PERSISTENT HOMOLOGY OF FUNCTION SPACES

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Abstract. We can view the Lipschitz constant as a height function on the space of maps between two manifolds and ask (as Gromov did nearly 30 years ago) what its "Morse landscape" looks like: are there high peaks, deep valleys and mountain passes? A simple and relatively well-studied version of this question: given two points in the same component (homotopic maps), does a path between them (a homotopy) have to pass through maps of much higher Lipschitz constant? Now we also consider similar questions for higher-dimensional cycles in the space. We make this precise using the language of persistent homology and give some first results.

1. Introduction

1.1. **Background.** The purpose of this paper is to give some information about the Morse landscape of function spaces, with the Lipschitz constant acting as a height function.

Suppose X and Y are finite complexes, and we give them "reasonable" metrics (e.g. path metrics compatible with a linear structure on simplices); then all continuous functions can be approximated by Lipschitz ones, and indeed the inclusion of the space $\mathrm{Lip}(X, Y)$ of Lipschitz functions in the continuous ones $\mathcal{C}(X, Y)$ is a weak homotopy equivalence. However, Lipschitz functions have a natural notion of complexity, namely the Lipschitz constant: for example, the space of functions from X to Y with Lipschitz constant $\leq L$ is compact. Moreover, if Y is a locally $\mathrm{CAT}(K)$ space for some finite K (any compact Riemannian manifold fits the bill), then the image of this space in the functions with constant $\leq L + \varepsilon$ has finitely generated homology (Theorem 2.6).

This suggests a program of trying to understand: at what level are particular homology classes in $H_i(\mathrm{Lip}(X, Y))$ born? More generally, we can try to understand how the topology of the spaces of L-Lipschitz maps evolves as L increases.

Persistent homology (see e.g. [ELZ02] for the paper that first used this term and [PRSZ20] for a monograph on some of its applications in pure mathematics) provides a natural language, called "barcodes" [ZC05], for describing this structure—at least with field coefficients[1]. For a Morse function filtering a manifold by sublevel sets, each critical value gives rise to either the beginning or end of a "bar", indicating either the birth or death of a homology class.

The set of barcodes admits a natural metric (based on throwing away some bars of length $< 2\varepsilon$ and aligning the remaining bars up to ε-errors), and C^0-close functions have close barcodes [CSEH07]. Since the different linear metrics on a finite complex Y are Lipschitz-related (i.e. the identity map is a bilipschitz equivalence), the function $\log \mathrm{Lip}$ on $\mathrm{Lip}(X, Y)$ changes by a uniformly bounded amount when one changes the metric—which means that the persistence barcodes of function spaces are (up to finite distance) topologically invariant; indeed, they turn out to be homotopy invariant (Theorem 2.8). Of course the infinite bars are just the ordinary homology of the function space, but some extra information is given by

[1]Although our results will be phrased in terms of persistent homology, the obvious rephrasing in terms of homology and induced maps applies in integral homology.

where they start (when the total homology of the function space is infinitely generated), and the information about finite length bars is entirely new.

These questions were first raised in a vague form by Gromov in [Gro99b] and [Gro99a, 7.20], although he made precise the question about PH_0: how much must one increase the Lipschitz constant to obtain a homotopy between homotopic L-Lipschitz maps? We study this problem below for all PH_i, though some of our results are new even for PH_0. Other papers that deal with these questions are [Gro78, FW13, NR13, CDMW18, CMW18, Man19, BM22, BGM24].

1.2. **Main results.** We write $PH_*(A, f)$ for the persistent homology of a space A equipped with a filtration by sublevel sets of a function $f : A \to \mathbb{R}$. We leave coefficients implicit, although they must be a field for "bars" to be taken literally.

The following theorem is a quite elementary consequence of convexity properties of distance functions on nonpositively curved spaces, and covering space theory.

Theorem A. *If Y is a complete locally CAT(0) metric space (e.g. a complete nonpositively curved manifold) then all of the bars in $PH_*(\mathrm{Lip}(X, Y), \mathrm{Lip})$ are of infinite length.*

Note that this result is highly metric-dependent: if Y is a compact hyperbolic manifold, one can easily give Y a metric where, say, for $X = S^1$, there are infinitely many finite-length bars; indeed the number (measured from where their bottoms lie) grows exponentially with the Lipschitz constant L. If X is higher-dimensional, the number of such bars will typically be $\exp(O(L^{\dim(X)}))$ (as is implicit in [Gro99a, p. 36]). However, when Y is compact, for all Riemannian metrics on Y (and indeed on compact manifolds homotopy equivalent to Y), all the bars are of uniformly bounded size with respect to $\log \mathrm{Lip}$.

For general Y, there may be infinitely many bars of arbitrarily long finite length. For example, this is necessarily the case when X is S^1 and Y has fundamental group with unsolvable word problem (or even superexponential Dehn function [Wei11, Wei05]). In order to avoid the difficulties caused by the fundamental group and concentrate on the impact of homotopy theory when studying our function spaces, we will assume now that Y is simply connected.

In this case, using methods from [Gro99a] and the remarkable paper [NR13], we get, for maps of a circle into Y:

Theorem B. *Let Y be a simply connected finite complex and let ΩY and ΛY denote the spaces of based and free Lipschitz loops, respectively. Then all of the finite length bars in $PH_i(\Omega Y, \mathrm{Lip})$ and $PH_i(\Lambda Y, \mathrm{Lip})$ are uniformly bounded in length; the bound is linear in i.*

Gromov, in [Gro78, §4] and [Gro99a, Theorem 7.3], shows that for both free and based loop spaces the bottoms of the infinite bars in PH_i lie within a range $C_1 i \leq \mathrm{Lip} \leq C_2 i$; this suggests that for the finite bars, too, one may not be able to do better than a linear bound.

For the more general simply connected case, where the domain is not necessarily the circle, we have available the tools of rational homotopy theory [Qui69, Sul77, GM81]. Indeed, one could well be guided towards results of our general sort by Quillen's algebraicization of rational homotopy theory via Lie algebras, which suggests a connection to the deep theory of filling functions on nilpotent Lie groups. However, we will largely make use of the Sullivan theory, which uses differential forms, and the connection between these and the Lipschitz functional given by the "shadowing principle" of [Man19].

Philosophically, the situation of ΩY is an unusually simple (though important) case because its cohomology is a Hopf algebra, so one knows a lot a priori about the infinite length bars. However, we have not been able to exploit this to give nearly as strong information as Theorem

B. Indeed, Theorems A and B are the only two we are aware of where one can get information about Lip, as opposed to log Lip.

When the domain is higher-dimensional we find that uniform boundedness of persistence intervals depends strongly on the rational homotopy type of Y. We emphasize, though, that we are not restricting ourselves to rational coefficients and that the persistent homology groups in the theorems below are not rationally invariant.

Theorem C. *If Y is a simply connected finite complex with positive weights (e.g. Y is formal or a homogeneous manifold), then for all finite X, all of the finite length bars in $PH_i(\mathrm{Lip}(X,Y)_0, \log_+ \mathrm{Lip})$ are of uniformly bounded length. Here $\mathrm{Lip}(X,Y)_0$ is the connected component of the function space that includes the constant maps.*

Theorem D. *If Y is a simply connected finite complex rationally equivalent to an H-space (e.g. an odd-dimensional sphere or a Lie group), then for all finite X, all of the finite length bars in $PH_i(\mathrm{Lip}(X,Y), \log_+ \mathrm{Lip})$ are of uniformly bounded length.*

Here $\log_+(x) = \max\{\log x, 0\}$; we use $\log_+ \mathrm{Lip}$ because we do not wish to emphasize the behavior of functions with very small Lipschitz constant.

In these theorems, as in Theorem B, we do not have uniformity in i; the statements are about uniformity for a fixed i in the persistence parameter log Lip. Moreover, outside the case of loop spaces we do not know how to study the question of growth with respect to i.

We show that at least some of these hypotheses are necessary, and that finite bars of unbounded length can exist for simply connected targets:

Theorem E. *For arbitrarily large L, there are homotopic pairs of L-Lipschitz maps f_L and $g_L : \mathbb{C}P^2 \times S^3 \to S^3 \vee S^3$ for which any homotopy must go through maps with Lipschitz constant $> cL^{4/3}$, giving bars of linearly growing length in $PH_0(\mathrm{Lip}(\mathbb{C}P^2 \times S^3, S^3 \vee S^3), \log_+ \mathrm{Lip})$.*

Remark 1.1. All of the results in this paper are true for "persistent homotopy" with identical proofs. We preferred to express our results in terms of homology because of its relationship to Morse theory.

1.3. **Open questions.** This paper is the beginning of a theory and there are many gaps left to fill in. We highlight some of these here:

Question 1.2. Is the assumption of positive weights necessary in Theorem C?

While spaces without positive weights are relatively complicated, the proof of Theorem E may provide a guide to what to look for in a potential counterexample.

Question 1.3. Can Theorem D be extended to "two-stage" spaces (spaces that are rationally fibers of maps between products of Eilenberg–MacLane spaces, such as homogeneous manifolds)?

We show in Theorem 9.1 that this is true for PH_0 under the additional assumption that Y is *scalable* in the sense of [BM22] (e.g. a symmetric space, but not every homogeneous manifold), but it is unclear how to extend the result to higher PH_i. On the other hand, the proof of Theorem E uses the presence of three "stages" in an essential way.

Rational homotopy theory works not only for simply connected spaces, but more generally for *nilpotent spaces*: those which have a nilpotent fundamental group which acts nilpotently on all higher homotopy groups. The shadowing principle of [Man19] is extended to nilpotent spaces in Kyle Hansen's thesis [Han25]; with this, the proof of Theorem C extends verbatim to nilpotent spaces with positive weights, including for example nilmanifolds whose fundamental group admits a (not necessarily Carnot) grading. On the other hand, the following question is wide open:

Question 1.4. Can Theorem B be extended to nilmanifolds (at least with respect to $\log \mathrm{Lip}$)?

A positive answer for PH_0 is due to Riley [Ril02], see also [BRS07, II, Theorem 4.1.1].

Similarly, one may ask whether an example similar to that of Theorem E can be found for a nilmanifold target, placing it within the domain of geometric group theory.

1.4. **Outline of the paper.** We start in §2 by explaining aspects of persistent homology relevant to our setting. The rest of the paper, however, can be understood without most of the technical material there. We next prove Theorem A in §3 and two proofs of variants of Theorem B in §4 and §5. In §6 we introduce prior results from quantitative rational homotopy theory which are used in the remainder of the paper. In §7 we use these to prove Theorems C, and D, as well as a more general, but weaker result in the spirit of Theorem B. In §8 we prove Theorem E, and in §9 we give some more results about PH_0 which show that this example is in some sense both as simple as it can be and sharp.

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2. Definitions and general facts

We will be studying the filtration given by the sublevel sets of the Lipschitz constant on the set of Lipschitz functions $\mathrm{Lip}(X, Y)$ between two metric spaces X and Y. We denote the subset of L-Lipschitz functions by $\mathrm{Lip}_L(X, Y)$. In this section we introduce the language and machinery of persistent homology and explain how it applies in this setting. Although this is necessary to justify using the language of barcodes in the rest of the paper, the technical details are largely irrelevant to the proofs of our main results and the reader should feel free to skip them.

2.1. **Persistent homology.** Let A be a topological space, and fix a filtration $\{A_t : t \in \mathbb{R}\}$ of A (i.e. $A_t \subseteq A_s \subseteq A$ for $t \leq s$); we can think of this as a functor from the order category $(\mathbb{R}, \leq)$ to the category of topological spaces. We would like to study how the topology of the sets in the filtration evolves over the time parameter. Intuitively, topological features may "persist" over various amounts of time; transient features are thought of as accidental and can be discounted, while long-lived ones tell us something meaningful about the nature of the filtration.

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This frame of reference originates in topological data analysis, where one attempts to understand the underlying topology implied by a discrete data set; different times correspond to different scales at which the data set can be viewed, and a topological feature visible at a large range of scales is hypothesized to be present in the underlying distribution rather than an artifact of the data. In the main example we will be studying, $A = \mathrm{Lip}(X, Y)$ and $A_t = \mathrm{Lip}_t(X, Y)$. In this case, persistent topological features correspond to families of L-Lipschitz maps which cannot be contracted through maps of Lipschitz constant close to L.

Purely formally, the functor $H_n(-; R)$ induces a family of groups $\{H_n(A_t; R)\}$ and maps

$$i_*(t, s) : H_n(A_t; R) \to H_n(A_s; R), \qquad \text{for } t \leq s,$$

i.e. a functor from $(\mathbb{R}, \leq)$ to R-modules, which contain all the homological information about the filtration. Such a functor is called a *persistence module*, and the persistence module consisting of the homology of A_t is called the *persistent homology* of A, notated $PH_n(A; R)$. Here the filtration is implicit. Our filtrations will typically consist of sublevel sets of continuous functions $f : A \to \mathbb{R}$; in this case, we will use the notation $PH_n(A, f; R)$.

So far this invariant is essentially tautological; to make it meaningful one needs to analyze the structure of persistence modules $\mathbf{M} = (\{M_t\}, \{\mu(t, s)\})$. Such an analysis is readily available under the following simplifying assumptions, which can be summarized as saying that $\mathbf{M}$ has *finite rank*:

(i) R is a field.
(ii) Each M_t is finite-dimensional.
(iii) The set $\{t_i\}$ of "critical times" for the filtration (i.e. those t_i for which for every $\varepsilon > 0$ the induced map $\mu(t_i - \varepsilon, t_i + \varepsilon)$ is not an isomorphism) is either finite or has the order type of $\mathbb{N}$.

These assumptions are satisfied, for example, for $H_n(A, f; R)$ when A is a manifold and f is a proper Morse function. In this case, one equips $\bigoplus_i H_n(A_{t_i}; R)$ with the operation

$$\mathbf{t} \cdot h = i_*(t_i, t_{i+1})(h), \qquad h \in H_n(A_{t_i}; R),$$

making it into an $R[\mathbf{t}]$-module. By the structure theorem for modules over a PID, such a module is isomorphic to a unique direct sum of modules of the form $R[\mathbf{t}]$ and $R[\mathbf{t}]/(\mathbf{t}^j)$ (*interval* or *cyclic modules*). These correspond to homology classes which are "born" at some time t_i (which may be $-\infty$) and "die" at time t_{i+j}, or live forever in the case of modules $R[\mathbf{t}]$.

The data of such a persistence module can also be expressed by a *persistence diagram* or *barcode*, a multisubset of $[-\infty, \infty) \times (-\infty, \infty]$ consisting of pairs (t_i, t_j) expressing the birth and death times—the lifetime interval—of the cyclic summands. These intervals are referred to as *bars*.

