauchy-Schwarz inequality.

Going back to the regularity of local minimizers of the Dirichlet energy, assume for simplicity that $0 \in \Omega$ and $\Omega = B_1$. Since $0 \leq N(r, u)$ and $N(r, u)$ is monotonic non-decreasing in $(0, 1)$, one concludes that the following limit exists:

$$N(0+) := \lim_{r \to 0+} N(r, u).$$

To use Almgren's monotonicity, we define the following rescalings of u: $u_r(x) = \frac{u(rx)}{r^{N(0+)}}$. The idea behind these rescalings is to "zoom in" close to 0, and analyze what one obtains in the limit as $r \to 0$. One first proves that $u_r \to u_0$, where $u_0 \neq 0$ is a global minimizer. Then, using the definition of the frequency function, one shows that for fixed s, as $r \to 0+$,

$$N(s, u_0) \leftarrow N(s, u_r) = N(sr, u) \to N(0+).$$

Almgren's monotonicity formula implies u_0 is $N(0+)$ homogeneous. The classification of $N(0+)$-homogenous global minimizers shows they are harmonic polynomials of degree $N(0+) \in \mathbb{N}$.

Heuristically, $N(0+)$ is the order of the first non-zero term in the analytic expansion of u at 0. The blow-up limit u_0 is the first non-zero term in that expansion. One then repeats the process with $u - u_0$, and so on, to obtain the analytic expansion of u.

3 Obstacle problems

3.1 The classical obstacle problem

Going back to the ideas from Section 2, notice that finding a solution in $W^{1,2}_{\psi}(\Omega)$ to

$$\begin{cases} \Delta u &= f \quad \text{in } \Omega, \\ u &= \psi \quad \text{on } \partial\Omega \end{cases}$$

is equivalent to minimizing the energy functional $J(v) = \int_\Omega \left(|\nabla v|^2 + 2fv\right)$ among all $v \in W^{1,2}_{\psi}(\Omega)$. See Figure 3.1.

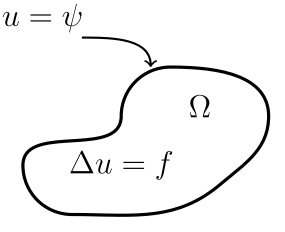

Figure 3.1

For example, in $\mathbb{R}$, minimizing $J(v) = \int_0^1 (v')^2$ among all v with $v(0) = -1$, $v(4) = 3$ leads to $u(x) = x - 1$, see Figure 3.2. Notice that $\Delta u = 0$ in $(0, 4)$. Modifying this problem by adding the condition that one must stay above an obstacle, in this case, say the function $\varphi(x) = 2 - (x - 2)^2$, one minimizes

$$J(v) = \int_0^1 (v')^2$$

among all v with $v(0) = -1$, $v(4) = 3$ and $v \geq 2 - (x - 2)^2$. We get, for certain a, b, that

$$u(x) = \begin{cases} x - 1, \text{ if } 0 \leq x \leq a \\ x - 1, \text{ if } b \leq x \leq 4 \\ 2 - (x - 2)^2, \text{ if } a \leq x \leq b \end{cases}$$

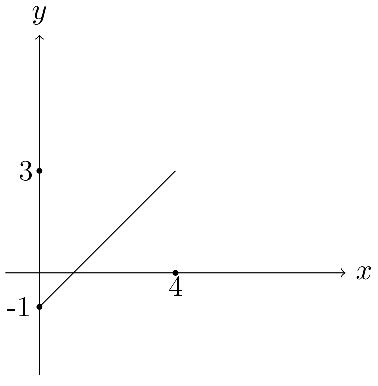

Figure 3.2: $u(x) = x - 1$

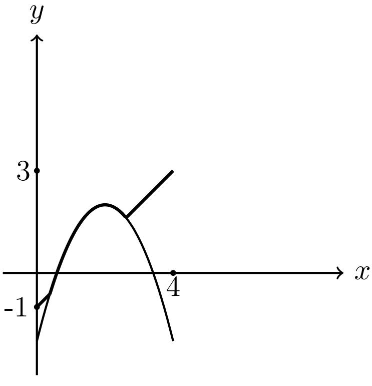

Figure 3.3: The obstacle problem

Notice that $\Delta u = 0$ in $(0, a)$ and in $(b, 4)$, which is the region where $u > 2 - (x-2)^2$. See Figure 3.3.

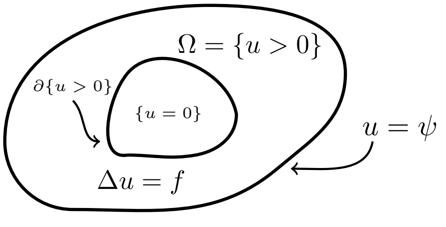

Figure 3.4

This leads us to the classical obstacle problem, an example of a free boundary problem.

One version of the classical obstacle problem consists of minimizing

$$J(v) = \int_U \left(|\nabla v|^2 + 2fv\right)$$

among all v with $v = \psi$ in ∂U, and $v \geq 0$ on the bounded domain U, see Figure 3.4. In this example, the obstacle is the function constant equal to 0.

This is a *free boundary problem.* Solving it means not only finding the function u, but also the free boundary $\Gamma = \partial\{u > 0\} \cap U$. The first fundamental question in the study of the classical obstacle problem is the regularity of u, which is $C^{1,1}_{\text{loc}}(U)$.

The second fundamental question is how smooth the free boundary is. In 1977, Kinderlherer and Nirenberg proved in [35] that, if the free boundary is a C^1 hypersurface close to a free boundary point, then it is

C^∞ in a neighborhood of that point. The groundbreaking work of Caffarelli [10] proved that the free boundary is C^1 near flat points. This completely settled the regularity of the so-called regular points of the free boundary. More generally, when the obstacle is φ and $u \geq \varphi$ on U, one can show that at every free boundary point $x_0 \in \Gamma = \partial\{u > \varphi\} \cap U$, for r small,

$$0 < cr^2 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^2.$$

The free boundary points can be divided into two groups: the set of regular points, and the set of singular points. The so-called regular set is an open subset of the free boundary, and it is C^∞. The singular points are those at which the contact set $\{u = \varphi\}$ has density zero, and those points (assuming they exist), are locally contained in a $(n-1)$-dimensional C^1 manifold. The interested reader is pointed to [8, 11, 24, 25, 41, 53], among others, and also to the beautiful survey [46] on obstacle problems.

3.2 The thin obstacle problem

Let us now consider a situation in which the obstacle is defined only on a portion of the boundary of the domain. For simplicity of notation, assume the initial domain is the upper unit half ball $U = B_1^+ = \{x \in \mathbb{R}^n : |x| < 1\}$.

Let $\mathcal{M} = B_1' = B_1 \cap \{x_n = 0\} \subset \partial U$, which represents a co-dimension one manifold, part of the boundary of our domain. Let $\varphi : \mathcal{M} \to \mathbb{R}$ be the obstacle, and $\psi : \partial U \to \mathbb{R}$ be our boundary data. We want to minimize

$$\int_U |\nabla u|^2 dx, \tag{3.1}$$

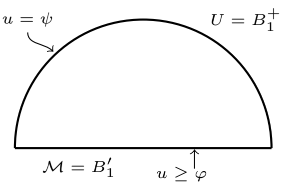

Figure 3.5

over the convex set

$$\mathcal{K} = \{u \in W^{1,2}(U) \mid u = \psi \text{ on } \partial U \setminus \mathcal{M}, u \geq \varphi \text{ on } \mathcal{M}\}.$$

See Figure 3.5. If one starts with a co-dimension one manifold which is not flat, but smooth, one can locally flatten it. This process changes the operator: instead of working with the Laplacian, one needs to consider variable coefficient, divergent form operators. To account for this, we minimize

$$\min_{v\in\mathcal{K}} \int_{B_1} \langle A(x)\nabla v, \nabla v\rangle,$$

over $\mathcal{K} = \{v \in W^{1,2}(B_1) \mid v = \psi \text{ on } S_1 = \partial B_1, v \geq \varphi \text{ on } B_1'\}$, where $A(x) = [a_{ij}(x)]$ is a matrix such that $A(0) = I$, $A(x)$ is symmetric, it is uniformly elliptic (that is, there exist $\lambda, \Lambda > 0$ such that $\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$ for all x, ξ), and it has Lipschitz coefficients.

