Isotropic-Nematic Phase Transition and Liquid Crystal Droplets

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Abstract

Liquid crystal droplets are of great interest from physics and applications. Rigorous mathematical analysis is challenging as the problem involves harmonic maps (or Oseen-Frank energy minimizers in general), free interfaces and topological defects which could be either inside the droplet or on its surface along with some intriguing boundary anchoring conditions for the orientation configurations. In this paper, through a study of the phase transition between the isotropic and nematic states of liquid crystal based on the Ericksen model, we can show, when the size of droplet is much larger in comparison with the ratio of the Frank constants to the surface tension, a Γ-convergence theorem for minimizers. This Γ-limit is in fact the sharp interface limit for the phase transition between the isotropic and nematic regions when the small parameter ε, corresponding to the transition layer width, goes to zero. This limiting process not only provides a geometric description of the shape of the droplet as one would expect, and surprisingly it also gives the anchoring conditions for the orientations of liquid crystals on the surface of the droplet depending on material constants. In particular, homeotropic, tangential, and even free boundary conditions as assumed in earlier phenomenological modelings arise naturally, provided the surface tension, Frank-Ericksen constants are in suitable ranges.

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1 Introduction and statement of results

1.1 Isotropic and nematic phase transitions

Liquid crystal is a state of matter between liquid and solid, where the molecules tend to align locally in a common direction and form an anisotropic structure. This orientational order produces an anisotropic complex fluid with remarkable optical features, which have profound applications in optical and display devices. There are many phases in liquid crystals including isotropic, nematic, and smectic phases. Perhaps the most common one is the nematic phase, where the molecules exhibit an orientational order in the absence of translational order. Under the influences of either external electric-magnetic fields, or thermal changes, or compositional changes, liquid crystals often undergo phase transitions. In the process of phase transitions, there form regions of different phases and thin fast transitional layers around sharp surfaces, across which the nematic order parameter becomes discontinuous.

For simple fluids, the phase transitions between two mixed fluids are usually driven by the interface tensions so that the geometric shape of a sharp free interface takes the form of either an area minimizing surface or surface of constant mean curvature that minimizes areas with volume constraint of enclosed regions. This phase transition problem has been extensively studied by many authors, including Modica-Mortola [43], Modica [42], Sternberg [49], Kohn-Sternberg [39], through the technique of De Giorgi's Gamma-convergence in the framework of (scalar-valued) Allen-Cahn energy functional with double-well potentials. Fonseca-Tartar [14], Sternberg [50], and Andre-Shafrir [4] studied the gradient theory of phase transitions involving Allen-Cahn type energy functionals with potential wells of points or curves in $\mathbb{R}^2$. More recently, partly motivated by the celebrated Keller-Rubinstein-Sternberg problem [30], Lin-Pan-Wang [34] have made a systematic study of the vectorial singular perturbation problem of general high dimensional wells, see also [35].

In contrast with simple fluids, the anisotropic structure of liquid crystals implies that both elastic constants of liquid crystal materials in the nematic region and anchoring angles of nematic liquid crystal director fields on the transitional interface will play important roles in determining the shape, possible defect structures and the stability of the interfaces. These are mathematically very challenging problems. There have been numerous works in the literature, including modeling and experiments, modular simulations and numerics, on phase transitions in liquid crystals by physicists and engineers, see [45, 46]. The study of the isotropic-nematic interface based on the Ginzburg-Landau-de Gennes (LGdG) theory was initiated in a paper by de Gennes [6], where the structure of the infinite, flat isotropic-nematic interface in a uniform uniaxial ansatz solution was analyzed. In general, nematic ordering is strongly influenced by confining surfaces, which can impose and favour a particular anchoring condition (e.g. homeotropic, planar, or oblique anchoring) on the nematic state. It turns out that the relative sizes of different elastic coefficients also play important roles on anchoring conditions on the interface, see Kamil-Bhattacharjee-Adhikari-Menon [27, 28]. There are proposed forms of the surface energy by Chanderashka [5], Ericksen [11] based on a phenomenological theory. Besides some earlier works by Virga [51] and Lin-Poon [37], there is an obvious lack of mathematical understanding of these problems. In [7], Dio and Kuzuu studied the structure of the interface between the coexisting isotropic and nematic phases of a lyotropic liquid crystal, and found an explicit formula of the interfacial tensions in terms of anchoring angles and the length and diameter of the liquid crystal molecule, which favour the planar anchoring. It is known that the nematic structure in the interfacial region can differ substantially from the nematic structure in the bulk. For example, it was shown by Popa-Nita-Sluckin-Wheeler [47] a region proximate to the interface can exhibit biaxiality within the LGdG theory, even if the stable nematic phase is purely uniaxial, provided planar anchoring is enforced on the interface. Such a biaxiality is absent if the anchoring is homeotropic [6], see also [27, 28].

The Landau-Ginzburg-de Gennes model is certainly more flexible and may also be more consistent from both mathematical and physical point of views. For example, it can be derived rigorously from microscopic (molecular/kinetic) models. It can be used to describe more complex defect structures, both uniaxial and biaxial. In fact, purely uniaxial solutions are very rare in the Landau-Ginzburg-de Gennes model though in many situations they are well approximated by uniaxial ones, see Henno-Majumdar [21] and Majumdar-Zarnescu [29]. The Landau-Ginzburg-de Gennes model in principle may also lead to anisotropic surface energies. This would result in different shapes of droplets and defect patterns within them. But for the analysis of Landau-Ginzburg-de Gennes model, the complexity is formidable. If the energy density functions are quadratic in gradient of Q with coefficients that are quadratic polynomials in Q, then there are 22 invariants (and 13 surface terms) along with 4 null-Lagrangians. If one would consider additional chiral effects one may need 2 to 4 additional terms, see [20] and [22]. On the other hand, with much simplified energy functionals as considered by various authors recently, we believe that the analysis in this current paper can be applied without essential difficulties. There are very few mathematical works on the phase transitions of nematic liquid crystals in high dimensions within the LGdG theory. Let us mention in this direction the work by Park-Wang-Zhang-Zhang [48] in dimension one, and Golovaty-Novack-Sternberg-Venkatraman [16, 17] in dimension two.

We can formally derive for these simplified models that in the isotropic-nematic sharp interface transitions, the biaxial property of solutions and their defects contribute only lower orders to the total energy of the system. Thus it does not affect too much the shape of droplet, rather the detailed structure of defects that could be biaxial in nature near defects. This is one of the important reasons that in this article, we shall adopt the reduced Landau-Ginzburg-de Gennes model, or the so-called Ericksen's model for the uniaxial nematic liquid crystals of variable degrees of orientations. This model is natural and relatively simple, and mathematically it is self-contained and consistent. It can accommodate point defects, disinclinations and domain walls in liquid crystals for which rigorous analysis is possible. Moreover, it can keep the classical Oseen-Frank model, favored by many physicists, intact. Since the surface tension in such highly viscous fluids is quite large, the ratio between Frank elastic constants and surface tension is often very small compared to the size of typical droplets. It leads us to the study of the phase transition problems formulated in a form of singular perturbations for the Ericksen energy functional and consequently a sharp interface limiting problem. We will need some Γ-convergence techniques that the authors have developed for vector-valued variational problems in [34, 35]. Here the coexisting isotropic and nematic states are separated by an interface in which order parameters rise from zero on the isotropic side of the interface to saturated, non-zero values on the nematic side. The nematic regions are what we have referred as droplets, and in this way we treat nematic droplets (positioned in an isotropic liquid), and their boundaries are the isotropic-nematic interfaces within the same framework of Ericksen's model. Naturally, one can also study droplets containing isotropic liquid immersed in a volume of nematic liquid crystals. Of particular interest is that the anchoring conditions for nematic liquid crystal configurations at the boundary of droplets are intrinsically determined by the material constants, and can be derived from this sharp interface limit instead of that it needs to be assumed in phenomenological models.

1.2 Technical descriptions of main theorems

In the framework of Ericksen's theory [12] (see also [32, 33]), a nematic liquid crystal configuration is described by a pair of parameters $(s,n) : \Omega \subset \mathbb{R}^3 \mapsto [-\frac{1}{2},1] \times \mathbb{S}^2$, where $s(x)$ denotes the degree of orientation and $n(x)$ denotes the average orientation field at a point $x \in \Omega$. In particular, at a point $x \in \Omega$ molecules are perfectly aligned in the direction $n(x)$ when $s(x) = 1$, while molecules are perpendicular to $n(x)$ when $s(x) = -\frac{1}{2}$.

The Ericksen energy density function is assumed to take the form [12]

$$\mathscr{W}(s,n,\nabla s,\nabla n) = \mathscr{W}_2(s,n,\nabla s,\nabla n) + \mathscr{W}_0(s),$$

where

$$\begin{aligned}
&\mathscr{W}_2(s,n,\nabla s,\nabla n)\\
&= s^2\left[k_1(\mathrm{div} n)^2 + k_2(n\cdot \mathrm{curl} n)^2 + k_3|n\wedge \mathrm{curl} n|^2\right.\\
&\quad \left. +(k_2+k_4)(\mathrm{tr}(\nabla n)^2 - (\mathrm{div} n)^2)\right] + \alpha s^2|\nabla n|^2\\
&+\beta|\nabla s|^2 + L_1(\nabla s\cdot n)^2 + L_2|\nabla s\wedge n|^2 + L_3(\nabla s\cdot n)(s\mathrm{div} n) + L_4 s\nabla s\cdot(\nabla n)n. \qquad (1.1)
\end{aligned}$$

Here $\alpha, \beta > 0$, $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$ are Frank elasticity constants, and k_4 is another constant that, throughout this paper, is assumed to satisfy

$$k_2 \geq |k_4| \text{ and } 2k_1 \geq k_2 + k_4,$$

and $L_1 \geq 0, L_2 \geq 0, L_3, L_4$ are also constants. $\mathscr{W}_0(s)$ is a Landau type bulk potential function that dictates isotropic and nematic phases.

It follows from the fact that $|n| = 1$ and direct calculations that

$$|n \wedge \mathrm{curl} n|^2 = |(\nabla n)n|^2,\ |\nabla s|^2 = (\nabla s \cdot n)^2 + |\nabla s \wedge n|^2.$$

Hence, as in Ericksen [12] and Lin-Poon [36, 37], we can reorganize the expression of $\mathscr{W}_2$ into the form

$$\begin{aligned}\mathscr{W}_2(s,n,\nabla s,\nabla n) &= s^2\big[\bar{k}_1|\mathrm{div} n|^2 + k_2(n\cdot \mathrm{curl} n)^2 + \bar{k}_3|n\wedge \mathrm{curl} n|^2 \\ &\quad +(k_2+k_4)(\mathrm{tr}(\nabla n)^2 - (\mathrm{div} n)^2)\big] + \alpha s^2|\nabla n|^2 + (\beta - L_2)|\nabla s|^2 \\ &\quad + k_5|\nabla s - (\nabla s\cdot n)n - \nu s(\nabla n)n|^2 + k_6|\nabla s\cdot n - \sigma s \mathrm{div} n|^2,\end{aligned} \tag{1.2}$$

where

$$\begin{cases}\sigma = -\frac{L_3}{2(\beta+L_1)},\ \ \nu = -\frac{L_4}{2(\beta+L_2)},\\ \bar{k}_1 = k_1 - \sigma^2 k_6 = k_1 - \frac{L_3^2}{4(\beta+L_1)},\\ \bar{k}_3 = k_3 - \nu^2 k_5 = k_3 - \frac{L_4^2}{4(\beta+L_2)},\\ k_5 = \beta + L_2,\ \ k_6 = \beta + L_1.\end{cases}$$

Based on the physical hypothesis of positivity of the energy density [12], it is usually assumed

$$\bar{k}_1 \geq 0,\ k_2 \geq 0,\ \bar{k}_3 \geq 0,\ k_2 \geq |k_4|,\ k_5 \geq 0,\ k_6 \geq 0,\ \beta > L_2. \tag{1.3}$$

Hence $\mathscr{W}_2(s,n,\nabla s,\nabla n)$ enjoys the coercivity in sn and quadratic growth of $\nabla(sn)$:

$$\lambda(|\nabla s|^2 + s^2|\nabla n|^2) \leq \mathscr{W}_2(s,n,\nabla s,\nabla n) \leq \Lambda(|\nabla s|^2 + s^2|\nabla n|^2) \tag{1.4}$$

for two positive constants $\lambda < \Lambda < \infty$ depending only on the coefficients in (1.1).

A sharp interface forms when the size of a liquid crystal droplet is much larger than the ratio of the Frank constants to the surface tension. In order to study the sharp interface formation between the isotropic phase, corresponding to $\{s = 0\}$, and the nematic phase, corresponding to $\{s = s_+\}$ for some $s_+ \in (0,1)$, we will assume the bulk potential function takes the form:

$$\mathscr{W}_0(s) = \frac{1}{\varepsilon^2}W(s),$$

where $\varepsilon > 0$ is a small parameter representing the width of the interfacial transition region, and the potential function $W \in C^\infty((-\frac{1}{2},1))$ is assumed to be nonnegative, and there exists a unique $s_+ \in (0,1)$ such that

$$\begin{cases} W(0) = W(s_+) = 0,\ W(s) = W(s_+ - s)\ \forall 0 \leq s \leq s_+,\\ \forall \delta_1 > 0, \exists \delta_2 > 0 \text{ such that } |s| \geq \delta_1 \text{ and } |s - s_+| \geq \delta_1 \Longrightarrow W(s) \geq \delta_2,\\ \lim\limits_{s\to(-\frac{1}{2})^+} W(s) = \lim\limits_{s\to 1^-} W(s) = +\infty,\end{cases} \tag{1.5}$$

In particular, W has two minimal wells of depth zero at 0 and s_+, and

$$W(s) \approx s^2 \ \text{ for } \ |s| << 1, \ \text{ and } \ W(s) \approx (s - s_+)^2 \ \text{ for } \ |s - s_+| << 1.$$

For $s \notin [-\frac{1}{2}, 1]$, we can simply let $W(s) = +\infty$ so that W is defined for all $s \in \mathbb{R}$.

By direct calculations we have the identity

$$\begin{aligned}\operatorname{div}\big(s^2((\nabla n)n - (\operatorname{div} n)n)\big) &= s^2\big(\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2\big)\\ &\quad + 2s\nabla s\big((\nabla n)n - (\operatorname{div} n)n\big).\end{aligned} \tag{1.6}$$

Also recall the null Lagrangian property of $\operatorname{div}\big(s^2((\nabla n)n - (\operatorname{div} n)n)\big)$ (see Hardt-Kinderlehrer-Lin [23]), that is,

$$\int_\Omega \operatorname{div}\big(s^2((\nabla n)n - (\operatorname{div} n)n)\big)\,dx, \quad sn \in H^1(\Omega, \mathbb{R}^3), \tag{1.7}$$

depends only on the value of (s,n) on $\partial\Omega$.

It turns out that both (1.6) and (1.7) will play a crucial role in our study of phase transitions between the isotropic and the nematic phases. From now on, we set

$$\mathscr{W}_\varepsilon(s, n, \nabla s, \nabla n) = \mathscr{W}_2(s, n, \nabla s, \nabla n) + \frac{1}{\varepsilon^2} W(s). \tag{1.8}$$

The problem of sharp interface formations between the isotropic and nematic phases depends on the relations between the Frank constants L_1 and L_2. From

$$|\nabla s|^2 = |\nabla s \cdot n|^2 + |\nabla s \wedge n|^2, \tag{1.9}$$

we see that

$$\begin{aligned}&\beta|\nabla s|^2 + L_1(\nabla s \cdot n)^2 + L_2|\nabla s \wedge n|^2\\ &= \begin{cases}(\beta + L_2)|\nabla s|^2 + (L_1 - L_2)(\nabla s \cdot n)^2, & L_1 > L_2,\\ (\beta + L_1)|\nabla s|^2 + (L_2 - L_1)|\nabla s \wedge n|^2, & L_1 < L_2,\\ (\beta + L_1)|\nabla s|^2, & L_1 = L_2.\end{cases}\end{aligned}$$

Hence we can reduce the case $L_1 > L_2$ into the case (**A**); the case $L_1 < L_2$ into the case (**B**); and the case $L_1 = L_2$ into the case (**C**). More precisely, we have

(**A**) $\boxed{L_1 > 0 \text{ and } L_2 = 0}$. By adding the null-Lagrangian term

$$-\frac{1}{2} L_4 \operatorname{div}\big(s^2((\nabla n)n - (\operatorname{div} n)n)\big)$$

to $\mathscr{W}_\varepsilon$ and applying (1.6) and (1.7), we can convert $\mathscr{W}_\varepsilon$ into

$$\begin{aligned}\widetilde{\mathscr{W}_\varepsilon}(s, n, \nabla s, \nabla n) &= s^2\widetilde{\mathscr{W}_2}(s, n, \nabla s, \nabla n) + \beta|\nabla s|^2 + L_1|\nabla s \cdot n|^2\\ &\quad + (L_3 - L_4)(\nabla s \cdot n)(s\operatorname{div} n) + \frac{1}{\varepsilon^2} W(s),\end{aligned}$$

where

$$\widetilde{\mathscr{W}_2}(s, n, \nabla s, \nabla n) = \mathscr{W}_2(s, n, \nabla s, \nabla n) - \frac{1}{2} L_4 s^2\big(\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2\big).$$

Thus, without loss of generality, we will further assume $\boxed{L_4 = 0}$.

(B) $\boxed{L_1 = 0 \text{ and } L_2 > 0}$. By adding the null-Lagrangian term

$$\frac{1}{2}L_3 \mathrm{div}\big(s^2((\nabla n)n - (\mathrm{div} n)n)\big)$$

to $\mathscr{W}_\varepsilon$, we can convert $\mathscr{W}_\varepsilon$ into

$$\begin{aligned}\widetilde{\mathscr{W}_\varepsilon}(s,n,\nabla s,\nabla n) &= s^2\widetilde{\mathscr{W}_2}(s,n,\nabla s,\nabla n) + \beta|\nabla s|^2 + L_2|\nabla s \wedge n|^2 \\ &\quad + (L_3+L_4)(s\nabla s)(\nabla n)n + \frac{1}{\varepsilon^2}W(s),\end{aligned}$$

where

$$\widetilde{\mathscr{W}_2}(s,n,\nabla s,\nabla n) = \mathscr{W}_2(s,n,\nabla s,\nabla n) + \frac{1}{2}L_3 s^2\big(\mathrm{tr}(\nabla n)^2 - (\mathrm{div} n)^2\big).$$

Thus, without loss of generality, we will further assume $\boxed{L_3 = 0}$.

(C) $\boxed{L_1 = L_2 = 0}$ We will only consider the following two subcases:

(**C**1) $\boxed{L_3 = L_4 = 0}$. Hence $\mathscr{W}_2$ can be rewritten as

$$\begin{aligned}\widetilde{\mathscr{W}_2}(s,n,\nabla s,\nabla n) &= s^2\big[k_1(\mathrm{div} n)^2 + k_2(n\cdot \mathrm{curl} n)^2 + k_3|n\wedge \mathrm{curl} n|^2 \\ &\quad + (k_2+k_4)(\mathrm{tr}(\nabla n)^2 - (\mathrm{div} n)^2)\big] \\ &\quad + \alpha s^2|\nabla n|^2 + \beta|\nabla s|^2.\end{aligned} \tag{1.10}$$

(**C**2) $\boxed{L_4 = -L_3}$. Hence after adding the null Lagrangian term

$$L_3 \mathrm{div}\big(s^2((\mathrm{div} n)n - (\nabla n)n)\big)$$

to $\mathscr{W}_2$, we can convert $\mathscr{W}_2$ to $\widetilde{\mathscr{W}_2}$

$$\begin{aligned}&\widetilde{\mathscr{W}_2}(s,n,\nabla s,\nabla n) \\ &= s^2\big[k_1(\mathrm{div} n)^2 + k_2(n\cdot \mathrm{curl} n)^2 + k_3|n\wedge \mathrm{curl} n|^2 \\ &\quad + (k_2+k_4+L_3)(\mathrm{tr}(\nabla n)^2 - (\mathrm{div} n)^2\big] \\ &\quad + \alpha s^2|\nabla n|^2 + \beta|\nabla s|^2.\end{aligned} \tag{1.11}$$

(**C**1) yields (**C**2) by replacing $(k_2+k_4+L_3)$ by (k_2+k_4).

We would like to summarize these preliminary discussions of the assumptions on coefficients L_i's of the Ericksen energy density $\mathscr{W}_2$ into the following:

Proposition 1.1. *Assume the strong positivity of the Ericksen energy density. Then, when the Ericksen constants satisfy $L_1 > L_2$, it can be reduced to an equivalent one with $(*) : L_1 > L_2 = L_4 = 0$, and $L_3^2 < 2\alpha L_1$ (strong positivity). If, instead, $L_1 < L_2$, then the model reduces to an equivalent one with $(**) : L_2 > L_1 = L_3 = 0$, and $L_4^2 < 4\alpha L_2$ (strong positivity). Henceforth, we shall refer to the case* (**A**) *if $(*)$ is valid, and the case* (**B**) *if $(**)$ is satisfied. Finally, If $L_1 = L_2$, then we shall only consider the situation which will be referred to as the case* (**C**)*, that is, $L_1 = L_2 = L_3 = L_4 = 0$.*

Through this paper, we denote by $\mathscr{H}^2$ the two dimensional Hausdorff measure in $\mathbb{R}^3$. Define the 1-dimensional minimal connecting energy by

$$\begin{aligned}\alpha_0 = \inf\Big\{ \int_{-\infty}^{\infty} \big(\beta \dot{s}(t)^2 + W(s(t))\big)\,dt \;\big|\; s \in C^{0,1}((-\infty,\infty),\mathbb{R}), \\ s(-\infty) = 0,\ s(\infty) = s_+ \Big\}.\end{aligned} \tag{1.12}$$

It is well-known that α_0 is attained by an $\xi \in C^\infty((-\infty,\infty),\mathbb{R})$, which satisfies

$$\sqrt{\beta}\xi'(t) = \sqrt{W(\xi(t))} \text{ in } (-\infty,\infty);\ \ \xi(-\infty) = 0,\ \xi(\infty) = s_+, \tag{1.13}$$

and

$$\begin{aligned}\alpha_0 &= 2\sqrt{\beta}\int_0^{s_+} \sqrt{W(t)}\,dt \\ &= \inf\Big\{ 2\int_{-\infty}^{\infty} \sqrt{\beta W(s(t))}|\dot{s}(t)|\,dt \;\big|\; s \in C^{0,1}((-\infty,\infty),\mathbb{R}), \\ &\qquad s(-\infty) = 0,\ s(\infty) = s_+ \Big\} \\ &= \inf\Big\{ 2\int_{-a}^{a} \sqrt{\beta W(s(t))}|\dot{s}(t)|\,dt \;\big|\; s \in C^{0,1}([-a,a],\mathbb{R}), \\ &\qquad s(-a) = 0,\ s(a) = s_+ \Big\},\end{aligned} \tag{1.14}$$

for any $a > 0$.

Now we state our first theorem. It concerns the Γ-convergence of minimizers of the Ericksen energy functional

$$\begin{aligned}&\mathscr{E}(s_\varepsilon, n_\varepsilon) := \int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s,n,\nabla s,\nabla n)\,dx, \\ &\text{where } \widetilde{\mathscr{W}_\varepsilon}(s,n,\nabla s,\nabla n) = \widetilde{\mathscr{W}_2}(s,n,\nabla s,\nabla n) + \frac{1}{\varepsilon^2}W(s),\end{aligned} \tag{1.15}$$

either under well prepared Dirichlet boundary values $(t_\varepsilon, g_\varepsilon)$ when $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, or under the volume constraint for nematic region when $\Omega = \mathbb{R}^3$, as $\varepsilon \to 0$. Notice that for any fixed $\varepsilon > 0$, the existence and regularity of minimizer $(s_\varepsilon, n_\varepsilon)$ to (1.15) have been studied by Lin [32, 33], Lin-Poon [36] and Ambrosio [2, 3].

For a bounded smooth $\Omega \subset \mathbb{R}^3$, we prescribe $(t_\varepsilon, g_\varepsilon) : \partial\Omega \to \mathbb{R} \times \mathbb{S}^2$ as follows. Let $\Sigma^\pm \subset \partial\Omega$ be two disjoint, connected open subset of $\partial\Omega$ such that

i) $\partial\Sigma^\pm = \Sigma^0$ is a smooth, closed curve of $\partial\Omega$, and $\partial\Omega = \Sigma^+ \cup \Sigma^- \cup \Sigma^0$.
ii) there exists $L > 0$ such that $t_\varepsilon \in H^1\big(\partial\Omega\big)$ satisfies

$$\big\|t_\varepsilon\big\|_{L^2(\Sigma^-)} \to 0 \text{ and } \big\|t_\varepsilon - s_+\big\|_{L^2(\Sigma^+)} \to 0, \text{ as } \varepsilon \to 0, \tag{1.16}$$

$$\sup_{0<\varepsilon<1} \int_{\partial\Omega} \big(\varepsilon(|\nabla_{\tan} t_\varepsilon|^2 + \frac{1}{\varepsilon}W(t_\varepsilon)\big)\,d\mathscr{H}^2 \le L, \tag{1.17}$$

and there exists a map $\widehat{n}_\varepsilon \in H^1(\Omega,\mathbb{S}^2)$ such that $\widehat{n}_\varepsilon = g_\varepsilon$ on $\partial\Omega$, and

$$\begin{cases} \widehat{n}_\varepsilon \cdot \nu_\Gamma\big|_\Gamma = 0 & \text{under the condition } (\mathbf{A}), \\ \widehat{n}_\varepsilon \wedge \nu_\Gamma\big|_\Gamma = 0 & \text{under the condition } (\mathbf{B}), \end{cases}$$
$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega |\nabla \widehat{n}_\varepsilon|^2\,dx = 0, \tag{1.18}$$

where $\Gamma \subset \Omega$ is an area minimizing surface such that $\partial\Gamma = \Sigma^0$.

