Double-Number Lines, Area Models, and Fifteen Years of the Common Core Standards for School Mathematics

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Abstract

In this paper, we examine changes in teachers' understandings of double number lines, possibly because of the Common Core. We revisit a paper we published in 2012 focused on teachers' understanding of the double-number line representation. Then, we compare that paper to our experiences in Summer 2024 in which teachers had no discernable issues with double-number lines. We ground our work in knowledge in pieces as well as a definition for mathematical understanding grounded in representations. We present our findings about current teachers' understanding of double-number lines and area models for reasoning about fractions and proportions, then conclude with discussion of the implications of our noticing for teacher preparation and teacher professional learning.

Purpose

In 2012, we wrote a paper about mathematics teachers' misunderstandings of double number lines used for solving proportional situations (BLINDED). Since that time, many changes have happened in mathematics teaching and learning, most notably the roll-out of the Common Core Standards for School Mathematics (CCSSM; NGA & CCSSO, 2010) and instructional materials to support students in meeting those standards. The CCSSM is important for our purposes because the representations that is lists include number lines and ratio tables, which are two components that can lead to understanding of double number lines.

In the initial paper (Orrill & Brown, 2012), we found that teachers struggled to coordinate the individual scales of the two number lines, they struggled to understand how to use it as a tool with which to reason, and they struggled to understand how it was showing an "equal" relationship because each line was on an independent scale. After a few class sessions of struggling with the double number line, the teachers were finally able to use their understanding of partitioning – an understanding they had come into professional learning with because of their comfort in using area models – to reason with the double number line.

We moved into other areas of research related to teachers' understandings. Until summer 2024 when we were confronted with two groups of teachers, several states apart, that had meaningful understanding of double number lines – as well as ratio tables – but lacked meaningful understanding of area models. In this paper, we present our experience as a rationale for our emerging hypotheses about what has happened. We also provide recommendations for working with teachers to support a broader set of understandings with which to support their students.

Perspective and Background

We rely on two primary perspectives to inform the hypotheses and recommendations presented here. First, we view teacher knowledge through a knowledge-in-pieces lens (e.g., diSessa, 1988, 2006; diSessa et al., 2004; diSessa et al., 2016). This lens allows us to conceptualize knowledge as being comprised of fine-grained understandings, called knowledge resources. These resources are interconnected in ways that allow them to be used in different combinations as one encounters new situations. From this perspective, one can learn by refining an existing knowledge resource, developing a new knowledge resource, or connecting knowledge resources to each other in new ways. We rely on knowledge-in-pieces because it offers an explanation for teachers' understandings at a grain size that allows us to think about specific knowledge resources they invoke to reason about tasks. For example, we can look at the teachers' use of representations and moves within those representations such as partitioning, coordinating, unitizing, etc. Knowledge-in-pieces also allows us a means to explain how teachers may appear to "have" knowledge in one situation but lack it in another. For example, if a teacher demonstrates the ability to partition within a unit an area model, but not to do that in a number line, we would explain that as the teacher not invoking such knowledge because of a lack of well-developed connections. Thus, coherent interconnection of knowledge resources is likely to result in a teacher being able to apply knowledge resources in more situations. This is similar to the ideas of expertise set forth by Chi and others (e.g., Bédard & Chi, 1992; Bransford et al., 1999; Chi, 2006) who noted that experts not only have more knowledge, but also knowledge that is organized in particular ways that differ from novice organization.

Next, we rely on a definition of mathematical understanding that is grounded in representations. Specifically, we draw on Lesh and colleagues (1987) who asserted that understanding mathematics means being able to reason about a situation across a variety of representations. We assert that this is knowledge that is critical knowledge for teachers because they need to understand mathematics enough to be able to connect students' invented methods to standard methods. This means that teachers need the kind of conceptual understanding that is needed to make sense of mathematics across representations.

Methods & Data Sources

Participants

Our participants participated in one of two professional learning workshops offered in summer 2024. One included 31 educators including math teachers, special education teachers, and school-based mathematics leaders from an urban district in the northeast US. The other included nine teachers (general education, math, and special education) from a city in the southeastern US. Both workshops included teachers in grades 3-8, with the larger one also including a small number of high school teachers. Each workshop was conducted at a site within the school district and lasted for 30 hours (6 hours per day for 5 days). The workshops were led by the first (northeast) and third (southeast) author. They focused on fractions operations and proportional reasoning, with a strong emphasis on the use of static and dynamic representations to support teachers' learning (cf. Orrill & Brown, 2023). All names used in this paper are pseudonyms.

