

On a variational problem of nematic liquid crystal droplets

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ABSTRACT. Let $\mu > 0$ be a fixed constant, and we prove that minimizers to the following energy functional

$$E_f(u, \Omega) := \int_{\Omega} |\nabla u|^2 + \mu P(\Omega)$$

exist among pairs (Ω, u) such that Ω is an M -uniform domain with finite perimeter and fixed volume, and $u \in H^1(\Omega, \mathbb{S}^2)$ with $u = \nu_{\Omega}$, the measure-theoretical outer unit normal, almost everywhere on the reduced boundary of Ω . The uniqueness of optimal configurations in various settings is also obtained. In addition, we consider a general energy functional given by

$$E_f(u, \Omega) := \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\partial^* \Omega} f(u(x) \cdot \nu_{\Omega}(x)) d\mathcal{H}^2(x),$$

where $\partial^* \Omega$ is the reduced boundary of Ω and f is a convex positive function on $\mathbb{R}$. We prove that minimizers of E_f also exist among M -uniform outer-minimizing domains Ω with fixed volume and $u \in H^1(\Omega, \mathbb{S}^2)$.

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1. Introduction

In this paper we study the existence of liquid crystal droplets (Ω_0, u_0) , consisting of a domain $\Omega_0 \subset \mathbb{R}^3$ representing the shape of a liquid crystal drop and a unit vector field $u_0 \in H^1(\Omega, \mathbb{S}^2)$ representing the average orientation field of liquid crystal molecules within the liquid crystal drop Ω , that minimizes the total energy

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functional, including both the elastic energy in the bulk and the interfacial energy defined by

$$E_f(u, \Omega) := \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\partial^* \Omega} f(u(x) \cdot \nu_{\Omega}(x)) d\mathcal{H}^2(x), \quad (1.1)$$

among all pairs (Ω, u) , where Ω is a domain of finite perimeter with a fixed volume that is compactly contained in the ball $B_{R_0} \subset \mathbb{R}^3$ with center 0 and radius R_0 for some fixed constant $R_0 > 0$, and $u \in H^1(\Omega, \mathbb{S}^2)$, which is defined by

$$H^1(\Omega, \mathbb{S}^2) \equiv \left\{ v \in H^1(\Omega, \mathbb{R}^3) : |v(x)| = 1 \text{ a.e. } x \in \Omega \right\}.$$

The functional $E_f(u, \Omega)$ should be understood in the sense that the surface integral is taken over the reduced boundary $\partial^* \Omega$ of Ω , $u|_{\partial^* \Omega}$ is the trace of u on $\partial^* \Omega$, ν_{Ω} is the measure theoretical outer unit normal of $\partial^* \Omega$, and f is usually assumed to have a nonnegative lower bound (with a typical choice of $f(t) = \mu(1 + wt^2)$, $t \in [-1, 1]$, for some constants $\mu > 0$ and $-1 < w < 1$).

We will study the following minimization problem of (1.1).

Problem A. Find a pair (Ω, u) that minimizes $E_f(u, \Omega)$ over all pairs (Ω, u) where Ω is a domain of finite perimeter in a fixed ball $B_{R_0} \subset \mathbb{R}^3$, with a fixed volume $V_0 > 0$, and $u \in H^1(\Omega, \mathbb{S}^2)$, when $f : [-1, 1] \rightarrow \mathbb{R}$ is a nonnegative, continuous convex function.

We are also interested in the case when there is a constant contact angle condition between the liquid crystal orientation field u and the reduced boundary of liquid crystal drop $\partial^* \Omega$, i.e., $u \cdot \nu_{\Omega} \equiv c$ on $\partial^* \Omega$, for some constant $c \in [-1, 1]$. In this case, the energy functional $E_f(u, \Omega)$ in (1.1) reduces to

$$E(u, \Omega) := \int_{\Omega} |\nabla u(x)|^2 dx + \mu \mathcal{H}^2(\partial^* \Omega) \quad (1.2)$$

for some constant $\mu \geq 0$. Problem A can be reformulated as follows.

Problem B. Find a pair (Ω, u) that minimizes $E(u, \Omega)$ over all pairs (Ω, u) where Ω is a domain of finite perimeter in a fixed ball $B_{R_0} \subset \mathbb{R}^3$, with a fixed volume $V_0 > 0$, and $u \in H^1(\Omega, \mathbb{S}^2)$ satisfies $u \cdot \nu_{\Omega} \equiv c$ on $\partial^* \Omega$ for some $c \in [-1, 1]$.

We would like to mention that the contact angle condition in Problem B is referred as

- (i) the planar anchoring condition when the constant $c = 0$, and
- (ii) the homeotropic anchoring condition when the constant $c = 1$.

We would like to point out that recently Geng and Lin in a very interesting paper [23] studied Problem B under the planar anchoring condition (i) in dimension two, and proved the existence of a minimizer (Ω, u) such that the optimal shape $\partial \Omega$ of the droplet is a chord-arc curve with two cusps, which can be parametrized in $H^{\frac{3}{2}}$ and has its unit normal vector field ν_{Ω} belongs to VMO.

Because the homeotropic anchoring condition is an important physical condition, we are also interested in the following problem.

Problem C. Find a solution to Problem B when the contact angle condition corresponds to $c = 1$.

Motivation. The main difficulty of the minimization problems A, B, and C lies in showing the sequential lower semicontinuity of $E_f(u, \Omega)$ (or $E(u, \Omega)$) when both domains Ω and vector fields $u \in H^1(\Omega, \mathbb{S}^2)$ vary. It is even a difficult question to ask

whether the configuration space is closed under weak convergence of liquid crystal pairs (Ω, u) .

In [34], under the assumption that all admissible domains $\Omega \subset B_{R_0}$ are *convex* domains, Lin and Poon have proved that there exists a minimizing pair (Ω_0, u_0) of Problem A. Moreover, u_0 enjoys a partial regularity property similar to that of minimizing harmonic maps by Schoen and Uhlenbeck [40, 41]. It was further proven by [34] that, up to translations, $(\Omega_0, u_0) = (B_R, \frac{x}{|x|})$ is a unique minimizer of Problem C among convex domains with $|B_R| = V_0$.

We would like to point out that the convexity assumption of admissible domains Ω plays a crucial role in [34], since a minimizing sequence (Ω_i, u_i) of *convex* domains $\Omega_i \subset B_{R_0}$ with $|\Omega_i| = V_0$ has a subsequence $\Omega_{i_k} \rightarrow \Omega$ in L^1 , for some bounded convex domain $\Omega \subset B_{R_0}$ with $|B_{R_0}| = V_0$, such that $\mathcal{H}^2(\partial\Omega_{i_k}) \rightarrow \mathcal{H}^2(\partial\Omega)$ and $\nu_{\Omega_{i_k}} \rightarrow \nu_\Omega$ almost everywhere with respect to a spherical coordinate system¹. Moreover, there exists $u \in H^1(\Omega, \mathbb{S}^2)$ such that $\nabla u_{i_k} \chi_{\Omega_{i_k}} \rightarrow \nabla u \chi_\Omega$ weakly in $L^2(\mathbb{R}^3)$. The uniqueness of minimizer of Problem C among convex domains relies on the following important inequalities:

$$\int_{\Omega} |\nabla u(x)|^2 dx \geq \int_{\partial\Omega} H(x) d\mathcal{H}^2(x), \quad \forall u \in H^1(\Omega, \mathbb{S}^2) \text{ with } u = \nu_\Omega \text{ a.e. on } \partial\Omega, \quad (1.3)$$

and

$$\int_{\partial\Omega} H(x) d\mathcal{H}^2(x) \geq \sqrt{4\pi\mathcal{H}^2(\partial\Omega)} \text{ for convex } \Omega, \text{ equality holds iff } \Omega = B_R, \quad (1.4)$$

where H denotes the mean curvature of $\partial\Omega$. In [34], (1.3) is derived for any $\Omega \in W^{2,1}$, while (1.4) is proven by the Brunn-Minkowski inequality for convex domains.