Definition 3.1. We call

$$\Lambda_\varphi(u) = \{x \in B_1' \mid u(x) = \varphi(x)\}$$

the coincidence set, and

$$\Gamma_\varphi(u) = \partial_{\mathbb{R}^{n-1}}\Lambda_\varphi(u)$$

the free boundary.

We will assume that the obstacle $\varphi \in C^{1,1}(B_1')$. Similarly to what was done in the case of the Dirichlet energy, one analyzes which equations a minimizer solves. It can be shown that

$$Lu := \operatorname{div}(A\nabla u) = 0 \text{ in } B_1^+ \cup B_1^-.$$

Moreover, $u \geq \varphi$ in B_1'. Finally, letting $\nu_\pm$ denote the outer unit normals to B_1^+ and B_1^-,

$$\begin{cases} \langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle \geq 0 & \text{in } B_1', \\ (u-\varphi)(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) = 0 & \text{in } B_1'. \end{cases}$$

The thin obstacle problem, also known as the Signorini problem, was first formulated in 1959 by Signorini in [47], where he studied the equilibrium of an elastic body resting on a rigid surface $\mathcal{M}$. Afterward, Fichera proved in [23] the existence of a unique variational solution to the problem. See also

[35]. The Signorini problem has numerous applications: it models the flow of a saline concentration through a semipermeable membrane, when the flows occurs in a preferred direction, [19]. In mathematical finance, it arises when the random variation of an underlying asset changes discontinuously, [15, 48]. For a beautiful introduction to obstacle-type problems, we refer the reader to [43].

In terms of regularity, Caffarelli proved in 1979 (see [9]) that when $\mathcal{M}$ is a hyperplane (say $\{x_n = 0\}$, as before), φ is $C^{2,\alpha}$ for some $0 < \alpha < \frac{1}{2}$, and $a_{ij} \in C^{1,1}_{loc}$, the solution is $C^{1,\alpha}_{\mathrm{loc}}(B_1^\pm \cup \mathcal{M})$. Arkhipova and Uraltseva [4, 5] obtained the same conclusion assuming $a_{ij} \in W^{1,p}_{\mathrm{loc}}$ and $\varphi \in W^{2,p}_{\mathrm{loc}}$, for some $p > n$. This includes, in particular, the case described in this survey, where $a_{ij} \in W^{1,\infty}_{\mathrm{loc}} = C^{0,1}_{loc}$.

Notice that even when $A(x) \equiv I$, $\mathcal{M}$ is flat and $\varphi = 0$ the best one can hope for in terms of optimal regularity is $C^{1,\frac{1}{2}}_{loc}(B_1^\pm \cup \mathcal{M})$. One has in fact the following global solution to the Signorini problem with $\mathcal{M} = \{x_n = 0\}$, and $\varphi \equiv 0$:

$$u(x) = \Re(x_1 + i|x_n|)^{3/2} \in C^{1,\frac{1}{2}}_{loc}(B_1^\pm \cup \mathcal{M}).$$

The interested reader is directed to [43], which discusses the global solution to the Signorini problem, and much more. In 1978, Richardson, in his Ph.D. dissertation [45], proved optimal regularity when $A(x) = I$ and $n = 2$. His proof used complex analysis methods and could not be generalized to higher dimensions. In the ground-breaking paper [6], Athanasopoulos and Caffarelli proved optimal regularity when $A(x) = I$, $\mathcal{M}$ is flat and $\varphi = 0$ for $n \geq 3$. Subsequently, in [7], Athanasopoulos, Caffarelli and Salsa introduced a powerful new approach to the optimal regularity. Still under the assumption that $\varphi = 0$, they relied on an Almgren monotonicity formula to obtain, among other results, a new proof of the results in [6]. This new approach was further considered in [12], where Caffarelli, Salsa and Silvestre used an Almgren-type monotonicity formula to prove the optimal regularity of the solution in the Signorini problem for fractional powers of the Laplacian. In [32], Guillen extended the optimal $C^{1,1/2}$ regularity to the case of variable coefficient operators, assuming $A \in C^{1,\gamma}$

for some $\gamma > 0$. In 2014, the optimal regularity result was generalized by Garofalo and Smit Vega Garcia, see [31] to the case when $A \in C^{0,1}_{\text{loc}}$, for arbitrary n, assuming $\varphi \in C^{1,1}$ and the manifold $\mathcal{M}$ is flat. The key ingredient in this proof was, once again, an appropriately modified Almgren monotonicity formula. More precisely, when $\varphi = 0$, the theorem from [31] states that if $0 \in \Gamma(u)$, there exists $C > 0$ such that

$$N(r) := e^{Cr} \frac{r \int_{B_r} \langle A\nabla u, \nabla u\rangle}{\int_{S_r} u^2 \mu}$$

is monotone non-decreasing, where $\mu(x) = \langle A(x)x, x\rangle/|x|^2$. In particular, $\lim_{r\to 0+} N(r) = N(0+)$ exists. A generalization of this theorem was also proved in [31] for the case $\varphi \neq 0$.

To address the regularity of the free boundary, one lets $x_0 \in \Gamma(u)$. The first step is to shift this free boundary point to the origin, denoting the corresponding frequency function by N_{x_0}. The number $\kappa(x_0) = N_{x_0}(0+)$ is called the frequency at x_0. Then, $\kappa(x_0)$ is used to classify free boundary points:

$$\Gamma(u) = \cup_\kappa \Gamma_\kappa(u), \qquad \text{where } \Gamma_\kappa(u) = \{x_0 \in \Gamma \mid \kappa(x_0) = \kappa\}.$$

Since $\kappa(x_0)$ gives information on the geometry of the free boundary close to x_0, the first (and difficult) question to consider is: what are the possible homogeneities? When $n = 2$, one can prove that $\kappa = \frac{3}{2},\ 2,\ 3,\ \frac{7}{2},\ 4, \ldots, m,\ 2m - \frac{1}{2},\ 2m, \ldots$, see [43], for example. In [31], Garofalo and Smit Vega Garcia showed that when $A \in C^{0,1}_{\text{loc}}$, then $N(0+) = \frac{3}{2}$ or $N(0+) \geq 2$. The fact that $N_{x_0}(0+) \geq 3/2$ is the crucial ingredient in proving optimal ($C^{1,1/2}_{\text{loc}}$) regularity of u. Focardi and Spadaro proved in [26] that when $n \geq 3$, the possible homogeneities match the case $n = 2$, up to a set of Hausdorff dimension $(n - 3)$.

