
Two-Timescale Linear Stochastic Approximation: Constant Stepsizes Go a Long Way

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Abstract

Previous studies on two-timescale stochastic approximation (SA) mainly focused on bounding mean-squared errors under diminishing stepsize schemes. In this work, we investigate the *constant* stepsize schemes through the lens of Markov processes, proving that the iterates of both timescales converge to a unique joint stationary distribution in the Wasserstein metric. We derive explicit geometric and non-asymptotic convergence rates, as well as the variance and bias introduced by constant stepsizes in the presence of Markovian noise. Specifically, with two constant stepsizes $\alpha < \beta$, we show that the biases scale linearly with both stepsizes as $\Theta(\alpha) + \Theta(\beta)$ up to higher-order terms, while the variance of the slower iterate (resp., faster iterate) scales only with its own stepsize as $O(\alpha)$ (resp., $O(\beta)$). Unlike previous work, our results do not require additional assumptions such as $\beta^2 \ll \alpha$ or additional dependence on dimensions. These fine-grained characterizations allow tail-averaging and extrapolation techniques to reduce variance and bias, improving mean-squared error bound to $O(\beta^4 + \frac{1}{t})$ for both iterates.

1 Introduction

Stochastic Approximation (SA) is an iterative procedure to find the root of unknown operators from their noisy samples (Robbins and Monro, 1951). There has been a long line of work understanding the convergence behavior of SA both asymptotically (Borkar and Meyn, 2000; Harold et al., 1997) and in a finite-time (Srikant

and Ying, 2019), with a wide range of applications including stochastic optimization (Harold et al., 1997; Moulines and Bach, 2011) and reinforcement learning (Sutton and Barto, 2018; Lakshminarayanan and Szepesvari, 2018; Srikant and Ying, 2019).

Two-Timescale Stochastic Approximation (TTSA) is a variant of the SA algorithm, designed to find the root of two intertwined operators (Borkar, 1997). In particular, given two operators F and G , we aim to find the solution (x^*, y^*) satisfying the fixed-point equations

$$\begin{cases} F(x^*, y^*) = 0, \\ G(x^*, y^*) = 0. \end{cases}$$

This work considers linear TTSA with constant stepsizes driven by Markovian data as the following:

$$\begin{aligned} x_{t+1} &= x_t - \alpha_t(F(x_t, y_t) + w^x(x_t, y_t; \xi_t)), & t \geq 0, \\ y_{t+1} &= y_t - \beta_t(G(x_t, y_t) + w^y(x_t, y_t; \xi_t)), \end{aligned} \quad (1)$$

where $\alpha_t \equiv \alpha$, $\beta_t \equiv \beta > 0$ are constant stepsizes for slower and faster iterates respectively, F and G are linear operators, and w^x and w^y are linear Markovian noises driven by *exogenous* Markovian states ξ_t (see Section 2 for precise formulation).

The updates in (1) arise in many applications: examples include popular reinforcement learning algorithms such as actor-critic (Konda and Tsitsiklis, 1999; Haarnoja et al., 2018) and gradient temporal-difference (GTD) methods (Maei et al., 2009; Szepesvári, 2022), and iterative algorithms for stochastic Bilevel optimization (Colson et al., 2007; Ghadimi and Wang, 2018; Hong et al., 2023; Kwon et al., 2023). The core idea of TTSA is the use of different stepsizes for two iterations, which establishes a hierarchy between the two fixed-point equations. For example, in actor-critic algorithms (Haarnoja et al., 2018), the y -variable minimizes the temporal-difference (TD) error, while the x -variable represents policy parameters to maximize long-term rewards. To ensure that the policy parameters are updated based on accurate value estimates, we set $\beta \gg \alpha$, meaning that y converges faster, staying close to the minimizer of the TD-error given the current policy parameter x .

Classical results have established asymptotic convergence of TTSA with diminishing step sizes, $\alpha_t, \beta_t \rightarrow 0$, under the requirement of order-wise different timescales, *i.e.*, $\frac{\alpha_t}{\beta_t} \rightarrow 0$ (Borkar, 1997; Konda and Tsitsiklis, 2004; Mokkadem and Pelletier, 2006). With the recent advances in large-scale optimization, several papers have focused on analyzing the finite-time convergence of TTSA under similar vanishing step-size conditions. Earlier analyses reported suboptimal convergence rates of $O(t^{-2/3})$ (Dalal et al., 2018; Doan, 2022), which have been improved to the best possible rate of $O(t^{-1})$ in more recent studies as long as $\beta_t^2 \lesssim \alpha_t$ (Kaledin et al., 2020; Dalal et al., 2020; Haque et al., 2023; Doan, 2024; Han et al., 2024; Hu et al., 2024). The key to recent improvements lies in eliminating the need for diminishing stepsize ratios, achieved through a more refined analysis of the cross-correlations between the two intertwined iterations (Kaledin et al., 2020; Haque et al., 2023; Han et al., 2024).

More recently, SA with constant stepsizes has attracted attention due to its simplicity, fast convergence, and good empirical performance, both for single- and two-timescale cases (see Section 1.1 for details). However, existing results for TTSA are often limited to only providing upper bounds for $\mathbb{E}[\|x_t - x^*\|_2^2]$ and $\mathbb{E}[\|y_t - y^*(x_t)\|_2^2]$, *i.e.*, mean-squared errors (MSE) from the fixed point of operators, leaving the non-asymptotic behavior of TTSA iterations with constant stepsizes unexplored. Through the lens of the Markov process on TTSA iterations, we break down the sources of MSE and demonstrate the advantages of a finer understanding, particularly when employing techniques like tail-averaging and extrapolation.

Our Contributions. We study the behaviors of Markovian TTSA iterations (1) with constant stepsizes. We focus on *linear* TTSA when the two operators F, G and Markovian noise fields w^x, w^y are linear in the iterates. Our contributions are summarized as follows:

- While the iterates do not converge pointwise with constant stepsizes, under the standard assumptions for TTSA, we show that the joint process (x_t, y_t, ξ_t) of iterates and Markovian noises converges to a unique *biased* stationary distribution.
- For the stationary distribution of slower iterates x_∞ , we show that its bias has a dominating term growing linearly with α and β , while its variance is $O(\alpha)$. Therefore, the asymptotic MSE of order $O(\alpha)$ for slower iterates reported in prior work (which requires the assumption $\beta^2 \leq \alpha$) in fact admits the following bias-variance decomposition:

$$\text{x-MSE} \asymp \underbrace{\|\mathbb{E}[x_\infty] - x^*\|_2^2}_{\text{squared-bias: } O(\alpha+\beta)^2} + \underbrace{\mathbb{E}[\|x_\infty - \mathbb{E}[x_\infty]\|_2^2]}_{\text{variance: } O(\alpha)}.$$

- Based on our distributional convergence results, we show the benefits of simple Polyak-Ruppert averaging (Polyak and Juditsky, 1992) and Richardson-Romberg Extrapolation (Stoer and Bulirsch, 2013) along with the use of constant stepsizes in TTSA iterations. Specifically, through combining the above techniques, we can achieve (1) exponentially-fast decaying optimization error, (2) variance decaying at $O(1/t)$ rate, and (3) order-wise improvement of asymptotic biases:

$$\mathbb{E}[\|\tilde{x}_t - x^*\|_2^2] \asymp \underbrace{\mathbb{E}[\|\tilde{x}_t - \tilde{x}_\infty\|_2^2]}_{\text{optimization error: } \exp(-\Theta(t))} + \underbrace{\mathbb{E}[\|\tilde{x}_\infty - \mathbb{E}[\tilde{x}_\infty]\|_2^2]}_{\text{variance: } O(1/t)} + \underbrace{\|\mathbb{E}[\tilde{x}_\infty] - x^*\|_2^2}_{\text{reduced-bias: } O(\beta^4)}.$$

We emphasize that our convergence results do not impose the restriction $\beta^2 \leq \alpha$, or involve additional dimension-dependent constants as prior work in Kaledin et al. (2020); Haque et al. (2023); Han et al. (2024).

1.1 Related Work

The literature on (two-timescale) SA is vast. Here we discuss prior work most relevant to us.

Weak Convergence of Constant Stepsize SA.

Recent studies have shown that under regularity conditions, SA iterates with constant stepsizes weakly converge to a stationary distribution (Bhandari et al., 2018; Dieuleveut et al., 2020; Durmus et al., 2024; Chen et al., 2024; Lauand and Meyn, 2023; Allmeier and Gast, 2024). In particular, a line of work has developed an approach based on the Wasserstein distance measure when operators are global contraction mapping (Dieuleveut et al., 2020; Durmus et al., 2021; Huo et al., 2023; Zhang and Xie, 2024; Lauand and Meyn, 2024). For cases where operators possess only local contraction or star-convexity properties, other studies have shown convergence in total variation distance under additional assumptions on the noise distribution's support (Yu et al., 2021; Vlatakis-Gkaragkounis et al., 2024). Our result adopts the approach based on Wasserstein metrics, providing a more explicit convergence rates without requiring assumptions on the noise support, even when the overall iterates (x_t, y_t) do not exhibit global contraction.

Existing Results for TTSA. TTSA arises as a popular iterative solution in various domain; from the classical iterate-averaging schemes (Polyak and Juditsky, 1992) and off-policy reinforcement learning algorithms (Sutton and Barto, 2018) to gradient descent-ascent algorithms for saddle-point problems (Jin et al.,

2020) and single-loop algorithms for Bilevel optimization (Hong et al., 2023). Asymptotic convergence and central limit theorems for TTSA with diminishing step sizes were initially established for linear cases with i.i.d. noise (Konda and Tsitsiklis, 2004), followed by extensions to non-linear and Markovian noise settings (Mokkadem and Pelletier, 2006; Hu et al., 2024).

More recent work has shifted focus to non-asymptotic results, deriving finite-time convergence rates for both linear (Dalal et al., 2018, 2020) and nonlinear cases (Kaledin et al., 2020; Han et al., 2024; Haque et al., 2023). However, these studies primarily address MSE bounds with diminishing stepsizes. In contrast, we investigate distributional convergence under constant stepsizes, providing explicit decoupling of biases and variances. Additionally, we establish new results for tail-averaging and extrapolation in TTSA schemes.

2 Problem Setup

Let $F : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}^{d_x}$ and $G : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}^{d_y}$ be linear mean-field operators in the following form:

$$\begin{aligned} F(x, y) &= J_{11}x + J_{12}y + b_1, \\ G(x, y) &= J_{21}x + J_{22}y + b_2, \end{aligned}$$

where J_{11}, \dots, J_{22} (resp., b_1, b_2) are fixed matrices (resp., vectors), and linear Markovian noises defined as the following:

$$\begin{aligned} w^x(x, y; \xi) &= W_{11}(\xi)x + W_{12}(\xi)y + u_1(\xi), \\ w^y(x, y; \xi) &= W_{21}(\xi)x + W_{22}(\xi)y + u_2(\xi). \end{aligned}$$

Let $J_{\max} := \max_{i,j \in \{1,2\}} \|J_{ij}\|_{\text{op}}$ be the smoothness parameter of the system. The first assumption is on the mean-field operators being Hurwitz:

Assumption 1. *The matrices $-J_{22}$ and $-\Delta := -J_{11} + J_{12}J_{22}^{-1}J_{21}$ are Hurwitz, that is, all real parts of the eigenvalues of J_{22} and Δ are strictly positive.*

Therefore, a fixed point in the slower timescale is uniquely defined $y^*(x) = -J_{22}^{-1}(J_{21}x + b_2)$ for every x , and the target joint fixed point (x^*, y^*) is given as:

$$\begin{aligned} x^* &= -\Delta^{-1}(b_1 - J_{12}J_{22}^{-1}b_2) \\ y^* &= -J_{22}^{-1}(J_{21}x^* + b_2). \end{aligned}$$

Assumption 1 is standard in the study of TTSA to ensure the stability of the system (Gupta et al., 2019; Doan, 2022). The main difference from single timescale SA is the star-type stability of slower iterations, *i.e.*, we only assume that $-H(x) := -F(x, y^*(x))$ is Hurwitz, while the entire operation $- \begin{bmatrix} F(x, y) \\ G(x, y) \end{bmatrix}$ may not. Therefore, existing results for single-timescale SA cannot be directly applied.

Next, we assume that the noise fields are controlled by a geometrically mixing exogenous (*i.e.*, state evolves independent of TTSA iterations) Markov chain $\{\xi_t\}_{t \geq 0}$:

Assumption 2. *Let $\{\xi_t\}_{t \geq 0}$ be an exogenous Markovian chain on a countable state-space Ξ with a transition kernel \mathcal{P} and a unique stationary distribution π . Furthermore, $\{\xi_t\}_{t \geq 0}$ is geometrically mixing:*

$$\|\mathcal{P}^n \pi_0 - \pi\|_1 \leq c_\rho \rho^n,$$

for some absolute constant $c_\rho > 0$, $\rho \in [0, 1)$ and any initial distribution $\xi_0 \sim \pi_0$ for all $n \geq 1$.

We also assume that the noise fields are bounded and unbiased at the stationary limit:

Assumption 3. *For all $j \in \{1, 2\}$ and $\xi \in \Xi$, we have*

$$\begin{aligned} \mathbb{E}_{\xi \sim \pi}[W_{ij}(\xi)] &= 0, & \forall i, j \in \{1, 2\}, \\ \mathbb{E}_{\xi \sim \pi}[u_i(\xi)] &= 0, & \forall i \in \{1, 2\}. \end{aligned}$$

Furthermore, for all $\xi \in \Xi$, the following holds:

$$\|W_{ij}(\xi)\|_{\text{op}} \leq W_{\max}, \quad \|u_i(\xi)\|_2 \leq u_{\max}, \quad \forall i, j \in \{1, 2\}.$$

for some constants $W_{\max}, u_{\max} \geq 0$. For simplicity, we further assume that $W_{\max} \leq J_{\max}$.

The above two assumptions are common in the analysis of SA schemes with Markovian noises (Dalal et al., 2020; Huo et al., 2023). We introduce the notion of noise variances in our setting:

$$\begin{aligned} \sigma_x^2 &:= \max_{\xi \in \Xi} \|W_{11}(\xi)x^* + W_{12}(\xi)y^* + u_1(\xi)\|_2^2, \\ \sigma_y^2 &:= \max_{\xi \in \Xi} \|W_{21}(\xi)x^* + W_{22}(\xi)y^* + u_2(\xi)\|_2^2, \end{aligned} \quad (2)$$

which reflect the mean-squared fluctuation of the stochastic update around the fixed point.

We study the convergence of TTSA iterations (1) via L^2 -Wasserstein distance (Villani et al., 2009). Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of square-integrable distributions on \mathbb{R}^d where $d := d_x + d_y$. Note that L_2 -Wasserstein distance between two distributions μ and ν in $\mathcal{P}_2(\mathbb{R}^d)$ is defined as the following:

$$\mathcal{W}_2(\mu, \nu) := \left(\inf_{\Gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|u - v\|_2^2 d\Gamma(u, v) \right)^{1/2},$$

where $\Pi(\mu, \nu)$ is a set of all possible couplings with marginal distributions μ and ν . To study the distribution convergence of the joint iterate-data sequence $(x_t, y_t, \xi_t)_{t \geq 0}$, we slightly extend the definition above to add hamming distance in Ξ . Let $\bar{\mathcal{P}}_2(\mathbb{R}^d \times \Xi)$ be the set of distributions $\bar{\mu}$ on $\mathbb{R}^d \times \Xi$ with the property that the marginal of $\bar{\mu}$ on \mathbb{R}^d is square-integrable.

Definition 1. For any two probability measures μ, ν in $\mathcal{P}_2(\mathbb{R}^{d_x+d_y} \times \Xi)$ over (x, y, ξ) , we define the distance between μ and ν as

$$\bar{\mathcal{W}}_2(\mu, \nu) := \inf_{\Gamma \in \Pi(\mu, \nu)} \left\{ \mathbb{E}_{((x_t, y_t, \xi_t), (x'_t, y'_t, \xi'_t)) \sim \Gamma} [\mathbb{1}\{\xi_t \neq \xi'_t\} + \|x_t - x'_t\|_2^2 + \|y_t - y'_t\|_2^2]^{1/2} \right\}, \quad (3)$$

where $\Pi(\mu, \nu)$ is a set of all possible couplings with marginal distributions μ, ν .

To establish the finite-time convergence of TTSA iterations (1), we define a few error metrics. Let $Q_x, Q_y \succ 0$ be the unique solutions of the Lyapunov equations

$$\begin{aligned} Q_x \Delta + \Delta^\top Q_x &= I, \\ Q_y J_{22} + J_{22}^\top Q_y &= I. \end{aligned}$$

The solutions Q_x, Q_y , which are guaranteed to exist since $-\Delta, -J_{22}$ are Hurwitz under Assumption 1 (Chen, 1984), are used for constructing the drift of potentials in our analysis. For the slower and faster iterates, we use $\|\cdot\|_{Q_x}$ and $\|\cdot\|_{Q_y}$ norms respectively, and define $\mu_x := \|Q_x\|_{\text{op}}^{-1}$ and $\mu_y := \|Q_y\|_{\text{op}}^{-1}$. Note that $\sigma_{\min}(\Delta) \geq \mu_x/2$ and $\sigma_{\min}(Q_x) \geq \|\Delta\|_{\text{op}}^{-1}/2$, and similarly for Q_y and μ_y . Consequently, we let the condition number of two iterations as $\kappa_x := \frac{\kappa_y J_{\max}}{\mu_x}$ and $\kappa_y := \frac{J_{\max}}{\mu_y}$.

Notation. For a positive definite matrix $Q \succ 0$ let $\|x\|_Q := \sqrt{x^\top Q x}$ for a vector x . With a general real-valued matrix A , we define $\|A\|_Q := \max_{\|x\|_Q=1} \|Ax\|_Q$. Let $\langle a, b \rangle_Q := a^\top Q b$ for two vectors a, b . For two real-valued matrices A, B , we denote $\langle A, B \rangle = \text{Tr}(A^\top B)$. We define 1-Schatten norm $\|A\|_1 := \sum_i |\sigma_i(A)|$ as the absolute sum of singular values (sometimes we call it S^1 -norm), and ∞ -Schatten norm $\|A\|_\infty := \max_i |\sigma_i(A)|$ be the maximum singular value, which is equivalent to matrix operator norm $\|A\|_{\text{op}}$. For a positive semi-definite matrix $Q \succeq 0$, $\|Q\|_1 = \text{Tr}(Q) = \sum_i Q_{ii}$ is the sum of diagonal elements. For a random vector x , we denote the covariance $\mathbb{V}(x) := \mathbb{E}[(x - \mathbb{E}[x])(x - \mathbb{E}[x])^\top]$. We often use shorthands $w_t^x := w^x(x_t, y_t; \xi_t)$ and $w_t^y := w^y(x_t, y_t; \xi_t)$. We denote the fixed point of the faster iterates given x as $y^*(x)$, such that $G(x, y^*(x)) = 0$. If we just write y^* , then it means $y^*(x^*)$. For two probability distributions p, q , we denote $\|p - q\|_1$ as the total-variation distance between p and q . We use the notation $O(\cdot)$ to hide absolute constants, and $O_P(\cdot)$ to omit up to polynomial factors in instance-dependent constants (smoothness, minimum eigenvalues, and noise variances) and up to logarithmic factors in stepsizes.

3 Main Results

We start with two conditions for stepsizes to ensure the stability of TTSA iterations:

Assumption 4. We assume that the stepsizes (α, β) satisfy the following:

$$\beta \tau_\alpha \leq \frac{c_1}{J_{\max} \kappa_y^2 \kappa_x^2}, \quad \frac{\alpha}{\beta} \leq \frac{c_2}{\kappa_y^3 \kappa_x}. \quad (4)$$

where $\tau_\alpha := \frac{\log(\alpha \mu_x / c_\rho)}{\log \rho}$ with some sufficiently small absolute constants $c_1, c_2 > 0$.

The first condition in (4) ensures β less than the inverse smoothness of operators, and the second condition bounds the ratio between two-timescale iterations. We mention that the dependence on the condition numbers is not fully optimized. In the sequel, we start with a fine-grained convergence in MSE in Section 3.1. We then show the convergence in distribution and characterize the biases and variances of the limit distribution in 3.2, which is followed by our final result on tail-averaging and extrapolation in Section 3.3.

3.1 Convergence in MSE

We analyze the MSE convergence of linear TTSA in terms of the centered iterates $\bar{x}_t := x_t - x^*$, $\bar{y}_t := y_t - y^*(x_t)$. To this end, we first rewrite the stochastic recursion as the following:

Lemma 3.1. Let $\bar{x}_t = x_t - x^*$, $\bar{y}_t = y_t - y^*(x_t)$. Then equation (1) can be rewritten as:

$$\begin{aligned} \bar{x}_{t+1} &= (I - \alpha \Delta) \bar{x}_t - \alpha J_{12} \bar{y}_t - \alpha w^x(x_t, y_t; \xi_t) \\ \bar{y}_{t+1} &= (I - \beta J_{22}) \bar{y}_t - \beta w^y(x_t, y_t; \xi_t) \\ &\quad - \alpha J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t + w^x(x_t, y_t; \xi_t)) \end{aligned} \quad (5)$$

Note that the slower iterates view the error in faster iterates as an additional noise. We are now ready to state our first main convergence theorem with constant step-sizes.

Theorem 3.2. Suppose Assumptions 1-3 hold, and the step sizes α, β satisfy Assumption 4. Then, for all $t \geq 0$ following the TTSA recursion (1), we have

$$\begin{aligned} \mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] &\leq \exp(-\alpha \mu_x t / 4) V_0 + O_P(\alpha \sigma_x^2 + (\alpha + \beta^2) \sigma_y^2), \\ \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] &\leq \exp(-\beta \mu_y t / 2) U_0 + O_P(\beta) \exp(-\alpha \mu_x t / 4) V_0 \\ &\quad + O_P((\alpha/\beta + \beta) \alpha \sigma_x^2 + \beta \sigma_y^2), \end{aligned}$$

where we define potential functions as $U_0 := \mathbb{E}[\|\bar{y}_0\|_{Q_y}^2] + O_P(\frac{\alpha}{\beta}) \|Q_y^{1/2} \mathbb{E}[\bar{y}_0 \bar{x}_0^\top]\|_1$, and $V_0 := \mathbb{E}[\|\bar{x}_0\|_{Q_x}^2] + O_P(\frac{\alpha^2}{\beta^2}) \mathbb{E}[\|\bar{y}_0\|_{Q_y}^2] + O_P(\frac{\alpha}{\beta}) \|Q_y^{1/2} \mathbb{E}[\bar{y}_0 \bar{x}_0^\top]\|_1$.

