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Envisioning the Future of Mathematics Education in Uncertain Times

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SYMBOLIC VARIATIONS ACROSS MATHEMATICAL SUBAREAS: EXPLORING CHALLENGES IN UNDERGRADUATE STUDENTS' INTERPRETATION OF MATHEMATICAL SYMBOLS

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This research explores how undergraduate students interpret mathematical symbols in new contexts when reading diverse mathematical texts across various subareas. Collaborating with experts in mathematical sciences, we collected proof-texts aligned with their specialized areas. These proof-texts were presented to undergraduate transition-to-proof students who had studied logic for mathematical proof while their experience of proofs in advanced mathematics topics was limited. Task-based interviews were conducted outside their regular classroom. This paper examined student encounters with curly bracket symbols in a graph theory context. Our findings suggest the nuanced relationship students have with symbols in proof- texts. While possessing familiarity with certain symbols, this pre-existing student knowledge could influence their accessibility to symbols introduced in unfamiliar contexts.

Keywords: Reasoning and proof, Mathematical Representations, Undergraduate Education

Introduction

Mathematical symbols serve as a fundamental language for mathematical representations, abstraction, argumentation, and communication (Cobb et al., 2000; Eckman, 2023; Harel & Kaput, 1991; Pape & Tchoshanov, 2001). Conventional symbols particularly play a crucial role in communication among individuals by representing normative meanings of mathematical ideas, formulas, and relationships (Pimm, 1995). Teachers and students can use conventional symbols to engage in a shared discourse in the mathematics classroom (Goos, 2004).

Despite the importance of symbolic representations in mathematics, numerous studies indicate that undergraduate students encounter challenges when confronted with reading mathematical expositions and proofs that include mathematical symbols (Dawkins & Zazkis, 2021; Inglis & Alcock, 2012; Shepherd & van de Sande, 2014; Weber & Mejia-Ramos, 2014). Mathematical texts often employ conventional symbols, especially those presenting theorem statements and their proofs. Moreover, advanced mathematics courses at the undergraduate level introduce new symbols for novel concepts or extend known ones in a different or broader context. Students may find these symbols challenging either because they represent newly introduced concepts or because their meanings are expanded to cover new areas. These challenges, arising from potentially unfamiliar or expanded-meaning symbols, may impact students' comprehension of the theorem statements and their proof-texts. This perspective resonates with the broader concept of 'symbol sense' discussed by Arcavi (1994, 2005), involving an individual's understanding, familiarity, and flexible use of (conventional) mathematical symbols.

In line with this standpoint, we address the following research question: *To what extent do undergraduate transition-to-proof students perceive and respond to mathematical symbols when encountering the familiar symbols in unfamiliar subareas of mathematics while reading proof-texts?* This question reflects earlier concerns about students' potential struggles in interpreting conventional symbols in proof-oriented mathematics courses. By investigating the awareness and responsiveness of undergraduate students to conventional symbols across different mathematical subareas, we aim to provide insights into the challenges students face. This study could also offer valuable implications for instructional practices and curriculum development for transition-to-proof mathematics courses.

Theoretical Framework

Our perspective on students' interpretation of conventional symbols aligns with radical constructivism, positing that symbols gain significance only when individuals attribute meanings shaped by their previous experiences (Glasersfeld, 1995). When facing a familiar symbol in an unfamiliar context, students assimilate, incorporating new information into their existing cognitive structures based on their past experiences. If assimilation proves insufficient, students engage in accommodation, adjusting their cognitive structure to integrate subtle distinctions in the meaning of the familiar symbol in the unfamiliar context. This perspective suggests that students who are not the creators of mathematical symbols may not bring the same meaning to symbols as the creator, especially when those symbols are introduced by authoritative creators, such as mathematicians, their classroom instructors, or textbook authors. In this situation, students may face challenges with what Hiebert (1988) suggested as the procedure of connecting symbols with mathematical objects or operations. Specifically, when students encounter a new conventional symbol for the first time, they may not have a connection with the mathematical objects or operations the symbol represents. Students face the challenge of deciphering the intended meaning behind conventional symbols, often without the opportunity to negotiate their meanings (Eckman & Roh, 2024). Far from indicating deficits, the interplay of assimilation and accommodation in response to these cognitive challenges serve as opportunities for deeper comprehension as students actively construct and expand the meaning of the symbols.