In 2008, Athanasopoulos, Caffarelli and Salsa and proved that when $A(x) \equiv I$, $\Gamma_{3/2}(u)$ (the so-called regular set) is locally a $C^{1,\alpha}$-regular $(n - 2)$-dimensional surface, assuming $\mathcal{M}$ is flat and $\varphi \in C^{2,1}$. The key ideas are to differentiate the equation in tangential directions, establish the nonnegativity of this directional derivative in a cone of directions, and use a boundary Harnack principle to prove the regularity of the regular set. These ideas, however, are not well-suited for variable coefficients. In 2016, Garofalo, Petrosyan and Smit Vega Garcia [30], considered the case $A(x) \in C^{0,1}_{\text{loc}}$, and proved that $\Gamma_{3/2}(u)$ is locally a $C^{1,\alpha}$-regular $(n-2)$-dimensional surface, assuming $\mathcal{M}$ is flat and $\varphi \in C^{1,1}$. The key ingredients of this proof were a Weiss monotonicity formula and an epiperimetric inequality, which are profoundly connected to Almgren's monotonicity formula.

Unrelated to Almgren monotonicity formula, in the constant-coefficient case, Koch, Petrosyan and Shi considered in [36] the higher regularity of the regular set for the thin obstacle problem for the Laplacian (when $A = I$), establishing its real analyticity by using a hodograph-type transformation and subelliptic estimates. In [18], De Silva and Savin proved the real analyticity of the regular set using a higher-order boundary Harnack principle in slit domains. Concerning the variable-coefficient case, in [37], Koch, Rüland and Shi used Carleman estimates to prove optimal regularity, and also the regularity of the regular set, assuming $A(x) \in W^{1,p}$, $p > 2n$, $\mathcal{M}$ to be flat and $\varphi \in W^{2,p}$, for some $p > 2n$.

Besides the set $\Gamma_{3/2}$, the regular set, one is also interested in understanding the remaining points of the free boundary. A subset of those is made of the so-called singular points:

Definition 3.2. $\Sigma(u) = \cup_{m=1}^{\infty} \Gamma_{2m}(u)$ is called the singular set.

The singular set is, actually, the collection of all points $x_0 \in \Gamma(u)$ for which the coincidence set has vanishing $(n-1)$-dimensional Hausdorff density at x_0, that is,

$$x_0 \in \Sigma(u) \iff \lim_{r \to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(u) \cap B'_r(x_0))}{\mathcal{H}^{n-1}(B'_r(x_0))} = 0,$$

where $\Lambda(u) = \{u(x', 0) = \varphi(x', 0)\}$. In 2009, Garofalo and Petrosyan [28] proved a stratification of the singular set when $A(x) = I$. More precisely, they proved that $\Sigma(u)$ is contained in a countable union of C^1 manifolds of dimensions $d = 0, \ldots, n-2$. This result was generalized in 2018 by Garofalo, Petrosyan and Smit Vega Garcia [29]. Assuming $A(x) \in C^{0,1}$ and $\varphi = 0$, $\Sigma(u)$ is contained in a countable union of C^1 manifolds of dimensions $d = 0, \ldots, n-2$. The key ingredients in this proof were Weiss and Monneau monotonicity formulas. The interested reader is also pointed to [21], [22], [20], and references therein.

4 Almost minimizers

In this section, we discuss another application of Almgren-type monotonicity formulas, this time for almost-minimizers. In [3], Anzellotti introduced the notion of almost minimizers for energy functionals.

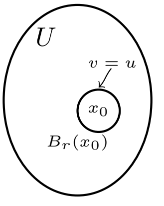

Figure 4.1

Definition 4.1. We say that u is an almost ω-minimizer of $\int_U |\nabla u|^2$ if $u \in W^{1,2}_{\mathrm{loc}}(U)$ and for all $B_r(x_0) \Subset U$ and for all $v \in W^{1,2}_u(B_r(x_0))$

$$\int_{B_r(x_0)} |\nabla u|^2 \leq (1 + \omega(r)) \int_{B_r(x_0)} |\nabla v|^2$$

where ω, the so-called gauge function, satisfies $\omega(r) \nearrow$ and $\omega(0+) = 0$. See Figure 4.1.

To discuss almost minimizers for the thin obstacle problem, we assume $U = B_1$, $\varphi = 0$, $\mathcal{M} = \{x_n = 0\}$, and $\omega(r) \leq M r^\alpha$, for some $0 < \alpha < 1$. See Figure 4.2

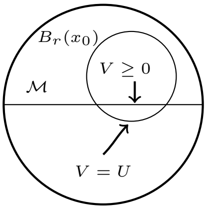

Figure 4.2

Definition 4.2. U is an almost minimizer for the Signorini problem if $U \in W^{1,2}_{\mathrm{loc}}(B_1)$, $U \geq 0$ on $\mathcal{M} \cap B_1$, and for all $B_r(x_0) \Subset B_1$

$$\int_{B_r(x_0)} |\nabla U|^2 \leq (1 + \omega(r)) \int_{B_r(x_0)} |\nabla V|^2,$$

for all $V \in W^{1,2}_U(B_r(x_0))$ such that $V \geq 0$ on $\mathcal{M} \cap B_r(x_0)$.

To generalize this definition for the variable coefficient case, let us assume $A(x) = (a_{ij}(x)) \in M^{n\times n}$ is symmetric, it is uniformly elliptic, that is, there exist $\lambda, \Lambda > 0$ such that $\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$ for all x, ξ, and the coefficients a_{ij} are α-Hölder continuous.

Definition 4.3. *U is an almost minimizer for the A-Signorini problem if $U \in W^{1,2}_{\mathrm{loc}}(B_1)$, $U \geq 0$ on $\mathcal{M} \cap B_1$, and for every ellipsoid $E_r(x_0) = A^{1/2}(x_0)(B_r) + x_0 \Subset B_1$,*

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle \leq (1+\omega(r)) \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle$$

for all $V \in W^{1,2}(E_r(x_0))$ such that $V \geq 0$ on $\mathcal{M} \cap E_r(x_0)$ and $V = U$ on $\partial E_r(x_0)$.

As in the minimizing case, one is interested in the regularity of almost minimizers, and also the regularity of the free boundary: $\Gamma(u) = \partial_{\mathbb{R}^{n-1}}\{u(\cdot, 0) > 0\}$.

The A-Signorini almost-minimization problem presents several challenges. In particular, defining the natural transformation $T_{x_0}(x) = A^{-1/2}(x_0)(x - x_0)$ (so that $E_r(x_0) = T^{-1}_{x_0}(B_r)$), and calling $\Pi = \mathbb{R}^{n-1} \times \{0\}$, one obtains that $\Pi_{x_0} = T_{x_0}(\Pi)$. In general, Π_{x_0} is tilted, which introduces technical difficulties in the proof of the regularity of the free boundary. To overcome these obstacles, one works with a new orthonormal basis $\{e^{x_0}_1, \dots, e^{x_0}_n\}$ of $\mathbb{R}^n$ for which, if $O_{x_0}(e_i) = e^{x_0}_i$, then $O^{-1}_{x_0}(\Pi_{x_0}) = \Pi$. Define $\overline{T}_{x_0} = O^{-1}_{x_0} \circ T_{x_0}$ and $u_{x_0} = U \circ \overline{T}^{-1}_{x_0}$.