In this paper, we would like to relax the convexity assumption from [34] and investigate Problems A, B, and C over a larger class of domains possibly containing *non-convex* domains with *less regular* boundaries. The class of domains contains Sobolev extension domains with some uniform parameters, as well outer minimal domains.

The main theorems of this paper arose from the Ph.D. thesis of the first author [32]. The interested reader can refer to [32] for more related results.

Outline of this paper:

In section 2, we will review certain classes of domains in $\mathbb{R}^n$, including M -uniform domains, which are Sobolev extension domains with constants depending on M and n ; the outer minimal domains, which are a generalization of convex domains.

In section 3, we will show in Theorem 3.5 that, up to a set of measure zero, the L^1 -limit of M -uniform domains is M -uniform. A few other results on the relation between L^1 -convergence and Hausdorff convergence are also derived.

In section 4, we will establish the weak lower semicontinuity of bulk elastic energy of (Ω, u) for two classes of domains: a) the admissible sets of M -uniform domains, and b) the admissible sets of outer minimal M -uniform domains. It is more subtle to prove the lower semicontinuity of surface energy for Problem A. We will only consider outer minimal sets and our proof is inspired by Reshetnyak's lower semicontinuity theorem (see [36, Theorem 20.11]) and the perimeter convergence

¹For example, one can parametrize $\partial\Omega_{i_k}$ and $\partial\Omega$ over the unit sphere $\mathbb{S}^2$.

Lemma 4.1. Thus combining the compactness of M -uniform domain results and the lower-semicontinuity results, the existence Theorem 4.2 on problems A, B and C is proved among these classes of admissible sets.

In section 5, we will apply results by [10], [16], [26] and [27] to show $(B_R, \frac{x}{|x|})$ is the unique minimizer of Problem C over strictly star-shaped mean convex $C^{1,1}$ -domains, $C^{1,1}$ -outer minimal sets, and $C^{1,1}$ -revolutionary domains, see Theorem 5.5 and Remark 5.6.

2. Prerequisite: sets of finite perimeter and traces of functions

We first stipulate some notations. Let $V_0 > 0$ be the fixed volume in the Problems A, B, C. Since the admissible domains in the problems have this fixed volume, we will use the convention that any minimizing sequences have their diameters larger than a universal constant $c_0 = c_0(V_0) > 0$ because of the isodiametric inequality (see [13, Theorem 2.2.1]).

We will denote by $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$ and $B_r := B_r(0)$. Throughout this paper all sets under consideration are contained in a large ball B_{R_0} , where $R_0 > 0$ is fixed. For any set $A \subset \mathbb{R}^n$, denote by A_ϵ the interior ϵ -neighborhood $\{x \in A : B_\epsilon(x) \subset A\}$, and A^ϵ the exterior ϵ -neighborhood $\bigcup_{x \in A} B_\epsilon(x)$. Denote by $\text{int}(A)$ the topological interior part of A , $A^c = \mathbb{R}^n \setminus A$, and $\text{diam}(A)$ the diameter of A . For $0 \leq d \leq n$, $\mathcal{H}^d$ denotes the d -dimensional Hausdorff measure in $\mathbb{R}^n$. Let $d^H(\cdot, \cdot)$ denote the Hausdorff distance in $\mathbb{R}^n$. $P(A; D)$ denotes the distributional perimeter of A in $D \subset \mathbb{R}^n$. For a set A of finite perimeter, let ν_A denote the measure theoretical outer unit normal of the reduced boundary ∂^*A , and μ_A denotes the Gauss-Green measure of A , that is, $\mu_A = \nu_A \cdot \mathcal{H}^{n-1}|_{\partial^*A}$. Denote by ω_n the volume of unit ball in $\mathbb{R}^n$ and $|A|$ the Lebesgue measure of A . For any open set $\Omega \subset \mathbb{R}^n$ and $u \in BV(\Omega)$, denote by Du the distributional derivative of u , that is a vector-valued Radon measure, and $\|Du\|(\Omega)$ the total variation of u on Ω .

In this paper, “ $\lesssim_c$ ” denotes an inequality up to constant multiplier $c > 0$. For any measurable set E and $0 \leq \alpha \leq 1$, we define

$$E^\alpha = \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \alpha \right\},$$

and refer E^1 and E^0 as the measure theoretical interior and exterior part of E respectively. Denote by $\partial_*E := \mathbb{R}^n \setminus (E^0 \cup E^1)$ the measure theoretical boundary of E , which is also called the essential boundary. In this paper, we will need the following theorem, due to Federer (see [13, Chapter 5]).

THEOREM 2.1. *For any measurable set E , if $\mathcal{H}^{n-1}(\partial_*E) < \infty$, then E is a set of finite perimeter. Furthermore, if E is a set of finite perimeter, then $\mathbb{R}^n = E^0 \cup E^1 \cup \partial_*E$, $\partial^*E \subset E^{(1/2)} \subset \partial_*E$, and $\partial^*E = \partial_*E \pmod{\mathcal{H}^{n-1}}$.*

Next, we recall the definition of M -uniform domains.

DEFINITION 2.2. For $M \geq 1$, a domain $\Omega \subset \mathbb{R}^n$ is called an M -uniform domain, if for any two points $x, y \in \Omega$, there is a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$, and

$$\mathcal{H}^1(\gamma([0, 1])) \leq M|x - y|, \quad (2.1)$$

$$d(\gamma(t), \partial\Omega) \geq \frac{1}{M} \min \{|\gamma(t) - x|, |\gamma(t) - y|\}, \quad \forall t \in [0, 1]. \quad (2.2)$$

REMARK 2.3. P. Jones [29] introduced the notion of (ϵ, δ) -domain. One can check that any (ϵ, ∞) -domain is an M -domain, with $M = \frac{2}{\epsilon}$. On the other hand, any M -uniform domain is a $(\frac{1}{M^2}, \infty)$ -domain². It was also proven by [29] that any (ϵ, δ) domain is a Sobolev extension domain, and the converse is true when $n = 2$. We refer to [20] and [29] for more details on M -uniform domains.

Since we will study minimization problems involving traces of bounded H^1 vector fields in this paper, we will need the following Gauss-Green formula.

THEOREM 2.4. *Let Ω be a bounded uniform domain of finite perimeter in $\mathbb{R}^n$ and $u \in H^1(\Omega) \cap L^\infty(\Omega)$. Then for any $\phi \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$, we have*

$$\int_{\Omega} u \operatorname{div} \phi + \int_{\Omega} \phi D u = \int_{\partial^* \Omega} (\phi \cdot \nu_{\Omega}) u^* d\mathcal{H}^{n-1}, \quad (2.3)$$

where ν_{Ω} is the measure-theoretic unit outer normal to $\partial^* \Omega$, and u^* is given by the formula

$$\lim_{r \rightarrow 0} \frac{\int_{B_r(x) \cap \Omega} |u - u^*(x)|}{r^n} = 0, \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* \Omega. \quad (2.4)$$

PROOF. According to [29], we may let $\hat{u} \in H_0^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be an extension of u such that $\hat{u} = u$ in Ω and

$$\|\hat{u}\|_{H^1(\mathbb{R}^n)} \leq C(n, \Omega) \|u\|_{H^1(\Omega)}.$$

Hence $\hat{u} \in BV(\mathbb{R}^n)$, and thus according to [2, Theorem 3.77], the interior trace of $\hat{u}$, denoted by $\hat{u}^*$ here, is well-defined for $\mathcal{H}^{n-1}$ -a.e. on $\partial^* \Omega$, and equals to u^* , given by (2.4), for $\mathcal{H}^{n-1}$ -a.e. on $\partial^* \Omega$. Let $\tilde{u} = \hat{u} \chi_{\Omega}$. Since $\hat{u}$ is bounded, $u^* \in L^1(\partial^* \Omega)$ and thus by [2, Theorem 3.84], $\tilde{u} = \hat{u} \chi_{\Omega} \in BV(\mathbb{R}^n)$, with

$$D\tilde{u} = D\hat{u}|_{\Omega^1} - u^* \nu_{\Omega} \mathcal{H}^{n-1}|_{\partial^* \Omega}.$$

Hence for any $\phi \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \phi D\tilde{u} = \int_{\Omega^1} \phi D\hat{u} - \int_{\partial^* \Omega} (\phi \cdot \nu_{\Omega}) u^* d\mathcal{H}^{n-1}. \quad (2.5)$$

Since

$$\int_{\mathbb{R}^n} \phi D\tilde{u} = - \int_{\mathbb{R}^n} \tilde{u} \operatorname{div} \phi = - \int_{\Omega} u \operatorname{div} \phi,$$

from (2.5) we have

$$\int_{\Omega} u \operatorname{div} \phi + \int_{\Omega^1} \phi D\hat{u} = \int_{\partial^* \Omega} (\phi \cdot \nu_{\Omega}) u^* d\mathcal{H}^{n-1}. \quad (2.6)$$

Since Ω is equivalent to Ω^1 up to a set of Lebesgue measure zero and $\hat{u} \in H^1(\mathbb{R}^n)$, we have

$$D\hat{u}|_{\Omega^1} = D\hat{u}|_{\Omega} = Du|_{\Omega} \quad (2.7)$$

Hence (2.6) and (2.7) imply (2.3). $\square$

For the purpose later in this paper, we also introduce the following definition.

