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Collapsing geometry of hyperkähler 4-manifolds and applications

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Contents

1. Introduction

A Riemannian metric g on a smooth four manifold X is *hyperkähler* if its holonomy group is contained in $\mathrm{SU}(2)\subset\mathrm{SO}(4)$. The latter condition is equivalent to saying that we can choose an orientation so that the bundle Λ^+X of self-dual 2-forms is trivialized by parallel sections. In particular, on a hyperkähler 4-manifold, there is a triple of closed self-dual 2-forms $\boldsymbol{\omega}\equiv\{\omega_1,\omega_2,\omega_3\}$ satisfying

$$\omega_\alpha\wedge\omega_\beta=2\delta_{\alpha\beta}\,d\mathrm{vol}_g\quad\text{for all }\alpha,\beta\in\{1,2,3\}.$$

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Such a triple is called a *hyperkähler triple*. Notice that conversely a hyperkähler triple uniquely determines a hyperkähler Riemannian metric. It is an important fact that hyperkähler 4-manifolds have vanishing Ricci curvature; indeed they form the simplest non-trivial class of Ricci-flat metrics. In this paper we systematically study degenerations of hyperkähler 4-manifolds, focusing on the case when the volume is *collapsing*. Below we describe two main geometric applications.

The first application of our study is to the moduli compactification of hyperkähler metrics on the K3 manifold. Here, the K3 manifold $\mathcal{K}$ is by definition the oriented smooth 4-manifold underlying a complex K3 surface. We know the intersection form on $H^2(\mathcal{K};\mathbb{Z})$ has signature $(3,19)$. Denote by $\mathfrak{M}$ the set of equivalence classes of all unit-diameter hyperkähler metrics on $\mathcal{K}$ modulo the natural action of $\mathrm{Diff}(\mathcal{K})$, endowed with the Gromov–Hausdorff topology. This space has an explicit description in terms of the *period map*. Recall that a hyperkähler metric g has a *period*, which is the element in the Grassmannian of oriented maximal positive subspaces in $H^2(\mathcal{K};\mathbb{R})$ given by the space $\mathbb{H}^+(g)$ of self-dual harmonic forms. Taking into account of $\mathrm{Diff}(\mathcal{K})$ action we have a well-defined *period map* (see [53])

$$\mathcal{P}\colon \mathfrak{M} \longrightarrow \mathcal{D} \equiv \Gamma \backslash O(3,19)/(O(3)\times O(19)), \tag{1.1}$$

where Γ is the automorphism group of $H^2(\mathcal{K};\mathbb{Z})$ preserving the intersection form. By the global Torelli theorem, $\mathcal{P}$ is injective and maps onto an open dense subset of $\mathcal{D}$. Moreover, $\mathcal{P}$ extends to a bijection $\mathcal{P}\colon \mathfrak{M}' \to \mathcal{D}$, where $\mathfrak{M}'$ is the partial compactification of $\mathfrak{M}$ obtained by adding the volume-non-collapsing Gromov–Hausdorff limits of smooth hyperkähler triples ; the latter are known to be hyperkähler *orbifolds*, and their periods are maximal positive subspaces in $H^2(\mathcal{K};\mathbb{R})$ which annihilate at least one homology class with self-intersection -2 (see for example [68, Chapter 6]).

We are interested in understanding the full Gromov–Hausdorff compactification $\overline{\mathfrak{M}}$. The elements in $\overline{\mathfrak{M}}\backslash\mathfrak{M}'$ are volume *collapsing* Gromov–Hausdorff limits of hyperkähler metrics whose periods diverge to infinity in $\mathcal{D}$. We prove the following structural results for these limit spaces, and hence confirm a folklore conjecture (see for example [68, Proposition IV]).

THEOREM 1.1. *Any collapsed limit in* $\overline{\mathfrak{M}}\backslash\mathfrak{M}'$ *is isometric to one of the following*:

- (dimension 3) *a flat orbifold* $\mathbb{T}^3/\mathbb{Z}_2$;
- (dimension 2) *a singular special Kähler metric on* S^2 *with local integral monodromy*;
- (dimension 1) *a* 1-*dimensional unit interval.*

In this paper will actually consider the more refined notion of *measured Gromov–Hausdorff convergence*, which includes the extra structure of a *renormalized limit measure* on the limit spaces (cf. §2.2). From the proof we know that in the first two cases the limit measure is proportional to the Hausdorff measure, while in the third case the limit measure may be non-trivial and it encodes interesting topological information of the collapsing family (cf. [46], [78], [50], see also §3.3). Notice that in the more general context of collapsing 4-dimensional Ricci-flat metrics, Lott [58] has obtained some classification results of limit spaces under certain technical assumptions on the limit spaces.

There have been extensive recent work on constructing special examples of collapsing sequences in $\mathfrak{M}$, which can be viewed as partial converses to Theorem 1.1. See for instance [41], [32], [22], [46], [68], [24], [23]. In particular, any flat orbifold $\mathbb{T}^3/\mathbb{Z}_2$ is in $\overline{\mathfrak{M}}\backslash\mathfrak{M}'$; further work is needed in order to classify all 2-dimensional limit spaces in $\overline{\mathfrak{M}}\backslash\mathfrak{M}'$ explicitly. We also mention that Odaka–Oshima [68] proposed an interesting conjecture relating the Gromov–Hausdorff compactification $\overline{\mathfrak{M}}$ to certain Satake compactification of $\mathcal{D}$ as a locally symmetric space, and [68] made some progress toward the conjecture.

The second application of our study is concerned with the asymptotic structure of *gravitational instantons*. The latter are by definition complete non-compact hyperkähler 4-manifolds (X, g) with

$$\int_X |\mathrm{Rm}_g|^2 \, d\mathrm{vol}_g < \infty.$$

These spaces originated from physics, but they also involve very rich geometry and analysis. There are a variety of constructions in the literature, such as hyperkähler quotients, twistor theory, gauge theory, complex Monge–Ampère equation, etc. Gravitational instantons are important in understanding the singularity formation of collapsing of hyperkähler metrics, since they may arise as rescaled limits around points where curvature blows up. The next theorem gives a classification of the asymptotic geometry of gravitational instantons.

THEOREM 1.2. *A non-flat gravitational instanton (X, g) has a unique asymptotic cone (Y, d_Y, p_*) which is a flat metric cone of dimension $d\in\{1,2,3,4\}$. Moreover, the following classification holds*:

- $(d{=}4)$ (X, g) *is* ALE;
- $(d{=}3)$ (X, g) *is* ALF;
- $(d{=}2)$ (X, g) *is* ALG *or* ALG*;
- $(d{=}1)$ (X, g) *is* ALH *or* ALH*.

The precise definition of these spaces will be given in §6.4. The above classification result has been long sought. The most recent result is due to Chen–Chen [20] building upon ideas from earlier work of Minerbe [63], where one assumes the extra condition $|\mathrm{Rm}|{=}O(r^{-2-\varepsilon})$ for some $\varepsilon{>}0$, and obtains a classification into only the four classes above without the superscript *. This weaker result is proved by studying the behavior of "short geodesic loops" at infinity using ODE comparison, and the asymptotic fibration is constructed using these short loops; the hyperkähler property mainly enters as a control on the holonomy. Our proof is based on a completely different approach. First, we make essential use of the Cheeger–Fukaya–Gromov theory on $\mathcal{N}$-structures, which has the advantage of incorporating multi-scale collapsing phenomenon at infinity. Secondly, we manifest the role of the hyperkähler equation itself as an elliptic PDE. These ideas could potentially apply to more general situation.

It is also worth pointing out that there have been numerous works on the construction and classification of gravitational instantons with *given* asymptotics at infinity, see for example [21]–[24], [27], [33], [44], [46], [47], [54], [55], [64] and the references therein. In particular, it is known that all the families of gravitational instantons listed in Theorem 1.2 can be compactified in the complex-analytic sense. Together with these results, Theorem 1.2 has the following corollary, which confirms the compactification conjecture of S.-T. Yau [83] in our setting.

COROLLARY 1.3. *Given any gravitational instanton* (X, g), *there is a choice of a complex structure* J *such that* (X, J) *is bi-holomorphic to* $\overline{X}\backslash D$, *where* $\overline{X}$ *is an algebraic surface and* D *is an anti-canonical divisor.*

Now we outline some ideas involved in the proof of the above results. As mentioned before the central goal is to understand the collapsing geometry of hyperkähler 4-manifolds with bounded L^2-energy. The result of Cheeger–Tian [19] provides an ε-regularity theorem in our context, and as a consequence we know that the collapsing is with bounded curvature away from finitely many singularities. However, due to the lack of a suitable monotonicity formula, there has been no progress so far in understanding the structure of these singularities. This issue is unique compared to other geometric analytic problems. Our study depends on three key ingredients:

- Geometric structures over the regular region (§3): we analyze the structure on the regular region of the limit space coming from the hyperkähler structure. The analysis depends on the dimension d of the limit space; when d=1, this was already done previously in [50]. A byproduct of this analysis is a new and simple proof of the ε-regularity theorem in our context (see §3.4).

- Singularity structure of the limit space (§4, 5): we study in detail the singularity structure in the cases d=3 and d=2. In particular we show that there is always a unique tangent cone which is a metric cone. Theorem 1.1 follows from Theorem 4.3 and Theorem 5.2. Notice again that the case d=1 in Theorem 1.1 is easy (see [50]).

- Perturbation to invariant hyperkähler metrics (§3.5 and §6): The classical theory of nilpotent Killing structures due to Cheeger–Fukaya–Gromov [17] asserts that, over the regular region, the collapsing sequence inherits an *approximate* nilpotent symmetry along the collapsing directions. We combine this with the perturbation theory of hyperkähler metrics to obtain nearby hyperkähler metrics with *genuine* nilpotent symmetry. This is performed at both the local and global level. The local result improves our understanding of the collapsing fibers (§3.5), whereas the global result yields that a gravitational instanton with non-maximal volume growth at infinity must be asymptotic to a model end which admits a continuous symmetry (§6.3). These allow us to prove Theorem 1.2 in §6.4. The techniques needed here are closely related to those used in the gluing constructions in [32], [46].

Notation

- Given a metric space (M, d) and a closed subset $E \subset M$, we denote

$$\begin{aligned} B_r(E) &\equiv \{q \in M : d(q, E) < r\}, \\ S_r(E) &\equiv \{q \in M : d(q, E) = r\}, \\ A_{r_1,r_2}(E) &\equiv \{q \in M : r_1 < d(q, E) < r_2\}. \end{aligned}$$

- We have various notations for Gromov–Hausdorff convergence:

$$\begin{aligned} \xrightarrow{\text{GH}} &: \quad \text{(pointed) Gromov–Hausdorff convergence,} \\ \xrightarrow{\text{eqGH}} &: \quad \text{equivariant Gromov–Hausdorff convergence,} \\ \xrightarrow{\text{mGH}} &: \quad \text{measured Gromov–Hausdorff convergence.} \end{aligned}$$

- For a group G, we denote by $\mathfrak{Z}(G)$ the center of G.
- $\mathbb{R}_+ \equiv [0, \infty) \subset \mathbb{R}$.

