



# $L^p$ -regularity for fourth order elliptic systems with antisymmetric potentials in higher dimensions

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## Abstract

We establish an optimal  $L^p$ -regularity theory for solutions to fourth order elliptic systems with antisymmetric potentials in all supercritical dimensions  $n \geq 5$ :

$$\Delta^2 u = \Delta(D \cdot \nabla u) + \operatorname{div}(E \cdot \nabla u) + (\Delta \Omega + G) \cdot \nabla u + f \quad \text{in } B^n,$$

where  $\Omega \in W^{1,2}(B^n, so_m)$  is antisymmetric and  $f \in L^p(B^n)$ , and  $D, E, \Omega, G$  satisfy the growth condition (GC-4), under the smallness condition of a critical scale invariant norm of  $\nabla u$  and  $\nabla^2 u$ . This system was brought into lights from the study of regularity of (stationary) biharmonic maps between manifolds by Lamm–Rivière, Struwe, and Wang. In particular, our results improve Struwe’s Hölder regularity theorem to any Hölder exponent  $\alpha \in (0, 1)$  when  $f \equiv 0$ , and have applications to both approximate biharmonic maps and heat flow of biharmonic maps. As a by-product of our techniques, we also partially extend the  $L^p$ -regularity theory of approximate harmonic maps by Moser to Rivière–Struwe’s second order

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elliptic systems with antisymmetric potentials under the growth condition (GC-2) in all dimensions  $n \geq 2$  when  $p > \frac{n}{2}$ , which partially confirms an interesting expectation by Sharp.

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## 1 Introduction and main results

### 1.1 Background and motivation

In his landmark work [26], Rivière proposed the second order linear elliptic system

$$-\Delta u = \Omega \cdot \nabla u \quad \text{in } B^2 \subset \mathbb{R}^2, \quad (1.1)$$

with  $\Omega = (\Omega_{ij}) \in L^2(B^2, so_m \otimes \Lambda^1 \mathbb{R}^2)$  and  $u \in W^{1,2}(B^2, \mathbb{R}^m)$ , which models the Euler-Lagrange equations of critical points of all second order conformally invariant variational functionals over maps  $u \in W^{1,2}(B^2, N)$ , where  $B^2 \subset \mathbb{R}^2$  is the unit disk, and  $N \subset \mathbb{R}^m$  is an arbitrary compact Riemannian manifold. In particular, (1.1) includes the equation of weakly harmonic maps from  $B^2$  to  $N$  and the prescribed mean curvature equations. A crucial observation of [26] is a conservation law induced by the anti-symmetry of  $\Omega$ , from which the continuity of weak solutions to Eq. (1.1) follows. This gave an affirmative answer to the long standing conjectures of Hildebrandt and Heinz, and an alternate proof of Helein's celebrated regularity theorem on weak harmonic maps in dimension two. The technique developed in [26] has also profound applications beyond conformally invariant problems; see [27, 28] for a comprehensive overview.

Rivière and Struwe have further considered in [29] the same system as (1.1) in supercritical dimensions  $n \geq 3$ :

$$-\Delta u = \Omega \cdot \nabla u \quad \text{in } B^n \subset \mathbb{R}^n, \quad (1.2)$$

where  $\Omega = (\Omega_{ij}) \in L^2(B^n, so_m \otimes \Lambda^1 \mathbb{R}^n)$ . Although there is no conservation law associated with (1.2) for  $n \geq 3$ , Rivière and Struwe managed to transform (1.2) into a gauge equivalent system through Uhlenbeck's gauge construction associated with  $\Omega$ . It was established in [29] that a local Hölder regularity holds for any weak solution  $u$  to (1.2) under the smallness

condition

$$\sup_{x \in B_1^n, r > 0} \left( \frac{1}{r^{n-2}} \int_{B_r^n(x) \cap B} (|\nabla u|^2 + |\Omega|^2) dx \right)^{1/2} < \varepsilon(n, m).$$

As an application, they reproved the partial regularity theorem on stationary harmonic maps in dimensions  $n \geq 3$ , due to Evans [8] and Bethuel [3].

The techniques in [26, 29] have been subsequently extended to fourth order elliptic systems by Lamm and Rivière [18] in dimension  $n = 4$  and Struwe [33] for  $n \geq 5$  in the course of study of biharmonic maps. Recall that an extrinsic (or intrinsic resp.) biharmonic map from  $B^n$  into a closed Riemannian manifold  $N$  is a critical point of the energy functional

$$\int_{B^n} |\Delta u|^2 \quad (\text{or } \int_{B^n} |(\Delta u)^T|^2 \quad \text{resp.}) \quad \text{for } u \in W^{2,2}(B^n, N),$$

where  $(\Delta u)^T$  is the orthogonal projection of  $\Delta u$  onto the tangent space  $T_u N$ . In [18], the authors formulated the following system of 4th order linear elliptic equations

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + F \cdot \nabla u \quad \text{in } B^4, \quad (1.3)$$

where  $V, w$  belong to certain function spaces and  $F = \nabla \omega + W$  with  $\omega \in L^2(B^4, so_m)$  being antisymmetric. By constructing a corresponding conservation law for system (1.3), an everywhere continuity for weak solutions of (1.3) was established in [18]. The approach of [18] was further refined by Guo and Xiang in [11], where a local Hölder continuity for weak solutions of (1.3) was proven. The result of [18] has been applied to the theory of regularity for heat flow of biharmonic maps in dimension four. In [33], Struwe revisited biharmonic maps in supercritical dimensions  $n \geq 5$  and formulated the following fourth order linear elliptic system:

$$\Delta^2 u = \Delta(D \cdot \nabla u) + \operatorname{div}(E \cdot \nabla u) + (\Delta \Omega + G) \cdot \nabla u \quad \text{in } B^n, \quad (1.4)$$

where  $D, E, G$  belong to certain function spaces and  $\Omega$  is an  $so_m$ -valued function with entries in  $\mathbb{R}^n$ . We refer interested readers to [18, 33] for detailed computations of writing the equation of biharmonic maps in the form of (1.3) or (1.4). By extending the approach of Rivière and Struwe [29], Struwe established in [33] a partial regularity theory for (1.4), under the growth condition (GC-4) below, which in turn gave an alternate proof of the Hölder regularity theorems of Chang, Wang and Yang [5] and Wang [34, 35] for biharmonic maps. Because of structural similarities, it seems natural to extend the result of Rivière and Struwe [29] on the system of second order linear equations (1.2) to system of fourth order linear equations (1.3) and (1.4). Indeed, Struwe raised the following question in [33]:

**Struwe's Question** *It would be interesting to see if our method can be extended to general linear systems of fourth order that exhibit a structure similar to the one of equation (1.4), as is the case for second order systems (1.2), or in the “conformal” case  $n = 4$  considered in [18].*

Struwe's Question in the “conformal” case  $n = 4$  has recently been solved by Guo and Xiang in [12]. More precisely, it was proven in [12] that in critical dimensions  $n = 2k$  for any  $k \geq 2$ , a Hölder continuity holds for any weak solution  $u \in W^{k,2}(B^n, \mathbb{R}^m)$  of the  $2k$ -order linear elliptic system with antisymmetric potentials introduced by de Longueville and Gastel in [7]. [12] was built upon the ideas by Rivière-Struwe [29] and utilized both Uhlenbeck's gauge transformation and the duality of Lorentz spaces  $L^{p,1} - L^{p',\infty}$ , where  $1 < p < \infty$  and  $p' = p/(p-1)$ . However, when dimensions  $n \geq 5$ , the approach by [12] (see [12, Section 5]) encountered serious technical difficulties, which left open Struwe's Question in

supercritical dimensions  $n \geq 5$ . Another interesting problem, closely related to the regularity theory on (1.2) and Struwe's Question on (1.4), is to study the corresponding inhomogeneous system of (1.4) in dimensions  $n \geq 4$ . These problems lead us to ask

**Problem 1.1** *Establish a  $L^p$ -regularity theory for weak solutions of the fourth order inhomogeneous elliptic system of Lamm and Rivière [18] or Struwe [33]*

$$\Delta^2 u = \Delta(D \cdot \nabla u) + \operatorname{div}(E \nabla u) + F \cdot \nabla u + f \quad \text{in } B^n$$

in dimensions  $n \geq 4$ .

More specifically, Problem 1.1 asks that for  $f \in L^p(B^n, \mathbb{R}^m)$  with  $1 < p < \infty$ , if a  $W_{\text{loc}}^{4,p}$ -regularity holds for weak solutions of the linear systems (1.3) or (1.4), provided certain smallness conditions are imposed on both the linear coefficient functions and the solution. In the critical dimension  $n = 4$ , Problem 1.1 was solved by Guo, Xiang and Zheng in [13], where they proved that if  $f \in L^p$  for  $1 < p < 4/3$ , then  $u \in W_{\text{loc}}^{3,4p/(4-p)} \subset C_{\text{loc}}^{0,4(1-1/p)}$ . In particular, this implies that when  $n = 4$ , every weak solution of the system (1.3) or (1.4) is locally  $\alpha$ -Hölder continuous for all  $0 < \alpha < 1$ . A similar  $L^p$ -theory for general even order linear elliptic systems proposed by de Longueville and Gastel [7] was also established by [14] in critical dimensions. For applications to biharmonic maps, see Laurain-Lin [20] for energy convexity and Laurain-Rivière [21] and Wang-Zheng [36] for energy quantization. We also point out that the theory of biharmonic maps has been successfully applied in Cheng-Zhou's solution of the Rosenberg-Smith conjecture in their recent work [6]. We would like to mention that a positive answer to Problem 1.1 would solve Struwe's Question. However, Problem 1.1 remains open in supercritical dimensions  $n \geq 5$ . In this paper, we will make some partial progress towards Problem 1.1.

In the second order case, motivated by the study on approximate harmonic maps and heat flow of harmonic maps, It is natural to consider the following problem.

*For  $p > 1$ , develop a  $W^{2,p}$ -regularity theory for the inhomogeneous Rivière's system*

$$-\Delta u = \Omega \cdot \nabla u + f \quad \text{in } B^n, \quad (1.5)$$

where  $\Omega \in L^2(B^n, \operatorname{so}_m \otimes \mathbb{R}^n)$  and  $f \in L^p(B^n, \mathbb{R}^m)$ .

This problem was first considered by Sharp and Topping [32] in dimension  $n = 2$ . Utilizing the conservation law of Rivière [26], they proved that if  $f \in L^p(B^2, \mathbb{R}^m)$  for  $p \in (1, 2)$ , then every weak solution  $u \in W^{1,2}(B^2, \mathbb{R}^m)$  belongs to  $W_{\text{loc}}^{2,p}(B^2, \mathbb{R}^m) \subset C_{\text{loc}}^{0,2(1-1/p)}$ . In particular, any weak solution of (1.1) is locally  $\alpha$ -Hölder continuous for any  $0 < \alpha < 1$ . See Laurain-Rivière [22] and Lamm-Sharp [19] for some further related results.

For dimensions  $n \geq 3$ , in the course of studying the heat flow of harmonic maps, Moser [25] considered the  $L^p$ -regularity of the system of approximate harmonic maps  $u : B^n \rightarrow N$ :

$$-\Delta u = A(u)(\nabla u, \nabla u) + f, \quad (1.6)$$

and proved that for any  $1 < p < \infty$  if  $f \in L^p(B^n, \mathbb{R}^m)$ , then  $u \in W_{\text{loc}}^{2,p}(B^n, N)$ , under certain smallness condition on  $\nabla u$ . One crucial idea of [25] is to rewrite the system (1.6) via the Gauge transformation of Rivière and Struwe [29]. On the other hand, Sharp [31] established a Morrey-space regularity for the linear system (1.5) for  $p > \frac{n}{2}$ , namely,  $M^{\frac{2p}{n}, n-2}$ -regularity for  $\nabla^2 u$  holds under a smallness condition on  $\|\Omega\|_{M^{2,n-2}}$ . In view of Moser [25], Sharp made the following expectation in [31, Remark 1.3]:

**Sharp's expectation** *One would expect Moser's  $L^p$ -regularity on (1.6) remains to hold for the system (1.5) for any  $1 < p < \infty$ , under the additional condition*

$$|\Omega| \leq C|\nabla u|. \quad (\text{GC-2})$$

We will provide a partial answer to this expectation in Theorem 1.8 below, dealing with the case  $p > n/2$ .

## 1.2 Main results

Henceforth, we will assume  $m > 1, n \geq 5$ . Let  $B_r = \{x \in \mathbb{R}^n : |x| < r\}$  and  $u \in W^{2,2}(B_2, \mathbb{R}^m)$ . Consider the following inhomogeneous 4th order elliptic system

$$\Delta^2 u = \Delta(D \cdot \nabla u) + \operatorname{div}(E \nabla u) + F \cdot \nabla u + f \quad \text{in } B_2, \quad (1.7)$$

with  $F = \Delta \Omega + G$ , and

$$\begin{aligned} D &\in W^{1,2}(B_2, M_m \otimes \Lambda^1 \mathbb{R}^n), & E &\in L^2(B_2, M_m) \\ \Omega &\in W^{1,2}(B_2, so_m), & G &\in L^{\frac{4}{3},1}(B_2, M_m \otimes \Lambda^1 \mathbb{R}^n). \end{aligned} \quad (1.8)$$

In coordinates, (1.7) reads as

$$\Delta^2 u^i = \Delta(D_j^i \cdot \nabla u^j) + \operatorname{div}(E_j^i \nabla u^j) + F_j^i \cdot \nabla u^j + f^i, \quad 1 \leq i \leq m,$$

where the Einstein summation convention is used for repeated indices.

In this paper, we aim to establish an  $L^p$ -regularity theory for (1.7) under the following growth condition on  $D, E, G, \Omega$ :

$$\begin{aligned} |D| + |\Omega| &\leq C|\nabla u|, \\ |E| + |\nabla D| + |\nabla \Omega| &\leq C|\nabla^2 u| + C|\nabla u|^2, \\ |G| &\leq C|\nabla^2 u||\nabla u| + C|\nabla u|^3. \end{aligned} \quad (\text{GC-4})$$

Although the  $L^p$ -theory of (1.7) under condition (GC-4) does not answer Problem 1.1, it provides very interesting insights on attacking this challenging problem. From the analytic point of view, the nonlinearity under (GC-4) is of critical growth so that for a weak solution  $u \in W^{2,2}(B^n, \mathbb{R}^m)$  there merely holds  $|\nabla^2 u|^2 \in L^1(B^n)$  and the standard  $L^p$ -regularity theory is not applicable. Furthermore, the nonlinearity is so strong that it is also impossible to apply the standard bootstrapping argument, even if some improved regularity, e.g.  $\nabla^2 u \in L^{2+\epsilon}$  for a sufficiently small  $\epsilon > 0$ , is assumed.

Since (1.7) models biharmonic maps when  $f \equiv 0$  (see Lamm and Rivière [18] and Struwe [33]), our  $L^p$ -regularity theory, via the Sobolev embedding theorem, implies that  $u \in C_{\text{loc}}^\alpha(B^n)$  for any  $\alpha \in (0, 1)$ , which in turn improves Struwe's Hölder regularity theorem. Note that the system for both approximate biharmonic maps and heat flow of biharmonic maps do satisfy both (1.7) and the growth condition (GC-4), hence the  $L^p$ -regularity theory will have direct application to the study of biharmonic maps and heat flow of biharmonic maps in supercritical dimensions.

We denote by  $M^{p,\lambda}(B_r)$  and  $M_*^{p,\lambda}(B_r)$  the  $(p, r)$ -Morrey space and weak  $(p, r)$ -Morrey space respectively (see Sect. 2 for their definitions). Our first theorem is stated as follows.