²Since (2.1) and (2.2) imply

$$d(\gamma(t), \partial\Omega) \geq \frac{1}{M} \frac{|\gamma(t) - x| |\gamma(t) - y|}{\mathcal{H}^1(\gamma([0, 1]))} \geq \frac{1}{M^2} \frac{|\gamma(t) - x| |\gamma(t) - y|}{|x - y|}, \quad \forall t \in [0, 1].$$

DEFINITION 2.5. For any $c > 0$, we denote by $\mathcal{D}_c$ the class of bounded sets in $\mathbb{R}^n$ such that for any set $E \in \mathcal{D}_c$,

$$|B_r(x) \cap E| > cr^n \quad (2.8)$$

holds for any $x \in \partial E$ and $0 < r < \text{diam}(E)$.

Recall that two sets $E, F \subset \mathbb{R}^n$ are said to be $\mathcal{H}^n$ -equivalent, denoted by $E \approx F$, if $E \Delta F = (E \setminus F) \cup (F \setminus E)$ has zero Lebesgue measure. Note that by the Lebesgue density theorem, if $E \in \mathcal{D}_c$, then $|\partial E \cap E^c| = 0$. Hence $\partial E \subset E \pmod{\mathcal{H}^n}$ and $\overline{E} \approx E$. In particular, we have

REMARK 2.6. Any $E \in \mathcal{D}_c$ is equivalent to its closure $\overline{E}$.

We also have

REMARK 2.7. For $c > 0$, if $E \in \mathcal{D}_c$ is a set of finite perimeter, then there is $c' > 0$ depending only on c and n such that for any $x \in \overline{E}$ and $0 < r < \text{diam}(E)$, $|B_r(x) \cap E| \geq c'r^n$.

PROOF. For $x \in \overline{E}$ and $0 < r < \text{diam}(E)$, there are two cases:

(a) If $r \geq 2d(x, \partial E)$, then there is $z \in \partial E$ such that $B_{\frac{r}{2}}(z) \subset B_r(x)$. Hence

$$|B_r(x) \cap E| \geq |B_{\frac{r}{2}}(z) \cap E| \geq c\left(\frac{r}{2}\right)^n = \frac{c}{2^n}r^n.$$

(b) If $r \leq 2d(x, \partial E)$, then $B_{\frac{r}{2}}(x) \subset E$ and hence

$$|B_r(x) \cap E| \geq |B_{\frac{r}{2}}(x)| = \frac{\omega_n}{2^n}r^n.$$

Hence the conclusion holds with $c' = \min\{\frac{c}{2^n}, \frac{\omega_n}{2^n}\}$. $\square$

The following proposition shows that any M -uniform domain belongs to $\mathcal{D}_c$ for some $c > 0$.

PROPOSITION 2.8. For any $M \geq 1$ and $c_0 > 0$, if $\Omega \subset \mathbb{R}^n$ is an M -uniform domain, with $\text{diam}(\Omega) \geq c_0 > 0$, then $\Omega \in \mathcal{D}_c$ for some $c > 0$ depending only on M , n and c_0 .

PROOF. For any $x \in \partial\Omega$ and $0 < r < \text{diam}(\Omega)$, we claim that there is a constant $c_1 = c_1(M) > 0$ such that $B_r(x) \cap \Omega$ contains a ball of radius c_1r . Indeed, since $0 < r < \text{diam}(\Omega)$, there is $y \in \Omega \setminus B_{\frac{r}{2}}(x)$. Let γ be the curve joining x and y given by the definition of M -uniform domain. Choose $z \in \partial B_{\frac{r}{3}}(x) \cap \gamma$. Then we have that $z \in \Omega$ and

$$d(z, \partial\Omega) \geq \frac{1}{M} \min\{|z - x|, |z - y|\} \geq \frac{1}{M} \min\left\{\frac{r}{3}, \frac{r}{2} - \frac{r}{3}\right\} = \frac{r}{6M}.$$

Hence $B_{c_1r}(z) \subset \Omega$, with $c_1 = \frac{1}{6M}$. From this claim, we see that for any $x \in \partial\Omega$ and any $r < \text{diam}(\Omega)$,

$$|B_r(x) \cap \Omega| \geq |B_{c_1r}(z)| \geq \omega_n c_1^n r^n.$$

This completes the proof. $\square$

The following remark will be used in the proof of compactness of M -uniform domains.

REMARK 2.9. For $M > 0$ and $c_0 > 0$, if $\Omega \subset \mathbb{R}^n$ is an M -uniform domain, with $|\Omega| \geq c_0$, then there is $r_0 > 0$ depending only on M, n, c_0 such that Ω contains a ball of radius r_0 .

PROOF. It follows directly from the isodiametric inequality and Proposition 2.8. $\square$

Similar to $\mathcal{D}_c$, we also define the class $\mathcal{D}^c$ as follows.

DEFINITION 2.10. For $c > 0$, the set class $\mathcal{D}^c$ consists of all bounded set $E \subset \mathbb{R}^n$ such that

$$|B_r(x) \cap E^c| > cr^n \quad (2.9)$$

holds for any $x \in \partial E$ and $0 < r < \text{diam}(E)$.

The following proposition from [36, Proposition 12.19] yields that we can always find an $\mathcal{H}^n$ -equivalent set $\tilde{E}$ of any set E of finite perimeter with slightly better topological boundary.

PROPOSITION 2.11. *For any Borel set $E \subset \mathbb{R}^n$, there exists an $\mathcal{H}^n$ -equivalent set $\tilde{E}$ of E such that for any $x \in \partial \tilde{E}$ and any $r > 0$,*

$$0 < |\tilde{E} \cap B_r(x)| < \omega_n r^n. \quad (2.10)$$

In particular, $\text{spt} \mu_E = \text{spt} \mu_{\tilde{E}} = \partial \tilde{E}$.

In order to illustrate the construction of such an equivalent set, which is needed in later sections, we will sketch the proof.

PROOF. First, we define two disjoint open sets

$$A_1 := \{x \in \mathbb{R}^n \mid \text{there exists } r > 0 \text{ such that } |E \cap B_r(x)| = 0\},$$

and

$$A_2 := \{x \in \mathbb{R}^n \mid \text{there exists } r > 0 \text{ such that } |E \cap B_r(x)| = \omega_n r^n\}.$$

Then by simple covering arguments we have that $|E \cap A_1| = 0$ and $|A_2 \setminus E| = 0$. Set $\tilde{E} = (A_2 \cup E) \setminus A_1$. Then

$$|\tilde{E} \Delta E| \leq |A_2 \setminus E| + |E \cap A_1| = 0.$$

Moreover, since $A_2 \subset \text{int}(\tilde{E})$ and $\tilde{E} \subset \mathbb{R}^n \setminus A_1$, we have that $\partial \tilde{E} \subset \mathbb{R}^n \setminus (A_1 \cup A_2)$ and hence (2.10) holds. $\square$

We now recall the notion of outer minimal sets, which can be viewed as a subsolution of area minimizing sets. It is a generalization of convex sets, see for example [22, Definition 15.6] and related results therein.