To comprehend students' cognitive processes of interpreting conventional mathematical symbols in proof-texts, we introduce the construct of *Symbol Sensitivity*. Symbol sensitivity involves being aware of and responding to mathematical symbols, requiring a nuanced understanding of the semantic subtleties within mathematical contexts. There are empirical studies illustrating student challenges of symbol sensitivity, where students may not be sensitive to distinguishing various mathematical symbols and, therefore, not perceive the resulting semantic differences the authors of the given mathematical expositions intend to convey through the symbols (Eckman, 2023; Roh & Lee, 2011; Sellers et al., 2017).

In contrast, this paper focuses on another critical aspect of symbol sensitivity that we will call *symbol contextual interpretation* (SCI), which is an individual's ability to perceive and interpret distinct meanings of a symbol in different contexts. In certain instances, the same mathematical symbol is employed to convey different semantic nuances across various sub-areas of mathematics. It becomes crucial for individuals to recognize and interpret these distinct meanings based on the specific context in which the symbol is used. For instance, a student may encounter the equal symbol (=) in a mathematical expression involving two functions, f and g. While the equal symbol itself is not new to the student as they have been using it between two numerical values, its usage in the symbolic expression $f = g$ may introduce a new context. In this situation, students need to be aware that the equal sign in the function context conveys a different meaning than the equality between two numerical values. However, students may not always be sensitive to these variations when encountering a familiar symbol (=) in an unfamiliar mathematical context (functions). In some ways, this parallels McGowen and Tall's (2010) notion of *met-before*. That is, meanings often change in mathematics as new contexts are encountered, and a student's *met-befores* can serve to support or hinder. McGown and Tall (2010) illustrate this with the subtraction symbol (-), which is initially associated with a "take away" meaning; however, that meaning is not conveyed in other contexts, such as when dealing with negative numbers.

This paper centers explicitly on exploring students' symbol contextual interpretation (SCI) across various areas of mathematics. By closely examining students' SCI, we aim to gain valuable insights into student challenges in reading comprehension of mathematical texts involving mathematical symbols.

Methodology

Data Collection

As part of a more extensive project (NSF DUE #2141925) focused on curriculum development for transition-to-proof courses at the undergraduate level, we created twenty-eight (28) proof-texts by collaborating with nine researchers across various mathematical sciences sub-areas. In preparation for implementing these proof-texts in a classroom, we first tested them through task-based clinical interviews (Hunting, 1997) with undergraduate students at two large public universities in the United States during the Spring of 2023. Participants were students chosen from transition-to-proof courses or proof-oriented courses to ensure students' understanding of logic for mathematical proof while maintaining limited exposure to proofs across diverse subareas in mathematics. The students are encountering diverse subareas in mathematics for the first time, with proof-texts authored by experts from these new subareas. This presents a dual challenge, as students not only face unfamiliar subareas but also grapple with challenging and novel proof-texts for the first time. We paired students whenever possible to foster meaningful interaction between students and promote dynamic discourse. Each interview extended over 90 minutes, maintaining independence from the participants' course instructors.

Interview Tasks

In each interview, we provided students with one or two proof-texts, each spanning 2-3 pages, encompassing three main components: background information (e.g., definitions, notations, and examples), the theorem statement to be proven, and a proof of the theorem. The interviews were divided into dedicated sections: background information discussion, theorem statement exploration, proof analysis, and a collective reflection post-reading.

The interviewer initiated each component by inviting students to read independently and collaboratively discuss the proof-text with their peers. Students were encouraged to pose questions and use tablets as scratch paper whenever they wanted. Subsequently, the interviewer posed targeted questions that drew inspiration from the proof comprehension assessment model developed by Mejia-Ramos et al. (2012). These questions encompassed both local and holistic comprehension questions. The former involved inquiries about the meaning of terms and statements, identification of the proof framework, and the explicit explanation of implicit warrants in the proof. The latter focused on summarizing the proof, identifying the modular structure of the proof, transferring general ideas or methods to different contexts, and providing illustrations with examples. The primary goal of the interviews was to investigate ways to support students in making sense of these new and challenging proof-texts.

Data Analysis and Results

Our analysis commenced with a thorough review of video recordings of the interview data. The primary objective of the analysis was to identify instances where students encountered challenges while engaging with reading the proof-texts. Through an exhaustive examination of the entire video dataset, we discerned persistent instances where students observed notational usage within proof-texts, akin to recognizing misuses or typographical errors in the proof-texts.