Theorem 1.2. *Assume either the condition* $(\mathbf{A})$*, or* $(\mathbf{B})$*, or* $(\mathbf{C})$ *holds. Then the following statements hold:*

i) *If* $\Omega \subset \mathbb{R}^3$ *is a bounded smooth domain and* $(t_\varepsilon, g_\varepsilon) : \partial\Omega \to \mathbb{R}\times\mathbb{S}^2$ *satisfies* (1.16)*,* (1.17) *and* (1.18)*, then*

$$\lim_{\varepsilon\to 0} \inf\Big\{ \int_\Omega \varepsilon \widetilde{\mathscr{W}_\varepsilon}(s,n,\nabla s,\nabla n)\,dx \;\Big|\; (s,n):\Omega\to\mathbb{R}\times\mathbb{S}^2,$$
$$sn \in H^1(\Omega,\mathbb{R}^3),\ (s,n)\big|_{\partial\Omega} = (t_\varepsilon,g_\varepsilon)\Big\} = \alpha_0 \mathscr{H}^2(\Gamma), \tag{1.19}$$

where $\Gamma \subset \Omega$ *is an area minimizing surface with* $\partial\Gamma = \Sigma^0$*.*

ii) *If* $\Omega = \mathbb{R}^3$*, then*

$$\lim_{\varepsilon\to 0} \inf\Big\{ \int_{\mathbb{R}^3} \varepsilon \widetilde{\mathscr{W}_\varepsilon}(s,n,\nabla s,\nabla n)\,dx \;\Big|\; (s,n):\mathbb{R}^3\to\mathbb{R}\times\mathbb{S}^2,\ sn\in H^1(\mathbb{R}^3,\mathbb{R}^3),$$
$$\big|\{x\in\mathbb{R}^3 : s(x)\geq s_+\}\big| = |B_1|\Big\} = \alpha_0\mathscr{H}^2(\partial B_1), \tag{1.20}$$

where B_1 *is a ball of radius* 1.[1]

Theorem 1.2 can be proved in the framework of Γ-convergence:

1) First, under the conditions on the coefficients L_i's and α, we can show the energy is bounded below by
$$\int_\Omega \big(\varepsilon\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon}W(s_\varepsilon)\big)\,dx,$$
which becomes a scalar-valued Allen-Cahn functional so the technique in the BV function space, as in [43], or the isoperimetric inequality in $\mathbb{R}^3$ can be employed to show it is bounded below by $\alpha_0\mathscr{H}^2(\Gamma)$.
2) Secondly, we construct a comparison map $(s_\varepsilon, n_\varepsilon)$ by letting $n_\varepsilon = \widehat{n}_\varepsilon$ and by placing an almost optimal 1-dimensional orbit $s_\varepsilon(x) = \xi_L(\frac{d_\Gamma(x)}{\varepsilon})$ in the transversal direction to Γ within $L\varepsilon$-width (with $L >> 1$), and away from this region, s_ε is made to have very small spatial variations. It turns out that the contribution of anchoring energy $\int_{\Gamma_{L\varepsilon}} |\nabla s_\varepsilon\cdot\widehat{n}_\varepsilon|^2\,dx$ or $\int_{\Gamma_{L\varepsilon}} |\nabla s_\varepsilon\wedge\widehat{n}_\varepsilon|^2\,dx$ can be made arbitrarily small.

[1] the volume constraint $|\{x\in\mathbb{R}^3, s(x)>s_+\}| = |B_1|$ can be replaced by $|\{x\in\mathbb{R}^3, s(x)>s_+\}| = \lambda$ for any $\lambda>0$. For convenience, we choose $\lambda = |B_1|$.

It follows from Theorem 1.2 that the leading order term of $\mathscr{E}(s_\varepsilon, n_\varepsilon)$ for minimizers $(s_\varepsilon, n_\varepsilon)$ is $\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma)$, so that

$$\mathscr{E}(s_\varepsilon, n_\varepsilon) = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}(\varepsilon).$$

An important question is to ask for the asymptotic behavior of $\mathscr{D}(\varepsilon)$, which is our focus in this work. For this purpose, we will need to assume that the boundary value $(t_\varepsilon, g_\varepsilon)$ provides an almost optimal transition in the fast transition area on $\partial\Omega$ across the interfacial curve Σ^0.

To describe our results, we need to introduce some notations. Set the Oseen-Frank energy density for n, with $|n| = 1$, by

$$\begin{aligned} W_{OF}(n, \nabla n) \ = \ & k_1(\mathrm{div} n)^2 + k_2(n\cdot \mathrm{curl} n)^2 + k_3|n\wedge \mathrm{curl} n|^2 \\ & +(k_2+k_4)\big(\mathrm{tr}(\nabla n)^2 - (\mathrm{div} n)^2\big). \end{aligned} \tag{1.21}$$

For $\delta > 0$, define the δ-neighborhood of $\partial\Omega$ and Γ by

$$(\partial\Omega)_\delta = \big\{x\in\overline{\Omega}:\ \mathrm{d}(x,\partial\Omega) < \delta\big\},$$

and

$$\Gamma_\delta = \big\{x\in\overline{\Omega}:\ \mathrm{d}(x,\Gamma) < \delta\big\} = \cup_{-\delta<\lambda<\delta}\Gamma(\lambda),$$

where $\Gamma(\lambda) = \big\{x\in\overline{\Omega}:\ d_\Gamma(x) = \lambda\big\}$, and $d_\Gamma(x)$ is the signed distance function of x to Γ:

$$d_\Gamma(x) = \begin{cases} -d(x,\Gamma) & x\in\Omega^-, \\ \ \ d(x,\Gamma) & x\in\Omega^+. \end{cases}$$

The following notations will be used in the proofs of later sections. Let $\Omega^\pm\subset\Omega$ be the connected components such that $\partial\Omega^\pm = \Sigma^\pm\cup\Gamma$. Set

$$\Gamma_\delta^\pm = \Gamma_\delta\cap\Omega^\pm,\ \ U_\delta^\pm = \Omega^\pm\setminus\Gamma_\delta,\ \ \Sigma_\delta^\pm = \Sigma^\pm\setminus\Gamma_\delta,\ \ Q_\delta^\pm = U_\delta^\pm\cap(\partial\Omega)_\delta,$$

$$\Omega_\delta = \Omega\setminus(\partial\Omega)_\delta,\ \ \Omega_\delta^\pm = \Omega_\delta\cap\Omega^\pm,\ \ V_\delta^\pm = \Omega_\delta^\pm\setminus\Gamma_\delta,$$

and

$$W_\delta^\pm = \Omega_\delta^\pm\cap\Gamma_\delta,\ \ O_\delta = \Gamma_\delta\setminus(W_\delta^+\cup W_\delta^-).$$

It follows from the condition (1.5) and Proposition A.4 of [34] that for any $\gamma\in(\frac{1}{2},1)$, there exist an almost minimal orbit $\xi_{\varepsilon,\gamma}\in C^\infty([-\varepsilon^{\gamma-1},\varepsilon^{\gamma-1}],\mathbb{R})$ and $C_1, C_2, C_3 > 0$ independent of ε such that

$$\begin{cases} \xi_{\varepsilon,\gamma}(-\varepsilon^{\gamma-1}) = 0,\ \xi_{\varepsilon,\gamma}(\varepsilon^{\gamma-1}) = s_+, \\ \xi_{\varepsilon,\gamma}(t) = s_+ - \xi_{\varepsilon,\gamma}(-t),\ \forall t\in(-\varepsilon^{\gamma-1},\varepsilon^{\gamma-1}), \\ \max\limits_{|t|\le\varepsilon^{\gamma-1}}\Big|\beta|\xi_{\varepsilon,\gamma}'|^2 + W(\xi_{\varepsilon,\gamma}) - 2\sqrt{\beta W(\xi_{\varepsilon,\gamma})}|\xi_{\varepsilon,\gamma}'|\Big| \le C_2 e^{-C_1\varepsilon^{\gamma-1}}, \\ \displaystyle\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}}\big(\beta|\xi_{\varepsilon,\gamma}'|^2 + W(\xi_{\varepsilon,\gamma})\big)\,d\tau \le \alpha_0 + C_2 e^{-C_1\varepsilon^{\gamma-1}}, \\ \big|\xi_{\varepsilon,\gamma}'(t)\big| \le C_2 e^{-C_1 t},\ \forall |t|\ge C_3. \end{cases} \tag{1.22}$$

We assume that

$$
\begin{cases}
t_\varepsilon(x) = \xi_{\varepsilon,\gamma}(\frac{d_\Gamma(x)}{\varepsilon}),\ x \in \Sigma^+_{\varepsilon^\gamma} \cup \Sigma^-_{\varepsilon^\gamma} = \big\{x \in \partial\Omega : -\varepsilon^\gamma \le d_\Gamma(x) \le \varepsilon^\gamma\big\}, \\
\big\|t_\varepsilon - s_+\big\|_{L^2(\Sigma^+\setminus\Gamma_{\varepsilon^\gamma})} = O(\varepsilon) \ \text{ and } \ \big\|t_\varepsilon\big\|_{L^2(\Sigma^-\setminus\Gamma_{\varepsilon^\gamma})} = O(\varepsilon), \\
\lim_{\varepsilon\to 0} \varepsilon^\gamma \int_{\partial\Omega\setminus\Gamma_{\varepsilon^\gamma}} |\nabla_{\tan} t_\varepsilon|^2 \, d\mathscr{H}^2 = 0.
\end{cases} \tag{1.23}
$$

To simplify the technical presentation, we will assume that there exists a map $g \in H^1(\partial\Omega, \mathbb{S}^2)$ such that (i) $g_\varepsilon \to g$ in $H^1(\partial\Omega)$, and (ii)[2]

$$
g_\varepsilon = g \ \text{ on } \ \partial\Omega \cap \Gamma_{\varepsilon^\gamma}, \ \text{ and } \ \big\|g_\varepsilon - g\big\|_{L^2(\partial\Omega\setminus\Gamma_{\varepsilon^\gamma})} = o_\varepsilon(1)\varepsilon^\gamma. \tag{1.24}
$$

Recall that a minimal surface S is called strictly stable, if, in addition,

$$
\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{H}^2\big(\{x + t\phi(x)\nu_S(x) : \ x \in S\}\big) > 0, \ \forall 0 \not\equiv \phi \in C_0^\infty(S), \tag{1.25}
$$

where ν_S is a unit normal vector field of S.

The main contributions of our paper concern the characterization of the $O(1)$-term, $\mathscr{D}(\varepsilon)$, in the energy expansion of $\mathscr{E}(s_\varepsilon, n_\varepsilon)$. We divide our results into two separate theorems. The first one deals the case that Ω is a bounded domain in $\mathbb{R}^3$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. Assume that Γ is a unique, strictly stable, area minimizing surface spanned by Σ^0, and the boundary values $(t_\varepsilon, g_\varepsilon)$ satisfy conditions* (1.17)*,* (1.22)*,* (1.23)*, and* (1.24)*. Let $(s_\varepsilon, n_\varepsilon)$, $0 < \varepsilon < 1$, be minimizers of $\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s, n, \nabla s, \nabla n)\, dx$, subject to the boundary condition $(s_\varepsilon, n_\varepsilon) = (t_\varepsilon, g_\varepsilon)$ on $\partial\Omega$. Then we have the following:*

A) *Under the condition* $(\mathbf{A})$*,*

$$
\mathscr{E}(s_\varepsilon, n_\varepsilon) = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}_A + o_\varepsilon(1). \tag{1.26}
$$

Here $\mathscr{D}_A$ is given by

$$
\mathscr{D}_A = \inf\Big\{E(n;\Omega^+) = s_+^2 \int_{\Omega^+} \big(W_{OF}(n, \nabla n) + \alpha|\nabla n|^2\big)\, dx\Big\} \tag{1.27}
$$

among all maps $n \in H^1(\Omega^+, \mathbb{S}^2)$ satisfying the planar anchoring condition on Γ:

$$
n = g \ \text{ on } \ \Sigma^+; \ \ n \cdot \nu_\Gamma = 0 \ \text{ on } \ \Gamma, \tag{1.28}
$$

where ν_Γ is the outward unit normal of the nematic region $\Omega^+ \equiv \big\{x \in \Omega : s(x) = s_+\big\}$.

---

[2] this assumption is technically restrictive, which plays a role in constructing a comparison director map $n_\varepsilon : \Omega \to \mathbb{S}^2$ such that its contribution to the Oseen-Frank energy on a part of transition region $\Gamma_{\varepsilon^\gamma} \cap (\partial\Omega)_{\varepsilon^\gamma}$ is of order ε^γ, see (3.6) below.

B) *Under the condition* $(\mathbf{B})$,

$$\mathscr{E}(s_\varepsilon, n_\varepsilon) = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}_B + o_\varepsilon(1). \tag{1.29}$$

Here D_B *is given by*

$$\mathscr{D}_B = \inf\Big\{E(n;\Omega^+) = s_+^2 \int_{\Omega^+} \big(W_{OF}(n,\nabla n) + \alpha|\nabla n|^2\big)\,dx\Big\} \tag{1.30}$$

among all maps $n \in H^1(\Omega^+, \mathbb{S}^2)$ *satisfying the homeotropic anchoring condition on* Γ*:*

$$n = g \ \text{ on } \ \Sigma^+; \quad n \wedge \nu_\Gamma = 0 \ \text{ on } \ \Gamma. \tag{1.31}$$

C) *Under the condition* $(\mathbf{C})$,

$$\mathscr{E}(s_\varepsilon, n_\varepsilon) = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}_C + o_\varepsilon(1). \tag{1.32}$$

Here $\mathscr{D}_C$ *is given by*

$$\mathscr{D}_C = \inf\Big\{E(n;\Omega^+) = s_+^2 \int_{\Omega^+} \big(W_{OF}(n,\nabla n) + \alpha|\nabla n|^2\big)\,dx\Big\} \tag{1.33}$$

among all maps $n \in H^1(\Omega^+, \mathbb{S}^2)$ *satisfying the free boundary condition on* Γ *and the strong anchoring condition on* Σ^+*:*

$$n(\Gamma) \subset \mathbb{S}^2; \ \ n = g \ \text{ on } \ \Sigma^+. \tag{1.34}$$

We would like to remark that the interior regularity and boundary regularity near Σ^+ of minimizing harmonic maps $n \in H^1(\Omega^+, \mathbb{S}^2)$ achieving $\mathscr{D}_A$, or $\mathscr{D}_B$, or $\mathscr{D}_C$ has been studied by Hardt-Kinderlehrer-Lin [23]. For the boundary regularity of n near the interface Γ when the isotropic Oseen-Frank energy is considered, we refer to Hardt-Lin [25] and Duzaar-Steffen [8, 9] for partially constrained or free boundary conditions, and Day-Zarnescu [10] under the planar anchoring condition.

The second one considers the entire space $\Omega = \mathbb{R}^3$.

Theorem 1.4. *Let* $(s_\varepsilon, n_\varepsilon) : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{S}^2$, $0 < \varepsilon < 1$, *be minimizers of*

$$\int_{\mathbb{R}^3} \widetilde{\mathscr{W}_\varepsilon}(s, n, \nabla s, \nabla n)\,dx,$$

subject to the constraint:

$$\big|\{x \in \mathbb{R}^3 : \ s_\varepsilon \geq s_+\}\big| = |B_1|.$$

Then the following statements hold:

A1) *Under the condition* $(\mathbf{A})$,

$$\mathscr{E}(s_\varepsilon, n_\varepsilon) = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1) + \mathscr{D}_A + o_\varepsilon(1), \tag{1.35}$$

where $\mathscr{D}_A$ *is given by*

(1.36)

$$\mathscr{D}_A = \inf\Big\{E(n;B_1) = s_+^2 \int_{B_1} \big(W_{OF}(n,\nabla n) + \alpha|\nabla n|^2\big)\,dx \ \big| \ n \in H^1\big(B_1, \mathbb{S}^2\big)\Big\},$$

subject to the planar anchoring condition:

$$n(x)\cdot x=0 \ \text{ on } \ \partial B_1. \tag{1.37}$$

B1) *Under the condition* $(\mathbf{B})$,

$$\mathscr{E}(s_\varepsilon,n_\varepsilon)=\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1)+\mathscr{D}_B+o_\varepsilon(1), \tag{1.38}$$

where D_B is given by

$$\mathscr{D}_B=\inf\Big\{E(n;B_1)=s_+^2\int_{B_1}\big(W_{OF}(n,\nabla n)+\alpha|\nabla n|^2\big)\,dx \;\big|\; n\in H^1(B_1,\mathbb{S}^2)\Big\}, \tag{1.39}$$

subject to the homeotropic anchoring condition:

$$n(x)\wedge x=0 \ \text{ on } \ \partial B_1. \tag{1.40}$$

C1) *Under the condition* $(\mathbf{C})$,

$$\mathscr{E}(s_\varepsilon,n_\varepsilon)=\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1)+o_\varepsilon(1). \tag{1.41}$$

We would like to point out that for a bounded domain $\Omega\subset\mathbb{R}^n$, while the boundary conditions imposed on $(t_\varepsilon,g_\varepsilon)$ in Theorems 1.1, 1.2, and 1.3 are physically natural, mathematically they are rather technical to describe. On the other hand, if we consider the same type problems on a compact manifold M without boundary or a torus $\mathbb{T}^n$, then the natural condition would be the volume constraint on approximate nematic regions $\{s_\varepsilon\geq s_+\}$. Hence the problem can be significantly simplified because we will have the compactness of the space (in contrast with $\mathbb{R}^n$) and avoid the technical issues arising from both the physical boundary and the boundary values.

We would like to remark that the regularity of minimizing harmonic maps in the case (B1) was studied by [23] and [24, 25]. See [10] for some work related to the boundary regularity of minimizing harmonic maps in the case (A1).

While the approach to prove Theorem 1.3 and Theorem 1.4 is based on the technique of Γ-convergence, it is very delicate to obtain the exact characterization of $O(1)$-term in the expansion of $\mathscr{E}(s_\varepsilon,n_\varepsilon)$ especially when we deal with a bounded domain Ω with physical boundary data.

1) For the construction of a sharp upper bound, we need to place an almost minimal 1-dimensional orbit $\xi_{\varepsilon,\gamma}(\frac{d_\Gamma(x)}{\varepsilon})$ in the transversal direction of Γ within the width of $O(\varepsilon^\gamma)$, which guarantees the GL energy is of $\alpha_0+o_\varepsilon(1)\varepsilon$, while we have to utilize the decay property of $\xi'_{\varepsilon,\gamma}(t)$ ensuring $\int_{\Gamma_{\varepsilon\gamma}}|\nabla s_\varepsilon\cdot n_\varepsilon|^2\,dx$ (or $\int_{\Gamma_{\varepsilon\gamma}}|\nabla s_\varepsilon\wedge n_\varepsilon|^2\,dx$) is of order $o_\varepsilon(1)$.
2) To achieve a sharp lower bound, we need to extract a sequence of sets of finite perimeters $E_\varepsilon=\big\{x\in\Omega : s_\varepsilon(x)\geq\delta_\varepsilon\big\}$ with uniformly bounded perimeters such that $\mathscr{H}^2(\partial^*E_\varepsilon\lfloor\Omega)\approx\mathscr{H}^2(\Gamma)$, $\int_{E_\varepsilon}|\nabla n_\varepsilon|^2\,dx\leq C$, and
$$\int_{\partial^*E_\varepsilon\lfloor\Omega}\Big|\frac{\nabla s_\varepsilon}{|\nabla s_\varepsilon|}\cdot n_\varepsilon\Big|^2\,d\mathscr{H}^2 \ \Big(or\int_{\partial^*E_\varepsilon\lfloor\Omega}\Big|\frac{\nabla s_\varepsilon}{|\nabla s_\varepsilon|}\wedge n_\varepsilon\Big|^2\,d\mathscr{H}^2\Big)\leq C.$$

Then we adapt the techniques from [34] and some measure theoretic arguments to show that $E_\varepsilon \to \Omega^+$, and $n_\varepsilon \to n$ in SBV(Ω) for some $n \in H^1(\Omega^+, \mathbb{S}^2)$ with $n \cdot \nu_\Gamma = 0$ (or $n \wedge \nu_\Gamma = 0$).

3) Utilize the strict stability of Γ to show that the leading order coefficients of $\frac{1}{\varepsilon}$ in both lower and upper bound estimates match up to order $o_\varepsilon(1)\varepsilon$.

When dealing with the entire space $\Omega = \mathbb{R}^3$, we observe that the approximate nematic region E_ε constitutes a minimizing sequence of sets that approach optimality in the isoperimetric inequality so that we can apply the quantitative stability theorem by Fusco-Maggi-Pratelli [15] (see also Maggi [40]) to show, after suitable translations, E_ε converges to B_1 in L^1.

Theorem 1.3 is also related to the optimal shape problem of variational problems on liquid crystal droplets previously studied by Lin-Poon [37]. More precisely, Lin and Poon [37] considered the following minimization problem

$$\inf\Big\{\int_\Omega |\nabla n|^2\,dx + \mu\mathscr{H}^{n-1}(\partial\Omega) \mid n \in H^1(\Omega,\mathbb{S}^2), n(x) = \nu_{\partial\Omega},\ |\Omega| = |B_1|\Big\}. \tag{1.42}$$

Among the class of convex domains Ω, it was shown by [37] that $(\Omega,u) = (B_1, \frac{x}{|x|})$ is a unique minimizer of (1.42). Very recently, this result was extended by Li-Wang [38] to the class of star-shaped mean convex domains in $\mathbb{R}^3$.

The paper is organized as follows. In section 2, we will establish both lower and upper bounds of $\varepsilon\mathscr{E}_\varepsilon(s_\varepsilon, n_\varepsilon)$ and prove Theorem 1.2. In section 3, we will study the bounded domain case and establish both refined lower and upper bounds for $\mathscr{E}_\varepsilon(s_\varepsilon, n_\varepsilon)$ for all three cases and then prove Theorem 1.3. In section 4, we will study the case that Ω is the entire space $\mathbb{R}^3$ and prove Theorem 1.4.

2 Proof of Theorem 1.2

In this section, we will provide a proof of Theorem 1.2. It involves (a) a concrete construction of comparison map $(s_\varepsilon, n_\varepsilon)$ in which s_ε exhibits a fast transition near Γ with energy order $\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma)$; and (b) obtain the lower bound by typical arguments of singular perturbations of functions of bounded variation.

2.1 Lower bound estimates

For either a bounded $\Omega \subset \mathbb{R}^3$ or $\Omega = \mathbb{R}^3$ itself, we assume that

$$\Lambda = \liminf_{\varepsilon\to 0} \varepsilon\mathscr{E}(s_\varepsilon, n_\varepsilon) = \liminf_{\varepsilon\to 0}\int_\Omega \varepsilon\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx < \infty. \tag{2.1}$$

It follows from the condition (1.3) that

$$W_{OF}(n_\varepsilon, \nabla n_\varepsilon) \ge 0.$$

Observe that by Cauchy-Schwarz inequality, the following properties hold:

i) If $2L_3^2 \le 4L_1\alpha$, then

$$|L_3(\nabla s_\varepsilon \cdot n_\varepsilon)s_\varepsilon \mathrm{div} n_\varepsilon| \le \sqrt{3}|L_3||\nabla s_\varepsilon \cdot n_\varepsilon||s_\varepsilon||\nabla n_\varepsilon| \le L_1(\nabla s_\varepsilon \cdot n_\varepsilon)^2 + \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2, \tag{2.2}$$

where we have used the inequality $|\mathrm{div} n_\varepsilon| \le \sqrt{2}|\nabla n_\varepsilon|$ (see [31] for a proof).

ii) If $L_4^2 \le 4L_2\alpha$, then

$$\begin{aligned}|L_4 s_\varepsilon \nabla s_\varepsilon (\nabla n_\varepsilon) n_\varepsilon| &= |L_4|\big|s_\varepsilon\big(\nabla s_\varepsilon - (\nabla s_\varepsilon\cdot n_\varepsilon)n_\varepsilon\big)(\nabla n_\varepsilon)n_\varepsilon\big| \\ &\le L_2\big|\nabla s_\varepsilon - (\nabla s_\varepsilon\cdot n_\varepsilon)n_\varepsilon\big|^2 + \alpha s_\varepsilon^2|(\nabla n_\varepsilon)n_\varepsilon|^2 \\ &\le L_2|\nabla s_\varepsilon \wedge n_\varepsilon|^2 + \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2, \end{aligned} \tag{2.3}$$

where we have used the fact that $n_\varepsilon\cdot((\nabla n_\varepsilon)n_\varepsilon) = 0$, (1.9), and $|(\nabla n_\varepsilon)n_\varepsilon| \le |\nabla n_\varepsilon|$.

Since

$$\begin{aligned}\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) &= s_\varepsilon^2 W_{OF}(n_\varepsilon,\nabla n_\varepsilon) + \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + \beta|\nabla s_\varepsilon|^2 + L_1(\nabla s_\varepsilon\cdot n_\varepsilon)^2 \\ &\quad + L_2|\nabla s_\varepsilon\wedge n_\varepsilon|^2 + L_3(\nabla s_\varepsilon\cdot n_\varepsilon)s_\varepsilon \mathrm{div} n_\varepsilon + L_4 s_\varepsilon\nabla s_\varepsilon(\nabla n_\varepsilon)n_\varepsilon + \frac{1}{\varepsilon^2}W(s_\varepsilon),\end{aligned}$$

we obtain that

i) Under the condition (**A**),

$$\begin{aligned}&\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) \\ &\ge \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + \beta|\nabla s_\varepsilon|^2 + L_1(\nabla s_\varepsilon\cdot n_\varepsilon)^2 + L_3(\nabla s_\varepsilon\cdot n_\varepsilon)s_\varepsilon\mathrm{div} n_\varepsilon + \frac{1}{\varepsilon^2}W(s_\varepsilon) \\ &\ge \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon),\end{aligned}$$

ii) Under the condition (**B**),

$$\begin{aligned}&\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) \\ &\ge \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + \beta|\nabla s_\varepsilon|^2 + L_2|\nabla s_\varepsilon\wedge n_\varepsilon|^2 + L_4 s_\varepsilon\nabla s_\varepsilon(\nabla n_\varepsilon)n_\varepsilon + \frac{1}{\varepsilon^2}W(s_\varepsilon) \\ &\ge \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon),\end{aligned}$$

iii) Under the condition (**C**),

$$\begin{aligned}\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) &\ge \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon) \\ &\ge \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon).\end{aligned}$$

Now we proceed by dividing the discussion into two separate cases:

$\Omega\subset\mathbb{R}^3$ is a bounded domain

For any $\delta > 0$, define

$$\Omega^+_{\varepsilon,\delta} = \Big\{x\in\Omega:\ |s_\varepsilon - s_+| \le \delta\Big\};\ \ \Omega^-_{\varepsilon,\delta} = \Big\{x\in\Omega:\ |s_\varepsilon| \le \delta\Big\},$$

and

$$E_{\varepsilon,\delta} = \Omega\setminus\big(\Omega^+_{\varepsilon,\delta}\cup\Omega^-_{\varepsilon,\delta}\big) = \Big\{x\in\Omega:\ |s_\varepsilon| > \delta,\ \ |s_\varepsilon - s_+| > \delta\Big\}.$$

By the condition (1.5) and Federer's co-area formula, we have that for any $0<\delta<\frac{s_+}{2}$,

$$
\begin{aligned}
\varepsilon\mathscr{E}(s_\varepsilon,n_\varepsilon) &= \int_\Omega \varepsilon\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\\
&\geq \int_\Omega \big(\varepsilon\beta|\nabla s_\varepsilon|^2+\frac{1}{\varepsilon}W(s_\varepsilon)\big)\,dx\\
&\geq 2\sqrt{\beta}\int_\Omega \sqrt{W(s_\varepsilon)}|\nabla s_\varepsilon|\,dx\\
&\geq 2\sqrt{\beta}\Big(\int_{\frac{\delta}{2}}^{\delta}\sqrt{W(\tau)}\mathscr{H}^2(\partial\Omega^-_{\varepsilon,\tau}\cap\Omega)\,d\tau\\
&\quad+\int_{s_+-\delta}^{s_+-\frac{\delta}{2}}\sqrt{W(\tau)}\mathscr{H}^2(\partial\Omega^+_{\varepsilon,\tau}\cap\Omega)\,d\tau\Big)\\
&\geq 2\sqrt{\beta}C_\delta\Big(\int_{\frac{\delta}{2}}^{\delta}\mathscr{H}^2(\partial\Omega^-_{\varepsilon,\tau}\cap\Omega)\,d\tau+\int_{s_+-\delta}^{s_+-\frac{\delta}{2}}\mathscr{H}^2(\partial\Omega^+_{\varepsilon,\tau}\cap\Omega)\,d\tau\Big).
\end{aligned}
\tag{2.4}
$$

Therefore, by Fubini's theorem there exists $\delta_*\in(\frac{\delta}{2},\delta)$ such that

$$
\mathscr{H}^2(\partial\Omega^-_{\varepsilon,\delta_*}\cap\Omega)+\mathscr{H}^2(\partial\Omega^+_{\varepsilon,\delta_*}\cap\Omega)\leq C(\beta,\Lambda,\delta).
\tag{2.5}
$$

From (1.5), we know that there exists $C_\delta>0$ such that

$$
\big|E_{\varepsilon,\delta}\big|\leq\frac{1}{C_\delta}\int_\Omega W(s_\varepsilon)\,dx\leq\frac{\Lambda\varepsilon}{C_\delta}\to 0,\ \text{as}\ \varepsilon\to 0.
\tag{2.6}
$$

From (2.5), there exist two subsets $E^\pm\subset\Omega$ with finite perimeters in Ω such that, after passing to a subsequence,

$$
\chi_{\Omega^\pm_{\varepsilon,\delta_*}}\rightharpoonup\chi_{E^\pm}\ \text{ in }\ \mathrm{BV}(\mathbb{R}^3)\ \text{ and }\ \chi_{\Omega^\pm_{\varepsilon,\delta_*}}\to\chi_{E^\pm}\ \text{ in }\ L^1(\mathbb{R}^3).
$$

This and (2.6) imply that

$$
\big|\Omega\setminus(E^+\cup E^-)\big|=\big|E^+\cap E^-\big|=0
$$

so that $\Omega=E^+\cup E^-$ (modulo a set of zero Lebesgue measure).