Example 2.1. Consider a Morse function $f : \mathbb{R}^2 \to \mathbb{R}$ with two local minima and one saddle point. Intuitively, as t increases, the sublevel sets $f^{-1}(-\infty, t]$ have two components that are "born" and then "merge". The two cyclic submodules in $PH_0(\mathbb{R}^2, f)$ are one corresponding to the first component (an infinite bar) and one corresponding to the difference between the components (a finite bar which is born with the second component and dies when the two components merge).

Suppose that $X = S^1$ and Y is a closed Riemannian manifold. Then the (free or based) Lipschitz loop space $\mathrm{Lip}(X, Y)$ is weakly homotopy equivalent to the subset of loops that are arclength-parametrized. Moreover, on this subspace the Lipschitz constant is just a measure of length, and so is the energy, which famously behaves rather like a Morse function [Mil63]. Therefore, the above formalism is sufficient to analyze the persistent homology of loopspaces.

In general, though, the Lipschitz constant does not behave like a Morse function. There may not be a discrete set of critical points, and $\mathrm{Lip}_L(X,Y)$ need not have finite-dimensional homology groups. However, when X and Y are nice finite complexes, this filtration fits into a more general framework described in [CCBdS16, CdSGO16].

2.2. **Distance between persistence modules.** In order to describe this framework, we first need to introduce notions of what it means for persistence modules to be "similar". Let $\mathbf{M} = (\{M_t\}, \{\mu(t,s)\})$ and $\mathbf{N} = (\{N_t\}, \{\nu(t,s)\})$ be two persistence modules. They are *δ-interleaved* if there are $\varphi_t : M_t \to N_{t+\delta}$ and $\psi_t : N_t \to M_{t+\delta}$ such that

(i) φ_t and ψ_t are *δ-shifted homomorphisms*, that is,
$$\varphi_s \circ \mu(t,s) = \nu(t+\delta, s+\delta) \circ \varphi_t \quad \text{and} \quad \psi_s \circ \nu(t,s) = \mu(t+\delta, s+\delta) \circ \psi_t.$$
(ii) They are almost inverses, in the sense that
$$\psi_{t+\delta} \circ \varphi_t = \mu(t, t+2\delta) \quad \text{and} \quad \varphi_{t+\delta} \circ \psi_t = \nu(t, t+2\delta).$$

The *interleaving distance* between $\mathbf{M}$ and $\mathbf{N}$ is the infimum of the δ for which they are δ-interleaved. This is defined for all persistence modules (although it can be infinite if no δ-matching exists for any δ).

This is related to another measure of distance, defined between two barcodes, which (we recall) are multisets of pairs (t_i, s_i) of elements of $[-\infty, \infty]$ with $t_i \leq s_i$. Let us say that a *δ-matching* between two barcodes D_1 and D_2 is a partial matching between their elements such that:

(i) Every pair $(t,s) \in D_1$ with $s - t \geq 2\delta$ is matched to a pair in D_2, and vice versa.
(ii) Every matching is between pairs of L^∞-distance at most δ.

Intuitively, every bar (interval) is matched to another bar whose endpoints are at most δ away, and bars of length $< 2\delta$ can be matched to an empty interval and disappear.

The *bottleneck distance* between D_1 and D_2 is the least δ such that there is a δ-matching between them. It is easy to see that a δ-matching between the barcodes associated to two finite-rank persistence modules defines a δ-interleaving between them. It is a fundamental result that the reverse is also true:

Theorem 2.2 (Isometry theorem [CSEH07])**.** *The interleaving distance between two finite-rank persistence modules is equal to the bottleneck distance between their barcodes.*

Further down, we will see how this is extended beyond the finite-rank case.

2.3. **Theory of infinite rank persistence modules.** We will now relax the finiteness assumptions for persistence modules, following [CCBdS16, CdSGO16]. In the language of these papers, a persistence module $\mathbf{M} = (\{M_t\}, \{\mu(t,s)\})$ is *q-tame* if for every $t < s$, $\mu(t,s)(M_t) \subset M_s$ is finitely generated. We will also say that $\mathbf{M}$ is *ε-q-tame* if $\mu(t,s)(M_t)$ is finitely generated whenever $s - t > \varepsilon$. Evidently:

Proposition 2.3. *If $\mathbf{M}$ is q-tame and $\mathbf{N}$ is ε-interleaved with $\mathbf{M}$, then $\mathbf{N}$ is 2ε-q-tame.*

A q-tame persistence module need not be decomposable into cyclic modules. Nevertheless, they are well-understood from the point of view of discarding "ephemeral" structure (that which persists for zero time). Formally, $\mathbf{M} = (\{M_t\}, \{\mu(t,s)\})$ is an *ephemeral* persistence module if $\mu_{t,s} = 0$ for all $t < s$. Then we have:

Theorem 2.4 ([CCBdS16, Thm. 4.5 and Cor. 3.5]). *The following are equivalent for a pair of q-tame persistence modules:*

(i) They are isomorphic in the "observable category", that is, the category of persistence modules formally inverting morphisms with ephemeral kernel and cokernel.
(ii) Their interleaving distance is zero.

Moreover, every equivalence class of q-tame persistence modules under this relation contains a minimal element which is a direct sum of interval modules.

Thus q-tame persistence modules have a well-defined barcode. This barcode is given an alternate definition in [CdSGO16] which directly uses the data of the original module. That paper also generalizes the isometry theorem above:

Theorem 2.5 ([CdSGO16, Theorem 5.14]). *The interleaving distance between two q-tame persistence modules is equal to the bottleneck distance between their barcodes.*

In order to have some way of understanding ε-q-tame modules, we also consider the ε-*smoothing* of a persistence module $\mathbf{M} = (\{M_t\}, \{\mu(t,s)\})$, defined in [CdSGO16, §5.5] as the module $\mathbf{M}^\varepsilon = (\{M_t^\varepsilon\}, \{\mu^\varepsilon(t,s)\})$ with

$$M_t^\varepsilon = \mu(t-\varepsilon, t+\varepsilon)(M_{t-\varepsilon})$$
$$\mu^\varepsilon(t,s) = \mu(t+\varepsilon, s+\varepsilon)|_{M_t^\varepsilon}.$$

Evidently, $\mathbf{M}^\varepsilon$ is ε-interleaved with $\mathbf{M}$. Moreover, if $\mathbf{M}$ is q-tame, or ε'-q-tame for any $\varepsilon' < 2\varepsilon$, then $\mathbf{M}^\varepsilon$ has pointwise finite rank (i.e. each M_t^ε has finite rank) and hence decomposes into interval modules by [CB15]. This means that an ε'-q-tame module $\mathbf{M}$ has a well-defined "barcode of bars of length $> 2\varepsilon$", in bijection with the bars of $\mathbf{M}^\varepsilon$, despite being potentially very infinite at small scales. This can again be given a precise definition internal to $\mathbf{M}$ using the machinery of [CdSGO16].

2.4. **Persistent homology of Lipschitz functions.** Here we show that when X and Y are nice spaces, then the persistence module $PH_*(\mathrm{Lip}(X,Y), \log\mathrm{Lip}; R)$ fits well into the theory described above. We omit coefficients, which may be any field. We first describe situations in which the persistent homology is q-tame:

Theorem 2.6. *Let X be a compact Riemannian manifold with boundary or a finite simplicial complex with a simplexwise linear metric, and let Y be a finite simplicial complex with a locally CAT(K) metric for some $K > 0$. Then for each $L, \varepsilon > 0$ and every coefficient ring R, $i_*(L, L+\varepsilon)(H_*(\mathrm{Lip}_L(X,Y); R)$ is finitely generated, i.e., $PH_*(\mathrm{Lip}(X,Y), \mathrm{Lip}; R)$ is q-tame.*

The property of being q-tame is invariant under reparametrization, so our height function here can be either Lip or log Lip.

Note that the CAT(K) condition is satisfied for some $K > 0$ by every closed Riemannian manifold and, more generally, every compact Riemannian manifold with boundary with the induced length metric. (In the latter case, metric geodesics are concatenations of Riemannian geodesic segments in the interior and in the boundary.) Moreover, every simplicial complex admits a locally CAT(1) metric [AKP24, §12C].

It is *not* satisfied, for example, for surfaces with cone points of total angle $< 2\pi$.

Proof. If X is a Riemannian manifold, it is $(1+\varepsilon/2L)$-bilipschitz to a simplicial complex with a simplexwise linear metric (via a fine triangulation), so it suffices to assume X is of the latter type. We will show that, inside $\mathrm{Lip}_{L+\varepsilon}(X,Y)$, $\mathrm{Lip}_L(X,Y)$ can be deformed into the image of a finite simplicial complex.

This finite simplicial complex is constructed as follows. Let $n = \dim X$, and let r be a small radius depending on K and ε, to be determined later. Choose a subdivision $\mathcal{T}$ of X, depending on r and L, such that for each k-simplex τ of $\mathcal{T}$, the linear map $f : \Delta^k \to \tau$ satisfies

$$c(X)\frac{r}{L}d(p,q) \leq d(f(p), f(q)) \leq \frac{r}{L}d(p,q)$$

for a constant $c(X) > 0$.[3] Fix an ordering $\mathcal{T}^{(0)} = \{u_1, \ldots, u_M\}$ of the vertices of $\mathcal{T}$. Also fix a triangulation $\mathcal{T}'$ of Y such that the diameter of the simplices is at most $d_Y = \frac{\varepsilon c(X) r}{8nL}$. Then restrictions of functions $X \to Y$ to $\mathcal{T}^{(0)}$ form the complex $(\mathcal{T}')^M$, and the L-Lipschitz functions land in the subcomplex $\mathbb{D} \subseteq (\mathcal{T}')^M$ of tuples $(f_1, \ldots, f_M)$ with the property that if $\{u_i, u_j\}$ is an edge of $\mathcal{T}$, then the simplices of $\mathcal{T}'$ containing f_i and f_j are at distance at most $Ld(u_i, u_j)$.

Now we construct an embedding $F : \mathbb{D} \to \mathrm{Lip}(X, Y)$ such that

$$F(f_1, \ldots, f_M)(u_i) = f_i.$$

Denote the simplex of $\mathcal{T}$ spanned by $u_{i_0}, \ldots, u_{i_k}$ by $\tau(u_{i_0}, \ldots, u_{i_k})$, and assume that the indices are in ascending order. We inductively define

$$F(t_0 u_{i_0} + \cdots + t_k u_{i_k}) = \gamma(t_k)$$

where $\gamma : [0,1] \to Y$ is the geodesic from $F(\frac{t_0}{1-t_k}u_{i_0} + \cdots + \frac{t_{k-1}}{1-t_k}u_{i_{k-1}})$ to f_k. For small enough r, this geodesic is unique since Y is locally $\mathrm{CAT}(K)$.

Lemma 2.7. *We can choose r so that the function F lands in $\mathrm{Lip}_{L+\varepsilon/2}(X, Y)$.*

Proof. We would like to show that $f = F(f_1, \ldots, f_M)$ is an $(L + \varepsilon)$-Lipschitz function.

We start by estimating the length of γ by induction on k; we would like to make sure it is at most $(L + \varepsilon)\delta$ where

$$\delta = d\left(\frac{t_0}{1-t_k}u_{i_0} + \cdots + \frac{t_{k-1}}{1-t_k}u_{i_{k-1}}, u_{i_k}\right).$$

If $k = 1$, then γ is the geodesic between two vertices of $\mathcal{T}'$, and its length is at most

$$Ld(u_{i_0}, u_{i_1}) + \frac{\varepsilon}{4n}c(X)r \leq \left(L + \frac{\varepsilon}{4n}\right)d(u_{i_0}, u_{i_1}) \leq \left(1 + \frac{\varepsilon}{4nL}\right)r.$$

At the kth step, γ passes through a vertex and opposite edge of a triangle whose sides are geodesics constructed in the $(k-1)$st step. If these sides are short enough, depending on K, then γ is at most $1 + \varepsilon/4n$ times the length of the corresponding Euclidean geodesic. By induction, we have

$$\mathrm{length}(\gamma) \leq \left(1 + \frac{\varepsilon}{4nL}\right)^n L\delta = (L + \varepsilon/4 + O(\varepsilon^2/L))\delta$$

for a small enough initial $r = r(K, \varepsilon)$. This bounds the distances between points along γ.

Now we would like to bound the distances between a pair of points $f(p), f(q)$ lying on different $\gamma, \tilde{\gamma}$ in the same k-simplex of $f(\mathcal{T})$. Notice that $f(p)$ and $f(q)$ lie on the sides of a triangle formed by γ, $\tilde{\gamma}$, and a geodesic between two points in a $(k-1)$-simplex. By a similar induction, we have that

$$d(f(p), f(q)) \leq \left(L + \frac{\varepsilon}{4} \cdot \frac{n+1}{n} + O\left(\frac{\varepsilon^2}{L}\right)\right)d(p,q).$$ □

[3] There are several ways of choosing such subdivisions, see [CDMW18, §2].

Finally, given an L-Lipschitz map $f : X \to Y$, there is a corresponding map $\tilde{f} \in F(\mathbb{D})$ which coincides with f on the vertices of $\mathcal{T}$. If r is small enough, we can take a linear homotopy h_t between f and $\tilde{f}$. Moreover, this homotopy goes through $(L+\varepsilon)$-Lipschitz maps by a similar argument to the above. Let p, q be two points in the same simplex of $\mathcal{T}$. Their images under the homotopy lie on geodesic segments between $f(p)$ and $\tilde{f}(p)$ and between $f(q)$ and $\tilde{f}(q)$. These segments have length at most $r(1+\varepsilon/2L)$, and their endpoints are at most $Ld(p,q)$ and at most $(L+\varepsilon/2)d(p,q)$ apart, respectively. The CAT(K) condition bounds the degree to which distance between geodesics can be nonconvex, so if r is small enough,

$$d(h_t(p), h_t(q)) \leq (L+\varepsilon)d(p,q). \qquad \square$$

Now we show that things don't get much worse when we replace the spaces in Theorem 2.6 by Lipschitz homotopy equivalent ones:

Theorem 2.8. *Consider the space* $\mathrm{Lip}(X,Y)$ *of Lipschitz functions between two metric spaces* X *and* Y*.*

(a) If X' *is Lipschitz homotopy equivalent to* X*, then the interleaving distance between* $PH_*(\mathrm{Lip}(X,Y), \log \mathrm{Lip})$ *and* $PH_*(\mathrm{Lip}(X',Y), \log \mathrm{Lip})$ *is finite.*
(b) If Y' *is Lipschitz homotopy equivalent to* Y*, then the interleaving distance between* $PH_*(\mathrm{Lip}(X,Y), \log \mathrm{Lip})$ *and* $PH_*(\mathrm{Lip}(X,Y'), \log \mathrm{Lip})$ *is finite.*

As an immediate corollary, we get some invariance properties of the persistent homology with respect to Lipschitz homotopy equivalence:

Corollary 2.9.

