RANDOM FIBONACCI WORDS VIA CLONE SCHUR FUNCTIONS

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Abstract. We investigate positivity and probabilistic properties arising from the Young–Fibonacci lattice $\mathbb{YF}$, a 1-differential poset on binary words composed of 1's and 2's (known as Fibonacci words). Building on Okada's theory of clone Schur functions [Oka94], we introduce clone coherent measures on $\mathbb{YF}$ which give rise to random Fibonacci words of increasing length. Unlike coherent systems associated to classical Schur functions on the Young lattice of integer partitions, clone coherent measures are generally not extremal on $\mathbb{YF}$.

Our first main result is a complete characterization of Fibonacci positive specializations — parameter sequences which yield positive clone Schur functions on $\mathbb{YF}$. We connect Fibonacci positivity with total positivity of tridiagonal matrices, Stieltjes moment sequences, and orthogonal polynomials in one variable from the (q-)Askey scheme.

Our second family of results concerns the asymptotic behavior of random Fibonacci words derived from various Fibonacci positive specializations. We analyze several limiting regimes for specific examples, revealing stick-breaking-like processes (connected to GEM distributions), dependent stick-breaking processes of a new type, or discrete limits tied to the Martin boundary of the Young–Fibonacci lattice. Our stick-breaking-like scaling limits significantly extend the result of Gnedin–Kerov [GK00a] on asymptotics of the Plancherel measure on $\mathbb{YF}$.

We also establish Cauchy-like identities for clone Schur functions (with the right-hand side given by a quadridiagonal determinant), and construct and analyze models of random permutations and involutions based on Fibonacci positive specializations and a version of the Robinson–Schensted correspondence for $\mathbb{YF}$.

Contents

1. Introduction

1.1. **Overview.** Branching graphs — particularly the Young lattice $\mathbb{Y}$ of integer partitions — have long held a central position at the crossroads of representation theory, combinatorics, and probability. Indeed, the Young lattice powers the representation theory of symmetric and (to some extent) general linear groups, giving rise to Schur functions and driving profound connections to random matrix theory and statistical mechanics. In this landscape, the *Plancherel measure* on $\mathbb{Y}$ and its generalization, the *Schur measures*, have emerged as foundational objects. These probability measures (in the terminology of statistical mechanics, *ensembles of random partitions*) serve as a framework for exploring phenomena such as limit shapes, random tilings, and universal distributions governing eigenvalues in random matrix ensembles.

Despite the prominence of the Young lattice, kindred combinatorial structures remain relatively underexplored from a probabilistic perspective. One notable example is the *Young–Fibonacci lattice* $\mathbb{YF}$ [Fom88], [Sta88], [Oka94], [GK00b]. Like the Young lattice, $\mathbb{YF}$ possesses a 1-differential poset structure defined, not on integer partitions, but rather binary words composed of the symbols 1 and 2 (known as *Fibonacci words*). As a 1-differential poset, $\mathbb{YF}$ carries a *Plancherel measure* on Fibonacci words. The Young–Fibonacci lattice also mirrors other structures found in the classical Young lattice, such as versions of the Robinson-Schensted (RS) correspondence, multiparameter analogues of Schur functions, and a representation-theoretic framework introduced by Okada [Oka94]. At the same time, $\mathbb{YF}$ exhibits novel combinatorial and probabilistic behaviors, which are the main focus of the present work.

Our starting point is the theory of *biserial clone Schur functions* [Oka94], a family of functions $s_w(\vec{x}\,|\,\vec{y})$ indexed by Fibonacci words $w \in \mathbb{YF}$ and involving two sequences of parameters $\vec{x} = (x_1, x_2, x_3, \ldots)$ and $\vec{y} = (y_1, y_2, y_3, \ldots)$. Clone Schur functions, first introduced by Okada, parallel the classical Schur functions $s_\lambda(\vec{z})$, $\lambda \in \mathbb{Y}$, $\vec{z} = (z_1, z_2, z_3, \ldots)$, in several aspects. Both Schur and biserial clone Schur functions branch according to a *Pieri rule* and, more generally, obey a *Littlewood-Richardson rule* which reflects the structure of covering relations in the $\mathbb{Y}$ and $\mathbb{YF}$ lattices. As a consequence, both Schur and biserial clone Schur functions give rise to *harmonic* functions on $\mathbb{Y}$ and $\mathbb{YF}$, defined respectively by

$$\lambda \mapsto \frac{s_\lambda(\vec{z})}{s_\square^n(\vec{z})}, \qquad w \mapsto \frac{s_w(\vec{x}\,|\,\vec{y})}{x_1 \cdots x_n},$$

where n is the *rank* of $\lambda \in \mathbb{Y}$ or $w \in \mathbb{YF}$. The Plancherel harmonic function (associated with the Plancherel measure) for each lattice arises from a special choice of parameters in Schur or clone Schur functions, respectively.

Positive specializations of classical Schur functions (sequences $\vec{z}$ for which $s_\lambda(\vec{z})$ is positive for all $\lambda \in \mathbb{Y}$) are central in the study of the Young lattice. They are related to total positivity [AESW51], [ASW52], [Edr52], [Edr53], characters of the infinite symmetric group [Tho64], and asymptotic theory of characters of symmetric groups of increasing order [VK81].

One important distinction with the $\mathbb{YF}$-lattice is that the harmonic functions on $\mathbb{Y}$ associated with positive specializations of classical Schur functions are *extremal* (this property is also often called *ergodic*, or *minimal*). Extremality here means that the functions $\lambda \mapsto s_\lambda(\vec{z})/s_\square^n(\vec{z})$ cannot be expressed as nontrivial convex combinations of other nonnegative harmonic functions. In contrast, this extremality property *does not generally hold* for harmonic functions arising from Fibonacci positive specializations of biserial clone Schur functions. One of the initial motivations for our work was to investigate the broad question of how clone harmonic functions on $\mathbb{YF}$ decompose into extremal components. The classification of extremal harmonic functions on $\mathbb{YF}$ was established in [GK00b], and results on the boundary of $\mathbb{YF}$ were strengthened in the preprints [BE20], [Evt20].

Our first goal is to investigate conditions for which a specialization $(\vec{x}, \vec{y})$ is *Fibonacci positive* — in the sense that the biserial clone Schur functions $s_w(\vec{x}\,|\,\vec{y})$ are strictly postive for all Fibonacci words $w \in \mathbb{YF}$. Subsequently, we explore probabilistic and combinatorial properties of Fibonacci positive specializations and related ensembles of random Fibonacci words. The present work extends and completes several well-studied classical topics associated with the Young lattice $\mathbb{Y}$ and Schur functions, adapting them to the Young–Fibonacci lattice $\mathbb{YF}$ and clone Schur functions. Our contributions can be summarized as follows:

1. Characterization of Fibonacci positivity. We establish a complete classification of specializations $(\vec{x}, \vec{y})$ for which the clone Schur functions $s_w(\vec{x} \mid \vec{y})$ are positive for all $w \in \mathbb{YF}$. The concept of Fibonacci positivity strengthens the notion of total positivity of tridiagonal matrices whose subdiagonal consist entirely of 1's. We identify two classes of Fibonacci positive specializations, of *divergent* and *convergent* type.

2. Stieltjes moment sequences and orthogonal polynomials. The connection to tridiagonal matrices places the Fibonacci positivity problem into the context of the Stieltjes moment problem and Jacobi continued fractions. For many examples of Fibonacci positive specializations, the Borel measure on $[0, \infty)$ coming from the corresponding Stieltjes moment problem is related (via a change of variables) to an orthogonality measure for a family of orthogonal polynomials in one variable from the (q-)Askey scheme [KS96]. This includes the Charlier, Type-I Al-Salam–Carlitz, Al-Salam–Chihara, and q-Charlier polynomials. In the latter two examples, the Fibonacci positivity enforces the atypical condition $q > 1$, in contrast to the usual assumption $|q| < 1$ in the q-Askey scheme.

3. Asymptotics of random Fibonacci words. We investigate the behavior of random Fibonacci words, (originating from various Fibonacci positive specializations) in the limit as the word length grows. We find examples when the growing random words exhibit one of the following patterns:
- $w = 1^{r_1}21^{r_2}2\ldots$, where r_i scale proportionally to the word length;
- $w = 2^{h_1}12^{h_2}1\ldots$, where h_i scale proportionally to the word length;
- $w = 1^{\infty}v$, that is, the word has a single growing prefix of 1's, followed by a finite (random) Fibonacci word $v \in \mathbb{YF}$.

In the first two cases, the joint scaling limit of either $(r_1, r_2, \ldots)$ or $(h_1, h_2, \ldots)$ displays a "stick-breaking"-type behavior. In this way, we extend the result of [GK00a] showing that in a scaling limit, the sequence $(h_1, h_2, \ldots)$ corresponding to the Plancherel measure on $\mathbb{YF}$ converges to the GEM distribution with parameter $\theta = \frac{1}{2}$. We observe different stick-breaking processes, including the ones with *dependent* stick-breaking steps.

For random Fibonacci words almost surely behaving as $w = 1^{\infty}v$, we determine the probability law of the random finite word v in the limit in terms of the parameters $(\vec{x}, \vec{y})$ of the Fibonacci positive specialization. This distribution on the v's is the desired decomposition of the clone harmonic function $w \mapsto s_w(\vec{x} \mid \vec{y})/(x_1 \cdots x_{|w|})$ into the extremal components.

4. Clone Cauchy identities and random permutations. We establish *clone Cauchy identities* which are summation identities involving clone Schur functions, in parallel to the celebrated Cauchy identities for Schur functions. In the Young–Fibonacci setting, the right hand side of each Cauchy identity is expressed by a quadridiagonal determinant (and not a product, like for classical Schur functions). We employ clone Cauchy identities to study models of random permutations coming from random Fibonacci words and a Robinson–Schensted correspondence for the $\mathbb{YF}$-lattice introduced in [Nze09]. In particular, we compute the moment generating function for the number of two-cycles in a certain ensemble of random involutions, and explore its asymptotic behavior under a specific Fibonacci positive specialization. Other specializations may lead to interesting models of random permutations with pattern avoidance properties.

In the remainder of the Introduction, we formulate our main results in more detail. Further discussion of possible extensions and open problems is postponed to the last Section 8.

1.2. **Clone Schur functions and Fibonacci positivity.** A Fibonacci word w is a binary word composed of the symbols 1 and 2. Its weight is the sum of the symbols. For example, $|12112| = 7$. By $\mathbb{YF}_n$ we denote the set of Fibonacci words of weight n. The lattice structure on $\mathbb{YF}$ is defined through branching (covering) relations $w \nearrow w'$ on pairs of Fibonacci words, where $|w'| = |w| + 1$. This relation is recursively defined to hold if and only if either $w' = 1w$, or $w' = 2v$ with $v \nearrow w$. The base case is given by $\varnothing \nearrow 1$. Let $\dim(w)$ denote the number of saturated chains $\varnothing = w_0 \nearrow w_1 \nearrow \cdots \nearrow w_n = w$ in the Young–Fibonacci lattice starting at $\varnothing$ and ending at w. See Figure 1 for an illustration of the $\mathbb{YF}$ up to level $n = 5$. A function φ on $\mathbb{YF}$ is called harmonic if $\varphi(w) = \sum_{w'\colon w \nearrow w'} \varphi(w')$ for all $w \in \mathbb{YF}$.

Let $\vec{x} = (x_1, x_2, x_3, \ldots)$ and $\vec{y} = (y_1, y_2, y_3, \ldots)$ be two sequences of parameters. Define the $\ell \times \ell$ tridiagonal determinants by

$$A_\ell(\vec{x} \,|\, \vec{y}) \coloneqq \det \underbrace{\begin{pmatrix} x_1 & y_1 & 0 & 0 & \cdots \\ 1 & x_2 & y_2 & 0 & \\ 0 & 1 & x_3 & y_3 & \\ \vdots & & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}},$$

$$B_{\ell-1}(\vec{x} + r \,|\, \vec{y} + r) \coloneqq \det \underbrace{\begin{pmatrix} y_{r+1} & x_{r+1}y_{r+2} & 0 & 0 & \cdots \\ 1 & x_{r+3} & y_{r+3} & 0 & \\ 0 & 1 & x_{r+4} & y_{r+4} & \\ \vdots & & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}},$$

where nonzero elements in all rows in A_ℓ and all rows in $B_{\ell-1}$ except for the first one follow the pattern $(1, x_j, y_j)$. Here and throughout the paper, $\vec{x} + r$ and $\vec{y} + r$ denote the sequences with indices shifted by $r \in \mathbb{Z}_{\geq 0}$.

The clone Schur function $s_w(\vec{x} \,|\, \vec{y})$ is defined by the following recurrence:

$$s_w(\vec{x} \,|\, \vec{y}) \coloneqq \begin{cases} A_k(\vec{x} \,|\, \vec{y}), & \text{if } w = 1^k \text{ for some } k \geq 0, \\ B_k(\vec{x} + r \,|\, \vec{y} + r) \cdot s_u(\vec{x} \,|\, \vec{y}), & \text{if } w = 1^k 2u \text{ for some } k \geq 0 \text{ and } |u| = r. \end{cases}$$

The function

$$\varphi_{\vec{x},\vec{y}}(w) \coloneqq \frac{s_w(\vec{x} \,|\, \vec{y})}{x_1 x_2 \cdots x_{|w|}}$$

is harmonic on $\mathbb{YF}$. It is normalized so that $\varphi_{\vec{x},\vec{y}}(\varnothing) = 1$.

Our first main result is a complete characterization of the *Fibonacci positive* sequences $(\vec{x}, \vec{y})$ for which the clone Schur functions $s_w(\vec{x} \,|\, \vec{y})$ are positive for all $w \in \mathbb{YF}$:

Theorem 1.1 (Theorem 3.9)**.** *All Fibonacci positive sequences $(\vec{x}, \vec{y})$ have the form*

$$x_k = c_k(1 + t_{k-1}), \qquad y_k = c_k c_{k+1} t_k, \qquad k \geq 1,$$

where $\vec{c}$ is an arbitrary positive sequence, and $\vec{t} = (t_1, t_2, \ldots)$ (with $t_0 = 0$, for convenience) is a positive real sequence of one of the two types:

- *(divergent type) The infinite series*
$$1 + t_1 + t_1t_2 + t_1t_2t_3 + \dots \tag{1.1}$$
diverges, and $t_{m+1} \geq 1 + t_m$ *for all* $m \geq 1$*;*
- *(convergent type) The series* (1.1) *converges, and*
$$1 + t_{m+3} + t_{m+3}t_{m+4} + t_{m+3}t_{m+4}t_{m+5} + \dots \geq \frac{t_{m+1}}{t_{m+2}(1 + t_m - t_{m+1})}, \qquad \text{for all } m \geq 0.$$

The sequences $\vec{c}$ *and* $\vec{t}$ *are determined by* $(\vec{x}, \vec{y})$ *uniquely.*

The distinguished Plancherel harmonic function
$$\varphi_{\mathrm{PL}}(w) = \frac{\dim(w)}{n!}, \qquad w \in \mathbb{YF}_n,$$
is obtained from clone Schur functions by setting $x_k = y_k = k$, $k \geq 1$. Throughout the paper we are primarily concerned with two deformations of the Plancherel specialization, both of divergent type: the *shifted Plancherel* specialization $x_k = y_k = k + \sigma - 1$, $\sigma \in [1, \infty)$, and the *Charlier* specialization $x_k = k + \rho - 1$, $y_k = k\rho$, $\rho \in (0, 1]$. In Section 3.4, we describe other examples of Fibonacci positive specializations, both of divergent and convergent type.

1.3. **Stieltjes moment sequences and orthogonal polynomials.** As a corollary of the Fibonacci positivity of a specialization $(\vec{x}, \vec{y})$, we see that the infinite tridiagonal matrix with the diagonals $(1, 1, \dots)$, $(x_1, x_2, \dots)$, and $(y_1, y_2, \dots)$ is totally positive, that is, all its minors which are not identically zero are positive. It is known from [Fla80], [Vie83], [Sok20], [PSZ23] that totally positive tridiagonal matrices correspond to *Stieltjes moment sequences* $a_n = \int t^n \nu(dt)$, $n \geq 0$, where ν is a Borel measure on $[0, \infty)$. Moreover, the monic polynomials $P_n(t)$, $n \geq 0$, orthogonal with respect to ν can be determined directly in terms of the parameters $(\vec{x}, \vec{y})$:
$$P_{n+1}(t) = (t - x_{n+1})P_n(t) - y_nP_{n-1}(t), \quad n \geq 1, \qquad P_0(t) = 1, \quad P_1(t) = t - x_1.$$
We refer to Section 4.1 for a detailed discussion of the connection between total positivity of tridiagonal matrices and Stieltjes moment sequences.

According to the (q-)Askey nomenclature [KS96], in Section 4.2 we find several Fibonacci positive specializations whose orthogonal polynomials are (up to a change of variables and parameters):

- Charlier polynomials;
- Type-I Al-Salam–Carlitz polynomials;
- Al-Salam–Chihara polynomials;
- q-Charlier polynomials.

In these cases, we also explicitly determine the orthogonality measures ν. For example, in the Charlier case, the orthogonality measure is simply the Poisson distribution with the parameter ρ. For the Al-Salam–Chihara and q-Charlier polynomials, the Fibonacci positivity enforces the atypical condition $q > 1$, in contrast to the usual assumption $|q| < 1$ in the q-Askey scheme. Our fifth example, the shifted Plancherel specialization $x_k = y_k = k + \sigma - 1$, corresponds to the so-called associated Charlier polynomials [ILV88], [Ahb23]. The orthogonality measure ν in this case is not explicit, but we find its moment generating function (Proposition 4.8).

In each of the five examples, we also list a combinatorial interpretation of the Stieltjes moment sequence a_n itself. For example, in the Charlier case, a_n is known as the Bell (also called Touchard) polynomial in ρ:
$$a_n = B_n(\rho) \coloneqq \sum\nolimits_{\pi \in \Pi(n)} \rho^{\#\mathrm{blocks}(\pi)},$$

where the sum is over all set partitions π of $[n]$. In other examples, a_n is also expressed as a sum over set partitions, weighted by other statistics. There is a rich literature on such combinatorial interpretations [WW91], [Zen95], [Ans05], [KSZ06], [KZ06], [Jos11]. Most of the combinatorial interpretations in Section 4.2 essentially follow from these references.

While we have a complete description of Fibonacci positive specializations $(\vec{x}, \vec{y})$, explicitly describing the set of corresponding Borel measures ν within all Borel measures on $[0, \infty)$ remains an open problem.

1.4. **Asymptotics of random Fibonacci words.** We investigate asymptotic behavior of growing random Fibonacci words distributed according to clone coherent probability measures M_n on $\mathbb{YF}_n$:

$$M_n(w) := \dim(w)\, \varphi_{\vec{x},\vec{y}}(w) = \dim(w)\, \frac{s_w(\vec{x}\,|\,\vec{y})}{x_1 x_2 \cdots x_n}, \qquad w \in \mathbb{YF}_n.$$

The measures M_n are called *coherent* since they are compatible for varying n; see (2.5).

In Sections 5.3 and 5.4, we prove two limit theorems concerning the asymptotic behavior of random Fibonacci words under the Charlier and the shifted Plancherel specializations. For the Charlier specialization $x_k = k+\rho-1$, $y_k = k\rho$, we decompose the random word as $w = 1^{r_1}21^{r_2}2\ldots$.

Theorem 1.2 (Theorem 5.7)**.** *Fix $\rho \in (0, 1)$. Let $w \in \mathbb{YF}_n$ be distributed according to the Charlier clone coherent measure M_n. Then for each fixed $k \geq 1$, the joint distribution of runs $(r_1(w), \ldots, r_k(w))$ converges to*

$$\frac{r_j(w)}{n - \sum_{i=1}^{j-1} r_i(w)} \xrightarrow[n\to\infty]{d} \eta_{\rho;j}, \qquad j = 1, \ldots, k,$$

where $\eta_{\rho;1}, \eta_{\rho;2}, \ldots$ are i.i.d. copies of a random variable with the distribution

$$\rho\,\delta_0(\alpha) + (1-\rho)\,\rho(1-\alpha)^{\rho-1}\,d\alpha, \qquad \alpha \in [0, 1].$$

This distribution is a convex combination of the point mass at 0 and the Beta random variable $\mathrm{beta}(1, \rho)$*, with weights ρ and $1-\rho$.*

Equivalently, we have $\{r_j/n\}_{j\geq 1} \to X_j$, where $X_1 = U_1$ and $X_n = (1-U_1)\cdots(1-U_{n-1})U_n$ for $n \geq 2$, where $U_j = \eta_{\rho;j}$ are i.i.d. The representation of the vector $(X_1, X_2, \ldots)$ through the variables U_j is called a *stick-breaking* process.

Note that if U_j have the distribution $\mathrm{beta}(1, \theta)$, then the distribution of the vector $(X_1, X_2, \ldots)$ is called the Griffiths–Engen–McCloskey distribution $\mathrm{GEM}(\theta)$. We refer to [JKB97, Chapter 41] for further discussion and applications of GEM distributions.

We see that the runs of 1's under the Charlier specialization scale to the $\mathrm{GEM}(\rho)$ vector with additional zero entries inserted independently with density $1-\rho$.

For the shifted Plancherel specialization $x_k = y_k = k + \sigma - 1$, we decompose the random word as $w = 2^{h_1}12^{h_2}1\ldots$. Denote $\tilde{h}_j = 2h_j + 1$.

Theorem 1.3 (Theorem 5.13)**.** *Fix $\sigma \geq 1$. Under the shifted Plancherel clone coherent measure M_n, we have for the joint distribution $(\tilde{h}_1(w), \ldots, \tilde{h}_k(w))$ for each fixed $k \geq 1$:*

$$\frac{\tilde{h}_j(w)}{n - \sum_{i=1}^{j-1} \tilde{h}_i(w)} \xrightarrow[n\to\infty]{d} \xi_{\sigma;j}, \qquad j = 1, \ldots, k.$$

The joint distribution of $(\xi_{\sigma;1}, \xi_{\sigma;2}, \ldots)$ can be described as follows. Toss a sequence of independent coins with probabilities of success $1, \sigma^{-1}, \sigma^{-2}, \ldots$. Let N be the (random) number of successes *until the first failure. Then, sample N independent* beta$(1, \sigma/2)$ *random variables. Set $\xi_{\sigma;k}$, $k = 1, \ldots, N$, to be these random variables, while $\xi_{\sigma;k} = 0$ for $k > N$.*

When $\sigma > 1$, the random variables $\xi_{\sigma;k}$ are not independent, but $\xi_{\sigma;1}, \ldots, \xi_{\sigma;n}$ are conditionally independent given $N = n$. Almost surely, the sequence $\xi_{\sigma;1}, \xi_{\sigma;2}, \ldots$ contains only finitely many nonzero terms.

At $\sigma = 1$ (Plancherel measure), we have $N = \infty$ almost surely, so the random variables $\xi_{1;k}$ are i.i.d. beta$(1, \sigma/2)$. Thus, we recover the convergence to GEM(1/2) obtained in [GK00a].

Theorems 1.2 and 1.3 follow from product-like formulas for the joint distributions of $r_j(w)$ and $h_j(w)$, respectively. The product-like formulas are valid for arbitrary Fibonacci positive specializations, but they greatly simplify in the Charlier and shifted Plancherel cases.

Consider now specializations of convergent type, with an additional condition that the infinite product $\prod_{i=1}^{\infty}(1 + t_i)$ converges to a finite value.

Theorem 1.4 (Propositions 5.18 and 5.19)**.** *With the above assumption on the sequence $\vec{t}$ of convergent type, the random word $\mathbf{w} \in \mathbb{YF}_n$ under the corresponding clone coherent measure behaves in the limit as $n \to \infty$ as*

- *either $\mathbf{w} \to 1^{\infty}$,*
- *or $\mathbf{w} \to 1^{\infty}2\mathbf{v}$, where $\mathbf{v} \in \mathbb{YF}$ is a finite random Fibonacci word.*

The convergence of $\mathbf{w}$ is stabilization on the discrete set. The distribution μ_{I} of the limiting word belonging to $1^{\infty}\mathbb{YF} := \{1^{\infty}\} \cup \{1^{\infty}2u \colon u \in \mathbb{YF}\}$ is given by

$$\mu_{\mathrm{I}}\left(1^{\infty}\right) = \prod_{i=1}^{\infty}(1 + t_i)^{-1}, \qquad \mu_{\mathrm{I}}\left(1^{\infty}2u\right) = \left(\prod_{i=1}^{|u|-1}(1 + t_i)\right)\left(|u| + 1\right) M_{|u|}(u)\, \frac{B_{\infty}(|u|)}{\prod_{i=1}^{\infty}(1 + t_i)},$$

where $u \in \mathbb{YF}$ is arbitrary, and $B_{\infty}(m)$, $m \geq 0$, is an infinite series defined below in (3.6). Moreover, μ_{I} is a probability measure on $1^{\infty}\mathbb{YF}$.

In Corollary 5.20, we obtain the following decomposition of the clone harmonic function $\varphi_{\vec{x},\vec{y}}$ for specializations of convergent type satisfying $\prod_{i=1}^{\infty}(1 + t_i) < \infty$:

$$\varphi_{\vec{x},\vec{y}} = \mu_{\mathrm{I}}(1^{\infty})\, \Phi_{1^{\infty}} + \sum_{u \in \mathbb{YF}} \mu_{\mathrm{I}}(1^{\infty}2u)\, \Phi_{1^{\infty}2u}, \quad \Phi_{1^{\infty}2u}(w) := \begin{cases} \dfrac{\dim(w, 1^k 2u)}{\dim(2u)}, & \text{if } w \trianglelefteq 1^k u,\ k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\trianglelefteq$ denotes the partial order on $\mathbb{YF}$ (induced from the branching relation). The functions $\Phi_{1^{\infty}}$ and $\Phi_{1^{\infty}2u}$ for $v \in \mathbb{YF}$ are called *Type-I harmonic functions*, and they are extremal.

1.5. **Clone Cauchy identities and random permutations.** In Section 6, we derive *clone Cauchy identities* generalizing the classical Cauchy-type summation formulas for the usual Schur functions. Two identities are presented in Propositions 6.8 and 6.9, with the second being

$$\sum_{|w|=n} s_w(\vec{p}\,|\,\vec{q})\, s_w(\vec{x}\,|\,\vec{y}) = \det \underbrace{\begin{pmatrix} \mathrm{A}_1 & \mathrm{B}_1 & \mathrm{C}_1 & 0 & \cdots \\ 1 & \mathrm{A}_2 & \mathrm{B}_2 & \mathrm{C}_2 & \\ 0 & 1 & \mathrm{A}_3 & \mathrm{B}_3 & \\ 0 & 0 & 1 & \mathrm{A}_4 & \\ \vdots & & & & \ddots \end{pmatrix}}_{n \times n \text{ quadridiagonal matrix}}, \tag{1.2}$$

where $\mathrm{A}_k = p_k x_k$, $\mathrm{B}_k = q_k(x_k x_{k+1} - y_k) + y_k(p_k p_{k+1} - q_k)$, $\mathrm{C}_k = p_k x_k q_{k+1} y_{k+1}$.

The identity in (1.2) can be used to define clone analogues of *Schur measures*, extending the framework from harmonic functions on $\mathbb{YF}$. Indeed, when one of the specializations in (1.2) is Plancherel, $p_k = q_k = k$, identity (1.2) reduces to the normalizing identity for the clone harmonic function $\varphi_{\vec{x},\vec{y}}$. For the Young lattice, Schur measures were introduced in [Oko01] and generalized to Schur processes (measures on sequences of partitions) in [OR03]. They found extensive applications in random matrices, interacting particle systems, random discrete structures like tilings, geometry, and other areas [OP06], [ORV06], [BF14], [BG16], [BP14], [CH14]. We leave clone analogues of Schur measures and processes for future work.

In Section 7, we introduce ensembles of random permutations and involutions by utilizing the Young–Fibonacci RS correspondence [Nze09] and positive harmonic functions on $\mathbb{YF}$. In full generality, the distribution of a permutation or involution depends, respectively, on a triplet (π, φ, ψ) or a couple (π, φ) of harmonic functions. We do not treat the general case in the present work, but focus on the clone harmonic / Plancherel random involutions, that is, corresponding to setting $\pi = \varphi_{\vec{x},\vec{y}}$ and $\varphi = \varphi_{\text{PL}}$, where $(\vec{x}, \vec{y})$ is a Fibonacci positive specialization. Using clone Cauchy identities, we find the moment generating function for the number of two-cycles in a random involution $\boldsymbol{\sigma} \in \mathfrak{S}_n$ (Proposition 7.5):

$$\mathbb{E}\big[\tau^{\#\,\text{two-cycles}(\boldsymbol{\sigma})}\big] = (x_1 \cdots x_n)^{-1} \det \underbrace{\begin{pmatrix} x_1 & (1-\tau)y_1 & -\tau x_1 y_2 & 0 & \cdots \\ 1 & x_2 & (1-2\tau)y_2 & -2\tau x_2 y_3 & \\ 0 & 1 & x_3 & (1-3\tau)y_3 & \\ 0 & 0 & 1 & x_4 & \\ \vdots & & & & \ddots \end{pmatrix}}_{n\times n \ \text{quadridiagonal matrix}},$$

where τ is an auxiliary parameter.

When $(\vec{x}, \vec{y})$ is the shifted Plancherel specialization ($x_k = y_k = k + \sigma - 1$, $\sigma \in [1, \infty)$), the Young–Fibonacci shape $w \in \mathbb{YF}_n$ of a random involution $\boldsymbol{\sigma} \in \mathfrak{S}_n$ under the RS correspondence has the same distribution as a random Fibonacci word considered in Theorem 1.3 above. In this way, we can compare the asymptotic behavior of the total number of 2's in a random Fibonacci word (which is the same as the number of two-cycles), and the scaling limit of initial long sequences of 2's from Theorem 1.3. We establish a law of large numbers (Proposition 7.10) for the total number of 2's:

$$\lim_{n\to\infty} \frac{\#\text{two-cycles}(\boldsymbol{\sigma})}{n} = \frac{1}{\sigma + 1}. \tag{1.3}$$

For $\sigma > 1$, this value exceeds the expectation of the sum of the scaled quantities h_j in Theorem 1.3. This discrepancy reveals that additional digits of 2 remain hidden in the growing random Fibonacci word after long sequences of 1's. This behavior is unaccounted for in the scaling limit of Theorem 1.3 but contributes to the law of large numbers (1.3).

Outline of the paper. Section 2 provides the necessary background on the Young–Fibonacci lattice $\mathbb{YF}$ and clone Schur functions, introducing harmonic functions and coherent measures on the Young–Fibonacci lattice arising from specializations of clone Schur functions. In Sections 3 and 4, we define and characterize Fibonacci positivity and relate it to total positivity, Stieltjes moment sequences, and orthogonal polynomials. Section 5 examines the asymptotic behavior of coherent measures derived from various Fibonacci positive specializations. We analyze several limiting regimes for specific examples, revealing distributions resembling stick-breaking processes (associated with GEM distributions), dependent stick-breaking processes, or discrete limits tied to the Martin boundary of the Young–Fibonacci lattice. Section 6 discusses clone Cauchy identities, focusing on summation identities involving products of clone Schur functions. In Section 7, we investigate how clone Schur functions and a variant of the Robinson–Schensted correspondence for $\mathbb{YF}$ can define models of random permutations and involutions. Utilizing clone Cauchy identities, we compute the moment generating function for the number of two-cycles in a random involution (under a particular specialization), and explore its asymptotic behavior. Finally, Section 8 outlines several open problems, including combinatorial ergodicity, truncated Young–Fibonacci lattices, further inquiries into the asymptotics of coherent measures and random permutation models, and connections to nonsymmetric and quasisymmetric functions.

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2. Background. Young–Fibonacci lattice and clone Schur functions

In this background section we review the Young–Fibonacci lattice $\mathbb{YF}$ (which is often referred to as the Young–Fibonacci branching graph) [Fom88], [Sta88], [GK00b] and clone Schur functions introduced in [Oka94]. The (biserial) clone Schur functions are harmonic on $\mathbb{YF}$ and we use them to define coherent probability measures on Fibonacci words.

2.1. **Young–Fibonacci lattice and harmonic functions.** A *Fibonacci word* $w = w_1 \dots w_\ell$ is any binary word with letters $w_j \in \{1, 2\}$. The integer $|w| := w_1 + \ldots + w_\ell = n$ is called the *weight* of the word w. The total number of Fibonacci words of weight n is equal to the n-th

Fibonacci number,[1] hence the name. Denote the set of all Fibonacci words of weight n by $\mathbb{YF}_n$, where $n \geq 0$.

Definition 2.1. The *Young–Fibonacci lattice* $\mathbb{YF}$ is the union of all sets $\mathbb{YF}_n$, $n \geq 0$. In this lattice, $w \in \mathbb{YF}_n$ is connected to $w' \in \mathbb{YF}_{n+1}$ if and only if w' can be obtained from w by one of the following three operations:

F1. $w' = 1w$.
F2. $w' = 2^{k+1}v$ if $w = 2^k 1 v$ for some $k \geq 0$ and an arbitrary Fibonacci word v.
F3. $w' = 2^\ell 1 2^{k-\ell} v$ if $w = 2^k v$ for some $k \geq 1$ and an arbitrary Fibonacci word v. While **F1** and **F2** each generate at most one edge, this rule generates k edges indexed by $\ell = 1, \ldots, k$.

We denote this relation by $w \nearrow w'$ (equivalently, $w' \searrow w$). An example of the Young–Fibonacci lattice up to level $n = 5$ is given in Figure 1.

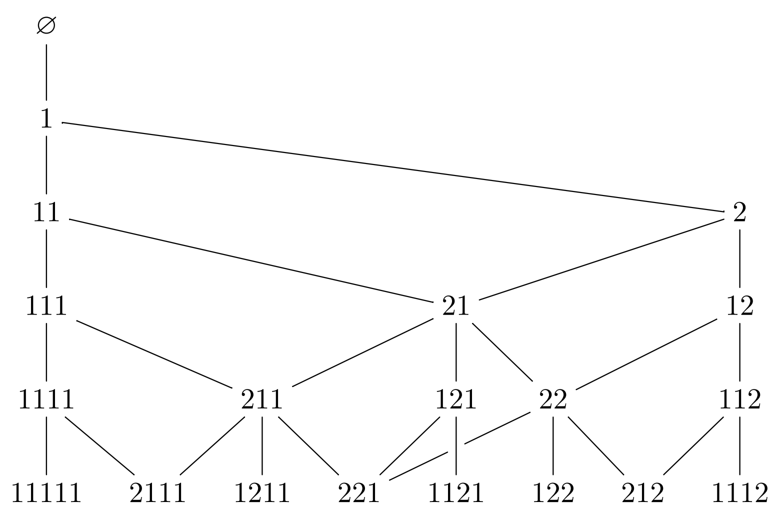


Figure 1. The Young–Fibonacci lattice up to level $n = 5$.

Definition 2.2. A function φ on $\mathbb{YF}$ is called *harmonic* if it satisfies

$$\varphi(w) = \sum_{w' \colon w' \searrow w} \varphi(w') \qquad \text{for all } w \in \mathbb{YF}.$$

A harmonic function is called *normalized* if $\varphi(\varnothing) = 1$.

For $w \in \mathbb{YF}$, denote by $\dim(w)$ the number of oriented paths (also known as *saturated chains*) from $\varnothing$ to w in the Young–Fibonacci lattice. Let $I_2(w)$ be the sequence of all positions of the letter 2 in w, reading from left to right. Then

$$\dim(w) = \prod_{i \in I_2(w)} \mathsf{d}_i(w), \qquad \text{where } \mathsf{d}_i(w) = |v| + 1 \text{ if } w = u2v \text{ is the splitting of } w \text{ at position } i. \tag{2.1}$$

[1]With the convention that $F_0 = F_1 = 1$.