Define an auxiliary function $\phi:(-\frac{1}{2},1)\to\mathbb{R}$ by letting

$$
\phi(t)=2\sqrt{\beta}\int_0^t\sqrt{W(\tau)}\,d\tau,\ t\in(-\frac{1}{2},1).
$$

Notice that the 1-dimensional minimal connecting energy $\alpha_0=\phi(s_+)$. It follows from (2.4) that

$$
\begin{aligned}
\int_\Omega|\nabla(\phi(s_\varepsilon))|\,dx &\leq 2\sqrt{\beta}\int_\Omega\sqrt{W(s_\varepsilon)}|\nabla s_\varepsilon|\,dx\\
&\leq\int_\Omega\varepsilon\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\leq\Lambda+o(1).
\end{aligned}
\tag{2.7}
$$

From the boundary condition (1.16), we know that

$$
\phi(s_\varepsilon)\to 0\ \text{ in }\ L^2(\Sigma^-)\ \text{ and }\ \phi(s_\varepsilon)\to\phi(s_+)\ \text{ in }\ L^2(\Sigma^+),\ \text{ as }\ \varepsilon\to 0.
\tag{2.8}
$$

In particular, we may assume that

$$\sup_{0<\varepsilon<1}\int_{\partial\Omega}|\phi(s_\varepsilon)|^2\,d\mathscr{H}^2\le C<\infty.$$

Hence by the Poincaré inequality we have that

$$\int_\Omega|\phi(s_\varepsilon)|\,dx\le C\Big(\int_\Omega|\nabla(\phi(s_\varepsilon))|+\int_{\partial\Omega}|\phi(s_\varepsilon)|\,d\mathscr{H}^2\Big)\le C(1+\Lambda). \tag{2.9}$$

It follows from (2.7) and (2.9) that there exists $\psi\in BV(\Omega)$ such that after passing to a subsequence, $\phi(s_\varepsilon)\to\psi$ weakly in $\mathrm{BV}(\Omega)$ and strongly in $L^1(\Omega)\cap L^1(\partial\Omega)$. By the lower semicontinuity, we have that

$$\begin{aligned}|D\psi|(\Omega)&\le\liminf_{\varepsilon\to0}\int_\Omega|\nabla(\phi(s_\varepsilon))|\,dx\le\liminf_{\varepsilon\to0}\int_\Omega\big(\varepsilon|\nabla s_\varepsilon|^2+\frac{1}{\varepsilon}W(s_\varepsilon)\big)\,dx\\&\le\liminf_{\varepsilon\to0}\int_\Omega\varepsilon\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\le\Lambda.\end{aligned} \tag{2.10}$$

It follows from (2.8) that $\psi=0$ on Σ^-, and $\psi=\phi(s_+)$ on Σ^+. We claim that

$$\psi(x)=\begin{cases}0 & x\in E^-,\\ \phi(s_+) & x\in E^+.\end{cases} \tag{2.11}$$

To see this, observe by Fatou's lemma that

$$\begin{aligned}&\int_{E^-}|\psi|^2\,dx=\int_{\mathbb{R}^3}|\psi|^2\chi_{E^-}\,dx\\&\le\liminf_{\varepsilon\to0}\int_{\mathbb{R}^3}|\phi(s_\varepsilon)|^2\chi_{\Omega^-_{\varepsilon,\delta_*}}\,dx\\&=\liminf_{\varepsilon\to0}\int_{\Omega^-_{\varepsilon,\delta_*}}|\phi(s_\varepsilon)|^2\,dx\\&\le C\big\|\phi'\big\|^2_{L^\infty([-\delta_*,\delta_*])}\liminf_{\varepsilon\to0}\int_{\Omega^-_{\varepsilon,\delta_*}}|s_\varepsilon|^2\,dx\\&\le C\liminf_{\varepsilon\to0}\int_{\Omega^-_{\varepsilon,\delta_*}}W(s_\varepsilon)\,dx\le C\Lambda\varepsilon\to0,\end{aligned}$$

where we have used the fact that $W(\tau)\approx\tau^2$ for $|\tau|\le\delta_*$. Thus $\psi=0$ a.e. in E^-.

Similarly, by using the fact that $W(\tau) \approx |\tau - s_+|^2$ for $|\tau - s_+| \le \delta_*$, we can estimate

$$\begin{aligned}
&\int_{E^+} |\psi - \phi(s_+)|^2\,dx = \int_{\mathbb{R}^3} |\psi - \phi(s_+)|^2 \chi_{E^+}\,dx\\
&\le \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} |\phi(s_\varepsilon) - \phi(s_+)|^2 \chi_{\Omega^+_{\varepsilon,\delta_*}}\,dx\\
&= \liminf_{\varepsilon\to 0} \int_{\Omega^+_{\varepsilon,\delta_*}} |\phi(s_\varepsilon) - \phi(s_+)|^2\,dx\\
&\le C\|\phi'\|_{L^\infty([s_+-\delta_*, s_++\delta_*])} \liminf_{\varepsilon\to 0} \int_{\Omega^+_{\varepsilon,\delta_*}} |s_\varepsilon - s_+|^2\,dx\\
&\le C \liminf_{\varepsilon\to 0} \int_{\Omega^+_{\varepsilon,\delta_*}} W(s_\varepsilon)\,dx \le C\Lambda\varepsilon \to 0,
\end{aligned}$$

this yields that $\psi = \phi(s_+)$ a.e. in E^+.

It follows from (2.11) that

$$E^+ = \big\{x \in \Omega:\ \psi(x) \ge t\big\},\ \forall t \in (0, \alpha_0). \tag{2.12}$$

In what follows, for a subset $E \subset \mathbb{R}^3$ we denote by $[[E]]$ the corresponding 3-dimensional current (through integration), and denote by $\partial[[E]]$ the boundary current of $[[E]]$.

Since $s_\varepsilon \in H^1(\partial\Omega, (-\frac{1}{2}, 1))$ satisfies (1.16),

$$\partial\big[\big[\partial[[\Omega^+_{\varepsilon,\delta_*}]]\lfloor\,\Omega\big]\big] = [[\{x \in \partial\Omega:\ s_\varepsilon(x) = s_+ - \delta_*\}]] \to [[\Sigma^0]],\ \text{as } \varepsilon \to 0,$$

holds as weak convergence of currents, we obtain that

$$\partial\big[\big[\partial[[E^+]]\lfloor\,\Omega\big]\big] = [[\Sigma^0]]. \tag{2.13}$$

It follows from (2.12), (2.13), and the area minimality of Γ that

$$\mathscr{H}^2\big(\partial^*\big\{x \in \Omega:\ \psi(x) \ge t\big\}\lfloor\,\Omega\big) \ge \mathscr{H}^2(\Gamma),\ \forall t \in [0, \alpha_0). \tag{2.14}$$

Here $\partial^* E$ denotes the reduced boundary of a set E of finite perimeter. By the co-area formula for BV functions and (2.14), we then have

$$\begin{aligned}
\Lambda = \lim_{\varepsilon_i\to 0} \varepsilon\mathscr{E}(s_{\varepsilon_i}, n_{\varepsilon_i}) \ge |D\psi|(\Omega) &= \int_{\mathbb{R}} \mathscr{H}^2\big(\partial^*\big\{x \in \Omega:\ \psi(x) > t\big\}\lfloor\Omega\big)\,dt\\
&\ge \int_0^{\alpha_0} \mathscr{H}^2\big(\partial^*\big\{x \in \Omega:\ \psi(x) > t\big\}\lfloor\Omega\big)\,dt\\
&\ge \alpha_0 \mathscr{H}^2(\Gamma).
\end{aligned} \tag{2.15}$$

This proves the part "$\ge$" of (1.19) in Theorem 1.2, when Ω is a bounded domain in $\mathbb{R}^3$.

$\Omega = \mathbb{R}^3$ **and** $\left|\{x \in \mathbb{R}^3 :\ s_\varepsilon(x) \geq s_+\}\right| = |B_1|$

First notice that for any $0 < \delta \leq s_+$,

$$\left|\left\{x \in \mathbb{R}^3 : s_\varepsilon(x) \geq \delta\right\}\right| \geq \left|\left\{x \in \mathbb{R}^3 : s_\varepsilon(x) \geq s_+\right\}\right| \geq |B_1| \tag{2.16}$$

As in (2.7) of the previous subsection, we can obtain that for any $0 < \delta < \frac{s_+}{2}$,

$$\begin{aligned}\Lambda + o(1) &\geq \int_{\mathbb{R}^3} \varepsilon \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx \\ &\geq 2\int_{\mathbb{R}^3} \sqrt{\beta W(s_\varepsilon)} |\nabla s_\varepsilon|\, dx \\ &\geq 2\int_0^{s_+} \sqrt{\beta W(\tau)} \mathscr{H}^2\left(\partial^*\left\{x \in \mathbb{R}^3 :\ s_\varepsilon(x) \geq \tau\right\}\right) d\tau\end{aligned}$$

By the isoperimetric inequality, we have that for any $0 < \tau \leq s_+$,

$$\begin{aligned}\mathscr{H}^2\left(\partial^*\left\{x \in \mathbb{R}^3 :\ s_\varepsilon(x) \geq \tau\right\}\right) &\geq (36\pi)^{\frac{1}{3}} \left|\left\{x \in \mathbb{R}^3 :\ s_\varepsilon(x) \geq \tau\right\}\right|^{\frac{2}{3}} \\ &\geq (36\pi)^{\frac{1}{3}} |B_1|^{\frac{2}{3}}\end{aligned}$$

Hence we obtain that

$$\Lambda + o(1) \geq \alpha_0 (36\pi)^{\frac{1}{3}} |B_1|^{\frac{2}{3}} = \alpha_0 \mathscr{H}^2(\partial B_1).$$

This proves "$\geq$" of Theorem 1.2 for $\Omega = \mathbb{R}^3$. $\square$

2.2 Upper bound estimates

The upper bound estimates are based on concrete constructions, similar to that by [34]. We will first discuss the construction for a bounded domain $\Omega \subset \mathbb{R}^3$.

$\Omega \subset \mathbb{R}^3$ is a bounded domain

We need to introduce some notations. Fix a large constant $L > 0$, whose value will be determined later, we may assume for simplicity that $Q^{\pm}_{L\varepsilon} := \left(\Omega^{\pm} \setminus \Gamma_{L\varepsilon}\right) \cap (\partial\Omega)_{L\varepsilon} \approx \Sigma^{\pm}_{L\varepsilon} \times [0, L\varepsilon]$. We will construct a function $\hat{s}_\varepsilon$ in $Q^-_{L\varepsilon}$, that is a linear interpolation of of $t_\varepsilon\big|_{\Sigma^-_{L\varepsilon} \times \{0\}}$ and $0\big|_{\Sigma^-_{L\varepsilon} \times \{L\varepsilon\}}$, i.e.,

$$\hat{s}_\varepsilon(x,t) = \frac{L\varepsilon - t}{L\varepsilon} t_\varepsilon(x),\ (x,t) \in \Sigma^-_{L\varepsilon} \times [0, L\varepsilon]. \tag{2.17}$$

Similarly, a function $\hat{s}_\varepsilon$ in $Q^+_{L\varepsilon}$ is constructed by a linear interpolation of $t_\varepsilon\big|_{\Sigma^+_{L\varepsilon} \times \{0\}}$ and $s_+\big|_{\Sigma^+_{L\varepsilon} \times \{L\varepsilon\}}$, i.e.,

$$\hat{s}_\varepsilon(x,t) = \frac{L\varepsilon - t}{L\varepsilon} t_\varepsilon(x) + \frac{t}{L\varepsilon} s_+,\ (x,t) \in \Sigma^+_{L\varepsilon} \times [0, L\varepsilon]. \tag{2.18}$$

Let $\widehat{n}_\varepsilon \in H^1(\Omega,\mathbb{S}^2)$ be given by (1.18). Then, by direct calculations and applying (1.16) and (1.5), we have that

$$\int_{Q^-_{L\varepsilon}} \big(\beta|\nabla \hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx \lesssim \varepsilon\int_{\Sigma^-}\big(|\nabla t_\varepsilon|^2+\frac{1}{\varepsilon^2}W(t_\varepsilon)\big)\,d\mathscr{H}^2$$
$$+\frac{1}{\varepsilon}\int_{\Sigma^-}|t_\varepsilon|^2\,d\mathscr{H}^2, \tag{2.19}$$

and

$$\int_{Q^+_{L\varepsilon}} (\beta|\nabla \hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)) \lesssim \varepsilon\int_{\Sigma^+}(|\nabla t_\varepsilon|^2+\frac{1}{\varepsilon^2}W(t_\varepsilon))\,d\mathscr{H}^2$$
$$+\frac{1}{\varepsilon}\int_{\Sigma^+}|t_\varepsilon-s_+|^2\,d\mathscr{H}^2. \tag{2.20}$$

By Fubini's theorem, there exists $L_1\in[L,2L]$ such that

$$\begin{cases} L_1\varepsilon\displaystyle\int_{\partial\Gamma^-_{L_1\varepsilon}\cap\Omega}\big(\beta|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,d\mathscr{H}^2 \lesssim \displaystyle\int_{Q^-_{L\varepsilon}}\big(\beta|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx,\\ L_1\varepsilon\displaystyle\int_{\partial\Gamma^+_{L_1\varepsilon}\cap\Omega}\big(\beta|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,d\mathscr{H}^2 \lesssim \displaystyle\int_{Q^+_{L\varepsilon}}\big(\beta|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx. \end{cases} \tag{2.21}$$

It follows from the regularity theorem of area minimizing surfaces (see [13] and [26]) that $\Gamma\in C^\infty(\overline{\Omega})$. Let $\xi\in C^\infty\big([-L_1,L_1]\big)$ be an almost minimal 1-dimensional connecting orbit, i.e., $\xi(-L_1)=0$, $\xi(L_1)=s_+$, and

$$\int_{-L_1}^{L_1}\big(\beta\dot{\xi}^2+W(\xi)\big)\,d\tau=\alpha_0+o_L(1),\ \lim_{L\to\infty}o_L(1)=0. \tag{2.22}$$

Define $\hat{s}_\varepsilon:\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}\to\mathbb{R}$ by letting

$$\hat{s}_\varepsilon(x)=\xi\big(\frac{d_\Gamma(x)}{\varepsilon}\big),\ \ x\in\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}. \tag{2.23}$$

By the co-area formula and the fact that $|\nabla d_\Gamma(x)|=1$ for $x\in\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}$, we can estimate

$$\begin{aligned}
&\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\big(\beta|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx\\
&=\frac{1}{\varepsilon^2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\big(\beta\dot{\xi}^2+W(\xi)\big)\big(\frac{d_\Gamma(x)}{\varepsilon}\big)\,dx\\
&=\frac{1}{\varepsilon}\int_{-L_1}^{L_1}\big(\beta\dot{\xi}^2+W(\xi)\big)(\tau)\mathscr{H}^2\big(\big\{x\in\Omega_{L\varepsilon}:\ d_\Gamma(x)=\varepsilon\tau\big\}\big)\,d\tau\\
&\le\frac{1}{\varepsilon}(\mathscr{H}^2(\Gamma)+o_\varepsilon(1))\int_{-L_1}^{L_1}\big(\beta\dot{\xi}^2+W(\xi)\big)\,d\tau\\
&\le\frac{1}{\varepsilon}(\mathscr{H}^2(\Gamma)+o_\varepsilon(1))(\alpha_0+o_L(1)),
\end{aligned} \tag{2.24}$$

where we have the fact that the surface $\{x \in \Omega : d_\Gamma(x) = \tau\}$ converges to Γ in C^2-norm, as $\tau \to 0$. Hence it holds that

$$\mathscr{H}^2\big(\{x \in \Omega_{L\varepsilon} : d_\Gamma(x) = \varepsilon\tau\}\big) \le \mathscr{H}^2(\Gamma) + o_\varepsilon(1),\ -L_1 < \tau < L_1,$$

where $\lim_{\varepsilon\to 0} o_\varepsilon(1) = 0$. It is not hard to check that

$$\int_{\partial\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}} \big(\beta|\nabla_{\tan}\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\, d\mathscr{H}^2 \le \frac{C}{\varepsilon}(\alpha_0 + o_L(1)). \tag{2.25}$$

In the regions $\Omega^\pm_{L\varepsilon} \setminus \Gamma_{L_1\varepsilon}$, we simply define

$$\hat{s}_\varepsilon = 0 \text{ in } \Omega^-_{L\varepsilon} \setminus \Gamma_{L_1\varepsilon};\ \ \hat{s}_\varepsilon = s_+ \text{ in } \Omega^+_{L\varepsilon} \setminus \Gamma_{L_1\varepsilon}, \tag{2.26}$$

so that

$$\int_{\Omega^\pm_{L\varepsilon}\setminus\Gamma_{L_1\varepsilon}} \big(\beta|\nabla\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\, dx = 0. \tag{2.27}$$

It remains to construct $\hat{s}_\varepsilon$ in the region $(\partial\Omega)_{L\varepsilon} \cap \Gamma_{L_1\varepsilon}$, which can be shown to be bi-Lipschitz equivalent to a ball of radius $L_1\varepsilon$ centered at $x_* \in \Omega$, with Lipshitz norms independent of ε. Hence we can do a homogeneous of degree zero extension of $\hat{s}_\varepsilon$ with respect to the center x_*, i.e.,

$$\hat{s}_\varepsilon(x) = \hat{s}_\varepsilon\big(L_1\varepsilon\frac{x - x_*}{|x - x_*|}\big),\ x \in (\partial\Omega)_{L\varepsilon} \cap \Gamma_{L_1\varepsilon}. \tag{2.28}$$

Hence by (2.21), (2.25), (2.19), and (2.20), we have that

$$\begin{aligned}
&\int_{(\partial\Omega)_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}} \big(\beta|\nabla\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\, dx \\
&\le L\varepsilon \int_{\partial[(\partial\Omega)_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}]} \big(\beta|\nabla_{\tan}\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\, d\mathscr{H}^2 \\
&= L\varepsilon\Big\{\int_{\partial\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}} + \int_{\partial\Omega\cap\Gamma_{L_1\varepsilon}} + \int_{\partial\Gamma^-_{L_1\varepsilon}\cap\Omega} \\
&\quad + \int_{\partial\Gamma^+_{L_1\varepsilon}\cap\Omega}\Big\}\big(\beta|\nabla_{\tan}\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\, d\mathscr{H}^2 \\
&\le CL(\alpha_0 + o_L(1)) + \Big\{\int_{Q^-_{L\varepsilon}} + \int_{Q^+_{L\varepsilon}}\Big\}\big(\beta|\nabla\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\, dx \\
&\quad + CL\varepsilon\int_{\partial\Omega\cap\Gamma_{L_1\varepsilon}} \big(\beta|\nabla_{\tan}t_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(t_\varepsilon)\big)\, d\mathscr{H}^2 \\
&\le CL + CL\varepsilon\int_{\partial\Omega} \big(\beta|\nabla_{\tan}t_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(t_\varepsilon)\big)\, d\mathscr{H}^2 \\
&\quad + \frac{C}{\varepsilon}\Big(\int_{\Sigma^-}|t_\varepsilon|^2\, d\mathscr{H}^2 + \int_{\Sigma^+}|t_\varepsilon - s_+|^2\, d\mathscr{H}^2\Big) \\
&\le C + \frac{o_\varepsilon(1)}{\varepsilon},
\end{aligned} \tag{2.29}$$

where we have used the conditions (1.16) and (1.17) in the last step.

Finally, by putting together (2.17), (2.18), (2.23), (2.26), and (2.28), we find an extension map $\hat{s}_\varepsilon : \Omega \to \mathbb{R}$. Furthermore, by the estimates (2.19), (2.20), (2.24), (2.27), (2.29), we see that $\hat{s}_\varepsilon$ satisfies the estimate:

$$\begin{aligned}\int_\Omega \big(\beta|\nabla \hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx \le\ & \frac{1}{\varepsilon}(\mathscr{H}^2(\Gamma)+o_\varepsilon(1))(\alpha_0+o_L(1))\\ &+C+\frac{o_\varepsilon(1)}{\varepsilon}.\end{aligned}\tag{2.30}$$

It follows from (1.18) that

$$\int_\Omega \hat{s}_\varepsilon^2(W_{OF}(\widehat{n}_\varepsilon,\nabla\widehat{n}_\varepsilon)+\alpha|\nabla\widehat{n}_\varepsilon|^2)\,dx \le Cs_+^2\int_\Omega |\nabla\widehat{n}_\varepsilon|^2\,dx \le \frac{o_\varepsilon(1)}{\varepsilon}.\tag{2.31}$$

To estimate the contributions from the terms involving the interactive energies between $\nabla\hat{s}_\varepsilon$ and $\widehat{n}_\varepsilon$, we proceed as follows:

i) Under the condition $(\mathbf{A})$, we can estimate

$$\begin{aligned}&\Big|\int_\Omega \big(L_1(\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon)^2 + L_3(\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon)(\hat{s}_\varepsilon \mathrm{div}\widehat{n}_\varepsilon)\big)\,dx\Big|\\ &\le C\Big(\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}} |\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon|^2\,dx + \int_{\Omega\setminus\Omega_{L\varepsilon}}|\nabla\hat{s}_\varepsilon|^2\,dx + \int_\Omega|\nabla\widehat{n}_\varepsilon|^2\,dx\Big).\end{aligned}$$

Notice that (2.19), (2.20), and (2.29) imply

$$\begin{aligned}\int_{\Omega\setminus\Omega_{L\varepsilon}}|\nabla\hat{s}_\varepsilon|^2\,dx \le\ & CL\varepsilon\int_{\partial\Omega}|\nabla t_\varepsilon|^2\,d\mathscr{H}^2\\ &+C\varepsilon^{-1}\Big(\int_{\Sigma^-}|t_\varepsilon|^2\,d\mathscr{H}^2+\int_{\Sigma^+}|t_\varepsilon-s_+|^2\,d\mathscr{H}^2\Big)\\ \le\ & C+\frac{o_\varepsilon(1)}{\varepsilon}.\end{aligned}$$

Since $|\dot{\xi}(t)|\le C_L$ in $t\in[-L,L]$ and $\widehat{n}_\varepsilon\cdot\nu_\Gamma=0$ on Γ, we can estimate

$$\begin{aligned}&\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}|\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon|^2\,dx = \frac{1}{\varepsilon^2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\dot{\xi}^2\Big(\frac{d_\Gamma(x)}{\varepsilon}\Big)(\nabla d_\Gamma(x)\cdot\widehat{n}_\varepsilon(x))^2\,dx\\ &\le C\varepsilon^{-2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}(\nabla d_\Gamma(x)\cdot\widehat{n}_\varepsilon(x))^2\,dx\\ &= C\varepsilon^{-2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}(\nabla d_\Gamma(x)\cdot\widehat{n}_\varepsilon(x)-\nabla d_\Gamma(\Pi_\Gamma(x))\cdot\widehat{n}_\varepsilon(\Pi_\Gamma(x)))^2\,dx\\ &\le C\varepsilon^{-2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\big(|\nabla d_\Gamma(x)-\nabla d_\Gamma(\Pi_\Gamma(x))|^2+|\widehat{n}_\varepsilon(x)-\widehat{n}_\varepsilon(\Pi_\Gamma(x))|^2\big)\,dx\\ &\le C\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\big(|\nabla^2 d_\Gamma(x)|^2+|\nabla\widehat{n}_\varepsilon(x)|^2\big)\,dx\\ &\le C\varepsilon+\frac{o_\varepsilon(1)}{\varepsilon},\end{aligned}\tag{2.32}$$

where $\Pi_\Gamma : \Gamma_{L_1\varepsilon} \to \Gamma$ is the (smooth) nearest point projection. Hence we have that

$$\Big|\int_\Omega \big(L_1(\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon)^2 + L_3(\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon)(\hat{s}_\varepsilon \mathrm{div}\widehat{n}_\varepsilon)\big)\,dx\Big| \le C(1+\varepsilon) + \frac{o_\varepsilon(1)}{\varepsilon}. \tag{2.33}$$

Adding (2.30), (2.31), and (2.33) together, we arrive at

$$\varepsilon\int_\Omega \widetilde{\mathscr{W}}(\hat{s}_\varepsilon,\widehat{n}_\varepsilon,\nabla\hat{s}_\varepsilon,\nabla\widehat{n}_\varepsilon)\,dx \le (\alpha_0+o_L(1))(\mathscr{H}^2(\Gamma)+o_\varepsilon(1)) + C\varepsilon + o_\varepsilon(1). \tag{2.34}$$