(a) If X *and* Y *are metric spaces Lipschitz homotopy equivalent to finite complexes, then* $PH_*(\mathrm{Lip}(X,Y), \log \mathrm{Lip})$ *is* ε*-q-tame for some* $\varepsilon < \infty$*.*
(b) The property that all bars, or all finite bars, in $PH_n(\mathrm{Lip}(X,Y), \log \mathrm{Lip})$ *are of bounded length is (Lipschitz) homotopy invariant.*

Moreover, for manifolds we get:

Corollary 2.10. *Let* X*,* Y*,* X'*, and* Y' *be compact Riemannian manifolds with boundary such that* $X \simeq X'$ *and* $Y \simeq Y'$*. Then the interleaving distance between* $PH_*(\mathrm{Lip}(X,Y), \log \mathrm{Lip})$ *and* $PH_*(\mathrm{Lip}(X',Y'), \log \mathrm{Lip})$ *is finite.*

Proof. It is easy to see that any homotopy equivalence between compact Riemannian manifolds can be deformed to a Lipschitz homotopy equivalence. This reduces the corollary to Theorem 2.8. $\square$

Proof of Theorem 2.8. We prove this for a Lipschitz homotopy equivalence between X and X'; the proof for a Lipschitz homotopy equivalence of the target space is identical. Let $X \underset{g}{\overset{f}{\rightleftarrows}} X'$ be a Lipschitz homotopy equivalence, with $g \circ f \simeq \mathrm{id}$ via a Lipschitz homotopy $H : X \times [0,1] \to X$, and $f \circ g \simeq \mathrm{id}$ via a Lipschitz homotopy $H' : X' \times [0,1] \to X'$.

Then f induces a map $\mathrm{Lip}(X',Y) \to \mathrm{Lip}(X,Y)$ which sends the sublevel set $\mathrm{Lip}_{\leq L}(X',Y)$ to $\mathrm{Lip}_{\leq \mathrm{Lip}(f)L}(X,Y)$. Similarly, g induces a map $\mathrm{Lip}(X,Y) \to \mathrm{Lip}(X',Y)$ which sends $\mathrm{Lip}_{\leq L}(X',Y)$ to $\mathrm{Lip}_{\leq \mathrm{Lip}(g)L}(X,Y)$. The composition of these induces a map

$$\varphi_{X'} : \mathrm{Lip}_{\leq L}(X',Y) \to \mathrm{Lip}_{\leq \mathrm{Lip}(g)\,\mathrm{Lip}(f)L}(X',Y).$$

This map $\varphi_{X'}$ is homotopic to the identity inclusion via the homotopy $\varphi_t(u) = u \circ H'_t$ inside the perhaps larger sublevel set $\mathrm{Lip}_{\leq L_{H'}L}(X', Y)$, where $L_{H'}$ is the maximal Lipschitz constant of a fiber $H'_t : X' \to X'$ of H'. Therefore $\varphi_{X'}$ and the identity inclusion induce the same homomorphism

$$H_*(\mathrm{Lip}_{\leq L}(X', Y)) \to H_*(\mathrm{Lip}_{\leq L_{H'}L}(X', Y)).$$

A similar computation holds going in the other direction, where we can similarly define

$$L_H = \max\{\mathrm{Lip}(H_t) : t \in [0, 1]\}.$$

Thus the persistence modules of the two filtrations are $\log(\max\{L'_H, L_H\})$-interleaved. □

3. Maps to manifolds of nonpositive curvature

In this section we discuss the case when the target space is a nonpositively curved manifold. In this case, there are no finite bars in the persistent homology of the mapping space:

Theorem 3.1. *Let Y be a complete manifold of nonpositive curvature (or, more generally, a complete locally CAT(0) metric space), and X any Riemannian manifold (or, more generally, a metric space admitting a universal cover). Then every cycle in $\mathrm{Lip}_{\leq L}(X, Y)$ which is trivial in $\mathrm{Lip}(X, Y)$ is trivial in $\mathrm{Lip}_{\leq L}(X, Y)$.*

Note that this property is not an invariant of the topology of Y. Take for example $X = S^1$ and Y a nonpositively curved surface. Adding a small bubble to the metric on Y immediately generates an infinite number of finite bars, one for each multiple of the resulting nullhomotopic closed geodesic. On the other hand, the proof of Theorem 2.8 shows that the property "no finite bars" is closed in the topology on $\mathrm{Met}(X) \times \mathrm{Met}(Y)$ induced by the Lipschitz distance.

Proof. We show the following stronger statement: the intersection of every connected component of $\mathrm{Lip}(X, Y)$ with $\mathrm{Lip}_{\leq L}(X, Y)$ is either empty or its inclusion into the corresponding component of $\mathrm{Lip}(X, Y)$ is a homotopy equivalence.

Since Y is nonpositively curved, its universal cover is contractible, i.e. Y is a $K(\Gamma, 1)$ for some group Γ. A connected component of the space of (unbased) maps $X \to Y$ corresponds to a conjugacy class of homomomorphisms $\varphi : \pi_1(X) \to \Gamma$; call this component $K_{[\varphi]}$.

Let $\tilde{X}$ and $\tilde{Y}$ be the universal covers of X and Y, and let $\widetilde{K_{[\varphi]}} \subset \mathrm{Lip}(\tilde{X}, \tilde{Y})$ be the subspace consisting of lifts of maps in $K_{[\varphi]}$. Two lifts of the same f, differing by $g \in \Gamma$, are homotopic if and only if $g\varphi g^{-1} = \varphi$, that is, if g lies in the centralizer $Z(\varphi(\pi_1 X)) \subseteq \Gamma$. (Conversely, if $\tilde{f}$ is a lift of f, then its translate $g \cdot \tilde{f}$ is a lift of f if and only if $g \in Z(\varphi(\pi_1 X))$; in general, $g \cdot \tilde{f}$ is a lift of a conjugate of f.)

Recall that $\tilde{Y}$ is uniquely geodesic, and moreover, if $\gamma, \gamma' : [0, 1] \to \tilde{Y}$ are geodesics, then

$$d(\gamma(t), \gamma'(t)) \leq \max\{d(\gamma(0), \gamma'(0)), d(\gamma(1), \gamma'(1))\}, \qquad t \in [0, 1].$$

In particular, for every $g \in \Gamma$ and lift $\tilde{f}$ of a map $f \in K_\varphi$, there is a "linear" equivariant homotopy between $\tilde{f}$ and $g \cdot \tilde{f}$, and this homotopy goes through maps of Lipschitz constant $\leq \mathrm{Lip}\, f$. This shows:

(1) Each connected component of $\widetilde{K_{[\varphi]}}$ is a convex, hence contractible set, and $K_{[\varphi]}$ is a quotient of this set by the action of $Z(\varphi(\pi_1 X))$.
(2) The same is true for the subset of $\widetilde{K_{[\varphi]}}$ consisting of L-Lipschitz maps, so long as one such map exists.

It follows that $K_{[\varphi]}$ and the set of L-Lipschitz maps inside it are both $K(Z(\varphi(\pi_1 X)), 1)$ spaces and the inclusion of one into the other is a homotopy equivalence. □

4. Loop spaces of closed manifolds following Nabutovsky–Rotman

Nabutovsky and Rotman [NR13] conducted a detailed analysis of what we now recognize as the persistent *homotopy* groups of loop spaces of simply connected closed manifolds. In this section, we use their results to prove a similar theorem for the persistent homology groups. Later, we will use a similar outline to prove Theorem C; we include this proof in part as a warmup for that argument.

The result we prove is as follows:

Theorem 4.1. *Let M be a simply connected closed n-manifold. Let d be its diameter, and let S be such that any loop of length $\ell \leq 2d$ can be contracted through curves of length at most $\ell + S$. Then for every $m \geq 1$:*

(i) Every infinite bar in $PH_m(\Omega M, \mathrm{len})$ has birth time at most $(6d + 2S)\binom{m}{2} + 5d + S$.
(ii) Every finite bar in $PH_{m-1}(\Omega M, \mathrm{len})$ has length at most $(6d + 2S)\binom{m}{2} + 5d + S$.

Note that the lengths of bars are with respect to the length, not the log-length functional. In other words, increasing the length of loops by an additive constant is sufficient to kill trivial cycles.

Theorem 4.1 is almost certainly not optimal: it should be possible to prove a result with constants linear in m using the *methods* of Nabutovsky and Rotman and not just their results. Indeed, a version of part (i) with a linear (albeit inexplicit) constant is due to Gromov [Gro78, §4], as is an analogous theorem for free loop spaces [Gro99a, Theorem 7.3]; in the next section, we extend his method to the relative case.

Proof. The proof relies on the following result:

Theorem 4.2 (Nabutovsky–Rotman [NR13, Thm. 8.2])**.** *Let M be a simply connected closed n-manifold. Let d be its diameter, and let S be such that any loop of length $\ell \leq 2d$ can be contracted through curves of length at most $\ell + S$. Denote the space of loops of length $\leq L$ by $\Omega^L M$. Then for every $\varepsilon > 0$:*

(i) Every map $S^m \to \Omega M$, $m \geq 1$, can be homotoped into $\Omega^{(6d+2S)m-d-S+\varepsilon} M$.
(ii) For $m \geq 1$, every nullhomotopy $f : (D^{m+1}, \partial D^{m+1}) \to (\Omega M, \Omega^L M)$ of a map $S^m \to \Omega^L M$ can be homotoped relative to ∂D^{m+1} into $\Omega^{\max\{L, 5d+S\}+(6d+2S)m+\varepsilon}(M)$. In the case $m = 0$, the constant is $L + 2d + S + \varepsilon$.

Now let $(Z, \partial Z)$ be an m-pseudomanifold with boundary, and consider a simplicial chain $(c, \partial c) : (Z, \partial Z) \to (\Omega M, \Omega^L M)$. (Here and further we work with integral chains, but the proofs make sense with any coefficient ring.) We apply Theorem 4.2 inductively to the simplices of this chain. Over the course of the induction, we create a sequence of homotopic maps c_k.

To build c_0, we homotope c so that each vertex of Z outside ∂Z maps to the constant loop. This homotopy can be extended to all of Z via the homotopy extension property.

Now we perform the inductive step, building a homotopy of c_k to c_{k+1}. By induction, c_k maps the k-skeleton of Z to the space of loops of length at most $L + (6d + 2S)\binom{k}{2} + 5d + S + \varepsilon$. We build c_{k+1} so that it is identical to c_k on $Z^{(k)} \cup \partial Z$; on $(k+1)$-cells, we apply Theorem 4.2 to ensure that c_{k+1} maps to $\Omega^{L+(6d+2S)\binom{k+1}{2}+5d+S+\varepsilon} M$; and on higher cells, we extend by the homotopy extension property. At the end of the induction, we have built c_m, homotopic to c rel ∂Z, which maps Z into the space of loops of length at most $L + (6d + 2S)\binom{m}{2} + 5d + S + \varepsilon$.

If ∂Z is empty, we can start the induction with $L = 0$. This gives a map into the space of loops of length at most $(6d + 2S)\binom{m}{2} + 5d + S + \varepsilon$. □

5. Loop spaces following Gromov

Now we use a different method—still heavily inspired by Nabutovsky–Rotman [NR13], but also by Gromov [Gro99a, Theorem 7.3]—to show the following:

Theorem 5.1. *Let Y be a simply connected finite simplicial complex with a metric bilipschitz to a linear one. Then there is a constant $C = C(Y)$ such that for every $m \geq 1$:*

(i) *Every infinite bar in $PH_m(\Omega Y, \operatorname{len})$ or $PH_m(\Lambda Y, \operatorname{len})$ has birth time at most $Cm + C$.*
(ii) *Every finite bar in $PH_{m-1}(\Omega Y, \operatorname{len})$ or $PH_{m-1}(\Lambda Y, \operatorname{len})$ has length at most $Cm + C$.*

Note that (i) is one half of [Gro78, Theorem 1.4] and [Gro99a, Theorem 7.3] (the other half is a corresponding linear lower bound). However, our proof will recover it for free.

Proof. Let $(Z, \partial Z)$ be an m-pseudomanifold with boundary, and consider a simplicial chain $(c, \partial c) : (Z, \partial Z) \to (\Omega Y, \Omega^L Y)$. We deform c relative to ∂Z to a $\tilde{c}$ which maps to $\Omega^{L+Cm+C}M$.

We do this in two steps. Consider the map $f_0 : (Z \times S^1, \partial Z \times S^1) \to Y$ induced by c; we can assume f_0 is Lipschitz after an arbitrarily small deformation. The first step is to deform f_0 (relative to $\partial Z \times S^1$) to a map f which behaves in a regular way on a fine triangulation of $Z \times S^1$. In particular, it will send most of the 1-skeleton of this triangulation to the basepoint. The second step is to reparametrize: choose a map $\Gamma : Z \times S^1 \to Z \times S^1$ (homotopic to the identity relative to $\partial Z \times S^1$) which maps every S^1-fiber mostly to the 1-skeleton. Since f sends most of the 1-skeleton to the basepoint, the composition $f \circ \Gamma$ will have short S^1-fibers. Then we can set

$$\tilde{c}(z)(t) = f \circ \Gamma(z, t).$$

We start with the first step. Given triangulations of Z and S^1, the induced cell structure on the product can be subdivided into a triangulation without adding any vertices. We choose fine enough triangulations so that:

(1) ∂Z is the full subcomplex of Z spanned by its vertices.
(2) The map sending each vertex v of $Z \times S^1$ to the vertex of Y nearest to $f_0(v)$ extends to a simplicial approximation of f_0.
(3) Fixing the cellwise linear metric d_Δ on $Z \times S^1$ in which all edges have length 1, f_0 is 1-Lipschitz on $\partial Z \times S^1$.

Now let f_t be a homotopy from f_0 to a map f_1 as follows:

Step 1: Apply the linear homotopy between f_0 and its simplicial approximation.
Step 2: Compose with a homotopy $Y \times [0,1] \to Y$ from the identity to a map sending $Y^{(1)}$ to the base point.

Then $f_t : Z \times S^1 \times [0,1] \to Y$ is $C(Y)$-Lipschitz with respect to the product metric $d_\Delta \times d_{[0,1]}$.

Finally, let $f(z,s) = f_{t(z)}(z,s)$, where

- $t(z) = 1$ if z is contained in a simplex disjoint from ∂Z;
- If the simplex containing z is $\Delta * \Delta'$, where $\Delta \subset \partial Z$ and Δ' is disjoint from ∂Z, then $t(z)$ is such that $z = (1 - t(z))v + t(z)v'$, for $v \in \Delta$ and $v' \in \Delta'$.