**Theorem 1.2** *Suppose  $f \in M^{1,n-4+\alpha}(B_2, \mathbb{R}^m)$  for some  $\alpha \in (0, 1)$  and  $u \in W^{2,2}(B_2, \mathbb{R}^m)$  is a solution of system (1.7) satisfying (GC-4). There exist constants  $\epsilon = \epsilon(m, n, \alpha)$  and  $C = C(m, n, \alpha) > 0$  such that if*

$$\|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|\nabla u\|_{M^{4,n-4}(B_1)} \leq \epsilon, \quad (1.9)$$

then

$$\nabla u \in M_*^{4,n-4+4\alpha}(B_{1/2}) \quad \text{and} \quad \nabla^2 u \in M_*^{2,n-4+2\alpha}(B_{1/2}).$$

Moreover,

$$\begin{aligned}\|\nabla u\|_{M_*^{4,n-4+4\alpha}(B_{1/2})} &\leq C \left( \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \right), \\ \|\nabla^2 u\|_{M_*^{2,n-4+2\alpha}(B_{1/2})} &\leq C \left( \|\nabla^2 u\|_{M_*^{2,n-4}(B_1)} + \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \right).\end{aligned}\quad (1.10)$$

Theorem 1.2, relaxing the  $L^p$ -assumption on  $f$  to a Morrey assumption on  $f$ , seems to be new even in the critical dimension  $n = 4$ . In connection with the Morrey smallness assumption (1.9), the Morrey assumption on  $f$  seems to be more compatible than the  $L^p$  assumption on  $f$ .

The smallness assumption (1.9) is natural in terms of both translation and dilation invariance. When  $f \equiv 0$ , the monotonicity formula for stationary biharmonic maps justifies this smallness assumption, see e.g. Wang [35], Struwe [33] and Moser [23]. For heat flow of biharmonic map flow (i.e.  $f = u_t$ ), a parabolic version of smallness assumptions can also be verified in some cases, see e.g. Hineman-Huang-Wang [15]. In view of the embedding  $M_*^{4,n-4+4\alpha}(B_1) \subset M^{q,n-q+q\alpha}(B_1)$  for any  $1 \leq q < 4$ , the estimate (1.10) can be viewed as a slight improvement of Struwe [33, Estimate (37)], where it was proved that  $\nabla u \in M^{q,n-q+q\alpha}(B_{1/2})$  for some  $1 < q < 2$ .

An immediate consequence of Theorem 1.2 is the following optimal Hölder regularity.

**Corollary 1.3** Suppose  $f \in M^{1,n-4+\alpha}(B_2, \mathbb{R}^m)$  for some  $\alpha \in (0, 1)$  and  $u \in W^{2,2}(B_2, \mathbb{R}^m)$  is a solution of system (1.7) satisfying (GC-4). There exist constants  $\epsilon = \epsilon(m, n, \alpha)$  and  $C = C(m, n, \alpha) > 0$  such that if (1.9) holds, then  $u \in C_{\text{loc}}^{0,\alpha}(B_1)$  with

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \leq C \left( \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \right).$$

When  $f \equiv 0$ , this implies that solutions of (1.4) are locally  $\alpha$ -Hölder continuous with any exponent  $0 < \alpha < 1$ , which improves the main result of Struwe [33]. See Rupflin [30] and Wang-Zheng [36] for some related results.

Theorem 1.2 provides the key technical tool to prove the following  $L^p$ -regularity result.

**Theorem 1.4** Suppose  $f \in L^p(B_1)$  for some  $n/4 < p < \infty$  and  $u \in W^{2,2}(B_1, \mathbb{R}^m)$  is a solution of system (1.7) satisfying (GC-4).

- (i) When  $p < n$ , there exists a constant  $\epsilon = \epsilon(m, n, p)$  such that if the assumption (1.9) holds, then  $u \in W_{\text{loc}}^{3, \frac{np}{n-p}}(B_1)$  and

$$\|\nabla^3 u\|_{L^{\frac{np}{n-p}}(B_{1/2})} \leq C_* \left( \|\nabla u\|_{M^{4,n-4}(B_1)} + \|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|f\|_{L^p(B_1)} \right).$$

Here  $C_* = c_*(1 + \epsilon + \|f\|_{L^p(B_1)})^{a_*}$  for constants  $c_*$  and  $a_*$  depending on  $n, m, p$ .

- (ii) When  $p \geq n$ , for any  $1 < q < \infty$ , there exists a constant  $\epsilon = \epsilon(m, n, p, q)$  such that if the smallness assumption (1.9) holds, then  $u \in W_{\text{loc}}^{3,q}(B_1)$  and

$$\|\nabla^3 u\|_{L^q(B_{1/2})} \leq D_* \left( \|\nabla u\|_{M^{4,n-4}(B_1)} + \|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|f\|_{L^p(B_1)} \right).$$

Here  $D_* = d_*(1 + \epsilon + \|f\|_{L^p(B_1)})^{b_*}$  for constants  $d_*$  and  $b_*$  depending on  $n, m, p, q$ .

We would like to remark that in Theorem 1.4, the constant  $C_*$  (and  $D_*$ ) depends not only on  $\epsilon, n, m, p$  (and  $q$ ) but also on  $\|f\|_{L^p(B_1)}$ . The reason for dependence on  $\|f\|_{L^p(B_1)}$  is that our system is nonlinear and in general one can only expect a priori estimates involving polynomial

dependency on  $\|\nabla u\|_{M^{4,n-4}(B_1)}$ ,  $\|\nabla^2 u\|_{M^{2,n-4}(B_1)}$  and  $\|f\|_{L^p(B_1)}$ , under the Morrey smallness assumption (1.9) on  $\nabla u$  and  $\nabla^2 u$ .

This result provides an affirmative answer to Problem 1.1 under the growth condition (GC-4). As a simple consequence of this theorem and the Sobolev embedding theorem, we can infer that the smallness assumption (1.9) implies

$$u \in \begin{cases} C_{\text{loc}}^{0,\alpha}(B_1), & \text{if } n/4 < p < n/3, \\ C_{\text{loc}}^{1,\alpha-1}(B_1), & \text{if } n/3 < p < n/2, \\ C_{\text{loc}}^{2,\alpha-2}(B_1), & \text{if } n/2 < p < n, \end{cases}$$

where  $\alpha = 4 - n/p$ , whenever  $p < n$ . As a further application, we have

**Corollary 1.5** Suppose  $f \in L^p(B_1)$  for some  $n/4 < p < \infty$  and  $u \in W^{2,2}(B_1, \mathbb{R}^m)$  a solution of system (1.7) satisfying (GC-4). Suppose further that

$$\begin{aligned} |\nabla D| &\leq C(|\nabla^2 u| + |\nabla u|^2), \\ |\nabla E| + |\nabla^2 \Omega| &\leq C(|\nabla^3 u| + |\nabla^2 u||\nabla u| + |\nabla u|^3). \end{aligned}$$

Then there exists a constant  $\epsilon = \epsilon(m, n, p)$  such that if the assumption (1.9) holds, then  $u \in W_{\text{loc}}^{4,p}(B_1)$  and

$$\|\nabla^4 u\|_{L^p(B_{1/2})} \leq C \left( \|\nabla u\|_{M^{4,n-4}(B_1)} + \|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|f\|_{L^p(B_1)} \right).$$

Here  $C = c(1 + \epsilon + \|f\|_{L^p(B_1)})^a$  for two constants  $c$  and  $a$  depending on  $n, m, p$ . In particular, any approximate biharmonic map  $u$  with drift term  $f \in L^p(B_1)$  for some  $n/4 < p < \infty$  belongs to  $W_{\text{loc}}^{4,p}(B_1)$ , provided the smallness condition (1.9) holds for a sufficiently small  $\epsilon$ .

Corollary 1.5 extends the corresponding results by Wang-Zheng [36] and Laurain-Rivière [21] in the critical dimension  $n = 4$ . It also leads to the following weak compactness and energy gap results.

**Corollary 1.6** There is a sufficient small constant  $\epsilon = \epsilon(m, n) > 0$  such that

(1) **Weak compactness** For any sequence  $u_k \in W^{2,2}(B_1, N)$  of biharmonic maps which converges weakly to a map  $u \in W^{2,2}(B_1, N)$ , if

$$\|\nabla^2 u_k\|_{M^{2,n-4}(B_1)} + \|\nabla u_k\|_{M^{4,n-4}(B_1)} \leq \epsilon, \quad \forall k \geq 1,$$

then up to a subsequence,  $u_k \rightarrow u$  strongly in  $W_{\text{loc}}^{2,2}(B_1, N)$ . In particular,  $u$  is a smooth biharmonic map; and

(2) **Energy gap** If  $u \in W^{2,2}(\mathbb{R}^n, N)$  is a biharmonic map satisfying

$$\|\nabla^2 u\|_{M^{2,n-4}(\mathbb{R}^n)} + \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)} \leq \epsilon,$$

then  $u \equiv p$  in  $\mathbb{R}^n$  for a point  $p \in N$ .

For geometric applications of this type of result, see Wang-Zheng [36] and Laurain-Rivière [21] on the energy identity of biharmonic maps in dimension  $n = 4$ .

The requirement  $p > n/4$  in Theorem 1.4 ensures that  $f \in M^{1,n-4+\alpha}(B_1)$  for some  $0 < \alpha < 1$ . It is natural to ask what happens when  $1 < p \leq n/4$ . Observe that the heat flow  $u$  of biharmonic maps can be viewed as (1.7) for  $f = u_t \in L^p$ , with  $p = 2 \leq n/4$  when dimensions  $n \geq 8$ , see e.g. Moser [24]. This motivates us to consider the case  $f \in L^p$  for  $1 < p \leq n/4$ . We can prove

**Theorem 1.7** Suppose  $f \in L^p \cap M^{1,n-4+\alpha}(B_1)$  for some  $1 < p \leq n/4$  and  $0 < \alpha < 1$  and  $u \in W^{2,2}(B_1, \mathbb{R}^m)$  is a weak solution to system (1.7) satisfying (GC-4). There exists  $\epsilon > 0$  such that if the smallness condition (1.9) holds, then

$$\nabla^2 u \in L^{p\eta} \cap M_*^{\eta, n-\eta(2-\alpha)}(B_{1/2}) \quad \text{and} \quad \nabla u \in L^{p\eta\chi} \cap M_*^{\eta\chi, n-\eta\chi(1-\alpha)}(B_{1/2}),$$

where  $\chi = (2 - \alpha)/(1 - \alpha) > 2$  and  $\eta = (4 - \alpha)/(2 - \alpha) > 2$ .

We would like to remark that with slight changes of arguments, all results stated as above remain to hold if the coefficient function  $F$  in Eq. (1.7) takes the form  $F = \nabla\omega + W$  of Eq. (1.3) and (GC-4) is replaced by a corresponding one.

Finally, as aforementioned, as a by-product of our method, we provide a partial answer to Sharp's interesting expectation.

**Theorem 1.8** Suppose  $f \in L^p(B_1)$  for some  $1 \leq n/2 < p < \infty$  and  $u \in W^{1,2}(B_1, \mathbb{R}^m)$  is a weak solution of system (1.5). If, in addition,  $\Omega \in L^2(B_1, so_m \otimes \wedge^1 \mathbb{R}^n)$  satisfies the growth condition (GC-2), then there exists  $\epsilon = \epsilon(m, n, p) > 0$  such that

$$u \in W_{\text{loc}}^{2,p}(B_1, \mathbb{R}^m),$$

provided  $\|\nabla u\|_{M^{2,n-2}(B_1, \mathbb{R}^m)} < \epsilon$ .

Theorem 1.8 extends a main theorem of Moser [25, Theorem 1.2] in the range  $p > n/2$  with also a different proof. However, due to the limitation of our method, it remains an interesting open question that how to confirm Sharp's expectation for the more difficult case  $1 < p \leq n/2$ .

### 1.3 Strategy and novelty of the proof

Roughly speaking, our proofs of main results follow the line of Sharp and Topping [32] in the large. More precisely, to derive Theorem 1.2, we shall first rewrite the system using the Gauge transform of Struwe [33], and then apply the Hodge decomposition to simplify the problem. Morrey type decay estimates then follow from a combination of Riesz potential theory and a decay property of harmonic functions. However, as there is no conservation law in our case anymore, and also due to the critical nonlinearity, we will have to overcome severe difficulties. To this end, some new ideas will be introduced.

To explain our strategy and novelty clearly, we first sketch the proof of Theorem 1.8.

**Proof of Theorem 1.8** We first consider the case  $\frac{n}{2} < p < n$ . Sharp [31, Theorem 1.2] has proved that  $\nabla u \in M_{\text{loc}}^{2,n-2+2\alpha}(B_1)$  with  $\alpha = 2 - n/p \in (0, 1)$ . It follows from the growth condition that  $\Omega \in M_{\text{loc}}^{2,n-2+2\alpha}(B_1)$ . Therefore  $\Omega \cdot \nabla u \in M_{\text{loc}}^{1,n-2+2\alpha}(B_1)$ .

First suppose  $\alpha < 1/2$ . Extend  $\Omega$ ,  $u$  and  $f$  from  $B_{1/2}$  into  $\mathbb{R}^n$  with compact support in  $B_2^n$  in a norm-bounded way. Let  $u_1 = I_2(\Omega \cdot \nabla u)$  and  $u_2 = I_2(f)$  such that  $h = u - u_1 - u_2$  is a harmonic function in  $B_{1/2}$ , where  $I_\alpha = c|x|^{|\alpha|-n}$  is the standard Riesz potential. Since  $\Omega \cdot \nabla u \in M^{1,n-2+2\alpha} \cap L^1(\mathbb{R}^n)$ , Adams' potential theory (see Proposition 2.5 below) implies

$$|\nabla u_1| \leq C I_1(\Omega \cdot \nabla u) \in L^{2\chi, \infty}(\mathbb{R}^n),$$

where

$$\chi = \frac{1}{2} \left( \frac{2 - 2\alpha}{1 - 2\alpha} \right) > 1.$$

By standard elliptic regularity theory,  $u_2 \in W^{2,p}(\mathbb{R}^n)$ . Hence  $\nabla u_2 \in L^{p^*}(\mathbb{R}^n)$ , where  $p^* = \frac{np}{n-p}$ .

Since  $h \in C^\infty$ , if  $2\chi > p^*$ , then we find that  $\nabla u \in L_{\text{loc}}^{p^*}(B_1)$ . If  $2\chi \leq p^*$ , we obtain  $\nabla u \in L^{2\chi,\infty}(B_{1/2})$ . Since  $2\chi > 2$ , we have  $\nabla u \in L_{\text{loc}}^q(B_1)$  for some  $q > 2$ . Then the growth condition (GC-2) implies that  $\Omega \in M_{\text{loc}}^{2,n-2+2\alpha} \cap L_{\text{loc}}^q(B_1)$ . It follows that

$$\Omega \cdot \nabla u \in M_{\text{loc}}^{1,n-2+2\alpha}(B^n) \cap L_{\text{loc}}^{q/2}.$$

Since  $q/2 > 1$ , using Adams' potential (see Proposition 2.6) again gives  $\nabla u_1 \in L^{\chi q}$ . If  $\chi q \leq p^*$ , we obtain  $\nabla u \in L^{\chi q}$ . Thus, we find the iteration:

$$\nabla u \in L^q \Rightarrow \nabla u \in L^{\chi q}.$$

Since  $\chi > 1$ , we can assume that  $\chi^k q \leq p^* < \chi^{k+1} q$  for some  $k \geq 1$ . After finitely many times iteration, we find that  $\nabla u \in L_{\text{loc}}^{p^*}(B_1)$ .

In the case  $\alpha \geq 1/2$ , we use embedding  $M^{2,n-2+2\alpha}(B_1) \subset M^{2,n-2+2\beta}(B_1)$  for any  $\beta < 1/2$  so as to obtain the same regularity as in the case  $\alpha < 1/2$ . As a consequence, we can always derive  $\nabla u \in L_{\text{loc}}^{p^*}(B_1)$ . Now the second order regularity  $u \in W_{\text{loc}}^{2,p}(B_1)$  follows from the usual elliptic regularity theory.