In this data analysis process, a recurrent theme emerged – several students faced similar challenges with understanding, interpreting, and using symbolic expressions in the given proof-texts. These challenges with symbols introduced in the proof-texts occurred multiple times, as exhibited in one of the universities part of the project (24 students with 14 interviews conducted), especially as students read to understand the background information such as definitions, theorems, and examples preceding a theorem to be proven and its proof. The symbols we focused on were those not unfamiliar to the students, but their appearance in unfamiliar contexts created student challenges.

In this paper, we suggest our construct, *symbol contextual interpretation* (SCI), as a type of symbol sensitivity. We use it to analyze an individual student's perception and responsiveness to distinct meanings of such symbols in varying proof-texts. We further delineated *contextual awareness* and *contextual adaptation* as characteristics of SCI. We refer to contextual awareness as an individual's awareness that a symbol can have multiple meanings in different contexts; and contextual adaptation as an individual's fluency in adapting a relevant meaning of a symbol in varying contexts. These characteristics laid the foundation for establishing three categories of student symbol sensitivity in recognizing and interpreting the same symbol's distinct meanings in different mathematics subareas. Table 1 summarizes the characteristics of each category with the number of instances where students exhibited the SCI category.

Table 1. Three Categories of Symbol Contextual Interpretation (SCI)

SCI	Contextual Awareness	Contextual Adaptation	Description	#(Instances)
SCI.0	X	X	An individual adapts only one meaning for a symbol, regardless of the various contexts in which the symbol is used, without indicating potentially different meanings.	9
SCI.1	O	X	An individual is aware that a symbol can convey different meanings in different	6

			contexts but has not developed the normative meaning in the relevant specific context.	
SCI.2	O	O	An individual is aware a symbol can convey different meanings in different contexts and exhibits fluency in adapting its normative meanings in varying contexts.	9

Results

In the rest of this section, we present an illustrative episode from an interview with Ernie and Sally. These students worked together to comprehend a theorem in graph theory, describing the condition for the degrees of the vertices of a graph that can determine the connectedness of a simple graph. As background information before the theorem statement, the proof-text introduced definitions pertinent to the theorem, such as graphs, vertices, edges, loops, parallel edges, degrees of vertices, etc. The curly brackets, {}, were also presented as symbols for the set of vertices, an edge, and the set of edges of a graph. A diagram of graph was provided as another representation, along with the symbolic expression of an example graph G, its vertex set $V(G) = \{a, b, c, d, e\}$ and edge set $E(G) = \{\{a, b\}, \{a, d\}, \{c, d\}, \{c, d\}, \{b, c\}, \{b\}\}$ (see Figure 1). The diagram illustrated five dots, labeled as $a, b, c, d,$ and e, representing five distinct vertices and 5 segments, representing 5 distinct edges of the example graph. Two of the edges connected the same vertices *c* and d, corresponding to the duplicates of two identical curly bracket symbols, $\{c, d\}$, in the edge set $E(G)$. The example graph G also included a loop, as an edge having one endpoint b, corresponding to the singleton set notation $\{b\}$, and a vertex, e, not connected to any of the other vertices of the graph.

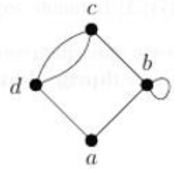
Definition. A **graph** G consists of two sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**.

Definition. Two vertices u, v are **adjacent** if $\{u, v\}$ is an edge. A vertex u and an edge e are **incident** if u is an endpoint of e. We also say that two edges are **incident** if they share an endpoint (vertex). A vertex is **isolated** if there are no edges incident to it.

Definition. The **order** of a graph is the cardinality of vertices, denoted $|V(G)|$. The **size** of a graph is the cardinality of edges, denoted $|E(G)|$.

Definition. A **loop** is an edge whose endpoints are equal. **Parallel edges** are edges having the same pair of endpoints. A **simple graph** is a graph that does not have any loops or parallel edges.

Example:
Consider the graph G below.



G

Graph G has vertex set $V(G) = \{a, b, c, d, e\}$ and edge set $E(G) = \{\{a, b\}, \{a, d\}, \{c, d\}, \{c, d\}, \{b, c\}, \{b\}\}$. The order of G is $|V(G)| = 5$, and the size of G is $|E(F)| = 6$. Graph G has two parallel edges and one loop.

Figure 4 The Excerpt from the Background Information in a Graph Theory Proof-text

The curly brackets, { }, were not new to Ernie and Sally because they had already been acquainted with the symbol when the concept of a set was introduced in transition-to-proof courses that they had taken. However, the proof-text in graph theory introduced the curly brackets in an unfamiliar context to the students, i.e., graph theory. We selected this episode from an earlier moment of the interview to illustrate how Ernie and Sally perceived and responded when encountering the symbol in an unfamiliar context.