Then f is again $C(Y)$-Lipschitz with respect to d_Δ.

Now f takes all internal edges of $Z \times S^1$ to the basepoint. So given a path $\gamma : [0,1] \to Z \times S^1$, the length of $f \circ \gamma$ is determined by the portion of γ that does not traverse internal edges (i.e. those disjoint from ∂Z). The next step is to build a map $\Gamma : Z \times S^1 \to Z \times S^1$ which is the identity on $\partial Z \times S^1$ and such that $f \circ \Gamma$ has short S^1-fibers.

To build intuition, assume first that Z is 1-dimensional, and consider an edge $e(v,w)$ between vertices $v, w \in Z$. Let $u_0, u_1, \ldots, u_N = u_0$ be the vertices of S^1 under our triangulation.

Let γ_0 be the S^1-fiber over v, γ_N be the S^1-fiber over w, and γ_i for $i = 1, \ldots, N-1$ be the piecewise linear loop in $Z \times S^1$ with vertices

$$(v, u_0), (v, u_1), \ldots, (v, u_i), (w, u_i), \ldots, (w, u_N), (v, u_N) = (v, u_0).$$

If v and w are both internal vertices, then $f \circ \gamma_i$ has length 0 for every i; if one is an external vertex, then it has length at most $L + 2C(Y)$. Moreover, γ_i and γ_{i+1} are homotopic via a homotopy that just traverses one grid square; these homotopies assemble into a map $\Gamma : [0, N]^2 \to e(v, w) \times [0, N]$, and each fiber of the homotopy $f \circ \Gamma$ has length at most $L + 3C(Y)$.

Now we generalize this construction to higher dimensions in a more abstract way, inspired by Gromov's proof of [Gro99a, Theorem 7.3]. We use an observation of Milnor [Mil56]: the set of piecewise linear paths in a simplicial complex admits a cell structure in which every path $v_1, \ldots, v_k$ with v_i contained in a simplex Δ_i is a point of the cell $\Delta_1 \times \cdots \times \Delta_k$. We consider this construction for $Z \times S^1$. For each simplex Δ of Z, consider the subcomplex of piecewise linear paths $\mathcal{P}_\Delta \subset \{\gamma : [0, N] \to \Delta \times S^1\}$ consisting of γ such that

(i) $\gamma([i-1, i]) \subset \Delta \times e_i$, where $e_i = [u_{i-1}, u_i]$;
(ii) $\gamma(0) = \gamma(N) \in \Delta \times \{0\}$.

This subcomplex is contractible, indeed convex; therefore, we can homotope the map

$$\Gamma_0 : Z \to \mathcal{P}(Z \times S^1)$$
$$\Gamma_0(z)(t) = (z, t)$$

via a linear homotopy Γ_s to a *cellular* map Γ_1 that still takes each Δ into $\mathcal{P}_\Delta$. Moreover, for $\gamma, \gamma' \in \mathcal{P}_\Delta$, we have

$$d_\Delta(\gamma(t), \gamma'(t)) \leq m + 1, \qquad \text{for all } t \in [0, N];$$

therefore Γ_s moves points by at most $m + 1$. Now we define a map $\Gamma : Z \to \mathcal{P}(Z \times S^1)$ by:

- On interior simplices of Z, $\Gamma = \Gamma_1$.
- For a simplex $\Delta * \Delta'$ of Z, where Δ is in ∂Z and Δ' is disjoint from ∂Z, write points as $z = (1-s)v + sv'$, where $v \in \Delta$ and $v' \in \Delta'$. Then, identifying S^1 with $[0, N]/0 \sim N$, we define

$$\Gamma(z)(t) = \begin{cases} \Gamma_1(v')(t) & |t - N/2| \leq (N/2 + 1)s - 1 \\ (v, t) & |t - N/2| \geq (N/2 + 1)s \\ \Gamma_{s'}((1-s')v + s'v')(t) & s' = (N/2+1)s - |t - N/2| \in (0, 1). \end{cases}$$

We claim that $\tilde{c}(z)(t) = f(\Gamma(z)(t))$ is the desired relative cycle. We need only bound the length of each path $\tilde{c}(z)$. But notice:

- $\Gamma_1(z)$ lies in the m-skeleton of the path space. This means that it lies in a cell $\Delta_1 \times \cdots \times \Delta_k$ where at most m of the Δ_i have dimension ≥ 1. Suppose now that z lies in an interior simplex of Z. Since f maps all interior edges to the base point, the curve $f \circ \Gamma_1(z)$ has length at most $m \operatorname{Lip}_{d_\Delta}(f)$.
- For $z \in \partial Z$, the curve $f(z, t)$ has length at most L.
- The speed of the remaining segment of $\Gamma(z)$ (when z is not contained in an interior or a boundary simplex) is always at most $m + 2$, and therefore that segment has length at most $2m + 4$.

In conclusion, $\operatorname{len}(\tilde{c}(z)) \leq L + (3m + 4) \operatorname{Lip}_{d_\Delta}(f)$. □

6. Tools from quantitative homotopy theory

The rest of the paper focuses on simply connected and, more generally, nilpotent target spaces Y, where Sullivan's model of rational homotopy theory applies. Here we are able to use a number of tools from quantitative homotopy theory developed in the last decade.

6.1. **Rational homotopy theory.** We start with a very brief review of Sullivan's theory of minimal models, focused on fixing notation. We refer the reader to [GM81, FHT01] for more details on the general background and [Man19, BM22] for treatments geared towards quantitative topology.

There is a *rationalization* functor on homotopy types of simply connected spaces. There are several ways to define this, but perhaps the easiest is by tensoring all spaces and maps in the Postnikov tower with $\mathbb{Q}$. We say two spaces X and Y are *rationally equivalent*, written $X \simeq_{\mathbb{Q}} Y$, if they have homotopy equivalent rationalizations.

Rational homotopy theory provides a way of translating the topology of simply connected spaces into algebraic language, which preserves the same information as the rationalization functor. There are several equivalent such languages, but the main one we will use is that of differential graded algebras, as developed by Sullivan.

A *(commutative) differential graded algebra*, or *DGA*, is a cochain complex over a field, typically $\mathbb{Q}$ or $\mathbb{R}$, with a graded-commutative multiplication satisfying the graded Leibniz rule. The prototypical examples are:

- The smooth forms $\Omega^*(X)$ on a smooth manifold X, or the simplexwise smooth forms on a simplicial complex.
- The *flat forms* $\Omega^*_\flat(X)$ on a smooth manifold or simplicial complex X. This is the closure of the smooth or piecewise smooth forms under the flat norm, defined by Whitney [Whi57, Ch. IX]. It has many of the same properties as smooth forms, and additionally is preserved under pullback by Lipschitz maps.
- Sullivan's *minimal DGA* $\mathcal{M}^*_Y(\mathbb{F})$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$) for a simply connected space Y. This is a free graded commutative algebra generated in degree n by a vector space of *indecomposable* elements $V_n = \operatorname{Hom}(\pi_n(Y); \mathbb{F})$ and with a differential which takes elements of V_n to elements of $\Lambda_{k=2}^{n-1} V_k$ and is dual to the k-invariants in the Postnikov tower of Y, $k_n \in H^{n+1}(Y_{n-1}; \pi_n(Y))$. We will write
$$\mathcal{M}^*_Y = \mathcal{M}^*_Y(\mathbb{R}) \cong \Lambda_{n=2}^{\infty} V_n,$$
noting that this isomorphism is non-canonical. We also write
$$\mathcal{M}^*_Y(n) = \Lambda_{k=2}^{n} V_k;$$
this is the minimal DGA of the nth Postnikov stage of Y.

A *quasi-isomorphism* between DGAs is a map inducing an isomorphism on cohomology. The existence of such a map between $\mathcal{A}$ and $\mathcal{B}$ is not an equivalence relation; therefore we say that two DGAs are *quasi-isomorphic* if they are connected by a zig-zag of one or more quasi-isomorphisms
$$\mathcal{A} \leftarrow \mathcal{C}_1 \rightarrow \cdots \leftarrow \mathcal{C}_k \rightarrow \mathcal{B}.$$

A *homotopy* between DGA homomorphisms $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism
$$\eta : \mathcal{A} \to \mathcal{B} \otimes \Lambda(t, dt),$$
where $\Lambda(t, dt)$ is formally generated by a degree 0 generator t and its differential, but can also be thought of as the algebra of polynomial differential forms on the unit interval. For

DGAs over $\mathbb{R}$, we get the same homotopy theory by using all differential forms on the interval instead.

If Y is a simply connected smooth manifold or simplicial complex, then it has a (non-unique) *minimal model*, that is, a quasi-isomorphism $m_Y : \mathcal{M}_Y^* \to \Omega^*(Y)$ realizing the generators of the minimal DGA as differential forms. The codomain of the minimal model may just as well be $\Omega_\flat^*(Y)$, see [BM22, §6]; in fact, we use flat forms implicitly throughout this paper so that we can pull them back along Lipschitz maps, but omit the $\flat$ symbol to save on notation.

We write $\mathcal{M}_Y^*(k)$ for the subalgebra generated by $V_{\leq k}$. This is the algebraic equivalent of the kth Postnikov stage, and maps from $\mathcal{M}_Y^*(k)$ admit an obstruction theory formally dual to that to rational Postnikov stages; see [GM81, Ch. 10 and 14] for details. We cite and use various obstruction-theoretic lemmas of this form in the course of our proofs.

6.2. **The shadowing principle.** The correspondence $f \mapsto f^* m_Y$ defines a mapping

$$\mathrm{Lip}(X, Y) \to \mathrm{Hom}(\mathcal{M}_Y^*, \Omega^* X)$$

sending genuine maps between X and Y to their "algebraicization". This induces a function between the corresponding sets of homotopy classes; we can think of the codomain of this function loosely as "homotopy classes tensored with the reals". However, here we focus on what happens inside a homotopy class. The main technical result of [Man19] shows that the image within a given homotopy class, if it is nonempty, is in some sense coarsely dense. That is, given a homomorphism $\varphi : \mathcal{M}_Y^* \to \Omega^* X$ which is homotopic to the image of a genuine map, one can produce another genuine map $X \to Y$ which is *nearby* φ. Moreover, the Lipschitz constant of this new map depends on geometric properties of φ.

To state this precisely, we first introduce some definitions. Let X and Y be finite simplicial complexes or compact Riemannian manifolds such that Y is simply connected and has a minimal model $m_Y : \mathcal{M}_Y^* \to \Omega^* Y$. Fix norms on the finite-dimensional vector spaces V_k of degree k indecomposables of $\mathcal{M}_Y^*$; then for homomorphisms $\varphi : \mathcal{M}_Y^* \to \Omega^*(X)$ we define the formal dilatation

$$\mathrm{Dil}(\varphi) = \max_{2 \leq k \leq \dim X} \|\varphi|_{V_k}\|_{\mathrm{op}}^{1/k},$$

where we use the L^∞ norm on $\Omega^*(X)$. Notice that if $f : X \to Y$ is an L-Lipschitz map, then $\mathrm{Dil}(f^* m_Y) \leq CL$, where the exact constant depends on the dimension of X, the minimal model on Y, and the norms. Thus (although no reverse inequality holds) the dilatation is an algebraic analogue of the Lipschitz constant.

Given a formal homotopy

$$\Phi : \mathcal{M}_Y^* \to \Omega^*(X \times [0, T]),$$

we can define the dilatation $\mathrm{Dil}_T(\Phi)$ in a similar way. The subscript indicates that we can always rescale Φ to spread over a smaller or larger interval, changing the dilatation; this is a formal analogue of defining separate Lipschitz constants in the time and space direction, although in the DGA world they are not so easily separable.

Now we can state some results from [Man19]. We start with a simplified version of [Man19, Thm. 4–1]:

Theorem 6.1 (Shadowing principle, non-relative version)**.** *Let X be a simplicial complex equipped with the standard simplexwise linear metric, and let Y be a simply connected compact Riemannian manifold or simplicial complex. Let $\varphi : \mathcal{M}_Y^* \to \Omega^*(X)$ be a homomorphism such that*

(i) $\mathrm{Dil}(\varphi) \leq L$*;*
(ii) φ *is formally homotopic to* $f^* m_Y$ *for some* $f : X \to Y$*.*

Then there is a $g : X \to Y$ *such that*

(i) g *is* $C(\dim X, Y)(L+1)$*-Lipschitz;*
(ii) g *is homotopic to* f*;*
(iii) $g^* m_Y$ *is homotopic to* φ *via a homotopy* Φ *satisfying* $\mathrm{Dil}_{1/L}(\Phi) \leq C(\dim X, Y)(L+1)$*.*

In other words, one can produce a genuine map by a small formal deformation of φ. Note that in the above result, X does not have to be compact. In fact, the constants depend only on the bounds on the local geometry of X.

The most general version of the [Man19, Thm. 4–1] is relative:

Theorem 6.2 (Shadowing principle, general version)**.** *Let* X *be a simplicial complex equipped with the standard simplexwise linear metric,* A *a subcomplex of* X*, and let* Y *be a simply connected compact Riemannian manifold or simplicial complex. Let* $\varphi : \mathcal{M}_Y^* \to \Omega^*(X)$ *be a homomorphism and* $f : X \to Y$ *be a map such that*

(i) $\mathrm{Dil}(\varphi) \leq L$*;*
(ii) $f|_A$ *is* L*-Lipschitz;*
(iii) $\varphi|_A = f^* m_Y|_A$*;*
(iv) φ *is formally homotopic to* $f^* m_Y$ *relative to* $\Omega^* A$*.*

Then there is a $g : X \to Y$ *such that*

(i) g *is* $C(\dim X, Y)(L+1)$*-Lipschitz;*
(ii) g *is homotopic to* f *relative to* A*;*
(iii) $g^* m_Y$ *is homotopic relative to* $\Omega^* A$ *to* φ *via a homotopy* Φ *satisfying* $\mathrm{Dil}_{1/L}(\Phi) \leq C(\dim X, Y)(L+1)$*.*

In the rest of the paper, we use both this result directly and a number of its consequences.

6.3. **Formality, scalability and positive weights.** Now we outline several distinguished classes of rational homotopy types. All of these have multiple characterizations, both in terms of algebraic properties of the minimal model and in terms of maps.

The broadest such class is that of ***spaces with positive weights***, a term attributed in [BMSS98] to Morgan and Sullivan. A minimal DGA $\mathcal{A}$ has positive weights if it satisfies the following equivalent characterizations:

(i) $\mathcal{A}$ admits a second grading with respect to which the differential has degree zero.
(ii) There is a family of automorphisms $\rho_t : \mathcal{A} \to \mathcal{A}$ and a basis for the indecomposables such that ρ_t multiplies every basis element by t^k for some k.