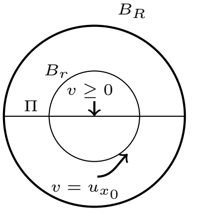

Figure 4.3

Definition 4.4. If U is an almost minimizer of the A-Signorini problem in B_1, $x_0 \in B'_1$, and $E_R(x_0) \subset B_1$, then $u_{x_0} = U \circ \overline{T}^{-1}_{x_0}$ satisfies the almost Signorini property at 0 in B_R, that is, $u_{x_0} \geq 0$ in $\Pi \cap B_R$ and for all $0 < r < R$ and all $v \in W^{1,2}_{u_{x_0}}(B_r)$ with $v \geq 0$ on $\Pi \cap B_r$, one has (see Figure 4.3)

$$\int_{B_r} |\nabla u|^2 \leq (1+\omega(r)) \int_{B_r} |\nabla v|^2.$$

Besides the particular challenges of the Signorini problem, whenever one works with almost minimizers, one is faced with the challenge that they do not satisfy a PDE. Furthermore, one does not have an explicit free boundary condition. In general, one can only rely on comparisons with competitors. In the case of almost minimizers of the A-Signorini problem, one employs Signorini replacements. That is, for $x_0 \in B_1'$, replace u_{x_0} in $B_r(x_0)$ with h, where

$$\begin{aligned}\Delta h = 0 \ \text{ in } \ B_r^{\pm}(x_0)\\ h \geq 0 \ \text{ on } \ B_r'(x_0)\\ \langle \nabla h, \nu_+\rangle + \langle \nabla h, \nu_-\rangle \geq 0,\ h(\langle \nabla h, \nu_+\rangle + \langle \nabla h, \nu_-\rangle) = 0 \ \text{ on } \ B_r'(x_0)\\ h = u_{x_0} \text{ on } \partial B_r(x_0).\end{aligned}$$

The use of Signorini replacements allows one to address the regularity of almost minimizers. In [33], Jeon and Petrosyan considered the case where $A = I$ and U is symmetric. Under these conditions, the authors proved that $U \in C^{1,\beta}_{\text{loc}}(B_1^{\pm} \cup B_1')$, for $\beta = \beta(\alpha, n) \in (0, 1)$. As in the original Signorini problem from Section 3.2, an Almgren monotonicity formula is crucial to prove this result. Subsequently, this result was generalized in [34] by Jeon, Petrosyan and Smit Vega Garcia, when the authors proved that $U \in C^{1,\beta}_{\text{loc}}(B_1^{\pm} \cup B_1')$, for $\beta = \beta(\alpha, n) \in (0, 1)$ assuming $A \in C^{0,\alpha}$, without any symmetry assumption on U.

Similarly to the Signorini minimization problem, one defines the free boundary for almost minimizers as $\Gamma(u) = \partial_{\mathbb{R}^{n-1}}\{u(\cdot, 0) > 0\}$. Once again, the regular set can be defined in terms of an Almgren-type monotonicity formula, which also holds in the setting of almost minimizers: $\Gamma_{3/2}(u) = \{x_0 \in \Gamma(u) \mid \kappa(x_0) = 3/2\}$. In [33], Jeon and Petrosyan proved that $\Gamma_{3/2}(u)$ is locally an $(n-2)$-dimensional $C^{1,\gamma}$-graph, for some $\gamma = \gamma(\alpha, n) > 0$, assuming $A = I$ and that U is symmetric. This result was generalized in [34]. There, the authors assumed $A \in C^{0,\alpha}$ and that U is quasisymmetric, which is the appropriate symmetry notion when dealing with variable coefficients. Under these conditions, it was proved in [34] that $\Gamma_{3/2}(u)$ is locally an $(n-2)$-dimensional $C^{1,\gamma}$-graph, for some $\gamma > 0$.

The notion of singular free boundary points can also be considered for almost minimizers, defining the singular set as $\Sigma(u) = \cup_{m=1}^{\infty} \Gamma_{2m}(u)$. Notice that, once again, this definition relies on an Almgren frequency functional. In [33], Jeon-Petrosyan assumed $A = I$ and U is symmetric, and proved that $\Sigma(u)$ is contained in a countable union of C^1 manifolds of dimensions $d = 0, \ldots, n-2$. This stratification result was generalized in [34] for the variable coefficient setting, assuming $A \in C^{0,\alpha}$, and U is quasisymmetric.

5 Parabolic Almgren

As seen in the previous sections, Almgren-type monotonicity formulas play a key role in the study of harmonic functions, of the regularity of solutions and the free boundary for the classical and thin obstacle problems, and also for almost minimizers. The same happens for a multitude of other free boundary problems.

Usually, proving such formulas for parabolic problems is much harder than for elliptic ones. In this section we describe how to prove parabolic Almgren monotonicity formulas as high-dimensional limits of families of elliptic Almgren-type monotonicity formulas, as done in [16] and [17]. This exemplifies how the technique of [17] can be used to prove other variable-coefficient parabolic results from a family of results in the elliptic setting.

We first describe the work of Davey [16] in Section 5.1, where the author proved an Almgren monotonicity formula for solutions of $\Delta u + \partial_t u = 0$ by analyzing the behavior of solutions to non-homogeneous equations of the form $\Delta v = h$ on the elliptic side. In Section 5.2, we describe the work of [17], where the authors proved Almgren monotonicity formulas for solutions to $\operatorname{div}(A\nabla u) + \partial_t u = 0$ on the parabolic side by studying solutions to non-homogeneous equations of the form $\operatorname{div}(\kappa \nabla v) = \kappa \ell$ on the elliptic side. Here, A has a certain structure, v and κ are defined in terms of u and A, and ℓ depends on u and A. One can also rewrite the elliptic equation as $\kappa^{-1}\text{div}(\kappa\nabla v) = \ell$, which means the associated operator is a special type of Witten Laplacian, or weighted Laplacian (see 2.4 in [32], and also [32], [27], [14], [39], [13], [41], [38], and [40]). That means the techniques presented from [17] allow one to obtain results for variable-coefficient parabolic operators from those for the Witten Laplacian.

The ideas of [17], described in Section 5.2, generalize the work [16] (see Section 5.1), where the author developed the framework to prove constant-coefficient parabolic theorems from appropriate elliptic counterparts. These ideas go back to Perelman [42], who considered parabolic theory as a high-dimensional limit of elliptic theory. This scheme was also discussed in the blog of Tao [52], and in course notes of Sverak [51].

The main idea is to use classical probabilistic formulas which go back to Wiener [54]. For each $n \in \mathbb{N}$, we construct a mapping of the form

$$\begin{aligned} F_{d,n} : \mathbb{R}^{d\times n} &\to \mathbb{R}^d \times \mathbb{R}_+ \\ y &\mapsto (x,t) \end{aligned} \tag{5.1}$$

that takes elements y in (high-dimensional) space $\mathbb{R}^{d\times n}$ to elements (x,t) in space-time $\mathbb{R}^d \times \mathbb{R}_+$. Given a function $u = u(x,t)$ defined on a space-time domain (a subset of $\mathbb{R}^d \times \mathbb{R}_+$), we use $F_{d,n}$ to define a function $v_n = v_n(y)$ on the space $\mathbb{R}^{d\times n}$ by setting $v_n(y) = u(F_{d,n}(y))$. If u is a solution to a backward parabolic equation, then each v_n is a solution to some non-homogeneous elliptic equation. As n becomes large, the function v_n behaves more and more like a solution to a homogeneous elliptic equation. As such, the transformation $F_{d,n}$ becomes more useful to our purposes as $n \to \infty$, thereby illuminating why the notion of a high-dimensional limit is pertinent here. From another perspective, when we use $F_{d,n}$ to push-forward measures on spheres and balls in $\mathbb{R}^{d\times n}$, we produce measures in space-time that are weighted by approximations to generalized Gaussians.