After first sending $\varepsilon\to 0$ and then $L\to\infty$, and using the minimality of $(s_\varepsilon, n_\varepsilon)$ we have that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_\Omega \varepsilon\widetilde{\mathscr{W}}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\\ &\le \limsup_{\varepsilon\to 0}\int_\Omega \varepsilon\widetilde{\mathscr{W}}(\hat{s}_\varepsilon,\widehat{n}_\varepsilon,\nabla\hat{s}_\varepsilon,\nabla\widehat{n}_\varepsilon)\,dx \le \alpha_0\mathscr{H}^2(\Gamma).\end{aligned} \tag{2.35}$$

ii) Under the condition $(\mathbf{B})$, we can bound

$$\begin{aligned}&\Big|\int_\Omega\big(L_2|\nabla\hat{s}_\varepsilon\wedge\widehat{n}_\varepsilon|^2 + L_4\hat{s}_\varepsilon\nabla\hat{s}_\varepsilon\cdot(\nabla\widehat{n}_\varepsilon)\widehat{n}_\varepsilon\big)\,dx\Big|\\ &\le C\Big(\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}|\nabla\hat{s}_\varepsilon\wedge\widehat{n}_\varepsilon|^2\,dx + \int_{\Omega\setminus\Omega_{L\varepsilon}}|\nabla\hat{s}_\varepsilon|^2\,dx + \int_\Omega|\nabla\widehat{n}_\varepsilon|^2\,dx\Big)\\ &\le C(1+\varepsilon) + \frac{o_\varepsilon(1)}{\varepsilon} + C\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}|\nabla\hat{s}_\varepsilon\wedge\widehat{n}_\varepsilon|^2\,dx.\end{aligned}$$

Since $\widehat{n}_\varepsilon\wedge\nu_\Gamma = 0$ on Γ, the last term in the right hand side can be estimated as follows.

$$\begin{aligned}&\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}|\nabla\hat{s}_\varepsilon\wedge\widehat{n}_\varepsilon|^2\,dx = \frac{1}{\varepsilon^2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\dot{\xi}^2\Big(\frac{d_\Gamma(x)}{\varepsilon}\Big)|\nabla d_\Gamma(x)\wedge\widehat{n}_\varepsilon(x)|^2\,dx\\ &\le C\varepsilon^{-2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}|\nabla d_\Gamma(x)\wedge\widehat{n}_\varepsilon(x)|^2\,dx\\ &= C\varepsilon^{-2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}|\nabla d_\Gamma(x)\wedge\widehat{n}_\varepsilon(x) - \nabla d_\Gamma(\Pi_\Gamma(x))\wedge\widehat{n}_\varepsilon(\Pi_\Gamma(x))|^2\,dx\\ &\le C\varepsilon^{-2}\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\big(|\nabla d_\Gamma(x)-\nabla d_\Gamma(\Pi_\Gamma(x))|^2 + |\widehat{n}_\varepsilon(x)-\widehat{n}_\varepsilon(\Pi_\Gamma(x))|^2\big)\,dx\\ &\le C\int_{\Omega_{L\varepsilon}\cap\Gamma_{L_1\varepsilon}}\big(|\nabla^2 d_\Gamma(x)|^2 + |\nabla\widehat{n}_\varepsilon(x)|^2\big)\,dx\\ &\le C\varepsilon + \frac{o_\varepsilon(1)}{\varepsilon},\end{aligned}$$

which yields that

$$\Big|\int_\Omega\big(L_2|\nabla\hat{s}_\varepsilon\wedge\widehat{n}_\varepsilon|^2 + L_4\hat{s}_\varepsilon\nabla\hat{s}_\varepsilon\cdot(\nabla\widehat{n}_\varepsilon)\widehat{n}_\varepsilon\big)\,dx\Big| \le C(1+\varepsilon) + \frac{o_\varepsilon(1)}{\varepsilon}. \tag{2.36}$$

Adding (2.30), (2.31), and (2.36) together yields (2.34). This, combined with the minimality of $(s_\varepsilon, n_\varepsilon)$, implies

$$\limsup_{\varepsilon\to 0}\int_\Omega \varepsilon\widetilde{\mathscr{W}}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx \le \alpha_0\mathscr{H}^2(\Gamma). \tag{2.37}$$

iii) Under the condition $(\mathbf{C})$, it is readily seen that (2.37) follows directly from (2.30) and (2.31).

Therefore the "$\le$" part of Theorem 1.2 is proven, when Ω is a bounded domain in $\mathbb{R}^3$. □

$\Omega = \mathbb{R}^3$ **and** $\left|\{x\in\mathbb{R}^3:\ s_\varepsilon(x)\ge s_+\}\right| = |B_1|$

The construction for the upper bound estimates for $\Omega = \mathbb{R}^3$ is rather simple. Here we sketch it as follows.

For a sufficiently large $L>0$, let $\xi_L \in C^\infty\big([-L,L]\big)$ be an almost 1-dimensional minimal connecting orbit, i.e., $\xi_L(-L)=s_+$, $\xi_L(L)=0$, and

$$\int_{-L}^{L}\big(\beta\dot{\xi}_L^2 + W(\xi_L)\big)\,d\tau = \alpha_0 + o_L(1). \tag{2.38}$$

Define $\hat{s}_\varepsilon : \mathbb{R}^3 \to \mathbb{R}_+$ by letting

$$\hat{s}_\varepsilon(x) = \begin{cases} s_+ & |x|\le 1,\\ \xi_L\big(\frac{|x|-(1+L\varepsilon)}{\varepsilon}\big) & 1\le |x|\le 1+2L\varepsilon,\\ 0 & |x|\ge 1+2L\varepsilon.\end{cases}$$

Notice that

$$\big\{x\in\mathbb{R}^3 : \hat{s}_\varepsilon(x)\ge s_+\big\} = B_1.$$

Direct calculations imply that

$$\int_{\mathbb{R}^3}\big(|\nabla\hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx \le \frac{1}{\varepsilon}(\mathscr{H}^2(\Gamma)+C\varepsilon)(\alpha_0+o_\varepsilon(1)). \tag{2.39}$$

Next, we will construct a map $\widehat{n}_\varepsilon \in H^1(B_1,\mathbb{S}^2)$ as follows:

i) Under the condition $(\mathbf{A})$, it is well-known that there exists a map $\widehat{n}_\varepsilon : B_{1+2L\varepsilon} \to \mathbb{S}^2$ such that $\widehat{n}_\varepsilon(x)\cdot x = 0$ on ∂B_1, and

$$\int_{B_{1+2L\varepsilon}} |\nabla\widehat{n}_\varepsilon|^2\,dx \le C(L).$$

Hence

$$\int_{\mathbb{R}^3}\hat{s}_\varepsilon^2\big(W_{OF}(\widehat{n}_\varepsilon,\nabla\widehat{n}_\varepsilon)+\alpha|\nabla\widehat{n}_\varepsilon|^2\big)\,dx \le C(L)s_+^2. \tag{2.40}$$

While

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\big(L_1(\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon)^2+L_3(\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon)(\hat{s}_\varepsilon\mathrm{div}\widehat{n}_\varepsilon)\big)\,dx\Big|\\
&\le C\Big(\int_{B_{1+2L\varepsilon}\setminus B_1}|\nabla\hat{s}_\varepsilon\cdot\widehat{n}_\varepsilon|^2\,dx+\int_{B_{1+2L\varepsilon}}|\nabla\widehat{n}_\varepsilon|^2\,dx\Big)\\
&\le C(L)+C\varepsilon^{-2}\int_{B_{1+2L\varepsilon}\setminus B_1}\Big|\widehat{n}_\varepsilon(x)-\widehat{n}_\varepsilon\Big(\frac{x}{|x|}\Big)\Big|^2\,dx\\
&\le C(L)+C\int_{B_{1+2L\varepsilon}\setminus B_1}\big|\nabla\widehat{n}_\varepsilon\big|^2\,dx\le C(L).
\end{aligned}
$$

Therefore we arrive at

$$
\begin{aligned}
\limsup_{\varepsilon\to0}\int_{\mathbb{R}^3}\varepsilon\widetilde{\mathscr{W}}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx &\le \limsup_{\varepsilon\to0}\int_{\mathbb{R}^3}\varepsilon\widetilde{\mathscr{W}}(\hat{s}_\varepsilon,\widehat{n}_\varepsilon,\nabla\hat{s}_\varepsilon,\nabla\widehat{n}_\varepsilon)\,dx\\
&\le \alpha_0\mathscr{H}^2(\partial B_1). \qquad (2.41)
\end{aligned}
$$

ii) Under the condition $(\mathbf{B})$, we simply let

$$
\widehat{n}_\varepsilon(x)=\frac{x}{|x|},\ x\in B_{1+2L\varepsilon}.
$$

Then it is straightforward to check that

$$
\int_{\mathbb{R}^3}\hat{s}_\varepsilon^2\big(W_{OF}(\widehat{n}_\varepsilon,\nabla\widehat{n}_\varepsilon)+\alpha|\nabla\widehat{n}_\varepsilon|^2\big)\,dx\le Cs_+^2L. \qquad (2.42)
$$

While

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\big(L_2|\nabla\hat{s}_\varepsilon\wedge\widehat{n}_\varepsilon|^2+L_4\hat{s}_\varepsilon\nabla\hat{s}_\varepsilon\cdot(\nabla\widehat{n}_\varepsilon)\widehat{n}_\varepsilon\big)\,dx\Big|\\
&\le C\varepsilon^{-2}\int_{B_{1+2L\varepsilon}\setminus B_1}\Big|\frac{x}{|x|}\wedge\frac{x}{|x|}\Big|^2\,dx=0. \qquad (2.43)
\end{aligned}
$$

Hence (2.41) holds.

iii) Under the condition $(\mathbf{C})$, we simply set $\widehat{n}_\varepsilon\equiv(0,0,1)\in\mathbb{S}^2$. In this case, it is easy to see that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\varepsilon\widetilde{\mathscr{W}}(\hat{s}_\varepsilon,\widehat{n}_\varepsilon,\nabla\hat{s}_\varepsilon,\nabla\widehat{n}_\varepsilon)\,dx &= \int_{B_{1+2L\varepsilon}\setminus B_1}\big(\varepsilon|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon}W(\hat{s}_\varepsilon)\big)\,dx\\
&\le \alpha_0(\mathscr{H}^2(\partial B_1)+C\varepsilon).
\end{aligned}
$$

Hence (2.41) holds.

The "$\le$" part of Theorem 1.2 is proven, when $\Omega=\mathbb{R}^3$. Combining these two subsections, we prove Theorem 1.2. $\square$

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3 for the case that Ω is a bounded smooth domain in $\mathbb{R}^3$. It involves refined estimates of both upper bounds and lower bounds of the total energy $\mathscr{E}(s_\varepsilon, n_\varepsilon)$ for minimizers $(s_\varepsilon, n_\varepsilon)$, in which the strict stability of Γ plays a crucial role in a perfect matching of the coefficients of leading order term or $O(\frac{1}{\varepsilon})$ term in the expansion of $\mathscr{E}(s_\varepsilon, n_\varepsilon)$.

3.1 Refined energy upper bounds

In this subsection, we will prove a sharp upper bound for the energy $\mathscr{E}(s_\varepsilon, n_\varepsilon)$ of minimizers $(s_\varepsilon, n_\varepsilon)$. This is done by utilizing the additional assumption on the boundary value $(t_\varepsilon, g_\varepsilon)$ to construct a comparison map such that s_ε is approximately a minimal connecting orbit in the transition region of Γ of width of $O(\varepsilon^\gamma)$, and n_ε is approximately a minimizing harmonic map in the corresponding configuration spaces in Ω_+.

We divide the estimates of refined upper bounds for the cases (**A**), (**B**), and (**C**) into three separate Lemmas.

Lemma 3.1. *Assume Γ and the boundary values $(t_\varepsilon, g_\varepsilon)$ satisfy the same assumptions as in Theorem 1.3. Under the condition* $(\mathbf{A})$*, it holds that*

$$\begin{aligned}
&\inf\Big\{ \int_\Omega \mathscr{W}_\varepsilon(s,n,\nabla s,\nabla n)\,dx \;\Big|\; (s_\varepsilon, n_\varepsilon) = (t_\varepsilon, g_\varepsilon) \text{ on } \partial\Omega \Big\} \\
&\le \frac{1}{\varepsilon}\int_0^{s_+} \sqrt{2\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau + \mathscr{D}_A + o_\varepsilon(1),
\end{aligned} \tag{3.1}$$

where $\mathscr{D}_A$ is given by (1.27) *and* (1.28) *of Theorem 1.3.*

Proof. We first construct an extension of s_ε from $\partial\Omega$ to Ω as follows. For $\gamma \in (0,1)$, let $\xi_{\varepsilon,\gamma} \in C^\infty([-\varepsilon^{\gamma-1}, \varepsilon^{\gamma-1}], \mathbb{R})$ be given by (1.22) and $t_\varepsilon : \partial\Omega \to \mathbb{R}$ satisfy (1.23). Define s_ε in the fast transition region $\Gamma_{\varepsilon^\gamma}$ by letting

$$s_\varepsilon(x) = \xi_{\varepsilon,\gamma}\Big(\frac{d_\Gamma(x)}{\varepsilon}\Big),\ \forall x \in \Gamma_{\varepsilon^\gamma}.$$

In the off-transition region $\Omega^+ \setminus \Gamma_{\varepsilon^\gamma}$, we perform a linear interpolation between s_ε and s_+ in a ε^γ-neighborhood of Σ^+. More precisely, decompose

$$\begin{aligned}
\Omega^\pm \setminus \Gamma_{\varepsilon^\gamma} &= \Big((\Omega^\pm \setminus \Gamma_{\varepsilon^\gamma}) \cap \{x \in \Omega : d(x,\Sigma^\pm) < \varepsilon^\gamma\}\Big) \\
&\quad \cup \Big((\Omega^\pm \setminus \Gamma_{\varepsilon^\gamma}) \setminus \{x \in \Omega : d(x,\Sigma^\pm) < \varepsilon^\gamma\}\Big) \\
&= E^\pm_{\varepsilon^\gamma} \cup F^\pm_{\varepsilon^\gamma}.
\end{aligned}$$

Define

$$s_\varepsilon(x) = \begin{cases} s_+, & x \in F^+_{\varepsilon^\gamma}, \\ \text{linear interpolation of } s_\varepsilon\big|_{\Sigma^+\setminus\Gamma_{\varepsilon^\gamma}} \text{ and } s_+\big|_{\{x\in\Omega^+\setminus\Gamma_{\varepsilon^\gamma}: d(x,\Sigma^+)=\varepsilon^\gamma\}}, & x \in E^+_{\varepsilon^\gamma}. \end{cases}$$

Similarly, define

$$s_\varepsilon(x) = \begin{cases} 0, & x \in F^-_{\varepsilon^\gamma}, \\ \text{linear interpolation of } \ s_\varepsilon\big|_{\Sigma^-\setminus\Gamma_{\varepsilon^\gamma}} \text{ and } 0\big|_{\{x\in\Omega^-\setminus\Gamma_{\varepsilon^\gamma}: d(x,\Sigma^-)=\varepsilon^\gamma\}}, & x \in E^-_{\varepsilon^\gamma}. \end{cases}$$

Then by the co-area formula and direct calculations (see Maggi [41]) we can estimate

$$\begin{aligned}
&\int_\Omega \big(\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)\big)\,dx \\
&= \Big\{\int_{\Gamma_{\varepsilon^\gamma}} + \int_{E^-_{\varepsilon^\gamma}} + \int_{E^+_{\varepsilon^\gamma}}\Big\}\big(\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)\big)\,dx \\
&\le \frac{1}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}} \big(\beta|\xi'_{\varepsilon,\gamma}(t)|^2 + W(\xi_{\varepsilon,\gamma}(t))\big)\mathscr{H}^2(\Gamma(\varepsilon t))\,dt \\
&\quad + C\varepsilon^\gamma \int_{\partial\Omega\setminus\Gamma_{\varepsilon^\gamma}} |\nabla_{\tan} t_\varepsilon|^2\,d\mathscr{H}^2 \\
&\quad + C\varepsilon^{-\gamma}\Big(\int_{\Sigma^+\setminus\Gamma_{\varepsilon^\gamma}} |t_\varepsilon - s_+|^2\,d\mathscr{H}^2 + \int_{\Sigma^-\setminus\Gamma_{\varepsilon^\gamma}} |t_\varepsilon|^2\,d\mathscr{H}^2\Big) \\
&\quad + C\int_{E^+_{\varepsilon^\gamma}\cup E^-_{\varepsilon^\gamma}} \frac{1}{\varepsilon^2}W(s_\varepsilon)\,dx \\
&\le \frac{1}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}} \big(\beta|\xi'_{\varepsilon,\gamma}(t)|^2 + W(\xi_{\varepsilon,\gamma}(t))\big)\mathscr{H}^2(\Gamma(\varepsilon t))\,dt + C\varepsilon^\gamma + o_\varepsilon(1).
\end{aligned} \tag{3.2}$$

Here we have applied (1.5) and (1.23) in the last step, which ensures

$$\begin{aligned}
\int_{E^+_{\varepsilon^\gamma}\cup E^-_{\varepsilon^\gamma}} \frac{1}{\varepsilon^2}W(s_\varepsilon)\,dx &\le C\varepsilon^\gamma\Big(\int_{\Sigma^+\setminus\Gamma_{\varepsilon^\gamma}} \frac{|t_\varepsilon - s_+|^2}{\varepsilon^2}\,d\mathscr{H}^2 + \int_{\Sigma^-\setminus\Gamma_{\varepsilon^\gamma}} \frac{|t_\varepsilon|^2}{\varepsilon^2}\,d\mathscr{H}^2\Big) \\
&\le C\varepsilon^\gamma.
\end{aligned}$$

Notice that the condition (1.22) implies that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}} \big(\beta|\xi'_{\varepsilon,\gamma}(t)|^2 + W(\xi_{\varepsilon,\gamma}(t))\big)\mathscr{H}^2(\Gamma(\varepsilon t))\,dt \\
&\le \frac{1}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}} \sqrt{2\beta W(\xi_{\varepsilon,\gamma}(t))}|\xi'_{\varepsilon,\gamma}(t)|\mathscr{H}^2(\Gamma(\varepsilon t))\,dt + C_2\varepsilon^{\gamma-2}e^{-C_1\varepsilon^{\gamma-1}} \\
&\le \frac{1}{\varepsilon}\int_0^{s_+} \sqrt{2\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi^{-1}_{\varepsilon,\gamma}(\tau))\,d\tau + o_\varepsilon(1).
\end{aligned} \tag{3.3}$$

Next we want to construct an extension map $n_\varepsilon : \Omega \to \mathbb{S}^2$ from $g_\varepsilon : \partial\Omega \to \mathbb{S}^2$. To do it, let $n \in H^1(\Omega^+, \mathbb{S}^2)$ achieve $\mathscr{D}_A$, i.e., $n = g$ on Σ^+ and $n \cdot \nu_\Gamma = 0$ on Γ, and

$$E(n;\Omega^+) = \mathscr{D}_A.$$

Recall that there exists a map $\widehat{n}_\varepsilon$ in the region $\Omega^+ \setminus \big(\Gamma_{2\varepsilon^\gamma} \cup \Omega^+_{\varepsilon^\gamma}\big)$ that is a linear interpolation between $g_\varepsilon\big|_{\Sigma^+\setminus\Gamma_{2\varepsilon^\gamma}}$ and $n\big|_{\partial\Omega^+_{\varepsilon^\gamma}\setminus\Gamma_{2\varepsilon^\gamma}}$, i.e.,

$$\widehat{n}_\varepsilon(r,\theta) = \frac{r}{\varepsilon^\gamma} g_\varepsilon(\theta) + \frac{\varepsilon^\gamma - r}{\varepsilon^\gamma} n(\varepsilon^\gamma, \theta),$$

where

$$x = (r,\theta) \in (\Sigma^+ \setminus \Gamma_{2\varepsilon^\gamma}) \times [0, \varepsilon^\gamma] \approx \Omega^+ \setminus \big(\Gamma_{2\varepsilon^\gamma} \cup \Omega^+_{\varepsilon^\gamma}\big).$$

Since $\widehat{n}_\varepsilon$ may not map into $\mathbb{S}^2$, we need to apply Hardt-Lin's extension Lemma to find a point $a \in \mathbb{R}^3$, with $|a| \le \frac{1}{2}$, such that the map $\Psi_a = \big(\Pi_a\big|_{\mathbb{S}^2}\big)^{-1} \circ \Pi_a$, with $\Pi_a(y) = \frac{y-a}{|y-a|} : \mathbb{R}^3 \to \mathbb{S}^2$, satisfies

$$\int_{\Omega^+\setminus(\Gamma_{2\varepsilon^\gamma}\cup\Omega^+_{\varepsilon^\gamma})} s_\varepsilon^2 \big|\nabla(\Psi_a(\widehat{n}_\varepsilon))\big|^2 \, dx \le C \int_{\Omega^+\setminus(\Gamma_{2\varepsilon^\gamma}\cup\Omega^+_{\varepsilon^\gamma})} s_\varepsilon^2 \big|\nabla \widehat{n}_\varepsilon\big|^2 \, dx.$$

Now we define $n_\varepsilon : \Omega^+ \to \mathbb{S}^2$ as follows. We refer the readers to page 8 above for definitions of various notations, e.g. $\Omega^\pm_\delta$ and Γ_δ, that will be used below. First, we define

$$n_\varepsilon(x) = \begin{cases} n(x) & x \in (\Omega^+ \cap \Gamma_{\varepsilon^\gamma}) \cup \Omega^+_{\varepsilon^\gamma}, \\ \Psi_a\big(\widehat{n}_\varepsilon(x)\big) & x \in \Omega^+ \setminus \big(\Gamma_{2\varepsilon^\gamma} \cup \Omega^+_{\varepsilon^\gamma}\big). \end{cases}$$

It is not hard to see that $(\Omega^+ \setminus \Omega^+_{\varepsilon^\gamma}) \cap (\Gamma_{2\varepsilon^\gamma} \setminus \Gamma_{\varepsilon^\gamma})$ is bi-Lipschitz equivalent to $B_{\varepsilon^\gamma}(x_*)$ (ball of radius ε^γ and centered at a point x_*), with Lipschitz norms independent of ε. Thus we can define $n_\varepsilon : (\Omega^+ \setminus \Omega^+_{\varepsilon^\gamma}) \cap (\Gamma_{2\varepsilon^\gamma} \setminus \Gamma_{\varepsilon^\gamma}) \to \mathbb{S}^2$ as the homogeneous degree zero extension, with respect to x_*, of the value of n_ε on $\partial\big((\Omega^+ \setminus \Omega^+_{\varepsilon^\gamma}) \cap (\Gamma_{2\varepsilon^\gamma} \setminus \Gamma_{\varepsilon^\gamma})\big)$.