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2. Premilinaries

2.1. Pointed Gromov–Hausdorff distance

The concept of pointed Gromov–Hausdorff convergence has been extensively used in the literature. For our purpose in this paper, it is convenient to exploit a metric space structure, which is likely well known, and we briefly recall the relevant notions. We refer the readers to [73], [48] for more details. Denote by $\mathcal{M}$et the collection of isometry classes of all pointed complete length spaces (M, d, p) such that every closed ball in M is compact.

Definition 2.1. Let $(M_1, d_1, p_1), (M_2, d_2, p_2) \in \mathcal{M}$et. The pointed Gromov–Hausdorff distance bwtween them is defined to be

$$d_{\mathrm{GH}}((M_1, d_1, p_1), (M_2, d_2, p_2)) \equiv \min\bigl\{\varepsilon_0, \tfrac{1}{2}\bigr\},$$

where $\varepsilon_0 \geqslant 0$ is the infimum of all $\varepsilon \in \bigl[0, \frac{1}{2}\bigr)$ such that there is a metric d on $M_1 \sqcup M_2$ extending d_i on M_i, with $d(p_1, p_2) \leqslant \varepsilon$, $B_{1/\varepsilon}(p_1) \subset B_\varepsilon(M_2)$ and $B_{1/\varepsilon}(p_2) \subset B_\varepsilon(M_1)$.

It is straightforward to verify that d_{GH} defines a metric on $\mathcal{M}$et. One can prove that $(\mathcal{M}et, d_{\mathrm{GH}})$ is a complete metric space. The convergence in this metric topology is the *pointed Gromov–Hausdorff convergence*. For simplicity of notation, in this paper we will omit the word *pointed* and simply refer to this as the *Gromov–Hausdorff convergence*. In the applications, one can also use the notion of ε-Gromov–Hausdorff approximation (see [73]), which gives essentially the same topology.

Let (X_j^n, g_j, p_j) be a sequence of n-dimensional Riemannian manifolds with

$$\mathrm{Ric}_{g_j} \geqslant K g_j$$

for some $K \in \mathbb{R}$ and for all j. Given a sequence of numbers $R_j > 0$, with $\overline{B_{R_j}(p_j)}$ compact, from Gromov's compactness theorem, by passing to a subsequence we may assume that

$$(\overline{B_{R_j}(p_j)}, g_j, p_j) \xrightarrow{\mathrm{GH}} (X_\infty, d_\infty, p_\infty)$$

for a complete length space $(X_\infty, d_\infty, p_\infty)$. If the sequence $\{R_j\}$ is unbounded, then X_∞ is non-compact. Fix such a limit space $(X_\infty, d_\infty, p_\infty)$. We consider the rescaled spaces $(X_\infty, \lambda d_\infty, p_\infty)$ and let $\lambda \to \infty$. Any Gromov–Hausdorff limit (Y, p^*) for a subsequence $\{\lambda_i\} \to \infty$ is called a *tangent cone* at p_∞. Recall that a tangent cone is known to be a metric cone under a volume-non-collapsing situation; but this is not always true in the volume-collapsing case. We will denote by $\mathcal{T}_{p_\infty} \subset \mathcal{M}$et the collection of isometry classes of all tangent cones at p_∞.

We now fix the above convergent subsequence $(\overline{B_{R_j}(p_j)}, g_j, p_j)$. Given any subsequence $\{\lambda_j\}\to\infty$, there is a further subsequence $\{\lambda_{m_j}\}$ such that

$$(\overline{B_{R_{m_j}}(p_{m_j})}, \lambda_{m_j}^2 g_{m_j}, p_{m_j}) \xrightarrow{\mathrm{GH}} (Z, d_Z, \bar{p}).$$

We call the space $(Z, d_Z, \bar{p})$ a *bubble limit* at p_∞ associated to the original convergent sequence. Denote by $\mathcal{B}_{p_\infty}$ the collection of isometry classes of all bubble limits at p_∞. Immediately, $\mathcal{T}_{p_\infty}\subset\mathcal{B}_{p_\infty}$.

Geometrically speaking, tangent cones describe the first order information of the singular behavior of the space X_∞ itself at p_∞, whereas bubble limits characterize more refined behavior for the singularity formation. The terminology should remind the readers the notion of a *bubble tree* structure in many geometric analytic problems. The following is a simple fact whose proof we leave as an exercise for the readers.

LEMMA 2.2. *For any $p_\infty\in X_\infty$, both $\mathcal{T}_{p_\infty}$ and $\mathcal{B}_{p_\infty}$ are compact in* $\mathcal{M}$et. *Moreover, $\mathcal{T}_{p_\infty}$ is connected.*

Later we will also use an analogous result for asymptotic cones. Let (X, d_X, p) be a complete Gromov–Hausdorff limit of a sequence of Riemannian manifolds with non-negative Ricci curvature. An asymptotic cone of X is, by definition, a complete metric space (Y, d, p_*) arising as the Gromov–Hausdorff limit of $(X, \lambda_j^{-1} d_X, p)$ for some sequence $\lambda_j\to\infty$. Clearly, this does not depend on the choice of the base point p. Denote by $\mathcal{T}_\infty(X)$ the collection of isometry classes of asymptotic cones X. Similar to above, we have the following result.

LEMMA 2.3. *$\mathcal{T}_\infty(X)$ is connected and compact in* $\mathcal{M}$et. *Moreover, it is invariant under rescaling, i.e., if (Y, d_Y, p_*) is in $\mathcal{T}_\infty(X)$, so is $(Y, \lambda d_Y, p_*)$ for all $\lambda>0$.*

We also mention that in this paper various other notions of convergence will also be used, such as Cheeger–Gromov convergence, equivariant Gromov–Hausdorff convergence, etc., and the mixture of them. For definitions of these notions we refer the readers to standard references.

2.2. Renormalized limit measure

As above, we let (X_j^n, g_j, p_j) be a sequence of n-dimensional Riemannian manifolds with $\mathrm{Ric}_{g_j}\geqslant K g_j$, such that $\overline{B_2(p_j)}$ is compact. Suppose that

$$(X_j^n, g_j, p_j) \xrightarrow{\mathrm{GH}} (X_\infty, d_\infty, p_\infty)$$

for some length space (X_∞, d_∞). We denote by

$$d\nu_j \equiv \frac{d\mathrm{vol}_{g_j}}{\mathrm{Vol}_{g_j}(B_1(p_j))}$$

the *renormalized measure* density on X_j^n. Then, by the work of Cheeger–Colding (see [15, Theorem 1.10]), we know that by passing to a further subsequence, there is a Radon measure ν_∞ on X_∞, called the *renormalized limit measure*, such that for any converging sequence of points $q_j \to q_\infty$ and for all $R>0$, we have $\nu_j(B_R(q_j)) \longrightarrow \nu_\infty(B_R(q_\infty))$. The metric measure space $(X_\infty, d_\infty, \nu_\infty, p_\infty)$ is called a *Ricci limit space*, and we have the *measured Gromov–Hausdorff convergence*

$$(X_j^n, g_j, \nu_j, p_j) \xrightarrow{\text{mGH}} (X_\infty, d_\infty, \nu_\infty, p_\infty). \tag{2.1}$$

It is known that ν_∞ satisfies the relative volume comparison, and the following volume estimate

$$\nu_\infty(B_r(x)) \leqslant C \cdot r \quad \text{for all } x \in X_\infty \text{ and all } r \in (0,1]. \tag{2.2}$$

See [15, Theorem 1.10 and Proposition 1.22], respectively.

Definition 2.4. Let $(X_\infty, d_\infty, \nu_\infty, p_\infty)$ be a (connected) Ricci limit space of

$$(X_j^n, g_j, \nu_j, p_j).$$

(1) We define the *regular set* $\mathcal{R}$ to be the set of points $q \in X_\infty$ such that there exist constants $r_0>0$ and $C_0>0$, and a sequence of points $q_j \in X_j^n$ converging to q with

$$\sup_{B_{r_0}(q_j)} |\mathrm{Rm}_{g_j}| \leqslant C_0 \quad \text{for all } j.$$

(2) We define the *smooth set* $\mathcal{G} \subset \mathcal{R}$ to be the set of points q such that X_∞ is a smooth Riemannian manifold in a neighborhood of q. We denote by g_∞ the Riemannian metric on $\mathcal{G}$.

(3) We define the *singular set* $\mathcal{S} \equiv X_\infty \setminus \mathcal{G}$.

Notice that these definitions depend on the convergent sequence X_j^n. Clearly, $\mathcal{R}$ is open. By Colding–Naber [28], there exist a subset $\mathcal{R}^\# \subset X_\infty$ and an integer $d \in \mathbb{Z}_+$ such that $\nu_\infty(X_\infty \setminus \mathcal{R}^\#)=0$ and every point in $\mathcal{R}^\#$ has a unique tangent cone which is isometric to $\mathbb{R}^d$. We call the integer d the *essential dimension* of X_∞, and we denote it by $\dim_{\mathrm{ess}}(X_\infty)$. It is obvious that $\mathcal{G} = \mathcal{R} \cap \mathcal{R}^\#$, so $\dim_{\mathrm{ess}}(X_\infty) = \dim \mathcal{G}$.

In this paper we are mainly interested in the collapsing situation so from now on we assume $d<n$. It is worth noting that neither $\mathcal{R}^\# \subset \mathcal{R}$ nor $\mathcal{R} \subset \mathcal{R}^\#$ necessarily holds in general. Nevertheless, in §3.1, we will show that $\mathcal{R} \subset \mathcal{R}^\#$ in our setting of collapsing 4-dimensional hyperkähler metrics.

Fukaya [35] showed that on $\mathcal{G}$ the measure ν_∞ has an explicit expression, namely, its density is

$$d\nu_\infty = \chi \cdot d\mathrm{vol}_{g_\infty}$$

for a smooth function χ determined as follows. Given $q \in \mathcal{G}$, we can find $q_j \in X_j$ converging to q and $\delta > 0$ such that the universal cover $(\widetilde{B_\delta(q_j)}, \tilde{g}_j, q_j, G_j)$ converges in the equivariant Cheeger–Gromov sense, to a smooth limit $(\widetilde{B}_\infty, \tilde{g}_\infty, q_\infty, G_\infty)$. Here, G_j is the fundamental group of $B_\delta(q_j)$, and the identity component of G_∞ is a nilpotent Lie group. Moreover, a neighborhood of q in X_∞ is isometric to the quotient $B_\infty \equiv \widetilde{B}_\infty / G_\infty$. In this context, we can identify the fiber $F_{q'}$ over any $q' \in B_\infty$ of the projection map $\widetilde{B}_\infty \to B_\infty$ locally with an open neighborhood in G_∞. Then, up to constant multiplication, χ is given by the ratio between the vertical Riemannian volume form on $F_{q'}$ (of the induced Riemannian metric from $\tilde{g}_\infty$) and a fixed left-invariant volume form on G_∞.

We often write $\chi = e^{-f}$. As observed by Lott (see [57, Theorem 2]), using O'Neill's formula, the Bakry–Émery–Ricci curvature lower bound is preserved on the limit, i.e., on $\mathcal{G}$ we have

$$\mathrm{Ric}_{g_\infty}^{n-d} \equiv \mathrm{Ric}_{g_\infty} + \nabla^2_{g_\infty} f - \frac{1}{n-d}\, df \otimes df \geqslant K g_\infty.$$

Although not needed in this paper, we notice the fact that globally one can say that $(X_\infty, d_\infty, \nu_\infty)$ has Ricci bounded below by K in the RCD sense, i.e., it is an $\mathrm{RCD}(K, n)$ space. See [1], [39] for details.