5.1 Constant-coefficient setting

The inherent motivation for the definition of the functions $F_{d,n} : \mathbb{R}^{d\times n} \to \mathbb{R}^d \times \mathbb{R}_+$ from (5.1) is a random walk. Consider d particles moving randomly in one space dimension, and assume they all start at the origin. Denote with $x_1, \dots, x_d$ their coordinates. We assume that if each x_i makes n random steps $y_{i,1}, y_{i,2}, \dots, y_{i,n}$, then

$$|y|^2 = \sum_{i=1}^{d} [y_{i,1}^2 + \ldots + y_{i,n}^2] = 2dt.$$

The new position of each particle after n steps is $x_i = y_{i,1} + \ldots + y_{i,n}$. The functions $F_{d,n} : \mathbb{R}^{d\times n} \to \mathbb{R}^d \times \mathbb{R}_{\geq 0}$ given by $F_{d,n}(y) = (x,t)$ serve as bridges between high-dimensional elliptic settings and the parabolic realm. Given $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$, we define $v(y) = u(F_{d,n}(y))$, so

$$\Delta v = n(\Delta u + u_t) + \frac{2}{d}(x,t) \cdot \nabla_{(x,t)} u_t.$$

Intuitively, this means that $F_{d,n}$ transforms the Laplacian in $\mathbb{R}^{d\times n}$ to a perturbation of the heat operator in $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$. One of the main results of [16] is the following parabolic version of Almgren's monotonicity formula:

Theorem 5.1 (Theorem 4 from [16], originally proved with different techniques in [44])**.** *Let $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$ be such that $\Delta u + u_t = 0$ in $\mathbb{R}^d \times (0,T)$. Define*

$$N(t,u) = \frac{t \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 G(x,t) dx}{\int_{\mathbb{R}^d} |u(x,t)|^2 G(x,t) dx}.$$

Then $N(t,u)$ is monotonically non-decreasing in t.

In [16], the author proved Theorem 5.1 through the use of the following family of Almgren-type formulas for non-homogeneous elliptic equations:

Lemma 5.2 (See Corolary 1 from [16])**.** *Let $v : \mathbb{R}^N \to \mathbb{R}$ with $\Delta v = h$ in $\mathbb{R}^N$, where h is bounded and measurable. Define*

$$L(r,v) = \frac{r \int_{B_r} |\nabla v|^2}{\int_{S_r} v^2}.$$

Then

$$L'(r,v) \geq 2 \frac{\int_{S_r} v \langle \nabla v, y \rangle \int_{B_r} hv}{\left(\int_{S_r} v^2 \right)^2} - 2 \frac{\int_{B_r} h \langle \nabla v, y \rangle}{\int_{S_r} v^2}.$$

To prove Theorem 5.1 in [16], Davey considered u a solution of $\Delta u + u_t = 0$ in $\mathbb{R}^d \times (0,T)$ and defined $v_n(y) = u(F_{n,d}(y))$. Then

$$\Delta v_n = n(\Delta u + u_t) + \frac{2}{d}(x,t) \cdot \nabla_{(x,t)} u_t = \frac{2}{d}(x,t) \cdot \nabla_{(x,t)} u_t = J(x,t).$$

Defining h_n such that $h_n(y) = J(F_{d,n}(y))$, one concludes that $\Delta v_n = h_n$. One can show that

$$\lim_{n\to\infty} L(\sqrt{2dt}, v_n) = 2N(t,u).$$

Applying Lemma 5.2 to the functions v_n with radius $\sqrt{2dt}$, one obtains a family of inequalities. Using several equalities which relate the elliptic and parabolic universes, one passes both sides of the family of inequalities to the limit as $n \to \infty$, and concludes that

$$\lim_{n\to\infty} \frac{d}{dt} L(\sqrt{2dt}, v_n) \geq 0.$$

With a little bit more technical work, one concludes that $N(t,u)$ is monotonic non-decreasing in t, as desired.

5.2 Variable-coefficient setting

In this section we describe how the authors of [17] were able to generalize the results of [16] to the variable coefficient setting. We let

$$y = (y_{1,1}, y_{1,2}, \dots, y_{1,n}, \dots, y_{d,1}, y_{d,2}, \dots, y_{d,n}) \in \mathbb{R}^{d\times n}$$

denote the variables that play the role of the "random steps" in the random walk. For some $t > 0$, assume that y satisfies

$$2dt = \sum_{i=1}^{d} \sum_{j=1}^{n} y_{i,j}^2.$$

The step size is not fixed; instead, we assume that y is uniformly distributed over the sphere of radius $\sqrt{2dt}$. Define

$$z_i = y_{i,1} + y_{i,2} + \ldots + y_{i,n} \quad \text{for } i = 1, \dots, d \tag{5.2}$$

so that $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$. We define $f_{d,n} : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ so that

$$z = f_{d,n}(y).$$

Now let $g : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible function with inverse $h : \mathbb{R}^d \to \mathbb{R}^d$. Let $x = g(z) \in \mathbb{R}^d$ so that

$$x_i = g_i(z) \quad \text{for } i = 1, \ldots, d \tag{5.3}$$

and then since $z = h(x)$ we have

$$z_i = h_i(x) \quad \text{for } i = 1, \ldots, d.$$

The Jacobian of $g = (g_1, \ldots, g_d)$ is a $d \times d$ invertible matrix function whose inverse matrix is the Jacobian of $h = (h_1, \ldots, h_d)$. Let G and H denote the Jacobian matrices of g and h, respectively. That is,

$$G(z) = \begin{bmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} \cdots & \frac{\partial g_1}{\partial z_d} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} \cdots & \frac{\partial g_2}{\partial z_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_d}{\partial z_1} & \frac{\partial g_d}{\partial z_2} \cdots & \frac{\partial g_d}{\partial z_d} \end{bmatrix}, \quad H(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \cdots & \frac{\partial h_1}{\partial x_d} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \cdots & \frac{\partial h_2}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_d}{\partial x_1} & \frac{\partial h_d}{\partial x_2} \cdots & \frac{\partial h_d}{\partial x_d} \end{bmatrix}. \tag{5.4}$$

Let $\gamma(z) = \det G(z)$ and $\eta(x) = \det H(x)$. Define $\kappa_n : \mathbb{R}^{d\times n} \to \mathbb{R}$ to satisfy

$$\kappa_n(y) = \gamma(f_{d,n}(y)) = \gamma(z) = \frac{1}{\eta(x)} = \frac{1}{\eta(g(f_{d,n}(y)))}. \tag{5.5}$$

Definition 5.3 (κ-weighted Sobolev space)**.** For $B_R \subset \mathbb{R}^N$, we say that a function $v : B_R \to \mathbb{R}$ belongs to $L^p(B_R, \kappa(y)\,dy)$, the space of *κ-weighted p-integrable functions*, if

$$\int_{B_R} \kappa(y)|v(y)|^p dy < \infty.$$

Moreover, if both v and $\nabla v \in L^2(B_R, \kappa(y)\,dy)$, then we say that v belongs to the *κ-weighted Sobolev space* and write $v \in W^{1,2}(B_R, \kappa(y)\,dy)$.