Ernie and Sally grappled with the concept of parallel edges represented in the edge set (which uses curly brackets) with repeated pairs. Specifically, unfamiliar with using this symbol to denote "parallel edges" in graph theory, these students found it challenging to interpret instances of the symbol occurring twice in the edge set $E(G)$. Ernie expressed concern about the repetition, while Sally imputed the repetition to two distinct curved segments in the diagram of the graph G, as representing distinct edges, which shared the endpoints c and d. See the transcript below for the students' utterances at the moment.

[1] Ernie: What I don't get, though, is how parallel edges work. If E [$E(G)$] is a set, right, then we can't have duplicate items [$\{c, d\}$] in a [the] set [$E(G)$].

[2] Sally: *(Grabs tablet and begins writing and speaking)* Cause maybe one of them is like pointing from *c* to *d* (*motions writing instrument counterclockwise from the top half of their imaginary circle)* and the other is *d* to *c* (*traces the lower half of the circle in the same counterclockwise direction)*.

[3] Ernie: But that's not ordered pairs though (*points to notation of edges on the proof-text)*. I guess it isn't a relation like that, so we don't have a vector, right?

Ernie's SCI regarding Contextual Awareness. Ernie interpreted the letter 'E' in the symbol $E(G)$ for 'the edge set' as the name of a set and extended his understanding of the curly brackets symbol to the definitions and the given example set G (see Figure 1). Ernie was also familiar with conventional rules for using curly brackets in mathematics to denote a set, including the avoidance of repeated elements within the same set or the consideration of repeated elements as representing the same elements (e.g., $\{c, d\} = \{c, d, c\}$). This suggests that Ernie associated the curly brackets with a mathematical meaning, viewing them as a conventional symbol for denoting a set. Despite grasping the mathematical symbol, Ernie encountered difficulties when transferring his principles with the curly brackets symbol to the graph theory context. Specifically, Ernie exhibited a limited awareness regarding specific conceptual nuances within the context. This limited contextual awareness is evident through three distinct instances.

Firstly, from the video recording, we noticed that Ernie directed his attention solely toward the curly brackets symbol in the provided example graph G, while overlooking the accompanying diagram (Figure 1). He did not exhibit any utterances or gestures to establish a representational connection between the two distinct edges in the diagram of the example graph G to the edges in the duplicated symbols $\{c, d\}$ in the edge set $E(G)$. Ernie's exclusive focus on the curly brackets did not position him to leverage the diagram, which may have provided more contextual information about the meaning of the edge set. In this instance, the presence of duplicates of the same symbol in the (edge) set was a barrier to supporting Ernie's comprehension of the concept of edges, rather than aiding his understanding of parallel edges.

Secondly, in the transcript, line 1, Ernie demonstrated a non-conventional principle to the curly brackets when denoting a set. Ernie noticed that in the example graph G, the symbol "$\{c, d\}$" was repeated twice in the symbol for the edge set of G, $E(G)$, and he asserted, "we can't have duplicate items [$\{c, d\}$] in a set." Ernie's utterance indicates that the edge set notation in the proof-text did not adhere to the conventional curly bracket rules for sets in set theory that he was familiar with. He was interpreting the curly brackets in the example not within the graph theory context but rather in the context of the transition-to-proof course where the students at his university initially learned about sets. Ernie is reasonable, bringing in his prior knowledge about avoiding duplicates within set notation. It is unlikely that he had experienced this requirement as a flexible conventional practice aimed at representing unique elements in a set.

Finally, in the transcript, line 3, Ernie responded to Sally's explanation of directional notations involving vertices c and d, by noting that $\{c, d\}$ is not an ordered pair or a vector from point c to point d. This suggests that Ernie expected Sally's directional interpretation to adhere to vector notation conventions rather than the use of curly brackets symbol $\{c, d\}$. Ernie would not allow duplicating an ordered pair, vector symbol, or any symbol within a set notation.