Simply connected spaces whose rational homotopy type has positive weights form a large class: for example, they include homogeneous spaces [BMSS98, Prop. 3.7] and smooth complex algebraic varieties [Mor78]. In fact, although in some sense "almost all" spaces do not have positive weights, it is somewhat difficult to find a non-example; as far as we know, the lowest-dimensional one is a complex given in [MT71, §4] constructed by attaching a 12-cell to $S^3 \vee \mathbb{C}P^2$.

A somewhat narrower class is that of ***formal spaces***, discussed by Sullivan in [Sul77, §12]. A minimal DGA $\mathcal{A}$ is formal if it satisfies the following equivalent characterizations:

(i) There is a quasi-isomorphism $\mathcal{A} \to H^*(\mathcal{A})$.
(ii) $\mathcal{A}$ admits a second grading with respect to which the differential has degree zero, and such that classes in $H^k(\mathcal{A})$ admit representative cycles of degree k.

(iii) There is a family of automorphisms $\rho_t : \mathcal{A} \to \mathcal{A}$ which induces the map $x \mapsto t^k x$ on $H^k(\mathcal{A})$.

Simply connected spaces whose rational homotopy type is formal include symmetric spaces [Sul77] and Kähler manifolds [DGMS75].

Finally, ***scalable spaces*** were introduced in [BM22] as a metric refinement of formal spaces. A simply connected Riemannian manifold with boundary X is scalable if it satisfies the following equivalent characterizations [BM22, BGM24]:

(i) It is formal, and there is an embedding $H^*(X;\mathbb{R}) \hookrightarrow \bigoplus_i \Lambda^*\mathbb{R}^{n_i}$ for some collection of n_i. (If X is a closed n-manifold, then the target can be $\Lambda^*\mathbb{R}^n$.)
(ii) There is an embedding $H^*(X;\mathbb{R}) \hookrightarrow \Omega^* X$ sending each cohomology class to a representative.
(iii) For an infinite family of $t \in \mathbb{Z}$, there is a family of $O(t)$-Lipschitz self-maps $r_t : X \to X$ which induce the map $x \mapsto t^k x$ on $H^k(X;\mathbb{R})$.

Scalable spaces include, once again, symmetric spaces, including in particular spheres and complex projective spaces. Products and wedge sums of scalable spaces are also scalable. Scalable spaces also include the connected sum of two or three (but not four) copies of $\mathbb{C}P^2$ or $S^2 \times S^2$.

Scalable spaces have a number of interesting geometric properties. Of these, the most relevant for this paper is a stronger version of the shadowing principle. Given a formal minimal DGA $\mathcal{A} = \Lambda_{k=2}^{\infty} V_k$, denote by $U_i, i = 0, 1, \ldots$ the subspaces comprising the second grading implied by formality. Then, given a homomorphism $\varphi : \mathcal{A} \to \Omega^* X$, where X is a Riemannian manifold or simplicial complex, define

$$\mathrm{Dil}^U(\varphi) = \max_{\substack{2 \le k \le \dim X \\ 2 \le i \le 2\dim X - 2}} \|\varphi|_{V_k \cap U_i}\|_{\mathrm{op}}^{1/i}.$$

We refer to this quantity as the *U-dilatation* of φ.

Example 6.3. It is easy to see that condition (ii) for formality implies condition (iii): given a second grading $\mathcal{A} = \bigoplus_i U_i$, there is an automorphism that multiplies elements of U_i by t^i. This automorphism has U-dilatation t.

For $S^n \vee S^n$, the second grading $\bigoplus_i U_i$ of the minimal model is unique up to isomorphism. In this case, the rational homotopy groups form a free Lie algebra generated by the inclusions of the two spheres under the Whitehead product operation. In particular, the ith order Whitehead products span $\pi_{i(n-1)+1}(S^n \vee S^n)$, and the dual $V_{i(n-1)+1}$ is also U_{in}. One can see by inductively computing that $dV_{i(n-1)+1} \subseteq U_{in+1}$, but also by considering the action of degree d self-maps on $S^n \vee S^n$ on the rational homotopy groups.

Theorem 6.4 (Improved shadowing principle for scalable targets [BM22, Lemma 8.3])**.** *Let X be a simplicial complex equipped with the standard simplexwise linear metric, and let Y be a scalable compact Riemannian manifold or simplicial complex. Let $\varphi : \mathcal{M}_Y^* \to \Omega^* X$ be a homomorphism and $f : X \to Y$ be a map such that*

(i) $\mathrm{Dil}^U(\varphi) \le L$*;*
(ii) $f|_A$ *is L-Lipschitz;*
(iii) $\varphi|_A = f^* m_Y|_A$*;*
(iv) φ *is formally homotopic to $f^* m_Y$ relative to $\Omega^* A$.*

Then there is a $g : X \to Y$ such that

(i) g is $C(\dim X, Y)(L+1)$-Lipschitz;

(ii) g is homotopic to f relative to A.

The main advantage of this result is that inductive algebraic constructions tend to produce homomorphisms with large dilatation, but small U-dilatation. Therefore the improved shadowing principle can be used to produce maps to a scalable space with the "best possible" geometry for their homotopy class; see Gromov [Gro99a, §7B]. For formal spaces that are not scalable, such a best-case scenario cannot happen, see [BM22, BGM24].

6.4. **Homotopies of maps from scalable spaces.** Now we prove a new lemma in the spirit of [Man19] which will be useful in §8. This lemma will allow us to find lower bounds on the size of all homotopies between two maps—a very large parameter space!—by studying a much more restrictive space of algebraic homotopies, parametrized by a few numerical variables and one-variable Lipschitz functions. It holds generally for maps from scalable domains:

Lemma 6.5. *Let Y be simply connected and X a scalable space. Let*

$$\alpha = \alpha(X, Y) = \prod_{n \le \dim X} \frac{n+1}{n},$$

where n ranges over degrees in which $\mathcal{M}_Y$ has generators with nontrivial differential.

(i) Let $f : X \to Y$ be a map. Then the diagram

$$\begin{array}{ccc} \mathcal{M}_Y & \overset{\varphi}{\dashrightarrow} & (H^*(X), d=0) \\ \downarrow{\scriptstyle m_Y} & & \downarrow{\scriptstyle \rho} \\ \Omega^* Y & \xrightarrow{f^*} & \Omega^*(X) \end{array}$$

can be completed up to a homotopy

$$\Phi : \mathcal{M}_Y \to \Omega^* X \otimes \Lambda(t, dt)$$

so that $\operatorname{Dil}(\varphi) = O((\operatorname{Lip} f)^\alpha)$ and $\operatorname{Dil}(\Phi) = O((\operatorname{Lip} f)^\alpha)$.

(ii) Let $h : X \times [0, S] \to Y$ be a homotopy, where X is a scalable space and Y is simply connected. Then the diagram

$$\begin{array}{ccc} \mathcal{M}_Y & \overset{\eta}{\dashrightarrow} & (H^*(X), d=0) \otimes \Omega^*[0,S] \\ \downarrow{\scriptstyle m_Y} & & \downarrow{\scriptstyle \rho \otimes \mathrm{id}} \\ \Omega^* Y & \xrightarrow{h^*} & \Omega^*(X \times [0,S]) \end{array}$$

can be completed up to a homotopy

$$\Phi : \mathcal{M}_Y \to \Omega^*(X \times [0, S]) \otimes \Lambda(t, dt)$$

so that $\operatorname{Dil}(\eta) = O((\operatorname{Lip} h)^\alpha)$ and $\operatorname{Dil}(\Phi) = O((\operatorname{Lip} h)^\alpha)$.

Moreover, $\eta|_{s=0}$ and $\eta|_{s=S}$ can be chosen to be any homomorphisms $\mathcal{M}_Y \to H^*(X)$ which are homotopic to h_0 and h_1, respectively, via a homotopy of dilatation $O((\operatorname{Lip} h)^\alpha)$.

We will apply part (ii) for $Y = S^3 \vee S^3$ and 7-dimensional X; in this case, the statement yields $\alpha = 48/35$. However, by employing a more fine-grained understanding of the minimal model of Y, we can improve this to $\alpha = 9/7$, as explained below the proof. This is the estimate we will actually use.

In order to prove the lemma, we recall [Man19, Prop. 3.9]:

Proposition 6.6. *Suppose that $\Phi_k : \mathcal{M}_Y^*(k) \to \Omega^* X \otimes \Lambda(t, dt)$ is a partially defined algebraic homotopy between $\varphi, \psi : \mathcal{M}_Y^* \to \Omega^* X$.*

(i) *The obstruction to extending Φ_k to a homotopy*

$$\Phi_{k+1} : \mathcal{M}_Y^*(k+1) \to \Omega^* X \otimes \Lambda(t, dt)$$

is a class in $H^{k+1}(X; V_{k+1})$ represented by a cochain

$$\sigma(v) = \psi(v) - \varphi(v) - \textstyle\int_0^1 \Phi_k(dv),$$

and therefore in general

$$\|\sigma\|_{\text{op}} \le C(k, d|_{V_{k+1}}) \operatorname{Dil}(\Phi_k)^{k+2} + \operatorname{Dil}(\varphi)^{k+1} + \operatorname{Dil}(\psi)^{k+1}.$$

(ii) *If this obstruction class vanishes, then we can choose a primitive $c(v)$ for $\sigma(v)$ and fix*

$$\Phi_{k+1}(v) = \varphi(v) + d(c(v) \otimes t) + \textstyle\int_0^t \Phi_k(dv),$$

so that in general

$$\big\|\Phi_{k+1}|_{V_{k+1}}\big\|_{\text{op}} \le (C_{\text{IP}} + 2)\big(C(k, d|_{V_{k+1}}) \operatorname{Dil}(\Phi_k)^{k+2} + \operatorname{Dil}(\varphi)^{k+1} + \operatorname{Dil}(\psi)^{k+1}\big),$$

where C_{IP} is the isoperimetric constant for $(k+2)$-forms in X.

Moreover, if for some subcomplex $A \subset X$ we have an existing homotopy

$$\chi : \mathcal{M}_Y^* \to \Omega^* A \otimes \Lambda(t, dt)$$

between $\varphi|_A$ and $\psi|_A$, and there is no obstruction to extending it, we can get an extension with similar bounds, using a relative isoperimetric constant and with an additional $O(\operatorname{Dil}(\chi))$ term.

Proof of Lemma 6.5. We first prove (i), building Φ by induction on degree. Suppose we have constructed a homotopy Φ_k in degrees up to k. In particular, since at time 1 the homotopy factors through ρ, we have $\Phi_k(dV_{k+1})|_{t=1} = 0$ and so applying $\Phi|_{t=1}$ to V_{k+1} should give us closed forms. Applying Prop. 6.6(i), we get that the obstruction to making $\Phi_{k+1}(V_{k+1})|_{t=1} = 0$ is bounded by $\operatorname{Dil}(\Phi_k)^{k+2}$. We then define $\Phi_{k+1}(V_{k+1})|_{t=1}$ by setting φ to the value of this obstruction. Then we apply Prop. 6.6(ii) to define $\Phi_{k+1}(V_{k+1})$ for all $t \in [0, 1]$.

For (ii), we first use (i) to construct Φ at all integer times $s \in [0, S]$. Then we extend it to intervals $[s, s+1]$. Note that going along three sides of the square already defines a homotopy between $\eta|_s$ and $\eta|_{s+1}$, so there is no obstruction to extending it. We apply the relative version of Prop. 6.6 to complete the extension.

From the proof it is clear that the choice of $\eta|_{s=0}$ and $\eta|_{s=S}$ doesn't affect the end result. □

The estimate coming from Prop. 6.6 can be improved if we have different estimates on the operator norms of Φ_k in different degrees. For example, in the case of $Y = S^3 \vee S^3$, we have $\|\Phi_3|_{V_3}\|_{\text{op}} = O((\operatorname{Lip} h)^3)$, since the differential at that stage is 0, and $\|\Phi_5|_{V_5}\|_{\text{op}} = O((\operatorname{Lip} h)^6)$. Now each term of the differential of elements of degree 7 is the product of a degree 3 and a degree 5 generator, so its operator norm is $O((\operatorname{Lip} h)^9)$. This differential is the source of the $\operatorname{Dil}(\Phi_k)^{k+2}$ term in the estimates in Prop. 6.6, and the other terms are smaller; thus this is also the right estimate for the primitive $\Phi_7|_{V_7}$. Therefore we get $\alpha = 9/7$ in this case.

7. Bounded length bars

In this section, we prove that for certain mapping spaces, the length of finite bars in persistent homology with respect to the log-Lipschitz constant is always bounded. In each case, we show this by proving a stronger result: that every chain in the mapping space whose boundary lies in L-Lipschitz functions can be deformed relative to this boundary into the subspace of CL-Lipschitz functions.

7.1. Maps from k-dimensional spaces to k-connected spaces.

Theorem 7.1. *Let Y be a compact, simply connected, rationally k-connected Riemannian manifold with boundary, and let X be a finite k-dimensional simplicial complex equipped with a standard simplexwise metric. Let $A \subset X$ be a subcomplex, and fix a Lipschitz map $f : A \to Y$. Write $\mathrm{Lip}(X,Y)_A$ to mean the space of Lipschitz maps $X \to Y$ extending f. Then there are constants $C(Y,m)$ such that:*

(i) Every infinite bar in $PH_n(\mathrm{Lip}(X,Y)_A, \log_+ \mathrm{Lip})$ has birth time in

$$[\log_+ \mathrm{Lip} f, \log_+ \mathrm{Lip} f + C(Y, n+k)].$$

(ii) Every finite bar in $PH_n(\mathrm{Lip}(X,Y)_A, \log_+ \mathrm{Lip})$ has length bounded by $C(Y, n+k)$.

As a special case we have a weak version of Theorem 5.1:

Corollary 7.2. *Let Y be a compact, simply connected Riemannian manifold with boundary. Then the lengths of finite bars in $PH_n(\Omega Y, \log \mathrm{len})$ and $PH_n(\Lambda Y, \log \mathrm{len})$ (the based and free loopspaces, respectively) are bounded by a constant $C(Y,n)$.*

Proof. Let $(Z, \partial Z)$ be an n-pseudomanifold with boundary, and consider a simplicial chain

$$(c, \partial c) : (Z, \partial Z) \to (\mathrm{Lip}(X,Y)_A, \mathrm{Lip}_L(X,Y)_f),$$

where Lip_L denotes the subspace of L-Lipschitz maps. We may assume, via a perturbation, that this map is Lipschitz with respect to the C^0-distance; equivalently, it defines a Lipschitz map $g_c : Z \times X \to Y$.