Then we can calculate

$$\begin{aligned} &\int_{(\Omega^+\cap\Gamma_{\varepsilon^\gamma})\cup\Omega^+_{\varepsilon^\gamma}} \big(s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) + \alpha s_\varepsilon^2 |\nabla n_\varepsilon|^2\big)\, dx \\ &\le (1 + o_\varepsilon(1)) s_+^2 \int_{\Omega^+} (W_{OF}(n, \nabla n) + \alpha|\nabla n|^2)\, dx \le (1 + o_\varepsilon(1)) \mathscr{D}_A, \end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
&\int_{\Omega^+\setminus(\Gamma_{2\varepsilon^\gamma}\cup\Omega^+_{\varepsilon^\gamma})}\left(s_\varepsilon^2 W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2\right)dx\\
&\le C\int_{\Omega^+\setminus(\Gamma_{2\varepsilon^\gamma}\cup\Omega^+_{\varepsilon^\gamma})} s_\varepsilon^2|\nabla n_\varepsilon|^2\,dx\\
&\le Cs_+^2\varepsilon^\gamma\big(\int_{\Sigma^+}|\nabla_{\tan} g_\varepsilon|^2\,d\mathscr{H}^2+\int_{\partial\Omega^+_{\varepsilon^\gamma}\cap\Omega^+}|\nabla_{\tan} n|^2\,d\mathscr{H}^2\big)\\
&\ +\frac{Cs_+^2}{\varepsilon^\gamma}\int_{\Sigma^+\setminus\Gamma_{\varepsilon^\gamma}}|g_\varepsilon(\theta)-n(\varepsilon^\gamma,\theta)|^2\,d\mathscr{H}^2\\
&\le C\varepsilon^\gamma+C\int_{\Omega^+\setminus\Omega^+_{2\varepsilon^\gamma}}|\nabla n|^2+\frac{C}{\varepsilon^\gamma}\int_{\Sigma^+\setminus\Gamma_{\varepsilon^\gamma}}|g_\varepsilon-g|^2\,d\mathscr{H}^2\\
&\ +C\int_{\Sigma^+}|g(\theta)-n(\varepsilon^\gamma,\theta)|^2\,d\mathscr{H}^2\\
&\le C\varepsilon^\gamma+o_\varepsilon(1)+C\int_{\Omega^+\setminus\Omega^+_{2\varepsilon^\gamma}}|\nabla n|^2\le C\varepsilon^\gamma+o_\varepsilon(1).
\end{aligned}\tag{3.5}
$$

where we have used (1.24) and the absolute continuity of $\int|\nabla n|^2\,dx$, and the inequality

$$\int_{\Sigma^+}|g(\theta)-n(\varepsilon^\gamma,\theta)|^2\,d\mathscr{H}^2\le\varepsilon^\gamma\int_{\Omega^+\setminus\Omega^+_{\varepsilon^\gamma}}|\nabla n|^2\,dx\le C\varepsilon^\gamma.$$

While

$$
\begin{aligned}
&\int_{(\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})\cap(\Gamma_{2\varepsilon^\gamma}\setminus\Gamma_{\varepsilon^\gamma})}\left(s_\varepsilon^2 W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2\right)dx\\
&\le Cs_+^2\int_{(\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})\cap(\Gamma_{2\varepsilon^\gamma}\setminus\Gamma_{\varepsilon^\gamma})}|\nabla n_\varepsilon|^2\,dx\\
&\le C\varepsilon^\gamma\int_{\partial\left((\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})\cap(\Gamma_{2\varepsilon^\gamma}\setminus\Gamma_{\varepsilon^\gamma})\right)}|\nabla_{\tan} n_\varepsilon|^2\,d\mathscr{H}^2\\
&\le C\varepsilon^\gamma\Big\{\int_{\Sigma^+\cap\Gamma_{2\varepsilon^\gamma}}+\int_{\partial\Gamma_{\varepsilon^\gamma}\cap(\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})}+\int_{\partial\Omega^+_{\varepsilon^\gamma}\cap(\Gamma_{2\varepsilon^\gamma}\setminus\Gamma_{\varepsilon^\gamma})}\\
&\quad+\int_{\partial\Gamma_{2\varepsilon^\gamma}\cap(\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})}\Big\}|\nabla_{\tan} n_\varepsilon|^2\,d\mathscr{H}^2\\
&\le C\Big(\varepsilon^\gamma\int_{\Sigma^+}|\nabla g_\varepsilon|^2\,d\mathscr{H}^2+\int_{\Omega^+\setminus\Omega^+_{\varepsilon^\gamma}}|\nabla n|^2\,dx\\
&\quad+\int_{\Omega^+\setminus(\Gamma_{2\varepsilon^\gamma}\cup\Omega^+_{\varepsilon^\gamma})}|\nabla n_\varepsilon|^2\,dx+\int_{\Omega^+_{\varepsilon^\gamma}\setminus\Omega^+_{2\varepsilon^\gamma}}|\nabla n|^2\,dx\Big)\\
&\le C(\varepsilon^\gamma+o_\varepsilon(1)+\int_{\Omega^+\setminus\Omega^+_{2\varepsilon^\gamma}}|\nabla n|^2\,dx)\le C(\varepsilon^\gamma+o_\varepsilon(1)),
\end{aligned}\tag{3.6}
$$

where we have used (3.6) in the last step, and we have applied Fubini's theorem which guarantees the following inequalities:

$$\varepsilon^\gamma \int_{\partial\Gamma_{\varepsilon^\gamma}\cap(\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})} |\nabla_{\tan} n_\varepsilon|^2 \, d\mathscr{H}^2 \le C \int_{\Omega^+\setminus\Omega^+_{\varepsilon^\gamma}} |\nabla n|^2 \, dx = o_\varepsilon(1),$$

$$\varepsilon^\gamma \int_{\partial\Gamma_{2\varepsilon^\gamma}\cap(\Omega^+\setminus\Omega^+_{\varepsilon^\gamma})} |\nabla_{\tan} n_\varepsilon|^2 \, d\mathscr{H}^2 \le C \int_{\Omega^+\setminus(\Gamma_{2\varepsilon^\gamma}\cup\Omega^+_{\varepsilon^\gamma})} |\nabla n_\varepsilon|^2 \, dx \le C(\varepsilon^\gamma + o_\varepsilon(1)),$$

and

$$\varepsilon^\gamma \int_{\partial\Omega^+_{\varepsilon^\gamma}\cap(\Gamma_{2\varepsilon^\gamma}\setminus\Gamma_{\varepsilon^\gamma})} |\nabla_{\tan} n_\varepsilon|^2 \, d\mathscr{H}^2 \le C \int_{\Omega^+\setminus\Omega^+_{2\varepsilon^\gamma}} |\nabla n|^2 \, dx = o_\varepsilon(1).$$

Hence

$$\begin{aligned} &\int_{\Omega^+} \big(s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) + \alpha s_\varepsilon^2 |\nabla n_\varepsilon|^2\big) \, dx \\ &\le (1 + o_\varepsilon(1))\mathscr{D}_A + C\big(\varepsilon^\gamma + o_\varepsilon(1)\big). \end{aligned} \tag{3.7}$$

The most difficult term to estimate is the interactive energy between ∇s_ε and n_ε. To do it, we proceed as follows.

$$\begin{aligned} &\int_{\Omega^+} \big(L_1 |\nabla s_\varepsilon \cdot n_\varepsilon|^2 + L_3 (\nabla s_\varepsilon \cdot n_\varepsilon)(s_\varepsilon \mathrm{div} n_\varepsilon)\big) \, dx \\ &\le \Big\{ \int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} + \int_{\Omega^+\setminus(\Omega^+_{\varepsilon^\gamma}\cup\Gamma_{\varepsilon^\gamma})} \Big\} \big(L_1 |\nabla s_\varepsilon \cdot n_\varepsilon|^2 + L_3 (\nabla s_\varepsilon \cdot n_\varepsilon)(s_\varepsilon \mathrm{div} n_\varepsilon\big) \, dx \\ &\le C\Big(\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\nabla s_\varepsilon \cdot n_\varepsilon|^2 + \big(\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\nabla s_\varepsilon \cdot n_\varepsilon|^2 \big)^{\frac{1}{2}} \Big) \\ &\quad + C\Big(\int_{E^+_{\varepsilon^\gamma}} |\nabla s_\varepsilon|^2 \, dx + \big(\int_{E^+_{\varepsilon^\gamma}} |\nabla s_\varepsilon|^2 \, dx \big)^{\frac{1}{2}} \Big) = I + II. \end{aligned}$$

From the estimate (3.2), we can see that

$$\int_{E^+_{\varepsilon^\gamma}} |\nabla s_\varepsilon|^2 \, dx \le C(\varepsilon^\gamma + o_\varepsilon(1))$$

so that

$$II \le C(\varepsilon^{\frac{\gamma}{2}} + o_\varepsilon(1)).$$

To estimate I, let $\Pi_\Gamma : \Gamma_{\varepsilon^\gamma} \to \Gamma$ be the smooth nearest point projection map. Since $\nabla d_\Gamma(x) = \nu_\Gamma(x)$ for $x \in \Gamma$, we have that $\nabla d_\Gamma(x) \cdot n(x) = 0$ for $x \in \Gamma$. Hence

$$\begin{aligned} &\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\nabla s_\varepsilon \cdot n_\varepsilon|^2 \, dx \le \frac{1}{\varepsilon^2} \int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} (\xi'_{\varepsilon,\gamma})^2 \big(\frac{d_\Gamma(x)}{\varepsilon}\big) |\nabla d_\Gamma(x) \cdot n(x)|^2 \, dx \\ &\le \frac{1}{\varepsilon^2} \int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} (\xi'_{\varepsilon,\gamma})^2 \big(\frac{d_\Gamma(x)}{\varepsilon}\big) |\nabla d_\Gamma(x) \cdot n(x) - \nabla d_\Gamma(\Pi_\Gamma(x)) \cdot n(\Pi_\Gamma(x))|^2 \, dx \\ &= \frac{1}{\varepsilon^2} \Big\{ \int_{\Omega^+\cap\Gamma_{L\varepsilon}} + \int_{\Omega^+\cap(\Gamma_{\varepsilon^\gamma}\setminus\Gamma_{L\varepsilon})} \Big\} (\xi'_{\varepsilon,\gamma})^2 \big(\frac{d_\Gamma(x)}{\varepsilon}\big) \\ &\qquad \cdot \big(|\nabla d_\Gamma(x) - \nabla d_\Gamma(\Pi_\Gamma(x))|^2 + |n(x) - n(\Pi_\Gamma(x))|^2 \big) \, dx \\ &= III + IV. \end{aligned}$$

III can be estimated similarly to (2.32) so that

$$III \le C\varepsilon + o_\varepsilon(1).$$

While we can utilize the decay property of $|\xi'|$ to estimate IV as follows.

$$\begin{aligned} IV &\le \frac{C}{\varepsilon^2}\int_{\Omega^+\cap(\Gamma_{\varepsilon^\gamma}\setminus\Gamma_{L\varepsilon})} e^{-\frac{C}{\varepsilon}d_\Gamma(x)}\big(|\nabla d_\Gamma(x)-\nabla d_\Gamma(\Pi_\Gamma(x))|^2+|n(x)-n(\Pi_\Gamma(x))|^2\big)\,dx \\ &\le C\big\|\nabla^2 d_\Gamma\big\|_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}\varepsilon^\gamma + C\varepsilon^{-2}\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} e^{-\frac{C}{\varepsilon}d_\Gamma(x)}|n(x)-n(\Pi_\Gamma(x))|^2\,dx. \end{aligned}$$

Notice that by identifying $\Omega^+\cap\Gamma_{\varepsilon^\gamma}$ with $\Gamma\times[0,\varepsilon^\gamma]$, we can bound

$$\begin{aligned} &\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} e^{-\frac{C}{\varepsilon}d_\Gamma(x)}|n(x)-n(\Pi_\Gamma(x))|^2\,dx \\ &\le C\int_\Gamma\int_0^{\varepsilon^\gamma} e^{-\frac{Ct}{\varepsilon}}|n(t,\theta)-n(0,\theta)|^2\,dt\,d\mathscr{H}^2 \\ &\le C\int_\Gamma\int_0^{\varepsilon^\gamma} te^{-\frac{Ct}{\varepsilon}}\Big(\int_0^{\varepsilon^\gamma}|n_t|^2(\tau,\theta)\,d\tau\Big)\,dt\,d\mathscr{H}^2 \\ &\le C\Big(\int_0^{\varepsilon^\gamma} te^{-\frac{Ct}{\varepsilon}}\,dt\Big)\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}|\nabla n|^2\,dx \\ &\le C\varepsilon^2\Big(\int_0^\infty te^{-Ct}\,dt\Big)\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}|\nabla n|^2\,dx \\ &\le C\varepsilon^2\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}|\nabla n|^2\,dx. \end{aligned}$$

Therefore we obtain that

$$IV \le C\varepsilon^\gamma + C\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}|\nabla n|^2\,dx \le C\varepsilon^\gamma + o_\varepsilon(1).$$

From the estimates of III and IV, we obtain that

$$\int_{\Omega^+}\big(L_1|\nabla s_\varepsilon\cdot n_\varepsilon|^2 + L_3(\nabla s_\varepsilon\cdot n_\varepsilon)(s_\varepsilon \mathrm{div} n_\varepsilon)\big)\,dx \le C\big(\varepsilon^{\frac{\gamma}{2}}+o_\varepsilon(1)\big). \tag{3.8}$$

Next we want to construct a map $n_\varepsilon:\Omega^-\to\mathbb{S}^2$ such that it enjoys an upper bound estimate similar to that in Ω^+. First let $\widehat{n}_\varepsilon:\Omega^-\to\mathbb{R}^3$ be such that

$$\begin{cases} \Delta\widehat{n}_\varepsilon = 0 & \text{in } \Omega^-, \\ \widehat{n}_\varepsilon = g_\varepsilon & \text{on } \Sigma^-, \\ \widehat{n}_\varepsilon = n & \text{on } \Gamma. \end{cases}$$

Then it is well-known that

$$\begin{aligned} \int_{\Omega^-}|\nabla\widehat{n}_\varepsilon|^2\,dx \le C\big\|\widehat{n}_\varepsilon\big\|^2_{H^{\frac12}(\partial\Omega^-)} &\le C\Big(\int_{\Sigma^-}|\nabla g_\varepsilon|^2\,d\mathscr{H}^2+\int_{\Omega^+}|\nabla n|^2\,dx\Big) \\ &\le C\Big(1+\int_{\Sigma^-}|\nabla g|^2\,d\mathscr{H}^2+\int_{\Omega^+}|\nabla n|^2\,dx\Big). \end{aligned}$$

Moreover, since $g_\varepsilon \to g$ in $H^1(\Sigma^-)$, there exists $\widetilde{n} \in H^1(\Omega^-)$ such that

$$\widehat{n}_\varepsilon \to \widetilde{n} \text{ in } H^1(\Omega^-). \tag{3.9}$$

Applying Hardt-Lin's extension Lemma again, there exists $a \in \mathbb{R}^3$ with $|a| \le \frac{1}{2}$ such that for $n_\varepsilon = \Psi_a(\widehat{n}_\varepsilon)$, where $\Psi_a = \left(\Pi_a\big|_{\mathbb{S}^2}\right)^{-1} \circ \Pi_a$, with $\Pi_a(y) = \frac{y-a}{|y-a|} : \mathbb{R}^3 \to \mathbb{S}^2$, satisfies

$$\int_{\Omega^-} |\nabla n_\varepsilon|^2\,dx \le C\int_{\Omega^-} |\nabla \hat{n}_\varepsilon|^2\,dx \le C\big(1 + \int_{\Sigma^-} |\nabla g|^2\,d\mathscr{H}^2 + \int_{\Omega^+} |\nabla n|^2\,dx\big),$$

and

$$n_\varepsilon \to \Psi_a(\widetilde{n}) \text{ in } H^1(\Omega^-). \tag{3.10}$$

With the help of (3.10), we can estimate

$$\begin{aligned}
&\int_{\Omega^-} \big(s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) + \alpha s_\varepsilon^2 |\nabla n_\varepsilon|^2\big)\,dx \le C\int_{\Omega^-} s_\varepsilon^2 |\nabla n_\varepsilon|^2\,dx \\
&\le C\big(\int_{\Omega^-\cap\Gamma_{\varepsilon\varepsilon}} s_\varepsilon^2 |\nabla n_\varepsilon|^2\,dx + \int_{E^-_{\varepsilon\gamma}} s_\varepsilon^2 |\nabla n_\varepsilon|^2\,dx\big) \\
&\le C\big(\int_{\Omega^-\cap\Gamma_{\varepsilon\gamma}} |\nabla n_\varepsilon|^2\,dx + \int_{E^-_{\varepsilon\gamma}} |\nabla n_\varepsilon|^2\,dx\big) \\
&\le C\int_{\Omega^-} |\nabla(n_\varepsilon - \Psi_a(\widetilde{n}))|^2\,dx + C\big(\int_{\Omega^-\cap\Gamma_{\varepsilon\gamma}} |\nabla\widetilde{n}|^2\,dx + \int_{E^-_{\varepsilon\gamma}} |\nabla\widetilde{n}|^2\,dx\big) \\
&= o_\varepsilon(1).
\end{aligned} \tag{3.11}$$

While

$$\begin{aligned}
&\int_{\Omega^-} \big(L_1|\nabla s_\varepsilon \cdot n_\varepsilon|^2 + L_3(\nabla s_\varepsilon \cdot n_\varepsilon)(s_\varepsilon \mathrm{div} n_\varepsilon)\big)\,dx \\
&\le \Big\{\int_{\Omega^-\cap\Gamma_{\varepsilon\gamma}} + \int_{\Omega^-\setminus(\Omega^-_{\varepsilon\gamma}\cup\Gamma_{\varepsilon\gamma})}\Big\}\big(L_1|\nabla s_\varepsilon \cdot n_\varepsilon|^2 + L_3(\nabla s_\varepsilon \cdot n_\varepsilon)(s_\varepsilon \mathrm{div} n_\varepsilon)\big)\,dx \\
&\le C\Big(\int_{\Omega^-\cap\Gamma_{\varepsilon\gamma}} |\nabla s_\varepsilon \cdot n_\varepsilon|^2 + \big(\int_{\Omega^-\cap\Gamma_{\varepsilon\gamma}} |\nabla s_\varepsilon \cdot n_\varepsilon|^2\big)^{\frac{1}{2}}\Big) \\
&\quad + C\Big(\int_{E^-_{\varepsilon\gamma}} |\nabla s_\varepsilon|^2\,dx + \big(\int_{E^-_{\varepsilon\gamma}} |\nabla s_\varepsilon|^2\,dx\big)^{\frac{1}{2}}\Big) = V + VI.
\end{aligned} \tag{3.12}$$

Again from the estimate (3.2), we see that

$$VI \le C\big(\varepsilon^{\frac{\gamma}{2}} + o(1)\big).$$

VI can be estimated similarly to that of I. In fact,

$$\begin{aligned}
&\int_{\Omega^-\cap\Gamma_{\varepsilon^\gamma}} |\nabla s_\varepsilon\cdot n_\varepsilon|^2\\
&\le C\varepsilon^{-2}\Big(\varepsilon^{2+\gamma}\|\nabla^2 d_\Gamma\|_{L^\infty(\Gamma_{\varepsilon^\gamma})}+\int_{\Omega^-\cap\Gamma_{\varepsilon^\gamma}}|\xi'|^2(\frac{d_\Gamma(x)}{\varepsilon})|n(x)-n(\Pi_\Gamma(x))|^2\,dx\Big)\\
&\le C\varepsilon^\gamma+o_\varepsilon(1)+C\varepsilon^{-2}\int_{\Omega^-\cap(\Gamma_{\varepsilon^\gamma}\setminus\Gamma_{L\varepsilon}}|n(x)-n(\Pi_\Gamma(x))|^2\,dx\\
&\le C\varepsilon^\gamma+o_\varepsilon(1).
\end{aligned}$$

Hence

$$VI\le C\varepsilon^{\frac{\gamma}{2}}+o_\varepsilon(1).$$

Substituting the estimates of III and IV into (3.12), we obtain

$$\int_{\Omega^-}\big(L_1|\nabla s_\varepsilon\cdot n_\varepsilon|^2+L_3(\nabla s_\varepsilon\cdot n_\varepsilon)(s_\varepsilon\mathrm{div}n_\varepsilon)\big)\,dx\le C\big(\varepsilon^{\frac{\gamma}{2}}+o_\varepsilon(1)\big). \tag{3.13}$$

Combining (3.2), (3.7), (3.8), (3.11), with (3.13) and (3.3), we obtain the upper bound (3.1). □

Lemma 3.2. *Assume Γ and the boundary value $(t_\varepsilon,g_\varepsilon)$ satisfies the same assumptions as in Theorem 1.3. Under the condition* $(\mathbf{B})$*, it holds that*

$$\begin{aligned}
&\inf\Big\{\int_\Omega \mathscr{W}_\varepsilon(s,n,\nabla s,\nabla n)\,dx:\ (s_\varepsilon,n_\varepsilon)=(t_\varepsilon,g_\varepsilon)\text{ on }\partial\Omega\Big\}\\
&\le\frac{1}{\varepsilon}\int_0^{s_+}\sqrt{2\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau+\mathscr{D}_B+o_\varepsilon(1),
\end{aligned}\tag{3.14}$$

where $\mathscr{D}_B$ is given by (1.30) *and* (1.31) *of Theorem 1.3.*

Proof. The proof of (3.14) can be done almost exactly as in Lemma 3.1. In fact, the construction of s_ε is exactly same as in Lemma 3.1. While the construction n_ε also follows the same procedure, except that we replace the map n, that is a minimizer of $\mathscr{D}_A$ in Lemma 3.1, by a map n that minimizes $\mathscr{D}_B$. Namely, $n\in H^1(\Omega^+,\mathbb{S}^2)$ satisfies $n=g$ on Σ^-, $n\wedge\nu_\Gamma=0$ on Γ, and

$$E(n;\Omega^+)=\mathscr{D}_B.$$

Since every other term in the integral $\int_\Omega \varepsilon\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx$ can be estimated in the same way as in Lemma 3.1, it suffices to sketch the estimate of the term

$$\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}\big(L_2|\nabla s_\varepsilon\wedge n_\varepsilon|^2+L_4 s_\varepsilon\nabla s_\varepsilon\cdot(\nabla n_\varepsilon)n_\varepsilon\big)\,dx.$$

Recall from the condition $n\wedge\nu_\Gamma=0$ on Γ that

$$\nabla d_\Gamma(\Pi_\Gamma(x))\wedge n(\Pi_\Gamma(x))=0,\ \forall x\in\Omega^+\cap\Gamma_{\varepsilon^\gamma}.$$

From the construction of $(s_\varepsilon, n_\varepsilon)$, we know that

$$\begin{aligned}
&\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} L_2|\nabla s_\varepsilon\wedge n_\varepsilon|^2\,dx \le C\varepsilon^{-2}\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\xi'_{\varepsilon,\gamma}|^2(\frac{d_\Gamma(x)}{\varepsilon})|\nabla d_\Gamma(x)\wedge n(x)|^2\,dx\\
&\le C\varepsilon^{-2}\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\xi'_{\varepsilon,\gamma}|^2(\frac{d_\Gamma(x)}{\varepsilon})|\nabla d_\Gamma(x)\wedge n(x)-\nabla d_\Gamma(\Pi_\Gamma(x))\wedge n(\Pi_\Gamma(x))|^2\,dx\\
&\le C\varepsilon^{-2}\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\xi'_{\varepsilon,\gamma}|^2(\frac{d_\Gamma(x)}{\varepsilon})(|\nabla d_\Gamma(x)-\nabla d_\Gamma(\Pi_\Gamma(x))|^2\\
&\ +|n(x)-n(\Pi_\Gamma(x))|^2)\,dx\\
&\le C\varepsilon^\gamma+o_\varepsilon(1).
\end{aligned}$$

This implies that

$$\begin{aligned}
&|\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} L_4 s_\varepsilon\nabla s_\varepsilon\cdot(\nabla n_\varepsilon)n_\varepsilon\,dx|\\
&\le C\Big(\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}} |\nabla s_\varepsilon\wedge n_\varepsilon|^2\,dx\Big)^{\frac12}\Big(\int_{\Omega^+\cap\Gamma_{\varepsilon^\gamma}}|\nabla n|^2\Big)^{\frac12}=o_\varepsilon(1).
\end{aligned}$$

Thus the estimate (3.14) holds. □

Lemma 3.3. *Assume Γ and the boundary values $(s_\varepsilon, n_\varepsilon)$ satisfy the same assumptions as in Theorem 1.3. Under the condition* $(\mathbf{C})$*, it holds that*

$$\begin{aligned}
&\inf\Big\{\int_\Omega \mathscr{W}_\varepsilon(s,n,\nabla s,\nabla n)\,dx:\ (s_\varepsilon,n_\varepsilon)=(t_\varepsilon,g_\varepsilon)\text{ on }\partial\Omega\Big\}\\
&\le\frac{1}{\varepsilon}\int_0^{s_+}\sqrt{2\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau+\mathscr{D}_C+o_\varepsilon(1),
\end{aligned}\tag{3.15}$$

where $\mathscr{D}_C$ is given by (1.33) *and* (1.34) *of Theorem 1.3.*

Proof. The proof of (3.15) can be done almost exactly as in Lemma 3.1. In fact, the construction and estimate of s_ε is exactly same as in that in Lemma 3.1. While in the construction of n_ε, we simply replace the minimizer n of $\mathscr{D}_A$ in Lemma 3.1 by a map n that minimizes $\mathscr{D}_C$. Namely, $n\in H^1(\Omega^+,\mathbb{S}^2)$ satisfies $n=g$ on Σ^+, $\frac{\partial n}{\partial\nu_\Gamma}=0$ on Γ, and

$$E(n;\Omega^+)=\mathscr{D}_C.$$

Since $L_1=L_2=L_3=L_4=0$,

$$\begin{aligned}
&\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\\
&=\int_\Omega\big(s_\varepsilon^2 W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2+\beta|\nabla s_\varepsilon|^2+\frac{1}{\varepsilon^2}W(s_\varepsilon)\big)\,dx
\end{aligned}$$

can be estimated as in (3.2), (3.7), and (3.11) of Lemma 3.1. □

3.2 Refined energy lower bounds for the case (A)

In this subsection, we will establish an improved lower bound of energy that matches the refined upper bound of energy, which ensures the planar anchoring condition for the limiting director field on the sharp interface Γ.

First, it follows from $2L_3^2 < 4L_1\alpha$ that there exists a positive number $\mu > 0$ such that $2L_3^2 \le 4(L_1-\mu)(\alpha-\mu)$ so that by Cauchy-Schwarz inequality we have

$$
\begin{aligned}
|L_3(\nabla s_\varepsilon \cdot n_\varepsilon)s_\varepsilon \mathrm{div} n_\varepsilon| &\le \sqrt{2}|L_3||\nabla s_\varepsilon \cdot n_\varepsilon||s_\varepsilon||\nabla n_\varepsilon| \\
&\le (L_1-\mu)(\nabla s_\varepsilon \cdot n_\varepsilon)^2 + (\alpha-\mu)s_\varepsilon^2|\nabla n_\varepsilon|^2. \qquad (3.16)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) &= (\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)) + s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) \\
&\quad + \big(\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + L_1(\nabla s_\varepsilon \cdot n_\varepsilon)^2 + L_3(\nabla s_\varepsilon \cdot n_\varepsilon)s_\varepsilon \mathrm{div} n_\varepsilon\big) \\
&\ge \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon) + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2 + \mu|\nabla s_\varepsilon \cdot n_\varepsilon|^2 \\
&= \big[(\beta+\gamma\cos^2\theta_\varepsilon)|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)\big] + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2 \\
&\ge \frac{2}{\varepsilon}|\nabla s_\varepsilon|\sqrt{W(s_\varepsilon)}\sqrt{\beta+\mu\cos^2\theta_\varepsilon} + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2,
\end{aligned}
$$

where $\cos\theta_\varepsilon = \frac{\nabla s_\varepsilon}{|\nabla s_\varepsilon|} \cdot n_\varepsilon$.

Thus by the co-area formula we obtain

$$
\begin{aligned}
&\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx \\
&\ge \int_\Omega \big(\frac{2}{\varepsilon}|\nabla s_\varepsilon|\sqrt{W(s_\varepsilon)}\sqrt{\beta+\mu\cos^2\theta_\varepsilon} + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2\big)\,dx \\
&\ge \mu\int_\Omega s_\varepsilon^2|\nabla n_\varepsilon|^2\,dx + \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{W(\tau)}\int_{\partial^*\mathscr{S}_\varepsilon(\tau)}\sqrt{\beta+\mu\cos^2\theta_\varepsilon}\,d\mathscr{H}^2\,d\tau \\
&= \mu\int_\Omega s_\varepsilon^2|\nabla n_\varepsilon|^2\,dx + \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{W(\tau)}\int_{\partial^*\mathscr{S}_\varepsilon(\tau)}\big(\sqrt{\beta+\mu\cos^2\theta_\varepsilon} - \sqrt{\beta}\big)\,d\mathscr{H}^2\,d\tau \\
&\quad + \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau, \qquad (3.17)
\end{aligned}
$$

where $\mathscr{S}_\varepsilon(\tau) = \big\{x \in \Omega :\ s_\varepsilon(x) \ge \tau\big\}$.

It follows from the assumption of s_ε on $\partial\Omega$ that for any $0 < \tau < s_+$, the enclosed surface $\mathscr{T}_\varepsilon(\tau) \subset \partial\Omega$ between $\partial^*\mathscr{S}_\varepsilon(\tau)$ and Σ is a strip with width at most $C\varepsilon$. Hence by the area minimality of Γ, we have

$$
\begin{aligned}
\mathscr{H}^2(\Gamma) &\le \mathscr{H}^2\big(\partial^*\mathscr{S}_\varepsilon(\tau) \cup \mathscr{T}_\varepsilon(\tau)\big) = \mathscr{H}^2\big(\partial^*\mathscr{S}_\varepsilon(\tau)\big) + \mathscr{H}^2\big(\mathscr{T}_\varepsilon(\tau)\big) \\
&\le \mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau)) + C\varepsilon,\ 0 < \tau < s_+. \qquad (3.18)
\end{aligned}
$$

This implies that

$$\frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau \geq \frac{1}{\varepsilon}\alpha_0\mathscr{H}^2(\Gamma)-C. \tag{3.19}$$

Notice that

$$\begin{aligned}\sqrt{\beta+\mu\cos^2\theta_\varepsilon}-\sqrt{\beta} &= \frac{\mu\cos^2\theta_\varepsilon}{\sqrt{\beta+\mu\cos^2\theta_\varepsilon}+\sqrt{\beta}}\\ &\geq \mu_*\cos^2\theta_\varepsilon,\end{aligned}$$

where $\mu_* = \dfrac{\mu}{\sqrt{\beta+\mu}+\sqrt{\beta}} > 0$.