DEFINITION 2.12. A set $E \subset \mathbb{R}^n$ of finite perimeter is an outer minimal set, if $P(E) \leq P(F)$ holds for any set $F \supset E$.

We would like to point out that an outer-minimal set is also called as a pseudo-convex set by [33]. Thus by [33, Corollary 7.16] we have

REMARK 2.13. If $E \subset \mathbb{R}^n$ is an outer-minimizing and $\text{spt} \mu_E = \partial E$, then $E \in \mathcal{D}^c$, for some $c > 0$ depending only on n and E . Consequently, $E = \text{int}(E) \pmod{\mathcal{H}^n}$.

REMARK 2.14. Since the boundary of an outer minimal set (domain) can have positive $\mathcal{H}^n$ measure (see [4]), an outer minimal domain may not be an M -uniform domain for any $M \geq 1$.

Combining Proposition 2.8 and Remark 2.13, we have

REMARK 2.15. Let Ω be an M -uniform outer minimal domain with $\text{spt}\mu_\Omega = \partial\Omega$, then $\Omega \in \mathcal{D}_c \cap \mathcal{D}^c$ for some $c > 0$, and hence $\partial_*\Omega = \partial\Omega$.

We would like to state the following proposition, which is a consequence of [24, Corollary 1.10], since for any $E \in \mathcal{D}_c$, $\mathcal{H}^{n-1}(\partial E \cap E^0) = 0$.

PROPOSITION 2.16. *Let $c > 0$ and $E \in \mathcal{D}_c$. Then there exists bounded smooth sets E_i such that $E_i \ni E$, $E_i \rightarrow E$ in L^1 and $P(E_i) \rightarrow P(E)$.*

3. Compactness of M -uniform domains

In this section, we will establish in Theorem 3.5 the L^1 -compactness property of M -uniform domains. We begin with

LEMMA 3.1. *For $c > 0$, suppose that $\{D_i\} \subset \mathcal{D}_c$ satisfies $D_i \rightarrow D$ in $L^1(\mathbb{R}^n)$ as $i \rightarrow \infty$. Then after modifying over a set of Lebesgue measure zero, $D \in \mathcal{D}_c$. Moreover, for any $\epsilon > 0$, there is $N = N(\epsilon) > 0$ such that for any $i > N$, the following properties hold:*

- (i) $D \subset D_i^\epsilon$.
- (ii) $(D_i)_\epsilon \subset D$.
- (iii) $D_i \subset D^\epsilon$.

In particular, $d^H(D_i, D) \rightarrow 0$ as $i \rightarrow \infty$.

PROOF. We first identify D with its $\mathcal{H}^n$ -equivalent set in the sense of Proposition 2.11. We argue by contradiction.

If (i) were false, then there would exist $\epsilon_0 > 0$, $x_0 \in D$ and a sequence $k \rightarrow \infty$ such that $B_{\epsilon_0}(x_0) \cap D_k = \emptyset$. Hence by the hypothesis and Proposition 2.11, we obtain that

$$0 = |B_{\epsilon_0}(x_0) \cap D_k| \rightarrow |B_{\epsilon_0}(x_0) \cap D| > 0,$$

this is impossible.

If (ii) were false, then there would exist $\epsilon_0 > 0$ and a sequence of points $x_i \in (D_i)_{\epsilon_0} \setminus D$. Assume that $x_i \rightarrow x_0$. Then $x_0 \in \partial D \cup D^c$. Hence by the proof of Proposition 2.11, we have that $\omega_n \epsilon_0^n > |B_{\epsilon_0}(x_0) \cap D|$. On the other hand, since $B_{\epsilon_0}(x_i) \subset D_i$, we have that

$$\begin{aligned} |B_{\epsilon_0}(x_0) \cap D| &= \lim_{i \rightarrow \infty} |B_{\epsilon_0}(x_i) \cap D| \geq \liminf_{i \rightarrow \infty} (|B_{\epsilon_0}(x_i) \cap D_i| - |D_i \Delta D|) \\ &= \omega_n \epsilon_0^n - \limsup_{i \rightarrow \infty} |D_i \Delta D| = \omega_n \epsilon_0^n. \end{aligned}$$

We get a desired contradiction.

If (iii) were false, then there would exist $\epsilon_0 > 0$ and a subsequence of $x_i \in D_i \setminus D^{\epsilon_0}$. Without loss of generality, assume $x_i \rightarrow x_0$ and thus $x_0 \in \mathbb{R}^n \setminus D^{\epsilon_0}$. By Remark 2.7, there is a $c' > 0$ depending only on c and n such that

$$c' \epsilon_0^n \leq |B_{\epsilon_0}(x_i) \cap D_i|.$$

On the other hand, it follows from $|B_{\epsilon_0}(x_0) \cap D| = 0$ that

$$\begin{aligned} \liminf_{i \rightarrow \infty} |B_{\epsilon_0}(x_i) \cap D_i| &\leq \limsup_{i \rightarrow \infty} (|B_{\epsilon_0}(x_i) \cap D| + |D \Delta D_i|) \\ &\leq |B_{\epsilon_0}(x_0) \cap D| + \limsup_{i \rightarrow \infty} |D_i \Delta D| = 0. \end{aligned}$$

This yields a desired contradiction.

It remains to show $D \in \mathcal{D}_c$. Indeed, by Proposition 2.11, $x \in \partial D$ implies that $x \in \text{spt}\mu_D$. Note $D_i \rightarrow D$ in $L^1(\mathbb{R}^n)$ implies that $\mu_{D_i} \xrightarrow{*} \mu_D$ as convergence of Radon measures. Hence there exists $x_i \in \text{spt}\mu_{D_i} \subset \partial D_i$ such that $x_i \rightarrow x$ so that for any $r > 0$, it holds that

$$|B_r(x) \cap D| = \lim_i |B_r(x_i) \cap D| \geq \liminf_i |B_r(x_i) \cap D_i| - \limsup_i |D_i \Delta D| \geq cr^n.$$

This implies $D \in \mathcal{D}_c$. $\square$

The following remark follows directly from (i) and (iii).

REMARK 3.2. If D_i and D satisfy the same assumptions as in Lemma 3.1, and if $\text{int}(D) \neq \emptyset$, then $\text{int}(D)$ is connected.

Similar to Lemma 3.1, for a set in the class $\mathcal{D}^c$ we have

LEMMA 3.3. *For $c > 0$, if $\{D_i\} \subset \mathcal{D}^c$ and $D_i \rightarrow D$ in $L^1(\mathbb{R}^n)$, then after modifying a set of zero $\mathcal{H}^n$ -measure, $D \in \mathcal{D}^c$. Moreover, for any $\epsilon > 0$, there is $N = N(\epsilon) > 0$ such that if $i > N$, the following properties holds:*

- (i) $D \subset D_i^\epsilon$.
- (ii) $(D_i)_\epsilon \subset D$.
- (iii) $D_\epsilon \subset D_i$.

The following corollary follows directly from Lemma 3.3.

COROLLARY 3.4. *For any $c > 0$ and a sequence $\{D_i\} \subset \mathcal{D}^c$ with uniformly bounded perimeters, there is an open set $D \in \mathcal{D}^c$ such that $D_i \rightarrow D$ in $L^1(\mathbb{R}^n)$. Moreover, D and D_i satisfy the properties (i), (ii) and (iii) of Lemma 3.3.*

Now we are ready to prove the main theorem of this section.

THEOREM 3.5. *For $M > 0$, $R_0 > 0$, and $c_0 > 0$, if $\{\Omega_i\}$ is a sequence of M -uniform domains in B_{R_0} such that $|\Omega_i| \geq c_0 > 0$ and $\Omega_i \rightarrow D$ in $L^1(\mathbb{R}^n)$, then there is an M -uniform domain Ω such that $\Omega_i \rightarrow \Omega$ in $L^1(\mathbb{R}^n)$.*

PROOF. As in Proposition 2.11, we assume $\text{spt}\mu_D = \partial D$. We first prove that $\text{int}(D) \neq \emptyset$. Indeed, notice that by Remark 2.9, there exists a $r_0 > 0$ depending only on c_0, n and M such that each Ω_i contains a ball of radius r_0 . Therefore, for each Ω_i , if $\epsilon < \frac{r_0}{2}$, then by definition $(\Omega_i)_\epsilon$ contains a ball of radius $\frac{r_0}{2}$. By Lemma 3.1 (ii), D also contains a ball of radius $\frac{r_0}{2}$ and hence $\text{int}(D) \neq \emptyset$.