Equivalently, $\dim(w)$ obeys the following recursion:

$$\dim(w) = \begin{cases} 1, & \text{if } w = \varnothing; \\ \dim(v), & \text{if } w = 1v \text{ for a Fibonacci word } v; \\ (|v|+1)\dim(v), & \text{if } w = 2v \text{ for a Fibonacci word } v. \end{cases} \tag{2.2}$$

For example, if $w = 22121$, then $I_2(w) = (1,2,4)$, and $\dim w = 70$. Since $\mathbb{YF}$ is a 1-differential poset, we have [Sta88, Corollary 3.9], (see also [Fom94]):

$$\sum_{w\in\mathbb{YF}_n} \dim^2(w) = n!. \tag{2.3}$$

With any *nonnegative* normalized harmonic function we can associate a family of probability measures M_n on $\mathbb{YF}_n$ as follows:

$$M_n(w) := \dim(w)\cdot\varphi(w), \qquad w \in \mathbb{YF}_n. \tag{2.4}$$

The fact that $\sum_{w\in\mathbb{YF}_n} M_n(w) = 1$ follows from the normalization of φ, and the harmonicity of φ translates into the *coherence property* of the measures M_n:

$$M_n(w) = \sum_{w' \colon w'\searrow w} M_{n+1}(w')\,\frac{\dim(w)}{\dim(w')}, \qquad w \in \mathbb{YF}_n. \tag{2.5}$$

The set of all nonnegative normalized harmonic functions on $\mathbb{YF}$ forms a simplex $\Upsilon(\mathbb{YF})$. The set of extreme points of this simplex (the ones not expressible as a nontrivial convex combination of other points) is denoted by $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$. In general, $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$ is a subset of the *Martin boundary*, denoted by $\Upsilon_{\mathrm{Martin}}(\mathbb{YF})$. The latter consists of harmonic functions which can be obtained by finite rank approximation. The Martin boundary of the Young–Fibonacci lattice is described in [GK00b]. Recently, it was shown in the preprints [BE20], [Evt20] that the Martin boundary coincides with the set of extreme points $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$.

For any coherent family of measures M_n on $\mathbb{YF}_n$, $n = 0,1,2,\ldots$, there exists a unique probability measure μ on $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$ such that

$$M_n(w) = \int_{\Upsilon_{\mathrm{ext}}(\mathbb{YF})} \dim(w)\,\varphi_\omega(w)\,\mu(d\omega), \qquad w \in \mathbb{YF}_n. \tag{2.6}$$

Here φ_ω is the extremal harmonic function corresponding to $\omega \in \Upsilon_{\mathrm{ext}}(\mathbb{YF})$.

2.2. **Plancherel measure and its scaling limit.** An important example of a harmonic function on $\mathbb{YF}$ is the *Plancherel* function defined as

$$\varphi_{\mathrm{PL}}(w) := \frac{\dim(w)}{n!}, \qquad w \in \mathbb{YF}_n. \tag{2.7}$$

In [GK00a] it is shown that φ_{PL} belongs to $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$. Moreover, for the Plancherel measure $M_n(w) = \dim^2(w)/n!$ corresponding to φ_{PL} as in (2.4), [GK00a] establishes a $n \to +\infty$ scaling limit theorem for the positions of the 1's in the random Fibonacci word w which we now describe.

Represent $w \in \mathbb{YF}$ as a sequence of contiguous blocks of letters 2 separated by 1's. For example, $w = 122112 = (1)(221)(1)(2)$. Each block except possibly the rightmost one contains exactly one 1, which is its terminating letter. Denote by $\tilde h_1, \tilde h_2, \ldots$ the sequence of weights of the blocks, reading from left to right. For the example above, $\tilde h_1 = 1, \tilde h_2 = 5, \tilde h_3 = 1, \tilde h_4 = 2$, and $\tilde h_j = 0$ for $j \ge 5$. We have $\tilde h_1 + \tilde h_2 + \ldots = n$. We use the notation $\tilde h_k$ for consistency with the computations in Section 5.2 below.

Definition 2.3. The *GEM* (*Griffiths–Engen–McCloskey*) *distribution* with parameter $\theta > 0$ (denoted $\mathrm{GEM}(\theta)$) is a probability measure on the infinite-dimensional simplex

$$\Delta := \Big\{(x_1, x_2, \ldots) \colon x_j \ge 0, \quad \sum_{j=1}^{\infty} x_j \le 1\Big\} \tag{2.8}$$

obtained from the residual allocation model (also called the stick-breaking construction) as follows. By definition, a random point $X = (X_1, X_2, \ldots) \in \Delta$ under $\mathrm{GEM}(\theta)$ is distributed as

$$X_1 = U_1, \qquad X_n = (1 - U_1)(1 - U_2) \cdots (1 - U_{n-1})U_n, \quad n = 2, 3, \ldots,$$

where $U_1, U_2, \ldots$ are independent $\mathrm{beta}(1, \theta)$ random variables (i.e., with density $\theta(1 - u)^{\theta-1}$ on the unit segment $[0, 1]$). We refer to [JKB97, Chapter 41] for further discussion and applications of GEM distributions.

Theorem 5.1 in [GK00a] establishes the convergence in distribution as $n \to +\infty$:

$$\Big(\frac{\tilde h_1(w)}{n}, \frac{\tilde h_2(w)}{n}, \ldots\Big) \longrightarrow X = (X_1, X_2, \ldots), \qquad X \sim \mathrm{GEM}(1/2), \tag{2.9}$$

where $\tilde h_j(w)$ are the block sizes (described above) of the random Fibonacci word w distributed according to the Plancherel measure on $\mathbb{YF}_n$.

2.3. **Harmonic functions from clone Schur functions.** A rich family of non-extremal harmonic functions on $\mathbb{YF}$ comes from clone Schur functions [Oka94] which we now describe. Let $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ be two families of indeterminates. Define two sequences of tridiagonal determinants as follows:

$$A_\ell(\vec{x} \,|\, \vec{y}) := \det \underbrace{\begin{pmatrix} x_1 & y_1 & 0 & \cdots \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & \\ \vdots & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}}, \qquad B_{\ell-1}(\vec{x} \,|\, \vec{y}) := \det \underbrace{\begin{pmatrix} y_1 & x_1 y_2 & 0 & \cdots \\ 1 & x_3 & y_3 & \\ 0 & 1 & x_4 & \\ \vdots & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}}. \tag{2.10}$$

Here $\ell \ge 0$. For a sequence $\vec{u} = (u_1, u_2, \ldots)$, denote its *shift* by $\vec{u} + \ell = (u_{1+\ell}, u_{2+\ell}, \ldots)$, where $\ell \in \mathbb{Z}_{\ge 0}$.

Remark 2.4. When there is no risk of ambiguity, we'll abbreviate $A_\ell(\vec{x} \,|\, \vec{y})$ and $B_\ell(\vec{x} \,|\, \vec{y})$ as A_ℓ and B_ℓ, respectively. Moreover, we will use the shorthand notation $A_\ell(m) := A_\ell(\vec{x} + m \,|\, \vec{y} + m)$ and $B_{\ell-1}(m) := B_{\ell-1}(\vec{x} + m \,|\, \vec{y} + m)$ for the shifted determinants.

Definition 2.5. For any Fibonacci word w, define the (*biserial*) *clone Schur function* $s_w(\vec{x} \,|\, \vec{y})$ through the following recurrence:

$$s_w(\vec{x} \,|\, \vec{y}) := \begin{cases} A_k(\vec{x} \,|\, \vec{y}), & \text{if } w = 1^k \text{ for some } k \ge 0, \\ B_k\big(\vec{x} + |u| \,|\, \vec{y} + |u|\big) \cdot s_u(\vec{x} \,|\, \vec{y}), & \text{if } w = 1^k 2u \text{ for some } k \ge 0. \end{cases} \tag{2.11}$$

Note that these functions are not symmetric in the variables, and the order in the sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ is important. The clone Schur functions satisfy a $\mathbb{YF}$-version of the Littlewood–Richardson identity, whose simplest form is the following clone Pieri rule established in [Oka94]:

$$x_{|w|+1} \cdot s_w(\vec{x} \,|\, \vec{y}) \;=\; \sum_{w' \colon w' \searrow w} s_{w'}(\vec{x} \,|\, \vec{y}) \tag{2.12}$$

Let us briefly mention the background (developed in [Oka94]) behind the clone Schur functions. The biserial clone Schur functions $s_w(\vec{x}\,|\,\vec{y})$ arise as evaluations (depending on $\vec{x}$ and $\vec{y}$) of Okada's clone Schur functions $s_w(\mathbf{x}\,|\,\mathbf{y})$ which are noncommutative polynomials in the free algebra generated by two symbols $\mathbf{x}$ and $\mathbf{y}$. Both the clone and biserial clone Schur functions play a vital role vis-à-vie the representation theory of the Okada algebra(s): the multiplicative structure of the noncommutative clone Schur functions models the *induction product* for irreducible representations of Okada algebras, while the biserial clone Schur functions are matrix entries for the action of the generators in these representations. This amplifies the parallel with usual Schur functions where the Littlewood–Richardson rule for multiplying Schur functions describes the induction product of representations of the symmetric group.

To summarize, the usual Young lattice $\mathbb{Y}$ (or partitions ordered by inclusion) is simultaneously responsible for the branching of the representations of the symmetric groups S_n, and for the Pieri rule for Schur functions (the simplest of the Littlewood–Richardson rules). Similarly, the Young–Fibonacci lattice $\mathbb{YF}$ is simultaneously the branching lattice for Okada algebra representations, and is responsible for the clone Pieri rule (2.12) for the biserial clone Schur functions.

Let us proceed with a number of straightforward properties of the biserial clone Schur functions. For a complex-valued sequence $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \dots)$ one readily sees from Definition 2.5 that

$$s_w\big(\vec{\gamma}\cdot\vec{x}\;\big|\;\vec{\gamma}\cdot(\vec{\gamma}+1)\cdot\vec{y}\big) \;=\; \big(\gamma_1\cdots\gamma_{|w|}\big)\, s_w(\vec{x}\,|\,\vec{y}), \tag{2.13}$$

where $\vec{\gamma}\cdot\vec{x} = (\gamma_1 x_1,\, \gamma_2 x_2,\, \gamma_3 x_3,\, \dots)$ and $\vec{\gamma}+1 = (\gamma_2, \gamma_3, \gamma_4, \dots)$. In particular, the biserial clone Schur functions with the variables $(\vec{x}, \vec{y})$ scale as follows:

$$s_w(\gamma\vec{x}\,|\,\gamma^2\vec{y}) = \gamma^{|w|}\, s_w(\vec{x}\,|\,\vec{y}), \tag{2.14}$$

where $\gamma\vec{x}$ means that we multiplied all the variables x_i by $\gamma \in \mathbb{C}$, and similarly for $\gamma^2\vec{y}$.

Assume that $x_i \neq 0$ for all i, and define the following normalization:

$$\varphi_{\vec{x},\vec{y}}(w) \coloneqq \frac{s_w(\vec{x}\,|\,\vec{y})}{x_1\cdots x_{|w|}}, \qquad w \in \mathbb{YF}. \tag{2.15}$$

The formula 2.12 implies that these normalized clone Schur functions define a harmonic function on $\mathbb{YF}$ (see Definition 2.2):

Proposition 2.6 ([Oka94])**.** *Let the variables $\vec{x}$ and $\vec{y}$ be such that $x_i \neq 0$ for all i. Then*

$$\varphi_{\vec{x},\vec{y}}(w) = \sum_{w' \colon w' \searrow w} \varphi_{\vec{x},\vec{y}}(w') \qquad \text{for all } w \in \mathbb{YF}. \tag{2.16}$$

We call $\varphi_{\vec{x},\vec{y}}$ the *clone harmonic function*, and the corresponding coherent probability measures (2.4) the *clone measures*. At this point, we treat the measures as *formal* and do not require them to be nonnegative (just need their individual "probability" weights to sum to 1). We will discuss positivity of the weights in Section 3 below.

Example 2.7. For the particular choice $x_k = y_k = k$, $k \geq 1$, the clone harmonic function $\varphi_{\vec{x},\vec{y}}$ turns into the Plancherel harmonic function φ_{PL} (2.7). Indeed, this follows from

$$A_\ell(\vec{x}\,|\,\vec{y}) = 1, \qquad B_{\ell-1}(\vec{x}+r\,|\,\vec{y}+r) = r+1, \tag{2.17}$$

and so with these parameters we have $s_w(\vec{x}\,|\,\vec{y}) = \dim(w)$, see (2.1). Denote this choice of parameters by $\Pi = (\vec{x}\,|\,\vec{y})$ and call it the *Plancherel specialization*.

As Example 2.7 shows, clone harmonic functions can be nonnegative. In Section 3 below we characterize specializations $(\vec{x}, \vec{y})$ for which the corresponding clone harmonic function is positive on the whole $\mathbb{YF}$. We also present many new examples of positive clone harmonic functions.

3. Fibonacci positivity and examples of coherent measures

In this section, we characterize the specializations $(\vec{x}, \vec{y})$ under which the clone Schur functions $s_w(\vec{x}\,|\,\vec{y})$ are positive for all $w \in \mathbb{YF}$ (referred to as *Fibonacci positive specializations*). This proves Theorem 1.1 from the Introduction. Additionally, we develop many new examples of Fibonacci positive specializations.

3.1. Reduction to a single sequence parametrization.

Definition 3.1. A specialization $(\vec{x}, \vec{y})$, where $\vec{x} = (x_1, x_2, \ldots)$, $\vec{y} = (y_1, y_2, \ldots)$, and $x_i, y_j \in \mathbb{C}$, is called *Fibonacci nonnegative* if the clone Schur functions $s_w(\vec{x}\,|\,\vec{y})$ are nonnegative for all Fibonacci words $w \in \mathbb{YF}$. If $s_w(\vec{x}\,|\,\vec{y}) > 0$ for all $w \in \mathbb{YF}$, we say that $(\vec{x}, \vec{y})$ is *Fibonacci positive.*

One readily sees that Fibonacci positivity is equivalent to the positivity of the determinants $A_\ell(\vec{x}\,|\,\vec{y})$ and $B_\ell(\vec{x}+r\,|\,\vec{y}+r)$ for all $\ell, r \in \mathbb{Z}_{\ge 0}$. This, in turn, is equivalent to the *total positivity* of the following family of semi-infinite tridiagonal matrices,[2] where $r \ge 0$:

$$\mathcal{A}\left(\vec{x}\,\middle|\,\vec{y}\right) := \begin{pmatrix} x_1 & y_1 & 0 & \cdots \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & \\ \vdots & & & \ddots \end{pmatrix}, \qquad \mathcal{B}_r\left(\vec{x}\,\middle|\,\vec{y}\right) := \begin{pmatrix} y_{r+1} & x_{r+1}y_{r+2} & 0 & \cdots \\ 1 & x_{r+3} & y_{r+3} & \\ 0 & 1 & x_{r+4} & \\ \vdots & & & \ddots \end{pmatrix}. \tag{3.1}$$

Indeed, it is known (for example, see [FZ99]) that the total positivity of a tridiagonal matrix is equivalent to the positivity of its leading principal minors, namely those formed by several initial and consecutive rows and columns. The list of additional references on total positivity is vast, and we mention only a few sources here: [Edr53], [Kar68], [Sch88], [FZ00].

Since total positivity of $\mathcal{A}(\vec{x}\,|\,\vec{y})$ (3.1) is a necessary condition for a specialization $(\vec{x}, \vec{y})$ to be Fibonacci positive, we may restrict our attention to pairs of sequences $(\vec{x}, \vec{y})$ for which $\mathcal{A}(\vec{x}\,|\,\vec{y})$ is totally positive. Using a general factorizaton ansatz introduced in [FZ99] for elements in double Bruhat cells, we know that the matrix $\mathcal{A}(\vec{x}\,|\,\vec{y})$ is totally positive if and only if there exist auxiliary real parameters $c_k, d_k > 0$, $k \ge 1$, such that

$$x_k = c_k + d_{k-1} \quad \text{and} \quad y_k = c_k\, d_k \quad \text{for all } k \ge 1, \tag{3.2}$$

with the condition that $x_1 = c_1$. Moreover, $\{c_k\}, \{d_k\}$ are uniquely determined by $(\vec{x}, \vec{y})$.

Notice that formula (2.13) implies that for $(\vec{x}, \vec{y})$ depending on c_k, d_k as above, we have

$$s_w(\vec{x}\,|\,\vec{y}) = (c_1 \cdots c_n)\, s_w(\vec{u}\,|\,\vec{v}) \quad \text{for all } w \in \mathbb{YF}, \quad |w| = n, \tag{3.3}$$

where $u_k = 1 + d_{k-1}/c_k$ and $v_k = d_k/c_{k+1}$ for $k \ge 1$, with the agreement that $u_1 = 1$. Clearly, the positivity of the left- and right-hand sides of (3.3) for all $w \in \mathbb{YF}$ are equivalent to each other, and so the problem of characterizing Fibonacci positive specializations $(\vec{x}, \vec{y})$ can be reduced to the problem of identifying necessary and sufficient conditions under which the sequence

$$t_k := v_k = \frac{d_k}{c_{k+1}}, \qquad k \ge 1, \tag{3.4}$$

[2]We use the convention that a tridiagonal matrix is called *totally positive* provided that all its minors are strictly positive except those forced to vanish by the tridiagonal structure. In the literature, the phrase (*strictly*) *totally positive* is sometimes used for matrices all of whose minors are positive. See, e.g., the first footnote in [FJS17] for a comparison of terminology in references. In the present paper, however, we need to adapt the terminology to the tridiagonal structure of the matrix, and require that all minors which are not identically vanishing are strictly positive.

(with $t_0 = 0$ and $t_k > 0$ for all $k \geq 1$) makes the tridiagonal matrices $\mathcal{A}(\vec{u} \,|\, \vec{v})$ and $\mathcal{B}_r(\vec{u} \,|\, \vec{v})$ totally positive (for all $r \geq 1$). In the next Section 3.2, we will classify such sequences $\vec{t}$, which leads to a complete characterization of Fibonacci positivity.

In other words, note that the $\vec{t}$-sequences (where $t_k > 0$ for $k \geq 1$ and $t_0 = 0$) parametrize a fundamental domain

$$\mathcal{D} = \{(\vec{u}, \vec{v})\colon u_k = 1 + t_{k-1},\, v_k = t_k\}$$

within the overall set of totally positive (not necessarily Fibonacci positive) tridiagonal matrices. The fundamental domain is understood with respect to the action of the multiplicative group $\mathbb{R}^{\infty}_{>0}$ which rescales by the $\vec{c}$-parameters as in (3.2)–(3.3). Our goal in characterizing Fibonacci positive specializations is to identify the subset $\mathcal{D}^{\mathrm{Fib}} \subset \mathcal{D}$ which is also a fundamental domain for the set of all Fibonacci positive specializations under the action of $\mathbb{R}^{\infty}_{>0}$.

3.2. **Characterization of Fibonacci positivity.** From the discussion in Section 3.1 above, to address the question of Fibonacci positivity (Definition 3.1), it suffices to consider only the sequences $(\vec{u}, \vec{v})$ depending on a positive real sequence $\vec{t} = (t_1, t_2, t_3, \ldots)$ as

$$u_k = 1 + t_{k-1}, \qquad v_k = t_k, \qquad k \geq 1, \tag{3.5}$$

with the agreement that $t_0 = 0$. Let us define for all $m \geq 0$:

$$A_\infty(m) := 1 + \sum_{r=1}^{\infty} t_m t_{m+1} \cdots t_{m+r-1}, \qquad B_\infty(m) := t_{m+1} + (t_{m+1} - t_m - 1) t_{m+2} A_\infty(m+3). \tag{3.6}$$

Note that $A_\infty(m)$ and $B_\infty(m)$ are the respective expansions of $\det \mathcal{A}\big(\vec{u} + m \,\big|\, \vec{v} + m\big)$ and $\det \mathcal{B}_m\big(\vec{u} \,\big|\, \vec{v}\big)$ in the parameters t_k for $k \geq 1$ when treated as formal variables. Be aware that $A_\infty(0) = 1$.

Lemma 3.2. *The sum $A_\infty(m)$ (3.6) is convergent (resp., divergent) for some $m \geq 1$ if and only if it is convergent (resp., divergent) for all $m \geq 1$.*

Proof. We have

$$t_m^{-1} \sum_{r=1}^{K} t_m t_{m+1} \cdots t_{m+r-1} \;=\; 1 \;-\; t_{m+1} \cdots t_{m+K} \;+\; \sum_{r=1}^{K} t_{m+1} t_{m+2} \cdots t_{m+r}.$$

If the product $t_{m+1} \cdots t_{m+K}$ does not go to zero, then $A_\infty(m)$ diverges for all $m \geq 1$. Otherwise, we see that the partial sums of $A_\infty(m)$ and $A_\infty(m+1)$ diverge or converge simultaneously. □

Definition 3.3. We introduce two types of positive real sequences $\vec{t}$ based on the convergence of the $A_\infty(m)$'s:

1. A sequence $\vec{t}$ has *convergent type* if the series $A_\infty(m)$ is convergent and $B_\infty(m) \geq 0$ for all $m \geq 0$ (with the agreement that $t_0 = 0$).

2. A sequence $\vec{t}$ has *divergent type* if $t_{m+1} \geq 1 + t_m$ for all $m \geq 0$.
Note that for a sequence of divergent type, we have $t_m \geq m$, and so the series $A_\infty(m)$ diverge for all m.

We will also refer to the corresponding specialization $(\vec{u}, \vec{v})$ given by (3.5) as having convergent or divergent type.

We now present two general criteria for the Fibonacci positivity of the specialization $(\vec{u}, \vec{v})$ (determined by $\vec{t}$), based on the convergence or divergence of the series $A_\infty(m)$ (3.6).

Proposition 3.4. *Assume that the $A_\infty(m)$'s are convergent for some (all) $m \geq 1$. The specialization $(\vec{u}, \vec{v})$ (3.5) is Fibonacci positive if and only if $\vec{t}$ is a sequence of convergent type.*

Proof. Throughout the proof, we will use the notation of Remark 2.4. First, let $\vec{t}$ be a sequence of convergent type. One readily sees that $A_\ell(m) = 1 + t_m A_{\ell-1}(m+1)$ for all $\ell \geq 2$, so

$$A_\ell(m) = 1 + \sum_{r=1}^{\ell} t_m t_{m+1} \cdots t_{m+r-1} > 0, \qquad m \geq 1. \tag{3.7}$$

Note that the right-hand side of (3.7) is a partial sum of $A_\infty(m)$.

Next, let us consider the determinants $B_\ell(m)$. We have

$$\begin{aligned} B_0(m) &= t_{m+1} > 0, \\ B_1(m) &= t_{m+1} - (1 + t_m - t_{m+1}) t_{m+2} \end{aligned}$$

for all $m \geq 0$. If $1 + t_m - t_{m+1} \leq 0$, then this is already positive. Otherwise, we have

$$B_1(m) > B_\infty(m) = t_{m+1} - (1 + t_m - t_{m+1}) t_{m+2} A_\infty(m+3) \geq 0,$$

where the last inequality holds thanks to the convergent type assumption. The first strict inequality holds because the partial sums of $A_\infty(m+3)$ monotically increase to the infinite sum. Thus, $B_1(m) > 0$.

For larger determinants with $\ell \geq 3$, we have

$$\begin{aligned} B_{\ell-1}(m) &= v_m A_{\ell-1}(m+2) - u_m v_{m+1} A_{\ell-2}(m+3) \\ &= v_m \left(1 + t_{m+2} A_{\ell-2}(m+3)\right) - u_m v_{m+1} A_{\ell-2}(m+3) \\ &= t_{m+1} - (1 + t_m - t_{m+1}) t_{m+2} A_{\ell-2}(m+3). \end{aligned} \tag{3.8}$$

Similarly, if $1 + t_m - t_{m+1} \leq 0$, then this is already positive. Otherwise, we have $B_{\ell-1}(m) > B_\infty(m) \geq 0$, since $A_{\ell-2}(m+2) < A_\infty(m+3)$. The last nonnegativity again follows from the convergent type assumption. This implies that for a sequence $\vec{t}$ of convergent type, all clone Schur functions $s_w(\vec{u} \,|\, \vec{v})$ are positive.

Let us now consider the converse statement and assume that the specialization (3.5) is Fibonacci positive. The positivity of the t_k's implies that $A_\ell(m)$ is positive for all $\ell \geq 1$, $m \geq 0$, see (3.7). Assume that $\vec{t}$ is not of convergent type, that is, $t_{m_0+1} < (1 + t_{m_0} - t_{m_0+1}) t_{m_0+2} A_\infty(m_0+3)$ for some $m_0 \geq 0$ (this automatically implies that $1 + t_{m_0} - t_{m_0+1} > 0$). Since

$$A_\infty(m_0+3) = \lim_{\ell \to \infty} A_{\ell-2}(m_0+2),$$

there exists $\ell_0 \gg 1$ (depending on m_0) such that $t_{m_0+1} < (1 + t_{m_0} - t_{m_0+1}) t_{m_0+2} A_{\ell_0-2}(m_0+2)$. By (3.8), this shows that $B_{\ell_0-1}(m_0) < 0$, which violates the Fibonacci positivity. □

Proposition 3.5. *Assume that $A_\infty(m)$ is divergent for some (all) $m \geq 1$. The specialization $(\vec{u}, \vec{v})$ (3.5) is Fibonacci positive if and only if $\vec{t}$ is a sequence of divergent type.*

Proof. Here we use the notation of Remark 2.4. Assume that $\vec{t}$ is a sequence of divergent type. Similarly to the proof of Proposition 3.4, we see that $A_\ell(m) > 0$ for all $\ell \geq 1$, $m \geq 0$. We have

$$B_0(m) = t_{m+1} > 0, \qquad B_1(m) = t_{m+1} + (t_{m+1} - t_m - 1) t_{m+2},$$

and $t_{m+1} - t_m - 1 \geq 0$ for all $m \geq 0$ by the assumption. Thus, $B_1(m) > 0$ for all $m \geq 0$. Next, using (3.8), we similarly see that $B_{\ell-1}(m) > 0$ for all $\ell \geq 3$ and $m \geq 0$.

Let us now consider the converse statement and assume that the specialization (3.5) is Fibonacci positive. We still have $A_\ell(m) > 0$ for all $\ell \ge 0$, $m \ge 0$. Assume that $\vec{t}$ is not of divergent type, that is, there exists $m_0 \ge 0$ such that $t_{m_0+1} < 1 + t_{m_0}$. We have

$$B_{\ell-1}(m_0) = t_{m_0+1} + \underbrace{(t_{m_0+1} - t_{m_0} - 1)}_{<0} t_{m_0+2} A_{\ell-2}(m_0+3).$$

Since $A_{\ell-2}(m_0+3)$ is positive and unbounded as $\ell \to \infty$, we see that $B_{\ell_0-1}(m_0) < 0$ for some $\ell_0 \gg 1$ (depending on m_0). This violates the Fibonacci positivity, and completes the proof. □

Sequences of divergent type can be treated formally. Introduce variables ϵ_k for $k \ge 1$, and let $\epsilon^{\mathbf{i}} := \epsilon_1^{i_1} \cdots \epsilon_k^{i_k}$ be the monomial corresponding to an integer composition $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{Z}^k_{\ge 0}$. Define

$$t_k := k + \epsilon_1 + \cdots + \epsilon_k, \tag{3.9}$$

and let $\vec{u}, \vec{v}$ depend on $\vec{t}$ as in (3.5).

Corollary 3.6. *Let $\vec{t}$ be given by (3.9). Then, the semi-infinite, tridiagonal matrices $\mathcal{A}(\vec{u} \,|\, \vec{v})$ and $\mathcal{B}_r(\vec{u} \,|\, \vec{v})$ (3.1) for $r \ge 0$ are* coefficientwise *totally positive: Each minor (which does not identically vanish on the space of all semi-infinite, tridiagonal matrices) is a polynomial in $\mathbb{Z}[\epsilon_1, \epsilon_2, \ldots]$ with nonnegative coefficients, at least one of which is positive.*

Consequently, the clone Schur function $s_w(\vec{u} \,|\, \vec{v})$ expands as a polynomial in $\mathbb{R}[\epsilon_1, \epsilon_2, \ldots]$ with nonnegative integer coefficients.

Proof. The statements readily follow from the expansions (3.7)–(3.8) and the recursion for the clone Schur functions (2.11). □

Problem 3.7. How to combinatorially interpret the coefficients of the monomials in the expansion of a clone Schur function $s_w(\vec{u} \,|\, \vec{v})$ in terms of the ϵ-variables?

The problem of identifying matrices (with polynomial entries) that are coefficientwise totally positive has been the subject of recent activity. We refer the reader to [Sok14], [PSZ23], [CDD+21], and [DS23]. The formal specialization given in (3.9) is universal in the following sense:

Corollary 3.8. *Any Fibonacci positive sequence $\vec{t}$ of divergent type can be obtained by specializing the ϵ-variables in (3.9) to arbitrary positive real numbers. Moreover, the values of the ϵ_j's are uniquely determined by $\vec{t}$.*

Summarizing Section 3.1 and the results of Propositions 3.4 and 3.5, we have:

Theorem 3.9 (Characterization of Fibonacci positive specializations)**.** *All Fibonacci positive specializations $(\vec{x}, \vec{y})$ have the form*

$$x_k = c_k(1 + t_{k-1}), \qquad y_k = c_k c_{k+1} t_k, \qquad k \ge 1$$

(with $t_0 = 0$ by agreement), where $\vec{t}$ is a sequence of convergent or divergent type as in Definition 3.3, and $\vec{c}$ is an arbitrary positive real sequence. The sequences $\vec{c}$ and $\vec{t}$ are determined by $(\vec{x}, \vec{y})$ uniquely.

3.3. **Properties of Fibonacci positive specializations.** Here we formulate a number of necessary conditions on sequences $\vec{t}$ corresponding to Fibonacci positive specializations, and also present operations that preserve Fibonacci positivity. These observations mainly follow from Definition 3.3.

Proposition 3.10. *For any $m \geq 1$, none of the inequalities $t_m \geq t_{m+1} \leq t_{m+2}$ hold whenever $\vec{t}$ is a Fibonacci positive specialization.*

Proof. It is sufficient to examine the determinants

$$B_1(m) = t_{m+1} - (1 + t_m - t_{m+1})\, t_{m+2}, \qquad m \geq 1,$$

and verify for each $m \geq 0$ that any of the inequalities listed above are inconsistent with the positivity of $B_1(m)$. □

Proposition 3.10 implies that a Fibonacci positive sequence $\vec{t}$ can exhibit one of three behaviors (bearing in mind our convention $t_0 = 0$):

- The sequence $\vec{t}$ strictly increases, i.e., $t_k > t_{k-1}$ for all $k \geq 1$.
- There exists an $\ell \geq 1$ such that $t_k > t_{k-1}$ for $1 \leq k \leq \ell$, and thereafter $t_k < t_{k-1}$ for $k \geq \ell + 1$.
- There exists an $\ell \geq 1$ such that $t_k > t_{k-1}$ for $0 \leq k \leq \ell$, the sequence forms a plateau with $t_\ell = t_{\ell+1}$, and subsequently $t_k < t_{k-1}$ for all $k \geq \ell + 2$.

In particular, a Fibonacci positive sequence $\vec{t}$ must eventually either strictly increase or strictly decrease.

Lemma 3.11. *If $\vec{t}$ is a sequence of convergent type, then*

$$A_\infty(1) \geq A_\infty(2) > A_\infty(3) > \ldots .$$

Furthermore, $A_\infty(1) = A_\infty(2)$ if and only if $t_1 \in (0,1)$ and $A_\infty(2) = (1 - t_1)^{-1}$.

Proof. For $m \geq 0$, observe that

$$B_\infty(m) = A_\infty(m+1) - A_\infty(m+2) - t_m\, t_{m+2}\, A_\infty(m+3).$$

Consequently, $B_\infty(m) \geq 0$ if and only if

$$A_\infty(m+1) \geq A_\infty(m+2) + t_m\, t_{m+2}\, A_\infty(m+3) \geq A_\infty(m+2),$$

the latter inequality being strict whenever $m \geq 1$. Note that $A_\infty(1) = 1 + t_1 A_\infty(2)$, so $A_\infty(1) = A_\infty(2)$ holds if and only if $t_1 \in (0,1)$ and $A_\infty(2) = (1 - t_1)^{-1}$. This completes the proof. □

Proposition 3.12. *If $\vec{t}$ is a sequence of convergent type, then it cannot eventually weakly increase. In other words, there is no $m_0 \geq 1$ such that $t_m \leq t_{m+1}$ for all $m \geq m_0$.*

Proof. If such an m_0 exists, then for all $m \geq m_0$, we have $A_\infty(m) \leq A_\infty(m+1)$, which contradicts the conclusion of Lemma 3.11. This completes the proof. □

Proposition 3.12 shows that the sequence $\vec{t}$ of convergent type must have a limit. In fact, this limit is always zero:

Proposition 3.13. *Let $\vec{t}$ be a sequence of convergent type. Then $\lim_{m\to\infty} t_m = 0$.*

Proof. Denote $\gamma := \lim_{m\to\infty} t_m$, which exists since the sequence eventually weakly decreases. Using the fact that $A_\infty(m+3) \leq (1 - t_{m+3})^{-1}$ for $m \geq m_0$, we see that γ must be between 0 and 1. By Definition 3.3, we can write for all $m \geq m_0$:

$$t_{m+1} \geq (1 + t_m - t_{m+1}) t_{m+2}\, A_\infty(m+3) \geq (1 + t_m - t_{m+1}) t_{m+2}\, (1 - \gamma)^{-1}.$$

Taking the limit as $m \to \infty$, we get the inequality $\gamma \geq \gamma(1-\gamma)^{-1}$, which implies that $\gamma = 0$. □

Proposition 3.14. *Let $\vec{t}$ be a sequence of convergent type. Then $\limsup_{m\to\infty} m t_m \in [0,1]$, and similarly $\liminf_{m\to\infty} m t_m \in [0,1]$.*

Proof. Harmonicity (Definition 2.2) implies that

$$\begin{aligned}
1 &= \sum_{|w|=m+1} \dim(w)\,\varphi_{\vec{u},\vec{v}}(w) \\
&= \sum_{|w|=m} \dim(1w)\,\varphi_{\vec{u},\vec{v}}(1w) + \sum_{|w|=m-1} \dim(2w)\,\varphi_{\vec{u},\vec{v}}(2w) \\
&= \sum_{|w|=m} \dim(w)\varphi_{\vec{u},\vec{v}}(1w) + \sum_{|w|=m-1} \frac{mt_m}{(1+t_{m-1})(1+t_m)} \dim(w)\,\varphi_{\vec{u},\vec{v}}(w) \\
&= \sum_{|w|=m} \dim(w)\,\varphi_{\vec{u},\vec{v}}(1w) + \frac{mt_m}{(1+t_{m-1})(1+t_m)}.
\end{aligned}$$

Both $\sum_{|w|=m} \dim(w)\varphi_{\vec{u},\vec{v}}(1w)$ and $mt_m(1+t_{m-1})^{-1}(1+t_m)^{-1}$ are nonzero. Thus, we may conclude that $mt_m(1+t_{m-1})^{-1}(1+t_m)^{-1} \in (0,1)$ for all $m \geq 1$. By Proposition 3.13, $t_m \to 0$ as $m \to \infty$, and consequently,

$$1 \geq \limsup_{m\to\infty} \frac{mt_m}{(1+t_{m-1})(1+t_m)} = \limsup_{m\to\infty} mt_m \geq 0.$$

Similarly, $\liminf_{m\to\infty} mt_m \in [0,1]$. This completes the proof. □

Remark 3.15 (Non-example of convergent type specializations)**.** Let $0 < \alpha < 1$. By Proposition 3.14, the sequence $t_k = \varkappa k^{-\alpha}$, $k \geq 1$, is never of convergent type for any value $\varkappa > 0$, despite the fact that $t_m \to 0$ as $m \to \infty$.

Remark 3.16. The sequence mt_m might not converge to a limit under the assumptions that t_m is eventually decreasing to zero, that mt_m are bounded by, say, 1. Indeed, denote $f_n = nt_n$, then $t_n \geq t_{n+1}$ implies that $f_n - f_{n+1} \geq -1/n$. Thus, f_n may make steps in the interval $[0,1]$ of size at most $1/n$ in any direction. Since the series $\sum 1/n$ diverges, we can organize the steps in such a way that f_n has at least two subsequential limits.

This observation shows that we cannot easily strengthen Proposition 3.14 to the full convergence of mt_m. However, less evident properties following from the fact that $\vec{t}$ is a Fibonacci positive specialization of convergent type might imply the convergence of mt_m. We do not further investigate this question here.