The theorem states that after sufficiently large iterations $t \gg \alpha^{-1}$, the convergence of TTSA in MSE can be characterized as the following:

1. $\lim_{t \rightarrow \infty} \mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] = O_P(\alpha) \sigma_x^2 + O_P(\alpha + \beta^2) \sigma_y^2$.

$$2. \lim_{t \rightarrow \infty} \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] = O_P\left(\frac{\alpha^2}{\beta} + \alpha\beta\right) \sigma_x^2 + O_P(\beta)\sigma_y^2.$$

To our best knowledge, this is the first result that explicitly characterizes the fine-grained scaling of MSE w.r.t. the stepsizes and noise variances of each iteration. The work in [Dalal et al. \(2018\)](#); [Srikant and Ying \(2019\)](#) only obtained an $O(\beta^2/\alpha)$ asymptotic bound for the slower iterate. More recent work in [Kaledin et al. \(2020\)](#); [Haque et al. \(2023\)](#) obtained an $O(\alpha)$ asymptotic bound but required $\beta^2 \leq \alpha$, hence not strong enough to reveal the dependence on β . Our result shows that noises from slower iterates only change x_t by $O(\alpha)$, while noises from faster iterates influence x_t by $O(\alpha + \beta^2)$, without requiring $\beta^2 \leq \alpha$.

3.2 Convergence to a Limit Distribution

Now we state the distributional convergence of the process (x_t, y_t, ξ_t) in Wasserstein distance as defined in [Definition 1](#). We require a mild assumption on the fourth-order moments of initial distributions:

Assumption 5. *We assume that the fourth-order moments of the initial distribution are bounded, i.e., $\mathbb{E}[\|\bar{x}_0\|_2^4 + \|\bar{y}_0\|_2^4] < \infty$.*

Our main theorem establishes the linear convergence of the Markovian process $(x_t, y_t, \xi_t)_{t \geq 0}$ in $\bar{\mathcal{W}}_2$ -distance to a unique stationary distribution:

Theorem 3.3. *Suppose Assumptions 1-3 hold, and step sizes α, β satisfy Assumption 4. If we start from an arbitrary initial distribution $(x_0, y_0, \xi_0) \sim \mu_0$ satisfying Assumption 5, then there exists a unique stationary distribution μ such that the process $(x_t, y_t, \xi_t) \sim \mu_t$ linearly converges in $\bar{\mathcal{W}}_2$ -distance:*

$$\bar{\mathcal{W}}_2^2(\mu_t, \mu) \leq O_P(1) \cdot \exp(-\alpha\mu_x t/8).$$

Furthermore, there exists vectors \bar{b}_i^x, \bar{b}_i^y independent of α, β with $\|\bar{b}_i^x\|_2, \|\bar{b}_i^y\|_2 = O_P(1)$ for $i \in \{1, 2\}$, such that for $(x_\infty, y_\infty, \xi_\infty) \sim \mu$,

$$\begin{aligned} \mathbb{E}[x_\infty - x^*] &= \alpha\bar{b}_1^x + \beta\bar{b}_2^x + O_P(\beta^2), \\ \mathbb{E}[y_\infty - y^*(x_\infty)] &= \alpha\bar{b}_1^y + \beta\bar{b}_2^y + O_P(\beta^2), \end{aligned} \quad (6)$$

and variances of x_∞ and y_∞ are bounded by

$$\text{Tr}(\mathbb{V}(x_\infty)) = O_P(\alpha), \quad \text{Tr}(\mathbb{V}(y_\infty)) = O_P(\beta). \quad (7)$$

A few remarks follow below. First, the theorem states that any sequence following TTSA (1) converges to some unique stationary distribution depending on problem instances and step sizes. Given the existence of the unique stationary distribution μ , henceforth, we can define random variables from the limit distribution $(x_\infty, y_\infty, \xi_\infty) \sim \mu$.

Second, the limit distribution has a bias, whose dominating term grows linearly with the stepsizes. The β -wise growth in the bias of faster iterates y_t is not surprising in light of known results for Markovian single-timescale SA ([Lauand and Meyn, 2023](#); [Huo et al., 2023](#)). More interesting is the bias of the slower iterates x_t , which also grows linearly with β , even though the size of the update is only $O(\alpha)$ in each slow iteration. This is a unique phenomenon of two-timescale SA: the slower iterate effectively views the error from faster iterates, $y_t - y^*(x_t)$, as additional ‘biased’ noise.

Finally, the theorem shows that the limit distribution of slower iterates has an interesting property: the bias in x (slower iterates) is dominated by the faster step-size β , while its variance only scales with the slower step-size α . This is another key property of two-timescale SA that has been overlooked in prior work. In particular, we can deduce that the asymptotic MSE of slower iterates is resulted from two factors:

$$\mathbb{E}[\|x_\infty - x^*\|_2^2] \asymp \underbrace{\alpha(\sigma_x^2 + \sigma_y^2)}_{\text{variance}} + \underbrace{\beta^2\sigma_y^2}_{\text{squared bias}}$$

Focusing separately on the two iterates, we have the following more fine-grained convergence results:

Corollary 3.4. *Suppose Assumptions 1-5 hold. Define $U_0 := \mathbb{E}[\|x_0 - \mathbb{E}[x_\infty]\|_2^2] + \mathbb{E}[\|\bar{y}_0 - \mathbb{E}[\bar{y}_\infty]\|_2^2] + O_P(\beta)$, and $V_0 := \mathbb{E}[\|x_0 - \mathbb{E}[x_\infty]\|_2^2] + O_P(\alpha/\beta)U_0$. Then for all $t \geq 0$, we have the bounds*

$$\begin{aligned} \mathcal{W}_2^2(\mu_t(\bar{y}_t), \mu(\bar{y}_\infty)) &\leq O_P(\beta) \exp(-\alpha\mu_x t/8)V_0 \\ &\quad + O_P(1) \exp(-\beta\mu_y t/8)U_0, \\ \mathcal{W}_2^2(\mu_t(x_t), \mu(x_\infty)) &\leq O_P(1) \exp(-\alpha\mu_x t/8)V_0. \end{aligned}$$

This corollary explicitly states how the optimization error decays from arbitrary initial points, and will be used in showing the convergence of tail-averaging next.

3.3 Tail-Averaging and Extrapolation

Using the explicit characterization of bias and variance in [Theorem 3.3](#), we derive improved convergence rates for tail-averaging and extrapolation.

3.3.1 Averaging

We first consider the tail-averaging variant of Polyak-Ruppert averaging ([Jain et al., 2018](#)):

$$\tilde{x}_t := \frac{1}{t - t_0} \sum_{t'=t_0}^t x_{t'}, \quad \tilde{y}_t := \frac{1}{t - t_0} \sum_{t'=t_0}^t y_{t'}, \quad (8)$$

where $t_0 \gtrsim \alpha^{-1}$ is the length of the warm-up period. With the result from [Theorem 3.2](#), we can analyze the MSE of tail-averaged sequence:

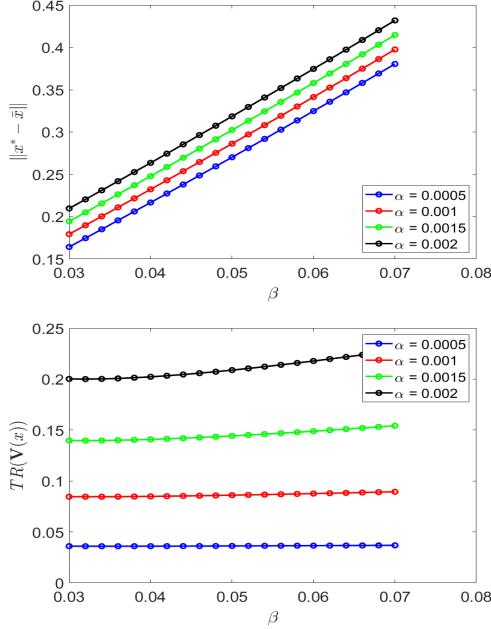


Figure 1: Bias (top) and variance (bottom) versus β at different α for the slower iterate x_t .

Theorem 3.5. Suppose Assumptions 1-5 hold and $t_0 > C(\alpha\mu_x)^{-1}$ for some sufficiently large absolute constant $C > 0$. Then for all $t > t_0$

$$\begin{aligned}\mathbb{E}[\|\tilde{x}_t - x^*\|_2^2] &= O_p(\beta^2) + \frac{O_p(1)}{t - t_0}, \\ \mathbb{E}[\|\tilde{y}_t - y^*\|_2^2] &= O_p(\beta^2) + \frac{O_p(1 + \sqrt{\beta^2/\alpha})}{t - t_0}.\end{aligned}$$

In the above result, we omitted an additional optimization error $\exp(-c\alpha\mu_x t_0)$ since it is dominated by other terms with $t_0 \gg 1/(\alpha\mu_x)$. As we can observe, $O(\beta^2)$ is attributed to the squared-bias, and $O(1/t)$ convergence is the variance decaying effect of tail-averaging. We also observe that the faster iterates has extra $O(\frac{1}{t}\sqrt{\beta^2/\alpha})$ -term. In part, this is because we measure the MSE of \tilde{y}_t from $y^* = y^*(x^*)$, not from $y^*(\tilde{x}_t)$. However, we are not fully aware whether this is an artifact of an analysis, or can be removed, and we leave the question as an open problem. Note that when $\beta^2 \leq \alpha$, both iterates enjoy the same $O(1/t)$ -decaying rate of variances as if the two iterates are decoupled.

3.3.2 Extrapolation

When tail-averaging can reduce the variance, extrapolation can reduce the biases of each iterate. As one example, using the fact that biases of iterates grow linearly with step sizes, we can extrapolate two sequences, $(x_t^{\alpha,\beta}, y_t^{\alpha,\beta})$ and $(x_t^{2\alpha,2\beta}, y_t^{2\alpha,2\beta})$ with pairing step sizes (α, β) and $(2\alpha, 2\beta)$. The extrapolated iterates

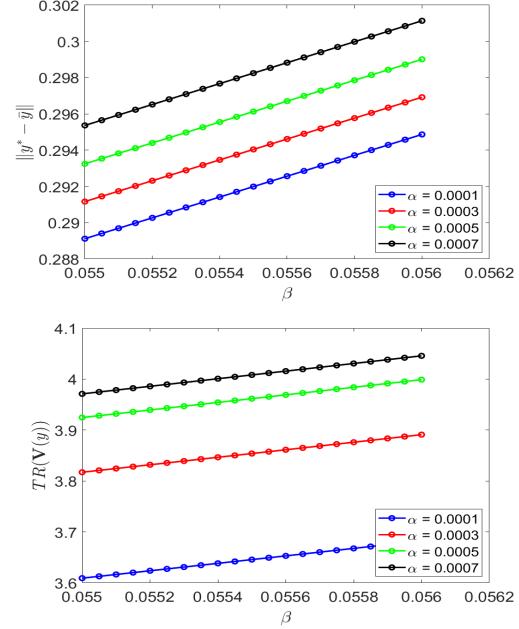


Figure 2: Bias (top) and variance (bottom) versus β at different α for the faster iterate y_t .

are computed as

$$\zeta_t^x := 2\tilde{x}_t^{\alpha,\beta} - \tilde{x}_t^{2\alpha,2\beta}, \quad \zeta_t^y := 2\tilde{y}_t^{\alpha,\beta} - \tilde{y}_t^{2\alpha,2\beta}.$$

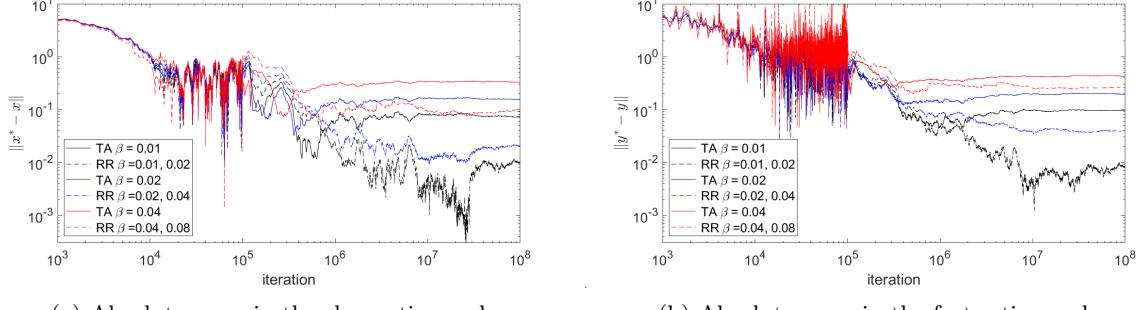
As a corollary of our main theorems, we have the following result characterizing the MSE of the extrapolated sequences. Extrapolation achieves reduced biases by canceling out the leading α and β terms in the asymptotic biases (6), improving the MSE bounds of both iterates from β^2 to β^4 .

Corollary 3.6. Suppose Assumptions 1-5 hold and $t_0 > C(\alpha\mu_x)^{-1}$ for some sufficiently large absolute constant $C > 0$. Then for all $t > t_0$,

$$\begin{aligned}\mathbb{E}[\|\zeta_t^x - x^*\|_2^2] &= O_p(\beta^4) + \frac{O_p(1)}{t - t_0}, \\ \mathbb{E}[\|\zeta_t^y - y^*\|_2^2] &= O_p(\beta^4) + \frac{O_p(1 + \sqrt{\beta^2/\alpha})}{t - t_0}.\end{aligned}$$

Remark 1. If one uses pairing stepsizes (α, β) and $(\alpha, 2\beta)$, then only the leading β terms in the asymptotic biases (6) are cancelled.

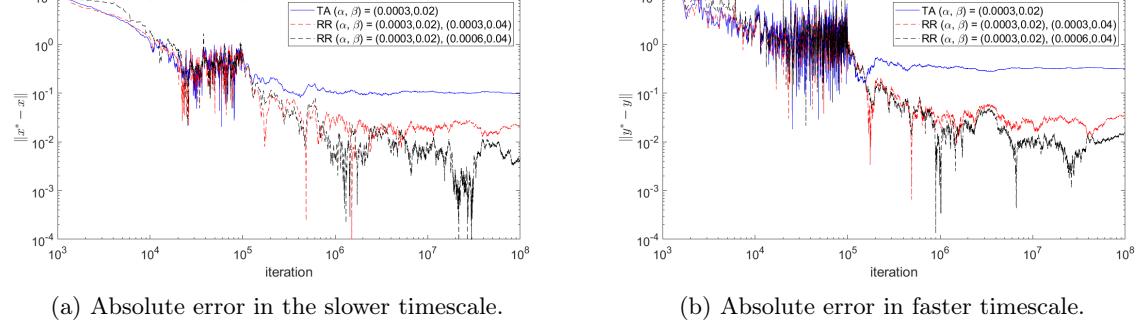
Remark 2. It is possible to further reduce the order of bias via higher-order extrapolation using more than two sets of stepsizes as in [Huo et al. \(2023, 2024\)](#), though it comes at the price of potentially slower convergence and higher variance due to using additional stepsizes [\(Durmus et al., 2021; Srikant and Ying, 2019\)](#).



(a) Absolute error in the slower timescale.

(b) Absolute error in the faster timescale.

Figure 3: Comparison of Tail-Averaging (TA) and Richard-Romberg (RR) extrapolation in β .



(a) Absolute error in the slower timescale.

(b) Absolute error in faster timescale.

Figure 4: Comparison of Tail-Averaging (TA), RR extrapolation in β , and RR extrapolation in both β and α .

4 Experiments

We consider the TTSA iteration (1) in dimension $d_x = d_y = 2$ driven by a 10-state, irreducible, aperiodic Markov chain. We construct the transition matrix randomly and choose $J_{11}, J_{12}, J_{21}, J_{22}$ such that Assumption 1 hold.

We tested the dependence of the bias and variance of both iterates with respect to α and β by varying each individually. After the tail-averaged iterates converged, we calculated the bias as the average distance between the averaged iterate and the true solution, and calculated the variance $\text{Tr } \mathbb{V}(\cdot)$ as the average square distance from the iterate to the sample mean of the iterates. For the dependence on β , we held α constant and varied β between 0.03 and 0.07. For the dependence on α , we held β constant and varied α between 0.0001 and 0.0005.

For the slower iterate x_t , Figure 1 shows that the bias scales with both β and α , while the variance is dependent mostly on α only. For the faster iterate y_t , Figure 2 shows that the bias depends both on β and α , and the variance depends on β . Both results are consistent with our theory.

We also tested the effects of tail-averaging (TA) and Richardson-Romberg (RR) extrapolation with a sim-

ilar setup. We fixed $\alpha = 0.0003$ and let $\beta = \{0.01, 0.02, 0.04, 0.08\}$. In Figure 3, for each β , we plotted the absolute errors achieved by tail-averaging at stepsize β (labeled as “TA $\beta = \text{stepsize}$ ”), as well as the errors achieved by RR extrapolation with stepsizes β and 2β (labeled as “RR $\beta = \text{stepsize}, 2*\text{stepsize}$ ”), which aims to cancel the β term in the bias. Compared to the TA iterates (solid line), the corresponding RR extrapolated iterate (the dashed line of the same color) achieved lower errors, corresponding to reduced asymptotic biases.

In addition, we examined the effectiveness of applying RR extrapolation to cancel *both* the α and β bias terms. Letting $\alpha = 0.0003, \beta = 0.02$, we compared RR extrapolating on only β (using stepsizes β, α and $2\beta, \alpha$) with RR extrapolating on both β and α (using stepsizes β, α and $2\beta, 2\alpha$). In Figure 4, we see that while the former (red curves) already reduced a large amount of the bias, the latter (black curves) reduced it even further, as predicted by our theoretical results.

5 Proof Outline

We outline the proofs of our main theorems. We focus on slower iterates; similar ideas apply to faster iterates.

5.1 Proof Outline of Theorem 3.2

The first step is to analyze the descent formula for each iterate separately. For the slower iterate, we have

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{t+1}\|_{Q_x}^2] &= \mathbb{E}[\|(I - \alpha\Delta)\bar{x}_t\|_{Q_x}^2] + 2\alpha \underbrace{\mathbb{E}[\langle \bar{x}_t, w_t^x \rangle_{Q_x}]}_{T_1} \\ &\quad + 2\alpha \underbrace{\mathbb{E}[\langle \bar{x}_t, J_{12}\bar{y}_t \rangle_{Q_x}]}_{T_2} + o(\alpha). \end{aligned}$$

The term T_1 would have been 0 if the noise sequence were martingale, and can be effectively handled with Markovian noises in a standard way by exploiting Assumption 2. More pressing issue is handling T_2 : with naively applying Young's inequality to bound (ii), *i.e.*, with $\langle \bar{x}_t, J_{12}\bar{y}_t \rangle \leq (c\|\bar{x}_t\|^2 + \frac{J_{\max}}{4c}\|\bar{y}_t\|^2)$, the asymptotic error easily end up being $O(\beta^2/\alpha)$ as in [Dalal et al. \(2018\)](#); [Gupta et al. \(2019\)](#), and such an approach can be improved up to at best $O(\beta)$ ([Doan, 2022](#)).

Recent results in [Kaledin et al. \(2020\)](#); [Haque et al. \(2023\)](#) directly analyzed the descent behavior of $\|T_2\|_{\text{op}}$, achieving $O(\alpha)$ asymptotic error for the slower iterate. However, using operator norm often results in extra dependence on dimensions d_x, d_y , despite the smoothness condition $J_{\max} = O(1)$ in operator norm.

Our tweak for this issue is simple: to track the convergence of cross-correlation norm, we employ the Schatten S^1 -measure for $\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1$, where $Q_y^{1/2}$ term is incorporated to ensure decreasing Lyapunov potential with asymmetric operators. The S^1 -norm is the best suited for exploiting the smoothness condition without incurring dimension dependence, thanks to the Holder's inequality for matrix Schatten norm:

$$\|AB\|_1 \leq \|A\|_1\|B\|_\infty = \|A\|_1\|B\|_{\text{op}}.$$

Leveraging this property, we can construct the potential function as the sum of three terms (omitting constants):

$$\mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] + \frac{\alpha(\alpha + \beta^2)}{\beta} \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \frac{\alpha}{\beta} \|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1.$$

With similar techniques for analyzing the faster iterates and cross-correlation norms, we can obtain a clean $O(\alpha)$ asymptotic error without additional dimension dependence. The full proof is given in [Appendix B](#).

5.2 Proof Outline of Theorem 3.3

Once we have the MSE convergence result, extending the strategies in prior work for the single-timescale SA ([Dieuleveut et al., 2020](#); [Huo et al., 2023](#)), we first consider two coupling sequences via sharing the common noise sequence (x_t^1, y_t^1, ξ_t) and (x_t^2, y_t^2, ξ_t) . The idea is to show that the coupled sequences $\delta_t^x := \bar{x}_t^1 - \bar{x}_t^2$,

$\delta_t^y := \bar{y}_t^1 - \bar{y}_t^2$ converge linearly ([Lemma B.1](#)),

$$\mathbb{E}[\|\delta_t^x\|_2^2] \lesssim \exp(-cat) \cdot \mathbb{E} \left[\|\delta_0^x\|_2^2 + \frac{\alpha}{\beta} \|\delta_0^y\|_2^2 \right].$$

Then we can design two sequences coupled in such a way that $(x_t^2, y_t^2, \xi_t) \stackrel{d}{=} (x_{t+1}^1, y_{t+1}^1, \xi_{t+1})$. Combining the two results, the sequence (x_t^1, y_t^1, ξ_t) converges in L_2 -Wasserstein distribution to a unique stationary point. The remaining details can be found in [Appendix B.2](#).