Set $\Omega = \text{int}(D)$. It suffices to show that Ω is an M -uniform domain, since the L^1 convergence of Ω_i to Ω follows directly from Remark 2.6, Proposition 2.8, and the fact $\Omega \subset D \subset \overline{\Omega}$.

Fix any $x, y \in \Omega$, then given any $N \gg M$, say $N > 2M$, we may choose $0 < \epsilon < \frac{1}{N}$ so small that $k\epsilon < d(x, \partial\Omega) \leq (k+1)\epsilon$, $k \gg N$ (say $k > (1+1/M)(N+1)$), and $|x - y| > 2(N+1)\epsilon$. From Lemma 3.1 (i) and (iii), and since $\text{int}(\Omega) \neq \emptyset$, we know that $d^H(\Omega_i, \Omega) \rightarrow 0$, hence we may choose $x_i, y_i \in \Omega_i \cap \Omega$, with $|x_i - x| < \epsilon, |y_i - y| < \epsilon$ for i large. By Lemma 3.1 (ii), we may also choose i large such that

$$(\Omega_i)_\epsilon \subset \Omega. \tag{3.1}$$

Also we choose $\gamma_i \subset \Omega_i$ to be the rectifiable curve connecting x_i and y_i in Ω_i as in the definition of M -uniform domain. For any $p \in \gamma_i$, if $p \in B_{N\epsilon}(x_i) \cup B_{N\epsilon}(y_i)$,

then clearly $p \in B_{(N+1)\epsilon}(x) \cup B_{(N+1)\epsilon}(y) \subset \Omega$. Moreover, this implies

$$d(p, \partial\Omega) \geq k\epsilon - (N+1)\epsilon > \frac{1}{M}(N+1)\epsilon \geq \frac{1}{M} \min\{|p-x|, |p-y|\}. \quad (3.2)$$

Clearly (3.2) also holds for any p on the line segment between x_i and x , and between y_i and y .

If $p \notin B_{N\epsilon}(x_i) \cup B_{N\epsilon}(y_i)$, then $d(p, \partial\Omega_i) \geq \frac{1}{M} \min\{|p-x_i|, |p-y_i|\} > \frac{1}{M}N\epsilon$, thus $p \in (\Omega_i)_{N\epsilon/M} \subset (\Omega_i)_\epsilon \subset \Omega \cap \Omega_i$. Moreover, let $r = d(p, \partial((\Omega_i)_\epsilon))$, then by (3.1), $B_r(p) \subset \Omega$, so $d(p, \partial\Omega) \geq r = d(p, \partial((\Omega_i)_\epsilon)) \geq d(p, \partial\Omega_i) - \epsilon$. Therefore,

$$\frac{d(p, \partial\Omega)}{\min\{|p-x_i|, |p-y_i|\}} \geq \frac{d(p, \partial\Omega_i) - \epsilon}{\min\{|p-x_i|, |p-y_i|\}} \geq \frac{1}{M} - \frac{\epsilon}{N\epsilon} \geq \frac{1}{M} - \frac{1}{N}. \quad (3.3)$$

Hence by the choice of ϵ and N we have that

$$d(p, \partial\Omega) \geq \left(\frac{1}{M} - \frac{1}{N}\right)(\min\{|p-x|, |p-y|\} - \epsilon) \geq \left(\frac{1}{M} - \frac{1}{N}\right)(\min\{|p-x|, |p-y|\}) - \frac{1}{MN}. \quad (3.4)$$

Therefore, we may let γ^N be the curve with three parts. The first part connects x and x_i with line segment, the second part connects x_i and y_i with γ_i as above and the third part connects y_i and y with line segment. It is clear that $\gamma^N \subset \Omega$ and γ^N connects x and y , then from (3.2) and (3.4) and the choice of ϵ , we obtain

- (i) $\mathcal{H}^1(\gamma^N) \leq M|x-y| + 2\frac{M+1}{N}$, and
- (ii) $d(p, \partial\Omega) \geq \left(\frac{1}{M} - \frac{1}{N}\right) \min\{|p-x|, |p-y|\} - \frac{1}{MN} \quad \forall p \in \gamma^N$.

Then by compactness of $(\overline{\Omega}, d^H)$, and since γ^N is connected, there is a compact connected set $E \subset \overline{\Omega}$ such that $d^H(\gamma^N, E) \rightarrow 0$ as $N \rightarrow \infty$. Then by [14, Theorem 3.18],

$$\mathcal{H}^1(E) \leq \liminf_{N \rightarrow \infty} \mathcal{H}^1(\gamma^N) \leq M|x-y|.$$

Then by [14][Lemma 3.12], E is path connected, thus we can choose a curve $\gamma \subset E$ joining x and y . For any $p \in \gamma$, we can choose sequence $p_N \in \gamma^N, p_N \rightarrow p$. Since

$$d(p_N, \partial\Omega) \geq \left(\frac{1}{M} - \frac{1}{N}\right) \min\{|p_N-x|, |p_N-y|\} - \frac{1}{2MN},$$

we have, after sending $N \rightarrow \infty$,

$$d(p, \partial\Omega) \geq \frac{1}{M} \min\{|p-x|, |p-y|\},$$

which also clearly implies $\gamma \subset \text{int}\Omega$. Then γ satisfies both properties in the definition of M -uniform domain, thus Ω is M -uniform. By Remark 3.2 and Proposition 2.8, Ω is a domain. This completes the proof. $\square$

REMARK 3.6. The full generality of compactness of M -uniform domains is obtained in [11, Theorem 1.2], where it is shown that any sequence of M -uniform domains with fixed volume must have uniformly bounded fractional perimeters, and thus have an L^1 limit up to a subsequence, and the limit is also M -uniform.

4. Existence of equilibrium liquid crystal droplets in Problem A-C

In this section we will study the existence of minimizers to Problems A-C, which can be extended in n -dimensions. We begin with the following Lemma, which plays a crucial role in Problems A-C over outer minimal sets.

LEMMA 4.1. *For $c > 0$, let $\{E_i\}_{i=1}^\infty \in \mathcal{D}_c$ be a sequence of outward-minimizing sets such that $E_i \rightarrow E$ in L^1 as $i \rightarrow \infty$. Then $E \in \mathcal{D}_c$ is also an outward-minimizing set. Moreover, $P(E_i) \rightarrow P(E)$ and $\mathcal{H}^{n-1}(\partial_* E_i) \rightarrow \mathcal{H}^{n-1}(\partial_* E)$ as $i \rightarrow \infty$.*

PROOF. Let $F \supset E$. Then by [2, Proposition 3.38(d)] and the outward-minimality of E_i we have

$$P(E_i \cap F) \leq P(F) + P(E_i) - P(E_i \cup F) \leq P(F).$$

This implies

$$P(E) = P(E \cap F) \leq \liminf_i P(E_i \cap F) \leq P(F).$$

Hence E is outward-minimizing. By Lemma 3.1 and Remark 2.13, $E \in \mathcal{D}_c \cap \mathcal{D}^c$. It follows from Proposition 2.16 that for any $\epsilon > 0$, there exists a smooth open set $O_\epsilon \ni E$ such that

$$P(O_\epsilon) \leq P(E) + \epsilon.$$

Applying Lemma 3.1 (iii), we have that there exists a sufficiently large $i_0 \geq 1$ such that

$$E_i \subset O_\epsilon, \forall i \geq i_0.$$

This, combined with the outward minimality of E_i , implies

$$P(E_i) \leq P(O_\epsilon) \leq P(E) + \epsilon, \quad \forall i \geq i_0.$$

Thus

$$\limsup_i P(E_i) \leq P(E).$$

On the other hand, by lower semicontinuity we have

$$P(E) \leq \liminf_i P(E_i).$$

Therefore $P(E_i) \rightarrow P(E)$ as $i \rightarrow \infty$.