Let us now describe a number of operations which preserve Fibonacci positivity. The first is straightforward:

Proposition 3.17. *For any integer $r \geq 0$, $(\vec{x}+r, \vec{y}+r) = (x_{1+r}, x_{2+r}, \ldots, y_{1+r}, y_{2+r}, \ldots)$ is a Fibonacci positive specialization whenever $(\vec{x}, \vec{y})$ is Fibonacci positive.*

Fibonacci positivity can be seen as a "snake" that eats its own tail because $\mathcal{B}_r(\vec{x}\,|\,\vec{y})$ provides a new Fibonacci positive specialization for each $r \geq 0$, whenever the pair $(\vec{x}, \vec{y})$ satisfies Fibonacci positivity. The following result introduces a form of plethystic substitution for clone Schur functions that preserves Fibonacci positivity:

Proposition 3.18 (Ouroboric Shift)**.** *Let $(\vec{x}, \vec{y})$ be a Fibonacci positive specialization. Then, for any $r \geq 0$, the specialization $(\vec{X}, \vec{Y})$ is also Fibonacci positive, where*

$$X_k := \begin{cases} y_{r+1}, & \text{if } k = 1, \\ x_{k+r+1}, & \text{if } k \geq 2, \end{cases} \qquad Y_k := \begin{cases} x_{r+1}\,y_{r+2}, & \text{if } k = 1, \\ y_{k+r+1}, & \text{if } k \geq 2. \end{cases}$$

Proof. We need only to prove that the semi-infinite matrix

$$\mathscr{B}_0\big(\,\vec{X}\,\big|\,\vec{Y}\,\big) = \begin{pmatrix} Y_1 & X_1Y_2 & 0 & \cdots \\ 1 & X_3 & Y_3 & \\ 0 & 1 & X_4 & \\ \vdots & & & \ddots \end{pmatrix} = \begin{pmatrix} x_{r+1}\,y_{r+2} & y_{r+1}\,y_{r+3} & 0 & \cdots \\ 1 & x_{r+4} & y_{r+4} & \\ 0 & 1 & x_{r+5} & \\ \vdots & & & \ddots \end{pmatrix} \tag{3.10}$$

is totally positive for all $r \geq 0$. Let us show that the corresponding determinants satisfy $B_\ell\big(\,\vec{X}\,\big|\,\vec{Y}\,\big) > 0$ for all $\ell \geq 0$. There is no issue when $\ell = 0$, since $B_0\big(\,\vec{X}\,\big|\,\vec{Y}\,\big) = x_{r+1}y_{r+1} > 0$. We can assume that $\ell \geq 1$. We have

$$\begin{aligned} B_\ell(\vec{X} \mid \vec{Y}\,) &= x_{r+1}\,y_{r+2}\,s_{1^\ell}(\vec{x}+r+3 \mid \vec{y}+r+3) - y_{r+1}\,y_{r+3}\,s_{1^{\ell-1}}(\vec{x}+r+4 \mid \vec{y}+r+4) \\ &= s_{1^\ell}(\vec{x}+r+3 \mid \vec{y}+r+3)\,s_{21}(\vec{x}+r \mid \vec{y}+r\,) \\ &\qquad - s_{1^{\ell-1}}(\vec{x}+r+4 \mid \vec{y}+r+4)\,s_{22}(\vec{x}+r \mid \vec{y}+r). \end{aligned} \tag{3.11}$$

By Proposition 3.17, it will be sufficient to restrict our analysis of formula (3.11) to the case where $r = 0$, since the four clone Schur functions that occur are each shifted by $r \geq 0$. We have

$$\begin{aligned} s_{1^\ell}(\vec{x}+3\,|\,\vec{y}+3)\,s_{21}(\vec{x}\,|\,\vec{y}) &- s_{1^{\ell-1}}(\vec{x}+4\,|\,\vec{y}+4)\,s_{22}(\vec{x}\,|\,\vec{y}) \\ &= s_{1^\ell 21}(\vec{x}\,|\,\vec{y}) + s_{1^{\ell-1}}(\vec{x}+4\,|\,\vec{y}+4)\,s_{211}(\vec{x}\,|\,\vec{y}). \end{aligned} \tag{3.12}$$

for all $\ell \geq 1$. Formula (3.12) follows as a direct consequence of the clone Littlewood-Richardson identity [Oka94]. Alternatively, it can be verified directly by induction. The base case, $\ell = 1$, reduces to a restatement of the clone Pieri rule for $s_{21}(\vec{x}\,|\,\vec{y})$:

$$s_1(\vec{x}+3\,|\,\vec{y}+3)\,s_{21}(\vec{x}\,|\,\vec{y}) - s_{22}(\vec{x}\,|\,\vec{y}) = s_{211}(\vec{x}\,|\,\vec{y}) + s_{121}(\vec{x}\,|\,\vec{y}).$$

As a result, we conclude that (3.12) is positive, as its right-hand side involves only sums and products of clone Schur functions. These functions, by definition, are positive since $(\vec{x}, \vec{y})$ is a Fibonacci positive specialization. This completes the proof. □

Proposition 3.19. *Let $\vec{t}$ be a divergent type sequence. Let $\vec{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \dots)$ be any positive real sequence such that $\sigma_k \leq \sigma_{k+1}$ for all $k \geq 0$. Then the specialization $(\vec{x}, \vec{y})$ defined by $x_k = 1 + t_{k-1} + \sigma_{k-1}$ and $y_k = t_k + \sigma_k$ is Fibonacci positive. In particular, the specialization $x_k = 1 + t_{k-1} + \sigma$ and $y_k = t_k + \sigma$ is Fibonacci positive for any $\sigma \geq 0$.*

Proposition 3.20. *Let $\vec{t}$ be a divergent type sequence. Let $\vec{\alpha}$ be a positive real sequence such that*

$$\alpha_k t_k - \alpha_{k+1} t_{k-1} \geq \alpha_k \alpha_{k+1}, \qquad k \geq 1. \tag{3.13}$$

Then the specialization $(\vec{x}, \vec{y})$ defined by $x_k = \alpha_k + t_{k-1}$ and $y_k = \alpha_k t_k$ is Fibonacci positive.

A particular case is when $\alpha_k = \rho \in (0, 1]$ for all k. Then (3.13) clearly holds, and for a sequence $\vec{t}$ of divergent type, the specialization $x_k = \rho + t_{k-1}$ and $y_k = \rho t_k$ is Fibonacci positive.

Proof of Proposition 3.20. Denote $r_k := t_k/\alpha_k$, $k \geq 1$ (with $r_0 = 0$). For all Fibonacci words w, we have $s_w(\vec{x}\,|\,\vec{y}) = (\alpha_1 \cdots \alpha_{|w|})\,s_w(\vec{x}\,'\,|\,\vec{y}\,')$, where $x'_k = 1 + r_{k-1}$ and $y'_k = r_k$. Note that (3.13) implies that $r_k \geq 1 + r_{k-1}$ for all $k \geq 1$. Thus, $\vec{s}$ is of divergent type, and the positivity follows. □

Proposition 3.21. *Let $\vec{t} = (t_1, t_2, t_3, \dots)$ be a strictly decreasing sequence of convergent type. Then the sequence $\gamma\vec{t} := (\gamma t_1, \gamma t_2, \gamma t_3, \dots)$ is of convergent type whenever $0 < \gamma \leq 1$.*

Proof. For $m \geq 0$, let

$$A_\infty(m,\gamma) := 1 + \gamma t_m + \gamma^2 t_m t_{m+1} + \gamma^3 t_m t_{m+1} t_{m+2} + \dots,$$
$$B_\infty(m,\gamma) := \gamma t_{m+1} - (1 + \gamma t_m - \gamma t_{m+1})\, \gamma t_{m+2} A_\infty(m+3,\gamma),$$
$$\phi_0(\gamma) := t_1 - (1-\gamma t_1)\, t_2 A_\infty(3,\gamma).$$

Clearly, $A_\infty(m,\gamma)$ is convergent for any $\gamma \geq 0$, so we only have to address the nonnegativity of $B_\infty(m,\gamma)$ whenever $0 < \gamma \leq 1$ and $m \geq 0$.

Consider two cases, $t_1 \leq 1$ and $t_1 > 1$. If $t_1 \leq 1$, we have

$$\phi_0(\gamma) \geq R(\gamma) \quad \text{for all} \quad \gamma \geq 0, \qquad \text{where} \qquad R(\gamma) := t_1 - \frac{(1-\gamma t_1)t_2}{1-\gamma t_3}.$$

Furthermore, $R(0) = t_1 - t_2 > 0$, and $R(\gamma)$ only vanishes at

$$\gamma_0 = \frac{t_1 - t_2}{t_1(t_3 - t_1)} < 0.$$

It follows that $R(\gamma) > 0$ for all $\gamma > 0$, which forces

$$\phi_0(\gamma) > 0 \quad \text{and} \quad B_\infty(0,\gamma) > 0 \quad \text{for all} \quad \gamma > 0.$$

If $t_1 > 1$, then $\phi_0(\gamma) \geq 0$ whenever $\frac{1}{t_1} \leq \gamma \leq 1$. For γ within the range $0 \leq \gamma < \frac{1}{t_1}$, the inequality $\phi_0(\gamma) \geq R(\gamma)$ is valid, and we may again conclude that $\phi_0(\gamma) > 0$ whenever $0 \leq \gamma < \frac{1}{t_1}$.

The nonnegativity of $B_\infty(m,\gamma)$ for $m \geq 1$ follows from

$$t_{m+1} \geq (1 + t_m - t_{m+1})\, t_{m+2} A_\infty(m+3),$$

together with $A_\infty(m) > A_\infty(m,\gamma)$ and $t_m - t_{m+1} \geq \gamma t_m - \gamma t_{m+1}$ whenever $0 \leq \gamma \leq 1$. This completes the proof. ☐

3.4. **Examples of Fibonacci positive specializations.** Here we present several Fibonacci positive specializations. For some of them, we consider scaling limits of the corresponding random Fibonacci words in Section 5 below. In what follows, we use the standard q-integer notation $[k]_q = (1-q^k)/(1-q)$.

Definition 3.22 (Examples of divergent type)**.** We introduce a list of Fibonacci positive specializations $(\vec{x}, \vec{y})$ related to sequences of divergent type. The naming of some of the specializations is motivated by connections with Stieltjes moment sequences and the Askey scheme, see Section 4.2 below. We consider the following specializations:

- **Shifted Plancherel.** $x_k = y_k = k + \sigma - 1$ for $\sigma \in [1,\infty)$. It reduces to the Plancherel specialization for $\sigma = 1$.
- **Charlier (deformed Plancherel).** $x_k = k + \rho - 1$, $y_k = k\rho$ for $\rho \in (0,1]$. It reduces to the Plancherel specialization for $\rho = 1$.
- **Shifted Charlier.** $x_k = k + \rho + \sigma - 2$, $y_k = (k + \sigma - 1)\rho$ for $\sigma \in [1,\infty)$, $\rho \in (0,1]$. This is a mixture of the previous two, and reduces to them for $\rho = 1$ or $\sigma = 1$.
- **Cigler–Zeng.** $x_k = q^{k-1}$, $y_k = q^k - 1$ for $q \in [\frac{3}{2},\infty)$. This specialization can also be deformed to $x_k = q^{k-1} + \rho - 1$, $y_k = \rho(q^k - 1)$ with $\rho \in (0,1]$. The name comes from the version of the q-Hermite polynomials introduced in [CZ11].
- **Type-I Al-Salam–Carlitz.** $x_k = \rho q^{k-1} + [k-1]_q$, $y_k = \rho q^{k-1}[k]_q$ for $\rho \in (0,1]$ and $q \in (0,1)$.
- **Al-Salam–Chihara.** $x_k = \rho + [k-1]_q$, $y_k = \rho[k]_q$ for $\rho \in (0,1]$ and $q \in [1,\infty)$.
- **q-Charlier.** $x_k = \rho q^{2k-2} + [k-1]_q\big(1 + \rho(q-1)q^{k-2}\big)$, $y_k = \rho q^{2k-2}[k]_q\big(1 + \rho(q-1)q^{k-1}\big)$ for $\rho, q \in (0,1]$.

Proposition 3.23. *The specializations in Definition 3.22 are Fibonacci positive.*

Proof. Let us check the Fibonacci positivity case by case using the representation (3.5) and the modifications to the specializations in Theorem 3.9 or Proposition 3.20, if required.

For the shifted Plancherel, we have $t_k = k+\sigma-1 > 0$, $k \ge 1$, and $t_0 = 0$. Clearly, the $A_\infty(m)$'s diverge. Moreover, $t_{k+1} - t_k = 1 \ge 1$ for all $k \ge 1$, and so $\vec{t}$ is of divergent type. The Charlier case follows from Proposition 3.20 with $\alpha_k = \rho$ and $t_k = k$. The shifted Charlier is obtained from the shifted Plancherel by applying Proposition 3.20 with $\alpha_k = \rho$.

For the Cigler–Zeng case, we have $t_k = q^k - 1$, so the $A_\infty(m)$'s clearly diverge. Then $t_{k+1} - t_k = q^{k+1}(q-1)$, which is ≥ 1 for $q \ge 3/2$. In fact, the precise threshold for q is the real root of the cubic equation $q^3 = q^2 + 1$ which is ≈ 1.47, but we take $3/2$ for simplicity.

For the Type-I Al-Salam–Carlitz case, take $t_k = (\rho q^k)^{-1}[k]_q$, so the $A_\infty(m)$'s clearly diverge. We have $t_{k+1} - t_k = (\rho q^{k+1})^{-1} \ge 1$, so $\vec{t}$ is of divergent type. To get the desired specialization, we use Theorem 3.9 with $c_k = \rho q^{k-1}$.

For the Al-Salam–Chihara case, take $t_k = [k]_q$, so the $A_\infty(m)$'s clearly diverge. We have $t_{k+1} - t_k = q^k \ge 1$, so $\vec{t}$ is of divergent type. To get the specialization, apply Proposition 3.20 with $\alpha_k = \rho$ for all k.

Finally, for the q-Charlier case, take $t_k = [k]_q(1+\rho(q-1)q^{k-1})/(\rho q^{2k})$ and $c_k = \rho q^{2k-2}$ in Theorem 3.9. The series $A_\infty(m)$ diverges for all m. Moreover,

$$t_{k+1} - t_k = \rho^{-1} q^{-2k-2}\big((\rho-1)q^{k+1} - \rho q^k + q + 1\big).$$

One can check that this expression is ≥ 1 for $0 < \rho \le 1$, $0 < q \le 1$. Thus, $\vec{t}$ is of divergent type. This completes the proof. $\square$

Let us now turn to examples of Fibonacci positive specializations of convergent type. We consider two examples of the form

$$t_k = \frac{\varkappa}{k^{\alpha}}, \qquad \alpha = 1, 2, \qquad k \ge 1, \tag{3.14}$$

where $\varkappa$ is a positive real parameter. We call these the *power specializations*. Note that we must have $\alpha \ge 1$, see Remark 3.15.

Proposition 3.24. *There exist upper bounds $\varkappa_1^{(\alpha)}$, $\alpha = 1, 2$, with $\varkappa_1^{(1)} \approx 0.844637$ and $\varkappa_1^{(2)} \approx 1.41056$, such that for all $0 < \varkappa < \varkappa_1^{(\alpha)}$, the specialization (3.14) is Fibonacci positive and of convergent type.*

In the proof and throughout the rest of the paper, we use the standard notation for the hypergeometric functions and Pochhammer symbols:

$${}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}, \qquad (a)_k = a(a+1)\cdots(a+k-1). \tag{3.15}$$

Proof of Proposition 3.24. We have for integer α:

$$\begin{aligned} A_\infty(m) = {} & 1 + \frac{\varkappa}{m^{\alpha}} + \frac{\varkappa^2}{m^{\alpha}(m+1)^{\alpha}} + \frac{\varkappa^3}{m^{\alpha}(m+1)^{\alpha}(m+2)^{\alpha}} + \cdots \\ = {} & \sum_{r=0}^{\infty} \left(\frac{\Gamma(m)}{\Gamma(m+r)}\right)^{\alpha} \varkappa^r = {}_1F_{\alpha}\big(1\,;\underbrace{m,\ldots,m}_{\alpha \text{ times}};\varkappa\big). \end{aligned}$$

The desired inequality

$$B_\infty(m) = t_{m+1} - \big(1 + t_m - t_{m+1}\big) t_{m+2}\, A_\infty(m+3) \geq 0$$

can be rewritten as

$$\frac{\varkappa}{(m+1)^\alpha} - \left(1 + \frac{\varkappa}{m^\alpha}\mathbf{1}_{m>0} - \frac{\varkappa}{(m+1)^\alpha}\right) \frac{\varkappa}{(m+2)^\alpha}\, {}_1F_\alpha\big(1\,;\underbrace{m+3,\ldots,m+3}_{\alpha \text{ times}};\,\varkappa\big) \geq 0.$$

As a function of $\varkappa$, one can check that $B_\infty(m)$ vanishes only at $\varkappa = 0$ and at a value $\varkappa_m^{(\alpha)} \in (0,\infty)$ for each $m \geq 0$. Furthermore, the sequence $\{\varkappa_m^{(\alpha)} : m \geq 0\}$ is strictly increasing, and consequently, the $\vec{t}$-sequence will be of convergent type if and only if $0 < \varkappa \leq \varkappa_1^{(\alpha)}$. The bounds are numerically found to be $\varkappa_1^{(1)} \approx 0.844637$ and $\varkappa_1^{(2)} \approx 1.41056$. □

4. Positive specializations and Stieltjes moment sequences

4.1. **Stieltjes moment sequences and related objects.** In this section we examine Fibonacci positivity in light of the well-known correspondence (due to [Fla80], [Vie83], [Sok20], [PSZ23]) between semi-infinite, totally positive, tridiagonal matrices and Stieltjes moment sequences. In this subsection, we recall the general setup related to Stieltjes moment sequences, continued fractions, tridiagonal matrices, orthogonal polynomials, Motzkin polynomials, and Toda flow. We specialize it to a number of examples coming from Fibonacci positive sequences in Section 4.2 below.

Recall that a sequence $\vec{a} = (a_0, a_1, a_2, \ldots)$ of real numbers is called a *strong Stieltjes moment sequence* if there exists a nonnegative Borel measure $\nu(dt)$ on $[0,\infty)$ with *infinite support* such that $a_n = \int_0^\infty t^n \nu(dt)$ for each $n \geq 0$. The following result may be found, e.g., in [Sok20]:

Theorem 4.1. *A sequence of real numbers $\vec{a} = (a_0, a_1, a_2, \ldots)$ is a strong Stieltjes moment sequence if and only if there exist two real number sequences, $\vec{x}$ and $\vec{y}$, such that the matrix $\mathcal{A}\big(\,\vec{x}\,|\,\vec{y}\,\big)$ defined in (3.1) is totally positive, and the (normalized) ordinary moment generating function of $\vec{a}$,*

$$M(z) = \sum_{n\geq 0} \frac{a_n}{a_0} z^n, \tag{4.1}$$

is expressed by the Jacobi continued fraction depending on $(\,\vec{x}\,|\,\vec{y}\,)$ as

$$M(z) = J_{\vec{x},\vec{y}}\,(z) := \cfrac{1}{1 - x_1 z - \cfrac{y_1 z^2}{1 - x_2 z - \cfrac{y_2 z^2}{1 - x_3 z - \cfrac{y_3 z^2}{\ddots}}}} \tag{4.2}$$

Moreover, the equality between the generating function $M(z)$ (4.1) and the continued fraction $J_{\vec{x},\vec{y}}\,(z)$ (4.2) is witnessed by the recursion

$$P_{n+1}(t) = (t - x_{n+1})P_n(t) - y_n P_{n-1}(t), \quad n \geq 1, \qquad P_0(t) = 1, \quad P_1(t) = t - x_1.$$

responsible for generating the polynomials $P_n(t)$ which are orthogonal with respect to the nonnegative Borel measure $\nu(dt)$ on $[0,\infty)$ whose moment sequence is $\vec{a}$.

A putative or "formal" moment sequence $\vec{a}$ can always be combinatorially determined from any pair of sequences $\vec{x}$ and $\vec{y}$ by calculating the associated *Motzkin polynomials*. Specifically, the ratio a_n/a_0 can be expressed as the generating function of all length-n Motzkin paths, where each up-step $\nearrow$ at height k is weighted by y_k, and each horizontal step $\to$ at height k is weighted by x_{k+1}. Figure 2 illustrates an example of a weighted Motzkin path of length seven.

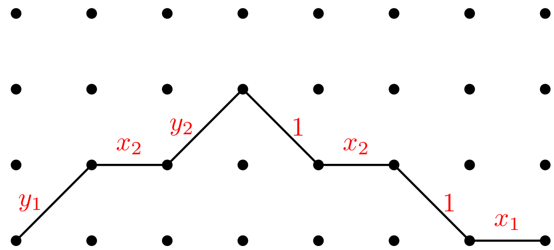


FIGURE 2. An example of a Motzkin path of weight $x_1x_2^2y_1y_2$.

Below we list the first four (normalized) formal moments which are the Motzkin polynomials:

$$\begin{aligned} a_1/a_0 &= x_1, \\ a_2/a_0 &= x_1^2 + y_1, \\ a_3/a_0 &= x_1^3 + 2x_1y_1 + x_2y_1, \\ a_4/a_0 &= x_1^4 + 3x_1^2y_1 + y_1^2 + 2x_1x_2y_1 + x_2^2y_1 + y_1y_2. \end{aligned} \tag{4.3}$$

By Theorem 4.1, the fact that the sequence $\vec{a}$, as in (4.3), is realized by an infinitely supported, nonnegative Borel measure is equivalent to the total positivity of $\mathscr{A}\left(\vec{x}\,|\,\vec{y}\right)$. Conversely, sequences $\vec{x}$ and $\vec{y}$ can be constructed from a Borel measure $\nu(dt)$ using the *Toda flow* [GS97, NZ04], which we now recall.

Having $\nu(dt)$, consider its exponential reweighting $e^{\varrho t}\nu(dt)$. The moments of the reweighted measure satisfy

$$a_n(\varrho) = \frac{d^n}{d\varrho^n} a_0(\varrho), \quad \text{where} \quad a_0(\varrho) = \int_{-\infty}^{\infty} e^{\varrho t}\,\nu(dt) = \sum_{n\ge 0} \frac{a_n}{n!}\,\varrho^n.$$

The sum on the far right is the exponential moment generating function of $\nu(dt)$. As functions of ϱ, the associated tridiagonal parameters $x_n(\varrho)$ and $y_n(\varrho)$ for $n \ge 1$ must obey the *Toda chain equations*, namely,

$$\begin{aligned} \frac{d}{d\varrho}\, x_n(\varrho) &= y_n(\varrho) - y_{n-1}(\varrho); \\ \frac{d}{d\varrho}\, y_n(\varrho) &= y_n(\varrho)\big(x_{n+1}(\varrho) - x_n(\varrho)\big). \end{aligned} \tag{4.4}$$

Their solutions are given by

$$\begin{aligned} x_n(\varrho) &= \frac{d}{d\varrho}\log\left(\frac{\Delta_n(\varrho)}{\Delta_{n-1}(\varrho)}\right) \\ &= \mathrm{Tr}\left(\mathrm{H}_n^{-1}(\varrho)\,\mathrm{H}_n^{(1)}(\varrho)\right) - \mathrm{Tr}\left(\mathrm{H}_{n-1}^{-1}(\varrho)\,\mathrm{H}_{n-1}^{(1)}(\varrho)\right), \\ y_n(\varrho) &= \frac{\Delta_{n-1}(\varrho)\,\Delta_{n+1}(\varrho)}{\Delta_n(\varrho)^2}. \end{aligned} \tag{4.5}$$

Here, $\Delta_n(\varrho) = \det \mathrm{H}_n(\varrho)$, and $\mathrm{H}_n(\varrho)$ and $\mathrm{H}_n^{(1)}(\varrho)$ are the Hankel matrices

$$\mathrm{H}_n(\varrho) \coloneqq \underbrace{\begin{pmatrix} a_0(\varrho) & a_1(\varrho) & a_2(\varrho) & \cdots \\ a_1(\varrho) & a_2(\varrho) & a_3(\varrho) & \cdots \\ a_2(\varrho) & a_3(\varrho) & a_4(\varrho) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{n\times n \text{ Hankel matrix}}, \quad \mathrm{H}_n^{(1)}(\varrho) \coloneqq \frac{d}{d\varrho}\mathrm{H}_n(\varrho) = \underbrace{\begin{pmatrix} a_1(\varrho) & a_2(\varrho) & a_3(\varrho) & \cdots \\ a_2(\varrho) & a_3(\varrho) & a_4(\varrho) & \cdots \\ a_3(\varrho) & a_4(\varrho) & a_5(\varrho) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{n\times n \text{ Hankel matrix}}.$$

In $\mathrm{H}_n^{(1)}(\varrho)$, we used the fact that $\frac{d}{d\varrho}a_k(\varrho) = a_{k+1}(\varrho)$. For example, the solutions for $x_k(\varrho)$ and $y_k(\varrho)$ for $k = 1, 2$ are

$$x_1(\varrho) = \frac{a_1(\varrho)}{a_0(\varrho)}, \qquad x_2(\varrho) = \frac{a_1^3(\varrho) - 2a_0(\varrho)a_1(\varrho)a_2(\varrho) + a_0^2(\varrho)a_3(\varrho)}{a_0\big(a_0(\varrho)a_2(\varrho) - a_1^2(\varrho)\big)},$$
$$y_1(\varrho) = \frac{a_0(\varrho)a_2(\varrho) - a_1^2(\varrho)}{a_0^2(\varrho)},$$
$$y_2(\varrho) = \frac{a_0(\varrho)\big(a_0(\varrho)a_2(\varrho)a_4(\varrho) + 2a_1(\varrho)a_2(\varrho)a_3(\varrho) - a_1^2(\varrho)a_4(\varrho) - a_0(\varrho)a_3^2(\varrho) - a_2^3(\varrho)\big)}{\big(a_0(\varrho)a_2(\varrho) - a_1^2(\varrho)\big)^2}.$$

The sequences $\vec{x}$ and $\vec{y}$ for the original measure $\nu(dt)$ can be obtained by setting $\varrho = 0$ in (4.5). We emphasize that the Toda flow preserves total positivity: Given two initial sequences $\vec{x}$ and $\vec{y}$ for which the matrix $\mathcal{A}(\vec{x}\,|\,\vec{y})$ is totally positive, the matrix $\mathcal{A}(\vec{x}(\varrho)\,|\,\vec{y}(\varrho))$ remains totally positive for any $\varrho \le 0$. Here, $\vec{x}(\varrho) = (x_1(\varrho), x_2(\varrho), \dots)$ and $\vec{y}(\varrho) = (y_1(\varrho), y_2(\varrho), \dots)$ are solutions of the Toda chain equations given by (4.5).

Example 4.2. Consider the Poisson distribution

$$\nu_{\mathrm{Pois}}^{(\rho)}(dt) \coloneqq e^{-\rho} \sum_{k\ge 0} \frac{\rho^k}{k!}\,\delta_k(dt), \tag{4.6}$$

where δ_k is the Dirac delta mass at k. This distribution is obtained by applying the Toda flow, with "time" $\varrho = \log(\rho)$, to the Poisson distribution $\nu_{\mathrm{Pois}}^{(1)}(dt)$, and then renormalizing by $e^{1-\rho}$. Indeed, the associated tridiagonal parameters have the form

$$x_n(\varrho) = n + \rho - 1 = n + e^{\varrho} - 1 \quad \text{and} \quad y_n(\varrho) = \rho\, n = e^{\varrho}\, n.$$

and satisfy the Toda chain equations (4.4). Note that for all $\rho \in (0, 1]$ these tridiagonal parameters are Fibonacci positive; equivalently, $(\vec{x}(\varrho), \vec{y}(\varrho))$ is Fibonacci positive when the Toda flow parameter satisfies $\varrho \in (-\infty, 0]$.

4.2. **Fibonacci positivity and Stieltjes moment sequences.** Fibonacci positivity is stronger than total positivity. This presents two natural questions:

Problem 4.3. What are the properties of moment sequences and nonnegative Borel measures $\nu_{\vec{x},\vec{y}}(dt)$ associated with Fibonacci positive specializations $(\vec{x}, \vec{y})$ by Theorem 4.1? Can these moment sequences and measures be characterized in a meaningful way?

Problem 4.4. Does the Toda flow preserve the space of Fibonacci positive specializations $(\vec{x}, \vec{y})$ for values of the deformation parameter ϱ within some interval $(-R, 0]$ with $R > 0$?

We do not address these problems in full generality here. In this subsection, for a number of Fibonacci positive examples considered in Section 3.4, we identify:

- Systems of orthogonal polynomials $P_n(t)$;
- Nonnegative Borel measures $\nu_{\vec{x},\vec{y}}(dt)$ on $[0,\infty)$ under which these polynomials are orthogonal;
- Combinatorial interpretations of the moment sequences a_n for these Borel measures.

The orthogonal polynomials we obtain come from the Askey scheme [KS96], which lends the names to our Fibonacci positive specializations.

Remark 4.5 (Toda flow)**.** Along with the Poisson measure (Example 4.2), the shifted Charlier specialization $x_k = \rho+\sigma+k-2$, $y_k = \rho(\sigma+k-1)$, $k \ge 1$, also satisfies the Toda chain equations (4.4), after the same change of variables $\rho = e^{\varrho}$. Thus, in the shifted Charlier case, the Toda flow preserves the Fibonacci positivity when $\varrho \in (-\infty, 0]$.

In contrast, the Type-I Al-Salam–Carlitz, Al-Salam–Chihara, and q-Charlier specializations we consider below in this subsection *do not* satisfy the Toda chain equations with the natural change of variables $\rho = \exp(\varrho)$. This is not evidence against a positive answer to Problem 4.4, but indicates that the associated Toda flow may require a different, more intricate parametrization.

4.2.1. *Charlier specialization.* For $\rho \in (0,1]$, set $x_k = \rho + k - 1$ and $y_k = \rho k$ for all $k \ge 1$. In this case, the orthogonal polynomials satisfy the three-term recurrence

$$P_{n+1}(t) = \bigl(t - \rho - n\bigr) P_n(t) - \rho n \, P_{n-1}(t).$$

These are readily recognized as the classical Charlier polynomials. The associated orthogonality measure is the Poisson distribution $\nu^{(\rho)}_{\mathrm{Pois}}$ (4.6) with the parameter ρ. This measure is supported on $\mathbb{Z}_{\ge 0}$. For more details on Charlier polynomials we refer to [KS96, Chapter 1.12].

The moments a_n of $\nu^{(\rho)}_{\mathrm{Pois}}$ are the Bell polynomials (sometimes called Touchard polynomials), which have the combinatorial interpretation

$$a_n = B_n(\rho) \coloneqq \sum\nolimits_{\pi \in \Pi(n)} \rho^{\#\mathrm{blocks}(\pi)}. \tag{4.7}$$

Here, $\Pi(n)$ are the set partitions of $\{1,\ldots,n\}$, and $\#\mathrm{blocks}(\pi)$ counts the number of blocks in π. The moment generating function $M(z)$ (4.1) is the confluent hypergeometric function ${}_1F_1\left(1; 1 - \frac{1}{z}; -\rho\right)$ (see (3.15) for the notation).

4.2.2. *Type-I Al-Salam–Carlitz specialization.* For $\rho, q \in (0,1]$, define $x_k = \rho\, q^{k-1} + [k-1]_q$ and $y_k = \rho\, q^{k-1}[k]_q$, where $k \ge 1$. The corresponding orthogonal polynomials

$$P_{n+1}(t) = \bigl(t - \rho q^n - [n]_q\bigr) P_n(t) - \rho\, q^{n-1}[n]_q P_{n-1}(t)$$

are known from [dMSW95], where they are denoted as $P_n(t) = C_n^{(\rho)}(t;q)$. These polynomials can be identified as Type-I Al-Salam–Carlitz polynomials $U_n^{(a)}(x;q) = U_n(x,a;q)$ through a change of variables and parameters. Namely,

$$P_n(t) = C_n^{(\rho)}(t;q) = \rho^n\, U_n\biggl(\frac{t}{\rho} - \frac{1}{\rho(1-q)}, \frac{-1}{\rho(1-q)}; q\biggr).$$

See [KS96, Chapter 3.24] for more details on the Al-Salam–Carlitz polynomials $U_n(x,a;q)$. In particular, their orthogonality measure is given by

$$\frac{\mathrm{e}_q(a)\,\mathrm{e}_q(q)\,\mathrm{e}_q\bigl(a^{-1}q\bigr)}{(1-q)} \sum_{k=0}^{\infty} q^k\, \mathrm{e}_q^{-1}\bigl(q^k\bigr)\, \Bigl(\mathrm{e}_q^{-1}\bigl(a^{-1}q^{k+1}\bigr)\, \delta_{q^k}(dx) \,-\, a\, \mathrm{e}_q^{-1}\bigl(aq^{k+1}\bigr)\delta_{aq^k}(dx)\Bigr),$$

where $\mathrm{e}_q(x)$ is the (little) discrete exponential function

$$\mathrm{e}_q(x) := \sum_{k\geq 0} \frac{x^k}{[k]_q!} = \frac{1}{(x(1-q);q)_\infty}. \tag{4.8}$$

Here, $q \in (0,1)$, and the parameter a must be negative. This is consistent with the change of variables used in [dMSW95], and with our range of values $\rho, q \in (0,1]$. In terms of the variable t, the nonnegative Borel measure corresponding to the Type-I Al-Salam–Carlitz Fibonacci positive specialization is supported by the discrete subset $\left\{[k]_q\,, \rho q^k + \frac{1}{1-q}\right\}_{k\geq 0} \subset \mathbb{R}_{\geq 0}$.

The n-th moment a_n of the orthogonality measure for $P_n(t)$ is given by a q-variant of the Bell polynomial, and can also be expressed as a generating function for set partitions:

$$a_n = B_{q,n}(\rho) := \sum\nolimits_{\pi\in\Pi(n)} \rho^{\#\mathrm{blocks}(\pi)} q^{\mathrm{inv}(\pi)},$$

which incorporates an additional q-statistic $\mathrm{inv}(\pi)$ counting inversions in the set partition π. We refer to [WW91] and [Zen95] for details.

4.2.3. *Al-Salam–Chihara specialization.* Take $\rho \in (0,1]$ and $q \in [1,\infty)$, and let $x_k = \rho + [k-1]_q$ and $y_k = \rho\,[k]_q$ for $k \geq 1$. In this case, the orthogonal polynomials satisfy the three-term recurrence

$$P_{n+1}(t) = \bigl(t - \rho - [n]_q\bigr) P_n(t) - \rho[n]_q P_{n-1}(t).$$

They appeared in [Ans05] and [KSZ06] under the notation $C_n(t,\rho;q)$. In the latter reference, $P_n(t) = C_n(t,\rho;q)$ were identified with the the Al-Salam–Chihara polynomials $Q_n(x;a,b\,|\,q)$, after rescaling and incorporating a change of variables as follows:

$$P_n(t) = C_n(t,\rho;q) = \left(\frac{\rho}{1-q}\right)^{n/2} Q_n\left(\frac{1}{2}\sqrt{\frac{1-q}{\rho}}\left(t-\rho-\frac{1}{1-q}\right);\, \frac{-1}{\sqrt{\rho(1-q)}}, 0 \,\middle|\, q\right).$$

See [KS96, Chapter 3.8] for more details on the Al-Salam–Chihara orthogonal polynomials Q_n. Note that our our parameter q is greater than one, while in [KS96] it is classically assumed that $|q| < 1$. Because of this, we cannot identify the nonnegative Borel measure $\nu(dt)$ (which exists by Theorem 4.1 and serves as the orthogonality measure for the $P_n(t)$'s) with the one coming from the Al-Salam–Chihara polynomials in [KS96].

The existence of different orthogonality measures for different ranges of parameters is a known phenomenon, see, e.g., [Ask89] or [Chr04]. In particular, for $q > 1$, the Al-Salam–Chihara polynomials admit a different orthogonality measure [Koo04]. Let us recall the necessary notation. Denote

$$\tilde{Q}_n(x;a,b\,|\,q) := i^{-n} Q_n(ix; ia, ib\,|\,q), \qquad i = \sqrt{-1}, \qquad n \geq 0.$$

For $q > 1$ we have, taking $b \to 0$ in [Koo04, (16)]:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1+q^{-2k}a^{-2}}{1+a^{-2}} \frac{(-a^2;q)_k}{(q;q)_k} a^{-4k} q^{\frac{3}{2}k(1-k)} \\ \times \tilde{Q}_n\bigl(\tfrac{1}{2}(aq^k - a^{-1}q^{-k}); a, 0\,|\,q\bigr)\, \tilde{Q}_m\bigl(\tfrac{1}{2}(aq^k - a^{-1}q^{-k}); a, 0\,|\,q\bigr) \\ = (-q^{-1}a^{-2}; q^{-1})_\infty (-1)^n (q;q)_n \mathbf{1}_{m=n}.\end{aligned} \tag{4.9}$$

Here $(z;q)_k := (1-z)(1-zq)\cdots(1-zq^{k-1})$ is the q-Pochhammer symbol. One can readily verify that the weights in (4.9) are nonnegative when $a = -1/\sqrt{\rho(1-q)}$ and $q > 1$. Moreover, matching the variables in the polynomials, we see that the measure $\nu(dt)$ is supported on the following discrete set:

$$\left\{[k]_q + (1-q^{-k})\,\rho\right\}_{k\geq 0} \subset \mathbb{R}_{\geq 0}.$$

The moment sequence a_n of $\nu(dt)$ can be derived from the continued fraction $M(z) = J_{\vec{x},\vec{y}}(z)$ (4.1)–(4.2). Moreover, in [KSZ06], the combinatorial interpretation of the a_n's was shown to be

$$a_n = B^{\mathrm{rc}}_{q,n}(\rho) := \sum\nolimits_{\pi\,\in\,\Pi(n)} \rho^{\#\mathrm{blocks}(\pi)} q^{\mathrm{rc}(\pi)},$$

which includes a certain q-statistic counting the number of restricted crossings in the set partition π. We refer to [KSZ06] for the definition of $\mathrm{rc}(\pi)$.