If  $p \geq n$ , then  $f \in L^q(B^n)$  for any  $n/2 < q < n$ . Running the previous argument we conclude that  $\nabla u \in L_{\text{loc}}^{\frac{nq}{n-q}}(B_1)$ . This implies that  $u_1 \in \bigcap_{1 < q < \infty} W_{\text{loc}}^{1,q}(B_1)$  and so finally  $u \in W_{\text{loc}}^{2,p}(B_1)$ . The proof is complete.  $\square$

Our proof of Theorem 1.4 follows a similar approach, but the analysis becomes much more involved. In a first step, we derive the weak Morrey decay estimate for solutions of system (1.7), that is, Theorem 1.2. Unlike the case of linear systems in [13, 14, 31, 32], this regularity improvement is not strong enough for iteration yet. To fill the gap, two observations are explored here:

- The weak Morrey regularity of  $\nabla^2 u$  automatically implies an improvement of  $\nabla u$ , i.e.,  $\nabla u \in L^{2\chi} \cap M_*^{2\chi,n-2\chi(1-\alpha)}(B_{1/4})$ , where  $\chi \equiv (2-\alpha)/(1-\alpha) > 2$ ;
- The growth condition implies a corresponding regularity improvement for the Gauge transformation (see Lemma 4.1 below).

By the first observation, we obtained an improved regularity of  $\nabla u$ . But this improvement itself is still not sufficient for the iteration method yet. To proceed, the new idea is to further track and improve the regularity of Gauge transforms, so as to fully make use of the gauge transforms. We then turn to construct an associated Gauge transform on smaller balls (half radius of the previous one) with improved regularity. This is realized by the second observation. Next, with these improved Gauge transforms at hand, the next new idea is to tracking both the Lebesgue integrability and Morrey regularity of  $\nabla u$  and  $\nabla^2 u$  simultaneously; an application of the Riesz potential theory gives a further improvement on integrability of  $\nabla u$  and  $\nabla^2 u$ . Finally, to obtain the optimal interior regularity, we run an iteration scheme by repeatedly constructing the gauge transforms on a sequence of shrinking balls and then using the gauge equivalent equations on shrinking balls. Our strategy works surprisingly well: in contrast to those infinite iteration on linear systems in [13, 14, 31, 32], our iteration process stops after finitely many steps, which also thanks to the nonlinear nature of the problems. We mention that a crucial harmonic analysis theory used in the proof is the boundedness of Riesz operators between weak Morrey spaces, which is due to Ho [16] (see also Proposition 2.7 below).

Our notations are standard. By  $A \lesssim B$ , we mean there is an absolute constant  $C > 0$  such that  $A \leq CB$ . The constant  $C$  may differ from line to line.

## 2 Preliminaries

In this section, we introduce some function spaces and the related Riesz potential theory between these function spaces. They play a central role in later proofs.

### 2.1 Morrey spaces

Let  $\Omega \subset \mathbb{R}^n$  be a smooth domain. For  $1 \leq p < \infty$ , let  $L^p(\Omega)$  be the usual  $L^p$  space on  $\Omega$  and  $L_*^p(\Omega)$  the weak  $L^p$  space on  $\Omega$ .

Let  $1 \leq p < \infty$  and  $0 \leq s \leq n$ . The Morrey space  $M^{p,s}(\Omega)$  consists of functions  $f \in L^p(\Omega)$  such that

$$\|f\|_{M^{p,s}(\Omega)} \equiv \sup_{x \in \Omega, 0 < r < \text{diam}(\Omega)} r^{-s/p} \|f\|_{L^p(B_r(x) \cap \Omega)} < \infty.$$

The weak Morrey space  $M_*^{p,s}(\Omega)$  consists of functions  $f \in L_*^p(\Omega)$  such that

$$\|f\|_{M_*^{p,s}(\Omega)} \equiv \sup_{x \in \Omega, 0 < r < \text{diam}(\Omega)} r^{-s/p} \|f\|_{L_*^p(B_r(x) \cap \Omega)} < \infty.$$

Note that  $M^{p,0}(\Omega) = L^p(\Omega)$  and  $M^{p,n}(\Omega) = L^\infty(\Omega)$ , and  $M_*^{p,0}(\Omega) = L_*^p(\Omega)$ . When  $\Omega$  is a bounded domain, it follows from Hölder's inequality and the simple embedding  $L_*^p(\Omega) \subset L^q(\Omega)$  ( $1 \leq q < p$ ) that,

$$L^p(\Omega) \subset M^{q,n(1-\frac{q}{p})}(\Omega), \quad \forall 1 \leq q < p$$

and

$$M_*^{p,s}(\Omega) \subset M^{1,n+\frac{s-n}{p}}(\Omega), \quad \forall 1 < p < \infty.$$

We shall need the following well-known Hölder's inequality for weak  $L^p$  functions.

**Proposition 2.1** *Let  $1 < p_1, p_2 < \infty$  be such that  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ . Then,  $f \in L_*^{p_1}(\Omega)$  and  $g \in L_*^{p_2}(\Omega)$  implies  $fg \in L_*^p(\Omega)$ . Moreover,*

$$\|fg\|_{L_*^p(\Omega)} \leq \|f\|_{L_*^{p_1}(\Omega)} \|g\|_{L_*^{p_2}(\Omega)}.$$

The following proposition concerns Hölder's inequalities in Morrey functions. The proof is straightforward and thus omitted.

**Proposition 2.2** *Let  $1 \leq p_1, p_2 \leq \infty$  and  $0 \leq q_1, q_2 \leq n$  be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad q = \frac{p}{p_1} q_1 + \frac{p}{p_2} q_2.$$

*Then, there hold*

$$\|fg\|_{M^{p,q}(\Omega)} \leq \|f\|_{M^{p_1,q_1}(\Omega)} \|g\|_{M^{p_2,q_2}(\Omega)}. \quad (2.1)$$

*and*

$$\|fg\|_{M_*^{p,q}(\Omega)} \leq \|f\|_{M_*^{p_1,q_1}(\Omega)} \|g\|_{M_*^{p_2,q_2}(\Omega)}. \quad (2.2)$$

As we are concerned with Hölder regularity theory, we need the following weak type of Morrey's Dirichlet growth theorem.

**Proposition 2.3** *Suppose  $\Omega$  is a bounded smooth domain and  $u \in L^1_{\text{loc}}(\Omega)$  such that  $\nabla u \in M^{p,n-p+p\alpha}_*(\Omega)$  holds for some  $1 < p < \infty$  and  $\alpha \in (0, 1)$ . Then  $u \in C^{0,\alpha}(\overline{\Omega})$  with*

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \|\nabla u\|_{M^{p,n-p+p\alpha}_*(\Omega)}$$

for some  $C = C(n, p, \Omega)$ .

**Proof** By Poincaré's inequality, for any  $x \in \Omega$  and  $0 < r < \text{diam}(\Omega)$ , there holds

$$\int_{B_r(x) \cap \Omega} |u - u_{r,x}| \leq Cr \int_{B_r(x) \cap \Omega} |\nabla u|.$$

Since  $p > 1$ , we have

$$\|\nabla u\|_{L^1(B_r(x) \cap \Omega)} \leq Cr^{n(1-1/p)} \|\nabla u\|_{L^p_*(B_r(x) \cap \Omega)} \leq C \|\nabla u\|_{M^{p,n-p+p\alpha}_*(\Omega)} r^{n-1+\alpha}.$$

Thus, for any  $x \in \Omega$  and  $0 < r < \text{diam}(\Omega)$ ,

$$\int_{B_r(x) \cap \Omega} |u - u_{r,x}| \leq C \|\nabla u\|_{M^{p,n-p+p\alpha}_*(\Omega)} r^\alpha.$$

This yields the conclusion by applying Campanato function space theory, see Giaquinta [10, Chapter III, Theorem 1.2].  $\square$

Higher order (weak) Morrey spaces will be useful in our later proofs. For any  $k \in \mathbb{N}$ , the  $k$ th order Morrey space  $M^{p,n-kp}_k(\Omega)$  consists of  $f \in W^{k,p}(\Omega)$  such that  $\nabla^l f \in M^{p,n-lp}(\Omega)$  for all  $0 \leq l \leq k$ , and we can similarly define the  $k$ th order weak Morrey space  $M^{p,n-kp}_{k,*}(\Omega)$ . It follows from [33, Proposition 3.2] that  $M^{p,n-2p}_2(B_1) \subset M^{2p,n-2p}_1(B_1)$  with  $1 < p < n/2$ , and

$$\|\nabla u\|_{M^{2p,n-2p}(B_1)}^2 \leq C \|\nabla u\|_{M^{1,n-1}(B_1)} (\|\nabla^2 u\|_{M^{p,n-2p}(B_1)} + \|\nabla u\|_{M^{p,n-p}(B_1)}). \quad (2.3)$$

In particular,  $u \in M^{2,n-4}_2(B_1)$  implies that  $\nabla u \in M^{4,n-4}(B_1)$ . Recall that the basic assumption of Struwe [33] is

$$R^{n-4} \int_{B_R} (|\nabla^2 u|^2 + |\nabla u|^4) < \epsilon,$$

which together with the monotonicity formula implies that  $u \in M^{2,n-4}_2(B_{R/2})$  and

$$\|\nabla^2 u\|_{M^{2,n-4}(B_{R/2})} + \|\nabla u\|_{M^{4,n-4}(B_{R/2})} < C\epsilon.$$

Thus, by (2.3), one may naturally assume that  $u \in M^{2,n-4}_2(B_2)$  satisfies

$$\|\nabla^2 u\|_{M^{2,n-4}(B_2)} + \|\nabla u\|_{M^{2,n-2}(B_2)} < \epsilon.$$

We shall frequently use (a special case of) the following Morrey-Sobolev extension<sup>1</sup> result due to Burenkov [4]; see also [9, Theorem 2.5] for a new proof.

<sup>1</sup> We would like to thank Prof. Pekka Koskela for pointing out the relevant literatures in this respect.

**Proposition 2.4** For any  $k \in \mathbb{N}$ ,  $1 \leq p$  and  $0 \leq s \leq n$ , there exists a bounded linear operator  $E: M_k^{p,s}(B_1) \rightarrow M_k^{p,s}(\mathbb{R}^n)$  such that if  $f \in M_k^{p,s}(B_1)$ , then  $Ef = f$  a.e. in  $B_1$  and there exists a constant  $C = C(k, p, s) > 0$  such that for all  $f \in M_k^{p,s}(B_1)$ , we have

$$\|Ef\|_{M_k^{p,s}(\mathbb{R}^n)} \leq C\|f\|_{M_k^{p,s}(B_1)}.$$

Furthermore, for each  $0 \leq l \leq k$ , there exists a constant  $C = C(l, p, s) > 0$  such that

$$\|\nabla^l Ef\|_{M^{p,s}(\mathbb{R}^n)} \leq C\|\nabla^l f\|_{M^{p,s}(B_1)}.$$

Similar extension results hold for the higher order weak Morrey-Sobolev spaces  $M_{k,*}^{p,s}(B_1)$  as well.

We also refer the interested readers to [17] for a different construction of the extension operator. Note that in [17], the authors only considered the higher order Morrey-Sobolev spaces  $M_k^{p,s}(B_1)$ . However, the proof works with minor changes (replacing the  $L^p$  estimates by corresponding weak  $L_*^p$  estimates) for the higher order weak Morrey-Sobolev spaces  $M_{k,*}^{p,s}(B_1)$ .

## 2.2 Riesz potentials

Let  $I_\alpha(x) = c_{\alpha,n}|x|^{\alpha-n}$ ,  $0 < \alpha < n$ , be the standard Riesz potentials in  $\mathbb{R}^n$ . The following two propositions are well-known; see Theorem 3.1, Proposition 3.2 and Proposition 3.1 of Adams [2].

**Proposition 2.5** Let  $0 < \alpha < n$  and  $0 \leq \lambda < n$ . For  $1 \leq p < (n - \lambda)/\alpha$ , set

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha}{n - \lambda}.$$

Then

(1) For every  $1 < p < (n - \lambda)/\alpha$ ,

$$I_\alpha: M^{p,\lambda}(\mathbb{R}^n) \rightarrow M^{\tilde{p},\lambda}(\mathbb{R}^n)$$

is a bounded linear operator;

(2) For  $p = 1$ ,

$$I_\alpha: M^{1,\lambda}(\mathbb{R}^n) \rightarrow M_*^{\frac{n-\lambda}{n-\lambda-\alpha},\lambda}(\mathbb{R}^n)$$

is also a bounded linear operator.

**Proposition 2.6** Let  $0 < \alpha < \beta \leq n$  and  $1 < p < \infty$ . Then there exists a constant  $C = C_{\alpha,\beta,n,p} > 0$  such that for  $f \in M^{1,n-\beta}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ , there holds

$$\|I_\alpha f\|_{\frac{p\beta}{\beta-\alpha},\mathbb{R}^n} \leq C\|f\|_{M^{1,n-\beta}(\mathbb{R}^n)}^\frac{\alpha}{\beta}\|f\|_{p,\mathbb{R}^n}^{1-\frac{\alpha}{\beta}}.$$

In view of the embedding  $M_*^{q,n-q\beta}(\mathbb{R}^n) \subset M^{1,n-\beta}(\mathbb{R}^n)$  for  $n/\beta \geq q > 1$ , there holds

$$\|I_\alpha f\|_{\frac{p\beta}{\beta-\alpha},\mathbb{R}^n} \leq C\|f\|_{M_*^{q,n-q\beta}(\mathbb{R}^n)}^\frac{\alpha}{\beta}\|f\|_{p,\mathbb{R}^n}^{1-\frac{\alpha}{\beta}}.$$

Concerning weak Morrey spaces, we will need the following proposition, which is a special case of Ho [16, Theorem 5.1].

**Proposition 2.7** *Let  $0 < \alpha, \lambda < n$  and  $1 < p < (n - \lambda)/\alpha$ . Set*

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha}{n - \lambda}.$$

*Then*

$$I_\alpha : M_*^{p,\lambda}(\mathbb{R}^n) \rightarrow M_*^{\tilde{p},\lambda}(\mathbb{R}^n)$$

*is a bounded linear operator.*

As a corollary of Propositions 2.6 and 2.7, for any  $\infty > p > 1$  and  $0 < \alpha < \beta < n/p$ , we have the following boundedness result:

$$I_\alpha : M_*^{p,n-p\beta} \cap L^p(\mathbb{R}^n) \rightarrow M_*^{\tilde{p},n-p\beta} \cap L^{\tilde{p}}(\mathbb{R}^n) \quad \text{where } \tilde{p} = \frac{\beta p}{\beta - \alpha}, \quad (2.4)$$

and

$$\|I_\alpha(f)\|_{L^{\tilde{p}}(\mathbb{R}^n)} + \|I_\alpha(f)\|_{M_*^{\tilde{p},n-p\beta}(\mathbb{R}^n)} \leq C \left( \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{M_*^{p,n-p\beta}(\mathbb{R}^n)} \right).$$

When the operator under consideration is a singular integral operator, there holds

**Proposition 2.8** (Theorem 8.1, [1]) *Let  $1 < p < \infty$  and  $0 < \lambda < n$ . The usual Calderon-Zygmund singular integral operators are bounded on  $M^{p,\lambda}(\mathbb{R}^n)$ .*

### 3 Morrey estimate and Hölder continuity

This section is devoted to prove Theorem 1.2. For simplicity, denote by  $B_r = B_r(0) \subset \mathbb{R}^n$  the open ball centered at origin with radius  $r$ . We shall need the following Gauge transform of Struwe [33, Lemma 3.3]; see also Lamm and Rivière [18, Theorem A.5] for an equivalent form.