Bias and Variance Turning to the stationary distributions of the iterates, we observe that x_∞ satisfies

$$\begin{aligned} \bar{x}_{\infty+1} &= (I - \alpha\Delta)\bar{x}_\infty - \alpha J_{12}\bar{y}_\infty - \alpha w_\infty^x, \\ \mathbb{E}[\bar{x}_{\infty+1}|\xi_{\infty+1} = \xi] &= \mathbb{E}[\bar{x}_\infty|\xi_\infty = \xi], \quad \forall \xi \in \Xi. \end{aligned}$$

Conditioned on the event $\xi_{\infty+1} = \xi$, we have $\xi_\infty \sim P^\dagger(\cdot|\xi)$, where P^\dagger is the adjoint of the transition kernel P . Using this relation, we can construct a stationary equation for $\mathbb{E}[\bar{x}_\infty|\xi_\infty = \xi]$, and find the explicit expression for biases by integrating the conditional expectation over a stationary distribution π , *i.e.*,

$$\mathbb{E}[\bar{x}_\infty] = \int_{\Xi} \mathbb{E}[\bar{x}_\infty|\xi_\infty = \xi] d\pi(\xi) = \alpha \bar{b}_1^x + \beta \bar{b}_2^x + O(\alpha\beta).$$

The variance of \bar{y}_∞ is relatively simple to bound:

$$\text{Tr}(\mathbb{V}(\bar{y}_\infty)) \leq \mathbb{E}[\|\bar{y}_\infty - y^*(\bar{x}_\infty)\|_2^2] \leq O(\beta).$$

However, showing the variance upper bound $O(\alpha)$ can not be derived in the same fashion since the MSE bound for \bar{x}_∞ is $O(\alpha + \beta^2)$. To derive this, we also construct a stationary equation for the covariance:

$$\mathbb{E}[\bar{x}_{\infty+1}\bar{x}_{\infty+1}^\top|\xi_{\infty+1} = \xi] = \mathbb{E}[\bar{x}_\infty\bar{x}_\infty^\top|\xi_\infty = \xi], \quad \forall \xi \in \Xi,$$

and show that S^1 -norm of the above is $O(\alpha)$. Using the inequality $\text{Tr}(A) \leq \|A\|_1$ completes the proof.

6 Conclusion

We analyze Markovian Linear TTSA with constant stepsizes. We show that both iterations converge geometrically to a unique stationary distribution, even in the absence of global contraction properties. We provide a precise characterization of the biases and variances in terms of the stepsizes, and establish the advantages of tail-averaging and extrapolation. Building off our results, several intriguing questions emerge. For instance, how would the behavior change if the Markovian states were dependent on the TTSA iterates, as in on-policy reinforcement learning with constant stepsizes? Additionally, can we extend our convergence guarantees in Wasserstein distance for a broader class of functions, such as those satisfying the Polyak-Łojasiewicz condition? These questions are remained to be explored in future works.

Acknowledgement

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Checklist

1. For all models and algorithms presented, check if you include:
 - (a) A clear description of the mathematical setting, assumptions, algorithm, and/or model. [Yes]
 - (b) An analysis of the properties and complexity (time, space, sample size) of any algorithm. [Yes]
 - (c) (Optional) Anonymized source code, with specification of all dependencies, including external libraries. [Yes]
2. For any theoretical claim, check if you include:

- (a) Statements of the full set of assumptions of all theoretical results. [Yes]
- (b) Complete proofs of all theoretical results. [Yes]
- (c) Clear explanations of any assumptions. [Yes]

3. For all figures and tables that present empirical results, check if you include:

- (a) The code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL). [Yes]
- (b) All the training details (e.g., data splits, hyperparameters, how they were chosen). [Not Applicable]
- (c) A clear definition of the specific measure or statistics and error bars (e.g., with respect to the random seed after running experiments multiple times). [Yes for measure and statistics / No for error bars]
- (d) A description of the computing infrastructure used. (e.g., type of GPUs, internal cluster, or cloud provider). [Not Applicable]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets, check if you include:

- (a) Citations of the creator If your work uses existing assets. [Not Applicable]
- (b) The license information of the assets, if applicable. [Not Applicable]
- (c) New assets either in the supplemental material or as a URL, if applicable. [Not Applicable]
- (d) Information about consent from data providers/curators. [Not Applicable]
- (e) Discussion of sensible content if applicable, e.g., personally identifiable information or offensive content. [Not Applicable]

5. If you used crowdsourcing or conducted research with human subjects, check if you include:

- (a) The full text of instructions given to participants and screenshots. [Not Applicable]
- (b) Descriptions of potential participant risks, with links to Institutional Review Board (IRB) approvals if applicable. [Not Applicable]
- (c) The estimated hourly wage paid to participants and the total amount spent on participant compensation. [Not Applicable]

A Technical Lemmas

Lemma A.1. For any two real matrices A, B , we have

$$\text{Tr}(A^\top B) \leq \|A^\top B\|_1 \leq \|A\|_\infty \|B\|_1 = \|A\|_{\text{op}} \|B\|_1.$$

Lemma A.2. For a positive definite matrix $Q \succ 0$ and any real matrix A , the following holds:

$$\|A\|_Q = \|Q^{1/2} A Q^{-1/2}\|_{\text{op}}.$$

Lemma A.3. For a positive definite matrix $Q \succ 0$, and for any vectors x, y and a matrix M ,

$$\begin{aligned} \langle x, y \rangle_Q &\leq \|x\|_Q \|y\|_Q, \quad \langle Mx, x \rangle_Q \leq \|M\|_{\text{op}} \|x\|_Q^2, \\ \|Mx\|_Q &\leq \|M\|_Q \|x\|_Q \leq \sqrt{\kappa(Q)} \|M\|_{\text{op}} \|x\|_Q, \end{aligned}$$

where $\kappa(Q) = \frac{\sigma_{\max}(Q)}{\sigma_{\min}(Q)}$ is the condition number of Q .

Lemma A.4 (Lemma C.13 in Haque et al. (2023)). Let $-A$ be a Hurwitz matrix and Q be the solution to

$$A^\top Q + QA = I. \quad (9)$$

Then for all $\epsilon \in \left[0, \frac{1}{\|Q\|_{\text{op}} \|A\|_Q^2}\right]$, for any matrix B , we have

$$\|(I - \epsilon A)B\|_Q \leq (1 - \mu\epsilon) \|B\|_Q,$$

where $\mu := \frac{1}{2\|Q\|_{\text{op}}}$. In particular, $\|I - \epsilon A\|_Q \leq 1 - \mu\epsilon$.

Lemma A.5. For any two positive definite matrices Q_1, Q_2 and a vector x , we have

$$\|x\|_{Q_1}^2 \leq \frac{\sigma_{\max}(Q_1)}{\sigma_{\min}(Q_2)} \cdot \|x\|_{Q_2}^2.$$

A.1 Auxiliary Lemmas

We list some useful facts and lemmas here.

Lemma A.6. For any $t \geq \tau$, for all $i, j \in \{1, 2\}$, we have

$$\begin{aligned} \mathbb{E}[\langle W_{ij}(\xi_t), v_{t-\tau} u_{t-\tau}^\top \rangle | \mathcal{F}_{t-\tau}] &= O(\rho^\tau W_{\max} \|v_{t-\tau}\|_2 \|u_{t-\tau}\|_2), \\ \mathbb{E}[\langle u_i(\xi_t), v_{t-\tau} \rangle | \mathcal{F}_{t-\tau}] &= O(\rho^\tau u_{\max} \|v_{t-\tau}\|_2). \end{aligned}$$

where v_t, u_t are any vectors that can be constructed at the t^{th} iteration.

Lemma A.7. Let two intermediate variables:

$$\begin{aligned} W_\Delta^x(\xi) &:= W_{11}(\xi) - W_{12}(\xi) J_{22}^{-1} J_{21}, \\ W_\Delta^y(\xi) &:= W_{21}(\xi) - W_{22}(\xi) J_{22}^{-1} J_{21}. \end{aligned}$$

Then, w_t^x, w_t^y can be rewritten as

$$\begin{aligned} w_t^x &= W_\Delta^x(\xi_t) \bar{x}_t + W_{12}(\xi_t) \bar{y}_t + W_\Delta^x(\xi_t) x^* + u_1(\xi_t), \\ w_t^y &= W_\Delta^y(\xi_t) \bar{x}_t + W_{22}(\xi_t) \bar{y}_t + W_\Delta^y(\xi_t) x^* + u_2(\xi_t). \end{aligned}$$

Lemma A.8. For any $t \geq \tau \geq \tau_\alpha$, we have

$$\begin{aligned} \|\bar{x}_t - \bar{x}_{t-\tau}\|_2 &\leq 4\alpha\tau (J_{\max}(\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) + \sigma_x + \beta J_{\max} \sigma_y), \\ \|\bar{y}_t - \bar{y}_{t-\tau}\|_2 &\leq 4\beta\tau (J_{\max}(\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) + \sigma_y) + 4\alpha\kappa_y \tau \sigma_x. \end{aligned}$$

The following corollary is convenient:

Corollary A.9. If $\alpha\tau\kappa_y \leq c_1\beta\tau \leq c_2/J_{\max}$ holds with absolute constants $c_1, c_2 > 0$, then for any $t \geq \tau \geq \tau_\alpha$,

$$\begin{aligned}\|\bar{x}_{t-\tau}\|_2 &\leq 2\|\bar{x}_t\|_2 + 8\alpha\tau J_{\max}\|\bar{y}_t\|_2 + 4\tau(\alpha\sigma_x + \alpha\beta J_{\max}\sigma_y), \\ \|\bar{y}_{t-\tau}\|_2 &\leq 4\beta\tau J_{\max}\kappa_y\|\bar{x}_t\|_2 + 2\|\bar{y}_t\|_2 + 4\tau(\alpha\kappa_y\sigma_x + \beta\sigma_y).\end{aligned}$$

Lemma A.10. For any $t \geq \tau \geq c \log(\frac{\kappa_y}{\alpha\mu_y})$ with an absolute constant $c > 0$, we have

$$\|\mathbb{E}[w_t^x \bar{y}_t^\top]\|_1 \leq \frac{\mu_y}{8\kappa_y} \mathbb{E}[\|\bar{y}_t\|_2^2] + O(J_{\max}^2 \kappa_y) \beta\tau \mathbb{E}[\|\bar{x}_t\|_2^2] + O(\tau)((\alpha^2/\beta)\kappa_y\sigma_x^2 + \beta\sigma_y^2).$$

Similarly, we can derive the same upper bound for $\|\mathbb{E}[w_t^y \bar{y}_t^\top]\|_1$.

Lemma A.11. For any $t \geq \tau \geq c \log(\frac{\kappa_x}{\alpha\mu_x})$ with an absolute constant $c > 0$, we have

$$\|\mathbb{E}[w_t^x \bar{x}_t^\top]\|_1 \leq \frac{\mu_x}{8\kappa_x} \mathbb{E}[\|\bar{x}_t\|_2^2] + O\left(\alpha\tau J_{\max}^2 + \beta^2\tau^2 \frac{J_{\max}^4 \kappa_x}{\mu_x}\right) \mathbb{E}[\|\bar{y}_t\|_2^2] + O(\tau)(\alpha\sigma_x^2 + \beta^2 J_{\max}\sigma_y^2).$$

Similarly, we can derive the same upper bound for $\|\mathbb{E}[w_t^y \bar{x}_t^\top]\|_1$.

B Proof of Main Theorems

We recall the definition of σ_x, σ_y in (2)

$$\begin{aligned}\sigma_x &:= \max_{\xi \in \Xi} \|u_1(\xi) + W_\Delta^x(\xi)x^*\|_2, \\ \sigma_y &:= \max_{\xi \in \Xi} \|u_2(\xi) + W_\Delta^y(\xi)x^*\|_2.\end{aligned}$$

Recall that we assume $\beta/\alpha \gg \kappa_y$ in Assumption 4, and $\|\Delta\|_{\text{op}} \leq J_{\max}\kappa_y$.

B.1 Proof of Theorem 3.2

The proof first investigates the convergence of three terms $\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2]$, $\mathbb{E}[\|\bar{x}_t\|_{Q_x}^2]$, $\|Q_y^{1/2}\mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1$ separately. Then, by constructing the potential function as the following:

$$\begin{aligned}V_t &= \mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] + \frac{O(1)J_{\max}^2\kappa_y\alpha(\alpha + \beta^2\tau^2 J_{\max})}{\mu_x\mu_y\beta} \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \frac{O(1)J_{\max}^{1/2}\kappa_y\alpha}{\mu_x\beta} \|Q_y^{1/2}\mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1, \\ U_t &= \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \frac{O(1)J_{\max}^{1/2}\kappa_y^{2.5}\alpha}{\mu_y\beta} \|Q_y^{1/2}\mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1,\end{aligned}\tag{10}$$

and show that they decay in exponential rates.

B.1.1 Convergence of \bar{y}_t

We first study the descent behavior of \bar{y}_t :

$$\begin{aligned}\mathbb{E}[\|\bar{y}_{t+1}\|_{Q_y}^2] &\leq \mathbb{E}\left[\|(I - \beta J_{22})\bar{y}_t\|_{Q_y}^2 + \alpha^2\|J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)\|_{Q_y}^2 + \beta^2\|w_t^y\|_{Q_y}^2\right] \\ &\quad + 2\alpha\left|\mathbb{E}[\langle(I - \beta J_{22})\bar{y}_t, -J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)\rangle_{Q_y}]\right| \\ &\quad + 2\beta\left|\mathbb{E}[\langle(I - \beta J_{22})\bar{y}_t, -w_t^y\rangle_{Q_y}]\right| + 2\alpha\beta\left|\mathbb{E}[\langle J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t - w_t^x), w_t^y\rangle_{Q_y}]\right|.\end{aligned}$$

We bound each term:

1. Using Lemma A.4, we have

$$\|(I - \beta J_{22})\bar{y}_t\|_{Q_y}^2 \leq (1 - \mu_y\beta)\|\bar{y}_t\|_{Q_y}^2.$$

2. Using the formula in Lemma A.7,

$$\begin{aligned}\|w_t^y\|_{Q_y}^2 &\leq O(1) \cdot \left(\|Q_y\|_{\text{op}} (W_{\max} \kappa_y)^2 \|\bar{x}_t\|_2^2 + \kappa_y W_{\max}^2 \|\bar{y}_t\|_{Q_y}^2 + \|Q_y\|_{\text{op}} \sigma_y^2 \right) \\ &\leq O(1) \cdot \left(J_{\max} \kappa_y^3 \|\bar{x}_t\|_2^2 + \kappa_y J_{\max}^2 \|\bar{y}_t\|_{Q_y}^2 + (1/\mu_y) \sigma_y^2 \right),\end{aligned}$$

where we also used Lemma A.3 to have $\|W_{22} \bar{y}_t\|_{Q_y}^2 \leq \kappa(Q_y) W_{\max}^2 \|\bar{y}_t\|_{Q_y}^2$, and $\kappa(Q_y) = O(\kappa_y)$.

3. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t + w_t^x)\|_{Q_y}^2 &\leq 3 \|J_{22}^{-1} J_{21} (J_{12} + W_{12}(\xi_t))\|_{Q_y}^2 \|\bar{y}_t\|_{Q_y}^2 + 3 \|Q_y\|_{\text{op}} \|J_{22}^{-1} J_{21} (\Delta + W_{\Delta}^x(\xi_t))\|_{\text{op}}^2 \|\bar{x}_t\|_2^2 \\ &\quad + 3 \|J_{22}^{-1} J_{21} (u_1(\xi_t) + W_{\Delta}^x(\xi_t) x^*)\|_{Q_y}^2 \\ &\leq O(1) \left(\kappa_y^3 J_{\max}^2 \|\bar{y}_t\|_{Q_y}^2 + J_{\max} \kappa_y^5 \|\bar{x}_t\|_2^2 + (\kappa_y^2 / \mu_y) \cdot \sigma_x^2 \right).\end{aligned}$$

4. We separate the cross-product term across \bar{y}_t and \bar{x}_t :

$$\begin{aligned}&|\mathbb{E}[\langle (I - \beta J_{22}) \bar{y}_t, -J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t + w_t^x) \rangle_{Q_y}]| \\ &\leq |\underbrace{\mathbb{E}[\langle \bar{y}_t, -J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t) \rangle_{Q_y}]|}_{(i)} + \beta |\underbrace{\mathbb{E}[\langle J_{22} \bar{y}_t, -J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t) \rangle_{Q_y}]|}_{(ii)} \\ &\quad + |\underbrace{\mathbb{E}[\langle (I - \beta J_{22}) \bar{y}_t, -J_{22}^{-1} J_{21} w_t^x \rangle_{Q_y}]|}_{(iii)}.\end{aligned}$$

For (i), we can derive that

$$\begin{aligned}-(i) &= \mathbb{E}[\text{Tr}(\bar{y}_t^\top Q_y J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t))] \\ &\leq \text{Tr}(\mathbb{E}[\bar{y}_t \bar{y}_t^\top Q_y^{1/2} Q_y^{1/2} J_{22}^{-1} J_{21} J_{12}]) + \text{Tr}(\mathbb{E}[\bar{x}_t \bar{y}_t^\top Q_y^{1/2} Q_y^{1/2} J_{22}^{-1} J_{21} \Delta]) \\ &\leq \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] \cdot \|Q_y^{1/2} J_{22}^{-1} J_{21} J_{12} Q_y^{-1/2}\|_{\text{op}} + \|Q_y^{1/2} \mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1 \|Q_y^{1/2} J_{22}^{-1} J_{21} \Delta\|_{\text{op}} \\ &\leq \kappa_y^{3/2} J_{\max} \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \kappa_y^{5/2} J_{\max}^{1/2} \cdot \|Q_y^{1/2} \mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1.\end{aligned}$$

For (ii), we can simply apply Cauchy-Schwarz inequality with $J_{22}^\top Q_y J_{22}^{-1} = J_{22}^{-1} - Q_y$, to get

$$\begin{aligned}-(ii) &= \mathbb{E}[\bar{y}_t^\top J_{22}^\top Q_y J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t)] \\ &= \mathbb{E}[\bar{y}_t^\top (J_{22}^{-1} - Q_y) J_{21} J_{12} \bar{y}_t] + \mathbb{E}[\bar{y}_t^\top (J_{22}^{-1} - Q_y) J_{21} J_{12} \bar{y}_t \Delta \bar{x}_t] \\ &\leq J_{\max} \kappa_y \mathbb{E}[\|\bar{y}_t\|_2^2] + \|\mathbb{E}[\bar{x}_t \bar{y}_t^\top Q_y^{1/2}]\|_1 \|Q_y^{-1/2} (J_{22}^{-1} - Q_y) J_{12} J_{21} \Delta\|_{\text{op}} \\ &\leq J_{\max}^2 \kappa_y \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + (\kappa_y^2 J_{\max}^{3/2}) \|Q_y^{1/2} \mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1.\end{aligned}$$

For (iii), we bound the term as

$$\begin{aligned}(iii) &= |\text{Tr}(\mathbb{E}[w_t^x \bar{y}_t^\top] (I - \beta J_{22})^\top Q_y J_{22}^{-1} J_{21})| \\ &\leq \|Q_y J_{22}^{-1} J_{21}\|_{\text{op}} \cdot \|\mathbb{E}[w_t^x \bar{y}_t^\top]\|_1 \\ &\leq (\kappa_y / \mu_y) \|\mathbb{E}[w_t^x \bar{y}_t^\top]\|_1.\end{aligned}$$

Combining (i)-(iii), we get

$$\begin{aligned}&|\mathbb{E}[\langle (I - \beta J_{22}) \bar{y}_t, -J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t + w_t^x) \rangle_{Q_y}]| \\ &\leq \kappa_y^{3/2} J_{\max} \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \kappa_y^{5/2} J_{\max}^{1/2} \cdot \|Q_y^{1/2} \mathbb{E}[\bar{y}_t \bar{x}_t^\top]\|_1 + (\kappa_y / \mu_y) \|\mathbb{E}[w_t^x \bar{y}_t^\top]\|_1.\end{aligned}$$

5. For the cross-product with noise, we get

$$|\mathbb{E}[\langle (I - \beta J_{22}) \bar{y}_t, w_t^y \rangle_{Q_y}]| \leq (1/\mu_y) \|\mathbb{E}[w_t^y \bar{y}_t^\top]\|_1.$$

6. For the last term, we simply apply Cauchy-Schwartz inequality and use inequalities used before:

$$2\alpha\beta|\mathbb{E}[\langle J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x), w_t^y \rangle_{Q_y}]| \leq \alpha^2\mathbb{E}[\|J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)\|_{Q_y}^2] + \beta^2\mathbb{E}[\|w_t^y\|_{Q_y}^2].$$

Hence to summarize, with $\alpha \ll \beta/\kappa_y^3$ and $\beta \ll 1/(J_{\max}\kappa_y^2)$, we get

$$\begin{aligned} \mathbb{E}[\|\bar{y}_{t+1}\|_{Q_y}^2] &\leq (1 - 3\beta\mu_y/4)\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + O(\kappa_y^3 J_{\max})\beta^2\mathbb{E}[\|\bar{x}_t\|_2^2] \\ &\quad + O(1/\mu_y)\beta^2\sigma_y^2 + O(\kappa_y^2/\mu_y)\alpha^2\sigma_x^2 + O(\kappa_y^{5/2}J_{\max}^{1/2})\alpha\|Q_y^{1/2}\mathbb{E}[\bar{x}_t\bar{y}_t^\top]\|_1 \\ &\quad + (2\kappa_y/\mu_y)\alpha\|\mathbb{E}[w_t^x\bar{y}_t^\top]\|_1 + (2/\mu_y)\beta\|\mathbb{E}[w_t^y\bar{y}_t^\top]\|_1. \end{aligned}$$

Then we can invoke Lemma A.10 with $\tau = O(\tau_\alpha)$, and noting that $\beta J_{\max}^2\kappa_y \ll \mu_y$ to conclude that

$$\begin{aligned} \mathbb{E}[\|\bar{y}_{t+1}\|_{Q_y}^2] &\leq (1 - \beta\mu_y/2)\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + O(\kappa_y^3 J_{\max})\beta^2\tau_\alpha\mathbb{E}[\|\bar{x}_t\|_2^2] \\ &\quad + O(\kappa_y^{5/2}J_{\max}^{1/2})\alpha\|Q_y^{1/2}\mathbb{E}[\bar{x}_t\bar{y}_t^\top]\|_1 + O(1/\mu_y)\tau_\alpha(\beta^2\sigma_y^2 + \kappa_y^2\alpha^2\sigma_x^2). \end{aligned} \quad (11)$$