2.3. Harmonic functions

In this subsection, we introduce some standard concepts and basic results about harmonic functions. For our purpose, we only state them on Ricci limit spaces, and we list the references in the general RCD setting. To begin with, let $(X_j^n, g_j, d\nu_j, p_j)$ be a sequence of n-dimensional Riemannian manifolds with $\mathrm{Ric}_{g_j} \geqslant 0$ and

$$d\nu_j \equiv \mathrm{Vol}_{g_j}(B_1(p_j))^{-1}\, d\mathrm{vol}_{g_j}$$

such that

$$(X_j^n, g_j, \nu_j, p_j) \xrightarrow{\mathrm{mGH}} (X_\infty, d_\infty, \nu_\infty, p_\infty).$$

A key ingredient in the definition of harmonic functions on a metric measure space is the following notion of *minimal weak upper gradient*, which plays the role of $|\nabla u|$ in the smooth case. To define this, let us first introduce the notation of the 2-modulus of a family of curves. Let Γ be a family of curves on X_∞. Then we define the 2-*modulus* $\mathrm{Mod}_2(\Gamma)$ of Γ by

$$\mathrm{Mod}_2(\Gamma)\equiv\inf\left\{\int_{X_\infty}\psi^2 d\nu_\infty : \psi\geqslant 0 \text{ is measurable and } \int_\gamma \psi\, ds\geqslant 1 \text{ for all } \gamma\in\Gamma\right\}.$$

Now, we are ready to define the minimal weak upper gradient.

Definition 2.5. (Minimal weak upper gradient) Let u be a measurable function on X_∞. A non-negative measurable function g on X_∞ is said to be a 2-*weak upper gradient* of a function u if, for any $z_1, z_2\in X_\infty$ and every rectifiable curve $\gamma:[0,\ell]\to X_\infty$ parameterized by arc length with $\gamma(0)=z_1$ and $\gamma(\ell)=z_2$, with the exception in a family of curves Γ with $\mathrm{Mod}_2(\Gamma)=0$, one has

$$|u(z_2)-u(z_1)|\leqslant\int_0^\ell g(\gamma(s))\, ds.$$

The *minimal weak upper gradient* $|\nabla u|$ of a function u is the 2-weak upper gradient such that, for all 2-weak upper gradient g, one has $|\nabla u|\leqslant|g|$ a.e. on X_∞.

Based on the notion of minimal weak upper gradient, the Cheeger energy of u is defined by

$$\mathrm{Ch}(u)\equiv\int_{X_\infty}|\nabla u|^2 d\nu_\infty,$$

and the $W^{1,2}$-Sobolev space is defined by

$$W^{1,2}(X_\infty)\equiv\{u\in L^2(X_\infty):\mathrm{Ch}(u)<\infty\}.$$

It is known that the Cheeger energy is quadratic on a Ricci limit space (see [1] and [39]). This enables us to define the following Dirichlet form

$$\mathcal{E}(u,v)=\int_{X_\infty}\langle\nabla u,\nabla v\rangle\, d\nu_\infty\equiv\frac{1}{2}(\mathrm{Ch}(u+v)-\mathrm{Ch}(u-v)),$$

where $u, v\in W^{1,2}(X_\infty)$. Note that $\langle\nabla u,\nabla v\rangle$ is a well-defined L^1-function, but ∇u itself is not defined in general. We also point out that $\mathcal{E}(u,v)$ coincides with the standard Dirichlet form in the smooth case.

Definition 2.6. (Harmonic function) Let $\Omega\subset X_\infty$ be an open set. A function $u\in W^{1,2}(\Omega)$ is said to be harmonic if $\mathcal{E}(u,\varphi)\equiv 0$ for all Lipschitz functions φ with compact support in Ω.

We will use the following weak Harnack inequality.

THEOREM 2.7. (Weak Harnack inequality) *Let* $(X_\infty, d_\infty, \nu_\infty, p_\infty)$ *be a Ricci limit space. For any* $p>0$, *there exists some constant* $C=C_p>0$ *depending only on* p *such that, if* u *is a non-negative harmonic function on* $B_2(p_\infty)$, *then*

$$\left(\fint_{B_2(p_\infty)} u^p \, d\nu_\infty\right)^{1/p} \leqslant C \cdot \operatorname*{ess\,inf}_{B_1(p_\infty)} u.$$

This theorem indeed holds in the very general context of metric measure spaces under appropriate assumptions; see [10, Theorem 4.21, Corollary 4.24 and Theorem 9.7]. For completeness, we briefly explain the crucial technical ingredients involved in the proof of Theorem 2.7. First, the space is required to support the $(1,p)$-Poincaré inequality, i.e., there exists some constant $C_{\mathrm{PI}}>0$ depending on p such that, for any function $u \in L^1(B_r(x))$ with $x \in X_\infty$ and $r>0$, and for all upper gradients g of u, it holds that

$$\fint_{B_r(x)} |u - u_{x,r}| \, d\nu_\infty \leqslant C_{\mathrm{PI}} \cdot r \cdot \left(\fint_{B_r(x)} g^p \, d\nu_\infty\right)^{1/p},$$

where

$$u_{x,r} \equiv \fint_{B_r(x)} u \, d\nu_\infty.$$

In the context of Ricci limit spaces, the above $(1,p)$-Poincaré inequality (for all $p \geqslant 1$) follows from Cheeger–Colding's segment inequality; see [16, Theorem 2.15]. Secondly, one needs to apply the technique of Moser's iteration, which requires a uniform Sobolev inequality. This follows from the Poincaré inequality and the volume comparison for the renormalized limit measure; see of [10, Theorem 4.21].

Now we consider the setting of §2.2. On the smooth set $\mathcal{G}$ we have a Riemannian metric g_∞, and a measure density $d\nu_\infty = e^{-f} \, d\mathrm{vol}_{g_\infty}$. Suppose $B_\delta(p_\infty) \subset \mathcal{G}$, then a function u on $B_\delta(p_\infty)$ is harmonic if and only if $\Delta_{\nu_\infty} u = 0$ on $B_\delta(p_\infty)$, where

$$\Delta_{\nu_\infty} u \equiv \Delta_{g_\infty} u - \langle \nabla_{g_\infty} f, \nabla_{g_\infty} u \rangle$$

is the Bakry–Émery Laplace operator. Locally, if we pull-back to $\widetilde{B}_\infty$, then it is easy to see that $\Delta_{\nu_\infty} u = \Delta_{\tilde{g}_\infty} u$. We have the following Cheng–Yau-type gradient estimate (see [71, Theorem 2.1])

THEOREM 2.8. (Gradient estimate) *Suppose that* $(\mathcal{G}, g_\infty, d\nu_\infty)$ *satisfies* $\mathrm{Ric}_{g_\infty}^{n-d} \geqslant 0$. *Then, there exists a constant* $C_0 = C_0(n) > 0$ *such that any positive harmonic function* u *defined on* $B_{2r}(x) \subset \mathcal{G}$ *satisfies*

$$\sup_{B_r(x)} |\nabla \log u| \leqslant C_0 r^{-1}.$$

2.4. Deformation of definite triples

Here we review [29], [32], [46]. Let X be an oriented smooth 4-manifold, possibly non-compact or with boundary, and fix a volume form $d\mathrm{vol}_0$. Let $\boldsymbol{\omega}\equiv\{\omega_1,\omega_2,\omega_3\}$ be a triple of closed 2-forms on X. Write $\omega_\alpha\wedge\omega_\beta=2Q_{\alpha\beta}\,d\mathrm{vol}_0$ for $1\leqslant\alpha,\beta\leqslant 3$.

Definition 2.9. $\boldsymbol{\omega}$ is called a *definite triple* if the matrix-valued function $Q\equiv(Q_{\alpha\beta})$ is positive definite everywhere on X.

A definite triple $\boldsymbol{\omega}$ uniquely determines a Riemannian metric $g_{\boldsymbol{\omega}}$ such that each ω_α is self-dual with respect to $g_{\boldsymbol{\omega}}$ and the volume form is given by $d\mathrm{vol}_{g_{\boldsymbol{\omega}}}=(\det(Q))^{1/3}\,d\mathrm{vol}_0$. Denote by Ω^+ the space of self-dual 2-forms (with respect to $g_{\boldsymbol{\omega}}$) on X. Below we will often identify an element in $\Omega^+\otimes\mathbb{R}^3$ (i.e., a triple of self-dual 2-forms) with a 3×3 matrix-valued function: $\boldsymbol{\eta}\in\Omega^+\otimes\mathbb{R}^3$ corresponds to $(A_{\alpha\beta})$ if $\eta_\alpha=\sum_\beta A_{\alpha\beta}\omega_\beta$.

Definition 2.10. A definite triple $\boldsymbol{\omega}$ is called *hyperkähler* if the metric $g_{\boldsymbol{\omega}}$ is hyperkähler, or equivalently, if the normalized determinant $Q_{\boldsymbol{\omega}}\equiv(\det(Q))^{-1/3}Q$ is the identity matrix.

Fix a definite triple $\boldsymbol{\omega}$. Consider a deformation $\boldsymbol{\omega}'=\boldsymbol{\omega}+\boldsymbol{\theta}$ for a triple $\boldsymbol{\theta}$ of closed 2-forms. Decompose $\boldsymbol{\theta}=\boldsymbol{\theta}^++\boldsymbol{\theta}^-$, where $\boldsymbol{\theta}^+$ is self-dual and $\boldsymbol{\theta}^-$ is anti-self-dual with respect to $g_{\boldsymbol{\omega}}$. Then, as above, we may identify $\boldsymbol{\theta}^+$ with a matrix-valued function $A=(A_{\alpha\beta})$. Define the matrix-valued function $S_{\boldsymbol{\theta}^-}\equiv(S_{\alpha\beta})$ via

$$\theta_\alpha^-\wedge\theta_\beta^-=2S_{\alpha\beta}\,d\mathrm{vol}_{g_{\boldsymbol{\omega}}},\quad 1\leqslant\alpha,\beta\leqslant 3.$$

If $\boldsymbol{\omega}'$ is definite, then the hyperkähler condition on $\boldsymbol{\omega}'$ can be expressed as

$$\mathrm{tf}(Q_{\boldsymbol{\omega}}A^T+AQ_{\boldsymbol{\omega}}+AQ_{\boldsymbol{\omega}}A^T)=\mathrm{tf}(-Q_{\boldsymbol{\omega}}-S_{\boldsymbol{\theta}^-}),\tag{2.3}$$

where we denote by $\mathrm{tf}(B)=B-\frac{1}{3}\,\mathrm{Tr}(B)\,\mathrm{Id}$ the trace-free part of a matrix B. Let $\mathscr{S}_0(\mathbb{R}^3)$ be the space of trace-free symmetric 3×3 matrices, and consider the non-linear map

$$\begin{aligned}\mathfrak{G}\colon\mathscr{S}_0(\mathbb{R}^3)&\longrightarrow\mathscr{S}_0(\mathbb{R}^3),\\ A&\longmapsto\mathrm{tf}(Q_{\boldsymbol{\omega}}A^T+AQ_{\boldsymbol{\omega}}+AQ_{\boldsymbol{\omega}}A^T).\end{aligned}$$

Then, $\mathfrak{G}$ is a local diffeomorphism near zero, and we denote by $\mathfrak{F}\colon U\to\mathscr{S}_0(\mathbb{R}^3)$ its local inverse, where U is a small neighborhood of zero. A sufficient condition for (2.3) to hold is

$$A=\mathfrak{F}(\mathrm{tf}(-Q_{\boldsymbol{\omega}}-S_{\boldsymbol{\theta}^-})).\tag{2.4}$$

Notice this is only a necessary condition, if we assume a priori that the matrix A above is symmetric and trace-free.