**Lemma 3.1** (Lemma 3.3, [33]) *There exist  $\epsilon = \epsilon(n, m) > 0$  and  $C = C(n, m) > 0$  with the following property: For every  $\Omega \in M_1^{2,n-4} \cap M^{4,n-4}(B_1, so_m \otimes \wedge^1 \mathbb{R}^n)$  with*

$$\|\nabla \Omega\|_{M^{2,n-4}(B_1)} + \|\Omega\|_{M^{4,n-4}(B_1)} \leq \epsilon,$$

*there exist  $P \in M_2^{2,n-4}(B_1, SO_m)$  and  $\xi \in M_2^{2,n-4}(B_1, so_m \otimes \wedge^{n-2} \mathbb{R}^n)$  such that*

$$PdP^{-1} + P\Omega P^{-1} = *d\xi \quad \text{in } B_1, \quad (3.1)$$

*and*

$$d * \xi = 0 \quad \text{in } B_1, \quad \xi = 0 \quad \text{on } \partial B_1.$$

*Moreover,*

$$\begin{aligned} \|\nabla P\|_{M^{4,n-4}(B_1)} + \|\nabla \xi\|_{M^{4,n-4}(B_1)} &\leq C \|\Omega\|_{M^{4,n-4}(B_1)} \leq C\epsilon, \\ \|\nabla^2 P\|_{M^{2,n-4}(B_1)} + \|\nabla^2 \xi\|_{M^{2,n-4}(B_1)} &\leq C \left( \|\nabla \Omega\|_{M^{2,n-4}(B_1)} + \|\Omega\|_{M^{4,n-4}(B_1)} \right) \leq C\epsilon. \end{aligned}$$

The last two estimates on  $P, \xi$  are not separated in the original statement of Struwe [33, Lemma 3.3], but they follow from the proofs there. Below let  $P, \xi$  be defined as in Lemma 3.1.

It follows from the growth condition (GC-4) on  $\Omega$  and (1.9) that

$$\begin{aligned} \|\nabla P\|_{M^{4,n-4}(B_1)} + \|\nabla \xi\|_{M^{4,n-4}(B_1)} &\leq C\|\nabla u\|_{M^{4,n-4}(B_1)} \leq C\epsilon. \\ \|\nabla^2 P\|_{M^{2,n-4}(B_1)} + \|\nabla^2 \xi\|_{M^{2,n-4}(B_1)} \\ &\leq C(\|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|\nabla u\|_{M^{4,n-4}(B_1)}) \leq C\epsilon. \end{aligned} \quad (3.2)$$

By [33, Formula (35)], the equation of  $P\Delta u$  on  $B_1$  is given by

$$\Delta(P\Delta u) = \operatorname{div}^2(D_P \otimes \nabla u) + \operatorname{div}(E_P \cdot \nabla u) + G_P \cdot \nabla u + *d\Delta\xi \cdot Pdu + Pf, \quad (3.3)$$

where the coefficient functions satisfy the growth condition

$$\begin{aligned} |D_P| &\leq C(|\nabla u| + |\nabla P|), \\ |\nabla D_P| + |E_P| &\leq C(|\nabla^2 u| + |\nabla u|^2 + |\nabla^2 P| + |\nabla P|^2), \\ |G_P| &\leq C(|\nabla^2 u| + |\nabla^2 P|)(|\nabla u| + |\nabla P|) + C(|\nabla u|^3 + |\nabla P|^3). \end{aligned} \quad (3.4)$$

For details, see the formula (36) of [33].

**Proof of Theorem 1.2** First apply the Hodge decomposition to derive

$$Pdu = d\tilde{u}_1 + d^*\tilde{u}_2 + \tilde{h} \quad \text{in } B_1,$$

where  $d^*\tilde{u}_1 = 0$ ,  $d\tilde{u}_2 = 0$  and  $\tilde{h}$  is a harmonic 1-form. Note that  $\Delta^2\tilde{u}_1 = \Delta d^*(Pdu)$ ,  $-\Delta\tilde{u}_2 = dP \wedge du$  and  $\Delta\tilde{h} = 0$  on  $B_1$ .

Next, we extend all the related functions  $u$ ,  $\xi$ ,  $P$  and  $D_P$ ,  $E_P$  and  $G_P$  from  $B_1$  into the whole space  $\mathbb{R}^n$  with compact supports in  $B_2$  in the same function space in a bounded way. Set  $f \equiv 0$  on  $B_1^c$ . For simplicity, we keep using the same notations for the extended functions. Then we define

$$u_{11} = I_4 \left( \operatorname{div}^2(D_P \otimes \nabla u) + \operatorname{div}(E_P \cdot \nabla u) + G_P \cdot \nabla u + *d\Delta\xi \cdot Pdu + \Delta(\nabla P \nabla u) \right), \quad (3.5)$$

$$u_{12} = I_4(Pf), \quad (3.6)$$

where  $I_4$  is the fundamental solution of  $\Delta^2$  in  $\mathbb{R}^n$  and define

$$u_2 = I_2(dP \wedge du), \quad (3.7)$$

where  $I_2$  is the fundamental solution of  $-\Delta$  in  $\mathbb{R}^n$ . It follows that

$$\Delta^2 u_{11} + \Delta^2 u_{12} = \Delta^2 \tilde{u}_1 \quad \text{and} \quad \Delta u_2 = \Delta \tilde{u}_2$$

on  $B_1$ . Set  $h = d\tilde{u}_1 - du_{11} - du_{12} + d^*\tilde{u}_2 - d^*u_2 + \tilde{h}$  so that

$$\Delta^2 h = 0 \quad \text{in } B_1.$$

We obtain the decomposition

$$Pdu = du_{11} + du_{12} + d^*u_2 + h \quad \text{in } B_1. \quad (3.8)$$

To obtain the Morrey decay estimates of  $\nabla u$  and  $\nabla^2 u$ , it suffices to estimate that of the components  $u_{11}$ ,  $u_{12}$  and  $u_2$ .

First we estimate  $\nabla u_{11}$ . From the definition (3.5) of  $u_{11}$ , it holds

$$\nabla u_{11} = \nabla I_4 * \left( \operatorname{div}^2(D_P \otimes \nabla u) + \operatorname{div}(E_P \cdot \nabla u) + G_P \cdot \nabla u + *d\Delta\xi \cdot Pdu + \Delta(\nabla P \nabla u) \right).$$

Let  $J_1 = I_4 (\operatorname{div}^2(D_P \otimes \nabla u) + \operatorname{div}(E_P \cdot \nabla u) + \Delta(\nabla P \nabla u))$ . Then

$$\nabla J_1 \approx \nabla^3 I_4(D_P \nabla u + \nabla P \nabla u) + \nabla^2 I_4(E_P \nabla u),$$

which implies that

$$|\nabla J_1| \lesssim I_1 (|D_P| |\nabla u| + |\nabla P| |\nabla u|) + I_2 (|E_P| |\nabla u|). \quad (3.9)$$

Applying the growth condition (3.4) gives

$$|D_P| |\nabla u| + |\nabla P| |\nabla u| \lesssim (|\nabla u| + |\nabla P|) |\nabla u|,$$

and

$$|E_P| |\nabla u| \lesssim (|\nabla^2 u| + |\nabla u|^2 + |\nabla^2 P| + |\nabla P|^2) |\nabla u|.$$

Since  $\nabla P, \nabla u \in M_{*}^{4,n-4}(\mathbb{R}^n)$  and  $\nabla^2 u, \nabla^2 P \in M_{*}^{2,n-4}(\mathbb{R}^n)$ , the Hölder inequality (2.2) implies that  $D_P \nabla u \in M_{*}^{2,n-4}(\mathbb{R}^n)$  and  $E_P \nabla u \in M_{*}^{4/3,n-4}(\mathbb{R}^n)$ , together with estimates

$$\begin{aligned} \|D_P \nabla u\|_{M_{*}^{2,n-4}(\mathbb{R}^n)} &\lesssim \left( \|\nabla P\|_{M_{*}^{4,n-4}(\mathbb{R}^n)} + \|\nabla u\|_{M_{*}^{4,n-4}(\mathbb{R}^n)} \right) \|\nabla u\|_{M_{*}^{4,n-4}(\mathbb{R}^n)} \\ &\lesssim \epsilon \|\nabla u\|_{M_{*}^{4,n-4}(\mathbb{R}^n)}, \end{aligned} \quad (3.10)$$

and

$$\|E_P \nabla u\|_{M_{*}^{4/3,n-4}(\mathbb{R}^n)} \lesssim \epsilon \|\nabla u\|_{M_{*}^{4,n-4}(\mathbb{R}^n)}. \quad (3.11)$$

Here we used the bounded extension of  $u, P$  from  $M_{2,*}^{2,n-4}(B_1)$  into  $M_{2,*}^{2,n-4}(\mathbb{R}^n)$  (see Proposition 2.4) and the smallness assumption (1.9). By Proposition 2.7,

$$I_1: M_{*}^{2,n-4}(\mathbb{R}^n) \rightarrow M_{*}^{4,n-4}(\mathbb{R}^n)$$

and

$$I_2: M_{*}^{4/3,n-4}(\mathbb{R}^n) \rightarrow M_{*}^{4,n-4}(\mathbb{R}^n)$$

are bounded operators. Thus from (3.9) and the above estimates we deduce

$$\|\nabla J_1\|_{M_{*}^{4,n-4}(\mathbb{R}^n)} \lesssim \epsilon \|\nabla u\|_{M_{*}^{4,n-4}(\mathbb{R}^n)}.$$

Using the bounded extension  $\|\nabla u\|_{M_{*}^{4,n-4}(\mathbb{R}^n)} \lesssim \|\nabla u\|_{M_{*}^{4,n-4}(B_1)}$ , it follows

$$\|\nabla J_1\|_{M_{*}^{4,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_{*}^{4,n-4}(B_1)}.$$

Let  $J_2 = I_4(G_P * \nabla u)$ . This is the most difficult term to estimate and we need to exploit the full nonlinearity of  $G_P$ . By (2.2) and the inequality (2.1), and the fact  $|\nabla u|, |\nabla P| \in M^{4,n-4}, |\nabla^2 u|, |\nabla^2 P| \in M^{2,n-4}$ , we infer that

$$|G_P \nabla u| \lesssim (|\nabla^2 u| + |\nabla^2 P|) (|\nabla u| + |\nabla P|) |\nabla u| + (|\nabla u|^3 + |\nabla P|^3) |\nabla u| \in M^{1,n-4}(\mathbb{R}^n)$$

with estimates

$$\begin{aligned} \| |G_P| |\nabla u| \|_{M^{1,n-4}(\mathbb{R}^n)} &\lesssim \|\nabla u\|_{M^{4,n-4}} \left( \|\nabla^2 u\|_{M^{2,n-4}} + \|\nabla^2 P\|_{M^{2,n-4}} \right) \left( \|\nabla u\|_{M^{4,n-4}} + \|\nabla P\|_{M^{4,n-4}} \right) \\ &\quad + \|\nabla u\|_{M^{4,n-4}} \left( \|\nabla u\|_{M^{4,n-4}}^3 + \|\nabla P\|_{M^{4,n-4}}^3 \right). \end{aligned} \quad (3.12)$$

Combining the estimate (3.2) of  $\nabla P$  with (3.12) yields

$$\| |G_P| |\nabla u| \|_{M^{1,n-4}(\mathbb{R}^n)} \lesssim (\|\nabla^2 u\|_{M^{2,n-4}} + \|\nabla u\|_{M^{4,n-4}}^2) \|\nabla u\|_{M^{4,n-4}}^2 \lesssim \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)}^2.$$

Therefore, applying the bounded operator  $I_3: M^{1,n-4}(\mathbb{R}^n) \rightarrow M_*^{4,n-4}(\mathbb{R}^n)$  by Proposition 2.5, we arrive at

$$\|\nabla J_2\|_{M_*^{4,n-4}(\mathbb{R}^n)} \lesssim \|G_P\|\|\nabla u\|_{M^{1,n-4}(\mathbb{R}^n)} \lesssim \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)}^2.$$

Thus we conclude

$$\|\nabla J_2\|_{M_*^{4,n-4}(B_1)} \lesssim \|\nabla u\|_{M^{4,n-4}(B_1)}^2.$$

Let  $J_3 = I_4(*d\Delta\xi \cdot Pdu)$ . Integrating by parts gives (up to signs)

$$J_3 = \int d\Delta\xi \wedge I_4 Pdu = \int \Delta\xi \wedge (dI_4 P + I_4 dP) \wedge du.$$

Thus

$$|\nabla J_3| \lesssim I_2(|\nabla^2\xi||\nabla u|) + I_3(|\nabla^2\xi||\nabla u||\nabla P|).$$

As  $|\nabla^2\xi||\nabla u| \in M_*^{2,n-4} \cdot M_*^{4,n-4} \subset M_*^{\frac{4}{3},n-4}$ , we infer that  $I_2(|\nabla^2\xi||\nabla u|) \in M_*^{4,n-4}$  as that of  $J_1$  with estimate

$$\|I_2(|\nabla^2\xi||\nabla u|)\|_{M_*^{4,n-4}(\mathbb{R}^n)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(\mathbb{R}^n)}.$$

For the second term, we have  $|\nabla^2\xi||\nabla u||\nabla P| \in M^{1,n-4}$ . As that of  $J_2$ , we obtain

$$\|I_3(|\nabla^2\xi||\nabla u||\nabla P|)\|_{M_*^{4,n-4}(\mathbb{R}^n)} \lesssim \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)}^2.$$

Consequently,

$$\|\nabla J_3\|_{M_*^{4,n-4}(\mathbb{R}^n)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(\mathbb{R}^n)} + \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)}^2.$$

Using the bounded extension of  $u$  gives

$$\|\nabla J_3\|_{M_*^{4,n-4}(B_1)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|\nabla u\|_{M^{4,n-4}(B_1)}^2.$$

Taking the three estimates involving  $\nabla J_1, \nabla J_2, \nabla J_3$ , we derive

$$\|\nabla u_{11}\|_{M_*^{4,n-4}(B_1)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|\nabla u\|_{M^{4,n-4}(B_1)}^2.$$

Applying the inequality (2.3) and the smallness assumption (1.9) and the embedding  $M_*^{4,n-4}(B_1) \subset M^{1,n-1}(B_1)$ , we find that

$$\|\nabla u\|_{M^{4,n-4}(B_1)}^2 \lesssim \epsilon\|\nabla u\|_{M^{1,n-1}(B_1)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(B_1)}.$$

Thus we obtain the estimate of  $u_{11}$  as

$$\|\nabla u_{11}\|_{M_*^{4,n-4}(B_1)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(B_1)}. \quad (3.13)$$

The estimate of  $u_{12}$  is standard. Since  $f \in M^{1,n-4+\alpha}$ ,  $|\nabla u_{12}| \approx I_3(Pf)$  and standard potential theory, Proposition 2.5, gives  $\nabla u_{12} \in M_*^{\frac{4-\alpha}{1-\alpha},n-4+\alpha}$ . Notice that for  $0 < \alpha < 1$ ,  $\frac{4-\alpha}{1-\alpha} > 4$  so we have

$$\|\nabla u_{12}\|_{M_*^{4,n-4}(B_r)} \lesssim r^\alpha \|\nabla u_{12}\|_{M_*^{\frac{4-\alpha}{1-\alpha},n-4+\alpha}} \lesssim \|f\|_{M^{1,n-4+\alpha}(B_1)} r^\alpha \quad (3.14)$$

for any  $r > 0$ . Here we have used the fact that  $f \equiv 0$  on  $B_1^c$ .