Hence, by matching the refined upper bound (3.1) with (3.17), we conclude that

$$\mu\int_\Omega (s_\varepsilon^2|\nabla n_\varepsilon|^2+|\nabla s_\varepsilon\cdot n_\varepsilon|^2)\,dx \leq \mathscr{D}_A + C + o_\varepsilon(1), \tag{3.20}$$

$$\frac{\mu_*}{\varepsilon}\int_0^{s_+}\sqrt{W(\tau)}\int_{\partial^*\mathscr{S}_\varepsilon(\tau)}\cos^2\theta_\varepsilon\,d\mathscr{H}^2\,d\tau \leq \mathscr{D}_A + C + o_\varepsilon(1), \tag{3.21}$$

and

$$\frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau \leq \frac{1}{\varepsilon}\alpha_0\mathscr{H}^2(\Gamma)+\mathscr{D}_A+o_\varepsilon(1). \tag{3.22}$$

For any fixed $\delta>0$ and for any $\varepsilon\in(0,1)$, applying Fubini's theorem to (3.21) and (3.22) we obtain that there exist $C>0$, that is independent of δ, and $\delta_\varepsilon\in(\delta,2\delta)$ such that

$$\int_{\partial^*\mathscr{S}_\varepsilon(s_+-\delta_\varepsilon)}\cos^2\theta_\varepsilon\,d\mathscr{H}^2 \leq C(\beta,\mathscr{D}_A)\frac{\varepsilon}{\delta}, \tag{3.23}$$

$$\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(s_+-\delta_\varepsilon)) \leq \mathscr{H}^2(\Gamma)+C\frac{\varepsilon}{\delta}. \tag{3.24}$$

Since it is straightforward to get (3.23), we only sketch how to obtain (3.24). From (3.18), we have

$$\begin{aligned}&\frac{2}{\varepsilon}\Big\{\int_0^{s_+-2\delta}+\int_{s_+-\delta}^{s_+}\Big\}\sqrt{\beta W(\tau)}\mathscr{H}^2\big(\partial^*\mathscr{S}_\varepsilon(\tau)\big)\,d\tau\\ &\geq \frac{2}{\varepsilon}\Big(\int_{[0,s_+-2\delta]\cup[s_--\delta,s_+]}\sqrt{\beta W(\tau)}\,d\tau\Big)\mathscr{H}^2(\Gamma)-C,\end{aligned}$$

so that

$$\begin{aligned}&\Big(\inf_{s_+-2\delta\leq\tau\leq s_+-\tau}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\Big)\frac{2}{\varepsilon}\int_{s_+-2\delta}^{s_+-\delta}\sqrt{\beta W(\tau)}\,d\tau\\ &\leq \frac{2}{\varepsilon}\int_{s_+-2\delta}^{s_+-\delta}\sqrt{\beta W(\tau)}\mathscr{H}^2\big(\partial^*\mathscr{S}_\varepsilon(\tau)\big)\,d\tau\\ &\leq \Big(\frac{2}{\varepsilon}\int_{s_+-2\delta}^{s_+-\delta}\sqrt{\beta W(\tau)}\,d\tau\Big)\mathscr{H}^2(\Gamma)+\mathscr{D}_A+C+o_\varepsilon(1).\end{aligned}$$

This, combined with the estimate $\int_{s_+-2\delta}^{s_+-\delta}\sqrt{\beta W(\tau)}\,d\tau \approx \delta$, yields (3.24).

If we set

$$\Omega^-_{\varepsilon,\delta} = \Big\{x\in\Omega:\ s_\varepsilon(x) < \delta\Big\},\ \ \Omega^+_{\varepsilon,\delta} = \Big\{x\in\Omega:\ s_\varepsilon(x) > s_+ - \delta\Big\},$$

and

$$E_{\varepsilon,\delta} = \Big\{x\in\Omega:\ \delta \le s_\varepsilon(x) \le s_+ - \delta\Big\},$$

then it follows from (1.5) and (3.20) that

$$\begin{cases} |E_{\varepsilon,\delta_\varepsilon}| \le \dfrac{C\varepsilon}{\delta^2}, \\ \displaystyle\int_{\Omega^+_{\varepsilon,\delta_\varepsilon}} |\nabla n_\varepsilon|^2 \le \dfrac{C(\alpha,\mathscr{D}_A)}{(s_+-\delta)^2}. \end{cases} \tag{3.25}$$

It follows from the energy upper bound (3.1) that

$$\int_{\Omega^+_{\varepsilon,\delta_\varepsilon}} \frac{1}{\varepsilon^2} W(s_\varepsilon)\,dx \le \frac{C}{\varepsilon},$$

this, combined with (1.5), implies that for a.e. $x\in\Omega^+_{\varepsilon,\delta_\varepsilon}$, $s_+ - \delta_\varepsilon \le s_\varepsilon(x) \le 1$ and $s_\varepsilon(x) \to s_+$ as $\varepsilon\to 0$. Hence for any $1<p<\infty$, it holds that

$$\int_{\Omega^+_{\varepsilon,\delta_\varepsilon}} |s_\varepsilon(x) - s_+|^p\,dx \to 0, \text{ as } \varepsilon\to 0. \tag{3.26}$$

Furthermore, from the Cauchy-Schwarz inequality and (3.19), we also have

$$\begin{aligned}\int_\Omega \Big(\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)\Big)\,dx &\ge \frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\,\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau \\ &\ge \frac{1}{\varepsilon}\alpha_0\mathscr{H}^2(\Gamma) - C.\end{aligned}$$

Matching with the refined upper bound (3.1), this also implies that

$$\begin{aligned}&\int_\Omega \big(s_\varepsilon^2 W_{OF}(n_\varepsilon,\nabla n_\varepsilon) + \alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 \\ &\quad + L_1(\nabla s_\varepsilon\cdot n_\varepsilon)^2 + L_3(\nabla s_\varepsilon\cdot n_\varepsilon)s_\varepsilon \mathrm{div} n_\varepsilon\big)\,dx \\ &\le \mathscr{D}_A + C + o_\varepsilon(1).\end{aligned} \tag{3.27}$$

From (3.23) and (3.25), there exist $\varepsilon_i \to 0$, $\delta_i = \delta_{\varepsilon_i} \in (\varepsilon_i^{\frac14}, 2\varepsilon_i^{\frac14})$, a set $\Omega_*\subset\Omega$ of finite perimeter such that for $\Omega_i = \Omega^+_{\varepsilon_i,\delta_i}$, it holds that

(a) $\chi_{\Omega_i} \rightharpoonup \chi_{\Omega_*}$ in $BV(\Omega)$, and $\chi_{\Omega_i} \to \chi_{\Omega_*}$ in $L^1(\mathbb{R}^3)$.
(b) We also view $\Gamma_i = \partial\Omega_i\lfloor\Omega$ and $\Gamma_* = \partial^*\Omega_*\lfloor\Omega$ as oriented boundaries and integral rectifiable 2-currents, and use the same notations, i.e., $\Gamma_i = [[\partial\Omega_i\lfloor\Omega]]$ and $\Gamma_* = [[\partial\Omega_*\lfloor\Omega]]$. Then

$$\Gamma_i \rightharpoonup \Gamma_*$$

weakly converges as oriented boundaries and integral rectifiable 2-currents. By the lower semicontinuity, we have that

$$\mathscr{H}^2(\Gamma) \le \mathscr{H}^2(\Gamma_*) \le \lim_i \mathscr{H}^2(\Gamma_i) = \mathscr{H}^2(\Gamma), \tag{3.28}$$

where the first inequality follows from the area minimality of Γ, since $\partial\Gamma = \partial\Gamma_* = [[\Sigma^0]]$.

Since Γ is assumed to be a unique area minimizing surface spanned by Σ^0, we have that $\Gamma_* = \Gamma$. Also, since

$$\partial\Omega_i \lfloor (\mathbb{R}^3 \setminus \Omega) \to \Sigma^+$$

as convergence of currents, we conclude that

$$\partial\Omega_* \lfloor (\mathbb{R}^3 \setminus \Omega) = \Sigma^+.$$

Therefore $\Omega_* = \Omega^+$. Next we need to show

Claim 1. For any $\eta > 0$, it holds

$$\lim_{\varepsilon_i \to 0} \mathscr{H}^2(\partial\Omega_i \cap \{x \in \Omega : d_\Gamma(x) \ge \eta\} = 0. \tag{3.29}$$

It follows from (3.28) that

$$\mathscr{H}^2 \lfloor (\partial\Omega_i \cap \Omega) \rightharpoonup \mathscr{H}^2 \lfloor \Gamma$$

as weak convergence of Radon measures. Hence by the lower semicontinuity,

$$\begin{aligned}
\mathscr{H}^2(\Gamma) &= \mathscr{H}^2\big(\Gamma \cap \{x \in \Omega : d_\Gamma(x) < \eta\}\big) \\
&\le \liminf_{\varepsilon_i \to 0} \mathscr{H}^2\big(\partial\Omega_i \cap \{x \in \Omega : d_\Gamma(x) < \eta\}\big) \\
&\le \liminf_{\varepsilon_i \to 0} \mathscr{H}^2(\partial\Omega_i \cap \Omega) = \mathscr{H}^2(\Gamma).
\end{aligned}$$

This clearly implies (3.29).

Claim 2. There exists a map $n \in \mathrm{SBV}(\Omega^+, \mathbb{S}^2)$ [3] *such that after passing to a subsequence,*

$$n_{\varepsilon_i} \chi_{\Omega_i} \rightharpoonup n \chi_{\Omega^+} \ \text{ in } \ \mathrm{BV}(\Omega), \ \text{ and } \ s_{\varepsilon_i} \chi_{\Omega_i} \to s_+ \chi_{\Omega^+} \ \text{ in } \ L^2(\Omega).$$

Furthermore, $n \in H^1(\Omega^+, \mathbb{S}^2)$.

To show Claim 2, we first observe that the absolutely continuous part of the distributional derivative of $v_i = n_{\varepsilon_i} \chi_{\Omega_i}$ is $\nabla v_i = \nabla n_{\varepsilon_i} \chi_{\Omega_i}$, which is uniformly bounded in L^2, i.e.

$$\int_\Omega |\nabla v_i|^2 \, dx = \int_{\Omega_i} |\nabla n_i|^2 \, dx \le C.$$

The jump part J_{v_i} of v_i satisfies

$$J_{v_i} \subset \partial\Omega_i \cap \Omega = \Gamma_i,$$

[3] Here $\mathrm{SBV}(\Omega)$ denotes the space of all BV (or bounded variations) functions such that the Cantor part of the distributional derivatives is zero. See Ambrosio [1] for more discussions.

so that

$$\mathscr{H}^2(J_{v_i}) \le \mathscr{H}^2(\Gamma_i) \le 2\mathscr{H}^2(\Gamma).$$

Moreover, we have that

$$\|v_i\|_{L^\infty(\Omega)} \le 1.$$

Thus it follows from [1] that $\{v_i\} \subset \mathrm{SBV}(\Omega)$ is a weakly compact sequence in $\mathrm{SBV}(\Omega)$. There exists a $n \in \mathrm{SBV}(\Omega)$ such that $v_i \rightharpoonup n$ in $\mathrm{BV}(\Omega)$ and strongly in $L^1(\Omega)$. Since $|v_i| = 1$ in Ω_i and $|v_i| = 0$ in $\Omega \setminus \Omega_i$, it follows that $|n| = 1$ in Ω^+ and $|n| = 0$ in $\Omega \setminus \Omega^+$ so that $n \in \mathrm{SBV}(\Omega^+, \mathbb{S}^2)$. From the lower semicontinuity, we have that

$$\int_{\Omega^+} |\nabla n|^2 \, dx \le \liminf_{\varepsilon_i \to 0} \int_{\Omega_i} |\nabla n_{\varepsilon_i}|^2 \, dx \tag{3.30}$$

Now we want to show its jump set has $\mathscr{H}^2$-measure zero. This follows from (3.29) and the lower semicontinuity:

$$\begin{aligned}
&\mathscr{H}^2\big(J_n \cap \{x \in \Omega^+ : \ d_\Gamma(x) > \eta\}\big) \\
&\le \liminf_{\varepsilon_i \to 0} \mathscr{H}^2\big(J_{n_{\varepsilon_i}} \cap \{x \in \Omega^+ : \ d_\Gamma(x) > \eta\}\big) \\
&\le \liminf_{\varepsilon_i \to 0} \mathscr{H}^2\big(\partial\Omega_i \cap \{x \in \Omega^+ : \ d_\Gamma(x) > \eta\}\big) = 0.
\end{aligned}$$

This, after sending $\eta \to 0$, yields $\mathscr{H}^2\big(J_n \cap \Omega^+\big) = 0$. Hence $n \in H^1(\Omega^+)$.

It follows from (3.30) that $\nabla n_{\varepsilon_i} \chi_{\Omega_i} \rightharpoonup \nabla n \chi_{\Omega^+}$ in $L^2(\Omega)$, and hence

$$s_{\varepsilon_i} \nabla n_{\varepsilon_i} \chi_{\Omega_i} \rightharpoonup s_+ \nabla n \chi_{\Omega^+} \ \text{ in } \ L^1(\Omega).$$

This, combined with the uniform H^1 bound (3.25), further implies

$$s_{\varepsilon_i} \nabla n_{\varepsilon_i} \chi_{\Omega_i} \rightharpoonup s_+ \nabla n \chi_{\Omega^+} \ \text{ in } \ L^2(\Omega). \tag{3.31}$$

We claim that

$$\nabla s_{\varepsilon_i} \cdot n_{\varepsilon_i} \to 0 \ \text{ in } \ \mathscr{D}'(\Omega^+). \tag{3.32}$$

To see this, let $\phi \in C_0^\infty(\Omega^+)$. Then by integration by parts we have

$$\begin{aligned}
\int_{\Omega^+} \nabla s_{\varepsilon_i} \cdot n_{\varepsilon_i} \phi \, dx &= -\int_{\Omega^+} s_{\varepsilon_i} \big(\mathrm{div} n_{\varepsilon_i} \phi + n_{\varepsilon_i} \nabla \phi\big) \, dx \\
&\to -\int_{\Omega^+} s_+ \big(\mathrm{div} n \phi + n \nabla \phi\big) \, dx \\
&= -\int_{\Omega^+} s_+ \mathrm{div}(n\phi) \, dx = 0, \ \text{as } \varepsilon_i \to 0.
\end{aligned}$$

Since $\int_{\Omega_i} |\nabla s_{\varepsilon_i} \cdot n_{\varepsilon_i}|^2 \, dx$ is uniformly bounded, it follows from (3.32) that

$$\nabla s_{\varepsilon_i} \cdot n_{\varepsilon_i} \chi_{\Omega_i} \rightharpoonup 0 \ \text{ in } \ L^2(\Omega). \tag{3.33}$$

It follows from (3.31) and the lower semicontinuity of $\int_{\Omega^+} s_{\varepsilon_i}^2 W_{OF}(n_{\varepsilon_i}, \nabla n_{\varepsilon_i})\,dx$ that

$$
\begin{aligned}
\int_{\Omega^+} s_+^2 W_{OF}(n,\nabla n)\,dx &= \int_{\Omega} s_+^2 W_{OF}(n,\nabla n)\chi_{\Omega^+}\,dx \\
&\le \liminf_{\varepsilon_i\to 0} \int_{\Omega} s_{\varepsilon_i}^2 W_{OF}(n_{\varepsilon_i},\nabla n_{\varepsilon_i})\chi_{\Omega_i}\,dx \\
&= \liminf_{\varepsilon_i\to 0} \int_{\Omega_i} s_{\varepsilon_i}^2 W_{OF}(n_{\varepsilon_i},\nabla n_{\varepsilon_i})\,dx. \qquad (3.34)
\end{aligned}
$$

Next we claim that

$$
\begin{aligned}
&\int_{\Omega^+} \alpha s_+^2 |\nabla n|^2\,dx \\
&\le \liminf_{\varepsilon_i\to 0} \int_{\Omega_i} \left(\alpha s_{\varepsilon_i}^2|\nabla n_{\varepsilon_i}|^2 + L_1(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})^2 + L_3(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i}) s_{\varepsilon_i}\mathrm{div} n_{\varepsilon_i}\right) dx. \qquad (3.35)
\end{aligned}
$$

For $\eta>0$, define $\Omega_\eta^+ = \left\{x\in\Omega^+ :\ d(x,\partial\Omega^+)>\eta\right\}$. Since $\Omega_i\to\Omega^+$ in Hausdorff distance, we may assume that for i sufficiently large, $\Omega_\eta^+\subset\Omega_i$ and hence

$$
s_{\varepsilon_i}\nabla n_{\varepsilon_i} \rightharpoonup s_+\nabla n \text{ in } L^2(\Omega_\eta^+),\ \ \nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i}\rightharpoonup 0 \text{ in } L^2(\Omega_\eta^+). \qquad (3.36)
$$

This and (3.16) imply that

$$
\begin{aligned}
\mathscr{D} &\equiv \liminf_{\varepsilon_i\to 0} \int_{\Omega_i} \left(\alpha s_{\varepsilon_i}^2|\nabla n_{\varepsilon_i}|^2 + L_1(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})^2 + L_3(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i}) s_{\varepsilon_i}\mathrm{div} n_{\varepsilon_i}\right) dx \\
&\ge \liminf_{\varepsilon_i\to 0} \int_{\Omega_\eta^+} \left(\alpha s_{\varepsilon_i}^2|\nabla n_{\varepsilon_i}|^2 + L_1(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})^2 + L_3(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i}) s_{\varepsilon_i}\mathrm{div} n_{\varepsilon_i}\right) dx \\
&= \liminf_{\varepsilon_i\to 0} \int_{\Omega_\eta^+} \Big[\alpha |s_{\varepsilon_i}(\nabla n_{\varepsilon_i}-\nabla n) + s_{\varepsilon_i}\nabla n|^2 + L_1(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})^2 \\
&\qquad\qquad + L_3(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})(s_{\varepsilon_i}(\mathrm{div} n_{\varepsilon_i}-\mathrm{div} n) + s_{\varepsilon_i}\mathrm{div} n)\Big]\, dx \\
&= \liminf_{\varepsilon_i\to 0} \Big\{ \int_{\Omega_\eta^+} \alpha s_{\varepsilon_i}^2 |\nabla n|^2\,dx \\
&\quad + \int_{\Omega_\eta^+} \Big[\alpha s_{\varepsilon_i}^2 |\nabla(n_{\varepsilon_i}-n)|^2 + L_1(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})^2 + L_3(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i}) s_{\varepsilon_i}\mathrm{div}(n_{\varepsilon_i}-n)\Big]\, dx \\
&\quad + \int_{\Omega_\eta^+} \Big(2\alpha s_{\varepsilon_i}\nabla(n_{\varepsilon_i}-n)(s_{\varepsilon_i}\nabla n) + L_3(\nabla s_{\varepsilon_i}\cdot n_{\varepsilon_i})(s_{\varepsilon_i}\mathrm{div} n)\Big)\, dx \Big\} \\
&= \liminf_{\varepsilon_i\to 0} (A_i+B_i+C_i).
\end{aligned}
$$

Applying (3.16) with n_ε replaced by $n_{\varepsilon_i}-n$, we have that $B_i\ge 0$. From (3.26) and (3.36), we see that $C_i\to 0$. On the other hand, since $s_{\varepsilon_i}\nabla n \rightharpoonup s_+\nabla n$ in $L^2(\Omega)$, we have

$$
\liminf_{\varepsilon_i\to 0} A_i \ge \int_{\Omega_\eta^+} \alpha s_+^2 |\nabla n|^2\,dx.
$$

Therefore we obtain that

$$\mathscr{D} \geq \int_{\Omega_\eta^+} \alpha s_+^2 |\nabla n|^2 \, dx.$$

Sending η to zero, this yields the claim (3.35), that is,

$$\mathscr{D} \geq \int_{\Omega^+} \alpha s_+^2 |\nabla n|^2 \, dx. \tag{3.37}$$

Combining (3.37) with (3.34), we arrive at

$$\begin{aligned} E(n, \Omega^+) &= \int_{\Omega^+} s_+^2 \left(W_{OF}(n, \nabla n) + \alpha |\nabla n|^2 \right) dx \\ &\leq \liminf_{\varepsilon_i \to 0} \int_\Omega \left(s_{\varepsilon_i}^2 W_{OF}(n_{\varepsilon_i}, \nabla n_{\varepsilon_i}) + \alpha s_{\varepsilon_i}^2 |\nabla n_{\varepsilon_i}|^2 \right. \\ &\quad \left. + L_1 (\nabla s_{\varepsilon_i} \cdot n_{\varepsilon_i})^2 + L_3 (\nabla s_{\varepsilon_i} \cdot n_{\varepsilon_i}) s_{\varepsilon_i} \mathrm{div} n_{\varepsilon_i} \right) dx. \end{aligned} \tag{3.38}$$

From the assumption on g_{ε_i} on Σ^+, we see that $n = g$ on Σ^+. Next we want to show the trace of n on Γ satisfies the planar anchoring condition:

$$n \cdot \nu_\Gamma = 0 \ \text{ on } \ \Gamma. \tag{3.39}$$

Sketch of proof of (3.39). We will show the planar anchoring condition of n on Γ as follows. For simplicity, write $n_i = n_{\varepsilon_i}$. First it is not hard to show that as $i \to \infty$, $\Omega_i \to \Omega^+$ in Hausdorff distance,

$$d\mathscr{H}^2 \lfloor \Gamma_i \rightharpoonup d\mathscr{H}^2 \lfloor \Gamma$$

as convergence of Radon measures, and

$$n_i \rightharpoonup n \ \text{ in } \ H^1(\Omega^+, \mathbb{S}^2).$$

This implies that

$$\mathrm{div}(n_i) \rightharpoonup \mathrm{div}(n) \text{ in } L^2(\Omega), \ \ \chi_{\Omega_i} \to \chi_{\Omega^+} \text{ in } L^2(\mathbb{R}^3).$$

Therefore

$$\begin{aligned} \int_{\Gamma_i} n_i \cdot \nu_{\Gamma_i} \, d\mathscr{H}^2 &= \int_{\partial\Omega_i} n_i \cdot \nu_{\partial\Omega_i} \, d\mathscr{H}^2 - \int_{\partial\Omega_i \cap \partial\Omega} g_i \cdot \nu_{\partial\Omega} \, d\mathscr{H}^2 \\ &= \int_{\Omega_i} \mathrm{div}(n_i) \, dx - \int_{\partial\Omega_i \cap \partial\Omega} g_i \cdot \nu_{\partial\Omega} \, d\mathscr{H}^2 \\ &= \int_{\mathbb{R}^3} \mathrm{div}(n_i) \chi_{\Omega_i} \, dx - \int_{\partial\Omega_i \cap \partial\Omega} g_i \cdot \nu_{\partial\Omega} \, d\mathscr{H}^2 \\ &\to \int_{\mathbb{R}^3} \mathrm{div}(n) \chi_{\Omega^+} \, dx - \int_{\partial\Omega^+ \cap \partial\Omega} g \cdot \nu_{\partial\Omega} \, d\mathscr{H}^2 \\ &= \int_{\partial\Omega^+} n \cdot \nu_{\partial\Omega^+} \, d\mathscr{H}^2 - \int_{\partial\Omega^+ \cap \partial\Omega} g \cdot \nu_{\partial\Omega} \, d\mathscr{H}^2 \\ &= \int_\Gamma n \cdot \nu_\Gamma \, d\mathscr{H}^2. \end{aligned} \tag{3.40}$$

It is readily seen that the planar anchoring condition of n on Γ follows from (3.40) and the following lower semicontinuity property: for any nonnegative convex function $f:\mathbb{R}\to\mathbb{R}_+$, it holds that

$$\int_\Gamma f\big(n\cdot\nu_\Gamma\big)\,d\mathscr{H}^2 \le \liminf_{i\to\infty}\int_{\Gamma_i} f\big(n_i\cdot\nu_{\Gamma_i}\big)\,d\mathscr{H}^2. \tag{3.41}$$

Indeed, if we choose $f(\theta)=\theta^2$, then (3.41) and (3.23) imply that

$$\int_\Gamma \big(n\cdot\nu_\Gamma\big)^2\,d\mathscr{H}^2 \le \liminf_{i\to\infty}\int_{\Gamma_i}\big(n_i\cdot\nu_{\Gamma_i}\big)^2\,d\mathscr{H}^2 = \liminf_{i\to\infty}\int_{\Gamma_i}\cos^2\theta_\varepsilon\,d\mathscr{H}^2=0.$$

This implies that $n\cdot\nu_\Gamma=0$ $\mathscr{H}^2$ a.e. on Γ.

Now we want to show (3.41) as follows. Define a family of Radon measures

$$\Theta_i(A)=\mathscr{H}^2(\Gamma_i\cap A)\text{ for } i\ge 1;\ \Theta(A)=\mathscr{H}^2(\Gamma\cap A),$$

and

$$\mu_i(A)=\int_A f\big(n_i\cdot\nu_{\Gamma_i}\big)\,d\Theta_i$$

for any measurable set $A\subset\mathbb{R}^3$.