4.2.4. *q-Charlier specialization.* For $\rho, q \in (0,1]$, let

$$x_k = \rho q^{2k-2} + [k-1]_q\big(1+\rho(q-1)q^{k-2}\big) \quad\text{and}\quad y_k = \rho q^{2k-2}[k]_q\big(1+\rho(q-1)q^{k-1}\big), \qquad k\geq 1.$$

The corresponding orthogonal polynomials satisfy the three-term recurrence

$$P_{n+1}(t) = \left(t - \rho q^{2n} - [n]_q\left(1+\rho(q-1)q^{n-1}\right)\right)P_n(t) - \rho q^{2n-2}[n]_q\left(1+\rho(q-1)q^{n-1}\right)P_{n-1}(t).$$

They appear in [Zen95] under the notation $V_n^{(\rho)}(t\,;q)$. Moreover, it follows from [Zen95] that these polynomials are related to the

After rescaling and a change of variables, these are exactly the q-Charlier polynomials from the Askey scheme [KS96, Chapter 3.23]. Namely, we have (using the notation $C_n(x;a,q)$ instead of $C_n(q^{-x};a,q)$ as in [KS96]):

$$P_n(t) = V_n^{(\rho)}(t;q) = (-\rho)^n q^{n(n-1)} C_n\left((q-1)t+1;\rho(1-q^{-1}),q^{-1}\right). \tag{4.10}$$

Like in the previous Al-Salam–Chihara case, here the q-Charlier polynomials contain the parameter $q^{-1}>1$. Therefore, the classical orthogonality measure [KS96, Chapter 3.23] does not correspond to our nonnegative Borel measure $\nu(dt)$. Instead, we have the following orthogonality for q^{-1}-Charlier polynomials:

$$\sum_{k=0}^{\infty}\frac{(-q)^k a^k}{(q;q)_k}C_n(q^k;a,q^{-1})\,C_m(q^k;a,q^{-1}) = \frac{q^n\,(q^{-1};q^{-1})_n\,(-a^{-1}q^{-1};q^{-1})_n}{(-a\,q;q)_\infty}\,\mathbf{1}_{m=n}.$$

The q-Pochhammer symbol $(-a\,q;q)_\infty$ in the denominator (as opposed to $(-a;q)_\infty$ in the numerator for $0<q<1$, see [KS96, (3.23.2)]) comes from normalizing the orthogonality measure to be a probability distribution.

In terms of our parameters, we have $a=\rho(1-q^{-1})<0$, which ensures that the orthogonality measure $\nu(dt)$ for the polynomials $P_n(t)$ is nonnegative. The support of $\nu(dt)$ in $\mathbb{R}_{\geq 0}$ consists of all q-integers $[k]_q$, where $k\in\mathbb{Z}_{\geq 0}$, as it should be due to Theorem 4.1.

The moments a_n of $\nu(dt)$ can be derived from the continued fraction (4.1)–(4.2), and their combinatorial interpretation is yet another q-variant of the Bell polynomials:

$$a_n = \widetilde{B}_{q,n}(\rho) := \sum\nolimits_{\pi\in\Pi(n)} \rho^{\#\mathrm{blocks}(\pi)}\, q^{\widetilde{\mathrm{inv}}(\pi)}.$$

Here the statistic $\widetilde{\mathrm{inv}}(\pi)$ is the number of so-called dual inversions of a set partition $\pi\in\Pi(n)$. We refer to [WW91] and [Zen95] for details.

4.2.5. *Shifted Charlier specialization.* Finally, consider the shifted Charlier specialization given by $x_k = \rho + \sigma + k - 2$ and $y_k = (\sigma + k - 1)\rho$, where $k \geq 1$ and $\rho \in (0, 1]$, $\sigma \in [1, \infty)$. The shifted Plancherel and Charlier specializations are obtained from this one by setting $\rho = 1$ and $\sigma = 1$, respectively. See Definition 3.22. The three-term recurrence for the orthogonal polynomials has the form

$$P_{n+1}(t) = (t - \rho - \sigma - n + 1)P_n(t) - \rho(\sigma + n - 1)P_{n-1}(t). \tag{4.11}$$

A similar recurrence is satisfied by the so-called *associated Charlier polynomials* [ILV88], [Ahb23]:

$$a\,\mathcal{C}_{n+1}(x; a, \gamma) = (n + \gamma + a - x)\mathcal{C}_n(x; a, \gamma) - (n + \gamma)\mathcal{C}_{n-1}(x; a, \gamma).$$

Namely, we have the following identification:

$$P_n(t) = (-\rho)^n\,\mathcal{C}_n(t; \rho, \sigma - 1). \tag{4.12}$$

The polynomials P_n (4.11) can be expressed through the hypergeometric function ${}_3F_2$ (see (3.15) for the notation). This follows from (4.12) and [Ahb23, (3.6)]:

$$P_n(t) = \sum_{k=0}^{n} (-\rho)^{n-k}\,\frac{(-n)_k(\sigma - 1 - t)_k}{k!}\,{}_3F_2\left(\begin{array}{c} -k, \sigma - 1, k - n \\ -n, \sigma - 1 - t \end{array}\middle|\, 1\right).$$

Let us now discuss the moment generating function $M(z) = J_{\vec{x},\vec{y}}(z)$ (4.1)–(4.2) for the shifted Charlier specialization. Define the fractional linear action of 2×2 matrices on power series $f(z)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) := \frac{a\,f(z) + b}{c\,f(z) + d}.$$

Lemma 4.6. *The moment generating function $M(z) = M(z; \rho, \sigma)$ satisfies the following functional equation:*

$$M(z; \rho, \sigma + 1) = \begin{pmatrix} 1 - (\sigma + \rho - 1)\,z & -1 \\ \sigma\,\rho\,z^2 & 0 \end{pmatrix} \cdot M(z; \rho, \sigma). \tag{4.13}$$

In terms of the series coefficients $a_n = a_n(\rho, \sigma)$, this yields the quadratic recurrence

$$a_{n+1}(\rho, \sigma) = (\sigma + \rho - 1)\,a_n(\rho, \sigma) + \rho\sigma \sum_{k=0}^{n-1} a_k(\rho, \sigma)\,a_{n-k-1}(\rho, \sigma + 1), \tag{4.14}$$

with the initial condition $a_0(\rho, \sigma) \equiv 1$.

In particular, when $\sigma = 1$, we know that $M(z; \rho, 1)$ is the moment generating function for the Poisson distribution (4.6), and thus $M(z; \rho, 1) = {}_1F_1(1; 1 - 1/z; -\rho)$. We thus have

$$M(z; \rho, k) = \begin{pmatrix} 1 - (k-1)z & -1 \\ (k-1)z^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 - z & -1 \\ z^2 & 0 \end{pmatrix} \cdot {}_1F_1(1; 1 - 1/z; -\rho). \tag{4.15}$$

We are grateful to Michael Somos and Qiaochu Yuan for helpful observations [Som22] leading to Lemma 4.6.

Proof of Lemma 4.6. The continued fraction for the shifted Charlier parameters has the form

$$M(z; \rho, \sigma) = \frac{1}{1 - (\sigma + \rho - 1)\,z - \sigma\,\rho\,z^2 M(z; \rho, \sigma + 1)}, \tag{4.16}$$

since the shifted sequences $(\vec{x}+1, \vec{y}+1)$ correspond to the specialization under the shift $\sigma \mapsto \sigma+1$. Identity (4.16) is equivalent to the desired functional equation (4.13).

The recurrence (4.14) follows by writing the equation (4.16) as

$$M(z; \rho, \sigma)\,(1 - (\sigma + \rho - 1)\,z) = 1 + \sigma\,\rho\,z^2 M(z; \rho, \sigma + 1)\,M(z; \rho, \sigma),$$

and comparing the coefficients by z^{n+1}. □

Remark 4.7. For integer values $\sigma = k \in \mathbb{Z}_{\ge 1}$, the generating function $M(z;\rho,k)$ is derived by applying a sequence of fractional linear transformations (4.15) to the meromorphic function $M(z;\rho,1) = {}_1F_1(1;1-1/z;-\rho)$. Thus, $M(z;\rho,k)$ is a meromorphic function of z. Consequently, the support of the measure $\nu(dt)$ is discrete, similarly to all the other specializations considered in this subsection. It is likely that for non-integer $\sigma > 1$, the measures $\nu(dt)$ remain atomic.

The existence of Fibonacci positive sequences $\vec{x}, \vec{y}$ such that their associated measures $\nu_{\vec{x},\vec{y}}(dt)$ are non-atomic remains unclear, and we do not address this question here.

We can solve the functional equation (4.13) for $M(z;\rho,\sigma)$ in terms of the confluent hypergeometric function ${}_1F_1$ (see (3.15) for the notation):

Proposition 4.8. *The moment generating function $M(z) = M(z;\rho,\sigma)$ of the shifted Charlier specialization is given by*

$$M(z;\rho,\sigma) = \frac{{}_1F_1\left(\sigma;\, \sigma - \frac{1}{z};\, -\rho\right)}{{}_1F_1\left(\sigma-1;\, \sigma - \frac{1}{z};\, -\rho\right) - z(\sigma-1)\,{}_1F_1\left(\sigma;\, \sigma - \frac{1}{z};\, -\rho\right)}. \tag{4.17}$$

Proof. The equation (4.13), rewritten as the recurrence (4.14) for the coefficients of the generating function in z has a unique solution. Therefore, we need to verify that the right-hand side of (4.17) is regular at $z = 0$ (and hence is expanded as a power series in z), and that it satisfies the functional equation (4.13). See also Remark 4.9 below for examples of other solutions to (4.13) which are not regular at $z = 0$.

We have

$${}_1F_1\left(\sigma;\, \sigma - \tfrac{1}{z};\, -\rho\right) = 1 + \sum_{r=1}^{\infty} \frac{\sigma(\sigma+1)\cdots(\sigma+r-1)}{(1-\sigma z)\cdots(1-(\sigma+r-1)z)}\, \frac{\rho^r}{r!}\, z^r,$$

and similarly, ${}_1F_1\left(\sigma-1;\sigma-\frac{1}{z};-\rho\right) - z(\sigma-1){}_1F_1\left(\sigma;\sigma-\frac{1}{z};-\rho\right)$ is a power series in z with constant coefficient 1. Therefore, the right-hand side of (4.17) is a power series in z.

After substituting the right-hand side of (4.17) into the functional equation (4.13) and cross-multiplying, we obtain

$$\left.\begin{aligned} &(1-\rho z){}_1F_1\left(\sigma;\sigma-\tfrac{1}{z};-\rho\right){}_1F_1\left(\sigma;\sigma+1-\tfrac{1}{z};-\rho\right)\\ &\quad - z\sigma{}_1F_1\left(\sigma;\sigma-\tfrac{1}{z};-\rho\right){}_1F_1\left(\sigma+1;\sigma+1-\tfrac{1}{z};-\rho\right)\\ &\quad - {}_1F_1\left(\sigma-1;\sigma-\tfrac{1}{z};-\rho\right){}_1F_1\left(\sigma;\sigma+1-\tfrac{1}{z};-\rho\right)\\ &\quad + z\sigma{}_1F_1\left(\sigma-1;\sigma-\tfrac{1}{z};-\rho\right){}_1F_1\left(\sigma+1;\sigma+1-\tfrac{1}{z};-\rho\right) \end{aligned}\right\} \stackrel{?}{=} 0.$$

This identity can be readily verified by applying two contiguous relations:

$$\begin{aligned} z\sigma{}_1F_1(\sigma+1;\sigma+1-\tfrac{1}{z};-\rho) &= {}_1F_1(\sigma;\sigma+1-\tfrac{1}{z};-\rho) + (z\sigma-1){}_1F_1(\sigma;\sigma-\tfrac{1}{z};-\rho),\\ \rho z{}_1F_1(\sigma;\sigma+1-\tfrac{1}{z};-\rho) &= (z\sigma-1){}_1F_1(\sigma-1;\sigma-\tfrac{1}{z};-\rho) - (z\sigma-1){}_1F_1(\sigma;\sigma-\tfrac{1}{z};-\rho). \end{aligned}$$

This completes the proof. □

Remark 4.9. Curiosly, the functional equation (4.13) has at least two solutions expressible as power series in z^{-1} with vanishing constant coefficient. If $\mathcal{M}(z;\rho,\sigma) = \sum_{n\ge 1} m_n(\rho,\sigma) z^{-n}$ is a solution, then the recurrence relations for the coefficients take a different form:

$$\begin{aligned} (1-\rho-\sigma)m_1(\rho,\sigma) \,-\, 1 &= \rho\sigma\, m_1(\rho,\sigma)\, m_1(\rho,\sigma+1), \\ m_{n-1}(\rho,\sigma) \,-\, (\rho+\sigma-1)m_n(\rho,\sigma) &= \rho\sigma \sum_{k=0}^{n-1} m_{k+1}(\rho,\sigma)\, m_{n-k}(\rho,\sigma+1), \qquad n\ge 2. \end{aligned} \tag{4.18}$$

Two choices of valid initial conditions for (4.18) are

$$m_1(\rho,\sigma) = (1-\sigma)^{-1}$$

and

$$m_1(\rho,\sigma) = \frac{\mathrm{U}\big(\sigma,\,\sigma,\,-\rho\big)}{(1-\rho)\,\mathrm{U}\big(\sigma,\,\sigma-1,\,-\rho\big) \,-\, \rho\sigma \mathrm{U}\big(\sigma+1,\,\sigma,\,-\rho\big)},$$

where

$$\mathrm{U}\big(\alpha,\,\beta,\,\xi\big) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\xi t}\, t^{\alpha-1}\,(1+t)^{\beta-\alpha-1}\,dt$$

is the Tricomi function.

The Tricomi initial condition yields the solution

$$\mathcal{M}(z;\rho,\sigma) = \frac{\mathrm{U}\big(\sigma,\,\sigma-1/z,\,-\rho\big)}{(1+z-z\rho)\,\mathrm{U}\big(\sigma,\,\sigma-1-1/z,\,-\rho\big) \,-\, z\rho\sigma\,\mathrm{U}\big(\sigma+1,\,\sigma-1/z,\,-\rho\big)}. \tag{4.19}$$

The fact that (4.19) yields a solution to (4.13) can be checked using contiguous relations similarly to the proof of Proposition 4.8.

Solution (4.19) also arises by first decoupling equation (4.13) through the Ansatz

$$\begin{aligned} \mathcal{P}(z;\rho,\sigma+1) &= \big(1-(\sigma+\rho-1)z\big)\mathcal{P}(z;\rho,\sigma) \,-\, \mathcal{Q}(z;\rho,\sigma), \\ \mathcal{Q}(z;\rho,\sigma+1) &= \rho\sigma z^2 \mathcal{P}(z;\rho,\sigma), \end{aligned} \tag{4.20}$$

assuming that $\mathcal{M}(z;\rho,\sigma) = \mathcal{P}(z;\rho,\sigma)/\mathcal{Q}(z;\rho,\sigma)$. We then apply the Fourier transform to (4.20), solve the resulting 2×2 system of ordinary differential equations, and finally apply the inverse Fourier transform to return to the original function.

It remains unclear whether other solutions to (4.13) exist, or how they might be classified.

Let us now describe a combinatorial interpretation of the moments $a_n(\rho,\sigma)$ for the shifted Charlier specialization. Recall that $\pi\in\Pi(n)$ denotes an arbitrary set partition of $\{1,\ldots,n\}$. It is always presented in canonical form, i.e.,

$$\pi = B_1\big|B_2\big|\cdots\big|B_r,$$

where the blocks $B_1,\ldots,B_r$ are ordered such that $\min B_1 < \cdots < \min B_r$. The non-maximal elements of a block are called *openers*; the set of all openers is denoted by $\mathcal{O}(\pi)$. The non-minimal elements of a block are called *closers* ($\mathcal{C}(\pi)$). Elements that are simultaneously openers and closers are called *transients* ($\mathcal{T}(\pi)$). Elements that are neither openers nor closers are called *singletons* ($\mathcal{S}(\pi)$).

If $i\in\mathcal{C}(\pi)$, let $\Gamma_i(\pi)$ denote the set of openers $a<i$ such that $i\le b$, where b is the closer succeeding a in π. Let $\gamma_i(\pi)$ be the position of the opener in $\Gamma_i(\pi)$ corresponding to i, where we list the elements $\Gamma_i(\pi)=\{a_1<\cdots<a_\ell\}$ in increasing order. Kasraoui and Zeng [KZ06] showed that a set partition $\pi\in\Pi(n)$ is uniquely determined by the tuple $(\mathcal{O},\mathcal{C},\mathcal{S},\mathcal{T})$ together with the integers $\gamma_i(\pi)$ for $i\in\mathcal{C}$. We need one more definition (not present in [KZ06]) for our interpretation of the moments $a_n(\rho,\sigma)$:

Definition 4.10 (Non-transient closers)**.** Let $\#\mathrm{ntc}(\pi)$ count the number of non-transient closers $i \in \mathcal{C}(\pi)$ such that $\gamma_i(\pi) = 1$.

As an example, consider the set partition $\pi = 135|29|4|678$ of $n = 9$. In this case

$$\begin{array}{lll} \mathcal{O}(\pi) = \{1,2,3,6,7\} & \Gamma_3(\pi) = \{1,2\} & \gamma_3(\pi) = 1 \\ \mathcal{C}(\pi) = \{3,5,7,8,9\} & \Gamma_5(\pi) = \{2,3\} & \gamma_5(\pi) = 2 \\ \mathcal{S}(\pi) = \{4\} & \Gamma_7(\pi) = \{2,6\} & \gamma_7(\pi) = 2 \\ \mathcal{T}(\pi) = \{3,7\} & \Gamma_8(\pi) = \{2,7\} & \gamma_8(\pi) = 2 \\ & \Gamma_9(\pi) = \{2\} & \gamma_9(\pi) = 1 \end{array}$$

and we see that $\mathrm{ntc}(\pi) = 1$.

Along with $\#\mathrm{ntc}(\pi)$, we introduce the following statistics. Let $\#\mathrm{blocks}^\star(\pi)$ denote the number of non-singleton blocks in π ($\#\mathrm{blocks}^\star(\pi) = 3$ in the example above), and $\#\mathcal{S}(\pi)$ be the number of singletons in π ($\#\mathcal{S}(\pi) = 1$ in the example above).

Proposition 4.11. *The n-th moment $a_n(\rho, \sigma)$ of the shifted Charlier specialization is given by the following variant of the Bell polynomial:*

$$a_n(\rho, \sigma) = B^{\mathrm{ntc}}_{\sigma,n}(\rho) \coloneqq \sum_{\pi \in \Pi(n)} \rho^{\#\mathrm{blocks}^\star(\pi)}\, \sigma^{\#\mathrm{ntc}(\pi)} \left(\rho + \sigma - 1\right)^{\#\mathcal{S}(\pi)}.$$

Proof. Our proof is an adaptation of the methods and results found in [Jos11] and [KZ06], and uses a well-known bijection between set partitions $\pi \in \Pi(n)$ and Charlier histoires.[3] A length-n *Charlier histoire* is a "colored" Motzkin path of length n, where each $\rightarrow$ step at height k is assigned a nonnegative integer color $c \in \{0, \ldots, k\}$, while each $\searrow$ step at height k is assigned a color $c \in \{1, \ldots, k-1\}$. Let $\mathfrak{H}_n$ denote the set of length-n Charlier histoires.

We can now define the bijection $\Pi(n) \to \mathfrak{H}_n$. Under this bijection, a Charlier histoire $\mathfrak{h}_\pi$ is constructed from left to right by converting, in order, each element $i \in \{1, \ldots, n\}$ of a set partition $\pi \in \Pi(n)$ into a (colored) step of type $\{\nearrow, \rightarrow, \searrow\}$ according to the following rules:

1. Each non-transient opener is converted into an $\nearrow$ step
2. Each singleton is converted into a $\rightarrow$ step with color $c = 0$
3. Each transient element is converted into a $\rightarrow$ step with color $\gamma_i(\pi)$.
4. Each non-transient closer is converted into a $\searrow$ step with color $\gamma_i(\pi)$.

As an example, the Charlier histoire $\mathfrak{h}_\pi$ corresponding to the set partition $\pi = 135|29|4|678$ under the bijection is depicted in Figure 3.

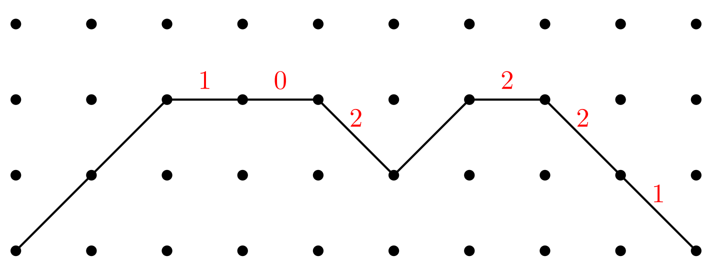


FIGURE 3. The Charlier histoire $\mathfrak{h}_\pi$ corresponding to $\pi = 135|29|4|678$. The weight of both π and $\mathfrak{h}_\pi$ is $\rho^2 \sigma (\rho + \sigma - 1)$.

[3]We are grateful to Dennis Stanton for explaining this relationship to us.

We define the weight $\omega(\mathfrak{h})$ of a Charlier histoire $\mathfrak{h} \in \mathfrak{H}_n$ as the product of the weights of its (colored) steps, where the weights are given as follows:

- $\nearrow$ step: ρ;
- $\rightarrow$ step: $\rho + \sigma - 1$ if $c = 0$, or 1 if $c > 0$;
- $\searrow$ step: σ if $c = 1$, or 1 if $c > 1$.

Let $\mathfrak{M}_n$ denote the set of all Motzkin paths of length n and let $\mathrm{pr}_n : \mathfrak{H}_n \to \mathfrak{M}_n$ be the projection map from Charlier histoires to Motzkin paths which simply "forgets" the histoire colors. Since $y_k = \rho(\sigma + k - 1)$ factors into ρ and $(\sigma + k - 1)$, we can modify the weighting scheme for Motzkin paths $\mathfrak{m} \in \mathfrak{M}_n$ given in Section 4.1, and assign weights to steps as follows:

- each $\nearrow$ step at height k is weighted ρ;
- each $\searrow$ step at height k and is weighted $\sigma + k - 1$;
- each $\rightarrow$ step at height k is weighted $\rho + \sigma + k - 1$.

We define the weight $\mathrm{wt}(\mathfrak{m})$ of a Motzkin path $\mathfrak{m} \in \mathfrak{M}_n$ to be the product of the weights of its steps. Note that $\mathrm{wt}(\mathfrak{m})$ coincides with the weight of $\mathfrak{m}$ as prescribed in Section 4.1. In particular, our shifted Charlier moments have the form $a_n(\rho, \sigma) = \sum_{\mathfrak{m} \in \mathfrak{M}_n} \mathrm{wt}(\mathfrak{m})$.

The technique of [KZ06] aligns the weights of Charlier histoires with the weights of their associated Motzkin paths under the projection map. That is,

$$\mathrm{wt}(\mathfrak{m}) = \sum_{\mathrm{pr}_n(\mathfrak{h}) = \mathfrak{m}} \omega(\mathfrak{h}).$$

This ensures that the total weight of all Motzkin paths of length n matches the total weight of all Charlier histoires of the same length. This total weight is $a_n(\rho, \sigma)$, which completes the proof. ∎

Remark 4.12 (Permutation statistics and Jacobi continued fractions)**.** Jacobi continued fractions and their associated moments are connected not only to set partitions, but also to permutation statistics. One of the most recent examples of these connections is the work [BS21], which connects a 14-parameter Jacobi continued fraction with permutation enumeration.

It would be very interesting to combine random permutations (arising from this 14-parameter enumeration) with the Young–Fibonacci RS correspondence which we describe in Section 7 below. The resulting measures on Fibonacci words may coincide with some of the clone Schur measures. We do not develop this direction further in the present work.

5. Asymptotic behavior of clone coherent measures

5.1. **Outline.** In this section, we examine scaling limits and other types of asymptotic behavior of clone coherent measures on Fibonacci words arising from various Fibonacci positive specializations $(\vec{x}, \vec{y})$ introduced in Section 3.4. This section is organized as follows.

In the preliminary Section 5.2, we obtain general identities for the joint distribution of sequences of 1's or 2's in the beginning of a random Fibonacci word distributed according to an arbitrary clone coherent measure.

In Sections 5.3 and 5.4, we examine coherent measures for two particular specializations of divergent type, where either sequences of 1's (respectively, 2's) become long in the corresponding random Fibonacci words. For the first model, the joint scaling limit of runs leads to a residual allocation (stick-breaking) type distribution. The limiting distribution we get differ from GEM(1/2) (see Definition 2.3), which appears in the Plancherel case [GK00a], not only in the value of the GEM parameter θ (which may not be 1/2), but also due to the random insertion of additional zeroes into the sequence $X = (X_1, X_2, \dots) \in \Delta$ (2.8). In the model with growing hikes of 2's, their joint scaling limit is a *dependent* stick-breaking process described in detail in Definition 5.10 and Remarks 5.12 and 5.14.

In Section 5.5, we consider specializations of the convergent type. We show that for them, the coherent measure is asymptotically supported on Fibonacci words of the form $1^\infty v$, with v being a finite Fibonacci word (i.e., words with a growing prefix of 1's). This asymptotic behavior contrasts sharply with the residual allocation type distributions arising in Sections 5.3 and 5.4. Finally, in Section 5.6, we consider examples of how clone coherent measures (of both divergent and convergent type) interact with words of the form $1^\infty v$.

5.2. **Initial runs and hikes under a general clone measure.** We begin by computing certain probabilities ("correlations") under general clone coherent measures on the Young–Fibonacci lattice. Observe that a Fibonacci word w can be parsed in two different ways. Looking at consecutive strings of 2's, define $(h_1, h_2, \dots)$ and $(\tilde{h}_1, \tilde{h}_2, \dots)$ by

$$w = 2^{h_1} 1 2^{h_2} 1 \cdots 1 2^{h_m}, \qquad h_j \in \mathbb{Z}_{\ge 0}; \qquad\qquad \tilde{h}_k := \begin{cases} 2h_k + 1, & k \le m-1, \\ 2h_m, & k = m. \end{cases} \tag{5.1}$$

The quantities $\tilde{h}_k$ appeared in Section 2.2 above. Alternatively, we can look at consecutive strings of 1's, and define $(r_1, r_2, \dots)$ and $(\tilde{r}_1, \tilde{r}_2, \dots)$ by

$$w = 1^{r_1} 2 1^{r_2} 2 \cdots 2 1^{r_p}, \qquad r_j \in \mathbb{Z}_{\ge 0}; \qquad\qquad \tilde{r}_k := \begin{cases} r_k + 2, & k \le p-1, \\ r_p, & k = p. \end{cases} \tag{5.2}$$

In (5.1) and (5.2), the sequences $(h_1, h_2, \dots)$ and $(r_1, r_2, \dots)$ are called the *hikes* and *runs* of the word w, respectively. We will use the shorthand notation $r_{[i,j]} := r_i + r_{i+1} + \dots + r_j$, and similarly for $\tilde{r}_{[i,j]}$, $h_{[i,j]}$, and $\tilde{h}_{[i,j]}$, and also for open and half-open intervals. In (5.1) and (5.2), the quantities m and p depend on w, and we have $\tilde{h}_{[1,m]} = \tilde{r}_{[1,p]} = |w|$.

Our goal is to obtain joint distributions for several initial runs or hikes r_j or h_j under a clone coherent measure

$$M_n(w) := \dim(w) \cdot \varphi_{\vec{x},\vec{y}}(w) = \dim(w) \cdot \frac{s_w(\vec{x} \,|\, \vec{y})}{x_1 \cdots x_n}, \qquad w \in \mathbb{YF}_n. \tag{5.3}$$

As always, we assume that $x_i \neq 0$ for all i. We start with runs:

Proposition 5.1. *Fix $k \in \mathbb{Z}_{\ge 1}$ and $r_1, \dots, r_k \in \mathbb{Z}_{\ge 0}$. Then for all $n \ge \tilde{r}_{[1,k]}$ we have*

$$M_n\left(w \colon r_1(w) = r_1, \dots, r_k(w) = r_k\right) = \prod_{j=1}^{k} \frac{(n_j - r_j - 1)\, B_{r_j}(n_j - r_j - 2)}{x_{n_j} x_{n_j - 1} \cdots x_{n_j - r_j - 1}}, \tag{5.4}$$

where we denoted $n_j := n - \tilde{r}_{[1,j)}$, and used the shorthand notation from Remark 2.4.

Remark 5.2. The sum over $r_1, \dots, r_k$ of the quantities (5.4) is strictly less than 1. Indeed, for example, if $k = 1$, then the word w must be of the form $1^{r_1} 2u$, where the Fibonacci word u is possibly empty. This excludes the possibility that $w = 1^n$. See also Lemma 5.8 below for an explicit example.

Proof of Proposition 5.1. We have $M_n(w) = \dim(w)\, s_w(\vec{x} \,|\, \vec{y}) / (x_1 \cdots x_n)$. Let $w = 1^{r_1} 2 \cdots 1^{r_k} 2u$, where u is a generic Fibonacci word with fixed weight $|u| = n - \tilde{r}_{[1,k]} \ge 0$. In particular, the event we consider in (5.4) requires the word w to have at least k letters 2, and $\tilde r_j = r_j + 2$ for all $j = 1, \dots, k$. Using the recurrent definition (2.11) of the clone Schur functions, we can write

$$\frac{s_w(\vec x \,|\, \vec y)}{x_1 \cdots x_n} = \frac{s_u(\vec x \,|\, \vec y)}{x_1 \cdots x_{|u|}} \prod_{j=1}^{k} \frac{B_{r_j}\bigl(n - \tilde r_{[1,j]}\bigr)}{x_{n-\tilde r_{[1,j)}} \cdots x_{n-\tilde r_{[1,j]}+1}}.$$

Applying this relation to the Plancherel specialization and using (2.17), we get

$$\dim(w) = \dim(u) \cdot \prod_{j=1}^{k} \bigl(n - \tilde r_{[1,j]} + 1\bigr).$$

Summing $M_n(w)$ over all words u eliminates the dependence on u thanks to the probability normalization, and we obtain the desired product. Note that in the product in (5.4), we changed the notation $n - \tilde r_{[1,j]} = n_j - r_j - 2$. □

Let us turn to hikes. Their joint distributions do not admit a simple product form like (5.4) due to runs of 1's arising for zero values of the hikes. Let us denote $d_j := n - \tilde h_{[1,j)}$ (with $d_0 = d_1 = n$), and recursively define for $j = 1, 2, \dots, m$:

$$c_j := \begin{cases} 0, & \text{if } j = 1; \\ c_{j-1} + 1, & \text{if } d_j = d_{j-1} - 1; \\ 1, & \text{otherwise.} \end{cases} \tag{5.5}$$

The condition $d_j = d_{j-1} - 1$ is equivalent to $h_{j-1} = 0$. For example, if $h = (2, 0, 0, 0, 0, 2, 0, 1)$, then the word w and the sequences d and c have the following form:

$$w = 221111122112, \qquad d = (17, 12, 11, 10, 9, 8, 3, 2), \qquad c = (0, 1, 2, 3, 4, 5, 1, 2). \tag{5.6}$$

Lemma 5.3. *Let a Fibonacci word $w = 2^{h_1} 1 \cdots 2^{h_m}$ be decomposed as in (5.1). Let $1 \le k \le m$ be such that $h_k > 0$. Then with the above notation d_j, c_j, we have*

$$s_w(\vec x \,|\, \vec y) = s_u(\vec x \,|\, \vec y) \cdot \Biggl(\prod_{i=1}^{k-1} \prod_{j=2}^{h_i} y_{d_i - 2j + 1} \Biggr) \prod_{j=1}^{k} \frac{B_{c_j}\bigl(d_j - 2\bigr)}{\mathbf{1}_{d_j \ne d_{j-1} - 1} + B_{c_{j-1}}\bigl(d_j - 1\bigr)\, \mathbf{1}_{d_j = d_{j-1} - 1}}, \tag{5.7}$$

where $u = 2^{h_k - 1} 1 2^{h_{k+1}} 1 \cdots 1 2^{h_m}$, and we used the shorthand notation from Remark 2.4.

For example, for the word in (5.6) and $k = 6$, the last product in (5.7) telescopes as

$$B_0(15)\, B_1(10)\, \frac{B_2(9)}{B_1(10)}\, \frac{B_3(8)}{B_2(9)}\, \frac{B_4(7)}{B_3(8)}\, \frac{B_5(6)}{B_4(7)} = B_0(15) B_5(6).$$

Proof of Lemma 5.3. This is established similarly to the proof of Proposition 5.1. The first product of the y_j's in (5.7) comes from the determinants B_0. The second product is telescoping to account for the recurrence involved in defining the clone Schur functions for words of the form $1^k 2 v$. Indeed, the entries of the sequence c are increasing by 1 when there is a run of 1's in the word w (see the example in (5.6)). This corresponds to the cases when $d_j = d_{j-1} - 1$ in the denominator. Once the run of 1's ends, the next element of the sequence c resets to 1. Then $d_j \ne d_{j-1} - 1$, the denominator is equal to 1, and the index of the remaining determinant B_{c_j} is precisely the length of the run of 1's in the word w. This completes the proof. □

Proposition 5.4. *Fix $k \in \mathbb{Z}_{\ge 1}$ and $h_1, \ldots, h_k \in \mathbb{Z}_{\ge 0}$. Then for all $n \ge \tilde{h}_{[1,k]} + 2$ we have*

$$M_n\left(w\colon h_1(w) = h_1, \ldots, h_k(w) = h_k, h_{k+1}(w) > 0\right)$$
$$= \Bigg(\prod_{i=0}^{\tilde{h}_{[1,k]}+1} x_{n-i}^{-1}\Bigg)\Bigg(\prod_{i=1}^{k}\prod_{j=2}^{h_i}(d_i - 2j + 1)y_{d_i-2j+1}\Bigg)\prod_{j=1}^{k+1}\frac{(d_j-1)B_{c_j}(d_j-2)}{\mathbf{1}_{d_j\ne d_{j-1}-1} + d_j\, B_{c_{j-1}}(d_j-1)\,\mathbf{1}_{d_j=d_{j-1}-1}}, \tag{5.8}$$

where we use the notation d_j, c_j introduced before Lemma 5.3.

Proof. Let $w = 2^{h_1}1\cdots 12^{h_k}12v$, where $v = 2^{h_{k+1}-1}1\cdots 12^{h_m}$. Here v is a generic Fibonacci word with fixed weight $|v| = n - \tilde{h}_{[1,k]} - 2 \ge 0$. In particular, the event we consider in (5.8) requires w to have at least k letters 1, and the number of hikes m in (5.1) satisfies $m \ge k+1$.

Applying Lemma 5.3 twice — once for $s_w(\vec{x}\,|\,\vec{y})/(x_1\cdots x_n)$, and once for $s_w(\Pi) = \dim(w)$, we obtain the desired product times $\dim(v)\cdot s_v(\vec{x}\,|\,\vec{y})/(x_1\cdots x_{|v|})$. Summing over the generic word v eliminates the dependence on v thanks to the probability normalization, and we obtain (5.8). □

Unlike for the runs in Proposition 5.1, the result of Proposition 5.4 does not uniquely determine the joint distribution of the hikes $h_1, \ldots, h_k$. Let us obtain an expression for the probability of the event $h_1 = 0$, which will be useful for the scaling limit in Section 5.4 below.