Combining the above estimates (3.13) and (3.14), we deduce that, for any  $0 < r \leq 1$ ,

$$\|\nabla u_{11}\|_{M_*^{4,n-4}(B_r)} + \|\nabla u_{12}\|_{M_*^{4,n-4}(B_r)} \lesssim \epsilon\|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} r^\alpha. \quad (3.15)$$

It remains to estimate  $u_2$  and  $h$ . Since  $u_2 = I_2(dP \wedge du)$ , we have  $|\nabla u_2| \lesssim I_1(|\nabla P| |\nabla u|)$ . As that of  $J_1$ , we obtain

$$\|\nabla u_2\|_{M_*^{4,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}. \quad (3.16)$$

Since  $h$  is biharmonic, for any  $x \in B_1$  with  $B_{2r}(x) \subset B_1$ , there holds

$$\max_{B_r(x)} |\nabla h| \leq C \int_{B_{2r}(x)} |\nabla h|.$$

So for any  $x \in B_{1/2}$  and  $0 < r < 1/2$ ,

$$\|\nabla h\|_{L^{4,\infty}(B_r(x))}^4 \leq \int_{B_r(x)} |\nabla h|^4 \lesssim r^n \max_{B_{1/2}} |\nabla h|^4 \lesssim r^n \left( \int_{B_1} |\nabla h| \right)^4 \lesssim r^n \|\nabla h\|_{M_*^{4,n-4}(B_1)}^4.$$

That is,

$$r^{\frac{n-4}{4}} \|\nabla h\|_{L^{4,\infty}(B_r(x))} \leq r \|\nabla h\|_{M_*^{4,n-4}(B_1)}.$$

Hence

$$\|\nabla h\|_{M_*^{4,n-4}(B_r)} = \sup_{x \in B_r, 0 < s < 2r} \left( s^{\frac{n-4}{4}} \|\nabla h\|_{L^{4,\infty}(B_s(x))} \right) \lesssim r \|\nabla h\|_{M_*^{4,n-4}(B_1)}. \quad (3.17)$$

Now we can obtain the decay estimate for  $\nabla u$ . For any  $0 < \tau < 1/2$ , combining (3.15), (3.16) and (3.17) gives

$$\begin{aligned} & \|\nabla u\|_{M_*^{4,n-4}(B_\tau)} \\ & \lesssim \|\nabla h\|_{M_*^{4,n-4}(B_\tau)} + \|\nabla u_{11}\|_{M_*^{4,n-4}(B_\tau)} + \|\nabla u_{12}\|_{M_*^{4,n-4}(B_\tau)} + \|\nabla u_2\|_{M_*^{4,n-4}(B_\tau)} \\ & \lesssim \tau \|\nabla h\|_{M_*^{4,n-4}(B_1)} + \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \tau^\alpha \\ & \lesssim \tau \left( \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|\nabla u_{11}\|_{M_*^{4,n-4}(B_1)} + \|\nabla u_{12}\|_{M_*^{4,n-4}(B_1)} \right. \\ & \quad \left. + \|\nabla u_2\|_{M_*^{4,n-4}(B_1)} \right) + \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \tau^\alpha \|f\|_{M^{1,n-4+\alpha}(B_1)} \\ & \leq C(\tau + \epsilon) \|\nabla u\|_{M_*^{4,n-4}(B_1)} + C\tau^\alpha \|f\|_{M^{1,n-4+\alpha}(B_1)} \end{aligned}$$

for some  $C > 0$  independent of  $\tau$  and  $\epsilon$ . Recall that  $0 < \alpha < 1$ . Take  $\beta \in (\alpha, 1)$ . Then take  $\tau = r_0$  small enough such that  $2Cr_0 < r_0^\beta$ , and then choose  $\epsilon \leq r_0$ . We obtain

$$\|\nabla u\|_{M_*^{4,n-4}(B_{r_0})} \leq r_0^\beta \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} r_0^\alpha.$$

Finally, using a standard scaling and translation and iteration argument, there holds, for any  $x \in B_{1/2}$  and  $0 < r < 1$ ,

$$\|\nabla u\|_{M_*^{4,n-4}(B_r(x))} \leq Cr^\alpha \left( \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \right).$$

In particular, this implies that for any  $x \in B_{1/2}$  and  $0 < r < 1$ ,

$$\|\nabla u\|_{L^{4,\infty}(B_r(x))}^4 \leq Cr^{n-4+4\alpha} \left( \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \right)^4.$$

Hence  $\nabla u \in M_*^{4,n-4+4\alpha}(B_{1/2})$  and the desired estimate (1.10) follows.

Next we derive the decay of  $\nabla^2 u$  and the proof is similar to the one given above. First estimate  $\nabla^2 u_{11}$ . Using the same notations, we have

$$\nabla^2 J_1 \approx \nabla^4 I_4(D_P \nabla u) + \nabla^3 I_4(E_P \nabla u).$$

Since  $\nabla^4 I_4$  is a singular integral operator, Proposition 2.8 implies that

$$\nabla^4 I_4: M^{p,\lambda}(\mathbb{R}^n) \rightarrow M^{p,\lambda}(\mathbb{R}^n)$$

is a bounded operator. Thus

$$\|\nabla^4 I_4(D_P \nabla u)\|_{M^{2,n-4}} \lesssim (\|\nabla P\|_{M^{4,n-4}} + \|\nabla u\|_{M^{4,n-4}}) \|\nabla u\|_{M^{4,n-4}} \lesssim \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)}^2,$$

where the second inequality follows from inequality (3.2). Using the embedding  $M^{2,n-4} \subset M_*^{2,n-4}$ , the inequality (2.3) and the smallness assumption (1.9) of  $u$  as before, we deduce

$$\|\nabla^4 I_4(D_P \nabla u)\|_{M_*^{2,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}.$$

For the second term, combining (3.11) and the boundedness of

$$I_1: M_*^{4/3,n-4}(\mathbb{R}^n) \rightarrow M_*^{2,n-4}(\mathbb{R}^n)$$

by Proposition 2.7, we infer

$$\|\nabla^3 I_4(E_P \nabla u)\|_{M_*^{2,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}.$$

Hence

$$\|\nabla^2 J_1\|_{M_*^{2,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}.$$

For  $J_2$ , we have

$$|\nabla^2 J_2| \lesssim I_2(|G_P| |\nabla u|).$$

Recall that  $G_P \nabla u \in M^{1,n-4}(\mathbb{R}^n)$  and estimate (3.12) holds. Hence  $\nabla^2 J_2 \in M_*^{2,n-4}(\mathbb{R}^n)$  by Proposition 2.5 with estimate

$$\|\nabla^2 J_2\|_{M_*^{2,n-4}} \lesssim \| |G_P| |\nabla u| \|_{M^{1,n-4}} \lesssim \|\nabla u\|_{M^{4,n-4}(\mathbb{R}^n)}^2.$$

Again, applying inequality (2.3) yields

$$\|\nabla^2 J_2\|_{M_*^{2,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}.$$

For  $J_3$ , we have

$$|\nabla^2 J_3| \lesssim I_1(|\nabla^2 \xi| |\nabla u|) + I_2(|\nabla^2 \xi| |\nabla u| |\nabla P|).$$

Similar to  $J_1$  and  $J_2$ , we derive

$$\|\nabla^2 J_3\|_{M_*^{2,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}.$$

All together we conclude that

$$\|\nabla^2 u_{11}\|_{M_*^{2,n-4}(B_1)} \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}. \quad (3.18)$$

For  $u_{12}$ , since  $f \in M^{1,n-4+\alpha}$  and  $|\nabla^2 u_{12}| \approx I_2(Pf)$ , Proposition 2.5 gives  $\nabla^2 u_{12} \in M_*^{\frac{4-\alpha}{2-\alpha},n-4+\alpha}$ . Notice that for  $0 < \alpha < 1$ ,  $\frac{4-\alpha}{2-\alpha} > 2$ . So similar to (3.14), we obtain for any  $0 < r < \infty$ ,

$$\|\nabla^2 u_{12}\|_{M_*^{2,n-4}(B_r)} \lesssim \|f\|_{M^{1,n-4+\alpha}(B_1)} r^\alpha. \quad (3.19)$$

For the term  $u_2$ , we have  $|\nabla^2 u_2| \lesssim I_0(|\nabla P| |\nabla u|) \in M^{2,n-4}$  with

$$\|\nabla^2 u_2\|_{M_*^{2,n-4}} \leq \|\nabla^2 g\|_{M^{2,n-4}} \lesssim \|\nabla u\|_{M^{4,n-4}}^2 \lesssim \epsilon \|\nabla u\|_{M_*^{4,n-4}(B_1)}. \quad (3.20)$$

Similarly dispose the biharmonic 1-form  $h$ , for any  $0 < r < 1$ , there holds

$$\|\nabla^2 h\|_{M_*^{2,n-4}(B_r)} \lesssim r \|\nabla^2 h\|_{M_*^{2,n-4}(B_1)}. \quad (3.21)$$

Combining estimates (3.18), (3.19), (3.20) and (3.21) yields, for any  $0 < r < 1$ ,

$$\begin{aligned} & \|\nabla^2 u\|_{M_*^{2,n-4}(B_r)} + \|\nabla u\|_{M_*^{4,n-4}(B_r)} \\ & \lesssim (r + \epsilon) \left( \|\nabla^2 u\|_{M_*^{2,n-4}(B_1)} + \|\nabla u\|_{M_*^{4,n-4}(B_1)} \right) + \|f\|_{M^{1,n-4+\alpha}(B_1)} r^\alpha. \end{aligned}$$

Similar iteration, scaling and translation arguments give

$$\begin{aligned} & \|\nabla^2 u\|_{M_*^{2,n-4}(B_r)} + \|\nabla u\|_{M_*^{4,n-4}(B_r)} \\ & \lesssim \left( \|\nabla^2 u\|_{M_*^{2,n-4}(B_1)} + \|\nabla u\|_{M_*^{4,n-4}(B_1)} + \|f\|_{M^{1,n-4+\alpha}(B_1)} \right) r^\alpha. \end{aligned}$$

The proof is complete.  $\square$

**Proof of Corollary 1.3** It follows from Theorem 1.2 and Proposition 2.3.  $\square$

## 4 $L^p$ regularity theory

In this section we prove Theorem 1.4 and Theorem 1.7. We will write

$$p_1 = \frac{np}{n-p}, \quad p_2 = \frac{np}{n-2p}, \quad p_3 = \frac{np}{n-3p},$$

whenever these are positive numbers. For  $p < n$ , set

$$\alpha = 4 - n/p. \quad (4.1)$$

Roughly speaking, Theorems 1.4 and 1.7 follow from the Morrey estimate of the previous section and an iteration argument. Along the iteration the constant  $\epsilon$  should become smaller and smaller. Fortunately, the iteration stops after finitely many steps. Thus we can always choose a sufficiently small  $\epsilon$  in the very beginning such that the whole iteration proceeds. As in the previous proofs, the Gauge transform plays a central role.

### 4.1 Case 1: $n/4 < p < n/3$

In this subsection we prove Theorem 1.4 in the case  $n/4 < p < n/3$ . Recall that our initial assumption is that  $\nabla u \in M^{4,n-4}(B_1)$ ,  $\nabla^2 u \in M^{2,n-4}(B_1)$  hold with the smallness assumption (1.9). Thus we can choose  $\epsilon$  sufficiently small such that we have the improvement

$$\nabla u \in M_*^{4,n-4+4\alpha}(B_{1/2}) \quad \text{and} \quad \nabla^2 u \in M_*^{2,n-4+2\alpha}(B_{1/2}),$$

where  $\alpha = 4 - n/p \in (0, 1)$ . At this moment, due to the strong nonlinearity, the regularity of the function

$$|G_P \nabla u| \lesssim (|\nabla^2 u| + |\nabla^2 P|) (|\nabla u| + |\nabla P|) |\nabla u| + (|\nabla u|^3 + |\nabla P|^3) |\nabla u|$$

will be too weak to iterate.

Fortunately we have the following two observations. The first one is that the second order weak Morrey regularity implies:

$$\nabla u \in L^{2\chi} \cap M_*^{2\chi, n-2\chi(1-\alpha)}(B_{1/4}) \quad (4.2)$$

where

$$\chi \equiv (2 - \alpha)/(1 - \alpha) > 2. \quad (4.3)$$

To find this, select  $\eta \in C_0^\infty(B_{1/2})$  with  $\eta \equiv 1$  on  $B_{1/4}$ . An elementary calculation shows that  $\nabla(u\eta) \in M_*^{4,n-4(1-\alpha)}(B_{1/2})$  and  $\nabla^2(\eta u) \in M_*^{2,n-2(2-\alpha)}(B_{1/2})$ . Set  $\eta u \equiv 0$  outside  $B_{1/2}$ . (2.4) implies that

$$\nabla(\eta u) = \nabla I_2(-\Delta(\eta u)) \approx I_1(\Delta u) \in L^{2\chi} \cap M_*^{2\chi,n-2\chi(1-\alpha)}(\mathbb{R}^n)$$

with estimates

$$\|\nabla(\eta u)\|_{L^{2\chi}} + \|\nabla(\eta u)\|_{M_*^{2\chi,n-2\chi(1-\alpha)}} \lesssim \|\Delta(\eta u)\|_{L^2} + \|\Delta(\eta u)\|_{M_*^{2,n-2(2-\alpha)}}.$$

This yields (4.2) for  $\nabla u$  with

$$\|\nabla u\|_{L^{2\chi}(B_{1/4})} + \|\nabla u\|_{M_*^{2\chi,n-2\chi(1-\alpha)}(B_{1/4})} \lesssim \|\nabla u\|_{M_*^{4,n-4(1-\alpha)}(B_{1/2})} + \|\nabla^2 u\|_{M_*^{2,n-2(2-\alpha)}(B_{1/2})}.$$

The second observation is:

**Lemma 4.1** *There exist  $\epsilon = \epsilon(n, m) > 0$  and  $C = C(n, m) > 0$  with the following property: For every  $\Omega \in M_1^{2,n-4} \cap M^{4,n-4}(B_{1/2}, so_m \otimes \wedge^1 \mathbb{R}^n)$  with*

$$\|\nabla \Omega\|_{M^{2,n-4}(B_{1/2})} + \|\Omega\|_{M^{4,n-4}(B_{1/2})} \leq \epsilon,$$

*there exist  $P \in W^{2,2}(B_{1/2}, SO_m)$  and  $\xi \in W^{2,2}(B_{1/2}, so_m \otimes \wedge^{n-2} \mathbb{R}^n)$  such that Lemma 3.1 holds on  $B_{1/2}$ .*

*In addition, if  $\Omega \in M_*^{4,n-4+4\alpha}(B_{1/2})$  and  $\nabla \Omega \in M_*^{2,n-4+2\alpha}(B_{1/2})$ , then we further have  $\nabla P, \nabla \xi \in M_*^{4,n-4+4\alpha}(B_{1/2})$ ,  $\nabla^2 P, \nabla^2 \xi \in M_*^{2,n-4+2\alpha}(B_{1/2})$  together with*

$$\|\nabla P\|_{M_*^{4,n-4+4\alpha}(B_{1/2})} + \|\nabla \xi\|_{M_*^{4,n-4+4\alpha}(B_{1/2})} \leq C \|\Omega\|_{M_*^{4,n-4+4\alpha}(B_{1/2})}, \quad (4.4)$$

and

$$\begin{aligned} & \|\nabla^2 P\|_{M_*^{2,n-4+2\alpha}(B_{1/2})} + \|\nabla^2 \xi\|_{M_*^{2,n-4+2\alpha}(B_{1/2})} \\ & \leq C \left( \|\nabla \Omega\|_{M_*^{2,n-4+2\alpha}(B_{1/2})} + \|\Omega\|_{M_*^{4,n-4+4\alpha}(B_{1/2})} \right). \end{aligned} \quad (4.5)$$

**Proof** The existence of  $P, \xi$  follows from the same method as that of Lemma 3.1. For the proof of estimates (4.4) and (4.5), see Lemma A.3 in the ‘‘Appendix’’.

Let  $P, \xi$  be obtained as in Lemma 4.1. By the first observation, we have

$$\nabla P, \nabla \xi \in L^{2\chi} \cap M_*^{2\chi,n-2\chi(1-\alpha)}(B_{1/4}) \quad (4.6)$$

and

$$\|\nabla P, \nabla \xi\|_{L^{2\chi} \cap M_*^{2\chi,n-2\chi(1-\alpha)}(B_{1/4})} \lesssim \|\nabla \Omega\|_{M_*^{2,n-4+2\alpha}(B_{1/2})} + \|\Omega\|_{M_*^{4,n-4+4\alpha}(B_{1/2})}.$$

Thus we deduce from the growth assumption on  $\Omega$  that

$$\|\nabla P, \nabla \xi\|_{L^{2\chi} \cap M_*^{2\chi,n-2\chi(1-\alpha)}(B_{1/4})} \lesssim \|\nabla^2 u\|_{M_*^{2,n-4+2\alpha}(B_{1/2})} + \|\nabla u\|_{M_*^{4,n-4+4\alpha}(B_{1/2})}.$$

We transform the system (1.7) on  $B_{1/4}$  to obtain the gauge equivalent system (3.3). Then we extend all the functions from  $B_{1/4}$  into  $\mathbb{R}^n$  with compact supports in  $B_2$  in a bounded way, and define similarly  $u_{11}, u_{12}, u_2$  and a biharmonic 1-form  $h$  on  $B_{1/4}$  as that of (3.5), (3.6) and (3.7) such that  $Pdu = du_{11} + du_{12} + d^*u_2 + h$  on  $B_{1/4}$ .