Since $E_i, E \in \mathcal{D}_c \cap \mathcal{D}^c$, the last statement follows from Theorem 2.1. $\square$

Now we are ready to state the main theorem of this section.

THEOREM 4.2. *The following statements hold:*

- i) *For $M \geq 1$, the infimum of Problem C in the class of M -uniform domains of finite perimeter is attained.*
- ii) *For $M > 1$, the infimum of Problems A, B, C can be attained in the class of M -uniform outer minimal domains.*

PROOF. We first prove i). For a minimizing sequence (Ω_i, u_i) , where Ω_i are M -uniform domains with finite perimeter and $u_i \in H^1(\Omega_i, \mathbb{S}^2)$. Let $\hat{u}_i \in H^1(B_{R_0}, \mathbb{R}^3)$ be an extension of u_i such that

$$\|\hat{u}_i\|_{H^1(B_{R_0})} \leq C(n, M)\|u_i\|_{H^1(\Omega_i)}.$$

Hence there is a $\hat{u} \in H^1(B_{R_0}, \mathbb{R}^3)$ such that

$$\hat{u}_i \rightharpoonup \hat{u} \text{ in } H^1(B_{R_0}).$$

By Theorem 3.5, there is an M -uniform domain $\Omega \subset B_{R_0}$ such that $\Omega_i \rightarrow \Omega$ in L^1 . Since $\nabla \hat{u}_i \rightharpoonup \nabla \hat{u}$ in $L^2(B_{R_0})$ and $\chi_{\Omega_i} \rightarrow \chi_\Omega$ in $L^1(B_{R_0})$, by the lower semicontinuity we have that

$$\int_\Omega |\nabla \hat{u}|^2 \leq \liminf_{i \rightarrow \infty} \int_{\Omega_i} |\nabla \hat{u}_i|^2 = \liminf_{i \rightarrow \infty} \int_{\Omega_i} |\nabla u_i|^2. \quad (4.1)$$

Denote $u = \hat{u}|_{\Omega}$. Then it is not hard to see $|u| = 1$ for a.e. $x \in \Omega$ so that $u \in H^1(\Omega, \mathbb{S}^2)$. In order to show (Ω, u) is a minimizer of Problem (C) among M -uniform domains of finite perimeter, we have to verify that $u^* = \nu_{\Omega}$ for $\mathcal{H}^{n-1}$ -a.e. on $\partial^* \Omega$. In fact, it follows from $\chi_{\Omega_i} \rightarrow \chi_{\Omega}$ in $L^2(B_{R_0})$ and $\operatorname{div}(\hat{u}_i) \rightarrow \operatorname{div}(\hat{u})$ in $L^2(B_{R_0})$ and Theorem 2.4 that

$$\begin{aligned} P(\Omega_i) &= \int_{\Omega_i} \operatorname{div}(u_i) = \int_{B_{R_0}} \chi_{\Omega_i} \operatorname{div}(\hat{u}_i) \rightarrow \int_{B_{R_0}} \chi_{\Omega} \operatorname{div}(\hat{u}) = \int_{\Omega} \operatorname{div}(u) \\ &= \int_{\partial^* \Omega} u^* \cdot \nu_{\Omega} d\mathcal{H}^{n-1} \leq P(\Omega). \end{aligned}$$

This, combined with the lower semicontinuity property of perimeter, implies that $u^* = \nu_{\Omega}$ for $\mathcal{H}^{n-1}$ -a.e. on $\partial^* \Omega$. Hence the proof of i) is complete.

Next, we prove ii). For Problem A in part ii), let (Ω_h, u_h) be a minimizing sequence among M -uniform, outer minimal domains and H^1 -unit vector fields on Ω_h . Since Ω_h are outward-minimizing sets in B_{R_0} , $P(\Omega_h)$ are uniformly bounded. By Lemma 4.1 and Theorem 3.5, we may assume that there exists an M -uniform, outer minimal domain Ω such that up to a subsequence, $\Omega_h \rightarrow \Omega$ in L^1 and $P(\Omega_h) \rightarrow P(\Omega)$. As in the proof of i) above, we may extend u_h in B_{R_0} , still denoted as u_h , so that $u_h \rightarrow u$ in $H^1(B_{R_0}, \mathbb{R}^3)$ for some $u \in H^1(B_{R_0}, \mathbb{R}^3)$. Thus we have

$$\int_{\Omega} |\nabla u|^2 \leq \liminf_h \int_{\Omega_h} |\nabla u_h|^2,$$

and $u(x) \in \mathbb{S}^2$ for a.e. $x \in \Omega$.

Since f is convex, we can write

$$f(x) = \sup_i (a_i x + b_i).$$

In the following, we do not distinguish u with u^* on $\partial^* \Omega$, and we do not distinguish $\partial^* \Omega_h, \partial^* \Omega$ with $\partial \Omega_h, \partial \Omega$ due to Remark 2.15. Define

$$\tau_h(A) := \mathcal{H}^{n-1}(\partial^* \Omega_h \cap A), \quad \tau(A) := \mathcal{H}^{n-1}(\partial^* \Omega \cap A), \quad \text{and} \quad \mu_h(A) := \int_A f(u_h \cdot \nu_h) d\tau_h,$$

for any measurable $A \subset \mathbb{R}^n$, where ν_h is the measure theoretical outer unit normal of Ω_h . Then Lemma 4.1 implies that

$$\tau_h(A) \rightarrow \tau(A) \quad \text{as } h \rightarrow \infty. \quad (4.2)$$

Since f is bounded and nonnegative, μ_h are nonnegative Radon measures so that we may assume there is a nonnegative Radon measure μ such that after passing to a subsequence, $\mu_h \rightarrow \mu$ as $h \rightarrow \infty$ as weak convergence of Radon measures. Decompose μ as $\mu = (D_{\tau} \mu) \tau + \mu^s, \mu^s \perp \tau$, and $\mu^s \geq 0$. Then

$$\liminf_{h \rightarrow \infty} \mu_h(A) \geq \mu(A) \geq \int_A D_{\tau} \mu d\tau. \quad (4.3)$$

It follows from Theorem 2.1 that $x \in \partial^* \Omega$ holds for τ -a.e. $x \in B_{R_0}$. Now any such $x \in \partial^* \Omega$, we claim that there exists $r_j \rightarrow 0$ such that for $B_j = B_{r_j}(x)$, it holds that

- (a) $\mathcal{H}^{n-1}(\partial B_j \cap \partial \Omega) = 0$ and $\mathcal{H}^{n-1}(\partial B_j \cap \partial \Omega_h) = 0, \forall h \geq 1$.
- (b) $\int_{\partial B_j \cap \Omega_h} u_h \cdot \nu_{B_j} d\mathcal{H}^{n-1} \rightarrow \int_{\partial B_j \cap \Omega} u \cdot \nu_{B_j} d\mathcal{H}^{n-1}$ as $h \rightarrow \infty$.
- (c) $\mu(\partial B_j) = 0$.

$$(d) \quad D_\tau \mu(x) = \lim_j \frac{\mu(B_j)}{\tau(B_j)} \text{ and } \lim_{j \rightarrow \infty} \frac{\int_{B_j} u \cdot \nu_{B_j} d\tau}{\tau(B_j)} = u(x) \cdot \nu(x).$$

Indeed, (a) and (c) are true because τ, τ_h , and μ are nonnegative Radon measures. (d) follows from the Lebesgue differentiation Theorem. To see (b), let $\tilde{u}_h = u_h \chi_{\Omega_h}$ and $\tilde{u} = u \chi_\Omega$. Since $\tilde{u}_h \rightarrow \tilde{u}$ in L^1 , we have

$$\int_{B_1(x)} |\tilde{u}_h - \tilde{u}| = \int_0^1 \int_{\partial B_r(x)} |\tilde{u}_h - \tilde{u}| d\mathcal{H}^{n-1} dr \rightarrow 0 \quad \text{as } h \rightarrow \infty.$$