Sally's SCI regarding Contextual Awareness. In contrast to Ernie, Sally exhibited contextual awareness when encountering the duplicates of the same symbol $\{c, d\}$ in the edge set notation. Sally's awareness of the graph theory context was evident in her consideration of both the curly brackets symbol and the diagram depicting the example graph G in Figure 1. By using both representations as resources to understand the parallel edges, she exhibited her nuanced understanding of the symbol in the graph theory context. Her remark in the transcript, line 2, accompanied by hand motions tracing each path in the diagram of the example graph G, illustrated awareness of the context by attending to the curly brackets and bracketed elements in relation to graph theory (and the diagram). Sally recognized that although both edges in the diagram share the same endpoints, they are distinct, the top edge "from c to d," and the other edge "from d to c." That is, they have directionality. Therefore, duplicating the symbol $\{c, d\}$ within the edge set $E(G)$ aligns with Ernie's rule, as each curly brackets symbol represents a distinct edge within the graph G.

Sally's SCI regarding Contextual Adaptation. While Sally demonstrated contextual awareness when encountering duplicates of the curly brackets symbol $\{c, d\}$ in the graph theory context, she exhibited potential interpretative challenges in adapting her interpretation of the symbol to a different graph theory context. Although Sally did not explicitly recognize this potential challenge, evidence of it emerged through her gestures and word choices in this episode. During her examination of the example graph G, Sally's hand motion traced two edges parallel to one another on the diagram for the graph G, attributing a distinct direction to each of them with the same pair of vertices (endpoints). In addition, she correlated these movements with the curly brackets symbols $\{c, d\}$ found in the edge set notation accompanying the diagram of graph G. Thus, Sally interpreted each instance of the symbol $\{c, d\}$ in the edge set symbol as representing a separate edge: one for the top edge and another for the bottom edge in the diagram. Sally's use of the phrase "from [c] to [d] … and from [d] to [c]" indicates that she may conceptualize edges as directed, with each edge having a specific associated direction. Sally appeared to be drawing on the same notions of set as Ernie, but perhaps adding this additional feature made the distinction between the same symbolically represented edge clear. As this is a non-normative distinction, Sally would likely need to continue to expand her contextual meaning if encountering a graph with more than two parallel edges.

A Couple More Examples While a detailed examination was conducted with two students to illustrate contrasting aspects of SCI, Table 2 provides a broader perspective by presenting concise examples across various subareas of mathematics. The table showcases instances of students with different SCI categories, each accompanied by a brief description explaining why their specific case corresponds to the identified SCI. This compilation not only enriches our understanding of SCI but also offers a valuable resource for educators and researchers seeking insights into the diverse manifestations of students' potential challenges with interpreting familiar symbols in unfamiliar mathematical contexts for the first time.

Table 2. More Examples of Students′ SCI

Student	Context	Symbol	Contextual Interpretation	SCI
Patty	Combinatorics	{ }	Perceived the notation within the context of the combinatorics proof-text and described the symbol's meaning using the objects from the combinatorial context, showing adaptivity from her previous transition to proof context to the new combinatorics context.	SCI.2
Nathan	Combinatorics	{ }	Described the use and meaning of the symbol within the contexts of a transition-to-proof course as opposed to the new combinatorics context.	SCI.1

Spencer	Topology	one-to-one	Perceived this term, a symbol, as it was used to describe functions in the context of Topology.	SCI.2
Cade	Topology	one-to-one	Described this term, a symbol, within the context of a ratio using the symbol colon (:).	SCI.0
Ronnie	Combinatorics	power set symbol P	Described the symbol script P as a power set as it has been denoted in transition-to-proof contexts.	SCI.0

Conclusion and Discussion

In the results section, we delve into the challenges experienced by Ernie and Sally as they grappled with a familiar symbol encountered in an unfamiliar context for the first time. Navigating novel situations beyond their prior experiences, the students faced challenges that demanded a nuanced understanding of symbols. We analyzed the students' Symbol Contextual Interpretation (SCI) to understand their sensitivity toward symbols in these contexts. Noteworthy is the collaborative effort exhibited by Ernie and Sally in making sense of these new symbols. This collaborative success suggests the viability of incorporating such challenging proof-texts into a transition-to-proof course. For mathematics education researchers, understanding students' comprehension of notation is crucial for informing the effective implementation of proof-texts in these courses. A key insight from our study emphasizes that introducing students to new symbols extends beyond providing them with texts and definitions; it requires careful consideration of their prior experiences and explicit elucidation of how symbols may take on different meanings. This study highlights the misconception that assumes students in mathematics courses can seamlessly discard prior meanings of symbols, emphasizing the need for a thoughtful approach to incorporating notations when used in new mathematical contexts.

> To emphasize this point further, we reference a quote by Kershner and Wilcox (1950): Whenever nonbasic mathematical words are introduced, they will, of course, be explicitly defined. Whenever technical use is made of these words, the reader must carefully eliminate any preconceptions concerning their meaning and think only of their definitions. This will be difficult, but it is absolutely necessary. Unless all suggestions conveyed by these words from past associations are persistently ignored, a multiplicity of meanings may arise. Our mathematical definitions will be unambiguous and complete (p. 17).