Let $m_Y : \mathcal{M}_Y \to \Omega^* Y$ be a minimal model for Y. Notice that the image of $g_c^* m_Y$ lies in forms of degree $\geq k+1$. For $i \geq k+1$, every i-vector in X is trivial. Therefore, if we rescale the metric on Z by a large factor $R = \mathrm{Dil}(g_c^* m_Y)$, under the new metric on $Z \times X$, $\mathrm{Dil}(g_c^* m_Y) \leq 1$. Now we apply the shadowing principle for maps $Z \times X \to Y$ relative to $\partial Z \times X \cup Z \times A$ with $f = g_c$ and $\varphi = g_c^* m_Y$. This gives us a map homotopic to g_c (relative to $\partial Z \times X \cup Z \times A$) whose Lipschitz constant in the X direction is bounded by $C(Y, n+k)(L+1)$.

The statement about infinite bars is given by the same proof with $\partial Z = \emptyset$. □

7.2. Nullhomotopic maps to spaces with positive weights.

Theorem 7.3. *Let Y be a compact, simply connected Riemannian manifold with boundary whose rational homotopy type has positive weights, and let X be a finite simplicial complex equipped with a standard simplexwise metric. Denote the component containing the constant maps in $\mathrm{Lip}(X,Y)$ by $\mathrm{Lip}(X,Y)_0$. Then there is a constant $C(Y, \dim X, n)$ such that:*

(i) Every infinite bar in $PH_n(\mathrm{Lip}(X,Y)_0, \log_+ \mathrm{Lip})$ has birth time below $C(Y, \dim X, n)$.

(ii) Every finite bar in $PH_n(\mathrm{Lip}(X,Y)_0, \log_+ \mathrm{Lip})$ has length bounded by $C(Y, \dim X, n)$.

Proof. The proof follows the same outline as that of Theorem 4.1, by inductive application of the following main lemma:

Lemma 7.4. *For $k \geq 1$, there is a constant $C_k = C(Y, \dim X, k)$ such that any map*

$$f : (D^k, \partial D^k) \to (\mathrm{Lip}(X,Y)_0, \mathrm{Lip}_L(X,Y)_0)$$

is homotopic relative to ∂D^k into $\mathrm{Lip}_{C_k(L+1)}(X,Y)_0$.

Assuming this lemma, let $(Z, \partial Z)$ be an m-pseudomanifold with boundary, and consider a simplicial chain

$$(c, \partial c) : (Z, \partial Z) \to (\mathrm{Lip}(X,Y)_0, \mathrm{Lip}_L(X,Y)_0).$$

We apply Lemma 7.4 inductively to the simplices of this chain. At the kth step of the induction, we create a map c_k which maps the k-simplices of Z into $\mathrm{Lip}_{C_1 \cdots C_k(L+1)}(X,Y)_0$.

To build c_0, we homotope c so that each vertex of Z outside ∂Z maps to the constant map, and then extend the homotopy to all of Z via the homotopy extension property. For the inductive step homotoping c_k to c_{k+1}, we leave the map the same on the k-skeleton, apply Lemma 7.4 to the $(k+1)$-simplices, and extend to higher skeleta by the homotopy extension property. After the nth step we get the desired map.

If ∂Z is empty, we can start the induction with $L = 0$ and by sending all the 0-simplices to the constant map. This gives a map into $\mathrm{Lip}_{C_1 \cdots C_n}(X,Y)_0$. □

It remains to prove the lemma.

Proof of Lemma 7.4. This is a combination of two auxiliary lemmas. We start with the following weaker result which is essentially proved in [Man19]:

Lemma 7.5. *For $k \geq 1$, there is a constant $C = C(Y, k+\dim X)$ such that any nullhomotopic map $g : S^{k-1} \to \mathrm{Lip}_L(X,Y)_0$ has an extension $\tilde{g} : S^{k-1} \times [0,1] \to \mathrm{Lip}_{C(L+1)}(X,Y)_0$ so that $\tilde{g}|_{t=0} = g$ and $\tilde{g}|_{t=1}$ sends every point to a constant map to a base point.*

Proof. Such a map can be thought of alternately as a map $G : S^{k-1} \times X \to Y$. By assumption it is Lipschitz on fibers over points in S^{k-1}, but by an arbitrarily small perturbation we can make G Lipschitz in the sphere direction as well, without control on the Lipschitz constant. After putting a round metric of sufficiently large diameter on S^{k-1}, we can even assume that G is $(L+\varepsilon)$-Lipschitz, for arbitrarily small $\varepsilon > 0$. Then [Man19, Theorem 5.6], $G|_{\partial \Delta^k}$ has a Lipschitz nullhomotopy through $C(Y, k+\dim X)(L+1)$-Lipschitz maps. We can reinterpret this nullhomotopy as a map $S^{k-1} \times [0,1] \to \mathrm{Lip}_{C(L+1)}(X,Y)$. □

Since the map $\tilde{g}$ is constant on $S^k \times \{1\}$, we may as well interpret it as a map from the disk. The reason this is insufficient is that we don't have control over the relative homotopy class of the resulting extension. The obstruction to building a relative homotopy between the extension we started with and this controlled one is an element $\alpha \in \pi_k(\mathrm{Map}(X,Y)_0)$. We resolve the issue by finding a controlled representative of this element:

Lemma 7.6. *We can represent every element of $\pi_k(\mathrm{Map}(X,Y)_0)$ by a map $S^k \times X \to Y$ whose Lipschitz constant in the X direction is bounded by some $C(Y, \dim X, k)$. More generally, every element of $\pi_k(\mathrm{Map}(X,Y)_f)$ for a fixed f has a representative whose Lipschitz constant in the X direction is bounded by $C(Y, \dim X, k)(\mathrm{Lip}\, f + 1)$.*

Proof. That this is true for a constant $C(Y, X, k)$ follows easily from the finite generation of $\pi_k(\mathrm{Map}(X,Y))$. To show that the constant only depends on the dimension of X, we use the shadowing principle (although it is possible that one could apply a more elementary obstruction-theoretic method).

Write $X \xrightarrow{\iota_X} S^k \times X \xrightarrow{\pi_X} X$ for the inclusion of the fiber at the base point and the projection, respectively. Rational obstruction theory [GM81, Prop. 10.5] shows that for any map $F : S^k \times X \to Y$ representing an element of $\pi_k(\mathrm{Map}(X, Y)_f)$, we can complete the diagram

$$\begin{array}{ccccc} \mathcal{M}_Y & \overset{\Phi}{\dashrightarrow} & \Lambda(e^{(k)})/(e^2) \otimes \Omega^* X & & \\ {\scriptstyle m_Y}\downarrow & & \downarrow{\scriptstyle d\,\mathrm{vol}_{S^k} \otimes \mathrm{id}} & \searrow & \\ \Omega^* Y & \xrightarrow{F^*} & \Omega^*(S^k \times X) & \xrightarrow{\iota_X^*} & \Omega^* X \end{array}$$

up to an algebraic homotopy $\mathcal{M}_Y \to \Omega^*(S^k \times X) \otimes \Lambda(t, dt)$ whose composition with ι_X^* is constant. In particular, for indecomposables $v \in \mathcal{M}_Y$, we can define $\varphi(v)$ by

$$\Phi(v) = \pi_X^* f^* m_Y(v) + e \otimes \varphi(v).$$

Now fix R sufficiently large so that for every indecomposable v,

$$\|d\,\mathrm{vol}_{S^k} \wedge \varphi(v)\|_{\mathrm{op}} \leq \|f^* m_Y(v)\|_{\mathrm{op}}$$

with respect to the product metric on $S^k_R \times X$, where S^k_R is a round sphere of radius R. Applying the shadowing principle to the diagram above gives us a map $S^k_R \times X \to Y$ homotopic to F with Lipschitz constant bounded by $C(Y, \dim X, k)(\mathrm{Lip}\, f + 1)$. □

Now we construct an extension of $f|_{S^{k-1}}$ to the disk by mapping an outer annulus via the nullhomotopy $\tilde{g}$ obtained in Lemma 7.5 and an inner disk via the controlled representative of the obstruction class as given by Lemma 7.6. This extension is homotopic to the original map f. □

7.3. **Maps to rational H-spaces.**

Theorem 7.7. *Let Y be a compact, simply connected Riemannian manifold with boundary which is rationally equivalent to a product of Eilenberg–MacLane spaces, and let X be a finite simplicial complex equipped with a standard simplexwise metric. Then there is a constant $C(Y, \dim X, n)$ such that:*

(i) Every infinite bar in $PH_n(\mathrm{Lip}(X, Y)_f, \log_+ \mathrm{Lip})$ has birth time in

$$\log_+ \mathrm{Lip}\, f, \log_+ \mathrm{Lip}\, f + C(Y, \dim X, n).$$

(ii) Every finite bar in $PH_n(\mathrm{Lip}(X, Y)_0, \log_+ \mathrm{Lip})$ has length bounded by $C(Y, \dim X, n)$.

A simply connected space is rationally equivalent to a product of Eilenberg–MacLane spaces if and only if it is rationally equivalent to an H-space. Such spaces include odd-dimensional spheres, connected Lie groups, as well as classifying spaces of connected Lie groups [FHT01, Prop. 15.15].

The proof of this theorem is identical to that of Theorem 7.3, except that Lemma 7.5 is replaced by the following fact:

Lemma 7.8. *For $k \geq 1$, there is a constant $C(Y, k + \dim X)$ such that any map $g : S^{k-1} \to \mathrm{Lip}_L(X, Y)$ has an extension $\tilde{g} : S^{k-1} \times [0, 1] \to \mathrm{Lip}_{C(L+1)}(X, Y)$ so that $\tilde{g}|_{t=0} = g$ and $\tilde{g}|_{t=1}$ sends every point to the same map.*

Just as Lemma 7.5 follows from [Man19, Theorem 5.6], this lemma follows from [CDMW18, Theorem B], or rather its more precise restatement in §4 of that paper.

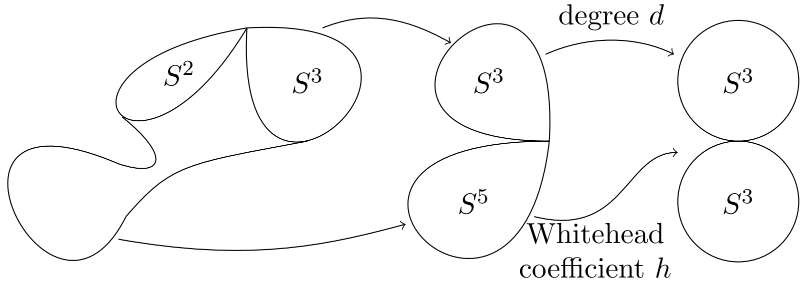

FIGURE 1. Construct maps $u_{d,h} : S^2 \times S^3 \to Y$ by "budding off" a small sphere and then projecting the rest onto the S^3 factor.

8. Examples with bars of unbounded length

Theorem 8.1. *Let $X = \mathbb{C}P^2 \times S^3$ or $S^2 \times S^2 \times (S^3 \vee S^3)$. Then there is a sequence of $L \to \infty$ and pairs of L-Lipschitz maps*

$$f_L, g_L : X \to S^3 \vee S^3$$

which are homotopic, but not through maps of Lipschitz constant $o(L^{4/3})$.

8.1. **Notation and conventions.** We will implicitly treat our spaces as CW complexes with the simplest possible CW structure. That is, S^n consists of a 0-cell and an n-cell, $\mathbb{C}P^2$ consists of a 0-cell, a 2-cell and a 4-cell, and products are equipped with the product cell structure. In particular, S^2 is unambiguously a subcomplex of $\mathbb{C}P^2$ and factors are subcomplexes of the product.

8.2. **Informal explanation.** We begin by describing for the case of $X = \mathbb{C}P^2 \times S^3$ the ordinary obstruction theory that gives some intuition for the source of these examples. We write $Y = S^3 \vee S^3$, with $\iota_1, \iota_2 : S^3 \to Y$ representing the inclusions of the two spheres.

As a warmup, consider the simpler domain space $S^2 \times S^3$. One way of producing maps $S^2 \times S^3 \to Y$ is to take the projection to S^3 followed by a map in the homotopy class $d_1[\iota_1] + d_2[\iota_2]$. One can also produce other maps by altering the map on the 5-cell of $S^2 \times S^3$ by an element of

$$\pi_5(S^3 \vee S^3) \cong \mathbb{Z} \oplus \text{finite},$$

where the $\mathbb{Z}$ factor is generated by the Whitehead product $[\iota_1, \iota_2]$; let h be the coefficient of $[\iota_1, \iota_2]$. Obstruction theory tells us that this gives a complete list of homotopy classes, and therefore d_1, d_2, and h determine the homotopy class up to finite ambiguity. In fact, relative to the S^2 factor, which all of these maps send to the basepoint, these are all distinct homotopy classes.

On the other hand, in the non-relative setting, if one of the d_i is nonzero, then h is only well-defined modulo $\gcd(d_1, d_2)$. (A similar observation was made in [CMW18, §5].) In fact, while a homotopy cannot change the d_i, a homotopy whose restriction to $S^2 \times [0, 1]$ winds a_i times around the ith sphere always satisfies

$$h_{\text{end}} - h_{\text{start}} = a_1 d_2 - d_1 a_2. \tag{8.1}$$

This can be seen geometrically by factoring the homotopy through a quotient map on the ends, $S^2 \times S^3 \times \{0, 1\}$, which pinches off a 5-sphere from the top cell of $S^2 \times S^3$, then projects the rest to S^3 (see Figure 1). On the whole interval, this extends to a quotient map

$$S^2 \times S^3 \times I \to S^3 \times S^3 \setminus \text{two open balls}, \tag{8.2}$$

where the first S^3 is the quotient of $S^2 \times [0,1]$. The sum of the boundary maps of the two balls is homotopic to the Whitehead product of the two S^3's, which implies (8.1).

In particular, let

$$u_{d,h} : S^2 \times S^3 \to S^3 \vee S^3$$

be a map which differs from $d[\iota_1] \circ \pi_{S^3}$ by Whitehead coefficient h. Then the above construction gives:

Proposition 8.2. *When $h - h' \equiv 0 \mod d$, then $u_{d,h}$ and $u_{d,h'}$ are homotopic.*

Now consider instead the domain space $X = \mathbb{C}P^2 \times S^3$, which naturally contains $S^2 \times S^3$ as a subcomplex. First notice:

Proposition 8.3. *When h is even, $u_{d,h}$ extends to a map $\tilde{u}_{d,h} : X \to S^3 \vee S^3$.*

Proof. Consider the space $X' = (\mathbb{C}P^2 \times S^3)/(\mathbb{C}P^2 \times *)$, and let $q : X \to X'$ be the quotient map. It's not hard to see that X' is homotopy equivalent to the third suspension $S^3(\mathbb{C}P^2 \sqcup *)$. This complex consists of cells in dimensions 3, 5 and 7, with attaching maps obtained by stabilizing those of the original complex. So the attaching map of the 5-cell is trivial and the attaching map of the 7-cell is the nontrivial element of $\pi_6(S^5) \cong \mathbb{Z}/2\mathbb{Z}$. Composing this element with a map $S^5 \to S^5$ of degree 2 trivializes it (since before suspending, it multiplies the Hopf invariant by 4.) Therefore there is a map $g : X' \to S^3 \vee S^5 \vee S^7$ which has degree 1 on the 3- and 7-cells and degree 2 on the 5-cell.