Data & Analysis

Data considered for this paper includes item interviews with 16 of the teachers who participated in the workshops (selected at random) as well as the full corpus of data collected in the workshops. The interviews were done via Zoom using a Jamboard that captured any marks the participant made as well as where they pointed their cursor. The workshop data included video recordings of each session on three video cameras in both whole-group and small-group discussions and written reflections completed by the teachers twice per day during the workshop.

Analysis to date has focused on transcripts of the interviews as well as the informal conversations had by each instructor in their course. We used hypothesis coding (Saldaña, 2013) to explore our hypotheses about double-number lines. Specifically, we hypothesized that teachers' comfort with double-number lines might be related to their comfort with number lines and ratio tables. We further hypothesized that the knowledge of double-number lines may have, in essence, replaced knowledge of area models that we had previously seen. Once we finished our hypothesis coding, we moved into assertion development (Erickson, 1986) – looking for data to confirm or disconfirm our assertions about current teachers' knowledge of the representations.

Results

We found that teachers in both workshop locations were comfortable and confident with double-number lines. The issues we had seen in the earlier data were not relevant for the teachers in either of the workshops this year. This seemed tied to two factors. The first is the prevalence of the ratio table. Particularly in the workshop in the southeast, participants reported that the double-number line reminded them of ratio tables. Using tables to reason about ratios is present in both the 6^{th} and 7^{th} grade standards for the CCSSM (NGA & CCSSO, 2010). The second was more comfort and confidence with number lines in general.

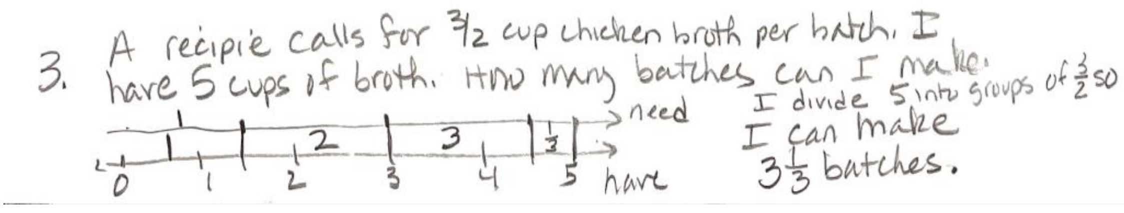
For example, in one reflection from the larger workshop, many teachers used number lines or very linear representations to model 5 ÷ 3/2 (Table 1). While the participants had worked with several representations in the workshop at this point – particularly area models – more than half the class used either a number line (Melanie and Wanda – or a strip model that they then divided the same way as a number line (Wendy), thus treating as a one dimensional measure like a line rather than in two dimensions as one would work with an area model.

Interestingly, unlike even one year before, several of the 2024 teachers were able to make sense of the item shown in Figure 1. Many participants were able to either identify that the model could be showing $\frac{5}{6} \div \frac{1}{4}$ or were able to explain how it was showing that relationship when specifically prompted. When specifically asked about whether it could show $\frac{1}{4} \div \frac{5}{6}$, they were more divided. In previous work with that item, most teachers were unable to reason about either relationship. Thus, we assert that number line reasoning within this group of teachers was stronger overall that in the past.

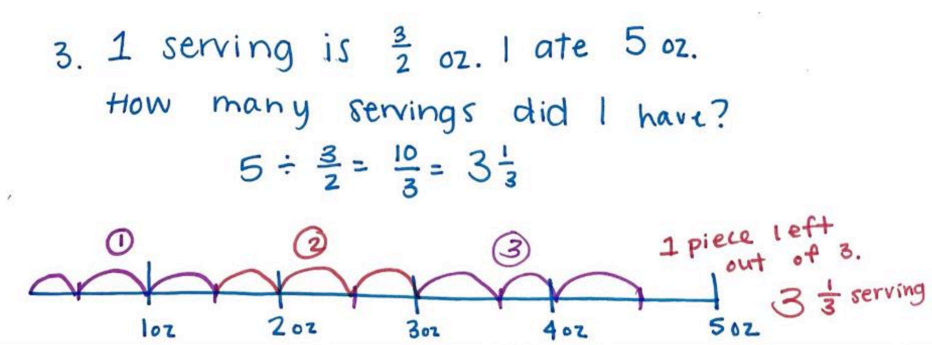
Table 1.

Examples of teacher's models when asked to "write a word problem…and then use a model to demonstrate the process of finding the correct answer" for 5 ÷ 3/2.