Lemma 5.5. *For an arbitrary clone Schur measure M_n, we have*

$$M_n(w\colon h_1(w) = 0) = 1 - \frac{(n-1)\,y_{n-1}}{x_{n-1}x_n}.$$

Proof. From the recurrent definition (2.11) of the clone Schur functions, we get for any $v \in \mathbb{YF}_{n-2}$:

$$M_n(w = 2v) = \frac{(n-1)\,y_{n-1}}{x_{n-1}x_n}\,M_{n-2}(v).$$

Summing over all v gives the probability that $h_1(w) > 0$, and the result follows. □

5.3. **Charlier (deformed Plancherel) specialization.** Consider the Charlier specialization (Definition 3.22)

$$x_k = k + \rho - 1 \quad\text{and}\quad y_k = k\rho, \qquad \rho \in (0,1]. \tag{5.9}$$

Definition 5.6. For any $0 < \rho < 1$, let η_ρ be a random variable on $[0,1]$ with the distribution

$$\rho\,\delta_0(\alpha) + (1-\rho)\,\rho(1-\alpha)^{\rho-1}\,d\alpha, \qquad \alpha \in [0,1]. \tag{5.10}$$

In words, η_ρ is the convex combination of the point mass at 0 and the Beta random variable $\mathrm{beta}(1,\rho)$, with weights ρ and $1-\rho$.

Recall the run statistics $r_k(w)$ (5.2), where w is a Fibonacci word.

Theorem 5.7. *Let $w \in \mathbb{YF}_n$ be a random Fibonacci word distributed according to the deformed Plancherel measure M_n (5.3), (5.9) with $0 < \rho < 1$. For any fixed $k \ge 1$, the joint distribution of the runs $(r_1(w), \ldots, r_k(w))$ has the scaling limit*

$$\frac{r_j(w)}{n - \sum_{i=1}^{j-1} r_i(w)} \xrightarrow[n\to\infty]{d} \eta_{\rho;j}, \qquad j = 1, \ldots, k,$$

where $\eta_{\rho;j}$ are independent copies of η_ρ.

Before proving Theorem 5.7, observe that we can reformulate this statement in terms of the residual allocation (stick-breaking) process, as in Definition 2.3:

$$\Big(\frac{r_1(w)}{n}, \frac{r_2(w)}{n}, \ldots\Big) \xrightarrow{d} X = (X_1, X_2, \ldots),$$

where $X_1 = U_1$, $X_k = (1-U_1)\cdots(1-U_{k-1})U_k$ for $k \geq 2$, and U_k are independent copies of η_ρ (see Definition 5.6). Unlike in the classical GEM distribution family, here the variables U_k can be equal to zero with positive probability ρ. Thus, the random Fibonacci word under the Charlier (deformed Plancherel) measure asymptotically develops hikes of 2's of bounded length (namely, these lengths are geometrically distributed with parameter ρ). On the other hand, if we remove all zero entries from the sequence $X = (X_1, X_2, \ldots)$, then the resulting sequence is distributed simply as $\mathrm{GEM}(\rho)$.

Note also that for $\rho = 1$, we have $U_k = 1$ almost surely. This corresponds to the fact that the deformed Plancherel measure reduces to the usual Plancherel measure. By [GK00a] (see Section 2.2), random Fibonacci words under the usual Plancherel measure have only a few 1's. Thus, for $\rho = 1$, the scaling limit of the runs of 1's is trivial, and instead one must consider the scaling limit of the hikes of 2's. This is the subject of the next Section 5.4.

In the rest of this subsection, we prove Theorem 5.7. First, by Proposition 5.1, we can express the joint distribution of finitely many initial runs of 1's in a random Fibonacci word in terms of a discrete distribution $\eta_\rho^{(m)}$ on $\{0, 1, \ldots, m-1\}$:

$$\mathbb{P}(\eta_\rho^{(m)} = r) = \begin{cases} \dfrac{(m-r-1)\,B_r\big(m-r-2\big)\,\Gamma(m+\rho-r-2)}{\Gamma(m+\rho)}, & r = 0, 1, \ldots, m-2, \\ \dfrac{\rho^m \Gamma(\rho)}{\Gamma(m+\rho)}, & r = m-1. \end{cases} \tag{5.11}$$

Here $B_r(m)$ are the determinants (2.10) with shifts (we use the notation of Remark 2.4). By Lemma 5.8 which we establish below, we have

$$\sum_{r=0}^{m-2} \frac{(m-r-1)\,B_r\big(m-r-2\big)\,\Gamma(m+\rho-r-2)}{\Gamma(m+\rho)} = 1 - \frac{\rho^m \Gamma(\rho)}{\Gamma(m+\rho)}, \tag{5.12}$$

so (5.11) indeed defines a probability distribution.

Proposition 5.1 states that the joint distribution of a finite number of initial runs of 1's under the deformed Plancherel measure has the product form

$$M_n\left(w \colon r_1(w) = r_1, \ldots, r_k(w) = r_k\right) = \prod_{j=1}^{k} \mathbb{P}(\eta_\rho^{(n_j)} = r_j), \tag{5.13}$$

where $n_j = n - \tilde{r}_{[1,j)} = n - (2j-2) - r_1 - \ldots - r_{j-1}$, and $0 \leq r_j \leq n_j - 2$ for all $j = 1, \ldots, k$. By (5.12), we know that the sum of the probabilities (5.13) over all r_j with $0 \leq r_j \leq n_j - 2$ is strictly less than 1 (see also Remark 5.2 and the proof of Lemma 5.8 below). To get honest probability distributions, we have artificially assigned the remaining probability weights $\rho^{n_j}\Gamma(\rho)/\Gamma(n_j+\rho)$ to $r_j = n_j - 1$. Since

$$\rho^{n_j} \frac{\Gamma(\rho)}{\Gamma(n_j+\rho)} = \frac{\rho^{n_j}}{\rho(\rho+1)\cdots(\rho+n_j-1)}$$

rapidly decays to 0 as $n_j \to \infty$, these additional probability weights can be ignored in the scaling limit. More precisely, by Lemma 5.9 which we establish below, each random variable $\eta_\rho^{(n_j)}$, scaled by n_j^{-1}, converges in distribution to η_ρ. Thanks to the product form of (5.13), the scaled random variables $r_j(w)/n_j$ become independent in the limit, and each of them converges in distribution to η_ρ. This completes the proof of Theorem 5.7 modulo Lemmas 5.8 and 5.9 which we now establish.

Lemma 5.8. *Let $B_r(m)$ be the determinants (2.10) with shifts (Remark 2.4), and consider the deformed Plancherel specialization (5.9) of the variables x_i, y_i. Then for any $m \ge 2$, we have*

$$\sum_{r=0}^{m-2} \frac{(m-r-1)\,B_r\big(m-r-2\big)\,\Gamma(m+\rho-r-2)}{\Gamma(m+\rho)} = 1 - \frac{\rho^m\Gamma(\rho)}{\Gamma(m+\rho)}. \tag{5.14}$$

Proof. Consider the random word $w \in \mathbb{YF}_m$ under the deformed Plancherel measure M_n. From Proposition 5.1, we know that the r-th summand in the left-hand side of (5.14) is the probability that this word has the form 1^r2u, for a (possibly empty) Fibonacci word u. Summing all these probabilities over $r = 0, 1, \ldots, m-2$, we obtain $1 - M_n(w = 1^m)$. We have

$$M_n(w = 1^m) = \frac{s_{1^m}(\Pi)\,s_{1^m}(\vec{x}\,|\,\vec{y})}{\rho(\rho+1)\cdots(\rho+m-1)} = \frac{A_m(\Pi)\,A_m(\vec{x}\,|\,\vec{y})}{\rho(\rho+1)\cdots(\rho+m-1)} = \frac{1\cdot\rho^m}{\rho(\rho+1)\cdots(\rho+m-1)}.$$

This completes the proof. □

Lemma 5.9. *Let $0 < \rho < 1$. Recall the distribution $\eta_\rho^{(m)}$ (5.11). We have*

$$\frac{\eta_\rho^{(m)}}{m} \xrightarrow{d} \eta_\rho, \qquad m \to \infty,$$

where η_ρ is described in Definition 5.6.

Proof. Since $\mathbb{P}(\eta_\rho^{(m)} = m-1)$ rapidly decays to zero as $m \to \infty$, we can ignore this probability it in the limit. For an arbitrary specialization $(\vec{x}, \vec{y})$, the determinants $B_k(m)$ satisfy the three-term recurrence:

$$B_k(m) = x_{m+k+2}B_{k-1}(m) - y_{m+k+1}B_{k-2}(m), \quad k \ge 2, \tag{5.15}$$

with initial conditions $B_0(m) = y_{m+1}$ and $B_1(m) = x_{m+3}y_{m+1} - x_{m+1}y_{m+2}$. Substituting (5.9), we obtain

$$\begin{aligned} &B_k(m) = (k+m+\rho+1)B_{k-1}(m) - \rho(k+m+1)B_{k-2}(m), \\ &B_0(m) = \rho(m+1), \qquad B_1(m) = \rho(m+2-\rho). \end{aligned} \tag{5.16}$$

This recurrence has a unique solution which has the form

$$\begin{aligned} B_k(m) = \rho^{k+1}(m+1) - \rho^{k+1}(1-\rho)(m+2)e^{-\rho}E_{m+3}(-\rho) \\ + \rho(1-\rho)\frac{(m+k+2)!}{(m+1)!}e^{-\rho}E_{m+k+3}(-\rho), \end{aligned} \tag{5.17}$$

where $E_r(z)$ is the exponential integral

$$E_r(z) = \int_1^\infty t^{-r}e^{-zt}dt, \qquad r \ge 0, \quad \mathrm{Re}(z) > 0. \tag{5.18}$$

Since our $z = -\rho < 0$, formula (5.18) needs to be analytically continued [NIS24, (8.19.8)]:

$$E_r(z) = \frac{(-z)^{r-1}}{(r-1)!}(\psi(r) - \ln z) - \sum_{\substack{k=0 \\ k\ne r-1}}^{\infty} \frac{(-z)^k}{k!\,(1-r+k)}, \qquad r = 1, 2, 3, \ldots. \tag{5.19}$$

Here $\psi(r) = \Gamma'(r)/\Gamma(r)$ is the digamma function. The logarithm $\ln z = \ln(-\rho) = \mathbf{i}\pi + \ln\rho$ has a branch cut, but all the summands in the remaining series are entire functions of z. Thus, formula (5.19) produces the desired analytic continuation of $E_r(-\rho)$.

Now, using the series representation (5.19) for the exponential integral, we can show that

$$\lim_{r\to+\infty} E_r(-\rho) = 0, \qquad \lim_{r\to+\infty} r\,E_{r+1}(-\rho) = e^{\rho}. \tag{5.20}$$

Indeed, $\psi(r)$ grows logarithmically with r, so the first summand is negligible as $r \to +\infty$ (even after multiplication by $r-1$). The series in (5.19) converges uniformly in z, so we can take the limit of the individual terms and conclude that $E_r(-\rho) \to 0$ as $r \to +\infty$. The second limit in (5.20) follows from the recurrence $r\,E_{r+1}(z) + z\,E_r(z) = e^{-z}$ [NIS24, (8.9.12)].

Assume that $r = \lfloor \alpha m \rfloor$, where $\alpha \in (0,1)$. By the standard Stirling asymptotics, the ratio $\Gamma(m+\rho-r-2)/\Gamma(m+\rho)$ in (5.11) decays to zero as $e^{-\alpha m \ln m}$. Using (5.17), (5.20), we see that

$$(m-r-1)B_r(m-r-2) \sim (m-r-1)^2\rho^{r+1} - (m-r-1)\rho^{r+1}(1-\rho) + \rho(1-\rho)\frac{(m-1)!}{(m-r-2)!}.$$

The first two summands decay exponentially and are thus negligible since $\rho < 1$. We have for the third summand:

$$\frac{(m-1)!}{(m-r-2)!}\,\frac{\Gamma(m+\rho-r-2)}{\Gamma(m+\rho)}\,\rho(1-\rho) \sim m^{-1}\rho(1-\rho)\,(1-\alpha)^{\rho-1}.$$

The prefactor m^{-1} corresponds to the scaling of our random variable $m^{-1}\eta_\rho^{(m)}$. Note that

$$\int_0^1 \rho(1-\rho)\,(1-\alpha)^{\rho-1}\,d\alpha = 1-\rho,$$

and the remaining mass is concentrated at 0 in the limit:

$$\mathbb{P}(\eta_\rho^{(m)} = 0) = \frac{(m-1)^2\rho}{(m+\rho-2)(m+\rho-1)} \to \rho, \qquad m \to \infty.$$

This completes the proof of Lemma 5.9, and finalizes the proof of Theorem 5.7. □

5.4. **Shifted Plancherel measure.** In this subsection, we consider the shifted Plancherel specialization (Definition 3.22)

$$x_k = y_k = k + \sigma - 1, \qquad \sigma \in [1,\infty). \tag{5.21}$$

Definition 5.10. Let

$$G(\alpha) \coloneqq 1 - (1-\alpha)^{\frac{\sigma}{2}}, \qquad g(\alpha) \coloneqq \frac{\sigma}{2}(1-\alpha)^{\frac{\sigma}{2}-1}, \qquad \alpha \in [0,1], \tag{5.22}$$

be the cumulative and density functions of the Beta distribution $\mathrm{beta}(1,\sigma/2)$. For any $\sigma \geq 1$, let $\xi_{\sigma;1}, \xi_{\sigma;2}, \ldots$ be the sequence of random variables with the following joint cumulative distribution function (cdf):

$$\mathbb{P}\left(\xi_{\sigma;1} \leq \alpha_1, \ldots, \xi_{\sigma;n} \leq \alpha_n\right) \coloneqq \sigma^{-n+1}G(\alpha_1)\cdots G(\alpha_n) + (\sigma-1)\sum_{j=1}^{n-1}\sigma^{-n+j}G(\alpha_1)\cdots G(\alpha_{n-j}). \tag{5.23}$$

Denote the right-hand side of (5.23) by $F_n^{(\sigma)}(\alpha_1,\ldots,\alpha_n)$.

Lemma 5.11. *The joint cdfs $F_n^{(\sigma)}$ for all $n \geq 1$ are consistent, and uniquely define the distribution of $\xi_{\sigma;1}, \xi_{\sigma;2}, \ldots$. The marginal distribution of each $\xi_{\sigma;k}$ is*

$$(1-\sigma^{-k+1})\,\delta_0(\alpha) + \sigma^{-k+1}g(\alpha)\,d\alpha.$$

In particular, $\xi_{\sigma;1}$ *is absolutely continuous and has the Beta distribution* $\mathrm{beta}(1,\sigma/2)$*, while* $\xi_{\sigma;k}$ *for each* $k \ge 0$ *has an atom at* 0 *of mass* $1-\sigma^{-k+1}$*, and the remaining mass is distributed according to* $\mathrm{beta}(1,\sigma/2)$*.*

When $\sigma = 1$, the random variables $\xi_{\sigma;k}$ reduce to a collection of independent identically distributed $\mathrm{beta}(1,1/2)$ random variables.

Proof of Lemma 5.11. Each $F_n^{(\sigma)}$ is a cdf, that is, it is continuous, increasing in each argument, satisfies the boundary conditions $F_n^{(\sigma)}(0,\ldots,0)=0$ and $F_n^{(\sigma)}(1,\ldots,1)=1$. The consistency

$$F_n^{(\sigma)}(\alpha_1,\ldots,\alpha_{n-1},1) = F_{n-1}^{(\sigma)}(\alpha_1,\ldots,\alpha_{n-1})$$

is straightforward.

Let us check the nonnegativity of the rectangle probabilities under $F_n^{(\sigma)}$. If a rectangle is n-dimensional, then we can use the fact that

$$\partial_{\alpha_1,\ldots,\alpha_n} F_n^{(\sigma)}(\alpha_1,\ldots,\alpha_n) = \sigma^{-n+1} g(\alpha_1)\cdots g(\alpha_n), \tag{5.24}$$

which produces nonnegative rectangle probabilities under $F_n^{(\sigma)}$ by integration of (5.24). If the rectangle $[a_1,b_1]\times\cdots\times[a_n,b_n]$ is of lower dimension, then it must contain zero values $a_m=b_m=0$ for each non-full axis, since under $F_n^{(\sigma)}$, there are no other lower-dimensional coordinate subspaces of positive mass. Observe that $F_n(\alpha_1,\ldots,\alpha_m,0,\alpha_{m+2},\ldots,\alpha_n)$ does not depend on $\alpha_{m+2},\ldots,\alpha_n$. Thus, it suffices to check the nonnegativity for each m-dimensional rectangle of the form

$$[a_1,b_1]\times\cdots\times[a_m,b_m]\times\{0\}\times\cdots\times\{0\}.$$

We have

$$\partial_{\alpha_1,\ldots,\alpha_m} F_n^{(\sigma)}(\alpha_1,\ldots,\alpha_m,0,\ldots,0) = (\sigma-1)\,\sigma^{-m} g(\alpha_1)\cdots g(\alpha_m),$$

which implies the nonnegativity.

We have shown that $F_n^{(\sigma)}$, $n\ge 1$, is a consistent family of cdfs, so by the Kolmogorov extension theorem, they uniquely determine the distribution of the family of random variables $\xi_{\sigma;1},\xi_{\sigma;2},\ldots$. The marginal distribution of each $\xi_{\sigma;k}$ readily follows from its cdf $F_k^{(\sigma)}(1,\ldots,1,x_k)$, and so we are done. ☐

Remark 5.12. Alternatively, the random variables $\xi_{\sigma;k}$ can be constructed iteratively as follows. Toss a sequence of independent coins with probabilities of success $1,\sigma^{-1},\sigma^{-2},\ldots$. Let N be the (random) number of successes until the first failure. We have

$$\mathbb{P}(N=n) = \sigma^{-\binom{n}{2}}(1-\sigma^{-n}), \qquad n\ge 1. \tag{5.25}$$

Then, sample N independent $\mathrm{beta}(1,\sigma/2)$ random variables. Set $\xi_{\sigma;k}$, $k=1,\ldots,N$, to be these random variables, while $\xi_{\sigma;k}=0$ for $k>N$. It is worth noting that the random variables $\xi_{\sigma;k}$ are not independent, but $\xi_{\sigma;1},\ldots,\xi_{\sigma;n}$ are conditionally independent given $N=n$.

Recall the hike statistics $h_k(w)$ and $\tilde h_k(w)$ (5.1), where w is a Fibonacci word.

Theorem 5.13. *Let* $w\in\mathbb{YF}_n$ *be a random Fibonacci word with distributed according to the shifted Plancherel measure* M_n (5.3), (5.21) *with* $\sigma\ge 1$*. For any fixed* $k\ge 1$*, the joint distribution of the hikes* $(\tilde h_1(w),\ldots,\tilde h_k(w))$ *has the scaling limit*

$$\frac{\tilde h_j(w)}{n-\sum_{i=1}^{j-1}\tilde h_i(w)} \xrightarrow[n\to\infty]{d} \xi_{\sigma;j}, \qquad j=1,\ldots,k,$$

where $\xi_{\sigma;j}$ *are given by Definition 5.10.*

Remark 5.14. In terms of the stick-breaking process, Theorem 5.13 states that

$$\Big(\frac{\tilde h_1(w)}{n},\frac{\tilde h_2(w)}{n},\ldots\Big)\xrightarrow{d} X=(X_1,X_2,\ldots),$$

where $X_1=U_1$, $X_k=(1-U_1)\cdots(1-U_{k-1})U_k$ for $k\ge 2$, and $U_k=\xi_{\sigma;k}$ are *dependent* random variables if $\sigma>1$. Due to the dependence structure of the $\xi_{\sigma;k}$'s (see Remark 5.12), a single zero in the sequence $\{X_j\}_{\ge 1}$ makes all subsequent X_j's zero. Thus, a growing random Fibonacci word under the shifted Plancherel measure has a growing number of 2's in (almost surely) finitely many initial hikes of lengths proportional to n. These initial hikes are then followed by a growing tail of 1's. We refer to the end of Section 7.5 for another approach to the asymptotics of the shifted Plancherel measure, and a detailed discussion of the limiting behavior of the total number of 2's.

When $\sigma=1$, the $\xi_{\sigma;k}$'s do not have the point mass at 0, and are independent and identically distributed as beta$(1,1/2)$. The sequence X almost surely has no zeroes, and is distributed simply as GEM$(1/2)$. In this special case, our Theorem 5.13 reduces to the result of [GK00a] recalled in Section 2.2.

Proof of Theorem 5.13. **Step 1.** We use Proposition 5.4 which expresses the joint distribution of initial hikes $h_j(w)$ (where $j=1,\ldots,k$) as a product (5.8). This product involves the determinants $B_k(m)$ which for the shifted Plancherel specialization take the same simple form for all sizes:

$$B_k(m)=m+\sigma,\qquad k\ge 0. \tag{5.26}$$

Indeed, one can deduce this from the three-term recurrence (5.15). Formula (5.26) implies that the factors in the product (5.8) are equal to

$$\begin{gathered}\prod_{i=0}^{\tilde h_{[1,k]}+1} x_{n-i}^{-1}=\frac{\Gamma(n+\sigma-\tilde h_{[1,k]}-2)}{\Gamma(n+\sigma)}=\prod_{i=1}^{k}\frac{\Gamma(\sigma+d_{i+1}-2\cdot\mathbf{1}_{i=k})}{\Gamma(\sigma+d_i)},\\ \prod_{i=1}^{k}\prod_{j=2}^{h_i}(d_i-2j+1)\,y_{d_i-2j+1}=\prod_{i=1}^{k}\frac{2^{2h_i-2}\Gamma(\frac{d_i}{2}-\frac12)\Gamma(\frac{d_i}{2}+\frac{\sigma}{2}-1)}{\Gamma(\frac{d_i}{2}-h_i+\frac12)\Gamma(\frac{d_i}{2}-h_i+\frac{\sigma}{2})},\end{gathered} \tag{5.27}$$

and the last product involving the determinants B_{c_j} is equal to

$$\prod_{i=1}^{k+1}\frac{(d_i-1)(d_i+\sigma-2)}{\mathbf{1}_{h_{i-1}>0\text{ or }i=1}+d_i\,(d_i+\sigma-1)\,\mathbf{1}_{h_{i-1}=0}\,\mathbf{1}_{i>1}}. \tag{5.28}$$

Here and in (5.27), we used the notation $d_i=n-\tilde h_{[1,i)}$, and the fact that the condition $d_i=d_{i-1}-1$ in (5.8) is equivalent to $h_{i-1}=0$ (for $i=1$, we have $d_1\ne d_0-1$, so $\mathbf{1}_{h_0>0}=1$). Note that for the shifted Plancherel specialization, the dependence on the quantities c_j (5.5) disappeared.

Step 2. Let us now consider the asymptotic behavior of (5.27), (5.28) as the d_j's grow to infinity. We examine two cases depending on whether the hike is zero or is also growing. We have for the factors in (5.28) for $i>1$:

$$\frac{(d_i-1)(d_i+\sigma-2)}{\mathbf{1}_{h_{i-1}>0}+d_i\,(d_i+\sigma-1)\,\mathbf{1}_{h_{i-1}=0}}\sim\begin{cases}d_i^2, & h_{i-1}>0,\\ 1, & h_{i-1}=0.\end{cases} \tag{5.29}$$

For the two products in (5.27), we have

$$\frac{\Gamma(\sigma+d_{i+1}-2\cdot\mathbf{1}_{i=k})}{\Gamma(\sigma+d_i)}\,\frac{2^{2h_i-2}\Gamma(\frac{d_i}{2}-\frac12)\Gamma(\frac{d_i}{2}+\frac{\sigma}{2}-1)}{\Gamma(\frac{d_i}{2}-h_i+\frac12)\Gamma(\frac{d_i}{2}-h_i+\frac{\sigma}{2})}\sim d_i^{-3-2\cdot\mathbf{1}_{i=k}}(1-\alpha_i)^{\frac{\sigma}{2}-1-2\cdot\mathbf{1}_{i=k}}, \tag{5.30}$$

where $h_i = \lfloor \alpha_i d_i/2 \rfloor$, $0 \le \alpha_i < 1$. Note that we inserted the factor $1/2$ since hikes count the 2's, so $\tilde{h}_i \sim \alpha_i d_i$.

Step 3. Consider first the situation when all $\alpha_1, \ldots, \alpha_k$ are strictly positive. Then the product of the quantities (5.29), (5.30) over all i (which is asymptotically equivalent to (5.8)) has the following behavior as $n \to +\infty$:

$$M_n\left(w\colon h_1(w) = \left\lfloor \frac{\alpha_1 d_1}{2} \right\rfloor, \ldots, h_k(w) = \left\lfloor \frac{\alpha_k d_k}{2} \right\rfloor, h_{k+1}(w) > 0\right) \sim \sigma^{-k} \prod_{i=1}^{k} (d_i/2)^{-1} \cdot \frac{\sigma}{2}(1-\alpha_i)^{\frac{\sigma}{2}-1}. \tag{5.31}$$

Here we used the fact that $d_{k+1} = d_k - 2h_k - 1$, so $d_{k+1}^2 d_k^{-2}(1-\alpha_k)^{-2} \sim 1$. Since the density $\frac{\sigma}{2}(1-u)^{\frac{\sigma}{2}-1}$ of $\mathrm{beta}(1, \sigma/2)$ integrates to 1 over $(0,1)$, we see that there is an asymptotic deficit of the probability mass equal to $1 - \sigma^{-k}$. This deficit mass is supported by the event

$$\bigcup_{i=1}^{k+1} \{w\colon h_i(w) = 0\}.$$

By Lemma 5.5, we have

$$M_n(w\colon h_1(w) = 0) = 1 - \frac{n-1}{n+\sigma-1} \to 0, \qquad n \to \infty.$$

In particular, $M_n(w\colon h_1(w) > 0) \to 1$, and the event $\{h_1 = 0\}$ is asymptotically negligible.

Now consider the case when some of the α_i's are zero in the left-hand side of (5.31). Then, due to (5.29), there is an extra factor of d_j^{-2} for each $j \ge 2$ with $\alpha_{j-1} = 0$. This means that the probability that at least one of the $h_j(w)$'s is zero (for some $1 \le j \le k$) while $h_{k+1}(w) > 0$ is negligible in the limit. Therefore, we conclude that

$$\lim_{n\to\infty} M_n(w\colon h_{k+1}(w) > 0) = \lim_{n\to\infty} M_n(w\colon h_1(w) > 0, \ldots, h_k(w) > 0, h_{k+1}(w) > 0) = \sigma^{-k}$$

for all $k \ge 0$. This implies that for all $k \ge 0$, all the deficit probability mass $1 - \sigma^{-k}$ from the left-hand side of (5.31) is supported on the event $\{h_{k+1} = 0\}$.

Step 4. Define for each $k \ge 1$ the joint cdf

$$F_k(\alpha_1, \ldots, \alpha_k) \coloneqq \lim_{n\to\infty} M_n\left(w\colon h_1(w) \le \lfloor \alpha_1 d_1/2 \rfloor, \ldots, h_k(w) \le \lfloor \alpha_k d_k/2 \rfloor\right)$$

of the scaled hikes $(\tilde{h}_1(w)/d_1, \ldots, \tilde{h}_k(w)/d_k)$. The observations in Step 3 imply that

$$F_{k-1}(\alpha_1, \ldots, \alpha_{k-1}) - F_k(\alpha_1, \ldots, \alpha_{k-1}, 0) = \sigma^{-k+1} G(\alpha_1) \cdots G(\alpha_{k-1}), \tag{5.32}$$

where $G(\cdot)$ is the cdf of the $\mathrm{beta}(1, \sigma/2)$ random variable given by (5.22). Note that $G(0) = 0$. We see that it remains to find the functions $F_k(\alpha_1, \ldots, \alpha_{k-1}, 0)$ for all $k \ge 1$. Iterating (5.32), we see that these functions are consistent as long as there is at least one zero, that is,

$$F_k(\alpha_1, \ldots, \alpha_{k-2}, 0, 0) = F_{k-1}(\alpha_1, \ldots, \alpha_{k-2}, 0),$$

and so on.

Differentiate (5.32) in $\alpha_1, \ldots, \alpha_{k-1}$. Then, because of the probability mass deficit at level $k-1$, we have

$$\partial_{\alpha_1, \ldots, \alpha_{k-1}} F_{k-1}(\alpha_1, \ldots, \alpha_{k-1}) = \sigma^{-k+2} g(\alpha_1) \cdots g(\alpha_{k-1}),$$

where $g(\alpha)$ is the density of the $\mathrm{beta}(1, \sigma/2)$ random variable (5.22). Therefore,

$$\partial_{\alpha_1, \ldots, \alpha_{k-1}} F_k(\alpha_1, \ldots, \alpha_{k-1}, 0) = (\sigma - 1)\sigma^{-k+1} g(\alpha_1) \cdots g(\alpha_{k-1}). \tag{5.33}$$

Using (5.33), we can now compute $F_k(\alpha_1, \ldots, \alpha_{k-1}, 0)$ by induction on k and iterative integration. We have $F_1(0) = 0$, then

$$\partial_{\alpha_1} F_2(\alpha_1, 0) = (\sigma - 1)\sigma^{-1} g(\alpha_1) \quad \Rightarrow \quad F_2(\alpha_1, 0) = (\sigma - 1)\sigma^{-1} G(\alpha_1) + F_2(0, 0),$$

but by consistency, $F_2(0, 0) = F_1(0) = 0$. For general k, the first integration in α_{k-1} yields

$$\partial_{\alpha_1,\ldots,\alpha_{k-2}} F_k(\alpha_1, \ldots, \alpha_{k-1}, 0) = (\sigma - 1)\sigma^{-k+1} g(\alpha_1) \cdots g(\alpha_{k-2}) G(\alpha_{k-1}) + F_{k-1}(\alpha_1, \ldots, \alpha_{k-2}, 0).$$

This procedure of iterative integration yields the unique solution for $F_k(\alpha_1, \ldots, \alpha_{k-1}, 0)$, and this leads to the formula for the joint cdf $F_k(\alpha_1, \ldots, \alpha_k) = F_k^{(\sigma)}(\alpha_1, \ldots, \alpha_k)$ (5.23) of the limit of the scaled hikes. This completes the proof of Theorem 5.13. □

5.5. **Specializations of convergent type and Type-I components.** Here we analyze the asymptotic behavior of random Fibonacci words under general convergent or divergent Fibonacci positive specializations (see Definition 3.3 and Theorem 3.9). We focus on words which have a growing prefix of 1's. Let us begin with a definition which follows [GK00b]:

Definition 5.15. A *Type-I* Fibonacci word[4] is an infinite Fibonacci word formed by appending a prefix consisting of infinitely many digits 1 to a Fibonacci word. A Type-I Fibonacci word can be uniquely expressed as either 1^∞ or $1^\infty 2w$, where w is a finite suffix in $\mathbb{YF}$. Denote the (countable) set of all Type-I words by $1^\infty\mathbb{YF}$, and the subset of all Type-I words of the form $1^\infty 2w$, $w \in \mathbb{YF}$, by $1^\infty 2\mathbb{YF} \subset 1^\infty\mathbb{YF}$.

A Type-I word $1^\infty w$ can be viewed as the equivalence class of infinite saturated chains $v_0 \nearrow v_1 \nearrow v_2 \nearrow \cdots$, starting at $v_0 = \varnothing$, with $v_n = 1^{n-m}w$ for all $n \geq m$, where $|w| = m$. We call infinite saturated chains of this kind *lonely paths*.

Definition 5.16. If $\varphi \colon \mathbb{YF} \to \mathbb{R}_{\geq 0}$ is a nonnegative, normalized harmonic function and $1^\infty w$ is a Type-I word, we define:

$$\mu_{\mathrm{I}}(1^\infty w) := \lim_{n\to\infty} M_{m+n}(1^n w),$$

where $w \in \mathbb{YF}_m$, and M_k denotes the coherent measure on $\mathbb{YF}_k$ associated to φ by (2.4). We call the (in general, sub-probability) measure $\mu_{\mathrm{I}}(\cdot)$ on $1^\infty\mathbb{YF}$ the *Type-I component* of the harmonic function φ.

Note that $0 \leq M_{n+m}(1^n w) \leq 1$. Moreover, the sequence $\{M_{n+m}(1^n w)\}_{n\geq 0}$ is weakly decreasing, and so the limit $\mu_{\mathrm{I}}(1^\infty w) \in [0, 1]$ exists.

Let $(\vec{x}, \vec{y})$ be a Fibonacci positive specialization

$$x_k = c_k(1 + t_{k-1}), \qquad y_k = c_k c_{k+1} t_k, \qquad k \geq 1,$$

which is guaranteed by Theorem 3.9. Here $\vec{t}$ is a sequence of either convergent or divergent type (Definition 3.3), and and $\vec{c}$ is any sequence of positive real numbers. Let $\varphi_{\vec{x},\vec{y}}$ be the corresponding clone harmonic function, and let μ_{I} be the associated Type-I component on $1^\infty\mathbb{YF}$.

Lemma 5.17. *We have*

$$\mu_{\mathrm{I}}(1^\infty) = \prod_{i=0}^{\infty} (1 + t_i)^{-1}. \tag{5.34}$$

Moreover, if $\vec{t}$ is of divergent type, then $\mu_{\mathrm{I}}(1^\infty)$ vanishes.

[4]Not to be confused with Type-I Al-Salam–Carlitz polynomials or similar specializations, as these objects are unrelated. The term "Type-I" in Fibonacci words comes from connections to Type-I factor representations of AF-algebras associated to the branching graph [GK00b, Section 4].

Proof. We have $\dim(1^n) = 1$ and $s_{1^n}(\vec{x}\,|\,\vec{y}) = c_1 \cdots c_n$ for all n. Using (2.15), we see that the coherent measures have the form

$$M_n(w) = \frac{s_w(\vec{x}\,|\,\vec{y})}{x_1 \cdots x_n} \dim w = \prod_{i=1}^{n-1} \frac{1}{1+t_i},$$

which converges to the desired infinite product (5.34).

For a divergent type sequence $\vec{t}$, we have (using the notation (3.6))

$$\infty = A_\infty(1) = 1 + t_1 + t_1 t_2 + t_1 t_2 t_3 + \ldots \le \prod_{i=1}^{\infty}(1+t_i),$$

where we used the fact that the t_i's are nonnegative. As the reciprocal of the product in (5.34) goes to infinity, we have $\mu_{\mathrm{I}}(1^\infty) = 0$. □

For a convergent type sequence $\vec{t}$, we either have $\mu_{\mathrm{I}}(1^\infty) = 0$ or $0 < \mu_{\mathrm{I}}(1^\infty) < 1$. The next statement discusses the latter case.

Proposition 5.18. *Let $\vec{t}$ be of convergent type, and let $\mu_{\mathrm{I}}(1^\infty) > 0$. Then*

$$\sum_{w \in \mathbb{YF}_m} \mu_{\mathrm{I}}(1^\infty 2w) = (m+1)\, B_\infty(m) \prod_{i=m}^{\infty} (1+t_i)^{-1}, \qquad m \ge 0, \tag{5.35}$$

where $B_\infty(m)$ is defined in (3.6). Moreover, $\mu_{\mathrm{I}}(1^\infty 2w) > 0$ for all $w \in \mathbb{YF}$.

Proof. For any $w \in \mathbb{YF}_m$, we have by (2.11):

$$s_{1^n 2w}(\vec{x}\,|\,\vec{y}) = B_{n-1}(m)\, s_w(\vec{x}\,|\,\vec{y}).$$

Therefore, by (2.2), we can write

$$M_{m+n+2}(1^n 2w) = \left(\prod_{i=1}^{m} x_i\right)(m+1)\, M_m(w) B_{n-1}(m) \prod_{i=1}^{m+n+2} x_i^{-1}. \tag{5.36}$$

The factor $B_{n-1}(m)$ converges as $n \to \infty$ to $B_\infty(m)$ (3.6). Thus, the limit as $n \to \infty$ of (5.36) is

$$\mu_{\mathrm{I}}(1^\infty 2w) = \lim_{n\to\infty} M_{m+n+2}(1^n 2w) = \left(\prod_{i=1}^{m-1}(1+t_i)\right)(m+1)\, M_m(w) \frac{B_\infty(m)}{\prod_{i=1}^{\infty}(1+t_i)}, \tag{5.37}$$

which is positive. This means that $\mu_{\mathrm{I}}(1^\infty 2w) > 0$ for all $w \in \mathbb{YF}$. Summing (5.37) over all $w \in \mathbb{YF}_m$ (which is a finite sum), we get the desired claim (5.35). □

Proposition 5.19. *Under the conditions of Proposition 5.18, we also have $\mu_{\mathrm{I}}(1^\infty \mathbb{YF}) = 1$.*

Proof. We rely on a result from Section 6 below, which is proven independently of the content of the present section. Multiply the identity (6.6) from Remark 6.5 by $\prod_{k=0}^{n}(1+t_k)^{-1}$ (recall that $t_0 = 0$):

$$\prod_{k=0}^{n-1}(1+t_k)^{-1} + \sum_{m=0}^{n-2}(m+1)\, B_{n-m-2}(m) \prod_{k=m}^{n-1}(1+t_k)^{-1} = 1. \tag{5.38}$$

We aim to take the limit as $n \to \infty$ inside the sum in the left-hand side of (5.38). This would yield

$$\begin{aligned}1 &= \prod_{k=0}^{\infty}(1+t_k)^{-1} + \sum_{m=0}^{\infty}(m+1)B_\infty(m)\prod_{k=m}^{\infty}(1+t_k)^{-1} \\ &= \prod_{k=0}^{\infty}(1+t_k)^{-1} + \sum_{m\ge 0}\sum_{|w|=m}(m+1)M_m(w)B_\infty(m)\prod_{k=m}^{\infty}(1+t_k)^{-1} \\ &= \mu_{\mathrm{I}}(1^\infty) + \sum_{w\in\mathbb{YF}}\mu_{\mathrm{I}}(1^\infty 2w),\end{aligned} \tag{5.39}$$

which is the desired result.