Our aim is to improve the regularity of  $\nabla^2 u$  through the gauge equivalent system (3.3).

**Claim 4.2** Let  $p_2 = \frac{np}{n-2p}$  and  $\chi$  be defined as in (4.3). Then

$$\nabla^2 u \in \begin{cases} L^\chi \cap M_*^{\chi, n-\chi(2-\alpha)} \left( B_{\frac{1}{4}} \right) & \text{if } \chi < p_2, \\ L^{p_2} \left( B_{\frac{1}{4}} \right) & \text{if } \chi \geq p_2, \end{cases} \quad (4.7)$$

**Proof** Hereafter all the norms are taken on the whole space  $\mathbb{R}^n$  unless specified. We first deduce the regularity of  $\nabla^2 u_{11}$ .

For the first term  $J_1$ , (4.2), (4.6) and Hölder's inequality (2.2) imply

$$(|\nabla^2 u| + |\nabla^2 P| + |\nabla u|^2 + |\nabla P|^2) (|\nabla u| + |\nabla P|) \in M_*^{\frac{2\chi}{\chi+1}, n-4+2\alpha} \cap L^{\frac{2\chi}{\chi+1}}.$$

Since

$$\nabla^2 J_1 \approx I_1 \left( (|\nabla^2 u| + |\nabla^2 P| + |\nabla u|^2 + |\nabla P|^2) (|\nabla u| + |\nabla P|) \right),$$

and by (2.4)

$$I_1: L^{\frac{2\chi}{\chi+1}} \cap M_*^{\frac{2\chi}{\chi+1}, n-4+2\alpha} \rightarrow L^\chi \cap M_*^{\chi, n-4+2\alpha}$$

is a bounded operator, we obtain  $\nabla^2 J_1 \in M_*^{\chi, n-4+2\alpha}(\mathbb{R}^n) \cap L^\chi(\mathbb{R}^n)$  with

$$\|\nabla^2 J_1\|_{M_*^{\chi, n-4+2\alpha} \cap L^\chi(\mathbb{R}^n)} \lesssim \|\nabla u\|_{L^\chi \cap M_*^{\chi, n-\chi(2-\alpha)}(B_{1/4})} \|\nabla^2 u\|_{L^2 \cap M_*^{2, n-2(2-\alpha)}(B_{1/4})}.$$

By the weak Morrey estimate,

$$\|\nabla u\|_{L^\chi \cap M_*^{\chi, n-\chi(2-\alpha)}(B_{1/4})} + \|\nabla^2 u\|_{L^2 \cap M_*^{2, n-2(2-\alpha)}(B_{1/4})} \lesssim \epsilon + \|f\|_{L^p(B_1)}.$$

This in turn leads

$$\|\nabla^2 J_1\|_{M_*^{\chi, n-4+2\alpha} \cap L^\chi(\mathbb{R}^n)} \lesssim (\epsilon + \|f\|_{L^p(B_1)}) \|\nabla^2 u\|_{L^2 \cap M_*^{2, n-2(2-\alpha)}(B_{1/4})}. \quad (4.8)$$

For the second term, we have  $|\nabla^2 J_2| \lesssim I_2(|G_P| |\nabla u|)$  and

$$|G_P| \leq (|\nabla^2 u| + |\nabla^2 P|) (|\nabla u| + |\nabla P|) + (|\nabla u|^3 + |\nabla P|^3).$$

Recall that  $\nabla u, \nabla P \in M_*^{2\chi, n-4+2\alpha} \cap L^{2\chi}$  and  $\nabla^2 u, \nabla^2 P \in M_*^{2, n-4+2\alpha} \cap L^2$ . So

$$\begin{aligned} (|\nabla^2 u| + |\nabla^2 P|) (|\nabla u| + |\nabla P|)^2 &\in M_*^{\frac{2\chi}{\chi+2}, n-4+2\alpha} \cap L^{\frac{2\chi}{\chi+2}}, \\ (|\nabla u| + |\nabla P|)^4 &= (|\nabla u| + |\nabla P|)^2 (|\nabla u| + |\nabla P|)^2 \in M_*^{\frac{2\chi}{\chi+2}, n-4+2\alpha} \cap L^{\frac{2\chi}{\chi+2}}. \end{aligned}$$

Here the first term can be regarded in the space  $M_*^{4, n-4+2\alpha}$  in view of the embedding  $M_*^{2\chi, n-4+2\alpha}(B_{1/2}) \subset M_*^{4, n-4+2\alpha}(B_{1/2})$ . Thus

$$G_P \nabla u \in M_*^{\frac{2\chi}{\chi+2}, n-4+2\alpha} \cap L^{\frac{2\chi}{\chi+2}}.$$

In the case  $\alpha < 2/3$ , we may apply (2.4) to deduce the boundedness of

$$I_2: M_*^{\frac{2\chi}{\chi+2}, n-4+2\alpha} \cap L^{\frac{2\chi}{\chi+2}} \rightarrow M_*^{\frac{2(2-\alpha)}{2-3\alpha}, n-4+2\alpha} \cap L^{\frac{2(2-\alpha)}{2-3\alpha}},$$

which implies  $\nabla^2 J_2 \in M_*^{\frac{2(2-\alpha)}{2-3\alpha}, n-4+2\alpha} \cap L^{\frac{2(2-\alpha)}{2-3\alpha}}$ . Similar to (4.8), we can obtain

$$\|\nabla^2 J_2\|_{M_*^{\frac{2(2-\alpha)}{2-3\alpha}, n-4+2\alpha} \cap L^{\frac{2(2-\alpha)}{2-3\alpha}}} \lesssim (\epsilon + \|f\|_{L^p(B_1)})^a \|\nabla u\|_{L^4 \cap M_*^{4, n-4(1-\alpha)}(B_{1/4})}. \quad (4.9)$$

for some  $a > 0$ .

For the third term  $|\nabla^2 J_3| \lesssim I_1(|\nabla^2 \xi| |\nabla u|) + I_2(|\nabla^2 \xi| |\nabla u| |\nabla P|)$ , the same estimates as that of  $J_1$  and  $J_2$  imply

$$I_1(|\nabla^2 \xi| |\nabla u|) \in M_*^{\chi, n-4+2\alpha}(\mathbb{R}^n) \cap L^\chi(\mathbb{R}^n)$$

and when  $\alpha < 2/3$

$$I_2(|\nabla^2 \xi| |\nabla u| |\nabla P|) \in M_*^{\frac{2(2-\alpha)}{2-3\alpha}, n-4+2\alpha} \cap L^{\frac{2(2-\alpha)}{2-3\alpha}}.$$

Note that if  $\alpha < 2/3$ , then

$$\frac{2(2-\alpha)}{2-3\alpha} = \frac{2-\alpha}{1-\frac{3}{2}\alpha} > \chi,$$

and if  $\alpha > \frac{2}{3}$ , then the regularity of  $\nabla^2 J_i$ ,  $i = 2, 3$ , become even better. All together, we may conclude

$$\nabla^2 u_{11} \in M_*^{\chi, n-4+2\alpha}(B_{1/4}) \cap L^\chi(B_{1/4}).$$

Since  $u_{12} \in W^{4,p}(\mathbb{R}^n)$ ,  $\nabla^2 u_{12} \in W^{2,p}(\mathbb{R}^n) \subset L^{p_2}$ . In particular, for any  $s > 0$ ,

$$\int_{B_s(x)} |\nabla^2 u_{12}|^\chi \lesssim \|\nabla^2 J_5\|_{L^{p_2}}^\chi s^{n-2\chi+\chi\alpha} \lesssim \|f\|_{L^p}^\chi s^{n-2\chi+\chi\alpha}.$$

That is

$$\nabla^2 u_{12} \in L^{p_2} \cap M^{\chi, n-2\chi+\chi\alpha}(\mathbb{R}^n).$$

Similar to the estimate of  $J_1$ , one deduces

$$\nabla^2 u_2 \in M_*^{\chi, n-4+2\alpha}(B_{1/4}) \cap L^\chi(B_{1/4}).$$

Note that the biharmonic 1-form  $h$  is always smooth. Hence Claim 4.2 holds if  $\chi \geq p_2$ . In the case  $\chi < p_2 = np/(n-2p) = n/(2-\alpha)$ , observe that  $n-4+2\alpha = n-2\chi+2\chi\alpha$ . So, for any  $w \in M_*^{\chi, n-2\chi+2\chi\alpha}(B_{1/4})$  and any  $0 < r < 1/2$ ,

$$\|w\|_{L^{\chi, \infty}(B_r(x))}^\chi \leq \|w\|_{M_*^{\chi, n-2\chi+2\chi\alpha}(B_{1/4})}^\chi r^{n-2\chi+2\chi\alpha} \leq \|w\|_{M_*^{\chi, n-2\chi+2\chi\alpha}(B_{1/4})}^\chi r^{n-2\chi+\chi\alpha}.$$

That is,

$$M_*^{\chi, n-4+2\alpha}(B_{1/4}) \subset M^{\chi, n-2\chi+\chi\alpha}(B_{1/4}).$$

Therefore,

$$\nabla^2 u_{11}, \nabla^2 u_2 \in L^\chi \cap M_*^{\chi, n-2\chi+\chi\alpha}(B_{1/4}).$$

The proof of Claim 4.2 is complete.  $\square$

Next we use iteration to derive the optimal regularity of  $\nabla u$  and  $\nabla^2 u$ .

**Claim 4.3** (Iteration lemma) Let  $\chi = \lambda_1 \leq \lambda_n < p_2$  and set

$$\lambda_{n+1} = \frac{\chi}{2} \lambda_n.$$

If

$$\nabla u \in L^{2\lambda_n} \cap M_*^{2\lambda_n, n-2\lambda_n(1-\alpha)}(B_1) \quad \text{and} \quad \nabla^2 u \in L^{\lambda_n} \cap M_*^{\lambda_n, n-\lambda_n(2-\alpha)}(B_1)$$

with sufficiently small norms and if

$$\lambda_{n+1} < \chi,$$

then

$$\nabla u \in L^{2\lambda_{n+1}} \cap M_*^{2\lambda_{n+1}, n-2\lambda_{n+1}(1-\alpha)}(B_{1/2}) \quad \text{and} \quad \nabla^2 u \in L^{\lambda_{n+1}} \cap M_*^{\lambda_{n+1}, n-\lambda_{n+1}(2-\alpha)}(B_{1/2}).$$

**Proof** The improvement of  $\nabla u$  on  $B_{1/2}$  follows as before. We also have the same regularity of  $P$ ,  $\xi$  as that of  $u$  by the same arguments as above. So we only need to deduce the regularity of  $\nabla^2 u$  and the arguments will be similar as in the previous step.

For the first term, we have

$$\nabla^2 J_1 \approx I_1 \left( (|\nabla^2 u| + |\nabla^2 P|) |\nabla u| + (|\nabla u| + |\nabla P|) |\nabla^2 u| \right).$$

Hölder's inequality gives

$$(|\nabla^2 u| + |\nabla^2 P|) |\nabla u| + (|\nabla u| + |\nabla P|) |\nabla^2 u| \in L^{\frac{\chi}{\chi+1} \lambda_n} \bigcap M_*^{\frac{\chi}{\chi+1} \lambda_n, n - \frac{\chi}{\chi+1} \lambda_n (3-2\alpha)}$$

and (2.4) gives the boundedness of

$$I_1: L^{\frac{\chi}{\chi+1} \lambda_n} \bigcap M_*^{\frac{\chi}{\chi+1} \lambda_n, n - \frac{\chi}{\chi+1} \lambda_n (3-2\alpha)} \rightarrow L^{\frac{3-2\alpha}{2-2\alpha} \frac{\chi}{\chi+1} \lambda_n} \cap M_*^{\frac{3-2\alpha}{2-2\alpha} \frac{\chi}{\chi+1} \lambda_n, n - \frac{\chi}{\chi+1} \lambda_n (3-2\alpha)}.$$

Note that

$$\frac{3-2\alpha}{2-2\alpha} \frac{\chi}{\chi+1} \lambda_n = \frac{\chi}{2} \lambda_n = \lambda_{n+1}.$$

Hence

$$\nabla^2 J_1 \in L^{\lambda_{n+1}} \cap M_*^{\lambda_{n+1}, n-\lambda_{n+1}(2-2\alpha)}.$$

For the second term, we have  $|\nabla^2 J_2| \lesssim I_2(|G_P| |\nabla u|)$  and

$$|G_P \nabla u| \leq (|\nabla^2 u| + |\nabla^2 P|) (|\nabla u| + |\nabla P|)^2 + (|\nabla u| + |\nabla P|)^4.$$

Note that  $2\lambda_{n+1} = \chi \lambda_n$  and so

$$\begin{aligned} (|\nabla^2 u| + |\nabla^2 P|) (|\nabla u| + |\nabla P|)^2 &\in L^{\frac{\chi}{\chi+2} \lambda_n} \bigcap M_*^{\frac{\chi}{\chi+2} \lambda_n, n - \frac{\chi}{\chi+2} \lambda_n (4-3\alpha)}, \\ (|\nabla u| + |\nabla P|)^4 &\in L^{\frac{\chi}{4} \lambda_n} \bigcap M_*^{\frac{\chi}{4} \lambda_n, n - \chi \lambda_n (1-\alpha)}. \end{aligned}$$

The second term has better regularity than the first one. In the case  $\alpha < 2/3$ , applying (2.4) gives boundedness of

$$I_2: L^{\frac{\chi}{\chi+2} \lambda_n} \bigcap M_*^{\frac{\chi}{\chi+2} \lambda_n, n - \frac{\chi}{\chi+2} \lambda_n (4-3\alpha)} \rightarrow L^{\frac{4-3\alpha}{2-3\alpha} \frac{\chi}{\chi+2} \lambda_n} \bigcap M_*^{\frac{4-3\alpha}{2-3\alpha} \frac{\chi}{\chi+2} \lambda_n, n - \frac{\chi}{\chi+2} \lambda_n (4-3\alpha)}.$$

So

$$\nabla^2 J_2 \in L^{\frac{4-3\alpha}{2-3\alpha} \frac{\chi}{\chi+2} \lambda_n} \bigcap M_*^{\frac{4-3\alpha}{2-3\alpha} \frac{\chi}{\chi+2} \lambda_n, n - \frac{\chi}{\chi+2} \lambda_n (4-3\alpha)}.$$

The third term can be splitted into a sum of two terms with the same regularity as that of  $J_1$  and  $J_2$ . Since  $\chi + 2 = (4 - 3\alpha)/(1 - \alpha)$ , we have

$$\tilde{\lambda}_{n+1} \equiv \frac{4-3\alpha}{2-3\alpha} \frac{\chi}{\chi+2} \lambda_n = \frac{2-\alpha}{2-3\alpha} \lambda_n = \frac{2(1-\alpha)}{2-3\alpha} \lambda_{n+1}.$$

Hence  $\tilde{\lambda}_{n+1} > \lambda_{n+1}$  and

$$n - \tilde{\lambda}_{n+1}(2 - 3\alpha) = n - \lambda_{n+1}(2 - 2\alpha) > n - \lambda_{n+1}(2 - \alpha).$$

This implies

$$L^{\tilde{\lambda}_{n+1}} \cap M_*^{\tilde{\lambda}_{n+1}, n - \tilde{\lambda}_{n+1}(2 - 3\alpha)}(B_{1/2}) \subset L^{\lambda_{n+1}} \cap M_*^{\lambda_{n+1}, n - \lambda_{n+1}(2 - \alpha)}(B_{1/2}).$$

Consequently, we obtain

$$\nabla^2 u_{11} \in L^{\lambda_{n+1}} \cap M_*^{\lambda_{n+1}, n - \lambda_{n+1}(2 - \alpha)}(B_{1/2}).$$

Note also that  $\nabla^2 u_{12} \in L^{p_2}$ . Thus if  $\lambda_{n+1} < p_2$ , then

$$\nabla^2 u_{12} \in L^{\lambda_{n+1}} \cap M_*^{\lambda_{n+1}, n - \lambda_{n+1}(2 - \alpha)}(B_{1/2}).$$

Similarly, we can deduce the result for  $u_2$  and the biharmonic part  $h$ . The proof of Claim 4.3 is complete.  $\square$

Since  $\chi > 2$ , Claims 4.2 and 4.3 imply that after finitely many steps, this iteration will stop, whence  $\nabla^2 u \in L_{\text{loc}}^{p_2}(B_1)$ . This in return implies that  $\nabla u \in L_{\text{loc}}^{p_3}(B_1)$  by the Sobolev embedding theorem.