B.1.2 Convergence of \bar{x}_t

We start with taking squared- $\|\cdot\|_{Q_x}$ norm for the slower iterates:

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{t+1}\|_{Q_x}^2] &\leq \mathbb{E}[\|(I - \alpha\Delta)\bar{x}_t\|_{Q_x}^2 + 2\alpha^2\|J_{12}\bar{y}_t\|_{Q_x}^2 + 2\alpha^2\|w_t^x\|_{Q_x}^2] \\ &\quad + 2\alpha|\mathbb{E}[\langle(I - \alpha\Delta)\bar{x}_t, J_{12}\bar{y}_t\rangle_{Q_x}]| + 2\alpha|\mathbb{E}[\langle(I - \alpha\Delta)\bar{x}_t, w_t^x\rangle_{Q_x}]| + 2\alpha^2|\mathbb{E}[\langle J_{12}\bar{y}_t, w_t^x\rangle_{Q_x}]|. \end{aligned}$$

Following the similar steps for the analysis of \bar{y}_t , we show the followings:

1. The main drift term satisfies

$$\|(I - \alpha\Delta)\bar{x}_t\|_{Q_x}^2 \leq (1 - \mu_x\alpha)\|\bar{x}_t\|_{Q_x}^2.$$

2. For the squared terms,

$$\begin{aligned} \|J_{12}\bar{y}_t\|_{Q_x}^2 &\leq \|Q_x\|_{\text{op}}J_{\max}^2\|\bar{y}_t\|_2^2, \leq (J_{\max}^2/\mu_x)\|\bar{y}_t\|_2^2, \\ \|w_t^x\|_{Q_x}^2 &\leq 3(\kappa_x(W_{\max}\kappa_y)^2\|\bar{x}_t\|_{Q_x}^2 + \|Q_x\|_{\text{op}}W_{\max}^2\|\bar{y}_t\|_2^2 + \|Q_x\|_{\text{op}}\sigma_x^2) \\ &\leq O(1)\cdot(\kappa_x\kappa_y^2J_{\max}^2\|\bar{x}_t\|_{Q_x}^2 + (J_{\max}^2/\mu_x)\|\bar{y}_t\|_2^2 + (1/\mu_x)\sigma_x^2). \end{aligned}$$

3. For the cross-product term,

$$\begin{aligned} |\mathbb{E}[\langle(I - \alpha\Delta)\bar{x}_t, J_{12}\bar{y}_t\rangle_{Q_x}]| &= |\text{Tr}(\mathbb{E}[\bar{y}_t\bar{x}_t^\top](I - \alpha\Delta)^\top Q_x J_{12})| \\ &\leq \|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1\|Q_x^{1/2}Q_x^{-1/2}(I - \alpha\Delta)^\top Q_x J_{12}Q_y^{-1/2}\|_{\text{op}} \\ &\leq (J_{\max}^{3/2}/\mu_x)\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1. \end{aligned}$$

4. For the product term with noise, we have

$$|\mathbb{E}[\langle(I - \alpha\Delta)\bar{x}_t, w_t^x\rangle_{Q_x}]| \leq \|\mathbb{E}[w_t^x\bar{x}_t^\top]\|_1\|Q_x\|_{\text{op}} \leq (1/\mu_x)\|\mathbb{E}[w_t^x\bar{x}_t^\top]\|_1.$$

Writing down the intermediate result, with $\alpha \ll 1/(J_{\max}\kappa_x\kappa_y^3)$, we have

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{t+1}\|_{Q_x}^2] &\leq (1 - \alpha\mu_x/2)\mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] + O(J_{\max}^2/\mu_x)\alpha^2\mathbb{E}[\|\bar{y}_t\|_2^2] + O(1/\mu_x)\alpha^2\sigma_x^2 \\ &\quad + O(J_{\max}^{3/2}/\mu_x)\alpha\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1 + (2/\mu_x)\alpha\|\mathbb{E}[w_t^x\bar{x}_t^\top]\|_1. \end{aligned}$$

Invoke Lemma A.11 with $\tau = O(\tau_\alpha)$, and we can conclude that

$$\begin{aligned} \mathbb{E}[\|\bar{x}_{t+1}\|_{Q_x}^2] &\leq (1 - \alpha\mu_x/2)\mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] + (J_{\max}^2/\mu_x)(\alpha^2 + \alpha\beta^2\tau_\alpha^2 J_{\max})\mathbb{E}[\|\bar{y}_t\|_2^2] \\ &\quad + \alpha(J_{\max}^{3/2}/\mu_x)\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1 + (1/\mu_x)(\alpha^2\tau_\alpha\sigma_x^2 + \alpha\beta^2\tau_\alpha^2 J_{\max}\kappa_x^2\sigma_y^2). \end{aligned} \quad (12)$$

B.1.3 Convergence of Cross-Correlations in S^1 -Norm

We start with unfolding the equation:

$$\begin{aligned}\bar{y}_{t+1}\bar{x}_{t+1}^\top &= (I - \beta J_{22})\bar{y}_t\bar{x}_t^\top(I - \alpha\Delta) - \alpha(I - \beta J_{22})\bar{y}_t(J_{12}\bar{y}_t + w_t^x)^\top \\ &\quad - \alpha J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)\bar{x}_t^\top - \beta w_t^y\bar{x}_t^\top \\ &\quad + \alpha^2 J_{22}^{-1}J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)(\Delta\bar{x}_t + J_{12}\bar{y}_t + w_t^x)^\top \\ &\quad + \alpha\beta \cdot w_t^y(\Delta\bar{x}_t + J_{12}\bar{y}_t + w_t^x)^\top.\end{aligned}$$

The target norm is $\|\cdot\|_1$ bound on the expectation of the cross-product term. The trick is to multiply $Q_y^{1/2}$ from left on both sides, and use identity $I = Q_y^{-1/2}Q_y^{1/2}$:

$$\begin{aligned}\|Q_y^{1/2}\mathbb{E}[\bar{y}_{t+1}\bar{x}_{t+1}^\top]\|_1 &\leq \|Q_y^{1/2}(I - \beta J_{22})Q_y^{-1/2}(Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top])(I - \alpha\Delta)\|_1 \\ &\quad + \alpha\|Q_y^{1/2}(I - \beta J_{22})Q_y^{-1/2}(Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{y}_t^\top]Q_y^{1/2})Q_y^{-1/2}J_{12}^\top\|_1 \\ &\quad + \alpha\|Q_y^{1/2}(I - \beta J_{22})Q_y^{-1/2}Q_y^{1/2}\mathbb{E}[w_t^x\bar{y}_t^\top]\|_1 \\ &\quad + \alpha\|Q_y^{1/2}J_{22}^{-1}J_{21}(J_{12}\mathbb{E}[\bar{y}_t\bar{x}_t^\top] + \Delta\mathbb{E}[\bar{x}_t\bar{x}_t^\top] + \mathbb{E}[w_t^x\bar{x}_t^\top])\|_1 + \beta\|Q_y^{1/2}\mathbb{E}[w_t^y\bar{x}_t^\top]\|_1 \\ &\quad + \alpha^2\|Q_y^{1/2}J_{22}^{-1}J_{21}\mathbb{E}[(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)(\Delta\bar{x}_t + J_{12}\bar{y}_t + w_t^x)^\top]\|_1 \\ &\quad + \alpha\beta\|\mathbb{E}[Q_y^{1/2}w_t^y(\Delta\bar{x}_t + J_{12}\bar{y}_t + w_t^x)^\top]\|_1.\end{aligned}$$

We observe the following:

1. $\|Q_y^{1/2}(I - \beta J_{22})Q_y^{-1/2}\|_{\text{op}} = \|I - \beta J_{22}\|_{Q_y} \leq 1 - \mu_y\beta$, and therefore

$$\begin{aligned}\|Q_y^{1/2}(I - \beta J_{22})Q_y^{-1/2}(Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top])(I - \alpha\Delta)\|_1 &\leq (1 - \mu_y\beta)(1 + \alpha J_{\max}\kappa_y)\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1 \\ &\leq (1 - \mu_y\beta/2)\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1.\end{aligned}$$

2. In all other terms, we use inequality $\|\mathbb{E}[uv^\top]\|_1 \leq \frac{1}{2}(\mathbb{E}[\|u\|_2^2] + \mathbb{E}[\|v\|_2^2])$.

We omit some algebraic details, and state the desired bounds:

$$\begin{aligned}\|Q_y^{1/2}\mathbb{E}[\bar{y}_{t+1}\bar{x}_{t+1}^\top]\|_1 &\leq (1 - \beta\mu_y/2)\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1 + \alpha J_{\max}\kappa_y^{3/2}\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1 \\ &\quad + (\alpha J_{\max}^{3/2})\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + (\alpha J_{\max}^{1/2}\kappa_y^{5/2})\mathbb{E}[\|\bar{x}_t\|_2^2] \\ &\quad + (\alpha/\sqrt{\mu_y})\|\mathbb{E}[w_t^x\bar{y}_t^\top]\|_1 + (\alpha\beta J_{\max}/\sqrt{\mu_y})\|\mathbb{E}[w_t^y\bar{y}_t^\top]\|_1 \\ &\quad + (\beta/\sqrt{\mu_y})\|\mathbb{E}[w_t^y\bar{x}_t^\top]\|_1 + (\alpha\beta/\sqrt{\mu_y})(\sigma_x^2 + \sigma_y^2).\end{aligned}$$

Applying Lemma A.11 and A.10, and using $\alpha \ll \beta/\kappa_y^2$ in Assumption 4, we can conclude that

$$\begin{aligned}\|Q_y^{1/2}\mathbb{E}[\bar{y}_{t+1}\bar{x}_{t+1}^\top]\|_1 &\leq (1 - \beta\mu_y/2)\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1 \\ &\quad + O(\alpha J_{\max}^{3/2} + \beta^2\tau^2 J_{\max}^{9/2}\kappa_x\kappa_y^{1/2}/\mu_x)\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + O\left(\alpha J_{\max}^{1/2}\kappa_y^{5/2}\right)\mathbb{E}[\|\bar{x}_t\|_2^2] \\ &\quad + O(\alpha\beta\tau\alpha/\sqrt{\mu_y})(\sigma_x^2 + \sigma_y^2) + (1/\sqrt{\mu_y})(\beta^3\tau^2 J_{\max}\kappa_x/\mu_x)\sigma_y^2.\end{aligned}\tag{13}$$

B.1.4 Overall Convergence

Recall the potential function V_t and U_t in (10):

$$\begin{aligned}V_t &= \mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] + \frac{O(1)J_{\max}^2\kappa_y\alpha(\alpha + \beta^2\tau^2 J_{\max})}{\mu_x\mu_y\beta}\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \frac{O(1)J_{\max}^{1/2}\kappa_y\alpha}{\mu_x\beta}\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1, \\ U_t &= \mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] + \frac{O(1)J_{\max}^{1/2}\kappa_y^{2.5}\alpha}{\mu_y\beta}\|Q_y^{1/2}\mathbb{E}[\bar{y}_t\bar{x}_t^\top]\|_1.\end{aligned}$$

We note that $\beta \ll 1/(\kappa_y^3 \kappa_x J_{\max})$ and $\beta/\alpha \ll 1/(\kappa_y^3 \kappa_x)$ in Assumption 4, and

$$\begin{aligned}\|\bar{x}_t\|_2^2 &\leq \frac{1}{\sigma_{\min}(Q_x)} \|\bar{x}_t\|_{Q_x}^2 \leq J_{\max} \kappa_y \|\bar{x}_t\|_{Q_x}^2, \\ \|\bar{y}_t\|_2^2 &\leq \frac{1}{\sigma_{\min}(Q_y)} \|\bar{y}_t\|_{Q_y}^2 \leq J_{\max} \|\bar{y}_t\|_{Q_y}^2.\end{aligned}$$

Putting altogether, we have

$$V_{t+1} \leq (1 - \alpha \mu_x/2) V_t + \frac{\kappa_y^{3/2}}{\mu_x} \left(\alpha^2 \tau_\alpha + \frac{J_{\max} \kappa_x \tau_\alpha^2}{\mu_x} \alpha \beta^2 \right) \sigma_y^2 + \frac{\tau_\alpha \kappa_y \kappa_x}{\mu_y} \alpha (\alpha + \beta^2 J_{\max} \tau_\alpha^2) \sigma_y^2 + \frac{\alpha^2 \tau_\alpha}{\mu_x} \sigma_x^2. \quad (14)$$

Solving this recursion,

$$\mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] \leq V_t \leq \exp(-\alpha \mu_x t/2) V_0 + \frac{\kappa_y^{1/2}}{\mu_x^2} (\alpha \kappa_y \tau_\alpha + \beta^2 J_{\max} \kappa_x^2 \tau_\alpha^2) \sigma_y^2 + \frac{\tau_\alpha \kappa_y \kappa_x}{\mu_y \mu_x} (\alpha + \beta^2 J_{\max} \tau_\alpha^2) \sigma_y^2 + \frac{\alpha \tau_\alpha}{\mu_x^2} \sigma_x^2,$$

for all t , hence for all sufficiently large $t \gg \alpha^{-1}$, we have bounds for $\mathbb{E}[\|\bar{x}_t\|_{Q_x}^2] \leq O_{\mathbb{P}}(\alpha) \sigma_x^2 + O_{\mathbb{P}}(\alpha + \beta^2) \sigma_y^2$.

Next, we consider the potential for faster iterates U_t . We see that

$$\begin{aligned}U_{t+1} &\leq (1 - \beta \mu_y/2) U_t + \beta^2 \kappa_y^3 J_{\max} \tau_\alpha \mathbb{E}[\|\bar{x}_t\|_2^2] + (\tau_\alpha / \mu_y) (\kappa_y^2 \alpha^2 \sigma_x^2 + \beta^2 \sigma_y^2) \\ &\leq (1 - \beta \mu_y/2) U_t + \beta^2 \kappa_y^3 J_{\max}^2 \tau_\alpha \exp(-\alpha \mu_x t/2) V_0 \\ &\quad + O\left(\frac{\kappa_y^2 \tau}{\mu_y} \alpha^2 + \frac{\tau^2 J_{\max}^2 \kappa_y^3}{\mu_x^2} \alpha \beta^2\right) \sigma_x^2 + O\left(\frac{\tau}{\mu_y} \beta^2\right) \sigma_y^2,\end{aligned} \quad (15)$$

which yields

$$\begin{aligned}\mathbb{E}[\|y_t\|_{Q_y}^2] &\leq U_t \leq \exp(-\beta \mu_y t/2) U_0 + \beta \kappa_y^4 \tau_\alpha \exp(-\alpha \mu_x t/4) V_0 \\ &\quad + O\left(\frac{\kappa_y^2 \tau}{\mu_y^2} \frac{\alpha}{\beta} + \frac{\tau^2 J_{\max} \kappa_y^4}{\mu_x^2} \beta\right) \alpha \sigma_x^2 + O\left(\frac{\tau}{\mu_y^2} \beta\right) \sigma_y^2,\end{aligned}$$

assuming $\beta \mu_y \gg \alpha \mu_x$. Thus for sufficiently large $t \gg \alpha^{-1} \log(1/\beta)$, we have $\mathbb{E}[\|\bar{y}_t\|_{Q_y}^2] = O_{\mathbb{P}}(\alpha^2/\beta + \alpha \beta) \sigma_x^2 + O_{\mathbb{P}}(\beta) \sigma_y^2$. This concludes the final error rates as $t \rightarrow \infty$.

B.2 Proof of Theorem 3.3

Showing the distributional convergence consists of two steps. First, we setup two sequences $\{(x_t^1, y_t^1, \xi_t)\}_{t \geq 0}$, $\{(x_t^2, y_t^2, \xi_t)\}_{t \geq 0}$ coupled with the same sequence of Markovian states $\{\xi_t\}_{t \geq 0}$. We show that these two sequences will converge in the squared- L_2 expectation sense:

Lemma B.1. *Under Assumptions 1-4, for any two sequences coupled with the same Markovian nosie (x_t^1, y_t^1, ξ_t) and (x_t^2, y_t^2, ξ_t) , the following holds:*

$$\begin{aligned}\mathbb{E}[\|x_t^1 - x_t^2\|_2^2] &\leq O_{\mathbb{P}}(1) \cdot \mathbb{E} \left[\|x_0^1 - x_0^2\|_2^2 + \frac{\alpha}{\beta} \|\bar{y}_0^1 - \bar{y}_0^2\|_2^2 \right] \exp(-\alpha \mu_x t/4), \\ \mathbb{E}[\|\bar{y}_t^1 - \bar{y}_t^2\|_2^2] &\leq O_{\mathbb{P}}(1) \cdot \mathbb{E} \left[\|x_0^1 - x_0^2\|_2^2 + \|\bar{y}_0^1 - \bar{y}_0^2\|_2^2 \right] \exp(-\beta \mu_y t/4) \\ &\quad + O_{\mathbb{P}}(1) \cdot \mathbb{E} \left[\|x_0^1 - x_0^2\|_2^2 + \frac{\alpha}{\beta} \|\bar{y}_0^1 - \bar{y}_0^2\|_2^2 \right] \beta \exp(-\alpha \mu_x t/4).\end{aligned}$$

We first prove the above lemma and use it to conclude that the distribution of iteration variables converges in Wasserstein distance to a unique stationary distribution.

B.2.1 Proof of Lemma B.1

Let us define $\delta_t^x = \bar{x}_t^1 - \bar{x}_t^2, \delta_t^y = \bar{y}_t^1 - \bar{y}_t^2$. Then the stochastic recursion (1) becomes

$$\delta_{t+1}^x = (I - \alpha\Delta)\delta_t^x - \alpha J_{12}\delta_t^y - \alpha\delta_t^{wx},$$

and for y , we have

$$\delta_{t+1}^y = (I - \beta J_{22})\delta_t^y - \alpha J_{22}^{-1}J_{21}(J_{12}\delta_t^y + \Delta\delta_t^x) - (\alpha J_{22}^{-1}J_{21}\delta_t^{wx} + \beta\delta_t^{wy}),$$

where the noise differences are given by:

$$\begin{aligned}\delta_t^{wx} &= W_\Delta^x(\xi_t)\delta_t^x + W_{12}(\xi_t)\delta_t^y, \\ \delta_t^{wy} &= W_\Delta^y(\xi_t)\delta_t^x + W_{22}(\xi_t)\delta_t^y,\end{aligned}$$

where we used the expression in Lemma A.7. This can be considered as the same TTSAs recursion with $\sigma_x = \sigma_y = 0$. Therefore, the remaining steps are equivalent to the pilot result with $\sigma_x = \sigma_y = 0$, and it leads to

$$\begin{aligned}\mathbb{E}[\|\delta_t^x\|_{Q_x}^2] &\leq \exp(-\alpha\mu_x t/2)V_0, \\ \mathbb{E}[\|\delta_t^y\|_{Q_y}^2] &\leq \exp(-\beta\mu_y t/2)U_0 + O_p(1)\beta\exp(-\alpha\mu_x t/4)V_0.\end{aligned}$$

where we define

$$\begin{aligned}V_t &= \mathbb{E}[\|\delta_t^x\|_{Q_x}^2] + \frac{O(1)J_{\max}^2\kappa_y\alpha(\alpha + \beta^2\tau_\alpha^2J_{\max})}{\mu_x\mu_y\beta}\mathbb{E}[\|\delta_t^y\|_{Q_y}^2] + \frac{O(1)J_{\max}^{1/2}\kappa_y\alpha}{\mu_x\beta}\|Q_y^{1/2}\mathbb{E}[\delta_t^y\delta_t^{x\top}]\|_1, \\ U_t &= \mathbb{E}[\|\delta_t^y\|_{Q_y}^2] + \frac{O(1)J_{\max}^{1/2}\kappa_y^{2.5}\alpha}{\mu_y\beta}\|Q_y^{1/2}\mathbb{E}[\delta_t^y\delta_t^{x\top}]\|_1,\end{aligned}$$

This shows that $(\bar{x}_t^1, \bar{y}_t^1), (\bar{x}_t^2, \bar{y}_t^2)$ converges exponentially fast with the noise coupling, which in turn means (x_t^1, y_t^1) and (x_t^2, y_t^2) couple exponentially fast since

$$\begin{aligned}\bar{x}_t^1 - \bar{x}_t^2 &= x_t^1 - x_t^2, \\ \bar{y}_t^1 - \bar{y}_t^2 &= (y_t^1 - y_t^2) - (y^*(x_t^1) - y^*(x_t^2)) = y_t^1 - y_t^2 + O_p(\|x_t^1 - x_t^2\|).\end{aligned}$$

B.2.2 Distributional Convergence via Coupling

The steps here mostly follows the proof steps in [Huo et al. \(2023\)](#), Appendix A.2.2. We first consider a sequence $(\xi_t^1, x_t^1, y_t^1)_{t \geq 0}$ that starts at $(x_0^1, y_0^1, \xi_0^1) \sim \mu_0$ sampled from some initial distribution μ_0 where $\xi_0^1 \sim \pi$ and (x_0^1, y_0^1) are *statistically independent*. Then, we similarly define the initial point distribution of the second sequence (x_{-1}^2, y_{-1}^2) as the same as (x_0^1, y_0^1) and set (x_0^2, y_0^2) be the result of one-step stochastic recursion (1), where $\xi_{-1}^2 \sim \mathcal{P}^\dagger(\cdot | \xi_0^1)$. Then we couple the Markovian states $\xi_t^1 = \xi_t^2$ for all $t \geq 0$. Now that we have

$$(\xi_t^2, x_t^2, y_t^2) \stackrel{d}{=} (\xi_{t+1}^1, x_{t+1}^1, y_{t+1}^1),$$

since $\xi_0^1 \stackrel{d}{=} \xi_1^1$ follows a stationary distribution π , and $\xi_t^1 = \xi_t^2$ is coupled. Then by definition of Wasserstein distance (with the optimal coupling), using Lemma B.1, we get

$$\bar{\mathcal{W}}_2^2((x_t^1, y_t^1, \xi_t^1), (x_{t+1}^1, y_{t+1}^1, \xi_{t+1}^1)) \leq C \exp(-\alpha\mu_x t/4), \forall t \geq 0,$$

and therefore (omitting superscript)

$$\sum_{t \geq 0} \bar{\mathcal{W}}_2^2((x_t, y_t, \xi_t), (x_{t+1}, y_{t+1}, \xi_{t+1})) \leq \sum_{t \geq 0} C \exp(-\alpha\mu_x t/4) < \infty,$$

with $\alpha > 0$. The probability space over $\Xi \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ equipped with $\bar{\mathcal{W}}_2$ -norm is known to be a Polish space where every Cauchy sequence converges ([Villani et al. \(2009\)](#), Theorem 6.18). Furthermore, convergence in Wasserstein distance implies weak convergence ([Villani et al. \(2009\)](#), Theorem 6.9), hence weak convergence to some distribution $\mu \in \mathcal{P}_2(\Xi \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_y})$.