Melanie's approach



Wanda's approach



Wendy's approach

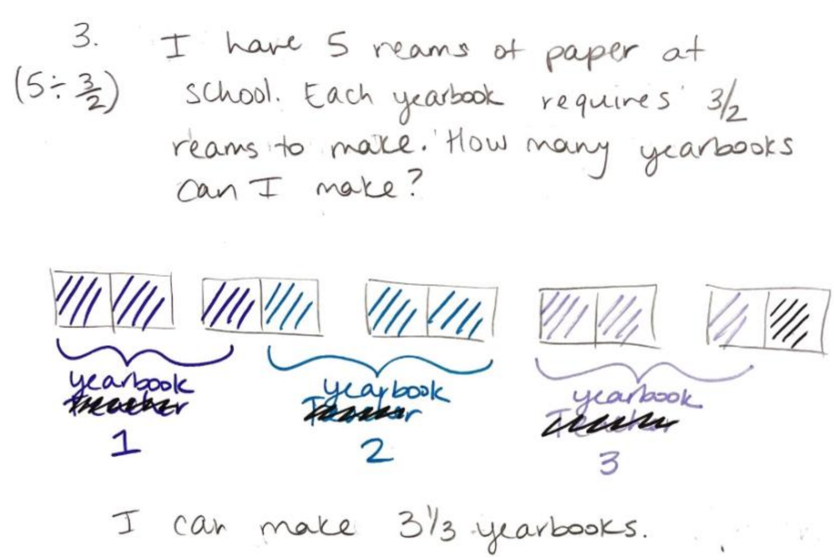
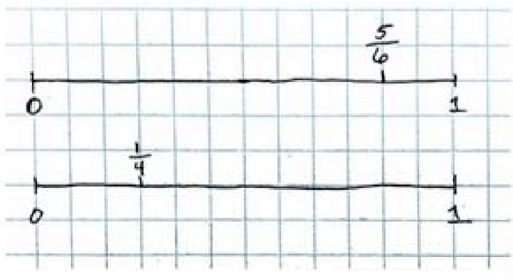


Figure 1.

Number line item asking teachers to make sense of how two separate number lines could be used to think about fraction division.

8. Below is a representation of two number lines that a student drew to better understand a fraction division problem.



Question 8.1:
What problem might the student be trying to represent?
How might this representation help the student understand that problem?

In contrast, teachers across both workshops struggled with area models. The struggles included both attitudinal and comprehension issues. A subset of the teachers participating, actively fought against the notion that area models are salient and important models for developing understanding. In addition to these arguments, we found that most teachers interpreted area models as ‘You draw lines one way. Then you draw the other way. Where they are colored in is the answer’ (Figure 2). Other teachers were unable to show multiplication with an area model at all (Figure 3), instead they tried to explain and replicate what they had seen in their textbooks. Several of these teachers mentioned in their interviews that they hoped to develop more meaning for these models in the workshop.

Figure 2.

Charles’ drawing for multiplication of 2/3 x 6/7 with an area model noting that the correct answer is where the colors overlap (outlined in green).

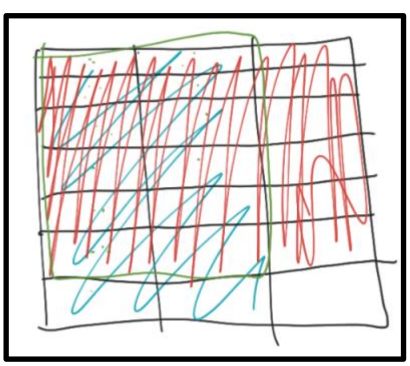
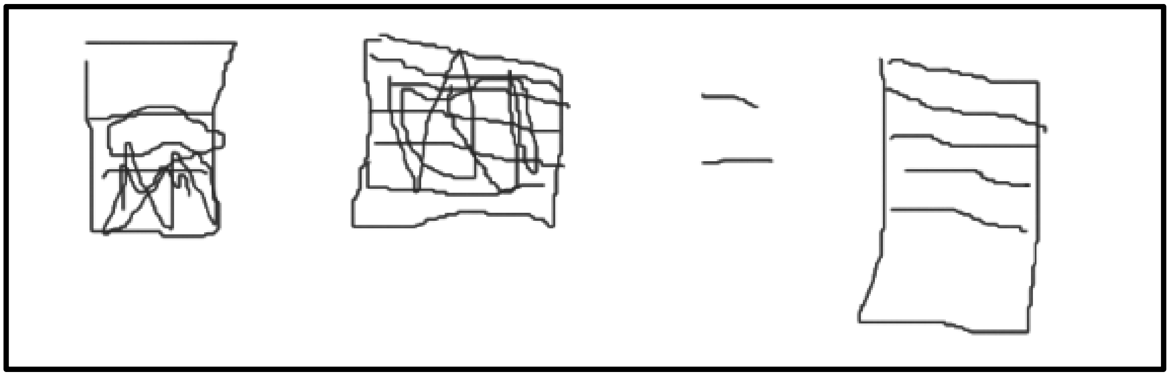


Figure 3.