We now impose the ansatz

$$\theta_\alpha = dd^*\left(\sum_\beta f_{\alpha\beta}\omega_\beta\right),$$

where $\boldsymbol{f}\equiv(f_{\alpha\beta})$ is a 3×3 matrix-valued function, and the Hodge $*$-operator is defined with respect to the metric $g_{\boldsymbol{\omega}}$. We can write this concisely as

$$\boldsymbol{\theta} = dd^*(\boldsymbol{f}\cdot\boldsymbol{\omega}).$$

Define the non-linear operator

$$\begin{aligned}\mathscr{F}\colon \Omega^+\otimes\mathbb{R}^3 &\longrightarrow \Omega^+\otimes\mathbb{R}^3,\\ \boldsymbol{f} &\longmapsto \mathscr{D}(\boldsymbol{f})+\mathscr{N}_0(\boldsymbol{f}),\end{aligned} \tag{2.5}$$

where $\mathscr{D}(\boldsymbol{f})\equiv d^+d^*(\boldsymbol{f}\cdot\boldsymbol{\omega})$ and $\mathscr{N}_0(\boldsymbol{f})\equiv -\mathfrak{F}(\mathrm{tf}(-Q_{\boldsymbol{\omega}}-S_{d^-d^*(\boldsymbol{f}\cdot\boldsymbol{\omega})}))$. Strictly speaking, $\mathscr{F}(\boldsymbol{f})$ is only well defined when $|\boldsymbol{f}|_{C^2_{\boldsymbol{\omega}}}$ is small (so that $\mathrm{tf}(-Q_{\boldsymbol{\omega}}-S_{d^-d^*(\boldsymbol{f}\cdot\boldsymbol{\omega})}(x))$ is in U for all $x\in X$). Clearly, (2.4) follows if $\mathscr{F}(\boldsymbol{f})=0$.

If $\boldsymbol{\omega}$ is hyperkähler, then $\nabla_{g_{\boldsymbol{\omega}}}\boldsymbol{\omega}\equiv\mathbf{0}$. In this case, we have $\mathscr{D}=-\Delta_{\boldsymbol{\omega}}$, where $\Delta_{\boldsymbol{\omega}}$ is the analyst's Laplace operator. In general, we have

$$d^+d^*(\boldsymbol{f}\cdot\boldsymbol{\omega}) = -(\Delta_{\boldsymbol{\omega}}\boldsymbol{f})\cdot\boldsymbol{\omega}+\nabla_{\boldsymbol{\omega}}\boldsymbol{f}\star\nabla_{\boldsymbol{\omega}}\boldsymbol{\omega}, \tag{2.6}$$

where $\star$ denotes a general tensor contraction. This follows from the following lemma.

Lemma 2.11. (See also [47]) *Given a closed self-dual 2-form γ and a function f, we have*

$$d^+d^*(f\gamma) = (-\Delta_{\boldsymbol{\omega}}f)\gamma+\nabla_{\boldsymbol{\omega}}f\star\nabla_{\boldsymbol{\omega}}\gamma.$$

Proof. Given a point p_0. We choose a local oriented orthonormal frame $\{e_i\}$ with dual co-frame $\{e^i\}$, such that $\nabla_{\boldsymbol{\omega}}e_i(p_0)=0$ and, in a neighborhood of p_0, the bundle $\Lambda^+(X)$ of self-dual 2-forms on X is spanned by $e^{12}+e^{34}$, $e^{13}+e^{42}$ and $e^{14}+e^{23}$, where $e^{ij}\equiv e^i\wedge e^j$. Since γ is closed and self-dual, we have $d^+d^*(f\gamma)=d^+*(df\wedge\gamma)$. We write $\gamma=\sum_{i<j}\gamma_{ij}e^{ij}$, with $\gamma_{12}=\gamma_{34}$, $\gamma_{13}=-\gamma_{24}$ and $\gamma_{14}=\gamma_{23}$. Then, the conclusion follows from straightforward computation, using the fact that

$$\Delta_{\boldsymbol{\omega}}f = -\sum_{i=1}^4 e_ie_if \quad \text{at } p_0. \qquad \square$$

We also notice that algebraically we have the pointwise estimate

$$|\mathscr{N}_0(\boldsymbol{f})-\mathscr{N}_0(\boldsymbol{g})| \leqslant C\left(|\nabla^2_{\boldsymbol{\omega}}\boldsymbol{f}|_{\boldsymbol{\omega}}+|\nabla^2_{\boldsymbol{\omega}}\boldsymbol{g}|_{\boldsymbol{\omega}}\right)\cdot|\nabla^2_{\boldsymbol{\omega}}(\boldsymbol{f}-\boldsymbol{g})|_{\boldsymbol{\omega}}, \tag{2.7}$$

as long as $\boldsymbol{f}$ and $\boldsymbol{g}$ are in the domain of definition of $\mathscr{F}$.

We now assume that $\mathscr{F}(\boldsymbol{f})=\mathscr{L}(\boldsymbol{f})+\mathscr{N}(\boldsymbol{f})$, where $\mathscr{L}$ is a linear operator. In our applications $\mathscr{L}$ will be a slight modification of $-\Delta_{\boldsymbol{\omega}}$. The following is an application of the standard quantitative implicit function theorem on Banach spaces.

Proposition 2.12. *Suppose that we have two Banach spaces* $(\mathfrak{A}, \|\cdot\|)$ *and* $(\mathfrak{B}, \|\cdot\|)$, *and numbers* $\eta>0$ *and* $L>0$, *such that the following statements hold*:

(1) $\mathfrak{A}\subset C^2(\Omega^+\otimes\mathbb{R}^3)$ *and* $\mathfrak{B}\subset C^0(\Omega^+\otimes\mathbb{R}^3)$;

(2) *for all* $\boldsymbol{f}\in B_\eta(0)\subset\mathfrak{A}$, *the triple* $\boldsymbol{\omega}+dd^*(\boldsymbol{f}\cdot\boldsymbol{\omega})$ *is definite, and*

$$\operatorname{tf}(Q_{\boldsymbol{\omega}}(x)+S_{d^- d^*(\boldsymbol{f}\cdot\boldsymbol{\omega})}(x))\in U \quad \textit{for all } x\in X;$$

(3) $\mathscr{L}$ *and* $\mathscr{N}$ *are both differentiable maps from* $B_\eta(0)\subset\mathfrak{A}$ *to* $\mathfrak{B}$;

(4) *there exists a bounded linear map* $\mathscr{P}:\mathfrak{B}\to\mathfrak{A}$ *with* $\mathscr{L}\circ\mathscr{P}=\mathrm{Id}$ *and* $\|\mathscr{P}\|\leqslant L$;

(5) $\|\mathscr{N}(\boldsymbol{f})-\mathscr{N}(\boldsymbol{g})\|\leqslant(3L)^{-1}\|\boldsymbol{f}-\boldsymbol{g}\|$ *for all* $\boldsymbol{f},\boldsymbol{g}\in B_\eta(0)\subset\mathfrak{A}$;

(6) $\|\mathscr{F}(\mathbf{0})\|\leqslant\eta(3L)^{-1}$.

Then, we can find an $\boldsymbol{f}\in\mathfrak{A}$ *that solves* $\mathscr{F}(\boldsymbol{f})=0$ *and satisfies*

$$\|\boldsymbol{f}\|\leqslant 2L\|\mathscr{F}(0)\|.$$

In particular, $\widetilde{\boldsymbol{\omega}}=\boldsymbol{\omega}+dd^*(\boldsymbol{f}\cdot\boldsymbol{\omega})$ *is a hyperkähler triple.*

In practice, condition (2) will be achieved by making sure that $|dd^*(\boldsymbol{f}\cdot\boldsymbol{\omega})|_{\boldsymbol{\omega}}$ is small for all $\boldsymbol{f}\in B_\eta(0)$, which in particular guarantees that $\widetilde{\boldsymbol{\omega}}$ is equivalent to $\boldsymbol{\omega}$. The above strategy was used for example by Foscolo [32] (see also [46]) to construct degeneration families of hyperkähler metrics on the K3 manifold with precise geometric information. The main technical issue, there, is to find a right inverse $\mathscr{L}$ with uniform estimates on suitable weighted spaces. In the setting of a K3 manifold due to topological reasons $\Delta_{\boldsymbol{\omega}}$ can not be surjective; this is circumvented by the extra freedom of adding a finite-dimensional space of self-dual harmonic forms.

For our purpose in this paper, we will work on manifolds with boundary (and possibly non-compact), so we do not encounter the topological obstruction to the surjectivity of $\Delta_{\boldsymbol{\omega}}$. We also make the trivial observation that, assuming that we have a compact group G acting on X preserving $\boldsymbol{\omega}$, if the assumptions (1)–(6) of Proposition 2.12 hold for G-invariant objects, then we can find a nearby hyperkähler triple $\widetilde{\boldsymbol{\omega}}$ which is also G-invariant.

Another remark is that the ansatz used in [32], [46] is slightly different from what is used above. Namely, in those situations, one would write $\boldsymbol{\theta}=d\boldsymbol{\eta}+\boldsymbol{\xi}$ for a triple of d^*-closed 1-forms $\boldsymbol{\eta}$ and a triple of self-dual harmonic forms $\boldsymbol{\xi}$. The corresponding linearized operator involves the Dirac operator d^*+d^+ acting on 1-forms. In our formulation above, we have further specified $\boldsymbol{\eta}=d^*(\boldsymbol{f}\cdot\boldsymbol{\omega})$. This provides some technical simplifications, since it allows us to only deal with elliptic operator on functions. To our knowledge this trick goes back to Biquard [7] (see also [47]).

3. Geometric structures over the regular region

In this section, we consider a measured collapsed Gromov–Hausdorff limit

$$(X^d_\infty, d_\infty, p_\infty, \nu_\infty)$$

of a sequence of hyperkähler 4-manifolds (X^4_j, g_j, p_j, ν_j) with

$$d \equiv \dim_{\mathrm{ess}}(X^d_\infty) < 4.$$

We will always fix a choice of hyperkähler triple $\boldsymbol{\omega}_j$ on X^4_j. Our goal is to understand the refined geometric structure on the regular region $\mathcal{R} \subset X^d_\infty$ inherited from the hyperkähler structure on X^4_j. Notice that, due to volume collapsing, one cannot make obvious sense of convergence of the hyperkähler triples $\boldsymbol{\omega}_j$. Instead, we will take the limit of $\boldsymbol{\omega}_j$ on local universal covers, which descends to a local structure on $\mathcal{R}$, then gluing them together yields certain global structure on $\mathcal{R}$. The precise structure we obtain on $\mathcal{R}$ depends on its dimension d.