B.2.3 Stationarity of the Limit Distribution

Next, we show that the sequence converges to a unique stationary distribution μ regardless of the initial distribution μ_0 . To do so, we first show that the sequence has bounded fourth-order moments:

Lemma B.2. *Suppose an initial distribution $(\xi_0, x_0, y_0) \sim \mu_0$ that satisfies Assumption 5. Then for all $t \geq 0$, we have*

$$\mathbb{E}[\|x_t\|_2^4 + \|y_t\|_2^4] < O_P(1) \cdot \mathbb{E}[\|x_0\|_2^4 + \|y_0\|_2^4] + O_P(1) \cdot (\sigma_x^4 + \sigma_y^4).$$

Then we consider two TTSA sequences starting from two arbitrary initial distributions μ_0^1, μ_0^2 . We start with the following lemma that is reminiscent of Lemma A.8 in [Huo et al. \(2023\)](#) for (1):

Lemma B.3. *For any two TTSA sequences $(x_t^1, y_t^1, \xi_t^1) \sim \mu_0^1$ and $(x_t^2, y_t^2, \xi_t^2) \sim \mu_0^2$ with bounded fourth-order moments satisfying Assumption 5, for all $t \geq 0$, we have*

$$\begin{aligned} \mathcal{W}_2^2(\mu_t^1(x_t^1), \mu_t^2(x_t^2)) &\leq O_P(1) \exp(-\alpha \mu_x t / 8) V_0, \\ \mathcal{W}_2^2(\mu_t^1(\bar{y}_t^1), \mu_t^2(\bar{y}_t^2)) &\leq O_P(1) (\beta \exp(-\alpha \mu_x t / 8) V_0 + \exp(-\beta \mu_y t / 8) U_0), \end{aligned} \quad (16)$$

where

$$\begin{aligned} V_0 &:= \mathcal{W}_2^2(\mu_0^1(x_0^1), \mu_0^2(x_0^2)) + \frac{\alpha}{\beta} \mathcal{W}_2^2(\mu_0^1(\bar{y}_0^1), \mu_0^2(\bar{y}_0^2)) + O_P(\alpha), \\ U_0 &:= \mathcal{W}_2^2(\mu_0^1(x_0^1), \mu_0^2(x_0^2)) + \mathcal{W}_2^2(\mu_0^1(\bar{y}_0^1), \mu_0^2(\bar{y}_0^2)) + O_P(\beta). \end{aligned}$$

Apply Lemma B.3, we have

$$\bar{\mathcal{W}}_2(\mu_t^1, \mu_t^2) < O_P(1) \cdot \exp(-\alpha \mu_x t / 8) \xrightarrow{t \rightarrow \infty} 0,$$

which in turn implies that all sequences converge to the unique limit distribution μ .

Lastly, we show that μ is an invariant distribution with $\mu(\xi) = \pi$. By the geometric mixing property of $(\xi_t)_{t \geq 0}$, the limit distribution must satisfy $\mu(\xi) = \pi$ (otherwise, we can derive a contradiction). Thus, for a sequence (x_t, y_t, ξ_t) starting from μ with marginal $\mu(\xi_0) = \pi$, we know that $\mu_t(\xi_t) = \pi$ for all $t \geq 0$. Thus, using the coupling results, we have

$$\begin{aligned} \bar{\mathcal{W}}_2(\mu_1, \mu) &\leq \bar{\mathcal{W}}_2(\mu_1, \mu_{t+1}) + \bar{\mathcal{W}}_2(\mu_{t+1}, \mu) \\ &\leq O_P(1) \bar{\mathcal{W}}_2(\mu_0, \mu_t) + \bar{\mathcal{W}}_2(\mu_{t+1}, \mu) \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

where we used $\mu_0 = \mu$.

B.2.4 Bias Characterization

For analyzing the bias if the limite distribution $(x_\infty, y_\infty, \xi_\infty) \sim \mu$, we start from sending $t \rightarrow \infty$ in (1)

$$\begin{aligned} \bar{x}_{t+1} &= (I - \alpha \Delta) \bar{x}_t - \alpha J_{12} \bar{y}_t - \alpha w^x(x_t, y_t; \xi_t), \\ \bar{y}_{t+1} &= (I - \beta J_{22}) \bar{y}_t - \alpha J_{22}^{-1} J_{21} (J_{12} \bar{y}_t + \Delta \bar{x}_t + J_{21} w^x(x_t, y_t; \xi_t)) + \beta w^y(x_t, y_t; \xi_t). \end{aligned}$$

Let

$$\begin{aligned} z^x(\xi) &= \mathbb{E}[x_\infty - x^* | \xi_\infty = \xi], \\ z^y(\xi) &= \mathbb{E}[y_\infty - y^*(x_\infty) | \xi_\infty = \xi], \end{aligned}$$

and

$$\begin{aligned} w^x(\xi) &= W_{11}(\xi) \mathbb{E}[x_\infty | \xi_\infty = \xi] + W_{12}(\xi) \mathbb{E}[y_\infty | \xi_\infty = \xi] + b_1(\xi), \\ w^y(\xi) &= W_{21}(\xi) \mathbb{E}[x_\infty | \xi_\infty = \xi] + W_{22}(\xi) \mathbb{E}[y_\infty | \xi_\infty = \xi] + b_2(\xi). \end{aligned}$$

We take conditional expectation on $\xi_{\infty+1} = \xi$, we have the backward conditional probability $\xi_\infty \sim P^\dagger(\cdot | \xi_{\infty+1} = \xi)$. Let $\mathcal{T} : \Xi \times \mathbb{R}^* \rightarrow \Xi \times \mathbb{R}^*$ an *unnormalized* Markov operator over ξ :

$$\mathcal{T}\{z\}(\xi') = \int_{\xi \in \Xi} z(\xi) d\mathcal{T}(\xi' | \xi).$$

Using the above notation, we can rewrite the recursion as

$$\begin{aligned} z^x &= \mathcal{P}^\dagger\{(I - \alpha\Delta)z^x - \alpha J_{12}z^y - \alpha w^x\}, \\ z^y &= \mathcal{P}^\dagger\{(I - \beta J_{22})z^y - \alpha J_{22}^{-1}J_{21}(J_{12}z^y + \Delta z^x + w^x) - \beta w^y\}. \end{aligned}$$

Let $\Pi = 1 \otimes \pi$, and note that $\Pi\{W_{ij}\} = 0$ for all $i, j \in \{1, 2\}$. We eventually want to characterize $\bar{z}^x := \mathbb{E}[x_\infty - x^*] = \pi\{z^x\} = \int_\xi z^x(\xi) d\pi(\xi)$ and $\bar{z}^y := \pi\{z^y\}$. Since $\pi P^\dagger = \pi$ by the time-reversing property of the geometrically mixing chain, this implies

$$\begin{aligned} \Delta \bar{z}^x + J_{12}\bar{z}^y + \pi\{w^x\} &= 0, \\ J_{22}\bar{z}^y + \pi\{w^y\} &= 0. \end{aligned} \tag{17}$$

To further proceed, let $\delta^x(\xi) = z^x(\xi) - \pi\{z^x\}$, $\delta^y(\xi) = z^y(\xi) - \pi\{z^y\}$, and since $(\mathcal{P}^\dagger - \Pi)\{z^x\} = (\mathcal{P}^\dagger - \Pi)\{\delta^x\}$, we can observe that

$$\begin{aligned} (I - P^\dagger + \Pi)\{\delta^x\} &= -\alpha(P^\dagger - \Pi)\{\Delta z^x + J_{12}z^y + w^x\} = -\alpha(P^\dagger - \Pi)\{\Delta \delta^x + J_{12}\delta^y + w^x\}, \\ (I - P^\dagger + \Pi)\{\delta^y\} &= -\beta(P^\dagger - \Pi)\left\{J_{22}z^y + \frac{\alpha}{\beta}J_{22}^{-1}J_{21}(J_{12}z^y + \Delta z^x + w^x) + w^y\right\} \\ &= (I - P^\dagger + \Pi)\{\delta^x\} - \beta(P^\dagger - \Pi)\{J_{22}\delta^y + w^y\}. \end{aligned} \tag{18}$$

Then we note that

$$\begin{aligned} w^y(\xi) &= W_{21}(\xi)z^x(\xi) + W_{22}(\xi)z^y(\xi) + u_2(\xi) + W_{21}(\xi)x^* - W_{22}(\xi)J_{22}^{-1}J_{21}(z^x(\xi) + x^*) \\ &= (W_{21}(\xi) - W_{22}(\xi)J_{22}^{-1}J_{21})z^x(\xi) + W_{22}(\xi)z^y(\xi) + u_2(\xi) + (W_{21}(\xi) - W_{22}(\xi)J_{22}^{-1}J_{21})x^* \\ &= W_\Delta^y(\xi)(\delta^x(\xi) + \bar{z}^x) + W_{22}(\xi)(\delta^y(\xi) + \bar{z}^y) + u_2(\xi) + W_\Delta^y(\xi)x^*, \\ w^x(\xi) &= (W_{11}(\xi) - W_{12}(\xi)J_{22}^{-1}J_{21})z^x(\xi) + W_{12}(\xi)z^y(\xi) + u_1(\xi) + (W_{11}(\xi) - W_{12}(\xi)J_{22}^{-1}J_{21})x^* \\ &= W_\Delta^x(\xi)(\delta^x(\xi) + \bar{z}^x) + W_{12}(\xi)(\delta^y(\xi) + \bar{z}^y) + u_1(\xi) + W_\Delta^x(\xi)x^*. \end{aligned}$$

Plugging this back into (17) yields

$$\begin{aligned} \Delta \bar{z}^x + J_{12}\bar{z}^y + \pi\{W_\Delta^x \circ \delta^x\} + \pi\{W_{12} \circ \delta^y\} &= 0, \\ J_{22}\bar{z}^y + \pi\{W_\Delta^y \circ \delta^x\} + \pi\{W_{22} \circ \delta^y\} &= 0, \end{aligned}$$

where we define $(a \circ b)(\xi) = a(\xi)b(\xi)$. In turn, we have

$$\begin{aligned} \bar{z}^y &= -J_{22}^{-1}(\pi\{W_\Delta^y \circ \delta^x\} + \pi\{W_{22} \circ \delta^y\}), \\ \bar{z}^x &= -\Delta^{-1}(\pi\{(W_\Delta^x - J_{12}J_{22}^{-1}W_\Delta^y) \circ \delta^x\} + \pi\{(W_{12} - J_{12}J_{22}^{-1}W_{22}) \circ \delta^y\}). \end{aligned} \tag{19}$$

Rearranging (18) yields

$$\begin{aligned} (I - \mathcal{P}^\dagger + \Pi)\{\delta^x\} &= -\alpha(\mathcal{P}^\dagger - \Pi)\{(\Delta + W_\Delta^x) \circ \delta^x + (J_{12} + W_{12}) \circ \delta^y + (u_1 + W_\Delta^x x^*)\} \\ &\quad - \alpha(\mathcal{P}^\dagger - \Pi)\{W_\Delta^x\}\bar{z}^x - \alpha(\mathcal{P}^\dagger - \Pi)\{W_{12}\}\bar{z}^y, \\ (I - \mathcal{P}^\dagger + \Pi)\{\delta^y - \delta^x\} &= -\beta(\mathcal{P}^\dagger - \Pi)\{W_\Delta^y \circ \delta^x + (J_{22} + W_{22}) \circ \delta^y + (u_2 + W_\Delta^y x^*)\} \\ &\quad - \beta(\mathcal{P}^\dagger - \Pi)\{W_\Delta^y\}\bar{z}^x - \beta(\mathcal{P}^\dagger - \Pi)\{W_{22}\}\bar{z}^y. \end{aligned} \tag{20}$$

The operation $(I - \mathcal{P}^\dagger + \Pi)$ is invertible (see Corollary B.6), and thus we can invert the operator $(I - \mathcal{P}^\dagger + \Pi)$. Putting all relationships together leads us to the recursion:

$$\delta^x = \alpha d^x + O_P(\alpha \delta^x + \alpha \delta^y),$$

$$\delta^y = \alpha d^x + \beta d^y + O_P(\beta \delta^x + \beta \delta^y), \quad (21)$$

where

$$\begin{aligned} d^x &:= -(I - \mathcal{P}^\dagger + \Pi)^{-1}(\mathcal{P}^\dagger - \Pi)\{u_1 + W_\Delta^x x^*\}, \\ d^y &:= -(I - \mathcal{P}^\dagger + \Pi)^{-1}(\mathcal{P}^\dagger - \Pi)\{u_2 + W_\Delta^y x^*\}, \end{aligned} \quad (22)$$

are independent of the choice of α, β . Next, we bound the norm of $\delta^x, \delta^y, d^x, d^y$, and thus the norm of \bar{z}^x, \bar{z}^y .

B.2.5 Additional Preliminaries for Bounding Norms

Before we proceed, we define the notion of norms that we use in the proof. For vector-valued quantities, let us define $\|v\|_{L^2(\pi)}$ as

$$\|v\|_{L^2(\pi)} = \sqrt{\int_{\Xi} \|v\|_2^2 d\pi(\xi)},$$

and for the Markov kernel \mathcal{T} ,

$$\|\mathcal{T}\|_{L^2(\pi)} := \sup_{\|v\|_{L^2(\pi)}=1} \|\mathcal{T}\{v\}\|_{L^2(\pi)}.$$

For matrices, we use the conjugate norm-pair $\|\cdot\|_1$ and $\|\cdot\|_\infty = \|\cdot\|_{\text{op}}$. Specifically, for matrix-valued quantities, we define $\|A\|_{S^1(\pi)}$ as

$$\|A\|_{S^1(\pi)} = \int_{\Xi} \|A\|_1 d\pi(\xi),$$

and

$$\|A\|_{S^\infty(\pi)} = \left(\int_{\Xi} \|A\|_\infty^\infty d\pi(\xi) \right)^{1/\infty} = \max_{\xi \in \Xi} \|A(\xi)\|_{\text{op}}.$$

The following holder's inequality is crucial to obtain dimension-free bounds on variances:

Lemma B.4. *For Markov kernel \mathcal{T} and conditional matrix $A(\xi)$, We have*

$$\|\mathcal{T}A\|_{S^1(\pi)} \leq \|\mathcal{T}\|_{S^\infty(\pi)} \|A\|_{S^1(\pi)},$$

where

$$\|\mathcal{T}\|_{S^\infty(\pi)} := \sup_{\|Y\|_{S^\infty(\pi)} \leq 1} \|\mathcal{T}Y\|_{S^\infty(\pi)}.$$

Using the results from Markov chain literature, we have the following lemma:

Lemma B.5 (Proposition 22.3.5 in [Douc et al. \(2018\)](#)). *Let \mathcal{P} be a Markov Kernel on a Boral state-space Ξ with invariant probability π . Under Assumption 2, we have*

$$\begin{aligned} \|(\mathcal{P} - \Pi)^k\|_{L^2(\pi)} &\leq \sqrt{2c_\rho} \rho^{k/2}, \\ \|(\mathcal{P} - \Pi)^k\|_{S^\infty(\pi)} &\leq 2c_\rho \rho^k. \end{aligned}$$

The following is the corollary:

Corollary B.6. *Under Assumption 2, we have*

$$\max \left(\| (I - \mathcal{P}^\dagger + \Pi)^{-1} \|_{L^2(\pi)}, \| (I - \mathcal{P}^\dagger + \Pi)^{-1} \|_{S^\infty(\pi)} \right) \leq 2c_\rho / (1 - \rho).$$

B.2.6 Norm-Bounds for Stationary Bias

We show that $\|\delta^x\|_{L^2(\pi)} = O_{\mathbb{P}}(\alpha)$, $\|\delta^y\|_{L^2(\pi)} = O_{\mathbb{P}}(\beta)$. First, we note that

$$\|\delta^x\|_{L^2(\pi)} \leq \alpha(C_1^x \|\delta^x\|_{L^2(\pi)} + C_2^x \|\delta^y\|_{L^2(\pi)} + C_3^x),$$

where

$$\begin{aligned} C(\mathcal{P}, \pi) &:= \|(I - \mathcal{P}^\dagger + \Pi)^{-1}\|_{L^2(\pi)} \|\mathcal{P}^\dagger - \Pi\|_{L^2(\pi)} \leq \frac{4c_\rho}{1 - \rho} = O(\tau_\alpha), \\ C_1 &:= C(\mathcal{P}, \pi) \|\Delta + W_\Delta^x\|_{L^2(\pi)} \leq C(\mathcal{P}, \pi) J_{\max} \kappa_y, \\ C_2 &:= C(\mathcal{P}, \pi) \|J_{12} + W_{12}\|_{L^2(\pi)} \leq C(\mathcal{P}, \pi) J_{\max}, \\ C_3 &:= C(\mathcal{P}, \pi) (\sigma_x + W_{\max} \kappa_y \|\bar{z}^x\|_{L^2(\pi)} + W_{\max} \|\bar{z}^y\|_{L^2(\pi)}) \\ &\leq C(\mathcal{P}, \pi) (\sigma_x + O_{\mathbb{P}}(\beta) W_{\max} \kappa_y). \end{aligned}$$

The last inequality is because

$$\begin{aligned} \|\bar{z}^y\|_{L^2(\pi)} &= \int_{\Xi} \|\mathbb{E}[\bar{y}_\infty | \xi] - \mathbb{E}[\bar{y}_\infty]\|^2 \pi(d\xi) \leq \int_{\Xi} \mathbb{E}[\|\bar{y}_\infty - \mathbb{E}[\bar{y}_\infty]\|^2 | \xi] \pi(d\xi) \\ &= \mathbb{E}[\|\bar{y}_\infty - \mathbb{E}[\bar{y}_\infty]\|^2] \leq \mathbb{E}[\|\bar{y}_\infty\|^2] = O_{\mathbb{P}}(\beta), \end{aligned}$$

by Theorem 3.2, and similarly, we can also show that $\|\bar{z}^x\|_{L^2(\pi)} = O_{\mathbb{P}}(\alpha + \beta^2)$. Furthermore,

$$\|\delta^y\|_{L^2(\pi)} \leq \beta(C_1 \|\delta^x\|_{L^2(\pi)} + C_2 \|\delta^y\|_{L^2(\pi)} + C_3) + \|\delta^x\|_{L^2(\pi)},$$

for the same problem-dependent constants C_1, C_2, C_3 as defined above. This concludes that for $\alpha \ll \beta \ll 1/\max(C_1, C_2)$, we have

$$\|\delta^x\|_{L^2(\pi)} = O_{\mathbb{P}}(\alpha), \|\delta^y\|_{L^2(\pi)} = O_{\mathbb{P}}(\beta).$$

Similarly, we can show that

$$\|d^x\|_{L^2(\pi)} \leq \frac{O(1)\sigma^x}{1 - \rho}, \quad \|d^y\|_{L^2(\pi)} \leq \frac{O(1)\sigma^y}{1 - \rho},$$

which implies that $\bar{b}_1^y, \bar{b}_2^y = O_{\mathbb{P}}(1)$ since

$$\|\bar{b}_2^y\|_2 = \left\| J_{22}^{-1} \int_{\Xi} W_{22} d^y d\pi(\xi) \right\|_2 \leq \kappa_y \left(\int_{\Xi} \|d^y\|_2 d\pi(\xi) \right) \leq \kappa_y \|d^y\|_{L^2(\pi)}.$$

Similarly, we have $\|\bar{b}_1^y\|_2 = O_{\mathbb{P}}(1)$ and $\bar{b}_i^x = O_{\mathbb{P}}(1)$ for $i = 1, 2$. We can plug this result back to (19) to conclude the bias part of Theorem 3.3.