Maya's attempt to show 2/3 x 6/7 with an area model. She tried to remember what she had seen in her textbook.



This work leads us to reflect on what has happened that the area models are no longer preferred by the teachers and the number line, and more interestingly, double number line, are more widely used. Our speculation from talking to the teachers and analyzing their work is that there are three intertwined things happening. First, they are getting more exposure to number lines and ratio tables because those are specifically named in the CCSSM and included in materials with which they teach. Second, perhaps because of teacher turnover or some other external factor, the current teachers were not exposed to area models in their teacher preparation or subsequent professional learning. This left them to make sense of the models in their textbooks on their own. The low level of understanding about the relationship of area models to the geometric concept of area or to the number concept of arrays suggests that the information taken away from the books may have been only procedural (e.g., how to create the area model rather than how to think with it). And third, the CCSSM (and state standards that have been developed since 2010) has privileged linear representations. Because the CCSSM content – particularly procedural knowledge – is what is on "the test", the limited attention to non-linear representations in the CCSSM may be showing up in teacher pedagogical content knowledge (PCK; Shulman, 1986).

Scholarly Significance

This significance of this work is that it highlights some of the factors that may be causing teachers' knowledge to change as well as highlighting an important change in teachers' PCK. This matters because having a knowledge of a variety of representations is necessary for teachers to be able to engage with students' invented strategies. Yet, in this group of 40 teachers from two states, we see both a strength in linear models and a weakness in area models. Both models are important for supporting students to develop conceptual knowledge across their K-12 mathematics careers. Thus, our argument is that teacher educators and professional developers need to be aware of this shifting landscape so they can design professional learning that will help the teachers connect what they know about lines and linear models to area models. Similarly, we assert that better guidance could be provided in the textbooks to support teachers in thinking about area models in more conceptual ways rather than procedurally. Finally, we assert that using representations to think about mathematics, rather than solve math problems might better support the development of both understandings.

Acknowledgement

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References

Bédard, J., & Chi, M. T. H. (1992). Expertise. *Current Directions in Psychological Science, 1*(4), 135-139.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (1999). *How people learn*. Washington, DC: National Academy Press.

Chi, M. T. H. (2006). Laboratory methods for assessing experts' and novices' knowledge. In K. Ericsson, N. Charness, P. Feltovich, & R. Hoffman (Eds.), *Cambridge handbook of expertise and expert performance* (pp. 167-184). Cambridge: Cambridge University Press.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49-70). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

diSessa, A. A. (2006). A history of conceptual change research: Threads and fault lines. In R. K. Sawyer (Ed.). *The Cambridge handbook of the learning sciences* (pp. 265-282). New York: Cambridge University Press.

diSessa, A. A., Gillespie, N. M., & Esterly, J. B. (2004). Coherence versus fragmentation in the development of the concept of force. *Cognitive Science, 28*, 843-900.

diSessa, A. A., Sherin, B. L., & Levin, M. (2016). Knowledge analysis: An introduction. In A. A. diSessa, M. Levin, & J. S. Brown (Eds.), *Knowledge and interaction: A synthetic agenda for the learning sciences,* pp. 30-71. New York: Routledge.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (#rd. ed, 119-161).

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), Problems on representation in the teaching and learning of mathematics (pp. 33–40). Lawrence Erlbaum Associates.

NGA (National Governor's Association Center for Best Practices) & CCSSO (Council of Chief State School Officers) (2010). *Common core state standards for mathematics*. National Governor's Association Center for Best Practices. https://www.thecorestandards.org/Math/

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*(5), 381-403. DOI: 10.1007/s10857-012-9218-z
Saldaña, J. (2013). *The coding manual for qualitative researchers* (2nd ed.). Sage Publications Ltd.

Orrill, C. H., & Brown, R. E. (2023). Using design-based research to develop a professional development model. In J. M. Spector, B. B. Lockee, & M. D. Childress (Eds.), *Learning, design, and technology: An international compendium of theory, research, practice, and policy*. Springer. https://doi.org/10.1007/978-3-319-17727-4_177-1

Shulman, L. S. (1986). Those who understanding: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.