However, in order to justify the passage from (5.38) to (5.39) (the interchange of the limit and the summation), we need the convergence

$$(m+1)B_{n-m-2}(m)\prod_{k=m}^{n-1}(1+t_k)^{-1} \to (m+1)B_\infty(m)\prod_{k=m}^{\infty}(1+t_k)^{-1}, \qquad n\to\infty,$$

to be uniform in m (and then apply the dominated convergence theorem for series). Note that in (5.38), we have $n \ge m+2$. That is, we can already turn the sum over $0 \le m \le n-2$ in (5.38) into an infinite sum, by adding the zero terms for $m > n-2$.

For the products, we have

$$\left|\prod_{k=m}^{n-1}(1+t_k)^{-1} - \prod_{k=m}^{\infty}(1+t_k)^{-1}\right| = \left|\prod_{k=1}^{n-1}(1+t_k)^{-1} - \prod_{k=1}^{\infty}(1+t_k)^{-1}\right|\prod_{k=1}^{m-1}(1+t_k), \tag{5.40}$$

where $\prod_{k=1}^{m-1}(1+t_k)$ is bounded in m, so (5.40) converges to zero uniformly in m.

It remains to establish the uniform convergence in m of (recall that $n \ge m+2$)

$$(m+1)B_{n-m-2}(m) = \begin{cases} 0, & n < m+2; \\ (m+1)t_{m+1}, & n = m+2; \\ (m+1)\left(t_{m+1} - (1+t_m-t_{m+1})t_{m+2}A_{n-m-3}(m+3)\right), & n > m+2, \end{cases}$$

as $n \to \infty$. Observe that $mt_m \to 0$ as $m \to +\infty$. Indeed, this follows from the convergence of the infinite product $\prod_{k=1}^{\infty}(1+t_k)$, which is equivalent to the convergence of the series $\sum_{k=1}^{\infty} t_k$ (since the t_k's are nonnegative). Moreover, since the t_m's eventually weakly decrease to zero (Propositions 3.12 and 3.13), we can use Cauchy condensation test to conclude that $mt_m \to 0$.

The convergence $mt_m \to 0$ implies that we can discard finitely many terms with $n-K-3 < m \le n-2$ from the sum in (5.38), as they converge to zero. Let us fix some $K > 5$ once and for all. For the remaining terms, we can write

$$\begin{aligned}&\left|(m+1)B_{n-m-2}(m) - (m+1)\left(t_{m+1} - (1+t_m-t_{m+1})t_{m+2}A_\infty(m+3)\right)\right| \\ &\qquad = (m+1)(1+t_m-t_{m+1})t_{m+2}\left(t_{m+3}\cdots t_n + t_{m+3}\cdots t_n t_{n+1} + \ldots\right).\end{aligned}$$

The factor $(m+3)t_{m+2}$ is bounded. Let the sequence t_m eventually decrease starting from $m = m_0$. Pick $\varepsilon > 0$, and find $N \ge m_0 + K$ such that $t_{N-K} \le \varepsilon$. Then for all $n \ge N+3$ and $m \ge m_0$ (with the condition $m \le n-K-3$):

$$t_{m+3}\cdots t_n + t_{m+3}\cdots t_n t_{n+1} + \ldots \le t_{N-K}\cdots t_{N+3} + t_{N-K}\cdots t_{N+3}t_{N+4} + \ldots \le \frac{\varepsilon}{1-\varepsilon},$$

which is small. This completes the proof. □

Let us restate the results of Lemma 5.17 and Proposition 5.18 in terms of the boundary of the Young–Fibonacci lattice [GK00b], [BE20], [Evt20]. Recall from Section 2.1 that the extremal (Martin) boundary $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$ is the set of all nonnegative, normalized, extremal harmonic functions on $\mathbb{YF}$. An arbitrary nonnegative, normalized harmonic function φ on $\mathbb{YF}$ can be represented as a Choquet integral (2.6) with respect to a probability measure μ on $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$. The measure μ is uniquely determined by φ.

The set $1^\infty\mathbb{YF}$ of all Type-I words (Definition 5.15) constitutes a part of the boundary $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$. Indeed, the extremal *Type-I harmonic functions* corresponding to Type-I words of the form $1^\infty 2w$ are given by [GK00b, Proposition 4.2]:

$$\Phi_{1^\infty 2w}(v) := \begin{cases} \dfrac{\dim(v, 1^k 2w)}{\dim(2w)}, & \text{if } v \trianglelefteq 1^k w \text{ for some } k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $w \in \mathbb{YF}$ is fixed, $\trianglelefteq$ denotes the partial order on $\mathbb{YF}$, and $\dim(u, v)$ is the number of saturated chains in the Young–Fibonacci lattice beginning at u and ending at v. Likewise, $\Phi_{1^\infty}(v)$ takes the values 1 if $v = 1^k$ for some $k \geq 0$, and 0 otherwise.

Corollary 5.20. *Let $(\vec{x}, \vec{y})$ be a Fibonacci positive specialization of convergent type such that $\mu_{\mathrm{I}}(1^\infty) \neq 0$. Then the measure μ on the boundary $\Upsilon_{\mathrm{ext}}(\mathbb{YF})$ coincides with its Type-I component μ_{I}. Consequently, the Choquet integral representation of the clone harmonic function $\varphi_{\vec{x},\vec{y}}$ involves only Type-I harmonic functions, and has the form:*

$$\varphi_{\vec{x},\vec{y}} \;=\; \mu_{\mathrm{I}}(1^\infty)\,\Phi_{1^\infty} \;+\; \sum_{w \,\in\, \mathbb{YF}} \mu_{\mathrm{I}}(1^\infty 2w)\,\Phi_{1^\infty 2w}.$$

Proof. This result follows from the fact that $\mu_{\mathrm{I}}(1^\infty\mathbb{YF}) = 1$ (Proposition 5.18), the Choquet integral representation (2.6), and the ergodicity of the Martin boundary established in the preprints [BE20], [Evt20]. □

5.6. **Type-I components in examples.** Here we consider the Al-Salam–Chihara specialization (which is of divergent type, see Definition 3.22), and the power specializations $t_k = \varkappa/k^\alpha$, $\alpha = 1, 2$ (given in (3.14)), which are of convergent type.

5.6.1. *Al-Salam–Chihara specialization.* Consider how the clone coherent measure M_n corresponding to the Al-Salam–Chihara specialization interacts with Type-I words. Recall that the parameters are (Definition 3.22): Recall that it is given by $x_k = \rho + [k-1]_q$, $y_k = \rho[k]_q$, $k \geq 1$, where $0 < \rho \leq 1$ and $q \geq 1$. Note that as $q \to 1$, we recover the Charlier specialization, and the asymptotic behavior of the corresponding clone coherent measures was considered in Section 5.3.

By Lemma 5.5, we have

$$M_n(w\colon h_1(w) = 0) = 1 - \frac{(n-1)(q-1)q^2\rho\,(q^n - q)}{(q^n + \rho q^2 - (\rho+1)q)\,(q^n + \rho q^3 - (\rho+1)q^2)}.$$

One readily checks that this expression converges to 1 as $n \to +\infty$ exponentially fast. This means that with probability exponentially close to 1, the growing random word $w \in \mathbb{YF}_n$ under the Al-Salam–Chihara clone coherent measure does not start with a 2.

For simplicity, let us only consider the case $\rho = 1/q$.

Proposition 5.21. *For the Al-Salam–Chihara specialization with $\rho = 1/q$, we have*

$$\mu_{\mathrm{I}}(1^\infty) = 0 \qquad \text{and} \qquad \mu_{\mathrm{I}}(1^\infty 2\mathbb{YF}) = 1.$$

This means that in the growing random word there will be finitely many (but at least one) occurrences of the digit 2.

Proof. Recall from the proof of Proposition 3.23 and Proposition 3.20 that we can take the $\vec{t}$-parameters to be $t_k = [k]_q/\rho = q\,[k]_q$, and then we must modify $x_k = 1+t_{k-1} = [k]_q$ and $y_k = [k]_q$. By Lemma 5.17, we have $\mu_{\mathrm{I}}(1^\infty) = \prod_{k=1}^{\infty}(1+q[k]_q)^{-1}$, which clearly diverges to zero. This proves the first claim.

Let us find the limiting distribution of $r_1(w)$, the initial run of 1's. By Proposition 5.1, we have

$$M_{n+m+2}(1^n 2\mathbb{YF}_m) = M_{n+m+2}(w\colon r_1(w) = n) = (m+1)B_n(m)\prod_{k=m+1}^{n+m+2} x_k^{-1}.$$

Using (3.8), we have

$$\begin{aligned} M_{n+m+2}(1^n 2\mathbb{YF}_m) &= (m+1)\left(t_{m+1} - (1+t_m - t_{m+1})t_{m+2}\,A_{n-1}(m+3)\right)\prod_{k=m+1}^{n+m+2} x_k^{-1} \\ &= (m+1)\,[m+1]_q\left(q + q(q-1)[m+2]_q\Big(1+\sum_{i=1}^{n-1} q^i [m+3]_q\cdots[m+2+i]_q\Big)\right)\prod_{k=m+1}^{n+m+2}[k]_q^{-1}. \end{aligned}$$

The product over k from $m+1$ to $n+m+2$ diverges to zero as $n\to\infty$. Therefore, the only possible nonzero contribution must include the sum over i:

$$(m+1)\,q(q-1)\,[m+1]_q[m+2]_q\left(\prod_{k=m+1}^{n+m+2}[k]_q^{-1}\right)\sum_{i=1}^{n-1} q^i[m+3]_q\cdots[m+2+i]_q. \tag{5.41}$$

We aim to show that

$$\lim_{n\to\infty}\left(\prod_{k=m+1}^{n+m+2}[k]_q^{-1}\right)\sum_{i=1}^{n-1} q^i[m+1]_q[m+2]_q\cdots[m+2+i]_q = (q-1)\,q^{-m-3} \tag{5.42}$$

for all $m\ge 0$. Indeed, after cancelling out, the sum in (5.42) becomes

$$\sum_{i=1}^{n-1} q^i \prod_{j=i+1}^{n}\frac{1}{[m+2+j]_q} = \sum_{i=1}^{n-1} q^{n-i}\prod_{j=n-i+1}^{n}\frac{1}{[m+2+j]_q}$$

All terms in the latter sum except the first one decay to zero exponentially fast as $n\to+\infty$. This is because for $i\ge 2$, there are at least two factors of the form $[n{+}\mathrm{const}]_q$ in the denominator, which cannot be compensated by q^{n-i} in the numerator. Therefore, we can exchange the summation and the limit, and immediately obtain the desired outcome (5.42).

Combined with (5.41), observe that the limiting quantities sum to 1 over m

$$\sum_{m=0}^{\infty}(m+1)\,q\,(q-1)^2\,q^{-m-3} = 1.$$

This implies that $\mu_{\mathrm{I}}(1^\infty 2\mathbb{YF}) = 1$, as desired. □

5.6.2. *Power specializations.* Throughout the current Section 5.6.2, we assume that $\alpha = 1$ or 2. Set $t_k = \varkappa/k^{\alpha}$, where $0 < \varkappa \le \varkappa_1^{(\alpha)}$ (see Proposition 3.24), and $x_k = 1 + t_{k-1}$, $y_k = t_k$.

For $\alpha = 1$, we have by Lemma 5.17:

$$\mu_{\mathrm{I}}(1^{\infty}) = \prod_{k=1}^{\infty} \left(1 + \frac{\varkappa}{k}\right)^{-1},$$

which diverges to zero for all $\varkappa$, $0 < \varkappa \le \varkappa_1^{(1)} \approx 0.844637$. Note that in Proposition 5.18 and Corollary 5.20 we assumed $\mu_{\mathrm{I}}(1^{\infty})$ to be positive, and this example shows that this assumption is not always satisfied for convergent type specializations. In particular μ_{I} is identically zero, and there is no Type-I support.

Remark 5.22. One expects, for the power specialization with $\alpha = 1$, that the run statistics r_k (5.2) admit a scaling limit, similarly to the Charlier specialization considered in Section 5.3. This is suggested by the characteristic quantity

$$\frac{y_k}{x_k x_{k+1}} = \frac{(k-1)\varkappa}{\left(k + \varkappa - 1\right)\left(k + \varkappa\right)},$$

which has a very similar form to the corresponding quantity in the Charlier case:

$$\frac{y_k}{x_k x_{k+1}} = \frac{k\rho}{\left(k + \rho - 1\right)\left(k + \rho\right)}.$$

The difference is only in the shift of the index k, and the renaming of ρ to $\varkappa$.

Turning to the case $\alpha = 2$, we have by Lemma 5.17:

$$\mu_{\mathrm{I}}(1^{\infty}) = \prod_{k=1}^{\infty} \left(1 + \frac{\varkappa}{k^2}\right)^{-1} = \frac{\pi\sqrt{\varkappa}}{\sinh(\pi\sqrt{\varkappa})} > 0,$$

which means that Proposition 5.18 applies for $\alpha = 2$, and $\mu_{\mathrm{I}}(1^{\infty}\mathbb{YF}) = 1$. Let us make the latter summation identity explicit. We have by (3.6):

$$A_{\infty}(m) = 1 + \sum_{r=1}^{\infty} \frac{\varkappa^r}{m^2(m+1)^2 \cdots (m+r-1)^2} = \sum_{r=0}^{\infty} \frac{r!}{(m)_r (m)_r r!}\,\varkappa^r = {}_1F_2(1; m, m; \varkappa).$$

Thus,

$$B_{\infty}(0) = \varkappa - \frac{\varkappa(1-\varkappa)}{4}\,{}_1F_2(1; 3, 3; \varkappa) = \frac{1}{\varkappa} + \frac{\varkappa - 1}{\varkappa}\, I_0(2\sqrt{\varkappa})$$

(where I_0 is the modified Bessel function of the first kind), and $B_{\infty}(m)$ for $m \ge 1$ is similarly defined by (3.6). We see that the general identity $\mu_{\mathrm{I}}(1^{\infty}\mathbb{YF}) = 1$ (equivalent to (5.39)) takes the form

$$\sum_{m=1}^{\infty} \left[\frac{\varkappa}{(m+1)} \left(1 - \frac{m^2(m+1)^2 + (2m+1)\varkappa}{m^2(m+2)^2}\,{}_1F_2(1; m+3, m+3; \varkappa) \right) \prod_{k=1}^{m-1} \left(1 + \frac{\varkappa}{k^2}\right) + \frac{(\varkappa - 1)\varkappa^{m-2}}{(m-1)!^2} \right] = \frac{\sinh(\pi\sqrt{\varkappa})}{\pi\sqrt{\varkappa}} - \frac{\varkappa + 1}{\varkappa}, \tag{5.43}$$

where we used the standard series representation for the Bessel function. Let us emphasize that (5.43) follows from Proposition 5.18. It is not clear how to prove this identity directly, without referring to the parameters $\vec{t}$ of the Fibonacci positive specialization.

6. Clone Cauchy identities

Here we discuss summation identities involving the clone Schur functions. These identities the classical summation identities for the usual symmetric functions, including the celebrated Cauchy identity.

6.1. Clone complete homogeneous functions and clone Kostka numbers.

In this subsection, $\vec{x} = (x_1, x_2, \dots)$ and $\vec{y} = (y_1, y_2, \dots)$ are two families of indeterminates.

Definition 6.1 ([Oka94]). Given a Fibonacci word $w \in \mathbb{YF}$, the biserial *clone homogeneous function* $h_w(\vec{x} \,|\, \vec{y})$ is the monomial defined recursively by

$$h_w(\vec{x} \,|\, \vec{y}) \coloneqq \begin{cases} x_{|v|+1}\, h_v(\vec{x} \,|\, \vec{y}), & \text{if } w = 1v; \\ y_{|v|+1}\, h_v(\vec{x} \,|\, \vec{y}), & \text{if } w = 2v, \end{cases} \tag{6.1}$$

starting with the base case $h_\varnothing(\vec{x} \,|\, \vec{y}) \coloneqq 1$.

The relationship between clone homogeneous and clone Schur functions is explained by the following statement involving a clone version of Kostka numbers:

Proposition 6.2 ([Oka94, Section 4]). *Given a Fibonacci word $v \in \mathbb{YF}$, the clone homogeneous function $h_v(\vec{x} \,|\, \vec{y})$ has an expansion into clone Schur functions given by*

$$h_v(\vec{x} \,|\, \vec{y}) = \sum_{|u|=|v|} K_{u,v}\, s_u(\vec{x} \,|\, \vec{y}), \tag{6.2}$$

where $K_{u,v}$ are nonnegative integers known as the clone Kostka numbers*. They can be calculated using the following four basic recursions:*

$K_{2u,2v} = K_{u,v}$	$K_{2u,1v} = \sum_{u \nearrow w} K_{w,v}$
$K_{1u,2v} = 0$	$K_{1u,1v} = K_{u,v}$

starting from the initial conditions $K_{\varnothing,\varnothing} = 1$ and $K_{1,1} = 1$.

We refer to [Oka94] for a combinatorial interpretation of these numbers in terms of chains in the Young–Fibonacci lattice.

Remark 6.3. The recursions for $K_{2u,1v}$ and $K_{1u,1v}$ imply that $K_{w,1^n} = \dim(w)$ for any Fibonacci word $w \in \mathbb{YF}_n$. This observation, together with the expansion given in (6.2), allows us to get the following identity (familiar from the normalization (2.15) of the clone Schur functions):

$$h_{1^n}(\vec{x} \,|\, \vec{y}) \coloneqq x_1 \cdots x_n = \sum_{|w|=n} \dim(w)\, s_w(\vec{x} \,|\, \vec{y}). \tag{6.3}$$

The next corollary allows us to conveniently interpret formula (6.3).

Corollary 6.4. *For any $n \geq 0$, we have*

$$s_{1^n}(\vec{x} \,|\, \vec{y}) + \sum_{m=0}^{n-2} (m+1)\,(x_1 \cdots x_m)\, s_{1^{n-m-2}2}(\vec{x} + m \,|\, \vec{y} + m) = x_1 \cdots x_n. \tag{6.4}$$

Proof. Using the expansion

$$x_1 \cdots x_m = \sum_{|w|=m} \dim(w) s_w(\vec{x} \,|\, \vec{y}),$$

we can rewrite

$$\mathrm{LHS}\,(6.4) = s_{1^n}(\vec{x} \,|\, \vec{y}) + \sum_{m=0}^{n-2} (m+1)\, s_{1^{n-m-2}2}(\vec{x}+m \,|\, \vec{y}+m) \sum_{|w|=m} \dim(w) s_w(\vec{x} \,|\, \vec{y}). \tag{6.5}$$

By Definition 2.5, we know that

$$s_{1^{n-m-2}2w}(\vec{x} \,|\, \vec{y}) = s_{1^{n-m-2}2}(\vec{x}+m \,|\, \vec{y}+m) s_w(\vec{x} \,|\, \vec{y}),$$

so substituting into (6.5) gives

$$\begin{aligned} &s_{1^n}(\vec{x} \,|\, \vec{y}) + \sum_{m=0}^{n-2} \sum_{|w|=m} (m+1) \dim(w) s_{1^{n-m-2}2w}(\vec{x} \,|\, \vec{y}) \\ &= s_{1^n}(\vec{x} \,|\, \vec{y}) + \sum_{m=0}^{n-2} \sum_{|w|=m} \dim\left(1^{n-m-2}2w\right) s_{1^{n-m-2}2w}(\vec{x} \,|\, \vec{y}) \\ &= \sum_{|w|=n} \dim(w) s_w(\vec{x} \,|\, \vec{y}) \\ &= x_1 \cdots x_n, \end{aligned}$$

as desired. □

Remark 6.5. Let us set $x_k = 1 + t_{k-1}$ and $y_k = t_k$ for all $k \geq 1$, where $\vec{t} = (t_1, t_2, t_3, \ldots)$ is a sequence of auxiliary indeterminates (with the agreement that $t_0 = 0$). This parametrization is natural from the point of view of Fibonacci positivity characterized in Section 3 above. Under this parametrization, (6.4) becomes

$$1 \;+\; \sum_{m=0}^{n-2} (m+1)\, B_{n-m-2}(m) \prod_{k=0}^{m-1} (1+t_k) \;=\; \prod_{k=0}^{n-1} (1+t_k), \tag{6.6}$$

where the $B_\ell(m)$'s are the determinants defined in Section 2.3 above. We used identity (6.6) in the proof of Proposition 5.19 in Section 5.5 above.

Remark 6.6. Continuing from the previous Remark 6.5, if we introduce a regulating parameter z and take the formal limit as $n \to \infty$, we obtain the following identity in the ring $\mathbb{C}[t_k \colon k \geq 1][\![z]\!]$ of formal power series in z with coefficients which are polynomials in the t_k's:

$$1 + \sum_{m \geq 0} (m+1) B_\infty(m; z) \prod_{k=0}^{m-1} (1 + t_k z^k) = \prod_{k=0}^{\infty} (1 + t_k z^k), \tag{6.7}$$

where

$$\begin{aligned} B_\infty(m;z) &\coloneqq t_{m+1} z^{m+1} + \left(t_{m+1} z^{m+1} - t_m z^m - 1\right) t_{m+2} z^{m+2} A_\infty(m+3; z), \\ A_\infty(m;z) &\coloneqq 1 + \sum_{r=1}^{\infty} t_m t_{m+1} \cdots t_{m+r-1} z^{rm + \binom{r}{2}}. \end{aligned}$$

Let $\mathbf{K}_n := (K_{u,v})_{u,v\in\mathbb{YF}_n}$ denote the matrix of clone Kostka numbers This matrix has the following block structure coming from the four recursions in Proposition 6.2:

$$\mathbf{K}_n = \left(\begin{array}{c|c} \mathbf{K}_{n-2} & \mathbf{D}_{n-1}\mathbf{K}_{n-1} \\ \hline \mathbf{0} & \mathbf{K}_{n-1} \end{array}\right), \tag{6.8}$$

where $\mathbf{D}_k$ is the $\mathbb{YF}_{k-1} \times \mathbb{YF}_k$ matrix of the k-th *down operator* for the Young–Fibonacci lattice, defined by

$$\mathcal{D}\,\delta_v \;=\; \sum_{u \nearrow v} \delta_u.$$

In (6.8), we order the Fibonacci words in $\mathbb{YF}_n$ lexicographically. For example, the Fibonacci words for $n = 5$ are ordered as follows:

w	221	212	2111	122	1211	1121	1112	11111
position	1	2	3	4	5	6	7	8

In Section 7.1 below, we provide the necessary references and discussion around the operator $\mathcal{D}$ (and its adjoint $\mathcal{U}$) in connection with the Robinson–Schensted-like correspondence for the Young–Fibonacci lattice.

The matrix $\mathbf{K}_n$ is invertible, and has an inverse given by the following recursion, which is straightforward from (6.8):

Lemma 6.7 (Recursion for inverse clone Kostka matrices)**.** *The inverse clone Kostka matrices* $\mathbf{K}_n^{-1} = (K^{u,v})$ *satisfy a three-step recursion with initial conditions*

$$\mathbf{K}_0^{-1} = \mathbf{K}_1^{-1} = (1) \quad \textit{and} \quad \mathbf{K}_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

and

$$\mathbf{K}_n^{-1} \;=\; \left[\begin{array}{c|c|c} \mathbf{K}_{n-2}^{-1} & \begin{array}{c} \mathbf{0} \\ \hline -\mathbf{K}_{n-3}^{-1} \end{array} & -\mathbf{K}_{n-2}^{-1} \\ \hline \mathbf{0} & \multicolumn{2}{c}{\mathbf{K}_{n-1}^{-1}} \end{array}\right]$$

6.2. **Clone Cauchy identities.** This subsection introduces two fundamental summation formulas involving biserial clone symmetric functions. The second formula serves as a clone analogue of the classical Cauchy identity for the usual Schur symmetric functions. It is worth emphasizing that these results are biserial specializations of formulas derived from the broader, noncommutative theory of clone symmetric functions. While our handling of the results of this subsection is self contained, we note that some important features of the noncommutative theory are lost in specialization. We plan to investigate noncommutative aspects further in a future work.

We need four families of indeterminates: $\vec{x} = (x_1, x_2, \dots)$ and $\vec{y} = (y_1, y_2, \dots)$ together with $\vec{p} = (p_1, p_2, \dots)$ and $\vec{q} = (q_1, q_2, \dots)$. Recall the clone homogeneous functions (6.1) and the clone Schur functions from Definition 2.5.

Proposition 6.8 (First clone Cauchy identity). *We have*

$$H_n\big(\vec{x},\vec{y}\,;\vec{p},\vec{q}\big) \coloneqq \sum_{|w|=n} h_w(\vec{p}\,|\,\vec{q})\,s_w(\vec{x}\,|\,\vec{y}) = \det \underbrace{\begin{pmatrix} \mathrm{A}_1' & \mathrm{B}_1' & -\mathrm{C}_1' & 0 & \cdots \\ 1 & \mathrm{A}_2' & \mathrm{B}_2' & -\mathrm{C}_2' & \\ 0 & 1 & \mathrm{A}_3' & \mathrm{B}_3' & \\ 0 & 0 & 1 & \mathrm{A}_4' & \\ \vdots & & & & \ddots \end{pmatrix}}_{n\times n \text{ quadridiagonal matrix}}, \tag{6.9}$$

where $\mathrm{A}_k' = p_k x_k$, $\mathrm{B}_k' = y_k(p_k p_{k+1} - q_k)$, *and* $\mathrm{C}_k' = q_k x_k y_{k+1} p_{k+2}$ *for all* $k \geq 1$.

Proof. For simplicity, let us use the shorthand s_w, h_w, and h_w' for $s_w(\vec{x}\,|\,\vec{y})$, $h_w(\vec{x}\,|\,\vec{y})$, and $h_w(\vec{p}\,|\,\vec{q})$, respectively. Begin by noticing that

$$\begin{aligned} H_0 &= 1, \\ H_1 &= p_1 x_1, \\ H_2 &= (x_1x_2 - y_1)p_1p_2 + q_1y_1. \end{aligned}$$

The expansion $s_v = \sum_{|u|=|v|} K^{u,v}\, h_u$, where $K^{u,v}$ is the $u \times v$ entry of the inverse clone Kostka matrix $\mathbf{K}_n^{-1}$, leads to

$$H_n = \sum_{u,v\in\mathbb{YF}_n} K^{u,v}\, h_u h_v'.$$

The recursive block-matrix decomposition of $\mathbf{K}_n^{-1}$ from Lemma 6.7, along with the following identities:

$$\begin{aligned} h_{1u} &= x_{n+1}\, h_u, & h_{1v}' &= p_{n+1}\, h_v', \\ h_{2u} &= y_{n+1}\, h_u, & h_{2v}' &= q_{n+1}\, h_v', \\ h_{11u} &= x_{n+1}x_{n+2}\, h_u, & h_{11v}' &= p_{n+1}p_{n+2}\, h_v', \\ h_{21u} &= x_{n+1}y_{n+2}\, h_u, & h_{12v}' &= p_{n+3}q_{n+1}\, h_v', \end{aligned} \tag{6.10}$$

with $|u| = |v| = n$, imply that H_n satisfies the following recursion:

$$H_n \;=\; x_np_n\, H_{n-1} \;+\; y_{n-1}\big(q_{n-1} - x_{n-1}x_n\big)H_{n-2} \;-\; p_nq_{n-2}x_{n-2}y_{n-1}\, H_{n-3}, \tag{6.11}$$

for all $n \geq 3$. Equivalently, the n-th kernel H_n can be expressed as the quadridiagonal determinant given by (6.9). This completes the proof. □

Proposition 6.9 (Second clone Cauchy identity). *We have*

$$S_n\big(\vec{x},\vec{y}\,;\vec{p},\vec{q}\big) \coloneqq \sum_{|w|=n} s_w(\vec{p}\,|\,\vec{q})\,s_w(\vec{x}\,|\,\vec{y}) = \det \underbrace{\begin{pmatrix} \mathrm{A}_1 & \mathrm{B}_1 & \mathrm{C}_1 & 0 & \cdots \\ 1 & \mathrm{A}_2 & \mathrm{B}_2 & \mathrm{C}_2 & \\ 0 & 1 & \mathrm{A}_3 & \mathrm{B}_3 & \\ 0 & 0 & 1 & \mathrm{A}_4 & \\ \vdots & & & & \ddots \end{pmatrix}}_{n\times n \text{ quadridiagonal matrix}}, \tag{6.12}$$

where

$$\mathrm{A}_k = p_k x_k, \quad \mathrm{B}_k = q_k(x_k x_{k+1} - y_k) + y_k(p_k p_{k+1} - q_k), \quad \mathrm{C}_k = p_k x_k q_{k+1} y_{k+1}.$$

Note that $\mathrm{A}_k, \mathrm{B}_k, \mathrm{C}_k$ differ from $\mathrm{A}'_k, \mathrm{B}'_k, \mathrm{C}'_k$ in Proposition 6.8, hence we use different notation.

Proof of Proposition 6.9. We apply inverse clone Kostka expansion twice:

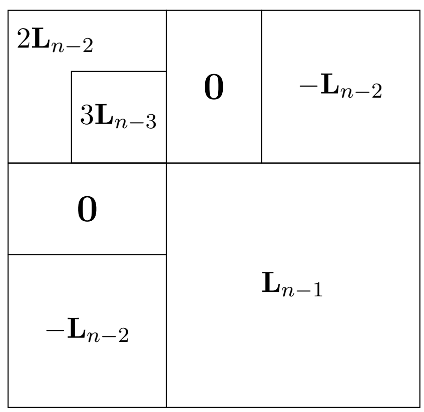
$$\begin{aligned} S_n\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) &= \sum_{w \in \mathbb{YF}_n} s_w(\vec{x}\,|\,\vec{y})\, s_w(\vec{p}\,|\,\vec{q}) \\ &= \sum_{w \in \mathbb{YF}_n} \sum_{u \in \mathbb{YF}_n} \sum_{v \in \mathbb{YF}_n} K^{u,w} K^{v,w}\, h_u(\vec{x}\,|\,\vec{y})\, h_v(\vec{p}\,|\,\vec{q}) \\ &= \mathbf{h}_n\, \mathbf{K}_n^{-1}\, \mathbf{K}_n^{-\mathrm{T}}. \end{aligned}$$

Here $\mathbf{K}_n^{-\mathrm{T}}$ is the inverse transpose of $\mathbf{K}_n$, and $\mathbf{h}_n$ is the row vector with entries $h_w(\vec{x}\,|\,\vec{y})\, h_w(\vec{p}\,|\,\vec{q})$ indexed by Fibonacci words $w \in \mathbb{YF}_n$, listed in increasing lexicographic order. Define $\mathbf{L}_n := \mathbf{K}_n^{-1}\, \mathbf{K}_n^{-\mathrm{T}}$. Observe that:

$$\mathbf{L}_0 = \mathbf{L}_1 = (1) \quad \text{and} \quad \mathbf{L}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

For $n \geq 3$, we have the following recursive block-matrix decomposition

$\mathbf{L}_n =$



It is important to emphasize that the rows and columns of $\mathbf{L}_n$ correspond to Fibonacci words $w \in \mathbb{YF}_n$ which are ordered lexicographically. For example, the hooked-shaped region labeled by $2\mathbf{L}_{n-2}$ in the upper left-hand corner corresponds to pairs of Fibonacci words $u \times v \in \mathbb{YF}_n \times \mathbb{YF}_n$ of the form $u = 2u'$ and $v = 2v'$, where $u', v' \in \mathbb{YF}_{n-2}$ and the prefixes of both u' and v' are not simultaneously equal to 1. Using (6.10) together with the block-decomposition of $\mathbf{L}_n$, we get the required three-step recurrence:

$$S_n\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) = \mathrm{A}_n\, S_{n-1}\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) + \mathrm{B}_{n-1} S_{n-2}\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) + \mathrm{C}_{n-2}\, S_{n-3}\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big)$$

for $n \geq 3$, where the initial values of S_n are given by:

$$\begin{aligned} S_0\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) &= 1, \\ S_1\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) &= p_1 x_1, \\ S_2\big(\vec{x}, \vec{y}\,; \vec{p}, \vec{q}\big) &= (p_1 p_2 - q_1)(x_1 x_2 - y_1) + q_1 y_1. \end{aligned}$$

The results of the proposition are consequences of this recurrence formula. □

7. Random permutations from clone Schur measures

In this section, we develop a model of random permutations and involutions based on the Young–Fibonacci Robinson–Schensted correspondence. This model incorporates transition and cotransition probabilities determined by clone Schur measures. These probability models exploit specific features of the Young–Fibonacci lattice that are absent in the Young lattice. Specifically, we introduce a system of cotransition probabilities defined by an arbitrary positive harmonic function $\varphi : \mathbb{YF} \to \mathbb{R}_{>0}$. This construction is valid on the Young lattice only when φ is the Plancherel harmonic function.

7.1. **The Young–Fibonacci Robinson–Schensted correspondence.** Both the Young–Fibonacci lattice $\mathbb{YF}$ and the Young lattice $\mathbb{Y}$ of integer partitions are examples of 1-*differential posets* [Sta88], [Fom94]. That is, they are:

1. Ranked, locally finite posets $(\mathbb{P}, \trianglelefteq)$ with a unique minimal element $\varnothing \in \mathbb{P}$.
2. Possess the *up* and *down* operators, denoted by $\mathcal{U}$ and $\mathcal{D}$, respectively, which satisfy the Weyl commutation relation $[\mathcal{D}, \mathcal{U}] = \mathbf{Id}$. Here $\mathcal{U}$, $\mathcal{D}$ act on the vector space $\mathbb{C}[\mathbb{P}]$ of complex-valued functions on $\mathbb{P}$ as follows:

$$\mathcal{U}\delta_v := \sum_{\substack{w \triangleright v \\ |w|=|v|+1}} \delta_w, \qquad \mathcal{D}\delta_v := \sum_{\substack{u \triangleleft v \\ |u|=|v|-1}} \delta_u,$$

where $\delta_v \colon \mathbb{P} \to \mathbb{C}$ is the indicator function supported at $v \in \mathbb{P}$.

An immediate consequence of the Weyl commutation relation is that $\mathcal{D}^n \mathcal{U}^n \delta_\varnothing = n!\delta_\varnothing$. This is equivalent to the assertion that

$$\sum_{|w|=n} \dim^2_{\mathbb{P}}(w) = n!, \tag{7.1}$$

where $|w|$ denotes the rank of $w \in \mathbb{P}$, and $\dim_{\mathbb{P}}(w)$ represents the number of saturated chains $w_0 \triangleleft \cdots \triangleleft w_n$ in $\mathbb{P}$, beginning at $w_0 = \varnothing$ and terminating at $w_n = w$. Formula (7.1) suggests a potential bijection between the set of saturated chains terminating at rank level $\mathbb{P}_n$, and permutations $\boldsymbol{\sigma} \in \mathfrak{S}_n$. In the case of the Young lattice $\mathbb{Y}$, such a bijection exists and is given by the celebrated Robinson–Schensted (RS) correspondence.

The theory of differential posets provides a framework that extends the RS correspondence beyond the combinatorics of integer partitions. Fomin [Fom94], [Fom95] demonstrated this generalization showing that an RS correspondence can be constructed for any differential poset using his concept of growth processes. Specifically, an explicit RS correspondence for the Young–Fibonacci lattice $\mathbb{YF}$ was developed in [Fom95], and later reformulated into a theory of *standard tableaux* by Roby [Rob91]. A subsequent variant was introduced by Nzeutchap in [Nze09], which circumvents the Fomin growth process. In this subsection, we briefly review Nzeutchap's version of the Young–Fibonacci RS correspondence, and employ it to get random permutations and involutions. We remark that other versions of the RS correspondence for $\mathbb{YF}$ are equally applicable for these purposes.