B.2.7 Dimension-Free Bounds for Variances

We note that the variance of x_∞ is measured by

$$\|\mathbb{V}(x_\infty)\|_1 = \text{Tr}(\mathbb{V}(x_\infty)) = \text{Tr}(\mathbb{E}[(x_\infty - \mathbb{E}[x_\infty])(x_\infty - \mathbb{E}[x_\infty])^\top]),$$

where the expectation is taken over the stationary distribution $(x_\infty, y_\infty, \xi_\infty) \sim \mu$, and thus we aim bound $\text{Tr}(\mathbb{V}(x_\infty))$. For y , it is sufficient to bound y_∞ by $O(\beta)$. To see this, note that

$$\begin{aligned} \text{Tr}(\mathbb{V}(y_\infty)) &= \mathbb{E}[\|y_\infty - \mathbb{E}[y_\infty]\|^2] = \mathbb{E}[\|y_\infty - y^*(x^*)\|^2 + \|\mathbb{E}[y_\infty] - y^*(x^*)\|^2] \\ &\leq 2\mathbb{E}[\|\bar{y}_\infty\|^2] + 2\mathbb{E}[\|y^*(x_\infty) - y^*(x^*)\|^2] + \|\mathbb{E}[y_\infty(x_\infty) - y^*(x^*)]\|^2 \\ &\leq 2\mathbb{E}[\|\bar{y}_\infty\|^2] + 3\kappa_y \mathbb{E}[\|\bar{x}_\infty\|^2] = O_{\mathbb{P}}(\beta). \end{aligned}$$

Similarly, we also have that

$$\text{Tr}(\mathbb{V}(x_\infty)) = O_{\mathbb{P}}(\alpha + \beta^2),$$

and

$$\|\mathbb{E}[(x_\infty - \mathbb{E}[x_\infty])\bar{y}_\infty^\top]\|_1 = O_{\mathbb{P}}(\alpha + \beta^2).$$

Next, we show that the variance of x_∞ is strictly in order $O(\alpha)$, without $\text{poly}(\beta)$ dependence. We first observe that

$$\begin{aligned} & (x_{\infty+1} - \mathbb{E}[x_\infty])(x_{\infty+1} - \mathbb{E}[x_\infty])^\top \\ &= (x_\infty - \mathbb{E}[x_\infty])(x_\infty - \mathbb{E}[x_\infty])^\top - \alpha(x_\infty - \mathbb{E}[x_\infty])(\Delta\bar{x}_\infty + J_{12}\bar{y}_\infty + w_\infty^x)^\top \\ &\quad - \alpha(\Delta\bar{x}_\infty + J_{12}\bar{y}_\infty + w_\infty^x)(x_\infty - \mathbb{E}[x_\infty])^\top + \alpha^2(\Delta\bar{x}_\infty + J_{12}\bar{y}_\infty + w_\infty^x)(\Delta\bar{x}_\infty + J_{12}\bar{y}_\infty + w_\infty^x)^\top. \end{aligned}$$

Let us define $\Sigma^x(\xi)$ and $\Sigma^{xy}(\xi)$ as the following:

$$\begin{aligned} \Sigma^x(\xi) &= \mathbb{E}[(x_\infty - \mathbb{E}[x_\infty])(x_\infty - \mathbb{E}[x_\infty])^\top | \xi_\infty = \xi], \\ \Sigma^{xy}(\xi) &= \mathbb{E}[(x_\infty - \mathbb{E}[x_\infty])\bar{y}_\infty^\top | \xi_\infty = \xi] = \mathbb{E}[(x_\infty - \mathbb{E}[x_\infty])(y_\infty - y^*(x_\infty))^\top | \xi_\infty = \xi], \\ \Sigma^y(\xi) &= \mathbb{E}[\bar{y}_\infty\bar{y}_\infty^\top | \xi_\infty = \xi]. \end{aligned}$$

We can then rewrite the recursion compactly:

$$\Sigma^x = \mathcal{P}^\dagger\{\Sigma^x - \alpha(A + A^\top) + \alpha^2B\},$$

where

$$\begin{aligned} A(\xi) &= \Sigma^x(\xi)\Delta + \Sigma^{xy}(\xi)J_{12}^\top + \mathbb{E}[(x_\infty - \mathbb{E}[x_\infty])^\top w_\infty^x | \xi_\infty = \xi] \\ &= \Sigma^x(\xi)(\Delta + W_\Delta^x(\xi))^\top + \Sigma^{xy}(\xi)(J_{12} + W_{12}(\xi))^\top + \delta^x(\xi)(u_1(\xi) + W_\Delta^x(\xi)\mathbb{E}[x_\infty])^\top, \\ B(\xi) &= (\Delta + W_\Delta^x(\xi))\Sigma^x(\xi)(\Delta + W_\Delta^x(\xi))^\top + (J_{12} + W_{12}(\xi))\Sigma^{xy}(\xi)(J_{12} + W_{12}(\xi))^\top \\ &\quad + (\Delta + W_\Delta^x(\xi))\Sigma^{xy}(\xi)(J_{12} + W_{12}(\xi))^\top + (J_{12} + W_{12}(\xi))\Sigma^{yx}(\xi)(\Delta + W_\Delta^x(\xi))^\top \\ &\quad + (W_\Delta^x(\xi)z^x(\xi) + u_1(\xi))(W_\Delta^x(\xi)z^x(\xi) + u_1(\xi))^\top + O(\delta^x(\xi) + \delta^y(\xi)). \end{aligned}$$

Let $\bar{\Sigma}^x = \pi\{\Sigma^x\} = \mathbb{E}[\Sigma^x]$, $D^x(\xi) := \Sigma^x(\xi) - \bar{\Sigma}^x$, and similarly define $\bar{\Sigma}^{xy}, D^{xy}$. The steady-state equation is given by

$$\begin{aligned} \bar{\Sigma}^x\Delta + \Delta\bar{\Sigma}^x + (\bar{\Sigma}^{xy}J_{12}^\top + J_{12}\bar{\Sigma}^{yx}) &= \alpha\pi\{(W_\Delta^x(\xi)z^x(\xi) + u_1(\xi))(W_\Delta^x(\xi)z^x(\xi) + u_1(\xi))^\top\} \\ &\quad + O_{\mathbb{P}}(\alpha)(\|\bar{\Sigma}^x\|_1 + \|\bar{\Sigma}^{xy}\|_1 + \|\bar{\Sigma}^y\|_1 + \|\delta^x\|_{L^2(\pi)} + \|\delta^y\|_{L^2(\pi)}) \\ &\quad + O_{\mathbb{P}}(\|D^x\|_{S^1(\pi)} + \|D^{xy}\|_{S^1(\pi)} + \|\delta^x\|_{L^2(\pi)}). \end{aligned} \tag{23}$$

We also note that $\mathcal{P}^\dagger\{\bar{\Sigma}^x\} = \bar{\Sigma}^x$ and $(\mathcal{P}^\dagger - \Pi)\{\Sigma^x\} = (\mathcal{P}^\dagger - \Pi)\{D^x\}$, and thus similarly to (18),

$$\begin{aligned} D^x &= -\alpha(I - \mathcal{P}^\dagger + \Pi)^{-1}(\mathcal{P}^\dagger - \Pi)\{O_{\mathbb{P}}(D^x + \bar{\Sigma}^x + D^{xy} + \bar{\Sigma}^{xy} + \delta^x)\} \\ &\quad + \alpha^2(I - \mathcal{P}^\dagger + \Pi)^{-1}(\mathcal{P}^\dagger - \Pi)\{O_{\mathbb{P}}(\bar{\Sigma}^x + \bar{\Sigma}^{xy} + \bar{\Sigma}^y + \delta^x + \delta^y + 1)\}. \end{aligned}$$

Taking $\|\cdot\|_{S^1(\pi)}$ of D^x , with Lemma B.4 and Corollary B.6, we can show that

$$\|D^x\|_{S^1(\pi)} \leq O_{\mathbb{P}}(\alpha)(\|D^x\|_{S^1(\pi)} + \|D^{xy}\|_{S^1(\pi)} + \|\bar{\Sigma}^x\|_1 + \|\bar{\Sigma}^{xy}\|_1) + \alpha^2O_{\mathbb{P}}(1),$$

where we also used $\|W_{ij}\|_{S^\infty(\pi)} \leq J_{\max}$ for $i, j \in \{1, 2\}$ by Assumption 3, and used a Cauchy-Schwarz inequality

$$\begin{aligned} \|AB\|_{S^1(\pi)} &\leq \int_{\xi} \|A(\xi)\|_1 \|B(\xi)\|_{\infty} d\pi(\xi) \leq \|A\|_{S^1(\pi)} \|B\|_{S^\infty(\pi)}, \\ \|uv^\top\|_{S^1(\pi)} &\leq \int_{\xi} \|u(\xi)\|_2 \|v(\xi)\|_2 d\pi(\xi) \leq \|u\|_{L^2(\pi)} \|v\|_{L^2(\pi)}, \end{aligned}$$

with $\|\delta^x\|_{L^2(\pi)} = O_{\mathbb{P}}(\alpha)$, $\|\delta^y\|_{L^2(\pi)} = O_{\mathbb{P}}(\beta)$. This suggests that as long as $\|D^{xy}\|_{S^1(\pi)}, \|\bar{\Sigma}^x\|_1, \|\bar{\Sigma}^{xy}\|_1 = o_{\mathbb{P}}(1)$, we have $\|D^x\|_{S^1(\pi)} = o_{\mathbb{P}}(\alpha)$.

To proceed, we also get the expression for Σ^{xy} :

$$\Sigma^{xy} = \mathcal{P}^\dagger \{ \Sigma^{xy} - \beta A' - \alpha B' + O(\alpha\beta) \},$$

where

$$\begin{aligned} A'(\xi) &= \Sigma^{xy}(\xi)(J_{22} + W_{22}(\xi))^\top + \Sigma^x(\xi)W_{21}(\xi) + \delta^x(\xi)(u_2(\xi) + W_\Delta^y(\xi)\mathbb{E}[x_\infty])^\top, \\ B'(\xi) &= (\Sigma^x(\xi)(\Delta + W_\Delta^x(\xi))^\top + \Sigma^{xy}(\xi)(J_{12} + W_{12}(\xi))^\top)(J_{22}^{-1}J_{21})^\top \\ &\quad + \Sigma^y(\xi)(J_{12} + W_{12}(\xi))^\top + W_\Delta^x(\xi)\Sigma^{yx}(\xi) + \delta^y(\xi)(W_\Delta^x(\xi)\mathbb{E}[x_\infty] + u_1(\xi))^\top. \end{aligned}$$

and C' is appropriately defined. The steady-state equation is

$$\begin{aligned} \bar{\Sigma}^{xy}J_{22}^\top + \frac{\alpha}{\beta}\bar{\Sigma}^yJ_{12}^\top &= \alpha\pi \{ (W_\Delta^x(\xi)z^x(\xi) + u_1(\xi))(W_\Delta^y(\xi)z^x(\xi) + u_2(\xi))^\top \} \\ &\quad + O_p(\|D^x\|_{S^1(\pi)} + \|D^{xy}\|_{S^1(\pi)} + \|\delta^x\|_{L^2(\pi)}) + O_p(\alpha), \end{aligned} \quad (24)$$

and the system equation is

$$D^{xy} = (I - \mathcal{P}^\dagger + \Pi)^{-1}(\mathcal{P}^\dagger - \Pi)\{\beta O_p(D^x + D^{xy} + \bar{\Sigma}^x + \bar{\Sigma}^{xy} + \delta^x) + \alpha O_p(D^y + \bar{\Sigma}^y + \delta^y) + O_p(\alpha\beta)\}.$$

Noting that $\|\delta^x\|_{L^2(\pi)} = O(\alpha)$, $\|\delta^y\|_{L^2(\pi)} = O(\beta)$, we can show that

$$\|D^{xy}\|_{S^1(\pi)} \leq O_p(\beta)(\|D^x\|_{S^1(\pi)} + \|D^{xy}\|_{S^1(\pi)} + \|\bar{\Sigma}^x\|_1 + \|\bar{\Sigma}^{xy}\|_1) + O_p(\alpha)(\|D^y\|_{S^1(\pi)} + \|\bar{\Sigma}^y\|_1) + O_p(\alpha\beta).$$

Combining these results, we can conclude that

$$\|D^{xy}\|_{S^1(\pi)}, \|D^x\|_{S^1(\pi)} = O_p(\alpha)(\|\bar{\Sigma}^x\|_1 + \|\bar{\Sigma}^{xy}\|_1) + O_p(\alpha\beta).$$

Now plugging this back to (24), we have

$$\|\bar{\Sigma}^{xy}\|_1 \leq \frac{\kappa_y\alpha}{\beta}\|\bar{\Sigma}^y\|_1 + O_p(\|D^x\|_{S^1(\pi)} + \|D^{xy}\|_{S^1(\pi)} + \|\delta^x\|_{L^2(\pi)}) + o_p(\alpha),$$

yielding $\|\bar{\Sigma}^{xy}\|_1 = O_p(\alpha)$ since $\|\bar{\Sigma}^y\|_1 = O_p(\beta)$. Then using these results, from (23), we can derive that

$$\begin{aligned} \text{Tr}(\bar{\Sigma}^x) + \text{Tr}(\Delta\bar{\Sigma}^x\Delta^{-1}) &= 2\text{Tr}(\bar{\Sigma}^x) = 2\|\bar{\Sigma}^x\|_1 \\ &\leq \|\bar{\Sigma}^{xy}J_{12}^\top\|_1\|\Delta^{-1}\|_{\text{op}} + O(\alpha)\|\Delta^{-1}\|_{\text{op}}\|W_\Delta^x \circ z^x + u_1\|_{L^2(\pi)}^2 + O_p(\alpha)\|\bar{\Sigma}^x\|_1 + o_p(\alpha). \end{aligned}$$

Therefore, we can conclude that $\|\bar{\Sigma}^x\|_1 = O_p(\alpha)$. Since $\|\bar{\Sigma}^x\|_1 = \text{Tr}(\bar{\Sigma}^x) = \text{Tr}(\mathbb{V}(x_\infty))$, we obtain the last part of the theorem.

B.2.8 Proof of Corollary 3.4

This is in fact a corollary of Lemma B.3. To see this, apply Lemma B.3 with $\mu_0^2 = \mu$, and then note that under optimal coupling between μ_0 and μ ,

$$\begin{aligned} \mathcal{W}_2^2(\mu_0(x_0), \mu(x_\infty)) &\leq \mathbb{E}[\|x_0 - x_\infty\|_2^2] \leq 2\mathbb{E}[\|x_0 - \mathbb{E}[x_\infty]\|_2^2] + \mathbb{E}[\|x_\infty - \mathbb{E}[x_\infty]\|_2^2] \\ &= 2\mathbb{E}[\|x_0 - \mathbb{E}[x_\infty]\|_2^2] + 2\text{Tr}(\mathbb{V}(x_\infty)). \end{aligned}$$

Similarly,

$$\mathcal{W}_2^2(\mu_0(\bar{y}_0), \mu(\bar{y}_\infty)) \leq 2\mathbb{E}[\|\bar{y}_0 - \mathbb{E}[\bar{y}_\infty]\|_2^2] + 2\text{Tr}(\mathbb{V}(\bar{y}_\infty)).$$

Then from Theorem 3.3, applying $\text{Tr}(\mathbb{V}(x_\infty)) = O_p(\alpha)$ and $\text{Tr}(\mathbb{V}(\bar{y}_\infty)) = O_p(\beta)$, we have the lemma.

B.3 Tail Averaging and Extrapolation

B.3.1 Proof of Theorem 3.5

First, let us define V_k, U_k as:

$$\begin{aligned} U_k &= \mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|^2] + \mathbb{E}[\|y_k - \mathbb{E}[y_\infty]\|^2] + O_{\mathbb{P}}(\beta), \\ V_k &= \mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|^2] + \frac{\alpha}{\beta} \mathbb{E}[\|y_k - \mathbb{E}[y_\infty]\|^2] + O_{\mathbb{P}}(\alpha). \end{aligned}$$

For all $k \geq t_0 \gg (\alpha\mu_x)^{-1} \log(1/(\alpha\mu_x))$, with Theorem 3.4, we ensure that under an optimal coupling,

$$\mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|^2] \leq \mathbb{E}[\|x_k - x_\infty\|^2] + \text{Tr}(\mathbb{V}(x_\infty)) \leq O_{\mathbb{P}}(\alpha),$$

and similarly, $\mathbb{E}[\|y_k - \mathbb{E}[y_\infty]\|^2] \leq O_{\mathbb{P}}(\beta)$.

Slower Iterate: We want to analyze

$$\mathbb{E}[\|\tilde{x}_t - x^*\|_2^2] = \mathbb{E}[\|\tilde{x}_t - \mathbb{E}[x_\infty] + (\mathbb{E}[x_\infty] - x^*)\|_2^2] \leq 2\mathbb{E}[\|\tilde{x}_t - \mathbb{E}[x_\infty]\|_2^2] + 2\mathbb{E}[\|\mathbb{E}[x_\infty] - x^*\|_2^2],$$

where $x_\infty \sim \mu(x)$. To show that this quantity is $O(\alpha)$, it suffices to bound $\mathbb{E}[\|\tilde{x}_t - \mathbb{E}[x_\infty]\|_{Q_x}^2]$ under the optimal coupling. Rewriting this term,

$$\mathbb{E}[\|\tilde{x}_t - \mathbb{E}[x_\infty]\|_2^2] = \frac{1}{(t-t_0)^2} \sum_{k=t_0}^t \mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|_2^2] + \frac{2}{(t-t_0)^2} \sum_{k=t_0}^t \sum_{l>k}^t \mathbb{E}[\langle x_k - \mathbb{E}[x_\infty], x_l - \mathbb{E}[x_\infty] \rangle].$$

We first note that by Theorem 3.4, under the optimal coupling between x_k and x_∞ , we get

$$\begin{aligned} \mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|_2^2] &\leq 2\mathbb{E}[\|x_k - x_\infty\|_2^2] + 2\mathbb{E}[\|x_\infty - \mathbb{E}[x_\infty]\|_2^2] \\ &\leq \exp(-\alpha\mu_x(k-t_0)/8)V_{t_0} + \text{Tr}(\mathbb{V}(x_\infty)). \end{aligned}$$

To proceed, we note that

$$\begin{aligned} \mathbb{E}[\|\tilde{x}_k - \mathbb{E}[x_\infty]\|_2^2] &\leq \frac{1}{(t-t_0)^2} \sum_{k=0}^{t-t_0} (\exp(-\alpha\mu_x k/8)V_{t_0} + \text{Tr}(\mathbb{V}(x_\infty))) \\ &\quad + \frac{2}{(t-t_0)^2} \sum_{k=t_0}^t \sum_{k'>0}^{t-k} \mathbb{E}[\mathbb{E}[\langle x_k - \mathbb{E}[x_\infty], x_{k+k'} - \mathbb{E}[x_\infty] \rangle | \mathcal{F}_k]] \\ &\leq \frac{1}{(t-t_0)^2} \sum_{k=0}^{t-t_0} (\exp(-\alpha\mu_x k/8)V_{t_0}) + \frac{\text{Tr}(\mathbb{V}(x_\infty))}{t-t_0} \\ &\quad + \frac{2}{(t-t_0)^2} \sum_{k=t_0}^t \sum_{k'>0}^{t-k} \mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|_2 \cdot \|\mathbb{E}[x_{k+k'} - x_\infty | \mathcal{F}_k]\|_2]. \end{aligned}$$

To bound the second term, we first note that for any $k' > 0$, we use an optimal coupling between $x_{k+k'} | \mathcal{F}_k$ and x_∞ , and again apply Theorem 3.4:

$$\mathbb{E}[\|\mathbb{E}[x_{k+k'} | \mathcal{F}_k] - \mathbb{E}[x_\infty]\|_2^2] \leq \mathbb{E}[\mathbb{E}[\|x_{k+k'} - x_\infty\|_2^2 | \mathcal{F}_k]] \leq \exp(-\alpha\mu_x k'/8)V_k.$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sum_{k=t_0}^t \sum_{k'>0}^{t-k} \mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|_2 \cdot \|\mathbb{E}[x_{k+k'} - x_\infty | \mathcal{F}_k]\|_2] \\ &\leq \sum_{k=t_0}^t \sqrt{\mathbb{E}[\|x_k - \mathbb{E}[x_\infty]\|_2^2]} \cdot \left(\sum_{k'>0}^{t-k} \sqrt{\mathbb{E}[\mathbb{E}[\|x_{k+k'} - x_\infty\|_2^2 | \mathcal{F}_k]]} \right) \end{aligned}$$

$$\leq \sum_{k=t_0}^t \sqrt{V_k} \cdot \frac{\sqrt{V_k}}{\alpha \mu_x} \leq \sum_{k=t_0}^t \frac{1}{\mu_x} = O_{\mathbb{P}}(t - t_0),$$

where we used that $V_k = O_{\mathbb{P}}(\alpha)$ for all $k \geq t_0$. Plugging this, we can conclude that

$$\mathbb{E}[\|\tilde{x}_k - \mathbb{E}[x_\infty]\|_2^2] \leq \frac{O_{\mathbb{P}}(1)}{t - t_0}.$$

Faster Iterate: In this case, we first note that

$$\begin{aligned} \|\tilde{y}_t - y^*\|_2^2 &\leq 2\|\tilde{y}_t - y^*(\tilde{x}_t)\|_2^2 + 2\|y^*(\tilde{x}_t) - y^*(x^*)\|_2^2 \\ &\leq 4\|\tilde{y}_t - \mathbb{E}[\bar{y}_\infty]\|_2^2 + 4\|\mathbb{E}[\bar{y}_\infty]\|_2^2 + 2\kappa_y^2\|\tilde{x}_t - x^*\|_2^2, \end{aligned}$$

where $\tilde{y}_k := \frac{1}{t-t_0} \sum_{t'=t_0}^t \bar{y}_t$. The second term is squared-bias in order $O_{\mathbb{P}}(\beta^2)$, and the third term inherits the error analysis from slower iterates. Thus, we focus on bounding the first term.