Now, we make the above description rigorous. Without loss of generality, we always assume $p_\infty \in \mathcal{R}$ in this section. Then, there exists some $\delta > 0$ independent of j such that the universal cover $\widetilde{B}_j$ of $B_j \equiv B_\delta(p_j)$ is volume-non-collapsing as $j \to \infty$. Let G_j be the deck transformation group and $\widetilde{\boldsymbol{\omega}}_j$ be the pullback of $\boldsymbol{\omega}_j$ to $\widetilde{B}_j$. The isometry action of G_j preserves $\widetilde{\boldsymbol{\omega}}_j$. Passing to a subsequence, we have the equivariant Gromov–Hausdorff convergence

$$\begin{array}{ccc}
(\widetilde{B}_j, \tilde{g}_j, G_j, \tilde{p}_j) & \xrightarrow{\text{eqGH}} & (\widetilde{B}_\infty, \tilde{g}_\infty, G_\infty, \tilde{p}_\infty) \\
\Big\downarrow{\scriptstyle \pi_j} & & \Big\downarrow{\scriptstyle \pi_\infty} \\
(B_j, g_j) & \xrightarrow{\text{GH}} & (B_\delta(p_\infty), g_\infty),
\end{array}$$

where $G_\infty \leqslant \mathrm{Isom}(\widetilde{B}_\infty)$ is a closed subgroup such that $B_\delta(p_\infty) = \widetilde{B}_\infty / G_\infty$. We refer the reader to [37, Definition 3.3] for the detailed definition of equivariant Gromov–Hausdorff convergence. Also, the standard regularity theory for non-collapsing Einstein metrics implies that the convergence of $\widetilde{B}_j$ can be improved to the C^k-convergence for any $k \in \mathbb{Z}_+$. Then, we obtain a smooth limit hyperkähler triple $\widetilde{\boldsymbol{\omega}}_\infty$ on $\widetilde{B}_\infty$ which is preserved by G_∞.

In the following proposition, we make the observation that any point in the regular set $\mathcal{R}$ is a manifold point.

Proposition 3.1. *In our setting, $\mathcal{R}\cap\mathcal{S}=\varnothing$. In particular, $\mathcal{R}\subset\mathcal{R}^{\#}$.*

Proof. To prove the proposition, we claim that G_∞ acts freely on $\widetilde{B}_\infty$. To see this, suppose otherwise that $\tilde{p}_\infty$ is a fixed point of some non-trivial element $\phi\in G_\infty$ such that there exists a sequence of $\phi_j\in G_j$ that converges to ϕ_j equivariantly.

Now, using the exponential map at $\tilde{p}_\infty$, we may identify the action of ϕ with the linear action $L=d\phi$ on $T_{\tilde{p}_\infty}\widetilde{B}_\infty$. As G_∞ preserves $\widetilde{\boldsymbol{\omega}}_\infty$, we may identify $T_{\tilde{p}_\infty}\widetilde{B}_\infty$ with the quaternions $\mathbb{H}$, so that $d\phi$ acts by left multiplication by a unit quaternion. In particular, 1 is not an eigenvalue of L. Now for any sufficiently large j, we may write $\phi_j\in G_j$ as

$$\phi_j=L+E_j,$$

where $\|E_j\|_{C^2}$ is small. By a simple application of the implicit function theorem, we see that, for j large, ϕ_j must also have a nearby fixed point. This contradicts the fact that the G_j action is free. □

Remark 3.2. Here, we used crucially the property that SU(2)(≅Sp(1)) acts freely on the unit sphere S^3. This proposition was also implicitly proved in [20] using a different argument.

Using similar arguments, we also obtain the following result.

Proposition 3.3. *The Lie group G_∞ is connected.*

Proof. Notice that the projection map $\widetilde{B}_\infty\to B_\infty$ has connected fibers, on which G_∞ acts transitively. Then, the conclusion follows from the fact that the G_∞ action is free. □

We now divide the discussion into three cases, depending on the dimension d. The case $d=1$ was studied in detail in [50]. So our main focus below is in the other two cases $d=2$ and $d=3$.

3.1. Case $d=3$

3.1.1. Geometric structure on $\mathcal{R}$

In this case $G_\infty=\mathbb{R}$. Choosing a generator of G_∞ gives a Killing field ∂_t on $\widetilde{B}_\infty$ which preserves the hyperkähler triple $\widetilde{\boldsymbol{\omega}}_\infty$. Shrinking $\widetilde{B}_\infty$ if necessary, we may assume that there is a triple of moment maps for ∂_t with respect to $\widetilde{\boldsymbol{\omega}}_\infty$ given by

$$\pi_\infty=(x,y,z)\colon \widetilde{B}_\infty\longrightarrow\mathbb{R}^3.$$

These serve as local coordinates on $\mathcal{R}$, and the Riemannian metric g_∞ can be written as

$$g_\infty=V(dx^2+dy^2+dz^2),$$

where $V=|\partial_t|^{-2}$ satisfies $\Delta_{V^{-1}g_\infty}V=0$. This is the well-known description of hyperkähler metrics with an $\mathbb{R}$ symmetry, i.e., the *Gibbons–Hawking ansatz*; see [**?**], [49] for details of the construction. We can write the hyperkähler triple on $\widetilde{B}_\infty$ as

$$\begin{cases} \widetilde{\omega}_{\infty,1}=V\,dx\wedge dy+dz\wedge\theta,\\ \widetilde{\omega}_{\infty,2}=V\,dy\wedge dz+dx\wedge\theta,\\ \widetilde{\omega}_{\infty,3}=V\,dz\wedge dx+dy\wedge\theta,\end{cases}$$

where θ is the 1-form dual to ∂_t. See [41, §2] or [46, §2] for more details. By the discussion in §2.2, the renormalized limit measure ν_∞ has the expression

$$d\nu_\infty=c\cdot e^{-f}\,d\mathrm{vol}_{g_\infty},\quad f=\tfrac{1}{2}\log V,\quad c\in\mathbb{R}_+. \tag{3.1}$$

Moreover, the Bakry–Émery–Ricci tensor is non-negative:

$$\mathrm{Ric}^1_{g_\infty}\equiv\mathrm{Ric}_{g_\infty}+\nabla^2_{g_\infty}f-df\otimes df\geqslant 0.$$

An immediate consequence of (3.1) is that the function V is well defined, up to a global multiplicative constant on each connected component of $\mathcal{R}$. Fixing a choice of V determines the Killing field ∂_t, and hence the exact frame $\{dx,dy,dz\}$, up to multiplication by ±1. In particular, $\mathcal{R}$ is endowed with an affine structure with monodromy contained in $\mathbb{R}^3\rtimes\mathbb{Z}_2\subset\mathrm{Aff}(\mathbb{R}^3)$. It is easy to see that V is a harmonic function on $\widetilde{B}_\infty$. Therefore, on $\mathcal{R}$, we have $\Delta_{\nu_\infty}V=0$.

Definition 3.4. A *special affine structure* on a 3-manifold Y^3 is an affine structure with monodromy contained in $\mathbb{R}^3\rtimes\mathbb{Z}_2$.

In particular, a special affine structure determines a flat Riemannian metric $g^\flat$ on Y^3, up to constant multiplication. In local special affine coordinates (x,y,z) we have that

$$g^\flat=C(dx^2+dy^2+dz^2).$$

Definition 3.5. A function u on a special affine 3-manifold is *harmonic* if

$$\Delta_{g^\flat}u=0.$$

Definition 3.6. A *special affine metric* on a 3-manifold Y^3 consists of a special affine structure together with a smooth Riemannian metric g such that $g=Vg^\flat$, for a positive harmonic function V on Y^3.

Here V is well defined, up to constant multiplication. A choice of V determines the flat metric $g^\flat=V^{-1}g$, which we call the *flat background geometry*; it also yields a measure ν with density $d\nu=V^{-1/2}\,d\mathrm{vol}_g$, so as a metric measure space we have $\mathrm{Ric}^1_g\geqslant0$. Our discussion above shows the following result.

PROPOSITION 3.7. *In case $d=3$, $\mathcal{R}$ is endowed with a special affine metric structure.*

Notice that, if we perform hyperkähler rotations, i.e., changing the choice of hyperkähler triples on each X_j^4, then the resulting metric is unchanged but the affine structure undergoes a rotation in SO(3).

3.1.2. Convergence of special affine metrics

Now, we discuss the convergence of special affine metrics. Suppose that we are given a sequence of special affine metrics (Y_i^3, g_i, p_i) such that $\overline{B_2(p_i)}$ is compact. We can first normalize the harmonic function V_i on Y_i by requiring $V_i(p_i)=1$. This fixes the measure $d\nu_i=V_i^{-1/2}\,d\mathrm{vol}_{g_i}$. As $\mathrm{Ric}^1_{g_i}\geqslant 0$, by Theorem 2.8, we have $0<C^{-1}\leqslant V_i\leqslant C$ uniformly on $B_{3/2}(p_i)$. Hence, the diameter of $B_1(p_i)$ with respect to the flat background metric $g_i^\flat$ is also uniformly bounded above and below. Then, by passing to a subsequence, we have

$$(B_1(p_i), g_i^\flat, p_i) \xrightarrow{\mathrm{GH}} (Z_\infty, d_\infty, p_\infty).$$

If $\dim_{\mathrm{ess}} Z_\infty=3$, then it is a flat 3-manifold. Since $\Delta_{g_i^\flat} V_i=0$, the uniform L^∞ bound on V_i gives uniform interior bounds on all derivatives. In particular, passing to a further subsequence, we may assume that the local frames $\{dx_i, dy_i, dz_i\}$ converge smoothly to a limit, giving a special affine structure on Z_∞, and the function V_i converges smoothly to a limit harmonic function V_∞. Globally it follows that in this case Y_∞^3 is also a smooth 3-manifold with a special affine metric, and the convergence of Y_i^3 to Y_∞^3 is smooth.

If $\dim_{\mathrm{ess}} Z_\infty<3$, then the flat metrics $g_i^\flat$ collapse. Using the fact that the monodromy is contained in $\mathbb{R}^3\rtimes\mathbb{Z}_2$, it is easy to see that, for i large, $g_i^\flat$ is locally isometric to a product $\mathbb{T}^k\times\mathbb{R}^{3-k}$, $k=1,2$, for some flat torus $\mathbb{T}^k$ and Euclidean space $\mathbb{R}^{3-k}$. Passing to local universal covers, we may assume that V_i still converges smoothly. Notice that, through each point in $B_1(p_i)\subset Y_i^3$, there is a unique flat totally geodesic $\mathbb{T}^k$ with respect to $g_i^\flat$, which are all isometric as the point varies.

The upshot of the above discussion is that we have a good understanding of convergence of special affine metrics. In particular, we always have a-priori interior curvature bound and its covariant derivatives.