The following Lemma follows from simple computations:

Lemma 5.4. *[See Lemma 2.2 from [17]] Given* $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$*, define* $v_n : B_{\sqrt{2dT}} \subset \mathbb{R}^{d\times n} \to \mathbb{R}$ *to satisfy*

$$v_n(y) = u(F_{d,n}(y)).$$

Define $B = B(z)$ *to be a* $d \times d$ *matrix function with entries*

$$b_{k,\ell} = \nabla_z g_k \cdot \nabla_z g_\ell = \sum_{i=1}^d \frac{\partial g_k}{\partial z_i}\frac{\partial g_\ell}{\partial z_i}.$$

That is, $B = GG^T$*. We then set* $A(x) = B(h(x)) = B(z)$ *so that* $A = H^{-1}(H^{-1})^T$*. Then*

$$\begin{aligned}
\frac{\partial v_n}{\partial y_{i,j}} &= \langle \nabla_x u, \frac{\partial g}{\partial z_i}\rangle + \frac{\partial u}{\partial t}\frac{y_{i,j}}{d} \\
y \cdot \nabla_y v_n &= \langle \nabla_x u, G(z) z\rangle + 2t\frac{\partial u}{\partial t} = \langle A(x)\nabla_x u, H(x)^T h(x)\rangle + 2t\frac{\partial u}{\partial t} \\
|\nabla_y v_n|^2 &= n\left|G(z)^T \nabla_x u\right|^2 + \frac{2}{d}\frac{\partial u}{\partial t}\left[\langle \nabla_x u, G(z)\, z\rangle + t\frac{\partial u}{\partial t}\right] \\
&= n\langle A(x)\nabla_x u, \nabla_x u\rangle + \frac{2}{d}\frac{\partial u}{\partial t}\left[\langle A(x)\nabla_x u, H(x)^T h(x)\rangle + t\frac{\partial u}{\partial t}\right].
\end{aligned}$$

Moreover, with κ_n *as in* (5.5)*,*

$$\begin{aligned}
\frac{div_y(\kappa_n(y)\nabla_y v_n)}{\kappa_n(y)} = n\left[div_x(A\nabla_x u) + \frac{\partial u}{\partial t}\right] + \frac{2}{d}\Big[\langle A\nabla_x \frac{\partial u}{\partial t}, H^T h\rangle \\
+ \frac{\partial u}{\partial t}\frac{tr(H\langle \nabla_z G(h), h\rangle)}{2} + t\frac{\partial^2 u}{\partial t^2}\Big],
\end{aligned}$$

where the expression on the right depends on x *and* t*.*

When u is a solution to a variable-coefficient backwards heat equation, we obtain the following consequence:

Corollary 5.5. *[See Corollary 2.3 from [17]] If* $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ *is a solution to* $div_x(A\nabla_x u) + \partial_t u = 0$ *and we define* $v_n : \mathbb{R}^{d\times n} \to \mathbb{R}$ *to satisfy*

$$v_n(y) = u(F_{d,n}(y)),$$

then

$$div_y(\kappa_n(y)\nabla_y v_n) = \kappa_n(y)\ell_n(y),$$

where

$$\ell_n = \frac{2}{d}\left[\langle A\nabla_x \frac{\partial u}{\partial t}, H^T h\rangle + \frac{1}{2}\frac{\partial u}{\partial t} tr(H\langle \nabla_z G(h), h\rangle) - t\, div_x\left(A\nabla_x \frac{\partial u}{\partial t}\right)\right].$$

The non-homogeneous version of Almgren's monotonicity formula proved in [17] is the following:

Proposition 5.6. *For some $R > 0$, let $B_R \subset \mathbb{R}^N$. Assume that for $\kappa : B_R \to \mathbb{R}_+$ it holds that $\nabla \log \kappa \cdot y \in L^\infty(B_R)$. Let $v \in W^{1,2}(B_R, \kappa dy)$ be a weak solution to $div(\kappa \nabla v) = \kappa \ell$ in B_R, where ℓ is integrable with respect to both $\kappa\, v$ and $\kappa \nabla v \cdot y$ on each B_r, for $r \in (0, R)$. For every $r \in (0, R)$, assuming that each $v|_{\partial B_r}$ is non-trivial, define*

$$H(r) = H(r; v, \kappa) = \int_{\partial B_r} \kappa(y)|v(y)|^2 d\sigma(y)$$
$$D(r) = D(r; v, \kappa) = \int_{B_r} \kappa(y)|\nabla v(y)|^2 dy$$
$$L(r) = L(r; v, \kappa) = \frac{rD(r; v, \kappa)}{H(r; v, \kappa)}.$$

Set $\widetilde{L}(r) = r^{2\Upsilon} L(r)$, where $\Upsilon \geq ||\nabla \log \kappa \cdot y||_{L^\infty(B_R)}$. Then for all $r \in (0, R)$, it holds that

$$\widetilde{L}'(r) \geq 2r^{2\Upsilon}\left[\frac{(\int_{B_r} \kappa\, \ell v\, dy)(\int_{\partial B_r} \kappa\, v\, \nabla v \cdot y\, d\sigma(y))}{(\int_{\partial B_r} \kappa |v|^2\, d\sigma(y))^2} - \frac{(\int_{B_r} \kappa\, \ell\, \nabla v \cdot y\, dy)}{(\int_{\partial B_r} \kappa |v|^2\, d\sigma(y))}\right].$$

Notice that if v is a solution to the homogeneous equation $\mathrm{div}(\kappa \nabla v) = 0$ in B_R, i.e. $\ell = 0$, then $\widetilde{L}(r)$ is non-decreasing in r. Moreover, if $\kappa = 1$, one recovers the non-homogenous elliptic result from [16, Corollary 1], which is the non-homogeneous version of Poon's result, [44]. In particular, one recovers the expected monotonicity formula for solutions to elliptic equations.

Now we describe how to use Proposition 5.6 to establish its parabolic counterpart. Before stating the result, we discuss the kinds of solutions that we work with.

Definition 5.7 (Moderate h-growth at infinity)**.** Let $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$ be a continuous function with locally integrable weak first order derivatives. With $h : \mathbb{R}^d \to \mathbb{R}^d$ as before and $A = H^{-1}(H^{-1})^T$, define

$$\mathcal{H}(t) = \mathcal{H}(t; u, h) = \int_{\mathbb{R}^d} |u(x,t)|^2 \exp\left(-\frac{|h(x)|^2}{4t}\right) dx$$
$$\mathcal{D}(t) = \mathcal{D}(t; u, h) = \int_{\mathbb{R}^d} \langle A(x)\nabla u(x,t), \nabla u(x,t)\rangle \exp\left(-\frac{|h(x)|^2}{4t}\right) dx$$
$$\mathcal{T}(t) = \mathcal{T}(t; u, h) = \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}(x,t)\right|^2 \exp\left(-\frac{|h(x)^2}{4t}\right) dx.$$

We say that such a function u has *moderate h-growth at infinity* if $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{T}$ belong to $L^1([0,T], t^{-\frac{d}{2}}dt)$.