Following the same process for slower iterates, we first note that

$$\begin{aligned} \mathbb{E}[\|\tilde{y}_t - \mathbb{E}[\bar{y}_\infty]\|_2^2] &= \frac{1}{(t-t_0)^2} \sum_{k=t_0}^t \mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2^2] + \frac{2}{(t-t_0)^2} \sum_{k=t_0}^t \sum_{l>k}^t \mathbb{E}[\langle \bar{y}_k - \mathbb{E}[\bar{y}_\infty], \bar{y}_l - \mathbb{E}[\bar{y}_\infty] \rangle] \\ &\leq \frac{1}{(t-t_0)^2} \sum_{k=t_0}^t \mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2^2] \\ &\quad + \frac{2}{(t-t_0)^2} \sum_{k=t_0}^t \sum_{l>k}^t \mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2 \cdot \|\mathbb{E}[\bar{y}_{k+k'} - \bar{y}_\infty | \mathcal{F}_k]\|_2]. \end{aligned}$$

For the first term, we invoke Corollary 3.4, under optimal coupling, we have

$$\begin{aligned} \mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2^2] &\leq 2\mathbb{E}[\|\bar{y}_k - \bar{y}_\infty\|_2^2] + 2\mathbb{E}[\|\bar{y}_\infty - \mathbb{E}[\bar{y}_\infty]\|_2^2] \\ &\leq \beta \exp(-\alpha \mu_x (k - t_0)/8) V_{t_0} + \exp(-\beta \mu_y (k - t_0)/8) U_{t_0} + \text{Tr}(\mathbb{V}(\bar{y}_\infty)). \end{aligned}$$

For the second term, with Corollary 3.4, we have

$$\mathbb{E}[\|\mathbb{E}[\bar{y}_{k+k'} | \mathcal{F}_k] - \mathbb{E}[\bar{y}_\infty]\|_2^2] \leq \mathbb{E}[\mathbb{E}[\|\bar{y}_{k+k'} - \bar{y}_\infty\|_2^2 | \mathcal{F}_k]] \leq \beta \exp(-\alpha \mu_x k'/8) V_k + \exp(-\beta \mu_y k'/8) U_k,$$

and again using Cauchy-Schwarz inequality, we can show that

$$\begin{aligned} &\sum_{k=t_0}^t \sum_{k'>0}^{t-k} \mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2 \cdot \|\mathbb{E}[\bar{y}_{k+k'} - \bar{y}_\infty | \mathcal{F}_k]\|_2] \\ &\leq \sum_{k=t_0}^t \sqrt{\mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2^2]} \cdot \left(\sum_{k'>0}^{t-k} \sqrt{\mathbb{E}[\mathbb{E}[\|\bar{y}_{k+k'} - \bar{y}_\infty\|_2^2 | \mathcal{F}_k]]} \right) \\ &\leq \sum_{k=t_0}^t \sqrt{U_k} \cdot \left(\frac{\sqrt{\beta V_k}}{\alpha \mu_x} + \frac{\sqrt{U_k}}{\beta \mu_y} \right) \leq O_{\mathbb{P}}(t - t_0) \left(\sqrt{\frac{\beta^2}{\alpha}} \frac{1}{\mu_x} + \frac{1}{\mu_y} \right). \end{aligned}$$

On the other hand, we can apply Cauchy-Schwarz inequality in different ways:

$$\begin{aligned} &\sum_{k=t_0}^t \sum_{k'>0}^{t-k} \mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2 \cdot \|\mathbb{E}[\bar{y}_{k+k'} - \bar{y}_\infty | \mathcal{F}_k]\|_2] \\ &\leq \sum_{k=t_0}^t \sqrt{\mathbb{E}[\|\bar{y}_k - \mathbb{E}[\bar{y}_\infty]\|_2^2]} \cdot \left(\sum_{k'>0}^{t-k} \sqrt{\mathbb{E}[\mathbb{E}[\|\bar{y}_{k+k'} - \bar{y}_\infty\|_2^2 | \mathcal{F}_k]]} \right) \\ &\leq \sqrt{t - t_0} \sum_{k=t_0}^t \sqrt{U_k} \cdot \left(\sqrt{\frac{\beta V_k}{\alpha \mu_x}} + \sqrt{\frac{U_k}{\beta \mu_y}} \right) \leq O_{\mathbb{P}}(t - t_0)^{3/2} \left(\frac{\beta}{\sqrt{\mu_x}} + \sqrt{\frac{\beta}{\mu_y}} \right). \end{aligned}$$

Summarizing the results, we can conclude that

$$\frac{1}{(t-t_0)^2} \mathbb{E}[\|(\tilde{y}_k - y^*(\tilde{x}_k)) - \mathbb{E}[\bar{y}_\infty]\|_2^2] \leq \frac{O_{\mathbb{P}}(1)}{t-t_0} + \min \left(\sqrt{\frac{\beta^2}{\alpha}} \frac{O_{\mathbb{P}}(1)}{t-t_0}, \sqrt{\frac{O_{\mathbb{P}}(\beta)}{t-t_0}} \right).$$

This concludes Theorem 3.5.

B.3.2 Proof of Corollary 3.6

We note that

$$\begin{aligned} \mathbb{E}[\|\zeta_t^x - x^*\|_2^2] &\leq 2\mathbb{E}[\|\zeta_t^x - \mathbb{E}[(2x_\infty^{2\alpha,2\beta} - x_\infty^{\alpha,\beta})]\|_2^2] + 2\|\mathbb{E}[(2x_\infty^{2\alpha,2\beta} - x_\infty^{\alpha,\beta})] - x^*\|_2^2 \\ &\leq 16\mathbb{E}[\|\tilde{x}_t^{2\alpha,2\beta} - \mathbb{E}[x_\infty^{2\alpha,2\beta}]\|_2^2] + 4\mathbb{E}[\|\tilde{x}_t^{\alpha,\beta} - \mathbb{E}[x_\infty^{\alpha,\beta}]\|_2^2] + 2\|\mathbb{E}[(2x_\infty^{2\alpha,2\beta} - x_\infty^{\alpha,\beta})] - x^*\|_2^2. \end{aligned} \quad (25)$$

Note that from Theorem 3.3,

$$2x_\infty^{\alpha,\beta} - x_\infty^{2\alpha,2\beta} - x^* = 2(x_\infty^{\alpha,\beta} - x^*) - (x_\infty^{2\alpha,2\beta} - x^*) = O_{\mathbb{P}}(\beta^2).$$

The first and second terms in (25) can be bounded by $O_{\mathbb{P}}(1)/(t-t_0)$, following exactly same steps in the proof of Theorem 3.5. The result for faster iterates can also be derived similarly.

C Deferred Proofs

C.1 Proof of Lemma 3.1

The stochastic approximation equation becomes

$$\begin{aligned} (x_{t+1} - x^*) &= (x_t - x^*) - \alpha F(x_t, y^*(x_t)) + \alpha(F(x_t, y^*(x_t)) - F(x_t, y_t)) - \alpha w^x(x_t, y_t; \xi_t) \\ &= (x_t - x^*) - \alpha H(x_t) - \alpha J_{12}(y_t - y^*(x_t)) - \alpha w_t^x(x_t, y_t; \xi_t), \\ (y_{t+1} - y^*(x_{t+1})) &= (y_t - y^*(x_t)) + (y^*(x_t) - y^*(x_{t+1})) - \beta G(x_t, y_t) - \beta w^y(x_t, y_t; \xi_t) \\ &= (y_t - y^*(x_t)) - J_{22}^{-1} J_{21}(x_t - x_{t+1}) - \beta G(x_t, y_t) - \beta w^y(x_t, y_t; \xi_t) \\ &= (y_t - y^*(x_t)) - \alpha J_{22}^{-1} J_{21}(F(x_t, y_t) + w^f(x_t, y_t; \xi_t)) - \beta G(x_t, y_t) - \beta w^y(x_t, y_t; \xi_t) \\ &= (y_t - y^*(x_t)) - \alpha J_{22}^{-1} J_{21}(F(x_t, y_t) - H(x_t) + H(x_t)) - \beta G(x_t, y_t) \\ &\quad - \alpha J_{22}^{-1} J_{21} w^x(x_t, y_t; \xi_t) - \beta w^y(x_t, y_t; \xi_t). \end{aligned}$$

Using $H(x^*) = 0$, $G(x, y^*(x)) = 0$, we can rewrite the recursion as (5).

C.2 Proof of Lemma B.2

We can start with a coarse bound on $\|\bar{y}_{t+1}\|_{Q_y}^2$:

$$\begin{aligned} \|\bar{y}_{t+1}\|_{Q_y}^2 &\leq \|(I - \beta J_{22})\bar{y}_t\|_{Q_y}^2 + \alpha^2 \|J_{22}^{-1} J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x)\|_{Q_y}^2 + \beta^2 \|w_t^y\|_{Q_y}^2 \\ &\quad + 2\alpha |\langle (I - \beta J_{22})\bar{y}_t, -J_{22}^{-1} J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t + w_t^x) \rangle_{Q_y}| \\ &\quad + 2\beta |\langle (I - \beta J_{22})\bar{y}_t, -w_t^y \rangle_{Q_y}| + 2\alpha\beta |\langle J_{22}^{-1} J_{21}(J_{12}\bar{y}_t + \Delta\bar{x}_t - w_t^x), w_t^y \rangle_{Q_y}| \\ &\leq (1 - \beta\mu_y/2) \|\bar{y}_t\|_{Q_y}^2 + O_{\mathbb{P}}(\beta^2) \|\bar{x}_t\|_2^2 + O_{\mathbb{P}}(\alpha)\sigma_x^2 + O_{\mathbb{P}}(\beta)\sigma_y^2. \end{aligned}$$

Thus, taking the square on both sides, we get

$$\begin{aligned} \|\bar{y}_{t+1}\|_{Q_y}^4 &\leq (1 - \beta\mu_y/2)^2 \|\bar{y}_t\|_{Q_y}^4 + 2(1 - \beta\mu_y/2) \|\bar{y}_t\|_{Q_y}^2 (O_{\mathbb{P}}(\beta^2) \|\bar{x}_t\|_2^2 + O_{\mathbb{P}}(\alpha)\sigma_x^2 + O_{\mathbb{P}}(\beta)\sigma_y^2) \\ &\quad + (O_{\mathbb{P}}(\beta^2) \|\bar{x}_t\|_2^2 + O_{\mathbb{P}}(\alpha)\sigma_x^2 + O_{\mathbb{P}}(\beta)\sigma_y^2)^2 \\ &\leq (1 - \beta\mu_y/4) \|\bar{y}_t\|_{Q_y}^4 + (O_{\mathbb{P}}(\beta^2) \|\bar{x}_t\|_2^4 + O_{\mathbb{P}}(\alpha)\sigma_x^4 + O_{\mathbb{P}}(\beta)\sigma_y^4). \end{aligned}$$

Similarly for \bar{x}_t , we have

$$\begin{aligned}\|\bar{x}_{t+1}\|_{Q_x}^2 &\leq \|(I - \alpha\Delta)\bar{x}_t\|_{Q_x}^2 + 2\alpha^2\|J_{12}\bar{y}_t\|_{Q_x}^2 + 2\alpha^2\|w_t^x\|_{Q_x}^2 \\ &\quad + 2\alpha|\langle(I - \alpha\Delta)\bar{x}_t, J_{12}\bar{y}_t\rangle_{Q_x}| + 2\alpha|\langle(I - \alpha\Delta)\bar{x}_t, w_t^x\rangle_{Q_x}| + 2\alpha^2|\langle J_{12}\bar{y}_t, w_t^x\rangle_{Q_x}| \\ &\leq (1 - \alpha\mu_x/2)\|\bar{x}_t\|_{Q_x}^2 + O_{\mathbb{P}}(\alpha)\|\bar{y}_t\|_2^2 + O_{\mathbb{P}}(\alpha)\sigma_x^2,\end{aligned}$$

and thus,

$$\|\bar{x}_{t+1}\|_{Q_x}^4 \leq (1 - \alpha\mu_x/4)\|\bar{x}_t\|_{Q_x}^4 + O_{\mathbb{P}}(\alpha)\|\bar{y}_t\|_2^4 + O_{\mathbb{P}}(\alpha)\sigma_x^4.$$

Taking potential $V_t = \mathbb{E}[\|\bar{x}_t\|_{Q_x}^4 + \frac{O_{\mathbb{P}}(1)\alpha}{\beta}\|\bar{y}_t\|_{Q_y}^4]$, we have

$$V_{t+1} \leq (1 - \alpha\mu_x/4)V_t + O_{\mathbb{P}}(\alpha)(\sigma_x^4 + \sigma_y^4),$$

which leads to

$$\mathbb{E}[\|\bar{x}_t\|_{Q_x}^4] \leq V_t \leq \exp(-\alpha\mu_x t/4)V_0 + O_{\mathbb{P}}(\sigma_x^4 + \sigma_y^4).$$

Plugging this back to the recursion for y , we also have

$$\mathbb{E}[\|\bar{y}_t\|_{Q_y}^4] \leq \exp(-\beta\mu_x t/4)\mathbb{E}[\|\bar{y}_0\|_{Q_y}^4] + O_{\mathbb{P}}(\sigma_x^4 + \sigma_y^4).$$

Converting $\|\cdot\|_{Q_x}$ and $\|\cdot\|_{Q_y}$ to $\|\cdot\|_2$ norm concludes the lemma.

C.3 Proof of Lemma B.3

The proof strategy is to consider three cases separately when t is small and large. Let $c_1 > 0$ be some sufficiently large absolute constant.

Case (i) $t \leq c_1 \cdot \tau_\alpha$: In this case, consider optimal coupling between μ_0^1, μ_0^2 , and apply Lemma A.8:

$$\begin{aligned}\mathbb{E}[\|x_t^1 - x_t^2\|_2^2] &\leq 3\mathbb{E}[\|x_0^1 - x_0^2\|_2^2] + 3\mathbb{E}[\|x_t^1 - x_0^1\|_2^2] + 3\mathbb{E}[\|x_t^1 - x_0^1\|_2^2] \\ &\leq 3\mathcal{W}_2^2(\mu_0^1(x_0^1), \mu_0^2(x_0^2)) + O_{\mathbb{P}}(\alpha^2\tau^2).\end{aligned}$$

Since in this case $\exp(-\alpha\mu_x t/4) > 1/2$ for $t < O(1)\tau_\alpha \ll 1/(\alpha\mu_x)$, the x part in the inequality (16) holds. The y part of (16) can be shown similarly.

Case (ii) $c_1 \cdot \tau_\alpha < t$: We consider a coupling on ξ_τ^1 and ξ_τ^2 first. Let ν^1, ν^2 be probability distributions over $\Xi \times \Xi$ such that

$$\begin{aligned}\nu^1(\xi^1, \xi^2) &\propto \mathbb{1}\{\xi^1 = \xi^2\} \cdot \min(\mu_\tau^1(\xi^1), \mu_\tau^2(\xi^2)), \\ \nu^2(\xi^1, \xi^2) &\propto \max(0, \mu_\tau^1(\xi^1) - \mu_\tau^2(\xi^1)) \times \max(0, \mu_\tau^2(\xi^2) - \mu_\tau^1(\xi^2)).\end{aligned}$$

The coupling distribution decides ν^1 with probability $1 - \text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2))$, and ν^2 with probability $\text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2))$ to sample (ξ_τ^1, ξ_τ^2) . Then it samples $(x_\tau^1, y_\tau^1) \sim \mu_\tau^1(\cdot|\xi_\tau^1)$ and $(x_\tau^2, y_\tau^2) \sim \mu_\tau^2(\cdot|\xi_\tau^2)$. When ν^1 is selected, we couple two sequences by setting $\xi_t^1 = \xi_t^2$ for all $t \geq 0$, and invoke Lemma B.1 to show that

$$\mathbb{E}[\|x_t^1 - x_t^2\|_2^2 | \nu^1] \leq O_{\mathbb{P}}(1) \cdot \mathbb{E}\left[\|x_\tau^1 - x_\tau^2\|_2^2 + \frac{\alpha}{\beta}\|y_\tau^1 - y_\tau^2\|_2^2 | \nu^1\right] \exp(-\alpha\mu_x(t - \tau)/4),$$

where we also took expectation over the optimal coupling for $(x_\tau^1, y_\tau^1)|\xi_\tau^1$ and $(x_\tau^2, y_\tau^2)|\xi_\tau^2$. When ν^2 is selected, we let the two sequences independently evolve, and using Lemma B.2 to show that

$$\begin{aligned}\mathbb{E}[\|x_t^1 - x_t^2\|_2^2 \cdot \mathbb{1}\{\nu^2\}] &\leq O_{\mathbb{P}}(1)\sqrt{\mathbb{E}[\|x_t^1 - x_t^2\|_2^4 + \|\bar{y}_t^1 - \bar{y}_t^2\|_2^4]} \cdot \sqrt{\text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2))} \\ &\leq O_{\mathbb{P}}(1)\sqrt{\mathbb{E}[\|x_t^1\|_2^4 + \|x_t^2\|_2^4 + \|\bar{y}_t^1\|_2^4 + \|\bar{y}_t^2\|_2^4]} \cdot \sqrt{\text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2))} \\ &\leq \sqrt{O_{\mathbb{P}}(1)\text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2))}.\end{aligned}$$

Given the above results, let $c_2 > 0$ be another sufficiently large absolute constant. Now if $t \leq c_2 \log(\beta/\alpha)/(\alpha\mu_x)$, then we set $\tau = c_1\tau_\alpha/4$ such that $t > 4\tau$. With $\text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2)) < \rho^\tau \ll (\mu_x\alpha)^{O(1)} \leq \exp(-\alpha\mu_x t/4)$, we have

$$\begin{aligned}\mathbb{E}[\|x_t^1 - x_t^2\|_2^2] &\leq O_{\mathbb{P}}(1) \cdot \mathbb{E} \left[\|x_\tau^1 - x_\tau^2\|_2^2 + \frac{\alpha}{\beta} \|y_\tau^1 - y_\tau^2\|_2^2 + O_{\mathbb{P}}(\alpha) \right] \exp(-\alpha\mu_x t/8) \\ &\leq O_{\mathbb{P}}(1) \cdot V_0 \exp(-\alpha\mu_x t/8).\end{aligned}$$

On the other hand, if $t > c_2 \log(\beta/\alpha)/(\alpha\mu_x)$, then we take $\tau = t/8$. In this case, we instead invoke the MSE result in Theorem 3.2, which gives

$$\begin{aligned}\mathbb{E}[\|x_\tau^1 - x_\tau^2\|_2^2] &\leq 2\mathbb{E}[\|\bar{x}_\tau^1\|_2^2] + 2\mathbb{E}[\|\bar{x}_\tau^2\|_2^2] \leq O_{\mathbb{P}}(1) \cdot \exp(-\alpha\mu_x \tau/4) \ll O_{\mathbb{P}}(\alpha), \\ \mathbb{E}[\|\bar{y}_\tau^1 - \bar{y}_\tau^2\|_2^2] &\leq 2\mathbb{E}[\|\bar{y}_\tau^1\|_2^2] + 2\mathbb{E}[\|\bar{y}_\tau^2\|_2^2] \\ &\leq O_{\mathbb{P}}(1) \cdot (\exp(-\beta\mu_y \tau/4) + \beta \exp(-\alpha\mu_x \tau/4)) \ll O_{\mathbb{P}}(\alpha).\end{aligned}$$

Together with $\text{TV}(\mu_\tau^1(\xi_\tau^1), \mu_\tau^2(\xi_\tau^2)) < \rho^{t/8} \leq \exp(-\alpha\mu_x t/4)$, we get the same conclusion that

$$\begin{aligned}\mathbb{E}[\|x_t^1 - x_t^2\|_2^2] &\leq O_{\mathbb{P}}(1) \cdot \mathbb{E} \left[\|x_\tau^1 - x_\tau^2\|_2^2 + \frac{\alpha}{\beta} \|y_\tau^1 - y_\tau^2\|_2^2 + O_{\mathbb{P}}(\alpha) \right] \exp(-\alpha\mu_x t/8) \\ &\leq O_{\mathbb{P}}(1) \cdot V_0 \exp(-\alpha\mu_x t/8).\end{aligned}$$

The inequality for y in (16) can also be similarly proven.

D Proof of Technical Lemmas

D.1 Proof of Lemma A.1

This result follows immediately from the fact that $\text{Tr}(AB) \leq \|AB\|_1$, and Hölder's inequality applied to matrix p-Schatten norm.

D.2 Proof of Lemma A.2

By definition of $\|A\|_Q$, we start from

$$\begin{aligned}\|A\|_Q^2 &= \max_{\|x\|_Q \leq 1} \|Ax\|_Q^2 = \max_{\|x\|_Q \leq 1} (x^\top A^\top Q A x) = \max_{\|z\|_2 \leq 1} (z^\top Q^{-1/2} A^\top Q A Q^{-1/2} z) \\ &= \max_{\|z\|_2 \leq 1} \|Q^{1/2} A Q^{-1/2} z\|_2^2 = \|Q^{1/2} A Q^{-1/2}\|_{\text{op}}^2.\end{aligned}$$

D.3 Proof of Lemma A.3

By definition for $\langle \cdot, \cdot \rangle_Q$,

$$\langle x, y \rangle_Q = x^\top Q y \leq \|x^\top Q^{1/2}\|_2 \|Q^{1/2} y\|_2 = \|x\|_Q \|y\|_Q.$$

Next, we observe that

$$\langle Mx, x \rangle_Q = x^\top M^\top Q x = \text{Tr}(x x^\top M^\top Q) \leq \|Q x x^\top\|_1 \|M\|_{\text{op}}.$$

Then since $Q x x^\top$ is a rank-1 matrix, we have

$$\|Q x x^\top\|_1 = \|Q x x^\top\|_2 = \sqrt{x^\top Q x} = \|x\|_Q.$$

Finally, by definition of $\|\cdot\|_Q$,

$$\|Mx\|_Q \leq \|M\|_Q \|x\|_Q.$$

Then, note that

$$\|M\|_Q^2 = \|Q^{1/2} M Q^{-1/2}\|_{\text{op}}^2 \leq \kappa(Q) \|M\|_{\text{op}}^2,$$

yielding the proof.

D.4 Proof of Lemma A.5

This can be shown from the definition of Q-norm:

$$\|x\|_{Q_1}^2 = x^\top Q_1 x \leq \sigma_{\max}(Q_1) \|x\|_2^2 \leq \frac{\sigma_{\max}(Q_1)}{\sigma_{\min}(Q_2)} \|x\|_{Q_2}^2.$$

D.5 Proof of Lemma A.6

Let $\pi_\tau = \mathbb{P}_{\xi_t}(\cdot | \mathcal{F}_{t-\tau})$ be a distribution over Ξ . By Assumption 2,

$$\|\pi_\tau - \pi\|_1 \leq c_\rho \rho^\tau.$$

Furthermore, we know that

$$\int_{\Xi} W_{ij}(\xi) d\pi(\xi) = 0.$$

Thus,

$$\begin{aligned} \mathbb{E}[\langle W_{ij}(\xi), u_{t-\tau} v_{t-\tau}^\top \rangle | \mathcal{F}_{t-\tau}] &\leq \mathbb{E}_{\xi \sim \pi}[\langle W_{ij}(\xi), u_{t-\tau} v_{t-\tau}^\top \rangle] + W_{\max} \|u_{t-\tau} v_{t-\tau}^\top\|_1 \|\pi_\tau - \pi\|_1 \\ &\leq W_{\max} \|u_{t-\tau}\|_2 \|v_{t-\tau}\|_2 \cdot c_\rho \rho^\tau. \end{aligned}$$

The second inequality also follows similarly.