It is readily seen that there exists a nonnegative Radon measure μ such that, after passing to a subsequence,

$$\Theta_i \rightharpoonup \Theta \ \text{ and } \ \mu_i\rightharpoonup\mu,$$

as convergence of Radon measures in $\mathbb{R}^3$. By the Radon-Nikodym theorem, we can decompose

$$\mu=(D_\Theta\mu)\Theta+\mu^s,\ \text{ with } \ \mu^s\perp\Theta.$$

Then we have

$$\int_A D_\Theta\mu\,d\Theta\le\mu(A)\le\liminf_{i\to\infty}\mu_i(A),$$

for any open set $A\subset\mathbb{R}^3$. Hence (3.41) follows, if we can show

$$f(\nu\cdot\nu_\Gamma)(x)\le(D_\Theta\mu)(x),\ \Theta-\text{a.e. } x\in\operatorname{supp}(\Theta)=\Gamma. \tag{3.42}$$

From the convexity of f, there exist $a_k,b_k\in\mathbb{R}$ such that

$$f(\theta)=\sup_k(a_k\theta+b_k). \tag{3.43}$$

For $x\in\Gamma$, we can find $r_j\to 0$ such that for each j, it holds that

$$\begin{cases}\lim_{i\to\infty}\int_{\partial B_{r_j}(x)\cap\Omega_i} n_i\cdot\frac{y-x}{|y-x|}\,d\mathscr{H}^2=\lim_{i\to\infty}\int_{\partial B_{r_j}(x)\cap\Omega^+} n\cdot\frac{y-x}{|y-x|}\,d\mathscr{H}^2,\\ \mu\big(\partial B_{r_j}(x)\big)=0.\end{cases} \tag{3.44}$$

Therefore we have that

$$\lim_{i\to\infty}\int_{B_{r_j}(x)\cap\Gamma_i} f(n_i\cdot\nu_{\Gamma_i})\,d\mathscr{H}^2=\lim_{i\to\infty}\mu_i(B_{r_j}(x))=\mu(B_{r_j}(x)). \tag{3.45}$$

Similar to (3.40), we have that for each j,

$$\begin{aligned}
&\lim_{i\to\infty}\int_{\partial(\Omega_i\cap B_{r_j}(x))} n_i\cdot\nu_{\partial(\Omega_i\cap B_{r_j}(x))}\,d\mathscr{H}^2\\
&=\lim_{i\to\infty}\int_{\Omega_i\cap B_{r_j}(x)} \mathrm{div} n_i\,dx\\
&=\int_{\Omega_i\cap B_{r_j}(x)} \mathrm{div} n\,dx\\
&=\int_{\partial(\Omega^+\cap B_{r_j}(x))} n\cdot\nu_{\partial(\Omega^+\cap B_{r_j}(x))}\,d\mathscr{H}^2.
\end{aligned}$$

This, combined with (3.44), implies that for each j it holds that

$$\lim_{i\to\infty}\int_{\Gamma_i\cap B_{r_j}(x)} n_i\cdot\nu_{\Gamma_i}\,d\mathscr{H}^2=\int_{\Gamma\cap B_{r_j}(x)} n\cdot\nu_\Gamma\,d\mathscr{H}^2.$$

Recall that for Θ a.e. $x\in\Gamma$, it holds that

$$D_\Theta\mu(x)=\lim_{j\to\infty}\frac{\mu(B_{r_j}(x))}{\Theta(B_{r_j}(x))},$$

and

$$\lim_{j\to\infty}\frac{\int_{\Gamma\cap B_{r_j}(x)} n\cdot\nu_\Gamma\,d\mathscr{H}^2}{\Theta(B_{r_j}(x))}=(n\cdot\nu_\Gamma)(x).$$

Applying (3.45), we obtain that for any fixed k,

$$\begin{aligned}
D_\Theta\mu(x) &= \lim_{j\to\infty}\lim_{i\to\infty}\frac{\int_{\Gamma_i\cap B_{r_j}(x)} f(n_i\cdot\nu_{\Gamma_i})\,d\mathscr{H}^2}{\Theta(B_{r_j}(x))}\\
&\ge \lim_{j\to\infty}\lim_{i\to\infty}\frac{\int_{\Gamma_i\cap B_{r_j}(x)} (a_k n_i\cdot\nu_{\Gamma_i}+b_k)\,d\mathscr{H}^2}{\Theta(B_{r_j}(x))}\\
&\ge \lim_{j\to\infty}\lim_{i\to\infty}\Big[a_k\frac{\int_{\Gamma_i\cap B_{r_j}(x)} n_i\cdot\nu_{\Gamma_i}\,d\mathscr{H}^2}{\Theta(B_{r_j}(x))}+b_k\frac{\Theta_i(B_{r_j}(x))}{\Theta(B_{r_j}(x))}\Big]\\
&\ge \lim_{j\to\infty}\Big[a_k\frac{\int_{\Gamma\cap B_{r_j}(x)} n\cdot\nu_\Gamma\,d\mathscr{H}^2}{\Theta(B_{r_j}(x))}+b_k\Big]\\
&= a_k(n\cdot\nu_\Gamma)(x)+b_k.
\end{aligned}$$

Taking supremum over $k\ge 1$, we conclude that for Θ a.e. $x\in\Gamma$,

$$D_\Theta\mu(x)\ge\sup_k\big(a_k(n\cdot\nu_\Gamma)(x)+b_k\big)=f\big(n\cdot\nu_\Gamma\big)(x).$$

This yields (3.41).

For $0 < \tau < s_+$, let $\mathscr{T}(\tau) \subset \Omega$ be an area minimizing surface spanned by $\Sigma(\tau) = \{x \in \partial\Omega : d_\Gamma(x) = t\} = \partial\Gamma(t)$:

$$\mathscr{H}^2(\mathscr{T}(\tau)) = \min\Big\{\mathscr{H}^2(S) : S \text{ is an integral 2-current in } \Omega,\ \partial S = \Sigma(t)\Big\}.$$

Then by putting all the above estimates together, we obtain the following lower bound:

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx\\ &\geq \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{T}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau + E(n,\Omega^+),\end{aligned}\tag{3.46}$$

where $E(n,\Omega^+)$ is the Oseen-Frank energy given by

$$E(n,\Omega^+) = s_+^2\int_{\Omega^+}\big(W_{OF}(n,\nabla n) + \alpha|\nabla n|^2\big)\,dx.$$

Finally, we want to show that $n \in H^1(\Omega^+,\mathbb{S}^2)$ is a minimizer of the Oseen-Frank energy $E(\cdot,\Omega^+)$, subject to the boundary condition: $n = g$ on Σ^+ and $n\cdot\nu_\Gamma = 0$ on Γ, i.e.,

$$E(n,\Omega^+) = \mathscr{D}_A.\tag{3.47}$$

In order to prove (3.47), we need to show that the leading order term in the lower bound estimate (3.46) exactly matches that in the refined upper bound estimate (3.1), i.e.,

$$\begin{aligned}&\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau\\ &\leq \int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{T}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau + C\varepsilon^2.\end{aligned}\tag{3.48}$$

The validity of (3.48) is a consequence of the strict stability of Γ, which ensures

Claim 3. Under the condition that Γ is a strictly stable, area minimizing surface, there exist $\eta_0 > 0$ and $C_0 > 0$, depending only on Γ and Ω, such that

$$\mathscr{H}^2(\Gamma(\lambda)) \leq \mathscr{H}^2(\mathscr{T}(\lambda)) + C_0\lambda^2,\ \forall\lambda\in[-\eta_0,\eta_0].\tag{3.49}$$

The proof of (3.49) is based on the second variation of surface areas and the strict stability of Γ, we refer the reader to [34] page 45-47.

It follows from (3.49) that

$$\begin{aligned}&\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau\\ &\leq \int_0^{s_+}\sqrt{\beta W(\tau)}\big(\mathscr{H}^2(\mathscr{T}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau)) + C(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))^2\big)\,d\tau\\ &\leq \int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{T}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau + C\varepsilon^2\int_0^{s_+}\sqrt{\beta W(\tau)}\big(\xi_{\varepsilon,\gamma}^{-1}(\tau)\big)^2\,d\tau\\ &\leq \int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{T}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau + C\varepsilon^2,\end{aligned}$$

since there exists $C > 0$, that is independent of ε, such that

$$\int_0^{s_+} \sqrt{\beta W(\tau)}\big(\xi_{\varepsilon,\gamma}^{-1}(\tau)\big)^2\,d\tau \le C\alpha_0.$$

Hence (3.48) holds. Now it is readily seen that the refined upper bound (3.1), the refined lower bound (3.46), and (3.48) imply $E(n,\Omega^+) \le \mathscr{D}_A$. On the other hand, since $n = g$ on Σ^+ and $n\cdot\nu_\Gamma = 0$ on Γ, we automatically have $E(n,\Omega_+) \ge \mathscr{D}_A$. Thus (3.47) holds, and

(3.50)

$$\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx = \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi^{-1}(\tau))\,d\tau + \mathscr{D}_A + o_\varepsilon(1).$$

Finally, we claim that (see also [34] page 46)

$$\frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + o_\varepsilon(1). \tag{3.51}$$

In fact, it is not hard to see that

$$\mathscr{H}^2(\Gamma(\lambda)) = \mathscr{H}^2(\Gamma) + a\lambda + O(\lambda^2),\ \lambda\in(-\varepsilon^\gamma,\varepsilon^\gamma),$$

where $a = \frac{d}{d\lambda}\big|_{\lambda=0}\mathscr{H}^2(\Gamma(\lambda))$. Thus we have that

$$\begin{aligned}
&\frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\Gamma(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau\\
&= \frac{2}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}}\sqrt{\beta W(\xi_{\varepsilon,\gamma}(t))}\mathscr{H}^2(\Gamma(\varepsilon t))|\xi'_{\varepsilon,\gamma}(t)|\,dt\\
&= \frac{2}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}}\sqrt{\beta W(\xi_{\varepsilon,\gamma}(t))}\big(\mathscr{H}^2(\Gamma) + a\varepsilon t + O(\varepsilon^2t^2)\big)|\xi'_{\varepsilon,\gamma}(t)|\,dt\\
&= \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + 2a\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}}\sqrt{\beta W(\xi_{\varepsilon,\gamma}(t))}|\xi'_{\varepsilon,\gamma}(t)|t\,dt + O(\varepsilon)\\
&= \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + O(\varepsilon),
\end{aligned}$$

where we have the fact that $\sqrt{\beta W(\xi_{\varepsilon,\gamma}(t))}|\xi'_{\varepsilon,\gamma}(t)|$ is an even function so that

$$\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}}\sqrt{\beta W(\xi_{\varepsilon,\gamma}(t))}|\xi'_{\varepsilon,\gamma}(t)|t\,dt = 0.$$

Thus we arrive at

$$\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}_A + o_\varepsilon(1). \tag{3.52}$$

This proves part (**A**) of Theorem 1.3. □

3.3 Refined lower bound for the case (B)

The refined lower bound in this case can be done similarly to that of the case (**A**). The major difference arises in showing the homeotropic boundary condition of the limiting map n on the sharp interface Γ, on which we will focus.

First, from $L_4^2 < 4L_2\alpha$ we can find a positive number $\mu > 0$ such that $L_4^2 \le 4(L_2-\mu)(\alpha-\mu)$. Hence by Cauchy-Schwarz inequality we have

$$\begin{aligned}
&|L_4 s_\varepsilon \nabla s_\varepsilon \cdot (\nabla n_\varepsilon) n_\varepsilon| \\
&= |L_4(\nabla s_\varepsilon - (\nabla s_\varepsilon \cdot n_\varepsilon)n_\varepsilon)\cdot(s_\varepsilon \nabla n_\varepsilon)| \\
&\le (L_2-\mu)|\nabla s_\varepsilon - (\nabla s_\varepsilon\cdot n_\varepsilon)n_\varepsilon|^2 + (\alpha-\mu)s_\varepsilon^2|\nabla n_\varepsilon|^2 \\
&= (L_2-\mu)|\nabla s_\varepsilon \wedge n_\varepsilon|^2 + (\alpha-\mu)s_\varepsilon^2|\nabla n_\varepsilon|^2.
\end{aligned} \tag{3.53}$$

This implies that

$$\begin{aligned}
\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) &= (\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)) + s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) \\
&+ \big(\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + L_2|\nabla s_\varepsilon\wedge n_\varepsilon|^2 + L_4(s_\varepsilon\nabla s_\varepsilon)\cdot(\nabla n_\varepsilon)n_\varepsilon\big) \\
&\ge \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon) + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2 + \mu|\nabla s_\varepsilon\wedge n_\varepsilon|^2 \\
&= \big[(\beta+\mu\sin^2\theta_\varepsilon)|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)\big] + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2 \\
&\ge \frac{2}{\varepsilon}|\nabla s_\varepsilon|\sqrt{W(s_\varepsilon)}\sqrt{\beta+\mu\sin^2\theta_\varepsilon} + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2,
\end{aligned}$$

where $\sin\theta_\varepsilon = \frac{\nabla s_\varepsilon}{|\nabla s_\varepsilon|}\wedge n_\varepsilon$.

Note that we also have

$$\begin{aligned}
\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon) &= (\beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)) + s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) \\
&\quad + \big(\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2 + L_2|\nabla s_\varepsilon\wedge n_\varepsilon|^2 + L_4(s_\varepsilon\nabla s_\varepsilon)\cdot(\nabla n_\varepsilon)n_\varepsilon\big) \\
&\ge \beta|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon) + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2 + \mu|\nabla s_\varepsilon\wedge n_\varepsilon|^2 \\
&\ge \frac{2}{\varepsilon}|\nabla s_\varepsilon|\sqrt{\beta W(s_\varepsilon)} + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2 + \mu|\nabla s_\varepsilon\wedge n_\varepsilon|^2.
\end{aligned}$$

Hence, by the co-area formula, this implies that

$$\begin{aligned}
&\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx \\
&\ge \int_\Omega \big(\frac{2}{\varepsilon}|\nabla s_\varepsilon|\sqrt{W(s_\varepsilon)}\sqrt{\beta+\mu\sin^2\theta_\varepsilon} + \mu s_\varepsilon^2|\nabla n_\varepsilon|^2\big)\,dx \\
&\ge \mu\int_\Omega s_\varepsilon^2|\nabla n_\varepsilon|^2\,dx + \frac{2\mu_*}{\varepsilon}\int_0^{s_+}\sqrt{W(\tau)}\int_{\partial^*\mathscr{S}_\varepsilon(\tau)}\sin^2\theta_\varepsilon\,d\mathscr{H}^2\,d\tau \\
&\quad + \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau,
\end{aligned} \tag{3.54}$$

where $\mu_* = \dfrac{\mu}{\sqrt{\mu+\beta}+\sqrt{\beta}}$, and

$$\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx$$

$$\geq \mu \int_\Omega \left(s_\varepsilon^2 |\nabla n_\varepsilon|^2 + |\nabla s_\varepsilon \wedge n_\varepsilon|^2\right)dx + \frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau))\,d\tau. \tag{3.55}$$

Matching (3.54) and (3.55) with the upper bound (3.14), we conclude that

$$\mu \int_\Omega (s_\varepsilon^2|\nabla n_\varepsilon|^2 + |\nabla s_\varepsilon \wedge n_\varepsilon|^2)\,dx \leq \mathscr{D}_B + C + o_\varepsilon(1), \tag{3.56}$$

$$\frac{\mu_*}{\varepsilon}\int_0^{s_+} \sqrt{W(\tau)} \int_{\partial^* \mathscr{S}_\varepsilon(\tau)} \sin^2\theta_\varepsilon\, d\mathscr{H}^2\, d\tau \leq \mathscr{D}_B + C + o_\varepsilon(1), \tag{3.57}$$

and

$$\frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau))\,d\tau \leq \frac{1}{\varepsilon}\alpha_0 \mathscr{H}^2(\Gamma) + \mathscr{D}_B + o_\varepsilon(1). \tag{3.58}$$

As in the previous section, for any small $\delta > 0$ there exist $C > 0$, independent of ε, and $\delta_\varepsilon \in (\delta, 2\delta)$ such that

$$\int_{\partial^* \mathscr{S}_\varepsilon(s_+ - \delta_\varepsilon)} \sin^2\theta_\varepsilon\, d\mathscr{H}^2 \leq C\frac{\varepsilon}{\delta}, \tag{3.59}$$

$$\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(s_+ - \delta_\varepsilon)) \leq \mathscr{H}^2(\Gamma) + C\frac{\varepsilon}{\delta}. \tag{3.60}$$

Moreover, from (3.57) we have

$$\int_{\Omega^+_{\varepsilon,\delta_\varepsilon}} \left(|\nabla n_\varepsilon|^2 + |\nabla s_\varepsilon \wedge n_\varepsilon|^2\right)dx \leq C. \tag{3.61}$$

As in the previous section, there exists $\varepsilon_i \to 0$ and $\delta_i \in (\varepsilon_i^{\frac{1}{4}}, 2\varepsilon_i^{\frac{1}{4}})$ such that $\Omega_i = \Omega^+_{\varepsilon_i,\delta_i}$ converges to Ω^+ weakly in $BV(\mathbb{R}^3)$,

$$\Gamma_i = \partial\Omega_i \lfloor \Omega \rightharpoonup \Gamma = \partial\Omega^+ \lfloor \Omega,$$

as convergence of measures and integral 2-currents. Furthermore, there exists a map $n \in H^1(\Omega^+, \mathbb{S}^2)$ such that

$$n_{\varepsilon_i}\chi_{\Omega_i} \rightharpoonup n\chi_{\Omega^+} \text{ in } \mathrm{BV}(\Omega), \text{ and } s_{\varepsilon_i}\chi_{\Omega_i} \to s_+\chi_{\Omega^+} \text{ in } L^2(\Omega),$$

and hence

$$\nabla n_{\varepsilon_i}\chi_{\Omega_i} \rightharpoonup \nabla n\chi_{\Omega^+}, \text{ and } s_{\varepsilon_i}\nabla n_{\varepsilon_i}\chi_{\Omega_i} \rightharpoonup s_+\nabla n\chi_{\Omega^+} \text{ in } L^2(\Omega).$$

As a consequence of these weak convergences and (3.61), we can deduce

$$\nabla s_{\varepsilon_i} \wedge n_{\varepsilon_i} \rightharpoonup 0 \text{ in } L^2(\Omega).$$

Similar to the proof of (3.34) and (3.35), we can obtain

$$\int_{\Omega^+} s_+^2 W_{OF}(n, \nabla n)\,dx \leq \liminf_{\varepsilon_i \to 0} \int_{\Omega_i} s_{\varepsilon_i}^2 W_{OF}(n_{\varepsilon_i}, \nabla n_{\varepsilon_i})\,dx, \tag{3.62}$$

and

$$\int_{\Omega^+} \alpha s_+^2 |\nabla n|^2 \, dx$$

$$\leq \liminf_{\varepsilon_i \to 0} \int_{\Omega_i} \left(\alpha s_{\varepsilon_i}^2 |\nabla n_{\varepsilon_i}|^2 + L_2 |\nabla s_{\varepsilon_i} \wedge n_{\varepsilon_i}|^2 + L_4 (s_{\varepsilon_i} \nabla s_{\varepsilon_i} \cdot (\nabla n_\varepsilon) n_\varepsilon \right) dx. \tag{3.63}$$

Adding (3.62) with (3.63), we arrive at

$$\begin{aligned} E(n,\Omega^+) &= \int_{\Omega^+} s_+^2 \left(W_{OF}(n,\nabla n) + \alpha |\nabla n|^2 \right) dx \\ &\leq \liminf_{\varepsilon_i \to 0} \int_{\Omega} \left(s_{\varepsilon_i}^2 W_{OF}(n_{\varepsilon_i}, \nabla n_{\varepsilon_i}) + \alpha s_{\varepsilon_i}^2 |\nabla n_{\varepsilon_i}|^2 \right. \\ &\quad \left. + L_2 |\nabla s_{\varepsilon_i} \wedge n_{\varepsilon_i}|^2 + L_4 s_{\varepsilon_i} \nabla s_{\varepsilon_i} \cdot (\nabla n_{\varepsilon_i}) n_{\varepsilon_i} \right) dx. \end{aligned} \tag{3.64}$$

Now we want to show the homeotropic condition of n on Γ, i.e.,

$$n \wedge \nu_\Gamma = 0 \text{ on } \Gamma. \tag{3.65}$$

In order to show (3.65), we first want to prove

$$\int_\Gamma n \wedge \nu_\Gamma \, d\mathscr{H}^2 = \lim_{i \to \infty} \int_{\Gamma_i} n_i \wedge \nu_{\Gamma_i} \, d\mathscr{H}^2. \tag{3.66}$$

In fact, since

$$\nabla \times n_i \rightharpoonup \nabla \times n \text{ in } L^2(\Omega^+), \text{ and } \chi_{\Omega_i} \to \chi_{\Omega^+} \text{ in } L^2(\mathbb{R}^3),$$

we must have

$$\lim_{i \to \infty} \int_{\Omega_i} \nabla \times n_i \, dx = \int_{\Omega^+} \nabla \times n \, dx.$$

This, combined with the divergence theorem, implies that

$$\begin{aligned} &\int_{\partial\Omega_i \cap \partial\Omega} n_i \wedge \nu_{\partial\Omega_i} \, d\mathscr{H}^2 + \int_{\Gamma_i} n_i \wedge \nu_{\Gamma_i} \, d\mathscr{H}^2 \\ &= \int_{(\partial\Omega_i \cap \partial\Omega) \cup \Gamma_i} n_i \wedge \nu_{\partial\Omega_i} \, d\mathscr{H}^2 = \int_{\Omega_i} \nabla \times n_i \, dx \\ &\to \int_{\Omega^+} \nabla \times n \, dx = \int_{(\partial\Omega^+ \cap \partial\Omega) \cup \Gamma} n \wedge \nu_{\partial\Omega^+} \, d\mathscr{H}^2 \\ &= \int_{\partial\Omega^+ \cap \partial\Omega} n \wedge \nu_{\partial\Omega} \, d\mathscr{H}^2 + \int_\Gamma n \wedge \nu_\Gamma \, d\mathscr{H}^2. \end{aligned}$$

On other hand, it follows from the assumption on g_{ε_i} on $\partial\Omega$ that

$$\begin{aligned} &\int_{\partial\Omega_i \cap \partial\Omega} n_i \wedge \nu_{\partial\Omega_i} \, d\mathscr{H}^2 = \int_{\partial\Omega_i \cap \partial\Omega} g_{\varepsilon_i} \wedge \nu_{\partial\Omega_i} \, d\mathscr{H}^2 \\ &\to \int_{\partial\Omega^+ \cap \partial\Omega} g \wedge \nu_{\partial\Omega} \, d\mathscr{H}^2 = \int_{\partial\Omega^+ \cap \partial\Omega} n \wedge \nu_{\partial\Omega} \, d\mathscr{H}^2. \end{aligned}$$

Thus (3.66) holds.

From (3.66), we can use a similar argument to show that for any nonnegative convex function $f : \mathbb{R} \to \mathbb{R}_+$, it holds that

$$\int_\Gamma f(n \wedge \nu_\Gamma)\, d\mathscr{H}^2 \le \liminf_{i\to\infty} \int_{\Gamma_i} f(n_i \wedge \nu_{\Gamma_i})\, d\mathscr{H}^2. \tag{3.67}$$

In particular, this implies that

$$\int_\Gamma |n \wedge \nu_\Gamma|^2\, d\mathscr{H}^2 \le \liminf_{i\to\infty} \int_{\Gamma_i} |n_i \wedge \nu_{\Gamma_i}|^2\, d\mathscr{H}^2 = \liminf_{i\to\infty} \int_{\Gamma_i} \sin^2\theta_{\varepsilon_i}\, d\mathscr{H}^2 = 0. \tag{3.68}$$

This yields (3.65).

For the convenience of readers, we will sketch the proof of (3.67) as follows. Define a family of Radon measures

$$\mathscr{M}_i(A) = \int_A f(n_i \wedge \nu_{\Gamma_i})\, d\Theta_i,$$

for any measurable set $A \subset \mathbb{R}^3$.

Without loss of generality, we can assume that there exists a Radon measure $\mathscr{M}$ in $\mathbb{R}^3$ such that

$$\mathscr{M}_i \rightharpoonup \mathscr{M}$$

as convergence of Radon measures on $\mathbb{R}^3$. Again by Radon-Nikodym theorem, we can decompose

$$\mathscr{M} = (D_\Theta \mathscr{M})\Theta + \mathscr{M}^s,\ \mathscr{M}^s \perp \mathscr{M}.$$

Hence for any open set $O \subset \mathbb{R}^3$, it holds that

$$\int_O D_\Theta \mathscr{M}\, d\Theta \le \mathscr{M}(O) \le \liminf_{i\to\infty} \mathscr{M}_i(O).$$

Now we want to show

$$D_\Theta \mathscr{M}(x) \ge f(n \wedge \nu_\Gamma)(x),\ \Theta \text{ a.e. } x \in \Gamma. \tag{3.69}$$

First, it is not hard to see that for any $x \in \Gamma$, it holds that for L^1 a.e. $r > 0$,

$$\lim_{i\to\infty} \int_{\partial B_r(x)\cap\Omega_i} n_i \wedge \frac{y-x}{|y-x|}\, d\mathscr{H}^2 = \int_{\partial B_r(x)\cap\Omega^+} n \wedge \frac{y-x}{|y-x|}\, d\mathscr{H}^2.$$

This, combined with

$$\begin{aligned}\int_{\partial(B_r(x)\cap\Omega_i)} n_i \wedge \nu_{\partial(B_r(x)\cap\Omega_i)} &= \int_{B_r(x)\cap\Omega_i} \nabla \times n_i \\ \to \int_{B_r(x)\cap\Omega^+} \nabla \times n &= \int_{\partial(B_r(x)\cap\Omega^+)} n \wedge \nu_{\partial(B_r(x)\cap\Omega^+)},\end{aligned}$$

yields that

$$\lim_{i\to\infty} \int_{B_r(x)\cap\Gamma_i} n_i \wedge \nu_{\Gamma_i}\, d\mathscr{H}^2 = \int_{B_r(x)\cap\Gamma} n \wedge \nu_\Gamma\, d\mathscr{H}^2.$$

It is readily seen that

$$\mathscr{M}(\partial B_r(x)) = 0$$

holds for L^1 a.e. $r > 0$.

Therefore, for any given $x \in \Gamma$,

$$
\begin{aligned}
\frac{\mathscr{M}(B_r(x))}{\Theta(B_r(x))} &= \lim_{i\to\infty} \frac{\mathscr{M}_i(B_r(x))}{\Theta(B_r(x))} \\
&\geq \lim_{i\to\infty} \frac{\int_{B_r(x)\cap\Gamma_i}(a_k n_i \wedge \nu_{\Gamma_i} + b_k)\,d\mathscr{H}^2}{\Theta(B_r(x))} \\
&= a_k \lim_{i\to\infty} \frac{\int_{B_r(x)\cap\Gamma_i} n_i \wedge \nu_{\Gamma_i}\,d\mathscr{H}^2}{\Theta(B_r(x))} + b_k \\
&= a_k \frac{\int_{B_r(x)\cap\Gamma} n \wedge \nu_{\Gamma}\,d\mathscr{H}^2}{\Theta(B_r(x))} + b_k
\end{aligned}
\tag{3.70}
$$

holds for any $k \geq 1$ and L^1 a.e. $r > 0$.

Since

$$
D_\Theta \mathscr{M}(x) = \lim_{r\to 0} \frac{\mathscr{M}(B_r(x))}{\Theta(B_r(x))},
$$

and

$$
(n\wedge\nu_\Gamma)(x) = \lim_{r\to 0} \frac{\int_{B_r(x)\cap\Gamma} n\wedge\nu_\Gamma\,d\mathscr{H}^2}{\Theta(B_r(x))}
$$

hold for Θ a.e. $x \in \Gamma$, after passing to the limit in (3.70) we obtain that for Θ a.e. $x \in \Gamma$,

$$
D_\Theta \mathscr{M}(x) \geq a_k (n\wedge\nu_\Gamma)(x) + b_k,\ \forall k \geq 1.
$$

Taking supremum over $k \geq 1$ and using (3.43), this yields (3.69).