3.2. Case $d=2$

3.2.1. Geometric structure on $\mathcal{R}$

In this case, we have $G_\infty=\mathbb{R}^2$. Fixing a basis of G_∞ yields two Killing fields $\{\partial_{t_1}, \partial_{t_2}\}$ on $\widetilde{B}_\infty$, which preserve the limit triple $\widetilde{\boldsymbol{\omega}}_\infty$. Choose moment maps $x_{\alpha j}$ for the vector field ∂_{t_α} with respect to the symplectic form $\widetilde{\omega}^j_\infty$. Since $[\partial_{t_1}, \partial_{t_2}]=0$, we have

$$d(\mathcal{L}_{\partial_{t_1}} x_{2j}) = \mathcal{L}_{\partial_{t_1}} dx_{2j} = 0,$$

so $\partial_{t_1} x_{2j}$ is a constant. It then follows that there is a unit vector $\boldsymbol{a}=(a_1, a_2, a_3)\in\mathbb{R}^3$ such that $\partial_{t_1}\sum_j a_j x_{2j}=0$. Rotating the hyperkähler triple by an element in SO(3), we may assume $\boldsymbol{a}=(1,0,0)$. Then, we have $\widetilde{\omega}^1_\infty(\partial_{t_1}, \partial_{t_2})\equiv 0$; in other words, the G_∞ orbit is Lagrangian with respect to $\widetilde{\omega}^1_\infty$. Notice the choice of $\boldsymbol{a}$ (hence of $\widetilde{\omega}^1_\infty$) is only unique up to SO(2) rotation, but we will fix a choice in the following discussion. Then, we obtain local moment maps $\pi=(x_1, x_2)\colon \widetilde{B}_\infty \to \mathbb{R}^2$ for the G_∞ action with respect to $\widetilde{\omega}^1_\infty$. We can view (x_1, x_2) as local coordinates on $\mathcal{R}$ which depend on the choice of the basis of G_∞, so are well defined, up to $\mathbb{R}^2 \rtimes \mathrm{GL}(2;\mathbb{R})$ action. We set

$$W^{\alpha\beta} \equiv \tilde{g}_\infty(\partial_{t_\alpha}, \partial_{t_\beta}), \quad 1 \leqslant \alpha, \beta \leqslant 2,$$

and let $(W_{\alpha\beta})$ be the inverse matrix of $(W^{\alpha\beta})$. Clearly these descend to $\widetilde{B}_\infty/G_\infty \subset \mathcal{R}$. As in [78, §2.5], it is easy to see, using the hyperkähler equation, that on $\widetilde{B}_\infty/G_\infty$ we have

$$\begin{cases} \det(W_{\alpha\beta}) = C > 0, \\ \partial_{x_\gamma} W_{\alpha\beta} = \partial_{x_\beta} W_{\alpha\gamma}, \quad 1 \leqslant \alpha, \beta, \gamma \leqslant 2. \end{cases} \tag{3.2}$$

Moreover, the metric g_∞ is given by

$$g_\infty = W_{\alpha\beta}\, dx_\alpha\, dx_\beta + W^{\alpha\beta}\theta_\alpha\theta_\beta, \tag{3.3}$$

where θ_α is the dual 1-form of the Killing field ∂_{t_α}.

The second equation in (3.2) implies that locally we can write $(W_{\alpha\beta})$ as the Hessian $(\phi_{\alpha\beta})$ of a convex function ϕ. We can rescale the coordinates $\{x_1, x_2\}$ simultaneously by a constant, so that $C=1$ in (3.2). In terms of the normalized local coordinates, we obtain an affine structure on $\mathcal{R}$ with monodromy group contained in $\mathbb{R}^2 \rtimes \mathrm{SL}(2;\mathbb{R})$, and a Riemannian metric $g_\infty = \phi_{\alpha\beta}\, dx_\alpha\, dx_\beta$, with $\det(\phi_{\alpha\beta})=1$.

The discussion in §2 implies that in this case the renormalized limit measure

$$\nu_\infty = dx_1 \wedge dx_2$$

is simply the volume measure of g_∞. Also, we have $\mathrm{Ric}_{g_\infty} \geqslant 0$. As $\mathcal{R}$ has real dimension 2, the metric g_∞ defines a complex structure J on $\mathcal{R}$ via the Hodge star operator:

$$J\, dx_1 = -\phi^{12} dx_1 + \phi^{11} dx_2 \quad \text{and} \quad J\, dx_2 = -\phi^{22} dx_1 + \phi^{12} dx_2.$$

The corresponding Kähler form is $\omega = dx_1 \wedge dx_2$. Let ∇ be the flat connection defined by the affine structure. Then, we have $\nabla\omega=0$ and $d^\nabla J=0$, where

$$d^\nabla J(\xi, \eta) \equiv (\nabla_\xi J)(\eta) - (\nabla_\eta J)(\xi).$$

We now recall the following definition.

Definition 3.8. ([34]) A *special Kähler manifold* is a Kähler manifold (M, ω, I) together with a torsion-free flat symplectic connection ∇ satisfying $d^{\nabla} I = 0$.

Therefore, we have proved the following result.

PROPOSITION 3.9. *If $d=2$, then $\mathcal{R}$ is endowed with a special Kähler structure.*

Remark 3.10. Notice that the construction depends on the symplectic form $\widetilde{\omega}^1_\infty$ that we choose at the beginning.

By [34], once we fix the choice of local affine coordinates x_1 and x_2, there i+s a pair of conjugate special holomorphic coordinates z and w, such that

$$\operatorname{Re}(z) = x_1 \quad \text{and} \quad \operatorname{Re}(w) = -x_2.$$

They are unique, up to transformations $z \mapsto z + c$ and $w \mapsto w + c'$ for $c, c' \in \sqrt{-1}\,\mathbb{R}$. With respect to these special holomorphic coordinates, the monodromy is then contained in $\sqrt{-1}\,\mathbb{R}^2 \rtimes \mathrm{SL}(2;\mathbb{R})$. Moreover, the Kähler form can be written as

$$\omega = \frac{\sqrt{-1}}{2} \operatorname{Im}(\tau)\, dz \wedge d\bar{z}, \tag{3.4}$$

where the local holomorphic function $\tau \equiv \partial w / \partial z$ satisfies $\operatorname{Im}(\tau) > 0$. We can view τ as a multi-valued holomorphic map from Z to the upper half-plane $\mathcal{H}$.

If we go around a loop γ, then we obtain new local special holomorphic coordinates $(\tilde{z}, \widetilde{w})$, with affine transformation given by

$$\begin{pmatrix} \tilde{z} \\ \widetilde{w} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{3.5}$$

In particular,

$$\tilde{\tau} = \frac{d\tau + c}{b\tau + a}.$$

For notational convenience, we call the following matrix the *monodromy matrix* along γ:

$$A_\gamma \equiv \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \mathrm{SL}(2;\mathbb{R})$$

So we have a monodromy representation

$$\begin{aligned} \rho \colon \pi_1(\mathcal{R}) &\longrightarrow \mathrm{SL}(2;\mathbb{R}), \\ \gamma &\longmapsto A_\gamma, \end{aligned}$$

which is well defined up to conjugation, i.e., up to the choice of the base point and the local special holomorphic coordinates around the base point.

Conversely, suppose that we are given a Riemann surface Z with a Kähler metric ω. If we can find local conjugate holomorphic coordinates (z, w) with $\tau = \partial w/\partial z$ satisfying $\mathrm{Im}(\tau) > 0$, such that (3.4) holds and the monodromy for (z, w) along any loop is contained in $\mathrm{SL}(2;\mathbb{R})$. Then, there is a unique special Kähler structure on Z associated to the metric ω, so that $\mathrm{Re}(dz)$ and $\mathrm{Re}(dw)$ are parallel with respect to the associated torsion-free connection ∇.

Definition 3.11. Let M be a special Kähler Riemann surface. We say that

- M has *integral monodromy* if the monodromy representation $\rho\colon \pi_1(M) \to \mathrm{SL}(2;\mathbb{R})$ is conjugate to a representation in $\mathrm{SL}(2;\mathbb{Z})$;
- M has *local integral monodromy* if the monodromy matrix associated to each loop γ in M is conjugate to an element in $\mathrm{SL}(2;\mathbb{Z})$.

In general, the two notions are not equivalent; see Remark 3.18. Now we give some singularity models.

Example 3.12. A flat metric cone

$$\omega = \frac{\sqrt{-1}}{2}\beta^2|\zeta|^{2\beta-2}d\zeta\wedge d\bar{\zeta}, \quad \beta\in(0,1)$$

on $\mathbb{C}^*$ induces a natural special Kähler structure, with local special holomorphic coordinates given by $z = \zeta^\beta$ and $w = \sqrt{-1}z$, so that $\tau = \sqrt{-1}$. The monodromy matrix around the generator of $\pi_1(\mathbb{C}^*)$ is

$$R_\beta = \begin{pmatrix} \cos(2\pi\beta) & -\sin(2\pi\beta) \\ \sin(2\pi\beta) & \cos(2\pi\beta) \end{pmatrix}$$

We denote by $\boldsymbol{C}_\beta$ such a special Kähler cone. By [34], the cotangent bundle $T^*\mathbf{C}_\beta$ admits a canonical flat hyperkähler metric.

Remark 3.13. [34] showed that, on a special Kähler manifold, there is a globally defined holomorphic cubic differential, given in local special holomorphic coordinates by

$$\Theta = \frac{\partial\tau}{\partial z}\, dz^{\otimes 3}.$$

Moreover, the scalar curvature satisfies $S = 4|\Theta|^2$. In particular, $\Theta\equiv 0$ if and only if the metric is flat. Using this, one can see that, if a special Kähler metric is flat, then the flat symplectic connection ∇ agrees with the Levi-Civita connection. So, the special Kähler structure is uniquely determined by the flat Kähler metric itself.

Example 3.14. Consider the metric on the punctured unit disk $(\mathbb{D}^*, \zeta)$ given by

$$\omega = -\frac{\sqrt{-1}}{4\pi}(\log|\zeta|)\, d\zeta \wedge d\bar{\zeta}.$$

This has a global special holomorphic coordinate $z=\zeta$, and local conjugate coordinate

$$w = -\frac{\sqrt{-1}}{2\pi} z\log z + \frac{\sqrt{-1}}{2\pi} z.$$

The period map is

$$\tau = -\frac{\sqrt{-1}}{2\pi}\log\zeta.$$

The monodromy around the generator of $\pi_1(\mathbb{D}^*)$ is given by

$$I_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The tangent cone is the flat space $\mathbb{R}^2$ with standard special Kähler structure.

Example 3.15. Consider the metric on $(\mathbb{D}^*, \zeta)$ given by

$$\omega = -\frac{\sqrt{-1}}{32\pi}|\zeta|^{-1}\log|\zeta| d\zeta \wedge d\bar{\zeta}.$$

We can use $z=\sqrt{\frac{1}{2}\zeta}$ to be a local special holomorphic coordinate. The period map is

$$\tau = -\frac{\sqrt{-1}}{2\pi}\log\zeta.$$

The monodromy is given by

$$I_1^* = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

The tangent cone is the flat cone $\mathbb{R}^2/\mathbb{Z}_2$, with monodromy $R_{\frac{1}{2}}$. Indeed, the metric here is a $\mathbb{Z}_2$ quotient of the metric in the previous example.

We will need the following elementary results on the classification of conjugacy classes in $\mathrm{SL}(2;\mathbb{R})$ and in $\mathrm{SL}(2;\mathbb{Z})$.