Then $u_{d,h}$ extends (up to homotopy) to a composition

$$X \xrightarrow{q} X' \xrightarrow{g} S^3 \vee S^5 \vee S^7 \xrightarrow{d[\iota_1] \vee \frac{h}{2}[\iota_1,\iota_2] \vee 0} S^3 \vee S^3. \qquad \square$$

As before, we can modify this map on the 7-cell by elements of $\pi_7(S^3 \vee S^3)$, whose rational part is generated by the two triple Whitehead products $[\iota_1, [\iota_1, \iota_2]]$ and $[\iota_2, [\iota_1, \iota_2]]$.

It turns out, although this is harder to see geometrically, that the homotopy described above between $f_{1,0}$ and $f_{1,h}$ also extends over the 7-cell in many cases, but the restrictions to the 7-cell of the maps on either side of the resulting homotopy differ by $h^2[\iota_2, [\iota_1, \iota_2]]$. It follows that the simplest lifts $\tilde{u}_{1,-h}$ and $\tilde{u}_{1,h}$ are homotopic (at least up to a torsion obstruction), but any homotopy between them passes through a map with complicated behavior on the 7-cell, and hence a large Lipschitz constant. We use rational homotopy theory to prove this below.

The example of $S^2 \times S^2 \times (S^3 \vee S^3)$ is more complicated, but again uses quadratic behavior of the homotopy class on the 7-cells. This time there are two 7-cells and two quadratic functions in two variables; we show that at some point during the homotopy, one of these functions must attain a large value.

8.3. **Algebraic model.** We now begin the formal proof of Theorem 8.1 for $X = \mathbb{C}P^2 \times S^3$. At the end of the section we will describe the analogous proof for $X = S^2 \times S^2 \times (S^3 \vee S^3)$.

We start by building simplified algebraic models of the maps and homotopies between them and showing that these algebraic homotopies must be large. Afterwards we will deduce the results for Lipschitz constants of genuine maps and homotopies between them.

Write $Y = S^3 \vee S^3$, write

$$\mathcal{M}_Y^* = (\Lambda(a_1^{(3)}, a_2^{(3)}, b^{(5)}, c_1^{(7)}, c_2^{(7)}, \ldots), da_i = 0, db = a_1 a_2, dc_i = ba_i, \ldots)$$
$$H^*(X) = \Lambda(x^{(2)}, y^{(3)})/(x^3),$$

and consider the homomorphisms $\varphi_L, \psi_L : \mathcal{M}_Y \to H^*(X)$ given by

$$\begin{aligned} \varphi_L(a_1) &= y & \psi_L(a_1) &= y \\ \varphi_L(a_2) &= 0 & \psi_L(a_2) &= 0 \\ \varphi_L(b) &= -L^6 yx & \psi_L(b) &= L^6 yx \\ \varphi_L(c_i) &= 0 & \psi_L(c_i) &= 0. \end{aligned}$$

These two homomorphisms are homotopic via the homotopy $\eta_L : \mathcal{M}_Y^* \to H^*(X) \otimes \Omega^*[0,1]$ given by:

$$\begin{aligned} &\eta_L(a_1) = y & &\eta_L(a_2) = -2L^6 x dt & &\eta_L(b) = L^6 yx(2t-1) \\ &\eta_L(c_1) = 0 & &\eta_L(c_2) = 2L^{12} yx^2 t(1-t). \end{aligned}$$

On the other hand:

Lemma 8.4. *Every homotopy between φ_L and ψ_L satisfies* $\mathrm{Dil}(\eta) = \Omega(L^{12/7})$.

Proof. We inductively explore the ways of defining such a homotopy, noting that the space of possibilities is rather small. First, we must have

$$\eta(a_1) = y - x d\alpha(t) \qquad \eta(a_2) = -x d\beta(t),$$

where $\alpha, \beta : [0,1] \to \mathbb{R}$ are Lipschitz functions. The relation $\eta d = d\eta$ forces us to set

$$\eta(b) = yx\beta(t) + x^2 d\gamma(t),$$

where $\gamma : [0,1] \to \mathbb{R}$ is again Lipschitz and the definitions of φ_L and ψ_L mean that $\beta(0) = -L^6$ and $\beta(1) = L^6$. Finally, again using $\eta d = d\eta$ and the definitions of the functions, we get

$$\eta(c_2) = yx^2(L^{12} - \frac{1}{2}\beta^2(t)).$$

By the intermediate value theorem, $\beta(t) = 0$ for some $t \in (0,1)$, so $\mathrm{Dil}(\eta) = \Omega(L^{12/7})$. □

8.4. **Construction of homotopic maps.** Now we define the maps

$$f_L : \quad X \xrightarrow{q} X' \xrightarrow{g} S^3 \vee S^5 \vee S^7 \xrightarrow{[\iota_1]\vee[-L^3\iota_1, L^3\iota_2]\vee 0} S^3 \vee S^3,$$

$$g_L : \quad X \xrightarrow{q} X' \xrightarrow{g} S^3 \vee S^5 \vee S^7 \xrightarrow{[\iota_1]\vee[L^3\iota_1, L^3\iota_2]\vee \alpha_L} S^3 \vee S^3,$$

where $\alpha_L \in \pi_7(S^3 \vee S^3)$ is a torsion element to be determined. By construction, f_L and g_L are $O(L)$-Lipschitz.

Lemma 8.5. *There are algebraic homotopies of dilatation $O(L^{6/5})$ from $f_L^* m_Y$ and $g_L^* m_Y$ to $\rho \circ \varphi_L$ and $\rho \circ \psi_L$, respectively, where $\rho : H^*(X) \to \Omega^*(X)$ is a way of realizing the cohomology algebra using differential forms.*

Proof. We give the proof for f_L and φ_L; the proof for g_L and ψ_L is analogous, if slightly complicated by the presence of α_L.

Note that f_L factors through maps

$$X \xrightarrow{u} S^3 \vee S^5 \xrightarrow{v_L} S^3 \vee S^3,$$

where u is independent of L. Applying Lemma 6.5(i) to v_L, we get an algebraic homotopy Φ of dilatation $O(L^{6/5})$ from $v_L^* m_Y$ to $\rho \circ \nu_L$, where

$$\nu_L : \mathcal{M}_Y \to H^*(S^3 \vee S^5)$$

sends $a_1 \mapsto [S^3]$, $a_2 \mapsto 0$, $b \mapsto -L^6[S^5]$.

Now $u^*\rho\nu_L$ is homotopic to $\rho\varphi_L$ via a homotopy of dilatation $O(L^{6/5})$: it suffices to extend the cohomologous pairs of closed forms $u^*\rho\nu_L(a_1), \rho\varphi_L(a_1)$ and $u^*\rho\nu_L(b), \rho\varphi_L(b)$ to closed forms on $X \times [0,1]$. Concatenating this homotopy with $(u^* \otimes \mathrm{id})\Phi$, we get the desired map. □

Lemma 8.6. *For an appropriate choice of α_L, f_L and g_L are homotopic.*

Proof. Clearly, $f_L|_{S^2\times S^3}$ is homotopic to the map $u_{1,-L^6}$ constructed earlier, and similarly, $g_L|_{S^2\times S^3}$ is homotopic to u_{1,L^6}. Thus by Proposition 8.2, we have a homotopy between these two restrictions which maps $S^2 \times [0,1]$ via $2L^6[\iota_2]$. In other words, so far we have a map defined on

$$(X \times \{0,1\}) \cup (S^2 \times S^3 \times [0,1]).$$

Moreover, this map takes the boundary of the extra 5-cell of $\mathbb{C}P^2 \times [0,1]$ to $S^3 \vee S^3$ via

$$2L^6[\iota_2] \circ (S\,\mathrm{Hopf}) : S^4 \to S^3 \vee S^3,$$

which is nullhomotopic in the second S^3, by the same argument as in the proof of Proposition 8.3. Thus the map extends to this 5-cell, yielding a map

$$F : (X \times [0,1])^{(7)} \to Y$$

which takes $\mathbb{C}P^2 \times [0,1]$ to the second S^3.

It remains to understand the obstruction in $\pi_7(S^3 \vee S^3)$ to extending this map over the 8-cell of $\mathbb{C}P^2 \times S^3 \times [0,1]$. Assuming for now that $\alpha_L = 0$, f_L and g_L both factor through $u : X \to S^3 \vee S^5$, and therefore F factors through a map $\overline{F} : Z^{(7)} \to Y$, where

$$Z = \frac{X \times [0,1] \sqcup (S^3 \vee S^5) \times \{0,1\}}{(x,t) \sim (u(x),t)}.$$

Note that the quotient map $X \times [0,1] \to Z$ extends that of (8.2), and in particular

$$Z^{(7)} = [(S^3 \times S^3) \setminus \text{two open balls}] \cup e^5$$

where the 5-cell represents the suspension of the 4-cell of $\mathbb{C}P^2$. It follows that

$$Z^{(7)} \simeq_{\mathbb{Q}} S^3 \vee S^3 \vee S^5 \vee S^5,$$

and the rational homotopy type of $\overline{F}$ is determined by the images of these four spheres. These are:

- $[\iota_1]$, for the S^3 factor of X;
- $2L^6[\iota_2]$, for the quotient of $S^2 \times [0,1]$;
- $-L^6[\iota_1, \iota_2]$, for the S^5 at time 0;
- 0, for the suspension of the 4-cell.

To understand the obstruction to extending $\overline{F}$ over the 8-cell, we construct a rational model. Consider the DGA

$$\mathcal{A} = \{(w, w_0', w_1') \in H^*(X) \otimes \Lambda(t, dt) \oplus H^*(S^3 \vee S^5)^2 \ : w|_{t=0} = u^*(w_0') \text{ and } w|_{t=1} = u^*(w_1')\}$$

representing the rational homotopy type of Z. (Here we are using the contravariance of the functor from spaces to DGAs: a quotient of a disjoint union becomes a subalgebra of a direct sum.) Since $S^3 \vee S^5$ and X are both scalable, there is an evaluation map $\mathrm{ev} : \mathcal{A} \to \Omega^* Z$. Now define a homomorphism

$$\overline{\eta}_L = (\eta_L, \nu_L, \nu_L') : \mathcal{M}_Y \to \mathcal{A}$$

by

$$\begin{aligned} a_1 &\mapsto (y, [S^3], [S^3]) & a_2 &\mapsto (-2L^6 x dt, 0, 0) \\ b &\mapsto (L^6 yx(2t-1), -L^6[S^5], L^6[S^5]) \\ c_1 &\mapsto (0,0,0) & c_2 &\mapsto (2L^{12} yx^2 t(1-t), 0, 0). \end{aligned}$$

To see that $(\mathrm{ev} \circ \overline{\eta}_L)|_{Z^{(7)}}$ is homotopic to $\overline{F}^* m_Y$, notice that the projections of $\overline{\eta}_L(a_1)$, $\overline{\eta}_L(a_2)$ and $\overline{\eta}_L(b)$ to $H^*(Z)$ are dual to the map described above.

It follows that there is no obstruction to extending $\overline{F}^* m_Y$ over the 8-cell of Z, and so any obstruction to extending $\overline{F}$ must be a torsion element of $\pi_7(S^3 \vee S^3)$. We can take this element to be α_L. □

Finally, we complete the proof of Theorem 8.1:

Lemma 8.7. *Every homotopy between f_L and g_L goes through maps of Lipschitz constant $\Omega(L^{4/3})$.*

Proof. By Lemma 8.4, any homotopy between φ_L and ψ_L has dilatation $\Omega(L^{12/7})$. By Lemma 6.5(ii), from an L'-Lipschitz homotopy between f_L and g_L we can obtain a homotopy between φ_L and ψ_L of dilatation $O((L')^{9/7})$. Therefore, any homotopy $h : X \times [0,T] \to Y$ between f_L and g_L has Lipschitz constant $\Omega(L^{\frac{12}{7}/\frac{9}{7}}) = \Omega(L^{4/3})$. Since we can stretch the time direction arbitrarily, this Lipschitz constant must be that of a time-slice of the homotopy. □

8.5. **Proof for $X = S^2 \times S^2 \times (S^3 \vee S^3)$.** The proof in this case follows the same outline. As before, we start with an algebraic argument. Write $X = S^2 \times S^2 \times (S^3 \vee S^3)$ and

$$H^*(X) = \Lambda(x_1, x_2, y_1, y_2)/(y_1 y_2, x_1^2, x_2^2),$$

and consider the homomorphisms $\varphi_L, \psi_L : \mathcal{M}_Y \to H^*(X)$ given by

$$\begin{aligned} \varphi_L(a_1) &= y_1 - y_2 & \psi_L(a_1) &= y_1 - y_2 \\ \varphi_L(a_2) &= 0 & \psi_L(a_2) &= 0 \\ \varphi_L(b) &= L^6(x_1 y_1 + x_2 y_2) & \psi_L(b) &= L^6(x_1 y_2 + x_2 y_1) \\ \varphi_L(c_i) &= 0 & \psi_L(c_i) &= 0. \end{aligned}$$

These two homomorphisms are homotopic via the homotopy $\eta_L : \mathcal{M}_Y^* \to H^*(X) \otimes \Omega^*[0,1]$ given by:

$$\begin{aligned} \eta_L(a_1) &= y_1 - y_2 & \eta_L(a_2) &= L^6(x_1 - x_2)dt \\ \eta_L(b) &= L^6[(x_1 y_1 + x_2 y_2)(1-t) + (x_1 y_2 + x_2 y_1)t] \\ \eta_L(c_1) &= 0 & \eta_L(c_2) &= L^{12} x_1 x_2 (y_1 - y_2) t(1-t). \end{aligned}$$

On the other hand:

Lemma 8.8. *Every homotopy between φ_L and ψ_L satisfies* $\mathrm{Dil}(\eta) = \Omega(L^{12/7})$.