From (3.65), we conclude that by

$$
E(n;\ \Omega^+) \geq \mathscr{D}_B,
$$

which, combined with (3.55) and (3.64), implies that

$$
\begin{aligned}
&\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx \\
&\geq \frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{S}_\varepsilon(\tau))\,d\tau + \mathscr{D}_B + o_\varepsilon(1).
\end{aligned}
\tag{3.71}
$$

This, combined with the inequality (3.48) and the upper bound (3.14), further implies that

$$
E(n;\ \Omega^+) = \mathscr{D}_B,
$$

$$
\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}_B + o_\varepsilon(1).
\tag{3.72}
$$

The part (**B**) of Theorem 1.3 is proven. □

3.4 Refined lower bound for the case (C)

This case is the easiest among the three cases we discuss in this paper. In fact, by (3.15) and direct calculations, we obtain that

$$
\begin{aligned}
&\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma)+\mathscr{D}_C+o_\varepsilon(1)\\
&\geq \int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\\
&=\int_\Omega \Big[(\beta|\nabla s_\varepsilon|^2+\frac{1}{\varepsilon^2}W(s_\varepsilon))+(s_\varepsilon^2 W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha s_\varepsilon^2|\nabla n_\varepsilon|^2)\Big]\,dx\\
&\geq \frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{S}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau\\
&\quad+(s_+-\delta)^2\int_{\Omega^+_{\varepsilon,\delta}}(W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha|\nabla n_\varepsilon|^2)\,dx\\
&\geq \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma)-C\\
&\quad+(s_+-\delta)^2\int_{\Omega^+_{\varepsilon,\delta}}(W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha|\nabla n_\varepsilon|^2)\,dx.
\end{aligned}
$$

This implies that for any $\delta>0$, there exists $\delta_\varepsilon\in(\delta,2\delta)$ such that

$$
\int_{\Omega^+_{\varepsilon,\delta_\varepsilon}}(W_{OF}(n_\varepsilon,\nabla n_\varepsilon)+\alpha|\nabla n_\varepsilon|^2)\,dx\leq\frac{1}{(s_+-\delta_\varepsilon)^2}\big(\mathscr{D}_C+C+o_\varepsilon(1)\big), \tag{3.73}
$$

and

$$
\mathscr{H}^2(\mathscr{S}_\varepsilon(s_+-\delta_\varepsilon))\leq\mathscr{H}^2(\Gamma)+C\frac{\varepsilon}{\delta}. \tag{3.74}
$$

As in the previous two cases, we can argue as follows. For $\varepsilon_i\to0$, there exists $\delta_{\varepsilon_i}\in(\varepsilon_i^{\frac12},2\varepsilon_i^{\frac12})$ such that $\Omega_i=\Omega^+_{\varepsilon_i,\delta_{\varepsilon_i}}\to\Omega^+$ weakly in $\mathrm{BV}(\mathbb{R}^3)$,

$$
\mathscr{S}_\varepsilon(s_+-\delta_{\varepsilon_i})=\partial\Omega_i\lfloor\Omega\rightharpoonup\Gamma=\partial\Omega^+\lfloor\Omega,
$$

weakly converges as measures and integral currents. Furthermore, there exists $n\in H^1(\Omega^+)$ with $n=g$ on Σ^+ such that

$$
s_{\varepsilon_i}\nabla n_{\varepsilon_i}\chi_{\Omega_i}\rightharpoonup s_+\nabla n\chi_{\Omega^+}\ \text{ in }\ L^2(\Omega).
$$

By the lower semicontinuity, we then have

$$
\begin{aligned}
\mathscr{D}_C&\leq\int_{\Omega^+}s_+^2(W_{OF}(n,\nabla n)+\alpha|\nabla n|^2)\,dx\\
&\leq\liminf_{\varepsilon_i\to0}\int_\Omega\big(s_{\varepsilon_i}^2W_{OF}(n_{\varepsilon_i},\nabla n_{\varepsilon_i})+\alpha s_{\varepsilon_i}^2|\nabla n_{\varepsilon_i}|^2\big)\,dx.
\end{aligned}\tag{3.75}
$$

On the other hand, it follows from the inequality (3.48) that

$$
\frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\mathscr{S}(\varepsilon\xi_{\varepsilon,\gamma}^{-1}(\tau))\,d\tau\geq\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma)-C\varepsilon.
$$

Thus we can conclude that

$$E(n;\,\Omega^+) = \mathscr{D}_C,$$

and

$$\int_\Omega \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\Gamma) + \mathscr{D}_C + o_\varepsilon(1).$$

This establishes the conclusion of part (C) in Theorem 1.3. □

4 Proof of Theorem 1.4

In this section, we will consider the asymptotic expansion of $\mathscr{E}(s_\varepsilon, n_\varepsilon)$ for $\Omega = \mathbb{R}^3$ and prove Theorem 1.4. We will first provide a sharp upper bound estimate for the cases (**A**), (**B**), and (**C**).

4.1 Refined upper bounds

The constructions are similar to those in the section 2.2.2, except that we need to use an almost minimal 1-dimensional connecting orbit in the fast transition region of width $O(\varepsilon^\gamma)$ around ∂B_1.

Let $\xi_{\varepsilon,\gamma} \in C^\infty([-\varepsilon^\gamma, \varepsilon^\gamma])$ be given by (1.22). Define $\hat{s}_\varepsilon : \mathbb{R}^3 \to \mathbb{R}_+$ by letting

$$\hat{s}_\varepsilon(x) = \begin{cases} s_+ & |x| \le 1, \\ \xi_{\varepsilon,\gamma}\left(\frac{|x|-(1+\varepsilon^\gamma)}{\varepsilon}\right) & 1 \le |x| \le 1+2\varepsilon^\gamma, \\ 0 & |x| \ge 1+2\varepsilon^\gamma. \end{cases}$$

Then we have

$$\left\{x \in \mathbb{R}^3 : \hat{s}_\varepsilon(x) \ge s_+\right\} = B_1,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|\nabla \hat{s}_\varepsilon|^2 + \frac{1}{\varepsilon^2} W(\hat{s}_\varepsilon)\right) dx \\ &= \frac{1}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}} \left(\beta|\xi'_{\varepsilon,\gamma}(t)|^2 + W(\xi_{\varepsilon,\gamma}(t))\right)\mathscr{H}^2(\partial B_{1+\varepsilon^\gamma+\varepsilon t})\,dt \\ &= \frac{4\pi}{\varepsilon}\int_{-\varepsilon^{\gamma-1}}^{\varepsilon^{\gamma-1}} \left(\beta|\xi'_{\varepsilon,\gamma}(t)|^2 + W(\xi_{\varepsilon,\gamma}(t))\right) \\ &\qquad \cdot[(1+\varepsilon^\gamma)^2 + 2\varepsilon(1+\varepsilon^\gamma)t + \varepsilon^2 t^2]\,dt \\ &= \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1) + C\varepsilon^{2\gamma-1} + o_\varepsilon(1) = \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1) + o_\varepsilon(1), \end{aligned} \tag{4.1}$$

since $\gamma > \frac{1}{2}$.

Now we divide the discussion into three different cases:

Case (**A**) Let $n \in H^1(B_1, \mathbb{S}^2)$ be such that $n \cdot \nu_{\partial B_1} = 0$ on ∂B_1, and

$$E(n; B_1) = \mathscr{D}_A, \tag{4.2}$$

where $\mathscr{D}_A$ is given by (1.36) and (1.37). Assume $\bar{n} \in H^1(B_{1+2\varepsilon^\gamma}, \mathbb{S}^2)$ be such that $\bar{n} = n$ in B_1, and

$$\int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\nabla \bar{n}|^2 \, dx = o_\varepsilon(1).$$

Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \hat{s}_\varepsilon^2 \big(W_{OF}(\bar{n}, \nabla \bar{n}) + \alpha |\nabla \bar{n}|^2\big)\, dx \\
&\leq \int_{B_1} s_+^2 \big(W_{OF}(n, \nabla n) + \alpha |\nabla n|^2\big)\, dx + C s_+^2 \int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\nabla \bar{n}|^2 \, dx \\
&= \mathscr{D}_A + o_\varepsilon(1).
\end{aligned} \tag{4.3}$$

While

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^3} \big(L_1 (\nabla \hat{s}_\varepsilon \cdot \bar{n})^2 + L_3 (\nabla \hat{s}_\varepsilon \cdot \bar{n})(\hat{s}_\varepsilon \operatorname{div} \bar{n})\big)\, dx \Big| \\
&= \Big| \int_{B_{1+2\varepsilon^\gamma}\setminus B_1} \big(L_1 (\nabla \hat{s}_\varepsilon \cdot \bar{n})^2 + L_3 (\nabla \hat{s}_\varepsilon \cdot \bar{n})(\hat{s}_\varepsilon \operatorname{div} \bar{n})\big)\, dx \Big| \\
&\leq C \Big(\int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\nabla \hat{s}_\varepsilon \cdot \bar{n}|^2 \, dx + \int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\nabla \bar{n}|^2 \, dx \Big) \\
&\leq o_\varepsilon(1) + C\varepsilon^{-2} \int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\xi'_{\varepsilon,\gamma}|^2 \Big(\frac{|x| - (1+\varepsilon^\gamma)}{\varepsilon} \Big) \Big| \bar{n}(x) - \bar{n}\Big(\frac{x}{|x|}\Big) \Big|^2 \, dx \\
&\leq o_\varepsilon(1) + C \int_{B_{1+2\varepsilon^\varepsilon}\setminus B_1} \big|\nabla \bar{n}\big|^2 \, dx = o_\varepsilon(1).
\end{aligned} \tag{4.4}$$

Therefore by putting together (4.1), (4.3), and (4.4), we arrive at

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx &\leq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(\hat{s}_\varepsilon, \bar{n}, \nabla \hat{s}_\varepsilon, \nabla \bar{n})\, dx \\
&\leq \frac{\alpha_0}{\varepsilon} \mathscr{H}^2(\partial B_1) + \mathscr{D}_A.
\end{aligned} \tag{4.5}$$

Case (**B**) Let $n \in H^1(B_1, \mathbb{S}^2)$ be such that $n \wedge \nu_{\partial B_1} = 0$ on ∂B_1, and

$$E(n; B_1) = \mathscr{D}_B, \tag{4.6}$$

where $\mathscr{D}_B$ is given by (1.39) and (1.40). Assume $\bar{n} \in H^1(B_{1+2\varepsilon^\gamma}, \mathbb{S}^2)$ be such that $\bar{n} = n$ in B_1, and

$$\int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\nabla \bar{n}|^2 \, dx = o_\varepsilon(1).$$

Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \hat{s}_\varepsilon^2 \big(W_{OF}(\bar{n}, \nabla \bar{n}) + \alpha |\nabla \bar{n}|^2\big)\, dx \\
&\leq \int_{B_1} s_+^2 \big(W_{OF}(n, \nabla n) + \alpha |\nabla n|^2\big)\, dx + C s_+^2 \int_{B_{1+2\varepsilon^\gamma}\setminus B_1} |\nabla \bar{n}|^2 \, dx = \mathscr{D}_B + o_\varepsilon(1).
\end{aligned} \tag{4.7}$$

While

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\big(L_2|\nabla\hat{s}_\varepsilon\wedge\bar{n}|^2+L_4(\hat{s}_\varepsilon\nabla\hat{s}_\varepsilon)(\nabla\bar{n})\bar{n}\big)\,dx\Big|\\
&=\Big|\int_{B_{1+2\varepsilon^\gamma}\setminus B_1}\big(L_2|\nabla\hat{s}_\varepsilon\wedge\bar{n}|^2+L_4(\hat{s}_\varepsilon\nabla\hat{s}_\varepsilon)(\nabla\bar{n})\bar{n}\big)\,dx\Big|\\
&\le C\Big(\int_{B_{1+2\varepsilon^\gamma}\setminus B_1}|\nabla\hat{s}_\varepsilon\wedge\bar{n}|^2\,dx+\int_{B_{1+2\varepsilon^\gamma}\setminus B_1}|\nabla\bar{n}|^2\,dx\Big)\\
&\le o_\varepsilon(1)+C\varepsilon^{-2}\int_{B_{1+2\varepsilon^\gamma}\setminus B_1}|\xi'_{\varepsilon,\gamma}|^2\Big(\frac{|x|-(1+\varepsilon^\gamma)}{\varepsilon}\Big)\Big|\bar{n}(x)-\bar{n}\Big(\frac{x}{|x|}\Big)\Big|^2\,dx\\
&\le o_\varepsilon(1)+C\int_{B_{1+2\varepsilon^\varepsilon}\setminus B_1}\big|\nabla\bar{n}\big|^2\,dx=o_\varepsilon(1).
\end{aligned}
\tag{4.8}
$$

Therefore by putting together (4.1), (4.7), and (4.8), we arrive at

$$
\begin{aligned}
\limsup_{\varepsilon\to0}\int_{\mathbb{R}^3}\widetilde{\mathscr{W}}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx &\le \limsup_{\varepsilon\to0}\int_{\mathbb{R}^3}\widetilde{\mathscr{W}}(\hat{s}_\varepsilon,\bar{n},\nabla\hat{s}_\varepsilon,\nabla\bar{n})\,dx\\
&\le \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1)+\mathscr{D}_B.
\end{aligned}
\tag{4.9}
$$

Case (**C**) Let $n\equiv(0,0,1)\in\mathbb{S}^2$. Then by (4.1) we see that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\widetilde{\mathscr{W}}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx &\le \int_{\mathbb{R}^3}\widetilde{\mathscr{W}}(\hat{s}_\varepsilon,\bar{n},\nabla\hat{s}_\varepsilon,\nabla\bar{n})\,dx\\
&= \int_{\mathbb{R}^3}\big(|\nabla\hat{s}_\varepsilon|^2+\frac{1}{\varepsilon^2}W(\hat{s}_\varepsilon)\big)\,dx\\
&\le \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1)+o_\varepsilon(1).
\end{aligned}
\tag{4.10}
$$

4.2 Refined lower bounds

In this subsection, we will sketch the proof of a sharp a lower bound estimate for the cases (**A**), (**B**), and (**C**). The ideas are similar to those presented in the section 3 for bounded domain cases, except that we will work on the entire space $\mathbb{R}^3$ where we only have the weak compactness property of $\mathrm{BV}(\mathbb{R}^3)$ locally in $\mathbb{R}^3$. We will focus on the case (**A**), and only sketch the cases (**B**) and (**C**).

Case (**A**) First, as in the discussion of the section 3.2, there exists $\mu>0$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon,n_\varepsilon,\nabla s_\varepsilon,\nabla n_\varepsilon)\,dx\\
&\ge\frac{2}{\varepsilon}\int_0^{s_+}\sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau+\mu\int_{\mathbb{R}^3}\big(s_\varepsilon^2|\nabla n_\varepsilon|^2+(\nabla s_\varepsilon\cdot n_\varepsilon)^2\big)\,dx,
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx \\
&\geq \mu \int_{\mathbb{R}^3} s_\varepsilon^2 |\nabla n_\varepsilon|^2\,dx + \frac{2\mu_*}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)} \int_{\partial^* \mathscr{S}_\varepsilon(\tau)} \cos^2\theta_\varepsilon \, d\mathscr{H}^2\, d\tau \\
&+\frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau))\,d\tau, \qquad (4.12)
\end{aligned}
$$

where $\mathscr{S}_\varepsilon(\tau) = \{x \in \mathbb{R}^3 : \ s_\varepsilon(x) \geq \tau\}$, $\cos\theta_\varepsilon = \frac{\nabla s_\varepsilon}{|\nabla s_\varepsilon|} \cdot n_\varepsilon$, and

$$
\mu_* = \frac{\mu}{\sqrt{\beta+\mu} + \sqrt{\beta}} > 0.
$$

Notice that by the isoperimetric inequality (see, e.g., Case (C) below) we have

$$
\frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau))\,d\tau \geq \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1). \qquad (4.13)
$$

It follows from (4.11), (4.12), (4.13) and (4.5) that there exists $\tau_\varepsilon \in (0, s_+)$ such that

$$
\int_{\mathscr{S}_\varepsilon(\tau_\varepsilon)} (s_\varepsilon^2 |\nabla n_\varepsilon|^2 + |\nabla s_\varepsilon \cdot n_\varepsilon|^2)\,dx \leq C(\mu,\ \mathscr{D}_A), \qquad (4.14)
$$

$$
\int_{\mathscr{S}_\varepsilon(\tau_\varepsilon)} \cos^2\theta_\varepsilon \, d\mathscr{H}^2 \leq C\varepsilon, \qquad (4.15)
$$

and

$$
\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau_\varepsilon)) \leq \mathscr{H}^2(\partial B_1) + o_\varepsilon(1). \qquad (4.16)
$$

By the isoperimetric inequality and the volume constraint condition, we have that

$$
|B_1| \leq \big|\mathscr{S}_\varepsilon(\tau_\varepsilon)\big| \leq |B_1|\Big(\frac{\big|\mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau_\varepsilon))\big|}{\mathscr{H}^2(\partial B_1)}\Big)^{\frac{3}{2}} \leq |B_1|(1 + o_\varepsilon(1)). \qquad (4.17)
$$

To simplify the presentation, we denote by $E_\varepsilon = \mathscr{S}_\varepsilon(\tau_\varepsilon)$. Although E_ε may not converge in $L^1(\mathbb{R}^3)$ due to the non-compactness of $\mathbb{R}^3$, we can apply the quantitative stability theorem by Fusco-Maggi-Pratelli [15] (see also [40]) to show that E_ε does converge in $L^1(\mathbb{R}^3)$ after suitable translations. In fact, if we set the isoperimetric deficit and Fraenkel asymmetry by

$$
A(E_\varepsilon) = \frac{\mathscr{H}^2(E_\varepsilon)}{3|B_1|^{\frac{1}{3}}|E_\varepsilon|^{\frac{2}{3}}} - 1, \text{ and } \ \delta(E_\varepsilon) = \inf\Big\{\frac{|E_\varepsilon \Delta B_r(x)|}{|E_\varepsilon|} : \ |B_r(x)| = |E_\varepsilon|\Big\}, \qquad (4.18)
$$

then it follows from [15] that

$$
A(E_\varepsilon) \leq C\delta(E_\varepsilon)^{\frac{1}{2}} \leq o_\varepsilon(1), \qquad (4.19)
$$

where we have used (4.16) and (4.17) in the second inequality of (4.19). From (4.17), there exist $x_\varepsilon \in \mathbb{R}^3$ and $r_\varepsilon = 1 + o_\varepsilon(1)$ such that after passing to a subsequence,

$$|E_\varepsilon \Delta B_{r_\varepsilon}(x_\varepsilon)| \to 0, \text{ as } \varepsilon \to 0,$$

or equivalently,

$$\widehat{E}_\varepsilon \equiv E_\varepsilon \setminus \{x_\varepsilon\} \to B_1 \text{ in } L^1(\mathbb{R}^3). \tag{4.20}$$

Since the problem is invariant under translations, for simplicity we may assume that $x_\varepsilon = 0$ so that $E_\varepsilon = \widehat{E}_\varepsilon$. The rest of argument can be done almost identically to the case (A) of Theorem 1.3 presented in section 3.2. For instance, we can show that there exists $n \in H^1(B_1, \mathbb{S}^2)$ such that

$$s_\varepsilon \to s_+ \text{ in } L^2(B_1), \ n_\varepsilon \rightharpoonup n \text{ in } H^1(B_1),$$

and

$$n(x) \cdot x = 0 \text{ on } \partial B_1.$$

Moreover, by the lower semicontinuity we have that

$$\begin{aligned}
&\mathscr{D}_A \le E(n; B_1) = s_+^2 \int_{B_1} (W_{OF}(n, \nabla n) + \alpha |\nabla n|^2)\, dx \\
&\le \liminf_{\varepsilon \to 0} \int_{E_\varepsilon} \big(s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) + \alpha |\nabla n_\varepsilon|^2 \\
&\ +L_1 (\nabla s_\varepsilon \cdot n_\varepsilon)^2 + L_3 (\nabla s_\varepsilon \cdot n_\varepsilon) s_\varepsilon \mathrm{div} n_\varepsilon\big)\, dx \\
&= \liminf \Big(\int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx - \int_{\mathbb{R}^3} \big(|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2} W(s_\varepsilon)\big)\, dx \Big) \\
&\le \mathscr{D}_A.
\end{aligned}$$

Hence

$$E(n; B_1) = \mathscr{D}_A, \ \int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx \ge \frac{\alpha_0}{\varepsilon} \mathscr{H}^2(\partial B_1) + \mathscr{D}_A + o_\varepsilon(1).$$

Case (**B**) First, as in the discussion of the section 3.2, there exists $\mu > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx \\
&\ge \frac{2\mu_*}{\varepsilon} \int_0^{s_+} \sqrt{W(\tau)} \int_{\partial^* \mathscr{S}_\varepsilon(\tau)} \sin^2 \theta_\varepsilon \, d\mathscr{H}^2 \\
&\ + \frac{2}{\varepsilon} \int_0^{s_+} \sqrt{\beta W(\tau)} \mathscr{H}^2(\partial^* \mathscr{S}_\varepsilon(\tau))\, d\tau,
\end{aligned} \tag{4.21}$$

where $\mu_* = \dfrac{\mu}{\sqrt{\mu+\beta}+\sqrt{\beta}}$ and $\sin^2\theta_\varepsilon = |\frac{\nabla s_\varepsilon}{|\nabla s_\varepsilon|}\wedge n_\varepsilon|^2$, and

$$\begin{aligned}
&\int_{\mathbb{R}^3} \widetilde{\mathscr{W}_\varepsilon}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx \\
&\geq \mu \int_{\mathbb{R}^3} \left(s_\varepsilon^2|\nabla n_\varepsilon|^2 + |\nabla s_\varepsilon \wedge n_\varepsilon|^2\right) dx \\
&+\frac{2}{\varepsilon}\int_0^{s_+} \sqrt{\beta W(\tau)}\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau))\,d\tau.
\end{aligned}\tag{4.22}$$

As in the case (**A**), we can find $\tau_\varepsilon \in (0, s_+)$ such that

$$\int_{\mathscr{S}_\varepsilon(\tau_\varepsilon)} (s_\varepsilon^2|\nabla n_\varepsilon|^2 + |\nabla s_\varepsilon \wedge n_\varepsilon|^2)\,dx \leq C(\mu,\ \mathscr{D}_B),\tag{4.23}$$

$$\int_{\mathscr{S}_\varepsilon(\tau_\varepsilon)} \sin^2\theta_\varepsilon\, d\mathscr{H}^2 \leq C\varepsilon,\tag{4.24}$$

and

$$\mathscr{H}^2(\partial^*\mathscr{S}_\varepsilon(\tau_\varepsilon)) \leq \mathscr{H}^2(\partial B_1) + o_\varepsilon(1).\tag{4.25}$$

As in the discussion of Case (**A**) above, we can apply the quantitative stability theorem of [15] to conclude that after passing to a subsequence,

$$S_\varepsilon(\tau_\varepsilon) \to B_1 \ \text{ in } \ L^1(\mathbb{R}^3).$$

Furthermore, by an argument similar to Case (**B**) of Theorem 1.3 presented in the section 3.3, there exists a $n \in H^1(B_1, \mathbb{S}^2)$ such that

$$s_\varepsilon \to s_+ \ \text{ in } \ L^2(B_1),\ \ n_\varepsilon \rightharpoonup n \ \text{ in } \ H^1(B_1),$$

$$n(x)\wedge x = 0 \ \text{ on } \ \partial B_1,$$

and by the lower semicontinuity,

$$\begin{aligned}
&\mathscr{D}_B \leq E(n; B_1) = s_+^2 \int_{B_1} (W_{OF}(n, \nabla n) + \alpha|\nabla n|^2)\,dx \\
&\leq \liminf_{\varepsilon\to 0} \int_{E_\varepsilon} \big(s_\varepsilon^2 W_{OF}(n_\varepsilon, \nabla n_\varepsilon) \\
&\ +\alpha|\nabla n_\varepsilon|^2 + L_2|\nabla s_\varepsilon \wedge n_\varepsilon|^2 + L_4 s_\varepsilon \nabla s_\varepsilon\cdot(\nabla n_\varepsilon)n_\varepsilon\big)\,dx \\
&= \liminf_{\varepsilon\to 0}\Big(\int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx - \int_{\mathbb{R}^3}\big(|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(s_\varepsilon)\big)\,dx\Big) \\
&\leq \lim_{\varepsilon\to 0}\Big(\frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1) + \mathscr{D}_B + o_\varepsilon(1) - \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1)\Big) = \mathscr{D}_B.
\end{aligned}$$

Hence

$$E(n; B_1) = \mathscr{D}_B,\ \ \int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\,dx \geq \frac{\alpha_0}{\varepsilon}\mathscr{H}^2(\partial B_1) + \mathscr{D}_B + o_\varepsilon(1).$$

Case (**C**) This case is the simplest, since it reduces to the iso-perimetric inequality:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx \geq \int_{\mathbb{R}^3} \left(|\nabla s_\varepsilon|^2 + \frac{1}{\varepsilon^2} W(s_\varepsilon)\right) dx \\
&\geq \frac{2}{\varepsilon} \int_{\mathbb{R}^3} \sqrt{\beta W(s_\varepsilon)} |\nabla s_\varepsilon|\, dx \\
&\geq \frac{2}{\varepsilon} \int_0^{s_+} \sqrt{\beta W(\tau)} \mathscr{H}^2\left(\partial^* \left\{x \in \mathbb{R}^3 : \ s_\varepsilon(x) \geq \tau\right\}\right) d\tau.
\end{aligned}
$$

Since

$$
\left|\left\{x \in \mathbb{R}^3 : \ s_\varepsilon(x) \geq \tau\right\}\right| \geq \left|\left\{x \in \mathbb{R}^3 : \ s_\varepsilon(x) \geq s_+\right\}\right| = |B_1|, \ \forall 0 < \tau < s_+,
$$

it follows from the isoperimetric inequality in $\mathbb{R}^3$ that for all $0 < \tau < s_+$,

$$
\begin{aligned}
\mathscr{H}^2\left(\partial^* \left\{x \in \mathbb{R}^3 : \ s_\varepsilon(x) \geq \tau\right\}\right) &\geq (36\pi)^{\frac{1}{3}} \left|\left\{x \in \mathbb{R}^3 : \ s_\varepsilon(x) \geq \tau\right\}\right|^{\frac{2}{3}} \\
&\geq (36\pi)^{\frac{1}{3}} |B_1|^{\frac{2}{3}} = \mathscr{H}^2(\partial B_1).
\end{aligned}
$$

Hence we obtain that

$$
\int_{\mathbb{R}^3} \widetilde{\mathscr{W}}(s_\varepsilon, n_\varepsilon, \nabla s_\varepsilon, \nabla n_\varepsilon)\, dx \geq \frac{\alpha_0}{\varepsilon} \mathscr{H}^2(\partial B_1). \tag{4.26}
$$

Completion of Proof of Theorem 1.4. It is readily seen that Theorem 1.4 follows by combining the arguments from both section 4.1 and section 4.2. □

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