This statement, though seemingly psychologically absurd, reflects expectations placed on students in mathematics courses. It highlights student challenges with isolating definitions from their past associations. Ernie's case exemplifies this challenge as he drew upon his prior understanding of the symbol "{ }" to interpret a new proof-text intending a different meaning. This situation underscores the importance of acknowledging the subtlety and complexity of interpreting symbols across various mathematical subareas in mathematics education literature.

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References

Arcavi, A. (1994). Symbol Sense: Informal Sense-Making in Formal Mathematics. *For the Learning of Mathematics*, *14*(3), 24–35.

Arcavi, A. (2005). Developing and Using Symbol Sense in Mathematics. *For the Learning of Mathematics*, *25*(2), 42–47. https://www.jstor.org/stable/40248497

Cobb, P., Yackel, E., & McClain, K. (Eds.). (2000). *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design*. Lawrence Erlbaum Associates.

Dawkins, P. C., & Zazkis, D. (2021). Using Moment-by-Moment Reading Protocols to Understand Students' Processes of Reading Mathematical Proof. *Journal for Research in Mathematics Education*, *52*(5), 510–538. https://doi.org/10.5951/jresematheduc-2020-0151

Eckman, D. (2023). *Students' Personal Algebraic Expressions as a Reflection of Their Meanings: The Case of Infinite Series*. ProQuest Dissertations Publishing.

Eckman, D., & Roh, K. (2024). A symbolizing activity for constructing personal expressions and its impact on a student's understanding of the sequence of partial sums. T*he Journal of Mathematical Behavior, 73, 101117.* https://doi.org/10.1016/j.jmathb.2023.101117

Glasersfeld, E. (1995). *Radical constructivism. A way of knowing and learning.* Falmer Press.

Goos, M. (2004). Learning Mathematics in a Classroom Community of Inquiry. *Journal for Research in Mathematics Education*, *35*(4), 258. https://doi.org/10.2307/30034810

Harel, G., & Kaput, J. (1991). The Role of Conceptual Entities and Their Symbols in Building Advanced Mathematical Concepts. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 82–94). Springer Netherlands. https://doi.org/10.1007/0-306-47203-1_6

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*(3), 333–355. https://doi.org/10.1007/BF00312451

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior*, *16*(2), 145–165. https://doi.org/10.1016/S0732-3123(97)90023-7

Inglis, M., & Alcock, L. (2012). Expert and Novice Approaches to Reading Mathematical Proofs. *Journal for Research in Mathematics Education*, *43*(4), 358–390. https://doi.org/10.5951/jresematheduc.43.4.0358

Kershner, R. B., & Wilcox, L. R. (1950). The anatomy of mathematics. New York, NY: Ronald Press.

McGowen, M.A., & Tall, D. O. (2010). Metaphor or met-before? The effects of previous experience on practice and theory of learning mathematics. *Journal of Mathematical Behavior, 29,* 169–169. DOI: 10.1016/j.jmathb.2010.08.002

Mejia-Ramos, J., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*, 3–18. DOI: 10.1007/s10649-011-9349-7

Pape, S. J., & Tchoshanov, M. A. (2001). *The role of Representation(s) in Developing Mathematical Understanding*. *40*(2), 118–127. https://www.jstor.org/stable/1477273

Pimm, D. (1995). *Symbols and meanings in school mathematics*. Routledge.

Roh, K.H., & Lee, Y. (2011). The Mayan activity: A way of teaching multiple quantifications in logical contexts. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *21*(*8*), 1–14. DOI: 10.1080/10511970.2010.485602

Sellers, M., Roh, K.H., & David, E. (2017). A comparison of calculus, transition-to-proof, and advanced calculus student quantifications in complex mathematical statements. *Proceedings for 20th annual Conference on Research in Undergraduate Mathematics Education* (pp. 285–297). San Diego, CA.

Shepherd, M. D., & Van De Sande, C. C. (2014). Reading mathematics for understanding—From novice to expert. *The Journal of Mathematical Behavior*, *35*, 74–86. https://doi.org/10.1016/j.jmathb.2014.06.003

Weber, K., & Mejia-ramos, J. P. (2014). Mathematics majors' beliefs about proof reading. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 89–103. https://doi.org/10.1080/0020739X.2013.790514