Now we can deduce the third order regularity of  $u$ . Rewrite the system (1.7) as

$$\Delta^2 u = \text{div}(I) + II, \quad (4.10)$$

where

$$\begin{aligned} I &= D\nabla^2 u + \nabla D \cdot \nabla u + E \cdot \nabla u + \nabla \Omega \cdot \nabla u, \\ II &= -\nabla \Omega \cdot \Delta u + G \cdot \nabla u + f. \end{aligned}$$

By the growth assumption (GC-4), we know

$$\begin{aligned} |I| &\leq C(|\nabla^2 u| + |\nabla u|^2)|\nabla u|, \\ |II| &\leq C(|\nabla^2 u|^2 + |\nabla u|^2|\nabla^2 u| + |\nabla u|^4) + f. \end{aligned} \quad (4.11)$$

Since we have proved that  $u \in W^{2, \frac{np}{n-2p}}$  and  $n \leq 4p$ , it follows that  $2p \leq p_2 \leq p_3/2$ . Hence  $I \in L_{\text{loc}}^{\frac{np}{2n-5p}} \subset L_{\text{loc}}^{\frac{np}{n-p}}$  and  $II \in L_{\text{loc}}^p$ . Here the least regular term of  $I$  and  $II$  are  $\nabla^2 u \nabla u$  and  $f$ , respectively.

Set  $\Delta^2 u_1 = \text{div}(I)$  and  $\Delta^2 u_2 = II$  in  $B_1$ . Standard elliptic regularity theory implies  $u_1 \in W_{\text{loc}}^{3, \frac{np}{n-p}}$  and  $u_2 \in W_{\text{loc}}^{4,p} \subset W_{\text{loc}}^{3, \frac{np}{n-p}}$ . As  $u - u_1 - u_2$  is a biharmonic function, we infer that

$$u \in W_{\text{loc}}^{3, \frac{np}{n-p}}(B_1).$$

Next we derive the apriori estimate of  $u$ . By the Hodge decomposition, we have the biharmonic 1-form  $h$  satisfying

$$h = Pdu - du_{11} - du_{12} - d^*u_2 \quad \text{in } B_1.$$

By the Morrey estimates (Theorem 1.2), we have

$$\|h\|_{M_*^{4, n-4+4\alpha}(B_{3/4})} \lesssim \epsilon + \|f\|_{L^p(B_1)}.$$

In particular, this implies that  $\|h\|_{L^1(B_{3/4})} \lesssim \epsilon + \|f\|_{L^p(B_1)}$ . Since  $h$  is biharmonic, we infer that

$$\|h\|_{L^{p_3}(B_{1/2})} \lesssim \|h\|_{L^1(B_{3/4})} \lesssim \epsilon + \|f\|_{L^p(B_1)}.$$

Returning to the Hodge decomposition, we have

$$\|\nabla u\|_{L^{p_3}(B_{1/2})} \lesssim \|h\|_{L^{p_3}(B_{1/2})} + \|\nabla u_{11}\|_{L^{p_3}(B_{1/2})} + \|\nabla u_{12}\|_{L^{p_3}(B_{1/2})} + \|\nabla u_2\|_{L^{p_3}(B_{1/2})}.$$

Using the potential theory, we can similarly estimate  $\|\nabla u_{11}\|_{L^{p_3}(B_{1/2})}$ ,  $\|\nabla u_{12}\|_{L^{p_3}(B_{1/2})}$  and  $\|\nabla u_2\|_{L^{p_3}(B_{1/2})}$  as that of (4.8) and (4.9). Hence

$$\begin{aligned} & \|du\|_{L^{p_3}(B_{1/2})} \\ & \leq c \left( \epsilon + \|f\|_{L^p(B_1)} \right)^a \left( \|\nabla u\|_{M_*^{4,n-4(1-\alpha)}(B_1)} + \|\nabla^2 u\|_{M_*^{2,n-2(2-\alpha)}(B_1)} + \|f\|_{L^p(B_1)} \right) \end{aligned}$$

for some  $a > 0$ . Similarly, we obtain

$$\begin{aligned} \|\nabla^2 u\|_{L^{p_2}(B_{1/2})} & \leq c \left( \epsilon + \|f\|_{L^p(B_1)} \right)^a \left( \|\nabla u\|_{M_*^{4,n-4(1-\alpha)}(B_1)} \right. \\ & \quad \left. + \|\nabla^2 u\|_{M_*^{2,n-2(2-\alpha)}(B_1)} + \|f\|_{L^p(B_1)} \right) \end{aligned}$$

for some  $a > 0$ . Here  $c$  and  $a$  are two constants that depend on  $n, m, p$ .

Now we derive the a priori estimate for  $\nabla^3 u$ . Applying the elliptic regularity theory to the Eq. (4.10), we obtain

$$\|\nabla^3 u\|_{L^{p_1}(B_{1/2})} \lesssim \|I\|_{L^{p_1}(B_{3/4})} + \|II\|_{L^p(B_{3/4})} + \|\nabla u\|_{L^4(B_{3/4})}.$$

By the growth property (4.11), we have

$$\begin{aligned} \|I\|_{L^{p_1}(B_{3/4})} & \lesssim \left( \|\nabla^2 u\|_{L^{\frac{n}{2}}(B_{3/4})} + \|\nabla u\|_{L^n(B_{3/4})}^2 \right) \|\nabla u\|_{L^{p_3}(B_{3/4})} \\ & \lesssim (\epsilon + \|f\|_{L^p(B_1)})^a \|\nabla u\|_{L^{p_3}(B_{3/4})}. \end{aligned}$$

Thus we obtain

$$\|I\|_{L^{p_1}(B_{1/2})} \lesssim (\epsilon + \|f\|_{L^p(B_1)})^a \left( \|\nabla u\|_{M^{4,n-4}(B_1)} + \|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|f\|_{L^p(B_1)} \right).$$

Similarly, we obtain

$$\|II\|_{L^p(B_{1/2})} \lesssim (\epsilon + \|f\|_{L^p(B_1)})^a \left( \|\nabla u\|_{M^{4,n-4}(B_1)} + \|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|f\|_{L^p(B_1)} \right).$$

In conclusion, we deduce

$$\begin{aligned} \|\nabla^3 u\|_{L^{p_1}(B_{1/2})} & \lesssim (\epsilon + \|f\|_{L^p(B_1)})^a \left( \|\nabla u\|_{M^{4,n-4}(B_1)} + \|\nabla^2 u\|_{M^{2,n-4}(B_1)} + \|f\|_{L^p(B_1)} \right) \\ & \quad + \|\nabla u\|_{L^4(B_1)}. \end{aligned}$$

This finishes the proof of Theorem 1.4 in the case  $n/4 < p < n/3$ .

## 4.2 Case 2: $n/3 \leq p < \infty$

In the remaining case  $n/3 \leq p < \infty$ , the result follows by an induction argument and a trivial iteration.

Since  $n/3 \leq p$ , it follows that  $f \in L^q(B_1)$  for any  $q < n/3$ . Choose  $\epsilon = \epsilon_q$  sufficiently small such that we can apply the previous result to obtain  $u \in W_{\text{loc}}^{3,q}$ , which then implies that

$$\nabla u \in L_{\text{loc}}^s, \nabla^2 u \in L_{\text{loc}}^{n-\delta} \quad \forall s < \infty, 0 < \delta \ll 1.$$

Write the equation as

$$\Delta^2 u = \text{div}(I) + II + f.$$

As a result,  $I \in L_{\text{loc}}^{n-\delta}$  and  $II \in L_{\text{loc}}^{n/2-\delta}$  for any  $\delta > 0$  small. Let

$$\Delta^2 u_1 = \text{div} I, \quad \Delta^2 u_2 = II, \quad \Delta^2 u_3 = f.$$

We find that  $u_1 \in W_{\text{loc}}^{3,n-\delta}$ ,  $u_2 \in W_{\text{loc}}^{4,\frac{n}{2}-\delta}$  and  $u_3 \in W_{\text{loc}}^{4,p}$ .

Case 2.1. If  $n/3 \leq p < n/2$ , then for  $\delta > 0$  sufficiently small  $W^{3,n-\delta} \subset W^{3,\frac{np}{n-p}}$ . Hence in this case  $u_1 + u_2 + u_3 \in W_{\text{loc}}^{3,\frac{np}{n-p}}$ . So

$$n/3 \leq p < n/2 \Rightarrow u \in W_{\text{loc}}^{3,\frac{np}{n-p}}.$$

Case 2.2. If  $p \geq n/2$ , then  $f \in L^q(B_1)$  for any  $q < n/2$ . Apply the above result gives  $u \in W_{\text{loc}}^{3,n-\delta}$ , which in turn implies that  $\nabla u \in L^\infty$  and  $\nabla^2 u \in L_{\text{loc}}^s$  for any  $s < \infty$ . Hence  $I, II \in L_{\text{loc}}^s$  for any  $s < \infty$ . This then gives  $u_1 \in W_{\text{loc}}^{3,s}$ ,  $u_2 \in W_{\text{loc}}^{4,s}$  for any  $s < \infty$ . However, recall that  $u_3 \in W_{\text{loc}}^{4,p}$ . So we can conclude that

$$\begin{cases} u \in W_{\text{loc}}^{3,\frac{np}{n-p}} & \text{if } n/2 \leq p < n, \\ u \in W_{\text{loc}}^{3,s} & \text{for any } s < \infty \text{ if } p \geq n. \end{cases}$$

The a priori estimates in this case can be derived similarly and thus omitted. The proof of Theorem 1.4 is complete.

### 4.3 Case 3: $1 < p \leq n/4$

**Proof of Theorem 1.7** The proof of this theorem is almost the same as that of Theorem 1.4, only with minor modifications in the arguments. First note that our Morrey estimate holds as well. So we can iterate. By the assumption of  $f$ , we have

$$I_2(f) \in L^{\eta q} \cap M_*^{\eta, n-\eta(2-\alpha)}(\mathbb{R}^n),$$

Remark that  $2 < \eta < \chi$ . This term determines how much regularity we can gain in the end.

If  $\eta q \leq \chi$ , the iteration stops at the first step, and gives

$$\nabla^2 u \in L^{\eta q} \cap M_*^{\eta, n-\eta(2-\alpha)}(B_{1/2}).$$

In case  $\eta q > \chi$ , using the same iteration method with slightly modification, we can obtain the same result. As a result, it follows from the potential theory that

$$\nabla u \in L^{q\eta\chi} \cap M_*^{\eta\chi, n-\eta\chi(1-\alpha)}(B_{1/2}).$$

We leave the details to interested readers. □

## 4.4 Proofs of other results

**Proof of Corollary 1.5** The proof is standard and omitted here; see for instance [13, Proposition 6.2].  $\square$

**Proof of Corollary 1.6** It follows easily from a contradiction argument; see for instance [13, Proof of Corollary 1.5].  $\square$

**Data Availability** Data sharing is not applicable to this article as no data sets were generated or analysed.

## Appendix A. Some apriori estimates concerning gauge transform

Lemma 4.1 can be proved following the strategy of Rivière [26] and Rivière-Struwe [29]. We sketch the proof for readers' convenience. Also, for future applications, we will prove a slightly more general result than that of Lemma 4.1.

Let  $D \subset \mathbb{R}^n$  be a bounded Lipschitz domain,  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $0 \leq s \leq n$ . We slightly extend the notion of Morrey spaces. Say that a function  $f$  belongs to the Lorentz–Morrey space  $LM^{p,q,s}(D)$ , if  $f$  belongs to the Lorentz space  $L^{p,q}(D)$ , and if

$$\|f\|_{LM^{p,q,s}(D)} \equiv \sup_{x \in D, 0 < r < d_D} (r^{-s/p} \|f\|_{L^{p,q}(B_r(x) \cap D)}) < \infty,$$

where  $d_D$  is the diameter of  $D$ . Note that

$$LM^{p,p,s}(D) = M^{p,s}(D) \text{ and } LM^{p,\infty,s}(D) = M_*^{p,s}(D).$$

When  $s = 0$ , we get the usual Lorentz space, i.e.,  $LM^{p,q,0}(D) = L^{p,q}(D)$ . When  $0 < s \leq n$  and  $D$  is a bounded domain, we have the continuous embedding  $LM^{p,q,s}(D) \subset L^{p,q}(D)$ . Moreover,

$$\|f\|_{L^{p,q}(D)} \leq d_D^{s/p} \|f\|_{LM^{p,q,s}(D)} \quad (\text{A.1})$$

**Lemma A.1** *Let  $D \subset \mathbb{R}^n$  be a bounded Lipschitz domain,  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $0 \leq s < n$ . Then, there exists a constant  $C > 0$  depending only on  $D$ ,  $p$ ,  $q$ ,  $s$  such that whenever  $u \in W_0^{1,p}(D)$  is the solution of the equation*

$$-\Delta u = \operatorname{div} f \quad \text{in } D,$$

*for some  $f \in LM^{p,q,s}(D, \mathbb{R}^n)$ , then  $\nabla u \in LM^{p,q,s}(D)$ . Moreover,*

$$\|\nabla u\|_{LM^{p,q,s}(D)} \leq C \|f\|_{LM^{p,q,s}(D)}.$$

**Proof** When  $s = 0$  and  $q = p$ , the result is well known. So the result follows from a standard interpolation arguments in the case  $s = 0$  and  $1 \leq q \leq \infty$ . In the below we suppose  $0 < s < n$ .

Let  $x_0 \in D$  be an arbitrary point in  $D$  and  $r > 0$ . Denote  $D_r = D \cap B_r(x_0)$ . Let  $v$  be the harmonic function in  $D_r$  with Dirichlet boundary value  $u$ . Then, the function  $w = u - v$  solves

$$\begin{cases} -\Delta w = \operatorname{div} f & \text{in } D_r, \\ w = 0 & \text{on } \partial D_r. \end{cases}$$

So apply the result for  $s = 0$ , we obtain

$$\|\nabla w\|_{L^{p,q}(D_r)} \leq C \|f\|_{L^{p,q}(D_r)}.$$

By the assumption, we find

$$\|\nabla w\|_{L^{p,q}(D_r)} \leq C\|f\|_{LM^{p,q,s}(D)}r^{s/p}.$$

On the other hand, for any  $0 < \rho < r$ ,

$$\|\nabla v\|_{L^{p,q}(D_\rho)}^p \leq C\left(\frac{\rho}{r}\right)^n \|\nabla v\|_{L^{p,q}(D_r)}^p.$$

Thus, for any  $0 < \rho < r$ , using a simple triangle inequality gives

$$\|\nabla u\|_{L^{p,q}(D_\rho)}^p \leq C\left(\frac{\rho}{r}\right)^n \|\nabla u\|_{L^{p,q}(D_r)}^p + C\|\nabla w\|_{L^{p,q}(D_r)}^p,$$

from which we deduce that

$$\|\nabla u\|_{L^{p,q}(D_\rho)}^p \leq C\left(\frac{\rho}{r}\right)^n \|\nabla u\|_{L^{p,q}(D_r)}^p + C\|f\|_{LM^{p,q,s}(D)}^p r^s.$$

Therefore, using an elementary lemma, we derive, for any  $0 < \rho < d_D$ ,

$$\|\nabla u\|_{L^{p,q}(D_\rho)}^p \leq C\rho^s \left( \frac{1}{d_D^s} \|\nabla u\|_{L^{p,q}(D)}^p + \|f\|_{LM^{p,q,s}(D)}^p \right).$$

Since  $x_0$  is arbitrary, this is equivalent to

$$\|\nabla u\|_{LM^{p,q,s}(D)} \leq C(\|\nabla u\|_{L^{p,q}(D)} + \|f\|_{LM^{p,q,s}(D)}).$$

Finally, note that by the result for  $s = 0$ , we have

$$\|\nabla u\|_{L^{p,q}(D)} \leq C\|f\|_{L^{p,q}(D)} \leq C\|f\|_{LM^{p,q,s}(D)}.$$

The second inequality follows from (A.1). Hence, we conclude from the above two estimates that the desired estimate holds. The proof is finished.  $\square$

Next we consider the following special Poisson equation.