Therefore by Fatou's Lemma,

$$\int_0^1 \liminf_{h \rightarrow \infty} \int_{\partial B_r(x)} |\tilde{u}_h - \tilde{u}| d\mathcal{H}^{n-1} dr = 0,$$

hence for almost every $r \in (0, 1)$ and for a subsequence of $h \rightarrow \infty$,

$$\begin{aligned} & \left| \int_{\partial B_r(x) \cap \Omega_h} u_h \cdot \nu_{B_r(x)} d\mathcal{H}^{n-1} - \int_{\partial B_r(x) \cap \Omega} u \cdot \nu_{B_r(x)} d\mathcal{H}^{n-1} \right| \\ & \leq \int_{\partial B_r(x)} |\tilde{u}_h - \tilde{u}| d\mathcal{H}^{n-1} \rightarrow 0. \end{aligned}$$

This finishes the proof of (b). Now we return to the proof of (v). By (c),

$$\mu(B_j) = \lim_{h \rightarrow \infty} \mu_h(B_j) = \lim_{h \rightarrow \infty} \int_{\partial \Omega_h \cap B_j} f(u_h \cdot \nu_h) d\mathcal{H}^{n-1}.$$

Also as $h \rightarrow \infty$, up to a subsequence we have

$$\begin{aligned} & \int_{\partial \Omega_h \cap B_j} u_h \cdot \nu_h d\mathcal{H}^{n-1} \\ &= \int_{\partial(\Omega_h \cap B_j)} u_h \cdot \nu_{\Omega_h \cap B_j} d\mathcal{H}^{n-1} - \int_{\partial B_j \cap \Omega_h} u_h \cdot \nu_{B_j} d\mathcal{H}^{n-1}, \\ &= \int_{\Omega_h \cap B_j} \operatorname{div} u_h - \int_{\partial B_j \cap \Omega_h} u_h \cdot \nu_{B_j} d\mathcal{H}^{n-1}, \\ &\rightarrow \int_{\Omega \cap B_j} \operatorname{div} u - \int_{\partial B_j \cap \Omega} u \cdot \nu_{B_j} d\mathcal{H}^{n-1}, \\ &= \int_{\partial(\Omega \cap B_j)} u \cdot \nu_{\Omega \cap B_j} d\mathcal{H}^{n-1} - \int_{\partial B_j \cap \Omega} u \cdot \nu_{B_j} d\mathcal{H}^{n-1} \\ &= \int_{\partial \Omega \cap B_j} u \cdot \nu_\Omega d\mathcal{H}^{n-1}. \end{aligned}$$

Therefore, for τ -a.e. $x \in B_{R_0}$, it follows

$$\begin{aligned}
D_\tau \mu(x) &= \lim_j \frac{\mu(B_j)}{\tau(B_j)} \\
&= \lim_j \lim_h \frac{\int_{\partial\Omega_h \cap B_j} f(u_h \cdot \nu_h) d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega \cap B_j)} \\
&\geq \lim_j \lim_h \frac{\int_{\partial\Omega_h \cap B_j} (a_i u_h \cdot \nu_h + b_i) d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega \cap B_j)} \\
&= \lim_j \frac{\int_{\partial\Omega \cap B_j} (a_i u \cdot \nu_\Omega + b_i) d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega \cap B_j)}, \quad \text{also by (4.2)} \\
&= a_i u(x) \cdot \nu_\Omega(x) + b_i.
\end{aligned} \tag{4.4}$$

Hence $D_\tau \mu \geq f(u \cdot \nu_\Omega)$ for τ -a.e. $x \in B_{R_0}$, and

$$\begin{aligned}
\liminf_h \int_{\partial\Omega_h} f(u_h \cdot \nu_h) d\mathcal{H}^{n-1} &= \liminf_h \mu_h(B_{R_0}) \geq \int_{B_{R_0}} D_\tau \mu d\tau \\
&\geq \int_{B_{R_0}} f(u \cdot \nu) d\tau = \int_{\partial\Omega} f(u \cdot \nu) d\mathcal{H}^{n-1}. \tag{4.5}
\end{aligned}$$

Therefore, (Ω, u) is a minimizer.

To complete the proof of statements in ii), it remains to show if (Ω_i, u_i) are a minimizing sequence in Problem (B) and converges weakly to (Ω, u) , then $u \cdot \nu = c$ for $\mathcal{H}^{n-1}$ -a.e. on $\partial^* \Omega$. This can be seen from

$$\liminf_{i \rightarrow \infty} \int_{\partial^* \Omega_i} f(u_i \cdot \nu_i) d\mathcal{H}^{n-1} \geq \int_{\partial^* \Omega} f(u \cdot \nu) d\mathcal{H}^{n-1}.$$

In fact, by choosing $f(t) = \mu(t - c)^2$ we have that

$$\int_{\partial^* \Omega} (u \cdot \nu - c)^2 d\mathcal{H}^{n-1} \leq \liminf_{i \rightarrow \infty} \int_{\partial^* \Omega_i} (u_i \cdot \nu_i - c)^2 d\mathcal{H}^{n-1} = 0. \tag{4.6}$$

Hence $u \cdot \nu \equiv c$ for $\mathcal{H}^{n-1}$ -a.e. on $\partial^* \Omega$. This completes the proof. $\square$

5. On the uniqueness of Problem C

In this section, we will show the uniqueness of Problem C in the class of $C^{1,1}$ -star-shaped, mean convex domains in $\mathbb{R}^3$. We will assume the domains has volume $V_0 = |B_1|$, where $B_1 \subset \mathbb{R}^3$ is the unit ball centered at 0. We begin with

LEMMA 5.1. *For any bounded $C^{1,1}$ -domain $\Omega \subset \mathbb{R}^3$,*

$$\inf \left\{ \int_{\Omega} |\nabla u|^2 \mid u \in H^1(\Omega, \mathbb{S}^2), u = \nu_\Omega \text{ on } \partial\Omega \right\} \geq \int_{\partial\Omega} H_{\partial\Omega} d\mathcal{H}^2, \tag{5.1}$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$.

PROOF. Let $u \in H^1(\Omega, \mathbb{S}^2)$, with $u = \nu_\Omega$ on $\partial\Omega$, be such that

$$\int_{\Omega} |\nabla u|^2 = \inf \left\{ \int_{\Omega} |\nabla u|^2 \mid u \in H^1(\Omega, \mathbb{S}^2), u = \nu_\Omega \text{ on } \partial\Omega \right\}.$$

Then by [40, 41], $u \in C^\infty(\Omega \setminus \{a_i\}_{i=1}^N, \mathbb{S}^2)$ for a finite set $\cup_{i=1}^N \{a_i\} \Subset \Omega$. Observe that

$$(\operatorname{div}(u))^2 - \operatorname{tr}(\nabla u)^2 = \operatorname{div}(\operatorname{div}(u)u - (\nabla u)u) \quad \text{in } \Omega \setminus \cup_{i=1}^N \{a_i\}.$$

By [35, Proposition 2.2.1], we have that

$$|\nabla u|^2 \geq (\operatorname{div} u)^2 - \operatorname{tr}(\nabla u)^2 \quad \text{in } \Omega \setminus \cup_{i=1}^N \{a_i\}.$$

By [3, Theorem 1.9], near each a_i , $u(x) \sim R(\frac{x-a_i}{|x-a_i|})$ for some rotation $R \in O(3)$. In particular, one has that for $r > 0$ sufficiently small,

$$\left| \int_{\partial B_r(a_i)} (\operatorname{div}(u)u - (\nabla u)u) \cdot \nu_{B_r(a_i)} d\mathcal{H}^2 \right| = O(r).$$

Hence

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 \\ & \geq \int_{\Omega \setminus \cup_{i=1}^N B_r(a_i)} (\operatorname{div}(u))^2 - \operatorname{tr}(\nabla u)^2 \\ & = \int_{\Omega \setminus \cup_{i=1}^N B_r(a_i)} \operatorname{div}((\operatorname{div} u)u - (\nabla u)u) \\ & = \int_{\partial \Omega} (\operatorname{div}(u)u - (\nabla u)u) \cdot \nu_{\Omega} d\mathcal{H}^2 \\ & \quad - \sum_{i=1}^n \int_{\partial B_r(a_i)} (\operatorname{div}(u)u - (\nabla u)u) \cdot \nu_{B_r(a_i)} d\mathcal{H}^2 \\ & \geq \int_{\partial \Omega} (\operatorname{div}(u) - ((\nabla u)\nu_{\Omega}) \cdot \nu_{\Omega}) d\mathcal{H}^2 - CNr \\ & = \int_{\partial \Omega} (\operatorname{div}_{\partial \Omega} \nu_{\Omega}) d\mathcal{H}^2 - CNr = \int_{\partial \Omega} H_{\partial \Omega} d\mathcal{H}^2 - CNr. \end{aligned}$$