Let $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$ have moderate h-growth at infinity. For every $t \in (0,T)$, assume first that u is sufficiently regular to define the functionals

$$\mathcal{I}(t) = \mathcal{I}(t; u, h) = \int_{\mathbb{R}^d} |u(x,t)||\langle A\nabla u, H^T h\rangle + 2t\,\partial_t u| \exp\left(-\frac{|h(x)|^2}{4t}\right) dx$$
$$\mathcal{J}(t) = \mathcal{J}(t; u, h) = \int_{\mathbb{R}^d} |J(x,t)||u(x,t)| \exp\left(-\frac{|h(x)|^2}{4t}\right) dx$$
$$\mathcal{K}(t) = \mathcal{K}(t; u, h) = \int_{\mathbb{R}^d} |J(x,t)||\langle A\nabla u, H^T h\rangle + 2t\,\partial_t u| \exp\left(-\frac{|h(x)|^2}{4t}\right) dx, \tag{5.6}$$

where

$$J(x,t) = J(x,t; u, h) = \frac{1}{d}\left[2\langle A\nabla \frac{\partial u}{\partial t}, H^T h\rangle + \frac{\partial u}{\partial t}\mathrm{tr}(H\langle \nabla_z G(h), h\rangle) + 2t\frac{\partial^2 u}{\partial t^2}\right] \tag{5.7}$$

and all derivatives are interpreted in the weak sense. Then we say that such a function u belongs to the function class $\mathfrak{A}(\mathbb{R}^d \times (0,T), h)$ if u has moderate h-growth at infinity (so is consequently continuous), and for every $t_0 \in (0,T)$, there exists $\epsilon \in (0,t)$ so that

$$\mathcal{I} \in L^\infty[t_0 - \epsilon, t_0]$$

and there exists $p > 1$ so that

$$\mathcal{J} \in L^p([0,t_0], t^{-\frac{d}{2}}dt), \quad \mathcal{K} \in L^p([0,t_0], t^{-\frac{d}{2}}dt).$$

Theorem 5.8. *Assume that* $tr(H\langle \nabla_z G(h), h\rangle) \in L^\infty(\mathbb{R}^d)$, *where* h, H, *and* G *are described by* (5.3), (5.2), *and* (5.4). *Define* $A = H^{-1}(H^{-1})^T : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ *and let* $u \in \mathfrak{A}(\mathbb{R}^d \times (0,T), h)$ *be a non-trivial solution to* $div(A\nabla u) + \partial_t u = 0$ *in* $\mathbb{R}^d \times (0,T)$. *For every* $t \in (0,T)$, *define*

$$\mathcal{H}(t) = \mathcal{H}(t;u,h) = \int_{\mathbb{R}^d} |u(x,t)|^2 e^{-\frac{|h(x)|^2}{4t}}\,dx$$

$$\mathcal{D}(t) = \mathcal{D}(t;u,h) = \int_{\mathbb{R}^d} \langle A(x)\nabla u(x,t), \nabla u(x,t)\rangle e^{-\frac{|h(x)|^2}{4t}}\,dx$$

$$\mathcal{L}(t) = \mathcal{L}(t;u,h) = \frac{t\mathcal{D}(t;u,h)}{\mathcal{H}(t;u,h)}.$$

Set $\widetilde{\mathcal{L}}(t) = t^{\Upsilon}\mathcal{L}(t)$, *where* $\Upsilon \geq ||tr(H\langle \nabla_z G(h), h)||_{L^\infty(\mathbb{R}^d)}$. *Then* $\widetilde{\mathcal{L}}(t)$ *is monotonically non-decreasing in* t.

Sketch of the proof: The main idea of the proof of Theorem 5.8 from [17] is to use Proposition 5.6, the Almgren-type monotonicity formula for solutions of elliptic equations of the form $\text{div}(B_n \nabla v_n) = h_n$. More precisely, given $n \in \mathbb{N}$ with $n \geq 2$, let $v_n : B_T^n \to \mathbb{R}$ satisfy

$$v_n(y) = u(F_{d,n}(y)).$$

An application of Corollary 5.5 shows that

$$\frac{1}{\kappa_n(y)}\text{div}(\kappa_n(y)\nabla v_n) = J(x,t),$$

where J is defined in (5.7) and does not depend on n. For every n, define $\ell_n : B_T^n \to \mathbb{R}$ so that

$$\ell_n(y) = J(F_{d,n}(y))$$

and then

$$\text{div}(\kappa_n \nabla v_n) = \kappa_n \ell_n.$$

First, defining $L_n(t) = \frac{1}{2}L(\sqrt{2dt}; v_n, \kappa_n)$, [17] shows that

$$\lim_{n\to\infty} L_n(t) = \mathcal{L}(t;u,h). \tag{5.8}$$

Then, the authors in [17] apply Proposition 5.6 to each v_n on any ball of radius $\sqrt{2dt}$ for $t < T$, obtaining

$$\begin{aligned}\frac{d}{dt}\widetilde{L}_n(t) \geq -\frac{d\mathcal{C}_d t^{\Upsilon-1}}{2\alpha_d \mathcal{H}_n(t)}\left[\int_0^t \left(\frac{s}{t}\right)^{\frac{dn-2}{2}} \mathcal{K}(s)\,ds + \frac{\mathcal{C}_d \mathcal{I}(t)}{\alpha_d \mathcal{H}_n(t)}\int_0^t \left(\frac{s}{t}\right)^{\frac{dn-2}{2}} \mathcal{J}(s)\,ds\right] \\ =: \widetilde{F}_n(t)\end{aligned}$$

where $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are defined in (5.6) and

$$\mathcal{H}_n(t) = \mathcal{H}_n(t; u, h) := \int_{\mathbb{R}^d} |u(x,t)|^2 \left(1 - \frac{|h(x)|^2}{2dnt}\right)^{\frac{dn-d-2}{2}} \chi_{B_{nt}}(h(x))dx.$$

To show that $\widetilde{\mathcal{L}}$ is monotone non-decreasing, it suffices to show that given any $t_0 \in (0, T]$, there exists $\delta \in (0, t_0)$ so that $\widetilde{F}_n$ converges uniformly to 0 on $[t_0-\delta, t_0]$. Indeed, since $\frac{d}{dt}\widetilde{L}_n(t) \geq \widetilde{F}_n(t)$, then for any $t \in [t_0-\delta, t_0]$, it holds that

$$\widetilde{L}_n(t_0) - \widetilde{L}_n(t) \geq \int_t^{t_0} \widetilde{F}_n(s)ds.$$

By definition and (5.8), $\widetilde{L}_n(t) = t^\Upsilon L_n(t)$ converges pointwise to $\widetilde{\mathcal{L}}(t) = t^\Upsilon \mathcal{L}(t; u, h)$, from which it follows that

$$\widetilde{\mathcal{L}}(t_0) - \widetilde{\mathcal{L}}(t) = \lim_{n\to\infty}[\widetilde{L}_n(t_0) - \widetilde{L}_n(t)] \geq \lim_{n\to\infty}\int_t^{t_0} \widetilde{F}_n(s)ds.$$

Assuming the local uniform convergence of $\widetilde{F}_n$ to 0 on $[t_0 - \delta, t_0] \supset [t, t_0]$, we see that

$$\lim_{n\to\infty}\int_t^{t_0} \widetilde{F}_n(s)ds = \int_t^{t_0} \lim_{n\to\infty} \widetilde{F}_n(s)ds = 0$$

and we conclude that $\widetilde{\mathcal{L}}(t_0) - \widetilde{\mathcal{L}}(t) \geq 0$, as desired. The local uniform convergence is proven in [17], concluding the proof.

References

[1] F. J. Almgren. Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. In *Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977)*, pages 1–6. North-Holland, Amsterdam-New York, 1979.

[2] F.J. Almgren. Almgrens big regularity paper. In *World Scientific Monograph Series in Mathematics*, volume 1. World Scientific Publishing Co. Inc., 2000.

[3] G. Anzellotti. On the $c^{1,\alpha}$-regularity of w-minima of quadratic functionals. *Boll. Un. Mat. Ital. C, 6*, 2(1):195–212, 1983.

[4] A.A. Arkhipova and N.N. Uraltseva. Regularity of solutions of a problem for elliptic and parabolic equations with bilateral constraints on the boundary. *Tr. Mat. Inst. Steklova*, pages 5–22, 1987.

[5] A.A. Arkhipova and N.N. Uraltseva. Regularity of the solution of a problem with a two-sided constraint on the boundary for elliptic and parabolic equations. *Proc. Steklov Inst. Math. 179*, 2:1–19, 1989.