**Lemma A.2** *Let  $D \subset \mathbb{R}^n$  be a bounded Lipschitz domain,  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $0 \leq s < n$ . Then, there exists a constant  $C > 0$  depending only on  $D$ ,  $p$ ,  $q$ ,  $s$  such that whenever  $u \in W_0^{1,p}(D, \wedge^{n-2}\mathbb{R}^n)$  is the solution of the equation*

$$-\Delta u = *(dP^{-1} \wedge dP) \text{ in } D,$$

*for some function  $P \in BMO(D)$  with  $dP \in LM^{p,q,s}(D)$ , then  $du \in LM^{p,q,s}(D)$ . Moreover,*

$$\|du\|_{LM^{p,q,s}(D)} \leq C\|P\|_{BMO(D)}\|dP^{-1}\|_{LM^{p,q,s}(D)}.$$

**Proof** (1) Suppose  $q = p$  and  $s = 0$ , i.e.,  $P \in BMO(D)$  and  $dP \in L^p(D)$ . Let  $F = |du|^{p-2}du \in L^{p'}(D, \wedge^{n-1}\mathbb{R}^n)$ . Hodge decomposition gives  $\psi \in W_T^{1,p'}(D, \wedge^{n-2}\mathbb{R}^n)$ ,  $\beta \in W_N^{1,p'}(D, \wedge^{n-2}\mathbb{R}^n)$  and an  $n-2$  harmonic form  $h \in \mathcal{H}^{n-2}(D, \mathbb{R}^n)$  such that  $F = d\psi + d^*\beta + h$  and

$$\|d\psi\|_{p'} + \|h\|_{p'} \leq C\|F\|_{p'} = C\|du\|_p^{p-1}.$$

Then

$$\int_D |du|^p = \int_D du \cdot (d\psi + d^*\beta + h) = \int_D du \cdot d\psi.$$

Here in last equality we used the boundary condition  $u = 0$  on  $\partial D$ . Therefore, we obtain

$$\int_D |du|^p = \int_D dP^{-1} \wedge dP \wedge \psi = \int_D dP^{-1}(P - P_D) \wedge d\psi,$$

] where  $P_D = \int_D P$ . Since  $dP^{-1} \wedge d\psi$  belongs to Hardy space, we obtain

$$\int_D |du|^p \leq C \|P\|_{BMO(D)} \|dP^{-1}\|_{L^p(D)} \|d\psi\|_{L^{p'}(D)}.$$

This gives

$$\|du\|_p \leq C \|P\|_{BMO(D)} \|dP^{-1}\|_{L^p(D)}.$$

(2) In the case  $s = 0$  and  $1 \leq q \leq \infty$ , we use the usual interpolation argument to obtain

$$\|du\|_{L^{p,q}(D)} \leq C \|P\|_{BMO(D)} \|dP^{-1}\|_{L^{p,q}(D)}.$$

(3) Now suppose  $0 < s < n$ . Use the same arguments as in the Lemma A.1. For any  $x_0 \in D$  and  $r > 0$ , denote  $D_r = D \cap B_r(x_0)$ . Let  $v$  be the harmonic function in  $D_r$  with Dirichlet boundary value  $u$ . Then, the function  $w = u - v$  solves

$$\begin{cases} -\Delta w = *(dP^{-1} \wedge dP) & \text{in } D_r, \\ w = 0 & \text{on } \partial D_r. \end{cases}$$

Thus using the result in the second step yields

$$\|dw\|_{L^{p,q}(D_r)} \leq C \|P\|_{BMO(D)} \|dP^{-1}\|_{L^{p,q}(D_r)}.$$

It follows

$$r^{-s/p} \|dw\|_{L^{p,q}(D_r)} \leq C \|P\|_{BMO(D)} \|dP^{-1}\|_{LM^{p,q,s}(D)}.$$

On the other hand, for any  $0 < \rho < r$ ,

$$\|dv\|_{L^{p,q}(D_\rho)}^p \leq C \left(\frac{\rho}{r}\right)^n \|dv\|_{L^{p,q}(D_r)}^p.$$

Therefore, a similar argument as in the previous lemma gives, for any  $x_0 \in D$  and  $0 < \rho < d_D$ ,

$$\|du\|_{L^{p,q}(D_\rho)} \leq C \rho^{s/p} (\|du\|_{L^{p,q}(D)} + \|P\|_{BMO(D)} \|dP^{-1}\|_{LM^{p,q,s}(D)}).$$

Since  $\|dP^{-1}\|_{L^{p,q}(D)} \leq C \|dP^{-1}\|_{LM^{p,q,s}(D)}$ , using the result in the second step together with the above estimate, we deduce the desired estimate. The proof is complete.  $\square$

Based on the above two lemmata, we can prove Lemma 4.1. We prove a slightly more general result here.

**Lemma A.3** *There exist  $\delta > 0$  and  $C > 0$  with the following property: Suppose that  $\Omega \in LM^{p,q,s}(B_{1/2})$  for some  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $0 < s < n$  such that there exist  $P, \xi \in LM^{p,q,s}(B_{1/2})$  satisfying the Eq. (3.1) of Lemma 3.1 on  $B_{1/2}$ , and*

$$\|dP\|_{M^{4,n-4}(B_{1/2})} + \|d\xi\|_{M^{4,n-4}(B_{1/2})} \leq \delta,$$

*then there hold*

$$\|dP\|_{LM^{p,q,s}(B_{1/2})} + \|d\xi\|_{LM^{p,q,s}(B_{1/2})} \leq C \|\Omega\|_{LM^{p,q,s}(B_{1/2})}.$$

If, in addition,  $\nabla\Omega \in M_{\frac{p}{2}, \frac{n-p+s}{2}}^{\frac{p}{2}, \frac{n-p+s}{2}}(B_{1/2})$ , then  $\nabla^2 P, \nabla^2 \xi \in M_{\frac{p}{2}, \frac{n-p+s}{2}}^{\frac{p}{2}, \frac{n-p+s}{2}}(B_{1/2})$ , and

$$\begin{aligned} & \|\nabla^2 P\|_{M_{\frac{p}{2}, \frac{n-p+s}{2}}^{\frac{p}{2}, \frac{n-p+s}{2}}(B_{\frac{1}{2}})} + \|\nabla^2 \xi\|_{M_{\frac{p}{2}, \frac{n-p+s}{2}}^{\frac{p}{2}, \frac{n-p+s}{2}}(B_{\frac{1}{2}})} \\ & \leq C \left( \|\nabla\Omega\|_{M_{\frac{p}{2}, \frac{n-p+s}{2}}^{\frac{p}{2}, \frac{n-p+s}{2}}(B_{\frac{1}{2}})} + \|\Omega\|_{LM^{p,q,s}(B_{\frac{1}{2}})} \right). \end{aligned}$$

In particular, (4.4) and (4.5) holds under the assumption  $\Omega \in M_*^{4,n-4+4\alpha}(B_{1/2})$  and  $\nabla\Omega \in M_*^{2,n-4+2\alpha}(B_{1/2})$ .

**Proof** By Eq. (3.1),

$$\begin{cases} \Delta\xi = *dP^{-1} \wedge dP + *(P^{-1}\Omega P) & \text{in } B_{1/2}, \\ \xi = 0 & \text{on } B_{1/2}. \end{cases}$$

Let  $\xi_1$  be the solution of

$$\begin{cases} \Delta\xi_1 = *dP^{-1} \wedge dP & \text{in } B_{1/2}, \\ \xi_1 = 0 & \text{on } B_{1/2}, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\xi_2 = *(P^{-1}\Omega P) & \text{in } B_{1/2}, \\ \xi_2 = 0 & \text{on } B_{1/2}. \end{cases}$$

Applying Lemma A.1 to  $\xi_2$  and Lemma A.2 to  $\xi_1$ , we deduce

$$\|d\xi_1\|_{LM^{p,q,s}(B_{1/2})} \leq C\delta\|dP\|_{LM^{p,q,s}(B_{1/2})}$$

and

$$\|d\xi_2\|_{LM^{p,q,s}(B_{1/2})} \leq C\|\Omega\|_{LM^{p,q,s}(B_{1/2})}.$$

Thus

$$\|d\xi\|_{LM^{p,q,s}(B_{1/2})} \leq C\delta\|dP\|_{LM^{p,q,s}(B_{1/2})} + \|\Omega\|_{LM^{p,q,s}(B_{1/2})}.$$

Directly from Eq. (3.1), we have

$$\|dP\|_{LM^{p,q,s}(B_{1/2})} \leq C\|d\xi\|_{LM^{p,q,s}(B_{1/2})} + \|\Omega\|_{LM^{p,q,s}(B_{1/2})}.$$

Combining the above two estimate together with a suitably chosen  $\delta < 1$  small enough, we obtain the first estimate.

The second estimate can be proved by the same method. We omit the details. The proof is complete.  $\square$

## References

1. Adams, R.C.: Morrey Spaces. Lecture Notes in Applied and Numerical Harmonic Analysis, xv+121 pp. Birkhäuser, Cham (2015)
2. Adams, D.R.: A note on Riesz potentials. Duke Math. J. **42**(4), 765–778 (1975)
3. Bethuel, F.: On the singular set of stationary harmonic maps. Manuscr. Math. **78**(4), 417–443 (1993)
4. Burenkov, V.I.: The continuation of functions with preservation and with deterioration of their differential properties. (Russian) Dokl. Akad. Nauk SSSR **224**(2), 269–272 (1975)
5. Chang, S.-Y.A., Wang, L., Yang, P.C.: A regularity theory of biharmonic maps. Commun. Pure Appl. Math. **52**(9), 1113–1137 (1999)
6. Cheng, D.R., Zhou, X.: Existence of constant mean curvature 2-spheres in Riemannian 3-spheres. Commun. Pure. Appl. Math. (2021) (to appear)

7. de Longueville, F.L., Gastel, A.: Conservation laws for even order systems of polyharmonic map type. *Calc. Var. Partial Differ. Equ.* **60**, 138 (2021)
8. Evans, C.L.: Partial regularity for stationary harmonic maps into spheres. *Arch. Ration. Mech. Anal.* **116**, 101–163 (1991)
9. Fanciullo, M.S., Lamberti, P.D.: On Burenkov's extension operator preserving Sobolev–Morrey spaces on Lipschitz domains. *Math. Nachr.* **290**(1), 37–49 (2017)
10. Giaquinta, M.: *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*. Annals of Mathematics Studies, vol. 105. Princeton University Press, Princeton (1983)
11. Guo, C.-Y., Xiang, C.-L.: Regularity of solutions for a fourth order linear system via conservation law. *J. Lond. Math. Soc.* (2) **101**(3), 907–922 (2020)
12. Guo, C.-Y., Xiang, C.-L.: Regularity of weak solutions to higher order elliptic systems in critical dimensions. *Trans. Am. Math. Soc.* **374**(5), 3579–3602 (2021)
13. Guo, C.-Y., Xiang, C.-L., Zheng, G.-F.: The Lamm–Rivière system I:  $L^p$  regularity theory. *Calc. Var. Partial Differ. Equ.* **60**, 213 (2021)
14. Guo, C.-Y., Xiang, C.-L., Zheng, G.-F.:  $L^p$  regularity theory for even order elliptic systems with antisymmetric first order potentials. *J. Math. Pures Appl.* **9**(165), 286–324 (2022)
15. Hineman, J., Huang, T., Wang, C.-Y.: Regularity and uniqueness of a class of biharmonic map heat flows. *Calc. Var. Partial Differ. Equ.* **50**(3–4), 491–524 (2014)
16. Ho, K.-P.: Sobolev–Jawerth embedding of Triebel–Lizorkin–Morrey–Lorentz spaces and fractional integral operator on Hardy type spaces. *Math. Nachr.* **287**, 1674–1686 (2014)
17. Lamberti, P.D., Violo, I.Y.: On Stein's extension operator preserving Sobolev–Morrey spaces. *Math. Nachr.* **292**(8), 1701–1715 (2019)
18. Lamm, T., Rivière, T.: Conservation laws for fourth order systems in four dimensions. *Commun. Partial Differ. Equ.* **33**, 245–262 (2008)
19. Lamm, T., Sharp, B.: Global estimates and energy identities for elliptic systems with antisymmetric potentials. *Commun. Partial Differ. Equ.* **41**(4), 579–608 (2016)
20. Laurain, P., Lin, L.: Energy convexity of intrinsic bi-harmonic maps and applications I: spherical target. *J. Reine Angew. Math.* **772**, 53–81 (2021)
21. Laurain, P., Rivière, T.: Energy quantization for biharmonic maps. *Adv. Calc. Var.* **6**(2), 191–216 (2013)
22. Laurain, P., Rivière, T.: Angular energy quantization for linear elliptic systems with antisymmetric potentials and applications. *Anal. PDE* **7**(1), 1–41 (2014)
23. Moser, R.: A variational problem pertaining to biharmonic maps. *Commun. Partial Differ. Equ.* **33**(7–9), 1654–1689 (2008)
24. Moser, R.: Weak solutions of a biharmonic map heat flow. *Adv. Calc. Var.* **2**(1), 73–92 (2009)
25. Moser, R.: An  $L^p$  regularity theory for harmonic maps. *Trans. Am. Math. Soc.* **367**(1), 1–30 (2015)
26. Rivière, T.: Conservation laws for conformally invariant variational problems. *Invent. Math.* **168**, 1–22 (2007)
27. Rivière, T.: The role of integrability by compensation in conformal geometric analysis. In: Baird, P., El Soufi, A., Fardoun, A., Regbaoui, R. (eds.) *Analytic Aspects of Problems in Riemannian Geometry: Elliptic PDEs, Solitons and Computer Imaging*, pp. 93–127. Sémin. Congr., 22, Soc. Math., Paris (2011)
28. Rivière, T.: *Conformally Invariant Variational Problems*. Lecture Notes at ETH Zurich (2012). <https://people.math.ethz.ch/~riviere/lecture-notes>
29. Rivière, T., Struwe, M.: Partial regularity for harmonic maps and related problems. *Commun. Pure Appl. Math.* **61**, 451–463 (2008)
30. Rupflin, M.: An improved uniqueness result for the harmonic map flow in two dimensions. *Calc. Var. Partial Differ. Equ.* **33**(3), 329–341 (2008)
31. Sharp, B.: Higher integrability for solutions to a system of critical elliptic PDE. *Methods Appl. Anal.* **21**(2), 221–240 (2014)
32. Sharp, B., Topping, P.: Decay estimates for Rivière's equation, with applications to regularity and compactness. *Trans. Am. Math. Soc.* **365**(5), 2317–2339 (2013)
33. Struwe, M.: Partial regularity for biharmonic maps, revisited. *Calc. Var. Partial Differ. Equ.* **33**, 249–262 (2008)
34. Wang, C.Y.: Biharmonic maps from  $R^4$  into a Riemannian manifold. *Math. Z.* **247**, 65–87 (2004)
35. Wang, C.Y.: Stationary biharmonic maps from  $R^m$  into a Riemannian manifold. *Commun. Pure Appl. Math.* **57**, 419–444 (2004)
36. Wang, C.Y., Zheng, S.Z.: Energy identity of approximate biharmonic maps to Riemannian manifolds and its application. *J. Funct. Anal.* **263**(4), 960–987 (2012)

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