This implies (5.1) after sending $r \rightarrow 0$. $\square$

The inequality (5.1) leads us to study the minimization of the total mean curvatures. It is well-known that

$$\int_{\partial \Omega} H_{\partial \Omega} d\mathcal{H}^2 \geq 4\sqrt{\pi P(\Omega)} \quad (5.2)$$

is true if Ω is convex, and the equality holds if and only if Ω is a ball. Very recently, Dalphin-Henrot-Masnou-Takahashi [10] proved that if Ω is a revolutionary solid and $H \geq 0$, then (5.2) is true, and the equality holds if and only if Ω is a ball. Without the mean convexity, (5.2) is false, see [10]. In the next lemma we present a proof that (5.2) is true if Ω is a $C^{1,1}$ star-shaped and mean convex domain. The key ingredient of the proof is based on the result by Gerhardt [21]. We remark that a more general version of (5.2) has been proven by Guan-Li [23]. Here we will sketch the proof, since it is elementary in $\mathbb{R}^3$.

LEMMA 5.2. *The inequality (5.2) holds, if Ω is $C^{1,1}$ -strictly star-shaped and mean convex.*

PROOF. By the remark below, we may assume $\Omega \in C^\infty$. By a standard argument, we can perturb Ω so that $H > 0$ everywhere. Indeed, represent $\partial \Omega$ as an embedding $F^0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ and consider the mean curvature flow $\{F_t : \mathbb{S}^2 \rightarrow \mathbb{R}^3 : t \in [0, T)\}$, which is a family of embeddings so that

$$\frac{\partial F}{\partial t} = H\nu_t \quad 0 < t < T; \quad F_0 = F^0,$$

where ν_t is the inward unit normal of the embedding F_t . It is well-known that the solution exists for a short time $T > 0$. If $t > 0$ is small, then $F_t(\mathbb{S}^2)$ remains to be star-shaped. The evolution of the mean curvature H of $F_t(\mathbb{S}^2)$ is given by

$$\frac{\partial H}{\partial t} = \Delta H + |A|^2 H,$$

where A is the second fundamental form of $F_t(\mathbb{S}^2)$. Then the strong maximum principle implies that $H > 0$ everywhere on $F_t(\mathbb{S}^2)$ for $t > 0$. It is clear that after a small perturbation in C^1 -norm, Ω is still strictly star-shaped.

Hence it suffices to prove (5.2) by assuming $H > 0$ everywhere on $\partial\Omega$. We argue it by contradiction. Suppose there were a strictly star-shaped domain Ω with $H > 0$ everywhere on $\partial\Omega$ such that

$$\frac{\int_{\partial\Omega} H d\mathcal{H}^2}{4\sqrt{\pi P(\Omega)}} < 1.$$

Representing $\partial\Omega$ as an embedding $G_0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$. Now consider the inverse mean curvature flow $\{G_t : \mathbb{S}^2 \rightarrow \mathbb{R}^3 : t \in [0, \infty)\}$, which is a family of embeddings that solves

$$\frac{\partial G}{\partial t} = \frac{1}{H} \nu_t,$$

where ν_t is the inward unit normal of the embedding G_t . It has been shown by Gerhard [\[21\]](#) that $S_t := G_t(\partial\Omega)$ converges to the unit sphere $\mathbb{S}^2$, up to rescalings by $e^{-t/2}$, as $t \rightarrow \infty$. Set

$$y(t) = \frac{\int_{S_t} H d\mathcal{H}^2}{4\sqrt{\pi \text{Area}(S_t)}}, \quad t > 0.$$

Observe that $y(t)$ is scaling-invariant. Therefore, $y(0) < 1$ and $y(t) \rightarrow 1$ as $t \rightarrow \infty$. On the other hand, using the evolution equations under the inverse mean curvature flow we have that

$$\frac{d}{dt} H = -\Delta H - \frac{|A|^2}{H},$$

and

$$\frac{d}{dt} \sqrt{g} = \sqrt{g},$$

where Δ is the surface Laplacian and g is the metric on surface S_t induced by Euclidean metric in $\mathbb{R}^3$. Direct calculations imply

$$\begin{aligned} \frac{d}{dt} \left(\frac{\int_{S_t} H d\mathcal{H}^2}{4\sqrt{\pi P(\Omega)}} \right) &= \left(\int_{S_t} \left(H - \frac{|A|^2}{H} \right) d\mathcal{H}^2 \right) \frac{1}{4\sqrt{\pi \text{Area}(S_t)}} - \frac{\int_{S_t} H d\mathcal{H}^2}{8\sqrt{\pi \text{Area}(S_t)}} \\ &= \frac{1}{4\sqrt{\pi \text{Area}(S_t)}} \left(\int_{S_t} \frac{2K}{H} d\mathcal{H}^2 - \frac{1}{2} \int_{S_t} H d\mathcal{H}^2 \right) \\ &= \frac{1}{4\sqrt{\pi \text{Area}(S_t)}} \int_{S_t} \frac{4K - H^2}{2H} d\mathcal{H}^2 \leq 0, \end{aligned}$$

since $H^2 \geq 4K$, here K is the Gauss curvature of S_t . Therefore, $y(t) \leq y(0) < 1$ for all $t > 0$. We get a desired contradiction. $\square$

REMARK 5.3. (5.2) is actually true for any C^1 -strictly star-shaped surface with bounded nonnegative generalized mean curvature, in particular for a $C^{1,1}$ -mean convex surface. Indeed, by [\[27, Lemma 2.6\]](#), we can find a family of smooth

strictly star-shaped mean convex hypersurfaces converging to the surface uniformly in $C^{1,\alpha} \cap W^{2,p}$ for $0 < \alpha < 1$ and $1 < p < \infty$ so that the total mean curvature of the smooth surfaces converges to the total mean curvature of the original surface. We refer the reader to [27] for the detail.

By Lemma 5.2 and the isoperimetric inequality $P(\Omega) \geq 4\pi(\frac{3}{4\pi}|\Omega|)^{2/3}$, we immediately have

COROLLARY 5.4. *It holds that*

$$\inf \left\{ \int_{\Omega} |\nabla u|^2 : \Omega \text{ is } C^{1,1}\text{-star-shaped, mean convex, } |\Omega| = |B_1|, u \in H^1(\Omega, \mathbb{S}^2), \right. \\ \left. u = \nu_{\Omega} \text{ on } \partial\Omega \right\} \geq 8\pi,$$

and the equality holds if and only if $\Omega = B_1$, up to translation and rotation.

As a consequence, we have

THEOREM 5.5. *The Problem (C) over $C^{1,1}$ -star-shaped and mean convex domains is uniquely achieved at $\Omega = B_1$ and $u(x) = \frac{x}{|x|}$.*

PROOF. By direct calculations,

$$\int_{B_1} \left| \nabla \left(\frac{x}{|x|} \right) \right|^2 = \int_{B_1} \frac{2}{|x|^2} = 8\pi.$$

Hence by the first statement in Corollary 5.4, (4.2) is attained at $(B_1, \frac{x}{|x|})$. The uniqueness follows from the last statement of Corollary 5.4 and [8, Theorem 7.1]. $\square$

REMARK 5.6. Huisken first proves that (5.2) holds if Ω is $C^{1,1}$ -outer minimal (not necessarily connected), though it seems that he didn't publish it. See also Freire-Schwartz [16, Theorem 5]. Hence the same result as in Theorem 5.5 holds in the class of $C^{1,1}$ -outer minimal open sets. By [10], the same result as in Theorem 5.5 holds in the class of smooth domains of revolution.

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