D.6 Proof of Lemma A.7

Note that $x_t = \bar{x}_t + x^*$ and $y_t = \bar{y}_t - J_{22}^{-1} J_{21} \bar{x}_t + J_{22}^{-1} J_{21} x^*$. Plugging these to $w_t^x = W_{11}(\xi_t) x_t + W_{12}(\xi_t) y_t + u_1(\xi_t)$ and similarly to w_t^y yields the expressions.

D.7 Proof of Lemma A.8

By the recursion in (5),

$$\begin{aligned} \|\bar{x}_{t+1}\|_2 &\leq (1 + \alpha \|\Delta + W_\Delta^x(\xi_t)\|_{\text{op}}) \|\bar{x}_t\|_2 + \alpha (\|J_{12} + W_{12}(\xi_t)\|_{\text{op}} \|\bar{y}_t\|_2 + \|W_\Delta^x(\xi_t) x^* + u_1(\xi_t)\|_2) \\ &\leq (1 + \alpha \kappa_y J_{\max}) \|\bar{x}_t\|_2 + \alpha (J_{\max} \|\bar{y}_t\|_2 + \sigma_x), \\ \|\bar{x}_{t+1} - \bar{x}_t\|_2 &\leq \alpha (\kappa_y J_{\max} \|\bar{x}_t\|_2 + J_{\max} \|\bar{y}_t\|_2 + \sigma_x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|\bar{y}_{t+1}\|_2 &\leq (1 + \beta J_{\max}) \|\bar{y}_t\|_2 + \beta (J_{\max} \kappa_y \|\bar{x}_t\|_2 + \sigma_y) + \kappa_y \alpha (\kappa_y J_{\max} \|\bar{x}_t\|_2 + J_{\max} \|\bar{y}_t\|_2 + \sigma_x), \\ \|\bar{y}_{t+1} - \bar{y}_t\|_2 &\leq \beta (J_{\max} \|\bar{y}_t\|_2 + J_{\max} \kappa_y \|\bar{x}_t\|_2 + \sigma_y) + \kappa_y \alpha (\kappa_y J_{\max} \|\bar{x}_t\|_2 + J_{\max} \|\bar{y}_t\|_2 + \sigma_x). \end{aligned}$$

Adding two equations,

$$\kappa_y \|\bar{x}_{t+1}\|_2 + \|\bar{y}_{t+1}\|_2 \leq (1 + 2\beta J_{\max}) (\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) + \beta \sigma_y + \alpha \kappa_y \sigma_x.$$

Solving this recursively, we get

$$\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2 \leq (1 + 2\beta \tau J_{\max}) (\kappa_y \|\bar{x}_{t-\tau}\|_2 + \|\bar{y}_{t-\tau}\|_2) + \tau (\beta \sigma_y + \alpha \kappa_y \sigma_x).$$

Using this result, we have

$$\begin{aligned} \|\bar{x}_t - \bar{x}_{t-\tau}\|_2 &\leq \sum_{i=t-\tau+1}^t \|\bar{x}_i - \bar{x}_{i-1}\|_2 \leq \alpha J_{\max} \sum_{i=t-\tau+1}^t (\kappa_y \|\bar{x}_i\|_2 + \|\bar{y}_i\|_2) + \alpha \tau \sigma_x \\ &\leq 2\alpha \tau J_{\max} (\kappa_y \|\bar{x}_{t-\tau}\|_2 + \|\bar{y}_{t-\tau}\|_2) + 2\alpha \tau \sigma_x + \alpha \beta J_{\max} \tau \sigma_y. \end{aligned}$$

Similarly,

$$\begin{aligned}\|\bar{y}_t - \bar{y}_{t-\tau}\|_2 &\leq \sum_{i=t-\tau+1}^t \|\bar{y}_i - \bar{y}_{i-1}\|_2 \leq (\beta + \alpha\kappa_y)J_{\max} \sum_{i=t-\tau+1}^t (\kappa_y \|\bar{x}_i\|_2 + \|\bar{y}_i\|_2) + \tau(\beta\sigma_y + \alpha\kappa_y\sigma_x) \\ &\leq 2\tau J_{\max} \beta (\kappa_y \|\bar{x}_{t-\tau}\|_2 + \|\bar{y}_{t-\tau}\|_2) + 2\tau(\alpha\kappa_y\sigma_x + \beta\sigma_y).\end{aligned}$$

Finally, from these two equations, note that

$$\kappa_y \|\bar{x}_t - \bar{x}_{t-\tau}\|_2 + \|\bar{y}_t - \bar{y}_{t-\tau}\|_2 \leq 8\beta J_{\max} \tau (\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) + 8(\alpha\kappa_y\sigma_x + \beta\sigma_y).$$

Plugging this back with $\|\bar{x}_{t-\tau}\|_2 \leq \|\bar{x}_t\|_2 + \|\bar{x}_t - \bar{x}_{t-\tau}\|_2$ and $\|\bar{y}_{t-\tau}\|_2 \leq \|\bar{y}_t\|_2 + \|\bar{y}_t - \bar{y}_{t-\tau}\|_2$, we get the lemma.

D.8 Proof of Lemma A.10

To begin with, we start with unfolding the expression as

$$\mathbb{E}[w_t^T \bar{y}_t^\top] = \mathbb{E}[W_{11}(\xi_t) \bar{x}_t \bar{y}_t^\top] + \mathbb{E}[W_{12}(\xi_t) \bar{y}_t \bar{y}_t^\top] + \mathbb{E}[u_1(\xi_t) \bar{y}_t^\top].$$

To proceed, we first note that

$$\mathbb{E}[W_{11}(\xi_t) \bar{x}_t \bar{y}_t^\top] = \mathbb{E}[W_{11}(\xi_t) \bar{x}_{t-\tau} \bar{y}_{t-\tau}^\top] + \mathbb{E}[W_{11}(\xi_t) (\bar{x}_t - \bar{x}_{t-\tau}) \bar{y}_{t-\tau}^\top] + \mathbb{E}[W_{11}(\xi_t) \bar{x}_t (\bar{y}_t - \bar{y}_{t-\tau})^\top].$$

For each term in the above, we have the following inequalities:

1. Using the mixing-time assumption, we can show that

$$\begin{aligned}\|\mathbb{E}[W_{11}(\xi_t) \bar{x}_{t-\tau} \bar{y}_{t-\tau}^\top]\|_1 &\leq \rho^\tau \cdot \mathbb{E}[\|\mathbb{E}[W_{11}(\xi_t) \bar{x}_{t-\tau} \bar{y}_{t-\tau}^\top]\|_1 | \mathcal{F}_{t-\tau}] \\ &\leq \rho^\tau W_{\max} \mathbb{E}[\|\bar{x}_{t-\tau} \bar{y}_{t-\tau}^\top\|_1] = \rho^\tau W_{\max} \mathbb{E}[\|\bar{x}_{t-\tau}\|_2 \|\bar{y}_{t-\tau}\|_2] \\ &\leq O(1) \rho^\tau W_{\max} \mathbb{E}[\|\bar{x}_t\|_2^2 + \|\bar{y}_t\|_2^2 + \alpha^2 \kappa_y^2 \tau^2 \sigma_x^2 + \beta^2 \tau^2 \sigma_y^2] \\ &\leq \frac{\mu_y}{32\kappa_y} \mathbb{E}[\|\bar{y}_t\|_2^2] + \rho^\tau W_{\max} \cdot O\left(\frac{\alpha^2 \mu_y^2}{\kappa_y^2} \mathbb{E}[\|\bar{x}_t\|_2^2] + \tau^2 (\alpha^2 \kappa_y^2 \sigma_x^2 + \beta^2 \sigma_y^2)\right).\end{aligned}$$

2. For the next term, we apply Lemma A.8 and Corollary A.9:

$$\begin{aligned}\|\mathbb{E}[W_{11}(\xi_t) (\bar{x}_t - \bar{x}_{t-\tau}) \bar{y}_{t-\tau}^\top]\|_1 &\leq W_{\max} \mathbb{E}[\|(\bar{x}_t - \bar{x}_{t-\tau}) \bar{y}_{t-\tau}^\top\|_1] \\ &\leq \alpha\tau J_{\max} W_{\max} \cdot \mathbb{E}[(\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) \|\bar{y}_{t-\tau}\|_2] + \alpha\tau W_{\max} (\sigma_x + \beta J_{\max} \sigma_y) \mathbb{E}[\|\bar{y}_{t-\tau}\|_2] \\ &\leq O(1) \cdot \alpha\tau J_{\max}^2 (\mathbb{E}[\beta J_{\max} \kappa_y^2 \|\bar{x}_t\|_2^2 + (\kappa_y \|\bar{x}_t\|_2 \|\bar{y}_t\|_2 + \|\bar{y}_t\|_2^2) + (\alpha^2 \kappa_y^2 \sigma_x^2 + \beta^2 \sigma_y^2)]) \\ &\quad + O(1) \cdot \alpha\tau J_{\max} (\sigma_x + \beta J_{\max} \sigma_y) (J_{\max} \beta \kappa_y \tau \|\bar{x}_t\| + \|\bar{y}_t\| + \tau(\alpha\kappa_y \sigma_x + \beta\sigma_y)) \\ &\leq \frac{\mu_y}{32\kappa_y} \mathbb{E}[\|\bar{y}_t\|_2^2] + O(1) J_{\max}^2 \kappa_y^2 \tau^2 \left(\beta^2 J_{\max} + \frac{\alpha^2 \kappa_y J_{\max}^2}{\mu_y}\right) \mathbb{E}[\|\bar{x}_t\|_2^2] \\ &\quad + O\left(\frac{J_{\max}^2 \kappa_y}{\mu_y}\right) (\alpha^2 \tau^2 \sigma_x^2 + J_{\max}^2 \alpha \beta^2 \tau^2 \sigma_y^2).\end{aligned}$$

We can simplify it further later, using the condition that $\alpha \ll \beta/\kappa_y$ and $W_{\max} \leq J_{\max}$.

3. For the last term, similarly,

$$\begin{aligned}\|\mathbb{E}[W_{11}(\xi_t) \bar{x}_t (\bar{y}_t - \bar{y}_{t-\tau})^\top]\|_1 &\leq W_{\max} \mathbb{E}[\|\bar{x}_t\|_2 \|\bar{y}_t - \bar{y}_{t-\tau}\|_2] \\ &\leq O(1) \beta \tau J_{\max}^2 \cdot \mathbb{E}[(\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) \|\bar{x}_t\|_2] + O(1) \tau J_{\max} \mathbb{E}[\|\bar{x}_t\|_2] (\alpha\kappa_y \sigma_x + \beta\sigma_y) \\ &\leq O(1) \beta \tau J_{\max}^2 \mathbb{E}[\kappa_y \|\bar{x}_t\|_2^2 + \|\bar{y}_t\|_2^2] + O(1) \tau ((\alpha^2/\beta) \kappa_y \sigma_x^2 + \beta\sigma_y^2).\end{aligned}$$

Combining these inequalities, and given that $\beta\tau \ll \kappa_y^{-2}/J_{\max}$ in Assumption 4, we can conclude that

$$\|\mathbb{E}[W_{11}(\xi_t) \bar{x}_t \bar{y}_t^\top]\|_1 \leq \frac{\mu_y}{16\kappa_y} \mathbb{E}[\|\bar{y}_t\|_2^2] + O(1) \beta \tau J_{\max}^2 \kappa_y \mathbb{E}[\|\bar{x}_t\|_2^2] + O(1) \tau ((\alpha^2/\beta) \kappa_y \sigma_x^2 + \beta\sigma_y^2).$$

Similarly, we can also show that

$$\|\mathbb{E}[W_{12}(\xi_t)\bar{y}_t\bar{y}_t^\top]\|_1 \leq \frac{\mu_y}{16\kappa_y} \mathbb{E}[\|\bar{y}_t\|_2^2] + O(1)\beta\tau J_{\max}^2\kappa_y \mathbb{E}[\|\bar{x}_t\|_2^2] + O(1)\tau((\alpha^2/\beta)\kappa_y\sigma_x^2 + \beta\sigma_y^2).$$

For the last one, we proceed as

$$\begin{aligned} \|\mathbb{E}[u_1(\xi_t)\bar{y}_t^\top]\|_1 &\leq \|\mathbb{E}[u_1(\xi_t)\bar{y}_{t-\tau}^\top]\|_1 + \|\mathbb{E}[u_1(\xi_t)(\bar{y}_t - \bar{y}_{t-\tau})^\top]\|_1 \\ &\leq \rho^\tau u_{\max} \mathbb{E}[\|\bar{y}_{t-\tau}\|_2] + u_{\max} \mathbb{E}[\|\bar{y}_t - \bar{y}_{t-\tau}\|_2] \\ &\leq O(1)(\rho^\tau + \beta\tau J_{\max})u_{\max} \mathbb{E}[\kappa_y\|\bar{x}_t\|_2 + \|\bar{y}_t\|_2] + \tau u_{\max}(\alpha\kappa_y\sigma_x + \beta\sigma_y) \\ &\leq O(1)\beta\tau J_{\max}^2 \mathbb{E}[\kappa_y\|\bar{x}_t\|_2^2 + \|\bar{y}_t\|_2^2] + O(1)\tau((\alpha^2/\beta)\kappa_y\sigma_x^2 + \beta\sigma_y^2), \end{aligned}$$

where in the last inequality, we use $u_{\max} \leq \sigma_y$. Combining all the above inequalities yields the lemma.

D.9 Proof of Lemma A.11

To begin with, we start with unfolding the expression as

$$\mathbb{E}[w_t^x \bar{x}_t^\top] = \mathbb{E}[W_{11}(\xi_t)\bar{x}_t \bar{x}_t^\top] + \mathbb{E}[W_{12}(\xi_t)\bar{y}_t \bar{x}_t^\top] + \mathbb{E}[u_1(\xi_t)\bar{x}_t^\top].$$

To proceed, we first note that

$$\mathbb{E}[W_{11}(\xi_t)\bar{x}_t \bar{x}_t^\top] = \mathbb{E}[W_{11}(\xi_t)\bar{x}_{t-\tau} \bar{x}_{t-\tau}^\top] + \mathbb{E}[W_{11}(\xi_t)(\bar{x}_t - \bar{x}_{t-\tau})\bar{x}_{t-\tau}^\top] + \mathbb{E}[W_{11}(\xi_t)\bar{x}_t(\bar{x}_t - \bar{x}_{t-\tau})^\top].$$

For each term in the above, we have the following inequalities:

1. Using the mixing-time assumption, we can show that

$$\begin{aligned} \|\mathbb{E}[W_{11}(\xi_t)\bar{x}_{t-\tau} \bar{x}_{t-\tau}^\top]\|_1 &\leq \rho^\tau \cdot \mathbb{E}[\mathbb{E}[\max_{\xi_t \in \Xi} \|W_{11}(\xi_t)\bar{x}_{t-\tau} \bar{x}_{t-\tau}^\top\|_1 | \mathcal{F}_{t-\tau}]] \\ &\leq \rho^\tau W_{\max} \mathbb{E}[\|\bar{x}_{t-\tau} \bar{x}_{t-\tau}^\top\|_1] = \rho^\tau W_{\max} \mathbb{E}[\|\bar{x}_{t-\tau}\|_2^2] \\ &\leq O(1)\rho^\tau W_{\max} \mathbb{E}[\|\bar{x}_t\|_2^2 + \alpha^2 J_{\max}^2 \|\bar{y}_t\|_2^2 + \alpha^2 \tau^2 \sigma_x^2] \\ &\leq \frac{\mu_x}{32\kappa_x} \mathbb{E}[\|\bar{x}_t\|_2^2] + \rho^\tau J_{\max} O(\alpha^2 \mu_x^2 \mathbb{E}[\|\bar{y}_t\|_2^2] + \tau^2 \alpha^2 \sigma_x^2). \end{aligned}$$

2. For the next term, we apply Lemma A.8 and Corollary A.9:

$$\begin{aligned} \|\mathbb{E}[W_{11}(\xi_t)(\bar{x}_t - \bar{x}_{t-\tau})\bar{x}_{t-\tau}^\top]\|_1 &\leq W_{\max} \mathbb{E}[\|(\bar{x}_t - \bar{x}_{t-\tau})\bar{x}_{t-\tau}^\top\|_1] \\ &\leq O(1)\alpha\tau J_{\max} W_{\max} \cdot \mathbb{E}[(\kappa_y\|\bar{x}_t\|_2 + \|\bar{y}_t\|_2)\|\bar{x}_{t-\tau}\|_2] + O(1)\alpha\tau W_{\max}(\sigma_x + \beta J_{\max} \sigma_y) \mathbb{E}[\|\bar{x}_{t-\tau}\|_2] \\ &\leq O(1) \cdot \alpha\tau J_{\max}^2 (\mathbb{E}[\kappa_y\|\bar{x}_t\|_2^2 + \|\bar{y}_t\|_2^2 + \tau^2 \alpha^2 (\sigma_x^2 + \beta^2 J_{\max}^2 \sigma_y^2)]) \\ &\leq \frac{\mu_x}{32\kappa_x} \mathbb{E}[\|\bar{x}_t\|_2^2] + O(1)\alpha\tau J_{\max}^2 \mathbb{E}[\|\bar{y}_t\|_2^2] + (\alpha\tau)^3 (\sigma_x^2 + \beta^2 J_{\max}^2 \sigma_y^2), \end{aligned}$$

where we use the condition that $\alpha\tau \ll 1/(J_{\max}\kappa_x^2)$ and $W_{\max} \leq J_{\max}$.

Combining these inequalities, and given that $\beta\tau \ll \kappa_y^{-2}/J_{\max}$ in Assumption 4, we can conclude that

$$\|\mathbb{E}[W_{11}(\xi_t)\bar{x}_t \bar{x}_t^\top]\|_1 \leq \frac{\mu_x}{16\kappa_x} \mathbb{E}[\|\bar{x}_t\|_2^2] + \alpha\tau J_{\max}^2 \mathbb{E}[\|\bar{y}_t\|_2^2] + \alpha\tau \sigma_x^2.$$

We also need to check the cross term:

$$\mathbb{E}[W_{12}(\xi_t)\bar{y}_t \bar{x}_t^\top] = \mathbb{E}[W_{12}(\xi_t)\bar{x}_{t-\tau} \bar{y}_{t-\tau}^\top] + \mathbb{E}[W_{12}(\xi_t)(\bar{y}_t - \bar{y}_{t-\tau})\bar{x}_{t-\tau}^\top] + \mathbb{E}[W_{12}(\xi_t)\bar{y}_t(\bar{x}_t - \bar{x}_{t-\tau})^\top].$$

First term can be bounded similarly using the geometric mixing assumption. For the second term,

$$\|\mathbb{E}[W_{12}(\xi_t)(\bar{y}_t - \bar{y}_{t-\tau})\bar{x}_{t-\tau}^\top]\|_1 \leq W_{\max} \mathbb{E}[\|(\bar{y}_t - \bar{y}_{t-\tau})\bar{x}_{t-\tau}^\top\|_1]$$

$$\begin{aligned}
&\leq O(1)\beta\tau J_{\max}^2 \cdot \mathbb{E}[(\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) \|\bar{x}_{t-\tau}\|_2] + O(1)\tau J_{\max}(\alpha\kappa_y\sigma_x + \beta\sigma_y)\mathbb{E}[\|\bar{x}_{t-\tau}\|_2] \\
&\leq \frac{\mu_x}{32\kappa_x}\mathbb{E}[\|\bar{x}_t\|_2^2] + O(1)\frac{\beta^2\tau^2 J_{\max}^4 \kappa_x}{\mu_x}\mathbb{E}[\|\bar{y}_t\|_2^2] + O\left(\frac{\alpha^2\tau^2 J_{\max}^2 \kappa_y^2 \kappa_x}{\mu_x}\sigma_x^2 + \frac{\kappa_x\beta^2\tau^2 J_{\max}^2}{\mu_x}\sigma_y^2\right).
\end{aligned}$$

For the third term, similarly, we have

$$\begin{aligned}
\|\mathbb{E}[W_{12}(\xi_t)\bar{y}_t(\bar{x}_t - \bar{x}_{t-\tau})^\top]\|_1 &\leq W_{\max}\mathbb{E}[\|\bar{y}_t(\bar{x}_t - \bar{x}_{t-\tau})^\top\|_1] \\
&\leq O(1)\alpha\tau J_{\max}^2 \cdot \mathbb{E}[(\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2) \|\bar{y}_t\|_2] + O(1)\tau J_{\max}(\alpha\sigma_x + \beta^2 J_{\max}\sigma_y)\mathbb{E}[\|\bar{y}_t\|_2] \\
&\leq \frac{\mu_x}{32\kappa_x}\mathbb{E}[\|\bar{x}_t\|_2^2] + O(1)\tau J_{\max}^2(\alpha + \beta^2 J_{\max})\mathbb{E}[\|\bar{y}_t\|_2^2] + O(\alpha\tau\sigma_x^2 + \beta^2\tau J_{\max}\sigma_y^2).
\end{aligned}$$

For the last one, we proceed as

$$\begin{aligned}
\|\mathbb{E}[u_1(\xi_t)\bar{x}_t^\top]\|_1 &\leq \|\mathbb{E}[u_1(\xi_t)\bar{x}_{t-\tau}^\top]\|_1 + \|\mathbb{E}[u_1(\xi_t)(\bar{x}_t - \bar{x}_{t-\tau})^\top]\|_1 \\
&\leq \rho^\tau u_{\max}\mathbb{E}[\|\bar{x}_{t-\tau}\|_2] + u_{\max}\mathbb{E}[\|\bar{x}_t - \bar{x}_{t-\tau}\|_2] \\
&\leq (\rho^\tau + \alpha\tau J_{\max})u_{\max}\mathbb{E}[\kappa_y \|\bar{x}_t\|_2 + \|\bar{y}_t\|_2] + \tau u_{\max}\alpha\sigma_x \\
&\leq \alpha\tau J_{\max}^2\mathbb{E}[\kappa_y \|\bar{x}_t\|_2^2 + \|\bar{y}_t\|_2^2] + \tau\alpha\sigma_x^2,
\end{aligned}$$

where in the last inequality, we use $u_{\max} \leq \sigma_x$. Combining all the above inequalities yields the lemma.