Like a partition, a Fibonacci word $w = a_1 \cdots a_k$ of rank $|w| = a_1 + \ldots + a_k = n$ can be visualized as an arrangement of boxes called a *Young–Fibonacci diagram*. This diagram consists of n boxes arranged from left to right into k adjacent columns, where the i-th column consists of a_i vertically stacked boxes. The following example, where $w = 12112211$, illustrates this concept in Figure 4.

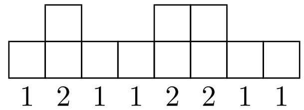


FIGURE 4. Young–Fibonacci diagram of $w = 12112211$.

A *standard Young–Fibonacci tableau* (SYFT) of shape $w \in \mathbb{YF}_n$ is a labeling of the boxes of the Young–Fibonacci diagram associated with w using indices from $\{1, \dots, n\}$ such that: (i) each index is used exactly once, (ii) box entries are strictly increasing in columns, and (iii) the top entry of any column has no entry greater than itself to is right. See Figure 5 for an example.

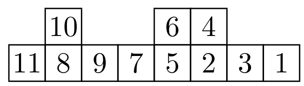
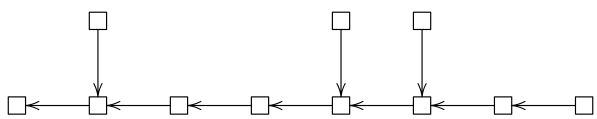


FIGURE 5. Example of a standard Young–Fibonacci tableaux of shape $w = 12112211$.

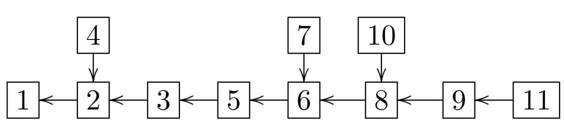
Remark 7.1. A Fibonacci word $w = a_1 \cdots a_k$ can equivalently be depicted by its *rooted tree* $\mathbb{T}_w$. This tree consists of a horizontal spine with k nodes, where the left-most node serves as the root. Additionally, a vertical leaf-node is attached to the i-th node on the spine whenever $a_i = 2$. Each edge of the tree is oriented towards the root, thereby inducing a partial order $\sqsubseteq$ on the nodes of the tree $\mathbb{T}_w$. Specifically, $a \sqsubset b$ represents a covering relation if and only if the nodes $a, b \in \mathbb{T}_w$ are joined by an edge directed from b to a. For example, the tree associated with $w = 12112211$ is illustrated in Figure 6.



(7.2)

FIGURE 6. Rooted tree $\mathbb{T}_w$ for $w = 12112211$

From this perspective, a SYFT T of shape w corresponds to a *linear extension* of $\mathbb{T}_w$. This is achieved by superimposing the cells (and entries) of T onto the nodes of $\mathbb{T}_w$, replacing each entry i with $n + 1 - i$, and then interchanging the top and bottom entries in each column of height two. The correspondence between SYFTs T of shape w and the linear extensions of $\mathbb{T}_w$ is bijective. See Figure 7 for an illustration.



(7.3)

FIGURE 7. A Llinear extension of $\mathbb{T}_w$ associated to the SYFT in Figure 5.

Equation (2.1) for $\dim(w)$ is a restatement of the general *hook-length* formula for counting linear extensions of a finite, rooted tree, applied to $\mathbb{T}_w$. Consequently, the number of SYFTs of shape w equals $\dim(w)$. This result can also be understood by constructing a bijection between SYFTs of shape $w \in \mathbb{YF}_n$ and saturated chains $w_0 \nearrow \cdots \nearrow w_n$, where $w_0 = \varnothing$ and $w_n = n$. Nzeutchap defines such a bijection using an *elimination map* $\mathcal{E}_n$. This map sends a SYFT T of shape $w \in \mathbb{YF}_n$ to a SYFT $\mathcal{E}_n[T]$ of shape $v \in \mathbb{YF}_{n-1}$ such that $w \searrow v$. See Figure 8 for an illustration. For details, see [Nze09].

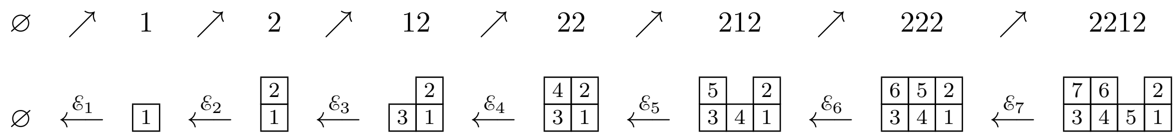


FIGURE 8. Example of elimination maps.

The *RS correspondence* for the Young–Fibonacci lattice $\mathbb{YF}$ is a bijection that maps a permutation $\boldsymbol{\sigma} \in \mathfrak{S}_n$ to an ordered pair $\mathrm{P}(\boldsymbol{\sigma}) \times \mathrm{Q}(\boldsymbol{\sigma})$ of SYFTs, both sharing the same shape $w \in \mathbb{YF}_n$.

Given a permutation $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n) \in \mathfrak{S}_n$, the *insertion tableau* $\mathrm{P}(\boldsymbol{\sigma})$ and *recording tableau* $\mathrm{Q}(\boldsymbol{\sigma})$ are constructed as follows:

1. Read $\boldsymbol{\sigma}$ from right to left.
2. For each index $\boldsymbol{\sigma}_k$ (proceeding from right to left), match it with the maximal unmatched index to its left (including itself if no such index exists).
3. To construct $\mathrm{P}(\boldsymbol{\sigma})$, place the matched indices into columns of height two (or leave single unmatched indices as columns of height one), in the order of reading $\boldsymbol{\sigma}$ from right to left. Place the larger value of each pair at the top of its column. Assemble these columns from left to right in the tableau.
4. To construct $\mathrm{Q}(\boldsymbol{\sigma})$, replace each entry $\boldsymbol{\sigma}_k \in \mathrm{P}(\boldsymbol{\sigma})$ with its index k. For columns of height two, swap the entries between the top and bottom positions.

Figure 9 illustrates this process for $\boldsymbol{\sigma} = (2, 7, 1, 5, 6, 4, 3)$.

FIGURE 9. The Young–Fibonacci RS correspondence for $\boldsymbol{\sigma} = (2, 7, 1, 5, 6, 4, 3)$.

The Young–Fibonacci RS correspondence enjoys many of the features of the classical RS correspondence together with many novel features. For example, we have [Nze09]:

1. $\mathrm{P}(\boldsymbol{\sigma}^{-1}) = \mathrm{Q}(\boldsymbol{\sigma})$ and $\mathrm{Q}(\boldsymbol{\sigma}^{-1}) = \mathrm{P}(\boldsymbol{\sigma})$.
2. $\mathrm{P}(\boldsymbol{\sigma}) = \mathrm{Q}(\boldsymbol{\sigma})$ if and only if $\boldsymbol{\sigma}$ is an involution. Furthermore, the cycle decomposition of an involution $\boldsymbol{\sigma}$ can be determined from the columns of $\mathrm{P}(\boldsymbol{\sigma})$ as follows:
   a. The number of two-cycles in $\boldsymbol{\sigma}$ is $\boldsymbol{h}(w)$, the total number of digits 2 in w.
   b. The number of fixed points of $\boldsymbol{\sigma}$ is $\boldsymbol{z}(w)$, the total number of digits 1 in w.
   Here, $w \in \mathbb{YF}$ is the shape of $\mathrm{P}(\boldsymbol{\sigma}) = \mathrm{Q}(\boldsymbol{\sigma})$.
3. $\mathrm{Q}(\boldsymbol{\sigma}') = \mathcal{E}_n\big[\mathrm{Q}(\boldsymbol{\sigma})\big]$, where $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n)$, and $\boldsymbol{\sigma}'$ is the *standardization* of $(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{n-1})$ (that is, the permutation $\boldsymbol{\sigma}' \in \mathfrak{S}_{n-1}$ preserves the relative order of the entries of $\boldsymbol{\sigma}$).

4. The lexicographically minimal, reduced factorization $s_{j_1\downarrow r_1}\cdots s_{j_k\downarrow r_k}$ of $\boldsymbol{\sigma}$ can be derived from the (appropriately defined) *inversions* in the tableaux $\mathrm{P}(\boldsymbol{\sigma})$ and $\mathrm{Q}(\boldsymbol{\sigma})$. Here we use the notations $s_j=(j,j+1)$ and $s_{j\downarrow r}=s_j\cdots s_{j-r+1}$. For further details, see [HS24].

7.2. **Transition and cotransition measures for the Young–Fibonacci lattice.** We first recall the standard construction of cotransition and transition probabilities for the Young–Fibonacci lattice. These notions are associated with general branching graphs (e.g., see [BO16]).

Let M_n on $\mathbb{YF}_n$ be a coherent family of measures associated with a positive normalized harmonic function φ by (2.4). The coherence property (2.5) is equivalent to the fact that the M_n's are compatible with the (*standard*) *cotransition probabilities*

$$\mu_{\mathrm{CT}}^{\mathrm{std}}(w,v)\coloneqq\frac{\dim v}{\dim w},\qquad v\in\mathbb{YF}_{n-1},\ w\in\mathbb{YF}_n. \tag{7.4}$$

If $w\not\searrow v$, we set $\mu_{\mathrm{CT}}^{\mathrm{std}}$ to zero. Note that $\mu_{\mathrm{CT}}^{\mathrm{std}}$ do not depend φ.

Using the cotransition probabilities (7.4), we can define the joint distribution on $\mathbb{YF}_{n-1}\times\mathbb{YF}_n$ with marginals M_{n-1} and M_n, whose conditional distribution from level n to $n-1$ is given by $\mu_{\mathrm{CT}}^{\mathrm{std}}$. The conditional distribution in the other direction is, by definition, given by the *transition probabilities*, which now depend on φ:

$$\mu_{\mathrm{T}}^{\varphi}(v,w)\coloneqq\frac{\varphi(w)}{\varphi(v)},\qquad v\in\mathbb{YF}_{n-1},\ w\in\mathbb{YF}_n \tag{7.5}$$

(and this is zero if $w\not\searrow v$).

Using the transition probabilities (7.5), we can define probability distributions on arbitrary saturated chains from $w_0=\varnothing$ to $\mathbb{YF}_n$:

$$\overline{\mu}_{\mathrm{T}}^{\varphi}\big(w_0\nearrow\cdots\nearrow w_n\big)\coloneqq\prod_{k=1}^{n}\mu_{\mathrm{T}}^{\varphi}\big(w_{k-1},w_k\big)=\varphi(w_n),\qquad w_n\in\mathbb{YF}_n. \tag{7.6}$$

Note that the distribution (7.6) is uniform for all chains that end at the same Fibonacci word w_n. This is known as the *centrality* property in the works of Vershik and Kerov (e.g., see [VK81]). The transition probabilities associated with a harmonic function φ define an infinite random walk on the Young–Fibonacci lattice starting from $\varnothing$. The probability that the random walk passes through a given Fibonacci word $w\in\mathbb{YF}_n$ is equal to $M_n(w)=\dim w\cdot\varphi(w)$.

Let us now define a new family of cotransition probabilities which are associated to a given positive normalized harmonic function φ. We emphasize that this construction is specific to the "reflective" nature of the Young–Fibonacci lattice; namely, that $v\searrow u$ if and only if $2u\searrow v$.

Definition 7.2 (Cotransition probabilities for an arbitrary harmonic function)**.** For $w\in\mathbb{YF}_n$ and $v\in\mathbb{YF}_{n-1}$, let us define the (*generalized*) *cotransition probabilities*

$$\mu_{\mathrm{CT}}^{\varphi}(w,v)\coloneqq\begin{cases}1, & \text{if } w=1v,\\ \dfrac{\varphi(v)}{\varphi(u)}, & \text{if } w=2u,\\ 0, & \text{if } w\not\searrow v.\end{cases} \tag{7.7}$$

One can readily check that in the Plancherel case $\varphi=\varphi_{\mathrm{PL}}$ (2.7), the cotransition probabilities (7.7) become the standard ones from (7.4).

Proposition 7.3. *Expression (7.7) indeed defines probabilities, that is,*

$$\sum_{v\in\mathbb{YF}_{n-1}} \mu^{\varphi}_{\mathrm{CT}}(w,v)=1, \qquad w\in\mathbb{YF}_n. \tag{7.8}$$

Proof. If w starts with 1, then there is only one possibility for v corresponding to $w=1v$, and (7.8) is evident. Otherwise, for $w=2u$, the edges $w\searrow v$ are in one-to-one correspondence with the edges $u\nearrow v$. The harmonicity of φ implies that

$$\sum_{v\in\mathbb{YF}_{n-1}} \varphi(v)=\varphi(u),$$

which is equivalent to (7.8). This completes the proof. □

Similarly to $\overline{\mu}^{\varphi}_{\mathrm{T}}$ (7.6), we can define the cotransition probabilities on all saturated chains that start at a fixed Fibonacci word $w_n\in\mathbb{YF}_n$ and terminate at $w_0=\varnothing$:

$$\overline{\mu}^{\varphi}_{\mathrm{CT}}(w_n\searrow\cdots\searrow w_1\searrow w_0):=\prod_{k=1}^{n}\mu^{\varphi}_{\mathrm{CT}}(w_k,w_{k-1}). \tag{7.9}$$

The measure (7.9) is uniform on all chains if and only if $\varphi=\varphi_{\mathrm{PL}}$, the Plancherel harmonic function. Let us write $\overline{\mu}^{\varphi}_{\mathrm{CT}}(T)=\overline{\mu}^{\varphi}_{\mathrm{CT}}(w_n\searrow\cdots\searrow w_1\searrow w_0)$ whenever T is the SYFT associated to the saturated chain $w_0\nearrow\cdots\nearrow w_n$ as in the example in Figure 4. We refer to [Nze09] for details, and examples of the generalized cotransition probabilities are given in Figure 10.

We will typically be interested in the case when $\varphi=\varphi_{\vec{x},\vec{y}}$ is a clone harmonic function coming from a Fibonacci positive specialization $(\vec{x},\vec{y})$.

7.3. **Building random permutations and involutions.** To construct a random permutation in $\mathfrak{S}_n$, we observe that the RS correspondence from Section 7.1 uniquely determines a permutation $\boldsymbol{\sigma}$ by three components, namely, a random shape $w\in\mathbb{YF}_n$, and two random saturated chains in $\mathbb{YF}$, both terminating at w. This construction proceeds as follows:

1. First, select a Fibonacci word $w\in\mathbb{YF}_n$ with probability $M_n(w)$, determined by a positive harmonic function π.
2. Next, generate two saturated chains terminating at w using the cotransition probabilities $\overline{\mu}^{\varphi}_{\mathrm{CT}}$ and $\overline{\mu}^{\psi}_{\mathrm{CT}}$, associated with two (possibly different) positive harmonic functions φ and ψ. The chains are conditioned to end at the previously chosen Fibonacci word w.
3. From these two chains (viewed as SYFTs), construct a permutation $\boldsymbol{\sigma}$ using the RS correspondence.

In this way, the triad (π,φ,ψ) of positive harmonic functions determines a random permutation $\boldsymbol{\sigma}\in\mathfrak{S}_n$ for every $n\geq 1$.

Similarly, to construct a random involution in $\mathfrak{S}_n$, we pick $w\in\mathbb{YF}_n$ according to $M_n(w)$ (determined by π), and generate a single saturated chain terminating at w, sampled according to the cotransition probabilities $\overline{\mu}^{\varphi}_{\mathrm{CT}}$.

Summarizing, we have the following probability measures on permutations and involutions in $\mathfrak{S}_n$ denoted by μ_n and ν_n, respectively:

$$\begin{aligned}\mu_n(\boldsymbol{\sigma})=\mu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi,\psi)&:=\dim(w)\,\pi(w)\,\overline{\mu}^{\varphi}_{\mathrm{CT}}\big(\mathrm{P}(\boldsymbol{\sigma})\big)\,\overline{\mu}^{\psi}_{\mathrm{CT}}\big(\mathrm{Q}(\boldsymbol{\sigma})\big),\\ \nu_n(\boldsymbol{\sigma})=\nu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi)&:=\dim(w)\,\pi(w)\,\overline{\mu}^{\varphi}_{\mathrm{CT}}\big(\mathrm{P}(\boldsymbol{\sigma})\big).\end{aligned} \tag{7.10}$$

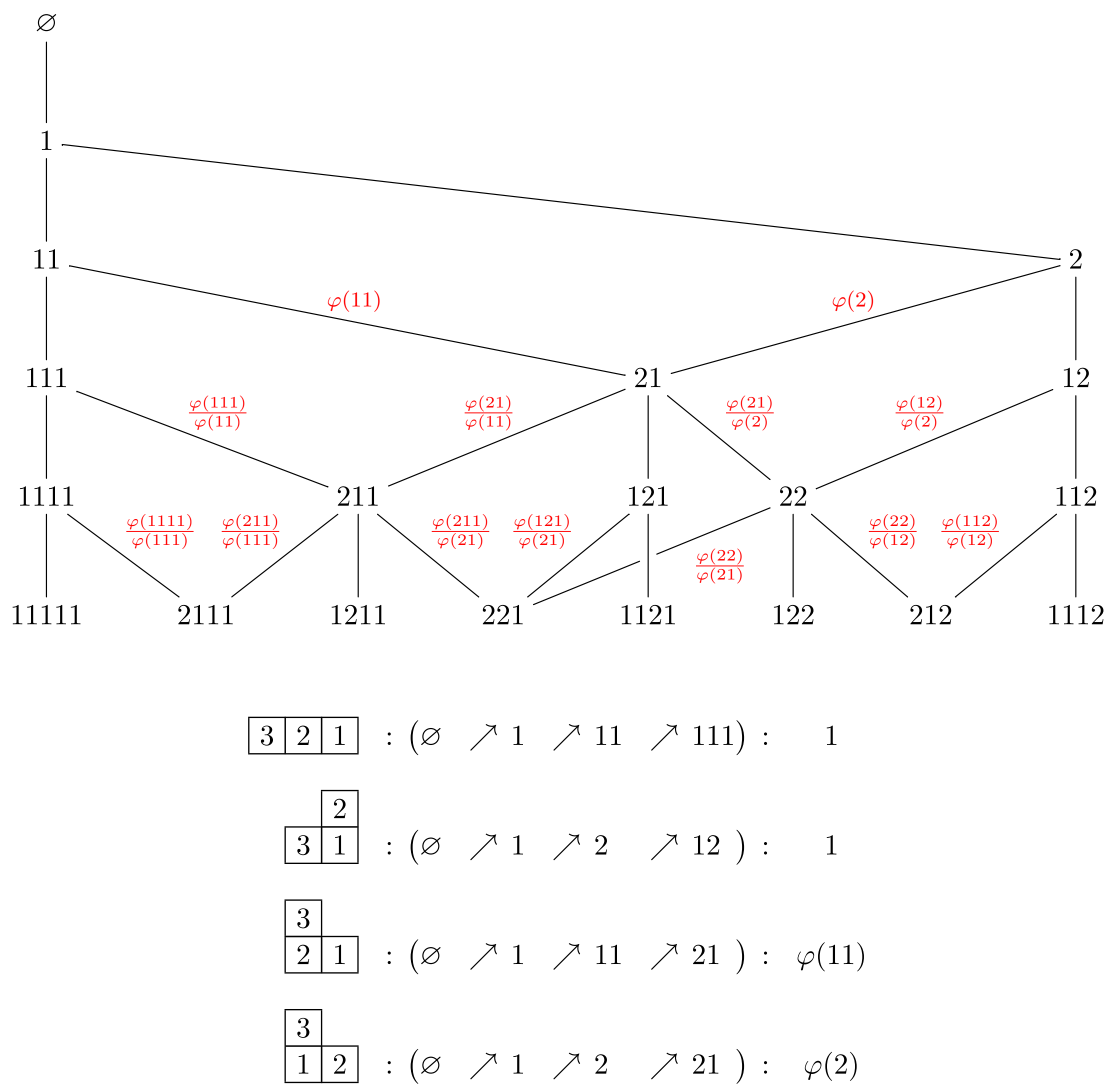


FIGURE 10. Top: Nonzero cotransition weights (in red with 1's omitted). Bottom: The four saturated chains which terminate in $\mathbb{YF}_3$, together with their associated SYFTs and cotransition weights.

For example, the distribution μ_3 on $\mathfrak{S}_3$ has the form (writing permutations in the one-line notation):

$$\begin{aligned} \mu_3(123) &= \pi(111), & \mu_3(213) &= \pi(12), \\ \mu_3(132) &= 2\pi(21)\varphi(11)\psi(11), & \mu_3(321) &= 2\pi(21)\varphi(2)\psi(2), \\ \mu_3(312) &= 2\pi(21)\varphi(11)\psi(2), & \mu_3(231) &= 2\pi(21)\varphi(2)\psi(11). \end{aligned}$$

Remark 7.4 (Plancherel cases)**.** When $\pi = \varphi = \psi = \varphi_{\mathrm{PL}}$, μ_n is simply the uniform measure on $\mathfrak{S}_n$. More generally, when $\varphi = \psi = \varphi_{\mathrm{PL}}$, each permutation $\boldsymbol{\sigma} \in \mathfrak{S}_n$ with the RS shape $w \in \mathbb{YF}_n$ occurs with probability $\mu_n(\boldsymbol{\sigma}) = \pi(w)/\dim(w)$.

For the model of random involutions, when $\varphi = \varphi_{\mathrm{PL}}$, each involution $\boldsymbol{\sigma} \in \mathfrak{S}_n$ with the RS shape $w \in \mathbb{YF}_n$ occurs with probability $\nu_n(\boldsymbol{\sigma}) = \pi(w)\dim(w)$.

7.4. **Observables from clone Cauchy identities.** As an illustration of the connection between random permutations and involutions with clone Schur functions, we calculate the expected numbers of fixed points and two-cycles of a random involution $\boldsymbol{\sigma} \in \mathfrak{S}_n$ distributed according to ν_n (7.10), where $\varphi = \varphi_{\mathrm{PL}}$, and $\pi = \varphi_{\vec{x},\vec{y}}$ for some Fibonacci positive specialization $(\vec{x}, \vec{y})$.

Recall from Section 7.1 that the fixed points and two-cycles of an involution $\boldsymbol{\sigma} \in \mathfrak{S}_n$ correspond, respectively, to the digits 1 and 2 (denoted by $\mathfrak{r}(w)$ and $\mathfrak{h}(w)$) in the shape $w \in \mathbb{YF}_n$ associated with $\boldsymbol{\sigma}$ under the Young–Fibonacci RS correspondence.

Rather than directly computing the expectations of $\mathfrak{r}(w)$ and $\mathfrak{h}(w)$, we introduce an auxiliary parameter τ and calculate the expectation values of $\tau^{\mathfrak{r}(w)}$ and $\tau^{\mathfrak{h}(w)}$. This approach leverages the first clone Cauchy identity (6.9) from Section 6.2. Since the $\mathfrak{r}(w) + 2\mathfrak{h}(w) = n$, it suffices to focus on two-cycles.

Proposition 7.5. *The expected value of $\tau^{\#\,\text{two-cycles}(\boldsymbol{\sigma})}$ for a random involution $\boldsymbol{\sigma} \in \mathfrak{S}_n$ distributed according to $\nu_n(\boldsymbol{\sigma} \,|\, \varphi_{\vec{x},\vec{y}}, \varphi_{\mathrm{PL}})$ (7.10) is given by*

$$\mathbb{E}_{\nu_n}\left[\tau^{\#\,\text{two-cycles}(\boldsymbol{\sigma})}\right] = (x_1 \cdots x_n)^{-1} \det \underbrace{\begin{pmatrix} x_1 & (1-\tau)y_1 & -\tau x_1 y_2 & 0 & \cdots \\ 1 & x_2 & (1-2\tau)y_2 & -2\tau x_2 y_3 & \\ 0 & 1 & x_3 & (1-3\tau)y_3 & \\ 0 & 0 & 1 & x_4 & \\ \vdots & & & & \ddots \end{pmatrix}}_{n\times n \text{ quadridiagonal matrix}}. \tag{7.11}$$

Proof. The left-hand side of (7.11) can be rewritten using clone Schur functions as

$$\sum_{|w|=n} \dim(w)\, \varphi_{\vec{x},\vec{y}}(w)\, \tau^{\mathfrak{h}(w)} = (x_1 \cdots x_n)^{-1} \sum_{|w|=n} \dim(w)\, s_w(\vec{x} \,|\, \vec{y})\, \tau^{\mathfrak{h}(w)}.$$

Setting $p_k = x_k^{-1}$ and $q_k = k\tau x_k^{-1} x_{k+1}^{-1}$ in the clone Cauchy identity (6.9) and noticing that $h_w(\vec{p} \,|\, \vec{q}) = (x_1 \cdots x_n)^{-1} \dim(w)\, \tau^{\mathfrak{h}(w)}$ under this specialization implies the desired quadridiagonal determinant. □

Remark 7.6. If we set $p_k = \tau x_k^{-1}$ and $q_k = k x_k^{-1} x_{k+1}^{-1}$ in the proof of Proposition 7.5, we would get the expected number of fixed points of a random involution distibuted according to $\nu_n(\boldsymbol{\sigma} \,|\, \varphi_{\vec{x},\vec{y}}, \varphi_{\mathrm{PL}})$.

The expected number of two-cycles can be computed in a standard way, by differentiating:

$$\mathbb{E}_{\nu_n}\left[\#\text{two-cycles}(\boldsymbol{\sigma})\right] = \frac{\partial}{\partial \tau}\bigg|_{\tau=1} \mathbb{E}_{\nu_n}\left[\tau^{\#\,\text{two-cycles}(\boldsymbol{\sigma})}\right].$$

This differentiation of a quadridiagonal determinant (7.11) is not explicit for a general Fibonacci positive specialization $(\vec{x}, \vec{y})$. In the next Section 7.5, we consider the particular case of the shifted Plancherel specialization $x_k = y_k = k + \sigma - 1$ for $\sigma \in [1, \infty)$ (Definition 3.22).

7.5. **Number of two-cycles under the shifted Plancherel specialization.** Consider

$$H_n(\sigma, \tau) := \mathbb{E}_{\nu_n}\left[\tau^{\#\,\text{two-cycles}(\boldsymbol{\sigma})}\right], \qquad \nu_n = \nu_n(\cdot \,|\, \varphi_{\vec{x},\vec{y}}, \varphi_{\mathrm{PL}}), \qquad x_k = y_k = k + \sigma - 1, \tag{7.12}$$

where $\sigma \in [1, \infty)$ is the parameter of the shifted Plancherel specialization (not to be confused with the random involution $\boldsymbol{\sigma}$). Denote also

$$G_n(\sigma) := \mathbb{E}_{\nu_n} [\#\text{two-cycles}(\boldsymbol{\sigma})] .$$

We side-step the differentiation of the quadridiagonal determinant, and instead work directly with the $H_n(\sigma, \tau)$'s, and their generating function

$$H(\sigma, \tau; z) := \sum_{n \geq 0} H_n(\sigma, \tau) z^n. \tag{7.13}$$

Lemma 7.7. *In the case of the shifted Plancherel specialization, the quantities $H_n(\sigma, \tau)$ (7.12) satisfy the inhomogeneous, two-step recurrence*

$$(n + \sigma - 1) H_n = H_{n-1} + \tau(n-1) H_{n-2} + (n + \sigma - 1) \varphi_{\vec{x},\vec{y}}(1^n) - \varphi_{\vec{x},\vec{y}}(1^{n-1}). \tag{7.14}$$

Proof. A crucial property of the shifted Plancherel specialization is that

$$\varphi_{\vec{x},\vec{y}}(1w) = \frac{\varphi_{\vec{x},\vec{y}}(w)}{x_n}, \qquad \varphi_{\vec{x},\vec{y}}(2v) = \frac{\varphi_{\vec{x},\vec{y}}(v)}{x_n}, \tag{7.15}$$

for any Fibonacci word $w \in \mathbb{YF}_{n-1}$ which does not consist entirely of 1-digits, and any Fibonacci word $v \in \mathbb{YF}_{n-2}$. Indeed, this is because for the shifted Plancherel specialization, we have for the second determinant in (2.10):

$$B_k(m) = m + \sigma = x_{m+1}, \qquad k \geq 0.$$

Moreover, $\dim(1w) = \dim(w)$. This implies (7.15). Now,

$$H_n(\sigma, \tau) = \sum_{|w|=n} \dim(w) \varphi_{\vec{x},\vec{y}}(w) \tau^{\hbar(w)}.$$

Split the sum into three parts: $w = 1^n$, $w = 1u$, and $w = 2v$. Rewriting the second two sums in terms of H_{n-1} and H_{n-2}, respectively, yields the desired recurrence (7.14). □

Note that for the shifted Plancherel specialization, we have

$$(n + \sigma - 1) \varphi_{\vec{x},\vec{y}}(1^n) - \varphi_{\vec{x},\vec{y}}(1^{n-1}) = \sigma - 1, \qquad n \geq 1. \tag{7.16}$$

Lemma 7.8. *The generating function $H(\sigma, \tau; z)$ (7.13) satisfies the first order ODE:*

$$z(1 - \tau z^2) \partial_z H(\sigma, \tau; z) + (\sigma - 1 - z - \tau z^2) H(\sigma, \tau; z) = \frac{\sigma - 1}{1 - z}$$

Proof. This immediately follows from the recurrence in Lemma 7.7 and the identity (7.16). □

Consider first the case $\sigma = 1$ (the usual Plancherel specialization $\pi = \varphi_{\mathrm{PL}}$). Then the ODE in Lemma 7.8 admits an explicit solution:

$$H(1, \tau; z) = \frac{1}{\sqrt{1 - \tau z^2}} \left(\frac{1 + \sqrt{\tau} z}{1 - \sqrt{\tau} z} \right)^{\frac{1}{2\sqrt{\tau}}}.$$

Taking the τ-derivative of the above expression at $\tau = 1$, we see that

$$\sum_{n \geq 0} G_n(1) z^n = \frac{z}{2(1 - z)^2} + \frac{1}{4(1 - z)} \log\left(\frac{1 - z}{1 + z} \right), \qquad |z| < 1. \tag{7.17}$$

The dominating singularity of this function is $z=1$. The first summand expands as $\sum_{n\ge 0}\frac{n}{2}\,z^n$, and one can readily check that the coefficients of the second summand are asymptotically bounded in n. We conclude that

$$\lim_{n\to\infty}\frac{G_n(1)}{n}=\frac{1}{2}. \tag{7.18}$$

Remark 7.9. The limit (7.18) aligns with the result of [GK00a], which states that under the Plancherel measure, the frequency of hikes of 2's in the random Fibonacci word (the number of two-cycles in the corresponding random permutation $\boldsymbol{\sigma}\in\mathfrak{S}_n$) scales proportionally to n. Moreover, asymptotically, 1's (fixed points of $\boldsymbol{\sigma}$) do not have a significance presence.

Consider now the general case $\sigma\in[1,\infty)$. It is not clear to the authors how to express solutions to the ODE of Lemma 7.8, even in terms of hypergeometric functions. Nevertheless, after differentiating the ODE in τ, setting $\tau=1$, and using the fact $H(\sigma,1;z)=(1-z)^{-1}$, we obtain an ODE for

$$G(\sigma;z)\coloneqq \partial_\tau|_{\tau=1}H(\sigma,\tau;z)=\sum_{n\ge 0}G_n(\sigma)\,z^n.$$

The new ODE has the form:

$$z(1-z^2)\,\partial_z G(\sigma;z)\,+\,(\sigma-1-z-z^2)\,G(\sigma;z)=\frac{z^2}{(1-z)^2}, \tag{7.19}$$

whose solution can be expressed through the hypergeometric functions:

$$\begin{aligned} G(\sigma;z)=\frac{z^{1-\sigma}\,(1+z)^{\sigma}}{2^{\sigma}(1+\sigma)\,(1-z)^2}\,{}_2F_1\left(-\frac{1+\sigma}{2},-\sigma;\frac{1-\sigma}{2};\frac{1-z}{1+z}\right)\\ -\frac{\Gamma(1+\sigma)\,\Gamma\left(\frac12-\frac{\sigma}{2}\right)}{2^{\sigma}(1+\sigma)\,\Gamma\left(\frac12+\frac{\sigma}{2}\right)}\,z^{1-\sigma}(1-z)^{-1}(1-z^2)^{(\sigma-1)/2}. \end{aligned} \tag{7.20}$$

The hypergeometric function makes sense unless σ is an odd positive integer. In the latter case, the singularities in the first and the second summand cancel out, and the whole function $G(\sigma;z)$ is well-defined for all $\sigma\in[1,\infty)$. We have $G(\sigma;0)=0$. One can also verify that as $\sigma\to 1$, the solution (7.20) reduces to the right-hand side of (7.17). Together with the known differentiation formula for the hypergeometric function, this implies that (7.20) is indeed a solution to (7.19).

Proposition 7.10. *The coefficients at $z=0$ of the generating function $G(\sigma;z)$ (7.20) scale as follows:*

$$\lim_{n\to\infty}\frac{G_n(\sigma)}{n}=\frac{1}{\sigma+1},\qquad \sigma\in[1,\infty).$$

Proof. We need to analyze the singularities of $G(\sigma;z)$ in z. There are two singularities closest to the origin, $z=1$ and $z=-1$. At $z=1$, the first summand in (7.20) clearly has a pole of order 2 and behaves as $(\sigma+1)^{-1}(z-1)^{-2}$. To complete the proof, it suffices to show that this is the dominant behavior. At $z=1$, the second summand in (7.20) behaves as $\mathrm{const}\cdot(z-1)^{\frac{\sigma-3}{2}}$, which is less singular than $(z-1)^{-2}$.

Consider now the singularity at $z=-1$. The second summand in (7.20) is regular at $z=-1$. For the first summand, transform the hypergeometric function as [NIS24, (15.8.1)]

$${}_2F_1\left(-\frac{1+\sigma}{2},-\sigma;\frac{1-\sigma}{2};\frac{1-z}{1+z}\right)=\left(\frac{2z}{1+z}\right)^{\frac{\sigma+1}{2}}\,{}_2F_1\left(-\frac{1+\sigma}{2},\frac{1+\sigma}{2};\frac{1-\sigma}{2};\frac{z-1}{2z}\right).$$

Now, the hypergeometric function becomes regular at $z = -1$. The power $(1+z)^{-\frac{\sigma+1}{2}}$, combined with the prefactor, is also regular. This implies that the singularity at $z = -1$ does not contribute to the leading behavior of the coefficients, and so we are done. $\square$

Remark 7.11. We see that for $\sigma > 1$, the asymptotic proportion of the 2's in a random Fibonacci word with the shifted Plancherel distribution is strictly less than $\frac{1}{2}$. This phenomenon agrees with the scaling limit of initial hikes of 2's in this Fibonacci word obtained in Theorem 5.13. Indeed, recall the random variables $\xi_{\sigma;k}$, $k \ge 1$, from Definition 5.10. See also Remark 5.12 for an alternative description using conditional independence on $N = n$, where N is defined by (5.25). Fix k. By Theorem 5.13, the scaling limit of the sum of the first k initial hikes in a random Fibonacci word with the shifted Plancherel distribution is given by

$$\lim_{n\to\infty} \frac{h_1(w) + \cdots + h_k(w)}{n} \to \frac{1}{2}\left(X_1 + \cdots + X_k\right) = \frac{1}{2} - \frac{1}{2}\prod_{j=1}^{k}(1 - \xi_{\sigma;j}), \tag{7.21}$$

where the X_j's are obtained from the $\xi_{\sigma;j}$'s by the stick-breaking construction (see Remark 5.14). Let us compute the expectation of the right-hand side of (7.21) with $k = \infty$ (one readily sees that the limit of (7.21) as $k \to \infty$ is well-defined). We have, using the fact that $\mathbb{E}(\mathrm{beta}(1, \sigma/2)) = \frac{2}{2+\sigma}$:

$$\mathbb{E}\left[\prod_{j=1}^{\infty}(1 - \xi_{\sigma;j})\right] = \sum_{m=1}^{\infty} \mathbb{P}(N = m)\left(\frac{\sigma}{2+\sigma}\right)^m = \sum_{m=1}^{\infty} \sigma^{-\binom{m}{2}}(1 - \sigma^{-m})\left(\frac{\sigma}{2+\sigma}\right)^m.$$

One can check that

$$\frac{1}{2}\mathbb{E}\left[\sum_{j=1}^{\infty} X_j\right] \le \frac{1}{\sigma + 1}, \tag{7.22}$$

with equality at $\sigma = 1$, where the difference between the two sides of the inequality is at most ≈ 0.015, and vanishes as $\sigma \to \infty$. See Figure 11 for an illustration of the two sides of the inequality. The discrepancy between the two sides of (7.22) is due to the interchange of the limits

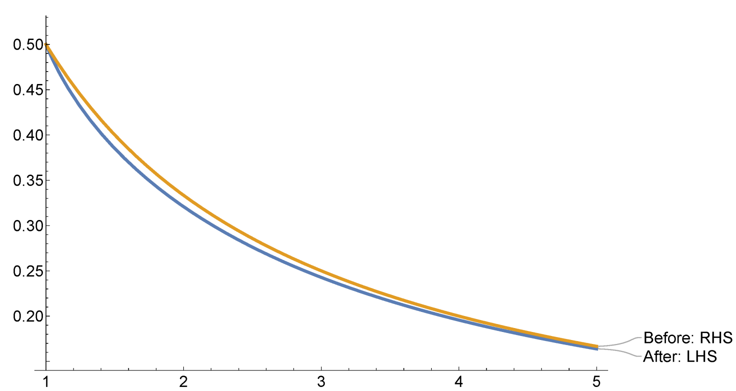


FIGURE 11. Comparing the two sides of (7.22), before and after the scaling limit of the initial hikes, for $1 \le \sigma \le 5$.

in n and k in (7.21). Specifically, taking the limit $k \to \infty$ first accounts for the total number of 2's in the random Fibonacci word, whereas sending $n \to \infty$ first considers only a finite number of initial hikes of 2's. This reveals that additional digits of 2 remain hidden in the growing random Fibonacci word after long sequences of 1's. These extra 2's contribute to the right-hand side of (7.22) but are absent from the left-hand side.