[6] I. Athanasopoulos and L.A. Caffarelli. *Optimal regularity of lower dimensional obstacle problems*, volume 35, pages 49–66. Zap. Naucn. Semin. POMI, 2004.

[7] I. Athanasopoulos, L.A. Caffarelli, and S. Salsa. The structure of the free boundary for lower dimensional obstacle problems. *Amer. J. Math. 130*, 130(2):485–498, 2008.

[8] L. Caffarelli and M. Rivière. Asymptotic behavior of free boundaries at their singular points. *Ann. of Math.*, pages 309–317, 1977.

[9] L.A. Caffarelli. Further regularity for the signorini problem. *Comm. Partial Differential Equations*, 4(9):1067–1075, 1979.

[10] L.A. Caffarelli. Harnack inequality approach to the regularity of free boundaries. part iii : existence theory, compactness, and dependence on x. *Annali della Scuola Normale Superiore di Pisa*, 4, 1988.

[11] L.A. Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.*, pages 383–402, 1998.

[12] L.A. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[13] B. Colbois and A. El Soufi. Spectrum of the Laplacian with weights. *Annals of Global Analysis and Geometry*, 55(03), 2019.

[14] B. Colbois, A. El Soufi, and A. Savo. Eigenvalues of the Laplacian on a compact manifold with density. *Communications in Analysis and Geometry*, 23(3):639–670, 2015.

[15] R. Cont and P. Tankov. *Financial Modeling with Jump Processes.* Chapman and Hall/CRC Financ, 2004.

[16] B. Davey. Parabolic theory as a high-dimensional limit of elliptic theory. *Arch Rational Mech Anal*, 228:159–196, 2018.

[17] B. Davey and M. Smit Vega Garcia. Variable-coefficient parabolic theory as a high-dimensional limit of elliptic theory. *Calc. Var.*, 63(40), 2024.

[18] D. De Silva and O. Savin. Boundary harnack estimates in slit domains and applications to thin free boundary problems. *Rev. Mat. Iberoam.*, 32(3):891–912, 2016.

[19] G. Duvaut and J.L. Lions. *Inequalities in Mechanics and Physics.* Springer-Verlag, 1976.

[20] X. Fernández-Real. he thin obstacle problem: A survey. *Publ. Mat.*, 66(1):3–55, 2022.

[21] X. Fernández-Real and Y. Jhaveri. On the singular set in the thin obstacle problem: higher order blow-ups and the very thin obstacle problem. *Anal. PDE*, 14:1599–1669, 2021.

[22] X. Fernández-Real and X. Ros-Oton. Free boundary regularity for almost every solution to the signorini problem. *Arch. Rat. Mech. Anal.*, pages 419–466, 2021.

[23] G. Fichera. Problemi elastostatici con vincoli unilaterali: Il problema di signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei*, 8(7):91–140, 1963/1964.

[24] A. Figalli, X. Ros-Oton, and J. Serra. Generic regularity of free boundaries for the obstacle problem. *Publ. Math. IHES*, 132:181–292, 2020.

[25] A. Figalli and J. Serra. On the fine structure of the free boundary for the classical obstacle problem. *Invent. Math.*, 215:311–366, 2019.

[26] M. Focardi and E. Spadaro. On the measure and the structure of the free boundary of the lower dimensional obstacle problem. *Arch Rational Mech Anal*, 230:125–184, 2018.

[27] A. Futaki, H. Li, and X.-D. Li. On the first eigenvalue of the Witten-Laplacian and the diameter of compact shrinking Ricci solitons. *Annals of Global Analysis and Geometry*, 44:105–114, 2013.

[28] N. Garofalo and A. Petrosyan. Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Invent. math.*, 177, 2009.

[29] N. Garofalo, A. Petrosyan, and M. Smit Vega Garcia. The singular free boundary in the Signorini problem for variable coefficients. *Indiana Univ. Math. J.*, 67:1893–1934, 2018.

[30] N. Garofalo and M. Petrosyan A. Smit Vega Garcia. An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients. *Journal de Mathématiques Pures et Appliquées*, 105(6):745–787, 2016.

[31] N. Garofalo and M. Smit Vega Garcia. New monotonicity formulas and the optimal regularity in the Signorini problem with variable coefficients. *Advances in Mathematics*, 262:682–750, 2014.

[32] A. Grigor'yan. Heat kernels on weighted manifolds and applications. In *The ubiquitous heat kernel*, volume 398, pages 93–191. Amer. Math. Soc., Providence, RI, 2006.

[33] S. Jeon and A. Petrosyan. Almost minimizers for the thin obstacle problem. *Calc. Var.*, 60(124), 2021.

[34] S. Jeon, A. Petrosyan, and M. Smit Vega Garcia. Almost minimizers for the thin obstacle problem with variable coefficients. *Interfaces Free Bound.*, 26(3), 2024.

[35] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Sc. Norm. Sup. Pisa*, pages 373–391, 1977.

[36] H. Koch, A. Petrosyan, and W. Shi. Higher regularity of the free boundary in the elliptic signorini problem. *Nonlinear Anal.*, 2014.

[37] H. Koch, A. Rüland, and W. Shi. The variable coefficient thin obstacle problem: optimal regularity and regularity of the regular free boundary. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 34(4):845–897, 2017.

[38] D. Le Peutrec and B. Nectoux. Small eigenvalues of the Witten Laplacian with Dirichlet boundary conditions: the case with critical points on the boundary. *Anal. PDE*, 14(8):2595–2651, 2021.

[39] W. X. Li. Compactness of the resolvent for the Witten Laplacian. *Ann. Henri Poincaré*, 19(4):1259–1282, 2018.

[40] W. X. Li and C. J. Xu. Subellipticity of some complex vector fields related to the Witten Laplacian. *Commun. Pure Appl. Anal.*, 20(7-8):2709–2724, 2021.

[41] R. Monneau. On the number of singularities for the obstacle problem in two dimensions. *J. Geom. Anal.*, pages 359–389, 2003.

[42] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv:math/0211159*, 2002.

[43] A. Petrosyan, H. Shahgholian, and N. Uraltseva. *Regularity of Free Boundaries in Obstacle-Type Problems*, volume 136 of *Graduate Studies in Mathematics*. American Mathematical Society, 2012.

[44] C.C. Poon. Unique continuation for parabolic equations. *Comm. Partial Differential Equations*, 21(3-4):521–539, 1996.

[45] D.J.A. Richardson. *Variational problems with thin obstacles*. PhD thesis, The University of British Columbia (Canada), 1978.

[46] X. Ros-Oton. Obstacle problems and free boundaries: an overview. *SeMA*, 75:399–419, 2018.

[47] A. Signorini. ignorini, questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. Appl.*, 5(18):95–139, 1959.

[48] L. Silvestre. Regularity of the obstacle problem for a fractional power of the laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[49] M. Smit Vega Garcia and S. Steinerberger. An almgren monotonicity formula for discrete harmonic functions. *arXiv:2311.11887*, 2023.

[50] L. Spolaor. Monotonicity formulas in the calculus of variation. *Notices of the American Mathematical Society*, 69:1731–1737, 2022.

[51] V. Sverak. Theory of pde course notes, 2011.

[52] T. Tao. 285g, lecture 9: Comparison geometry, the high-dimensional limit, and Perelman reduced volume, 2008.

[53] G. S. Weiss. A homogeneity improvement approach to the obstacle problem. *Invent. Math.*, 138:23–50, 1999.

[54] N. Wiener. Differential space. *J. Math. and Phys.*, 2:132–174, 1923.