LEMMA 3.16. *Let A be an element in* $\mathrm{SL}(2;\mathbb{R})$. *Then, the following holds*:

(1) *if A is* parabolic, *i.e.*, $|\mathrm{Tr}(A)|=2$, *then A is* $\mathrm{SL}(2;\mathbb{R})$ *conjugate to* Id, $-\mathrm{Id}$, I_1, I_1^{-1}, I_1^*, *or* $(I_1^*)^{-1}$;

(2) *if A is* elliptic, *i.e.*, $|\mathrm{Tr}(A)|<2$, *then A is* $\mathrm{SL}(2;\mathbb{R})$ *conjugate to R_β for some* $\beta \in (0,1)\setminus\{\frac{1}{2}\}$;

(3) *if A is* hyperbolic, *i.e.*, $|\mathrm{Tr}(A)|>2$, *then A is* $\mathrm{SL}(2;\mathbb{R})$ *conjugate to*

$$D_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$

for some $r \notin \{0, 1, -1\}$.

LEMMA 3.17. *Let A be an element in* $\mathrm{SL}(2;\mathbb{Z})$. *Then, the following holds*:

- *if A is elliptic, then A is* $\mathrm{SL}(2;\mathbb{Z})$ *conjugate to one of the following*:

$$\begin{aligned}&\widetilde{R}_{1/4}\equiv\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \widetilde{R}_{3/4}\equiv\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \widetilde{R}_{1/6}\equiv\begin{pmatrix}1&-1\\1&0\end{pmatrix},\\&\widetilde{R}_{1/3}\equiv\begin{pmatrix}0&-1\\1&-1\end{pmatrix},\quad \widetilde{R}_{2/3}\equiv\begin{pmatrix}-1&1\\-1&0\end{pmatrix},\quad \widetilde{R}_{5/6}\equiv\begin{pmatrix}0&1\\-1&1\end{pmatrix}.\end{aligned}\tag{3.6}$$

- *if A is parabolic, then A is* $\mathrm{SL}(2;\mathbb{Z})$ *conjugate to one of the following*:

$$I_n\equiv\begin{pmatrix}1&n\\0&1\end{pmatrix}\ (n\in\mathbb{Z}),\quad I_n^*\equiv\begin{pmatrix}-1&-n\\0&-1\end{pmatrix}\ (n\in\mathbb{Z}),$$
$$\widetilde{R}_1\equiv\mathrm{Id},\quad \widetilde{R}_{1/2}\equiv-\mathrm{Id}.$$

Notice that each $\widetilde{R}_\beta$ is $\mathrm{SL}(2;\mathbb{R})$ conjugate to R_β, so it is a rotation of $\mathbb{R}^2$ that preserves a lattice.

3.2.2. Convergence of special Kähler structures

Let (M_i,p_i) be a sequence of 2-dimensional manifolds with special Kähler metrics (ω_i,J_i), where $\overline{B_2(p_i)}$ is compact. Since the curvature is non-negative, passing to a subsequence we first obtain a Gromov–Haudorff limit $(M_\infty,d_\infty,p_\infty)$. For our purpose, we may assume that M_∞ is not a single point.

Let $\widetilde{U}_i$ be the universal cover of $\overline{B_2(p_i)}$, endowed with the induced special Kähler structure. Then, it has trivial monodromy representation. Let $\tilde{p}_i$ be a lift of p_i, and let (z_i,w_i) be a choice of special holomorphic coordinates on $\widetilde{U}_i$. Then, we can write

$$\widetilde{\omega}_i=\frac{\sqrt{-1}}{2}\,\mathrm{Im}(\tau_i)\,dz_i\wedge d\bar{z}_i$$

for some holomorphic function τ_i. Applying a linear transformation to (z_i,w_i) by an element in $\mathrm{SL}(2;\mathbb{Z})$ we may assume that

$$\mathrm{Re}(\tau_i(\tilde{p}_i))\in\left[-\frac{1}{2},\frac{1}{2}\right]\quad\text{and}\quad \mathrm{Im}(\tau_i(\tilde{p}_i))\geqslant\frac{\sqrt{3}}{2}.$$

Then, replacing (z_i,w_i) by $(\lambda_i^{-1}z_i,\lambda_iw_i)$ for a suitable $\lambda_i>0$, we may further assume that

$$\mathrm{Im}(\tau_i(\tilde{p}_i))\in\left[\frac{\sqrt{3}}{2},1\right].$$

Notice that $\mathrm{Im}(\tau_i)$ is a positive harmonic function on $\widetilde{U}_i$, so by Theorem 2.8 we have $|\log \mathrm{Im}(\tau_i)|\leqslant C$ uniformly on $B_{3/2}(\tilde{p}_i)$. Then, on this ball, the metric $\widetilde{\omega}_i$ is uniformly equivalent to the flat metric $\widetilde{\omega}_i^\flat=\mathrm{Im}(\tau_i)^{-1}\widetilde{\omega}_i$. Then, clearly, we have local smooth convergence of $\widetilde{\omega}_i^\flat$ by identifying each ball $(B_1(\tilde{p}_i), z_i)$ holomorphically with a domain in $(\mathbb{C}, z)$. Then, passing to a subsequence, we may assume that $\mathrm{Im}(\tau_i)$ converges smoothly, and hence we obtain a smooth limit Kähler metric $(B_1(\tilde{p}_\infty), \widetilde{\omega}_\infty)$.

Since $\mathrm{Im}(\tau_i)$ is harmonic and bounded, its derivative is uniformly bounded on $B_1(\tilde{p}_i)$. Using the Cauchy–Riemann equation and the assumption that $\mathrm{Re}(\tau_i(\tilde{p}_i))\in\left[-\frac{1}{2},\frac{1}{2}\right]$, one can show that $\mathrm{Re}(\tau_i)$ is also uniformly bounded on $B_1(\tilde{p}_i)$, so, passing to a further subsequence, we may assume that τ_i converges smoothly to a limit τ_∞. At the same time, since $\tau_i=\partial w_i/\partial z_i$, we may ensure the holomorphic function w_i also converges to a limit w_∞. Then, we can write

$$\widetilde{\omega}_\infty=\frac{\sqrt{-1}}{2}\,\mathrm{Im}(\tau_\infty)\,dz_\infty\wedge d\bar{z}_\infty.$$

In particular, there is a special Kähler structure on the limit space $B_1(\tilde{p}_\infty)$.

A consequence of the above discussion is that, for a special Kähler metric, we have a-priori interior curvature bound, as well as its covariant derivatives. It is worth mentioning that, even though not needed in this paper, the above arguments hold for special Kähler metrics in any dimension.

Now we divide into two cases.

Case 1. $\mathrm{Vol}(B_2(p_i))\geqslant\kappa$ uniformly for some $\kappa>0$. In this case, M_∞ is a smooth Riemann surface and the M_i's converge smoothly to M_∞ in the Cheeger–Gromov sense. We may assume the Kähler metrics ω_i converge smoothly to a limit ω_∞ on M_∞.

We claim that, by passing to a further subsequence, M_∞ can be naturally endowed with a special Kähler structure. To see this, first we may find a $\delta>0$ such that $B_\delta(p_i)$ is diffeomorphic to a ball, for all i large. Then, in the above discussion, we can directly work with the ball $B_\delta(p_i)$, and find special conjugate holomorphic coordinates (z_i, w_i) which converge to (z_∞, w_∞) on $B_\delta(p_\infty)$. Now, for any $q\in M_\infty$ which is the limit of $q_i\in M_i$, we can choose a path γ in M_∞ connecting p and q. Using the smooth convergence of M_i to M_∞, we may view γ as a path γ_i in M_i (for i large) connecting p_i and q_i. Then, we can analytically continue the special holomorphic coordinates (z_i, w_i) along γ_i to obtain special coordinates in a neighborhood of q_i. Applying the Harnack inequality for $\mathrm{Im}(\tau_i)$ along γ_i, we see that $|\log \mathrm{Im}(\tau_i)|$ is uniformly bounded along γ, which implies a uniform bound of $|\nabla_{\omega_i} z_i|$ and $|\nabla_{\omega_i} w_i|$ along γ. Passing to a subsequence, we may assume that (z_i, w_i) converges uniformly to (z_∞, w_∞) along γ. In particular, (z_∞, w_∞) serve as local conjugate holomorphic coordinates in a neighborhood of q. Now, these coordinates depend on the homotopy class of γ. But the fact that M_i has monodromy contained in $\mathrm{SL}(2;\mathbb{R})$ implies that the limit special holomorphic coordinates also have monodromy contained in $\mathrm{SL}(2;\mathbb{R})$. So, we obtain a global special Kähler structure on M_∞.

By the above construction, we also have the convergence of the conjugacy classes of monodromy representations. More precisely, if we fix a choice of the monodromy representation $\rho_i: \pi_1(M_i;p_i)\to\mathrm{SL}(2;\mathbb{R})$ (for example, by fixing a choice of special conjugate holomorphic coordinates in a neighborhood of p_i), then there exists some $P_i\in\mathrm{SL}(2;\mathbb{R})$ such that, for every $\sigma\in\pi_1(M_\infty;p_\infty)$,

$$\lim_{i\to\infty} P_i A_{\sigma,i} P_i^{-1} = A_{\sigma,\infty}, \tag{3.7}$$

where we denote by $A_{\sigma,i}$ and $A_{\sigma,\infty}$ the monodromy matrix of the special Kähler structure on M_i and M_∞ along the loop σ, respectively. An immediate consequence is that

$$\operatorname{Tr} A_{\sigma,\infty} = \lim_{i\to\infty} \operatorname{Tr} A_{\sigma,i}. \tag{3.8}$$

Case 2. $\mathrm{Vol}(B_2(p_i))\to 0$ as $i\to\infty$. Then, we know that M_i collapses with locally uniformly bounded curvature along circle fibrations. Let σ_i be a loop in M_i corresponding to the collapsing circle fiber. Since on the local universal cover we have smooth convergence of the special Kähler metrics, it follows easily that we can find $P_i\in\mathrm{SL}(2;\mathbb{R})$ with

$$\lim_{i\to\infty} P_i A_{\sigma_i,i} P_i^{-1} = \mathrm{Id}. \tag{3.9}$$

In particular, $A_{\sigma_i,i}$ must be conjugate to Id, I_1 or I_1^{-1}, for i large.

Remark 3.18. If M_i has integral monodromy for all i, then in the above Case 1 the limit M_∞ must have local integral monodromy. To see this, we make a choice of local conjugate special holomorphic coordinates (z_i,w_i) near p_i such that the induced monodromy matrices along all the loops are integral. Then, we look at more closely the above discussion. First, by a transformation in $\mathrm{SL}(2;\mathbb{Z})$, we may assume that $\mathrm{Re}(\tau_i(p_i))\in\left[-\frac{1}{2},\frac{1}{2}\right]$. Now, if $\mathrm{Im}(\tau_i(p_i)$ is bounded, then we may take $P_i=\mathrm{Id}$ in the above, and it follows that $A_{\sigma,\infty}\in\mathrm{SL}(2;\mathbb{Z})$ for all loops σ based at p_∞. In this case, M_∞ indeed has integral monodromy. If $\mathrm{Im}(\tau_i(p_i))$ is unbounded, then we may take

$$P_i = \begin{pmatrix} \lambda_i^{-1} & 0 \\ 0 & \lambda_i \end{pmatrix},$$

and we must have $\lambda_i\to 0$. Write

$$A_{\sigma,i} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}) \quad \text{and} \quad A_{\sigma,\infty} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{R}).$$

Then, (3.7) implies that, for i large, $a_i \equiv a$, $d_i \equiv d$ and

$$\lim_{i\to\infty} \lambda_i^2 c_i = c \quad \text{and} \quad \lim_{i\to\infty} \lambda_i^{-2} b_i = b.$$

So, for i large, we must have $b_i = b = 0$ and $a_i = d_i = \pm 1$. Therefore, we know that $A_{\sigma,\infty}$ is parabolic, and hence is conjugate to a matrix in $\mathrm{SL}(2;\mathbb{Z})$.