Proof. Any such homotopy must have

$$\begin{aligned} \eta(a_1) &= y_1 - y_2 + x_1 d\alpha_1(t) + x_2 d\alpha_2(t) \\ \eta(a_2) &= L^6(x_1 d\beta_1(t) + x_2 d\beta_2(t)), \end{aligned}$$

where $\alpha_i, \beta_i : [0,1] \to \mathbb{R}$ are Lipschitz functions. Then, normalizing so that $\beta_1(0) = \beta_2(0) = 0$, the relation $\eta d = d\eta$ forces us to have

$$\eta(b) = L^6 \left[(1-\beta_1(t))x_1y_1 + (1-\beta_2(t))x_2y_2 + \beta_1(t)x_1y_2 + \beta_2(t)x_2y_1\right] + x_1x_2 d\gamma(t),$$

from which we see that $\beta_1(1) = \beta_2(1) = 1$. Finally, again using $\eta d = d\eta$, we get

$$\eta(c_2) = L^{12} \left[\beta_2(t)(1-\beta_1(t))y_1 - \beta_1(t)(1-\beta_2(t))y_2\right] x_1x_2.$$

Now suppose that for all t,

$$|\beta_1(t)(1-\beta_2(t))| + |\beta_2(t)(1-\beta_1(t))| < \frac{1}{6}.$$

Then by the reverse triangle inequality, for all t,

$$|\beta_1(t) - \beta_2(t)| < 1/6.$$

But by the intermediate value theorem, there is a point where $\beta_1(t) = 1/2$. At this point, $\beta_2(t) < 2/3$, and therefore

$$\beta_1(t)(1-\beta_2(t)) > 1/6,$$

a contradiction. Therefore, $\mathrm{Dil}(\eta) = \Omega(L^{12/7})$. □

Now we define the maps f_L and g_L. Define

$$s_{h^2,d} : S^2 \times S^3 \to S^3 \vee S^3$$

to be the composition

$$S^2 \times S^3 \xrightarrow{f_0} S^5 \vee S^3 \xrightarrow{[i_1 r_h, i_2 r_h] \vee (i_1 r_d)} S^3 \vee S^3,$$

where

- f_0 is a map which pinches off a sphere and projects the remainder of the space onto the S^3 factor.
- i_1 and i_2 are the inclusions of the two spheres in the wedge.
- r_p is an $O(p^{1/3})$-Lipschitz self-map of S^3 of degree p.

Define $\pi_1, \pi_2 : S^2 \times S^2 \times S^3 \to S^2 \times S^3$ to be the projections that project onto the first and second copy of S^2, respectively. Then we define

$$f_L(x) = \begin{cases} s_{L^6,1} \circ \pi_1(x) & x \in S^2 \times S^2 \times S^3_1 \\ s_{L^6,-1} \circ \pi_2(x) & x \in S^2 \times S^2 \times S^3_2, \end{cases}$$

which is well-defined since both maps take the intersection of the two subdomains to the base point. Similarly, we define

$$g_L(x) = \begin{cases} s_{L^6,1} \circ \pi_2(x) & x \in S^2 \times S^2 \times S^3_1 \\ s_{L^6,-1} \circ \pi_1(x) & x \in S^2 \times S^2 \times S^3_2. \end{cases}$$

Clearly these maps are $O(L)$-Lipschitz.

A similar argument to Lemma 8.6 shows that these maps are homotopic (perhaps after modifying g_L on the two 7-cells by some torsion elements of $\pi_7(S^3 \vee S^3)$). A similar argument to Lemma 8.5 shows that they have algebraic homotopies of dilatation $O(L^{6/5})$ to $\rho \circ \varphi_L$ and $\rho \circ \psi_L$, respectively. The argument of Lemma 8.7 completes the proof for this case.

9. Optimality of the example

In this section we show, using related techniques, that the example above is optimal in two different senses. First, two homotopic L-Lipschitz maps from any 7-complex to $S^3 \vee S^3$ are always homotopic through $O(L^{4/3})$-Lipschitz maps. Secondly, if Y is a scalable space with nontrivial rational homotopy groups in only two degrees, then all pairs of homotopic L-Lipschitz maps to Y are homotopic through $O(L)$-Lipschitz maps; so our example with nontrivial rational obstruction groups in three degrees is in some sense the simplest possible.

We start with the second part:

Theorem 9.1. *Let Y be a compact, simply connected Riemannian manifold (with boundary) which is scalable and rationally equivalent to the total space of a principal fibration whose base and fiber are products of Eilenberg–MacLane spaces, and let X be a finite simplicial complex equipped with a standard simplexwise metric. Then every pair of homotopic L-Lipschitz maps $f, g : X \to Y$ is homotopic through $C(X, Y)(L + 1)$-Lipschitz maps. Moreover, the Lipschitz constant in the time direction of the homotopy takes the form $C(X, Y)(L + 1)^p$, where $p = p(X, Y)$ is some integer exponent.*

In other words, the finite bars of $PH_0(\mathrm{Lip}(X, Y), \log_+ \mathrm{Lip})$ have bounded length in all components. We note, however, that the constant $C(X, Y)$ depends not only on the local geometry of X, but also its global geometry. This means that Theorem 9.1 is insufficient to show, using the technique of Theorem 7.3, the boundedness of finite bars in all degrees.

While the conditions on Y are rather strong, they include all compact-type symmetric spaces: symmetric spaces are scalable, and all homogeneous manifolds satisfy the "two-stage" condition [FHT01, Prop. 15.16].

Proof. This proof is similar to the proofs of [Man19, Theorems 5.5 and 5.6]. We construct an algebraic homotopy and then apply the argument of [Man19, Theorem 5.7] to make sure that it lies in the relative homotopy class of a genuine homotopy between f and g. We then use the improved shadowing principle for scalable spaces (rather than the original shadowing principle as in the proofs we imitate) to construct a homotopy with controlled Lipschitz constant.

Fix a minimal model $m_Y : \mathcal{M}_Y^* \to \Omega^* Y$ which factors through $H^*(Y; \mathbb{R})$. By assumption, $\mathcal{M}_Y$ decomposes as

$$\mathcal{M}_Y = \Lambda(V \oplus W)$$

where $dV = 0$ and $dW \subseteq \Lambda V$, and $H^*(Y; \mathbb{R})$ is a quotient of ΛV. We also fix a linear map $\alpha : H^*(X; \mathbb{R}) \to \Omega^* X$ sending cohomology classes to representatives. (Note this may not be a ring homomorphism, which does not exist unless X is also scalable.)

We start by producing an algebraic homotopy between $f^* m_Y$ and $g^* m_Y$, which will in turn consist of two steps. First we build a homotopy

$$\Phi_1 : \mathcal{M}_Y^* \to \Omega^*(X \times [0, 1])$$

between $f^* m_Y$ and a homomorphism φ which coincides with $g^* m_Y$ on V. For $v \in V$ of degree k, we set

$$\Phi_1(v) = (1 - t^k) f^* m_Y(v) + t^k g^* m_Y(v) + c(v) \otimes dt,$$

where $c(v)$ is chosen so that $dc(v) = (-1)^{k+1}(g^* m_Y(v) - f^* m_Y(v))$. For $v \in W$, we then choose an extension using Prop. 6.6 whose operator norm is $O(L^{\deg v + 1})$. Thus $\mathrm{Dil}^U(\Phi_1) = O(L)$, and $\varphi = \Phi_1|_{t=1}$ sends W to closed forms.

Since $g^*m_Y|_W = 0$, $[\varphi|_W] \in \mathrm{Hom}(W, H^*(X;\mathbb{R}))$ measures the obstruction to extending the constant homotopy on V to a homotopy between φ and g^*m_Y. On the other hand, since we know a homotopy between these homomorphisms exists, by [GM81, Prop. 14.4] $[\varphi|_W]$ must be in the image of the homomorphism

$$[V, \Omega^* X \otimes \Lambda(e^{(1)})]_{\alpha f^*} \to \mathrm{Hom}(W, H^*(X;\mathbb{R}))$$

sending $[g^*m_Y + \eta \otimes e] \mapsto [\eta d|_W]$. The domain of this homomorphism is isomorphic to $\mathrm{Hom}(V, H^{*-1}(X;\mathbb{R}))$, and we can choose specific representatives using α; in this way, choose a preimage $g^*m_Y + \alpha\tilde{\varphi} \otimes e$ for φ (in a way that minimizes $\|\alpha\tilde{\varphi}\|_{\mathrm{op}}$). Now we define a homomorphism $\Phi_2 : \mathcal{M}_Y^* \to \Omega^*(X \times [0,T])$ by

$$\begin{aligned} \Phi_2(v) &= g^*m_Y(v) - dt/T \wedge \alpha\tilde{\varphi}(v) && v \in V \\ \Phi_2(w) &= (1 - t/T)\varphi(w) - dt/T \wedge c(w) && w \in W, \end{aligned}$$

where $dc(w) = \varphi(w) + \int_0^T \Phi_2(dw)$. Then Φ_2 is a homotopy from φ to g^*m_Y, and for

$$T = \sup_{v \in V} L^{-\deg v} \|\alpha\tilde{\varphi}(v)\|_{\mathrm{op}},$$

it has U-dilatation $O(L)$. (By an argument given in the proof of [Man19, Theorem 5.5(ii)], this estimate for T is in general polynomially bounded in L.)

The concatenation Φ of Φ_1 and Φ_2 is an algebraic homotopy between f^*m_Y and g^*m_Y, but it may not lie in the relative homotopy class of a genuine homotopy of maps. Fixing this is the purpose of [Man19, Theorem 5.7]; we summarize the proof here for convenience.

Build a homomorphism $\Sigma : \mathcal{M}_Y^* \to \Omega^*(X \times S^1)$ by a circular concatenation of Φ and h^*m_Y, where h is a genuine, but uncontrolled, homotopy between f and g. This homomorphism lifts up to homotopy to a homomorphism

$$g^*m_Y + \eta \otimes e : \mathcal{M}_Y^* \to \Omega^* X \otimes \Lambda(e^{(1)})$$

representing an element of $\pi_1(Y^X, g) \otimes \mathbb{R}$. By [MW20, Lemma 5.2(i)] and the surrounding discussion, we can find an η' such that $[g^*m_Y + \eta' \otimes e]$ is in the image of a genuine element of $\pi_1(Y^X)$ and $\|\eta' - \eta\|_{\mathrm{op}}$ is polynomial in $\mathrm{Lip}\, g$. Then define a homomorphism $\Psi : \mathcal{M}_Y^* \to \Omega^*(X \times [0,T'])$ by

$$\Psi(v) = g^*m_Y(v) + dt/T' \wedge (\eta' - \eta),$$

where T' is sufficiently large that $\mathrm{Dil}(\Psi) = O(L)$. Then the concatenation of Φ and Ψ lies in the relative homotopy class of a genuine homotopy between f and g and has U-dilatation $O(L)$. By the improved shadowing principle, we can find such a homotopy with Lipschitz constant $O(L)$. □

Now we show that the $L^{4/3}$ bound in the example is sharp:

Theorem 9.2. *Let X be any finite 7-complex and let $f, g : X \to S^3 \vee S^3$ be homotopic L-Lipschitz maps. Then there is a homotopy between them through $O(L^{4/3})$-Lipschitz maps, with implicit constants depending on X.*

Proof. We proceed as in the proof of Theorem 9.1. As before, write $Y = S^3 \vee S^3$ and

$$\mathcal{M}_Y^* = (\Lambda(a_1, a_2, b, c_1, c_2, \ldots), da_i = 0, db = a_1 a_2, dc_i = ba_i, \ldots).$$

Since X is 7-dimensional, these are the only generators on which the algebraic homotopy we construct will be nonzero.

We start by applying the construction in Theorem 9.1 to the subalgebra $\Lambda(a_1, a_2, b)$, using the same notation. We know that, for the homomorphism φ constructed in the first step, $\|\varphi(b)\|_\infty = O(L^6)$. Since $\tilde\varphi$ is constructed so that

$$[\varphi(b)] = \tilde\varphi(a_1) \cup g^* a_2 + g^* a_1 \cup \tilde\varphi(a_2) \in H^n(X;\mathbb{R}),$$

and the $g^* a_i$ lie in a lattice in $H^3(X;\mathbb{R})$, the forms $\alpha\tilde\varphi(a_i) \in \Omega^2(X)$ also satisfy $\|\alpha\tilde\varphi(a_i)\|_{\mathrm{op}} = O(L^6)$; moreover so does $\int_0^1 \Phi_2(db)$. This allows us to choose $T = L^3$ to obtain U-dilatation $O(L)$ for the homotopy Φ.

Now we extend Φ over c_1 and c_2 to create a homotopy $\tilde\Phi$ between $f^* m_Y$ and a map ψ which coincides with $g^* m_Y$ on $\Lambda(a_1, a_2, b)$. By the argument of Prop. 6.6, we can choose this so that

$$\|\tilde\Phi(c_i)\|_{\mathrm{op}} \leq C\|\tilde\Phi(a_i)\|_{\mathrm{op}} \cdot \|\tilde\Phi(b)\|_{\mathrm{op}} = O(L^{12}).$$

Next, we concatenate $\tilde\Phi$ with a second homotopy constructed similarly to Φ_2 in the proof of Theorem 9.1. Namely, we first consider the obstruction to extending the homotopy which is constant on $\Lambda(a_1, a_2, b)$ to the c_i. This obstruction is in the image of a homomorphism

$$[\Lambda(a_1, a_2, b), \Omega^* X \otimes \Lambda(e^{(1)})]_{g^* m_Y} \to \mathrm{Hom}(\langle c_1, c_2\rangle, H^*(X;\mathbb{R}))$$

sending $[g^* m_Y + \eta \otimes e] \mapsto [\eta d]$. Like before, choose a representative $g^* m_Y + \tilde\psi \otimes e$ that maps to $[\psi d]$. Then we define $\Phi_3 : \mathcal{M}_Y^* \to \Omega^* X \times [0, T'']$ by

$$\begin{aligned} \Phi_3(a_i) &= g^* m_Y(a_i) - dt/T'' \wedge \alpha\tilde\psi(a_i) && i = 1, 2 \\ \Phi_3(b) &= -dt/T \wedge \alpha\tilde\psi(b) \\ \Phi_3(c_i) &= (1 - t/T)\psi(c_i) - dt/T'' \wedge \sigma(c_i)_i && i = 1, 2, \end{aligned}$$

where $d\sigma_i = \psi(c_i) + \int_0^{T''} \Phi_3(dc_i)$. For T'' sufficiently large, $\|\Phi_3(c_i)\|_{\mathrm{op}} = O(L^{12})$ and therefore the U-dilatation of Φ_3 is $O(L^{4/3})$.

Finally, we proceed as in the previous proof to construct a self-homotopy Ψ of $g^* m_Y$ such that the concatenation of $\tilde\Phi$, Φ_3 and Ψ is in the relative homotopy class of a geometric homotopy between f and g; Ψ can be chosen to have dilatation $O(L)$ if one makes the time interval sufficiently large. By the improved shadowing principle, we can then find a geometric homotopy with Lipschitz constant $O(L^{4/3})$. □

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