8. Additional remarks and open problems

We conclude with a discussion of some possible directions for future work. Before exploring new open problems, we note that we have already raised two questions earlier in the text — Problem 4.3 concerning Fibonacci positive specializations and the corresponding Borel measures, and Problem 4.4 regarding the Toda flow. Beyond these, several other interesting directions remain to be explored, and we highlight a few of them here.

8.1. **Schützenberger promotion and combinatorial ergodicity.**

As explained in Remark 7.1, standard Young–Fibonacci tableaux (SYFTs) T of shape $w \in \mathbb{YF}$ are in bijection with linear extensions of a binary, rooted tree $\mathbb{T}_w$ constructed from w. Like for any finite poset, there is a $\mathbb{Z}$-action on the set of linear extensions of $\mathbb{T}_w$, which implements Schützenberger *promotion* [Sch72, Sta09]. This $\mathbb{Z}$-action can be transported to the set of SYFTs of shape $w \in \mathbb{YF}$ and, by extension, to saturated chains $w_0 \nearrow \cdots \nearrow w_n$ starting at $w_0 = \varnothing$ and terminating at $w_n = w$. It would be very interesting to study the interplay between this action and the probability distributions $\mu^{\varphi}_{\mathrm{CT}}$ and $\overline{\mu}^{\varphi}_{\mathrm{CT}}$, associated to a positive harmonic function φ. This study becomes particularly intriguing when viewed through the lens of J. Propp and T. Roby's notion of *combinatorial ergodicity*; see [PR15].

A related question concerns promotion and Type-I harmonic functions. Recall that for $v \in \mathbb{YF}_k$ and $w \in \mathbb{YF}_n$ with $k \leq n$, the measure $M_k(v) = \dim(v)\,\Phi_{1^\infty 2w}(v)$ represents the probability that $w_k = v$, where $w_0 \nearrow \cdots \nearrow w_n$ is a uniformly sampled random saturated chain starting at $w_0 = \varnothing$ and terminating at $w_n = w$. Now fix a saturated chain $\mathbf{u} = u_0 \nearrow \cdots \nearrow u_n$ which terminates at $u_n = w$. For $v \in \mathbb{YF}_k$ with $k \ll n$, consider the probability $\zeta_{\mathbf{u};k}(v)$ that $w_k = v$, where $w_0 \nearrow \cdots \nearrow w_n$ is a uniformly sampled random saturated chain from the promotion orbit $\mathcal{O}_{\mathbf{u}}$ of $\mathbf{u}$. The measures $\zeta_{\mathbf{u};k}$ are not coherent. However, in light of combinatorial ergodicity, one expects that $\zeta_{\mathbf{u};k}$ approximates M_k as $n \to \infty$.

8.2. **Truncations of the Young–Fibonacci lattice.**

The theory of biserial clone Schur functions, along with the constructions introduced in Sections 6.1 and 7.2, can be adapted to the k-th *truncation* $\mathbb{YF}^{(k)}$ of the Young–Fibonacci lattice; see [HS24]. From a representation-theoretic perspective, $\mathbb{YF}^{(k)}$ is the Young-Fibonacci counterpart of the poset $\mathbb{Y}^{(k)}$, which consists of partitions $\lambda \in \mathbb{Y}$ with at most k parts. Without going into detail, $\mathbb{YF}^{(k)}$ is an infinite, ranked poset that is part of an infinite filtration:

$$\mathbb{YF}^{(1)} \subset \mathbb{YF}^{(2)} \subset \mathbb{YF}^{(3)} \subset \cdots \subset \mathbb{YF},$$

where the Hasse diagram of $\mathbb{YF}^{(k)}$ sits inside $\mathbb{YF}^{(k+1)}$ as an induced subgraph. The first two truncations, $\mathbb{YF}^{(1)}$ and $\mathbb{YF}^{(2)}$, are respectively the half-Pascal and Pascal lattices. The next truncation, $\mathbb{YF}^{(3)}$, is illustrated in Figure 12.

None of the truncations are differential posets. Clone harmonic functions $\varphi_{\vec{x},\vec{y}}$ on the Young–Fibonacci lattice can be restricted to $\mathbb{YF}^{(k)}$, where they remain harmonic provided that the specialization $(\vec{x},\vec{y})$ stabilizes appropriately. Fibonacci positive specializations as well as positive normalized harmonic functions for $\mathbb{YF}^{(k)}$ are defined in a standard way. The space of Fibonacci positive specializations for $\mathbb{YF}^{(k)}$ is finite-dimensional and is expected to have a simple description.

In general, saturated chains in $\mathbb{YF}^{(k)}$ with a fixed endpoint are not known to be in bijection with linear extensions of any poset. Consequently, there is no Schützenberger promotion at our disposal. However, there exists a $\mathbb{Z}$-action that implements an *adic shift*, allowing issues related to combinatorial ergodicity to be explored in the truncated setting.

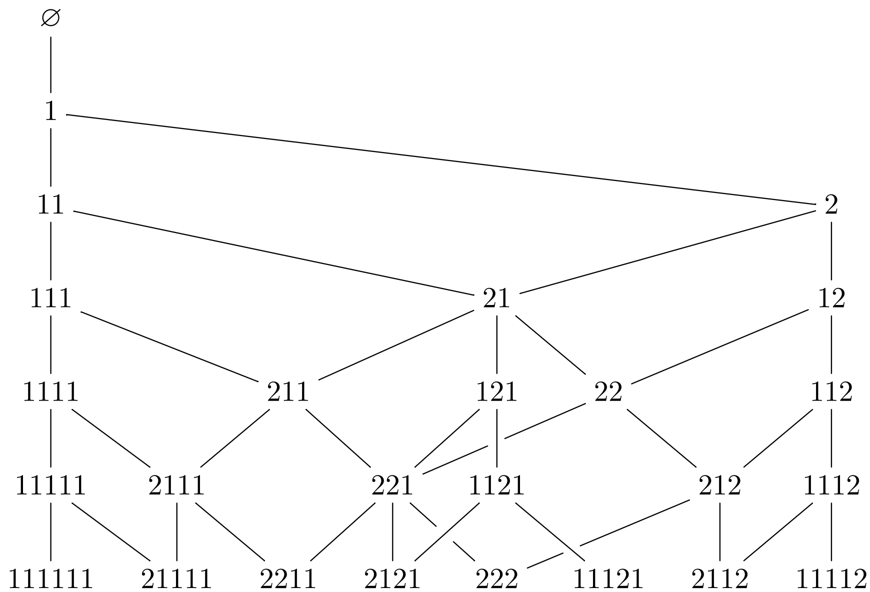


FIGURE 12. The truncated poset $\mathbb{YF}^{(3)}$ up to level $n = 6$ (compare to the full Young–Fibonacci lattice in Figure 1).

Each truncation $\mathbb{YF}^{(k)}$ supports a restricted version of the Young–Fibonacci RS correspondence, which involves pattern-avoiding permutations. Accordingly, random pattern-avoiding permutations can be studied using the framework set up in Section 7.3.

8.3. **Martin boundary, Fibonacci positive specializations, and stick-breaking.** The relationship between the Martin boundary $\Upsilon_{\text{Martin}}(\mathbb{YF})$ of the Young–Fibonacci lattice and the space of Fibonacci positive specialization is not fully understood. For a Fibonacci positive specialization $(\vec{x}, \vec{y})$ and its associated Type-I component μ_{I} (see Definition 5.16), it seems likely that $\mu_{\text{I}}(1^\infty\mathbb{YF})$ is equal to 0 or 1. A first step would be to confirm this observation or construct a counter-example.

Second, do the Fibonacci positive specializations $(\vec{x}, \vec{y})$ for which $\mu_{\text{I}}(1^\infty\mathbb{YF}) = 0$ coincide with those clone measures whose joint distributions, on either runs or hikes, give rise to continuous models of random sequences in $[0, 1]^\infty$? So far we have only encountered two types of continuous scaling distributions. One is a stick-breaking scheme (Theorem 5.7), and another one involves a family of conditionally independent beta random variables (Theorem 5.13).

It is natural to ask whether other continuous models of random sequences in $[0, 1]^\infty$ arise from clone measures in the $n \to \infty$ limit of the joint distributions of runs and/or hikes. Finally, it remains an open question whether there exist observables, beyond consecutive joint hike and/or run statistics, which asymptotically exhibit continuous behavior.

8.4. **Random permutation models.** In [GK00a], Gnedin and Kerov introduced a surjection called the *Fibonacci solitaire* which maps permutations in $\mathfrak{S}_n$ to saturated chains terminating in $\mathbb{YF}_n$. This solitaire is distinct from the surjection $\sigma \mapsto \mathrm{P}(\sigma)$ obtained by simply forgetting the recording tableaux $\mathrm{Q}(\sigma)$ in Nzeutchap's RS correspondence. The push-forward of the uniform measure on $\mathfrak{S}_n$ under the Fibonacci solitaire was shown in [GK00a] to be the Plancherel measure (i.e., $\nu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi)$ with $\pi = \varphi = \varphi_{\mathrm{PL}}$) on the set of saturated chains terminating within $\mathbb{YF}_n$. Beyond this example, it is not clear which measures on $\mathfrak{S}_n$ realize $\nu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi)$ as push-forward measures, even when $\varphi = \varphi_{\mathrm{PL}}$, $\pi = \varphi_{\vec{x},\vec{y}}$, and where $(\vec{x},\vec{y})$ is any of the Fibonacci positive specializations considered in Definition 3.22.

The probability models for random permutations and involutions described in Section 7.3 are constructed from the Young–Fibonacci side of the RS correspondence. For general Fibonacci positive specializations, or even specific ones such as the Charlier specialization, it would be helpful to identify natural multivariate statistics on permutations and involutions which realize the corresponding probability measures as Gibbs measures (that is, where the probability weight of a permutation is proportional to the exponent of a certain combination of these statistics).

Another key question is whether the distributions $\mu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi,\psi)$ and $\nu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi)$ can be understood in connection with the Stieltjes moment problem. For instance, the Plancherel specialization corresponds to the uniform distribution on permutations $\boldsymbol{\sigma} \in \mathfrak{S}_n$, as this is realized by $\mu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi,\psi)$ when $\pi = \varphi = \psi = \varphi_{\mathrm{PL}}$. It is known [Ful24], [AT92] that the distribution of fixed points of a uniformly sampled random permutation $\boldsymbol{\sigma} \in \mathfrak{S}_n$ tends to the Poisson distribution $\nu^{(1)}_{\mathrm{Pois}}(dt)$ on $[0,\infty)$ as $n \to \infty$. Notably, $\nu^{(1)}_{\mathrm{Pois}}(dt)$ is the Borel measure associated with the Plancherel Fibonacci positive specialization by Theorem 4.1. It is natural to ask whether this coincidence generalizes to other Fibonacci positive specializations.

To explore this possible connection, a first step would be to examine the asymptotic behavior of the number of fixed points in permutations $\boldsymbol{\sigma} \in \mathfrak{S}_n$ sampled according to $\mu_n(\boldsymbol{\sigma}\,|\,\pi,\varphi,\psi)$, where $\varphi = \psi = \varphi_{\mathrm{PL}}$, and where $\pi = \varphi_{\vec{x},\vec{y}}$ is a general clone harmonic function. Combinatorially, this requires counting permutations $\boldsymbol{\sigma} \in \mathfrak{S}_n$ that have exactly $k \geq 0$ fixed points and whose shape under the RS correspondence corresponds to a given Fibonacci word $w \in \mathbb{YF}_n$. This counting problem should be tractable, as fixed points of a permutation are straightforward to identify under the $\mathbb{YF}$-version of the RS correspondence. In particular, when $(\vec{x},\vec{y})$ is the Charlier specialization, one could test whether the asymptotic distribution of fixed points under $\mu_n(\boldsymbol{\sigma})$ aligns with the Poisson distribution $\nu^{(\rho)}_{\mathrm{Pois}}(dt)$, which is the Borel measure associated with the Charlier specialization (Section 4.2.1).

8.5. **Clone Cauchy identities and Okada's noncommutative theory.** The clone Cauchy identities from Section 6.2 allow one to define Gibbs measures on the $\mathbb{YF}$-lattice:

$$\mathrm{prob}_H(w) := \frac{h_w(\vec{p}\,|\,\vec{q}) \cdot s_w(\vec{x}\,|\,\vec{y})}{H\big(\vec{x},\vec{y}\,;\vec{p},\vec{q}\big)}, \qquad \mathrm{prob}_S(w) := \frac{s_w(\vec{p}\,|\,\vec{q}) \cdot s_w(\vec{x}\,|\,\vec{y})}{S\big(\vec{x},\vec{y}\,;\vec{p},\vec{q}\big)}, \qquad w \in \mathbb{YF}. \tag{8.1}$$

Here, the normalizing constants H and S are the sums over n of the right-hand sides of the first and second clone Cauchy identities (Propositions 6.8 and 6.9, respectively), and $(\vec{x},\vec{y})$ and $(\vec{p},\vec{q})$ are two independent Fibonacci positive specializations. The measures (8.1) are two natural clone analogues of *Schur measures on partitions* introduced in [Oko01]. The next step in this direction is to define and investigate *clone Schur processes* — measures on sequences of Fibonacci words whose joint distributions are expressed through suitable skew analogues of clone Schur functions.

Furthermore, the relationship between clone measures (8.1) and Okada's noncommutative theory [Oka94] may illuminate fundamental algebraic and combinatorial properties of the Young–Fibonacci lattice. Indeed, this noncommutative framework is particularly useful for understanding the clone Cauchy identities. In this setting, noncommutative clone Schur functions $\mathrm{s}_w(\mathbf{x}\,|\,\mathbf{y})$ form a basis for the free algebra $\mathbb{C}\langle \mathbf{x},\mathbf{y}\rangle$ generated by two noncommutative variables $\mathbf{x},\mathbf{y}$. To formulate the Cauchy identities, we introduce an auxiliary pair of noncommutative variables $\mathbf{p},\mathbf{q}$ which independently commute with both $\mathbf{x}$ and $\mathbf{y}$. The noncommutative version of the quadridiagonal matrix in formula (6.12) is given by:

$$S_n(\mathbf{x},\mathbf{y};\mathbf{p},\mathbf{q}) := \underbrace{\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & 0 & \cdots \\ 1 & \mathbf{A} & \mathbf{B} & \mathbf{C} & \\ 0 & 1 & \mathbf{A} & \mathbf{B} & \\ 0 & 0 & 1 & \mathbf{A} & \\ \vdots & & & & \ddots \end{pmatrix}}_{n\times n \text{ quadridiagonal matrix}},$$

where the entries are:

$$\mathbf{A} = \mathbf{p}\mathbf{x}, \qquad \mathbf{B} = \mathbf{q}(\mathbf{x}^2 - \mathbf{y}) + (\mathbf{p}^2 - \mathbf{q})\mathbf{y}, \qquad \mathbf{C} = \mathbf{q}\mathbf{p}\mathbf{y}\mathbf{x}.$$

These entries are valued in the free algebra $\mathfrak{A}\langle \mathbf{x},\mathbf{y}\rangle$ whose coefficient ring is the free algebra $\mathfrak{A} = \mathbb{C}\langle \mathbf{p},\mathbf{q}\rangle$.

We conjecture that all matrix minors (quasi-determinants) of $S_n(\mathbf{x},\mathbf{y};\mathbf{p},\mathbf{q})$ are *coefficient-wise* clone Schur positive. That is, the coefficient of each noncommutative clone Schur function $\mathrm{s}_w(\mathbf{x}\,|\,\mathbf{y})$ in the expansion of any such matrix minor must be a nonnegative integer linear combination of noncommutative clone Schur functions $\mathrm{s}_v(\mathbf{p}\,|\,\mathbf{q})$ in $\mathfrak{A} = \mathbb{C}\langle \mathbf{p},\mathbf{q}\rangle$. This property reflects another manifestation of total positivity, which is not evident under the biserial specialization considered in Section 3.

8.6. **Quasisymmetric versions of clone Schur functions.** Quasisymmetric functions emerge naturally from Nzeutchap's Robinson–Schensted–Knuth correspondence for the Young–Fibonacci lattice. This correspondence is an injective map from positive integer sequences $\mathbb{N}^\infty$ to pairs (P,Q) of standard and semi-standard Young–Fibonacci tableaux sharing a common shape in $\mathbb{YF}$. One can define a naive quasisymmetric analogue Q_w of the clone Schur function $\mathrm{s}_w(\mathbf{x}\,|\,\mathbf{y})$ as the generating function of all semi-standard tableaux of a fixed shape $w \in \mathbb{YF}$. However, this analogue loses key properties: its branching rule no longer follows the $\mathbb{YF}$-lattice's covering relations, and its expansion into Gessel fundamental quasisymmetric functions deviates from the expected clone version of [Sta01, Theorem 7.19.7] for Schur functions.

A more speculative approach leverages the graded Hopf algebra duality between noncommutative symmetric functions (**NSym**) and quasisymmetric functions (QSym); see [GKL+95], [Ges84], [MR95]. We identify **NSym** with the free algebra $\mathbb{C}\langle \mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{\Psi}_3, \ldots\rangle$, equipped with the multiplicative basis of noncommutative monomials $\mathbf{\Psi}_\alpha = \mathbf{\Psi}_1^{\alpha_1}\cdots\mathbf{\Psi}_k^{\alpha_k}$ indexed by integer compositions $\alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbb{N}^k$ for $k \geq 0$. In the terminology of [GKL+95], these are the *Type-I noncommutative power symmetric functions*. The Hopf algebra structure is standard for free algebras. In particular, the generators $\mathbf{\Psi}_k$ are primitive: $\Delta(\mathbf{\Psi}_k) = \mathbf{1}\otimes\mathbf{\Psi}_k + \mathbf{\Psi}_k\otimes\mathbf{1}$ for all $k \geq 1$.

The algebra QSym has a basis $\{\psi_\alpha\}$ of *Type-I quasisymmetric power functions* [BDH+20] indexed by integer compositions, which is dual to $\mathbf{\Psi}_\alpha$. While non-multiplicative, this basis determines the commutative product and co-product structures through *shuffling* and *de-concatenation*:

$$\psi_\alpha\psi_\beta = \sum_{\gamma\in\alpha\sqcup\!\sqcup\beta} \psi_\gamma, \qquad \Delta(\psi_\gamma) = \sum_{\alpha\cdot\beta=\gamma} \psi_\alpha\otimes\psi_\beta.$$

Moreover, each ψ_α expands into monomial quasisymmetric functions:

$$\psi_\alpha = \sum_{\beta\succeq\alpha} \frac{M_\beta}{\pi(\alpha,\beta)},$$

where the sum ranges over compositions $\beta \models n$ coarsening α.

Okada's clone ring $\mathbb{C}\langle \mathbf{x}, \mathbf{y} \rangle$ embeds naturally into **NSym** by identifying $\mathbf{x}, \mathbf{y}$ with $\mathbf{\Psi}_1, \mathbf{\Psi}_2$, and the Hopf algebra structure on **NSym** restricts to the standard Hopf algebra structure on $\mathbb{C}\langle \mathbf{x}, \mathbf{y} \rangle$ viewed as a rank two free algebra. Interpreting Fibonacci words $w \in \mathbb{YF}$ as compositions consisting only of 1's and 2's yields a multiplicative basis $\{\mathbf{\Psi}_w : w \in \mathbb{YF}\}$ for the Okada's clone ring. The clone Kostka numbers mediate the expansion into clone Schur functions:

$$\mathbf{\Psi}_w = \sum_{|v|=|w|} K_{v,w} \, \mathrm{s}_v(\mathbf{x} \,|\, \mathbf{y}).$$

The quasisymmetric counterpart of $\mathbb{C}\langle \mathbf{x}, \mathbf{y} \rangle$ is the subalgebra of QSym generated by ψ_1, ψ_2, or, equivalently, the span of ψ_w for $w \in \mathbb{YF}$ (viewed as integer compositions consisting only of 1's and 2's). The quasisymmetric dual thus inherits a Hopf-algebra structure (closed under shuffling), and admits functions dual to $\mathrm{s}_w(\mathbf{x} \,|\, \mathbf{y})$:

$$Q_w^{(\mathrm{I})} := \sum_{|v|=|w|} K_{v,w} \psi_v.$$

Notably, $Q_w^{(\mathrm{I})}$ expands into monomial quasisymmetric functions M_β, but here the compositions β may contain integers larger than 2. The function $Q_w^{(\mathrm{I})}$ is a quasisymmetric analogue of the clone Schur function s_w.

Several natural questions arise. A parallel theory of Type-II power functions yields another quasisymmetric analogue of clone Schur functions, $Q_w^{(\mathrm{II})}$, raising questions about the relationship between Q_w, $Q_w^{(\mathrm{I})}$, and $Q_w^{(\mathrm{II})}$. Understanding these connections may more shed light on the Young–Fibonacci RSK correspondence, as well as on potential quasisymmetric versions of the clone Cauchy identities. The Hopf-algebra duality between **NSym** and QSym can also be brought to bear in order to understand algebraic features, such as the (common) branching structure and clone *Littlewood-Richardson* coefficients of $Q_w^{(\mathrm{I})}$ and $Q_w^{(\mathrm{II})}$.

References

[AESW51] M. Aissen, A. Edrei, I. J. Schoenberg, and A. Whitney, *On the generating functions of totally positive sequences*, Proc. Nat. Acad. Sci. U. S. A. **37** (1951), 303–307. ↑2

[Ahb23] K. Ahbli, *New results on the associated Meixner, Charlier, Laguerre, and Krawtchouk polynomials*, arXiv preprint (2023). arXiv:2306.05371 [math.CA]. ↑5, 29

[Ans05] M. Anshelevich, *Linearization coefficients for orthogonal polynomials using stochastic processes*, Ann. Probab. **33** (2005), no. 1, 114–136. arXiv:math/0301094 [math.CO]. ↑6, 27

[Ask89] R. Askey, *Continuous q-Hermite Polynomials when $q > 1$*, q-Series and Partitions, 1989, pp. 151–158. ↑27

[ASW52] M. Aissen, I. J. Schoenberg, and A. Whitney, *On the generating functions of totally positive sequences I*, J. Analyse Math. **2** (1952), 93–103. ↑2

[AT92] R. Arratia and S. Tavaré, *The Cycle Structure of Random Permutations*, Ann. Probab. **20** (1992), 1567–1591. ↑66

[BDH+20] C. Ballantine, Z. Daugherty, A. Hicks, S. Mason, and E. Niese, *Quasisymmetric power sums*, J. Combin. Theory Ser. A **175** (2020), no. 1, 105273. arXiv:1710.11613 [math.CO]. ↑67

[BE20] I. Bochkov and V. Evtushevsky, *Ergodicity of the Martin boundary of the Young–Fibonacci graph. I*, arXiv preprint (2020). arXiv:2012.07447 [math.CO]. ↑2, 11, 46

[BF14] A. Borodin and P. Ferrari, *Anisotropic growth of random surfaces in 2+1 dimensions*, Commun. Math. Phys. **325** (2014), 603–684. arXiv:0804.3035 [math-ph]. ↑8

[BG16] A. Borodin and V. Gorin, *Lectures on integrable probability*, Probability and Statistical Physics in St. Petersburg, 2016, pp. 155–214. arXiv:1212.3351 [math.PR]. ↑8

[BO16] A. Borodin and G. Olshanski, *Representations of the Infinite Symmetric Group*, Cambridge Studies in Advanced Mathematics, vol. 160, Cambridge University Press, 2016. ↑57

[BP14] A. Borodin and L. Petrov, *Integrable probability: From representation theory to Macdonald processes*, Probab. Surv. **11** (2014), 1–58. arXiv:1310.8007 [math.PR]. ↑8

[BS21] N. Blitvić and E. Steingrímsson, *Permutations, moments, measures*, Trans. AMS **374** (2021), no. 8, 5473–5508. arXiv:2001.00280 [math.CO]. ↑33

[CDD+21] X. Chen, B. Deb, A. Dyachenko, T. Gilmore, and A. D. Sokal, *Coefficientwise total positivity of some matrices defined by linear recurrences*, Sém. Lothar. Combin. **85B** (2021), no. 30, 12. arXiv:2012.03629 [math.CO]. ↑17

[CH14] I. Corwin and A. Hammond, *Brownian Gibbs property for Airy line ensembles*, Invent. math. **195** (2014), no. 2, 441–508. arXiv:1108.2291 [math.PR]. ↑8

[Chr04] J. S. Christiansen, *Indeterminate Moment Problems within the Askey-scheme*, Ph.D. Thesis, Institute for Mathematical Sciences, University of Copenhagen, 2004. ↑27

[CZ11] J. Cigler and J. Zeng, *A curious q-analogue of Hermite polynomials*, J. Comb. Theory Ser. A **118** (2011), no. 1, 9–26. arXiv:0905.0228 [math.CO]. ↑21

[dMSW95] A. de Medicis, D. Stanton, and D. White, *The Combinatorics of q-Charlier Polynomials*, J. Combin. Theory Ser. A **69** (1995), 87–114. arXiv:math/9307208 [math.CA]. ↑26, 27

[DS23] B. Deb and A. D. Sokal, *Coefficientwise Hankel-total positivity of the Schett polynomials*, arXiv preprint (2023). arXiv:2311.11747 [math.CO]. ↑17

[Edr52] A. Edrei, *On the generating functions of totally positive sequences. II*, J. Analyse Math. **2** (1952), 104–109. ↑2

[Edr53] A. Edrei, *On the generating function of a doubly infinite, totally positive sequence*, Trans. AMS **74** (1953), 367–383. ↑2, 14

[Evt20] V. Evtushevsky, *Ergodicity of the Martin boundary of the Young–Fibonacci graph. II*, arXiv preprint (2020). arXiv:2012.08107 [math.CO]. ↑2, 11, 46

[FJS17] S. Fallat, C. R. Johnson, and A. D. Sokal, *Total positivity of sums, Hadamard products and Hadamard powers: Results and counterexamples*, Linear Algebra Appl. **520** (2017), 242–259. arXiv:1612.02210 [math.AC]. ↑14

[Fla80] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Math. **32** (1980), 125–161. ↑5, 23

[Fom88] S. V. Fomin, *Generalized Robinson-Schensted-Knuth correspondence*, J. Math. Sci. **41** (1988), no. 2, 979–991. Translated from Zap. Nauchn. Semin. LOMI, vol. 155, pp. 156–175, 1986. ↑2, 9

[Fom94] S. Fomin, *Duality of graded graphs*, J. Algebr. Comb. **3** (1994), no. 4, 357–404. ↑11, 54

[Fom95] S. Fomin, *Schensted algorithms for dual graded graphs*, J. Algebr. Comb. **4** (1995), no. 1, 5–45. ↑54

[Ful24] J. Fulman, *Fixed points of non-uniform permutations and representation theory of the symmetric group*, arXiv preprint (2024). arXiv:2406.12139 [math.RT]. ↑66

[FZ00] S. Fomin and A. Zelevinsky, *Total positivity: Tests and parametrizations*, The Mathematical Intelligencer **22** (2000), no. 1, 23–33. arXiv:math/9912128 [math.RA]. ↑14

[FZ99] S. Fomin and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380. arXiv:math/9802056 [math.RT]. ↑14

[Ges84] I. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Contemp. Math. **34** (1984), 289–301. ↑67

[GK00a] A. Gnedin and S. Kerov, *The Plancherel measure of the Young–Fibonacci graph*, Math. Proc. Camb. Philos. Soc. **129** (2000), no. 3, 433–446. ↑1, 3, 7, 11, 12, 33, 37, 41, 62, 65

[GK00b] F. Goodman and S. Kerov, *The Martin Boundary of the Young-Fibonacci Lattice*, Jour. Alg. Comb. **11** (2000), 17–48. arXiv:math/9712266 [math.CO]. ↑2, 9, 11, 43, 46

[GKL+95] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), 218–348. arXiv:hep-th/9407124. ↑67

[GS97] M. I. Gekhtman and M. Z. Shapiro, *Completeness of real Toda flows and totally positive matrices*, Math. Z. **226** (1997), 51–66. ↑24

[HS24] F. Hivert and J. Scott, *Diagram Model for the Okada Algebra and Monoid*, Sém. Lothar. Combin. (2024). Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2024), Bochum, Article 25, 12 pp. arXiv:2404.16733 [math.RT]. ↑57, 64

[ILV88] M. Ismail, J. Letessier, and G. Valent, *Linear Birth and Death Models and Associated Laguerre and Meixner Polynomials*, J. Approx. Theory **55** (1988), 337–348. ↑5, 29

[JKB97] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Discrete multivariate distributions*, Vol. 165, Wiley, New York, 1997. ↑6, 12

[Jos11] Josuat-Vergès, M. and Rubey, M., *Crossings, Motzkin paths, and moments*, Discret. Math. **311** (2011), no. 18-19, 2064–2078. arXiv:1008.3093 [math.CO]. ↑6, 32

[Kar68] S. Karlin, *Total positivity, Vol.1*, Stanford, 1968. ↑14

[Koo04] T.H. Koornwinder, *On q^{-1}-Al-Salam-Chihara polynomials* (2004). Informal note, version of January 29, 2004, available at: https://staff.fnwi.uva.nl/t.h.koornwinder/art/informal/ASC.pdf. ↑27

[KS96] R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Technical report, Delft University of Technology and Free University of Amsterdam (1996). arXiv:math/9602214 [math.CA], report no. OP-SF 20 Feb 1996. Updated version available at https://fa.ewi.tudelft.nl/~koekoek/documents/as98.pdf. ↑3, 5, 26, 27, 28

[KSZ06] D. Kim, D. Stanton, and J. Zeng, *The Combinatorics of the Al-Salam-Chihara q-Charlier polynomials*, Sém. Lothar. Combin. **54** (2006), article B54i. ↑6, 27, 28

[KZ06] A. Kasraoui and J. Zeng, *Distribution of crossings, nestings and alignments of two edges in matchings and partitions*, Electron. J. Combin. **13** (2006), no. 1, R33. arXiv:math/0601081 [math.CO]. ↑6, 31, 32, 33

[MR95] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), 967–982. ↑67

[NIS24] NIST, *NIST Digital Library of Mathematical Functions*, 2024. Olver, F. W. J., Olde Daalhuis, A. B., Lozier, D. W., Schneider, B. I., Boisvert, R. F., Clark, C. W., Miller, B. R., Saunders, B. V., Cohl, H. S., McClain, M. A., eds. Release 1.2.0 of 2024-03-15. Available at: https://dlmf.nist.gov/. ↑38, 39, 62

[NZ04] Y. Nakamura and A. Zhedanov, *Toda Chain, Sheffer Class of Orthogonal Polynomials and Combinatorial Numbers*, Proc. Inst. Math. NAS Ukraine **50** (2004), 450–457. ↑24

[Nze09] J. Nzeutchap, *Young-Fibonacci insertion, tableauhedron, and Kostka numbers*, J. Comb. Theory, Ser. A **116** (2009), 143–167. ↑3, 8, 54, 56, 58

[Oka94] S. Okada, *Algebras associated to the Young-Fibonacci lattice*, Trans. Amer. Math. Soc. **346** (1994), 549–568. ↑1, 2, 9, 12, 13, 20, 49, 66

[Oko01] A. Okounkov, *Infinite wedge and random partitions*, Selecta Math. **7** (2001), no. 1, 57–81. arXiv:math/9907127 [math.RT]. ↑8, 66

[OP06] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, Ann. Math. **163** (2006), no. 2, 517–560. arXiv:math/0204305 [math.AG]. ↑8

[OR03] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, Jour. AMS **16** (2003), no. 3, 581–603. arXiv:math/0107056 [math.CO]. ↑8

[ORV06] A. Okounkov, N. Reshetikhin, and C. Vafa, *Quantum Calabi-Yau and Classical Crystals*, Progress in mathematics, 2006, pp. 597–618. arXiv:hep-th/0309208. ↑8

[PR15] J. Propp and T. Roby, *Homomesy in products of two chains*, Electron. J. Comb. **22** (2015), no. 3. arXiv:1310.5201 [math.CO]. ↑64

[PSZ23] M. Pétréolle, A. D. Sokal, and B.-X. Zhu, *Lattice Paths and Branched Continued Fractions: An Infinite Sequence of Generalizations of the Stieltjes–Rogers and Thron–Rogers Polynomials, with Coefficientwise Hankel-Total Positivity*, Mem. Amer. Math. Soc., vol. 291, AMS, 2023. arXiv:1807.03271 [math.CO]. ↑5, 17, 23

[Rob91] T. Roby, *Applications and Extensions of Fomin's Generalization of Robinson-Schensted Correspondence to Differential Posets*, Ph.D. Thesis, MIT, 1991. ↑54

[Sch72] M.-P. Schützenberger, *Promotion des morphismes d'ensembles ordonnés*, Discrete Math. **2** (1972), 73–94. ↑64

[Sch88] I.J. Schoenberg, *Contributions to the Problem of Approximation of Equidistant Data by Analytic Functions. Part A.—On the Problem of Smoothing or Graduation. A First Class of Analytic Approximation Formulae*, I.J. Schoenberg Selected Papers (1988), 3–57. ↑14

[Sok14] A. D. Sokal, *Coefficientwise total positivity (via continued fractions) for some Hankel matrices of combinatorial polynomials*, 2014. Talk at the Séminaire de Combinatoire Philippe Flajolet, Institut Henri Poincaré, Paris, 5 June 2014; transparencies available at http://semflajolet.math.cnrs.fr/index.php/Main/2013-2014. ↑17

[Sok20] A. D. Sokal, *The Euler and Springer numbers as moment sequences*, Expo. Math. **38** (2020), no. 1, 1–26. arXiv:1804.04498 [math.CO]. ↑5, 23

[Som22] M. Somos, *Math.StackExchange answer to "Strange polynomial analog of the Bell numbers"*, 2022. https://math.stackexchange.com/q/4541543 (version: 2022-10-01). ↑29

[Sta01] R. Stanley, *Enumerative Combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 2001. With a Foreword by Gian-Carlo Rota and Appendix 1 by Sergey Fomin. ↑67

[Sta09] R. P. Stanley, *Promotion and Evacuation*, Electron. J. Comb. **16** (2009), no. 2. arXiv:0806.4717 [math.CO]. ↑64

[Sta88] R. Stanley, *Differential posets*, Jour. AMS **1** (1988), no. 4, 919–961. ↑2, 9, 11, 54

[Tho64] E. Thoma, *Die unzerlegbaren, positive-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe*, Math. Zeitschr **85** (1964), 40–61. ↑2

[Vie83] X. G. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux*, UQAM, Montréal, Montréal, Canada, 1983. Lecture Notes, 219 pages. ↑5, 23

[VK81] A. Vershik and S. Kerov, *Asymptotic theory of the characters of the symmetric group*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 15–27, 96. ↑2, 57

[WW91] M. Wachs and D. White, *p, q-Stirling numbers and set partition statistics*, J. Combin. Theory Ser. A **56** (1991), no. 1, 27–46. ↑6, 27, 28

[Zen95] J. Zeng, *The q-Stirling numbers, continued fractions and the q-Charlier and q-Laguerre polynomials*, J. Comput. Appl. Math. **57** (1995), no. 3, 413–424. ↑6, 27, 28

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