On the other hand, without extra assumptions, one cannot expect M_∞ to have integral monodromy globally. For example, consider the punctured domain $\Omega = \mathbb{D} \setminus \{0, \frac{1}{2}\}$ endowed with the special Kähler metrics

$$\omega_{m,n} = \left(-m \log |z| - n \log \left| z - \frac{1}{2} \right| \right) \frac{\sqrt{-1}}{2}\, dz \wedge d\bar{z}.$$

These obviously have integral monodromy. Now, we take a sequence $m_j, n_j \to \infty$ such that the ratio m_j/n_j converging to an irrational number. Then the limit of $m_j^{-1} \omega_{m_j,n_j}$ has local integral monodromy but the global monodromy is not integral.

3.2.3. Singular special Kähler metric

We refer the reader to [12], [43] for discussion on local models of more general singularities of special Kähler metrics. For the convenience of our later discussion, we introduce the notion of a singular special Kähler metric adapted to our context.

Definition 3.19. A singular special Kähler metric on a 2-dimensional Riemann surface M is a smooth special Kähler metric ω on $M \setminus \{p_1, ..., p_k\}$ such that, near each p_i, there exists $\delta > 0$ and a holomorphic embedding $B_\delta(p_i) \setminus \{p_i\}$ into a domain in $(\mathbb{C}^*, \zeta)$ which extends to a topological embedding of $B_\delta(p_i)$ into $\mathbb{C}$ such that one of the following holds:

- (Type I) $z = \zeta$ is a special holomorphic coordinate on $B_\delta(p_i)$, the local period map is given by

$$\tau = -\frac{\sqrt{-1}}{2\pi} \log \zeta + f(\zeta)$$

for f holomorphic across zero, and

$$\omega = \frac{\sqrt{-1}}{4\pi} (-\log |\zeta| + \mathrm{Im}(f))\, d\zeta \wedge d\bar{\zeta}.$$

In thic case, the monodromy matrix around the counterclockwise generator of the group $\pi_1(B_\delta(p_i) \setminus \{p_i\})$ is given by I_1.

- (Type II) $z = \zeta^{1/2}$ is local special holomorphic coordinate, the local period map is of the form

$$\tau = -\frac{\sqrt{-1}}{4\pi} \log \zeta + f$$

for f holomorphic across zero, and

$$\omega=\frac{\sqrt{-1}}{32\pi}(-\log|\zeta|+\mathrm{Im}(f))|\zeta|^{-1}\,d\zeta\wedge d\bar{\zeta}.$$

In this case, the monodromy matrix around the counterclockwise generator of the group $\pi_1(B_\delta(p_i)\setminus\{p_i\})$ is given by I_1^*.

- (Type III) $\zeta=(z-\sqrt{-1}w)^{1/\beta}$ for local conjugate special holomorphic coordinates (z,w) (for some $\beta\in\left\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}\right\}$), and we may locally write

$$\omega=\tfrac{1}{8}\sqrt{-1}(1-|\xi|^2)\beta^2|\zeta|^{2\beta-2}\,d\zeta\wedge d\bar{\zeta},$$

where ξ is a multi-valued holomorphic function and is related to the local period map τ by the formula

$$\xi=\frac{\tau-\sqrt{-1}}{\tau+\sqrt{-1}}.$$

Moreover,

– if $\beta=\frac{1}{2}$, then ξ is a holomorphic function of ζ;
– if $\beta\in\left\{\frac{1}{4},\frac{3}{4}\right\}$, then $\xi=F(\zeta)^{1/2}$ for a holomorphic function F with $F(0)=0$;
– if $\beta\in\left\{\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{2}{3}\right\}$, then $\xi=F(\zeta)^{1/3}$ for a holomorphic function F with $F(0)=0$.

In particular, a singular special Kähler metric is asymptotic to one of the model singularities in Examples 3.12, and has local integral monodromy. It is also easy to check that there is always a unique tangent cone at the singularity given by a flat cone of angle in $(0,2\pi]$. More general examples of singular special Kähler metrics satisfying the above conditions are given by the base of an elliptic fibration with singular fibers (see for example [44]).

3.3. Case $d=1$

In this case, the group G_∞ is either the abelian group $\mathbb{R}^3$ or the Heisenberg group $\mathscr{H}_1$ with Lie algebra $\mathfrak{h}_1$, where

$$\mathscr{H}_1\equiv\left\{\begin{pmatrix}1 & x & t\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}: x,y,t\in\mathbb{R}\right\}\quad\text{and}\quad \mathfrak{h}_1\equiv\left\{\begin{pmatrix}0 & x & t\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix}: x,y,t\in\mathbb{R}\right\}.$$

This case has already been studied in detail in [50]. The analysis is simpler than the discussion in the case $d>1$. We briefly summarize the results here, and the readers may refer to [50] for proofs.

If $G_\infty=\mathbb{R}^3$, then the local universal cover converges to the flat metric on $\mathbb{R}^4=\mathbb{R}\times\mathbb{R}^3$. The limit metric is locally isometric to a 1-dimensional interval $(a,b)_z$ endowed with the standard metric $g_\infty=dz^2$ and the renormalized limit measure $\nu_\infty=cV\,dz$ for a constant $c>0$, and $V\equiv1$.

If $G_\infty=\mathscr{H}_1$, then the local universal cover converges to $\mathscr{H}_1\times(a,b)_z$, and the limit hyperkähler metric is given by applying the Gibbons–Hawking ansatz to a linear function $V=z+l$ with $l\in\mathbb{R}$, such that $V=|\partial_t|^{-2}$ for some generator ∂_t of the center $\mathfrak{z}(\mathfrak{h}_1)$. Here, z is the moment map for the action of the center $\mathfrak{Z}(\mathscr{H}_1)$, and is well defined up to an affine linear transformation of the type $z\mapsto\lambda z+\mu$. The limit metric $g_\infty=V\,dz^2$ and the renormalized limit measure $\nu_\infty=cV\,dz$ for a constant $c>0$. The lemma below follows from direct computation, and we omit the details.

Lemma 3.20. *The second fundamental form of the limit $\mathscr{H}_1$-fibers satisfies*

$$|\mathrm{II}_\infty|=\frac{\sqrt{3}}{2}V^{-3/2},$$

and the Bakry–Émery Laplace operator is given by

$$\Delta_{\nu_\infty}=V^{-1}\partial_z^2.$$

Notice that, since $d=1$, the limit space X_∞ globally must be a 1-dimensional manifold, possibly with boundary. The singular set $\mathcal{S}$ consists of finitely many points in X_∞. The main result of [50] is that these local affine structures indeed patch together to define a global affine coordinate z on X_∞, such that $g=V\,dz^2$ and $\nu_\infty=cV\,dz$, for a concave piecewise linear function $V=V(z)$. Furthermore, in [50] some conjectures are posed on the structure of V in the case when X_∞ is the collapsing limit of hyperkähler metrics on the K3 manifold. Odaka [67] and Oshima [70] have made connections with the algebro-geometric study of type-II degenerations of K3 surfaces.

3.4. ε-regularity theorem

The following was proved by Cheeger–Tian [19] for general Einstein metrics in dimension 4. In the hyperkähler setting, we provide a simple alternative proof, as an application of the study in this section.

Theorem 3.21. (ε-regularity theorem) *There are universal constants $\varepsilon>0$ and $\mathfrak{C}>0$ such that, if a hyperkähler 4-manifold (X^4,g,p) with $\overline{B_{10}(p)}$ compact satisfies*

$$\int_{B_{10}(p)}|\mathrm{Rm}_g|^2\,d\mathrm{vol}_g\leqslant\varepsilon,$$

then $\sup_{B_1(p)}|\mathrm{Rm}_g|\leqslant\mathfrak{C}$.

An immediate consequence is the following.

COROLLARY 3.22. *Let* (X_j^4, g_j, p_j) *be a sequence of hyperkähler* 4*-manifolds converging to a Gromov–Hausdorff limit* $(X_\infty, d_\infty, p_\infty)$*. If there exists a* $C>0$ *such that*

$$\int_{X_j^4} |\mathrm{Rm}_{g_j}|^2 \, d\mathrm{vol}_{g_j} \leqslant C \quad \textit{for all } j,$$

then the singular set $\mathcal{S}$ *consists of at most finitely many points.*

To prove the theorem, we denote $A=|\mathrm{Rm}_g(p)|^{1/2}$, and denote by $\mathcal{G}_p$ the set of all $q\in B_2(p)$ such that

$$|\mathrm{Rm}_g(q)| \geqslant A^2 \quad \text{and} \quad |\mathrm{Rm}_g(q')| \leqslant 4|\mathrm{Rm}_g(q)|$$

for all q' with

$$d(q', q) \leqslant A|\mathrm{Rm}_g(q)|^{-1/2} \leqslant 1.$$

The following point-selection lemma is well known.

LEMMA 3.23. *We have* $\mathcal{G}_p\neq\varnothing$.

Proof. If not, then we can find a sequence q_j, $j=0, 1, ...$, with $q_0=p$, such that

$$d(q_{j+1}, q_j) \leqslant |\mathrm{Rm}_g(q_j)|^{-1/2} A \quad \text{and} \quad |\mathrm{Rm}_g(q_{j+1})| \geqslant 4|\mathrm{Rm}_g(q_j)| \geqslant 4A.$$

So, we have

$$|\mathrm{Rm}_g(q_j)| \geqslant 4^j A^2 \quad \text{and} \quad d(q_j, p) < 2$$

for all j. Clearly, we get a contradiction if $j\to\infty$. □

LEMMA 3.24. *There exists* $A_0>0$ *and* $\kappa>0$ *such that, if* $|\mathrm{Rm}_g(p)|\geqslant A_0$*, then for any* $q\in\mathcal{G}_p$ *we have* $\mathrm{Vol}(B_r(q))\geqslant\kappa r^4$ *with* $r=|\mathrm{Rm}_g(q)|^{-1/2}$.

Proof. Suppose otherwise, then there is a sequence (X_j, g_j, p_j) and $q_j\in\mathcal{G}_{p_j}$ with $A_j\equiv|\mathrm{Rm}_{g_j}(p_j)|\to\infty$ and $\mathrm{Vol}(B_{Q_j^{-1/2}}(q_j))\leqslant j^{-1}Q_j^{-2}$, where $Q_j\equiv|\mathrm{Rm}(q_j)|^{1/2}\geqslant A_j$. Then, consider the rescaled sequence $(X_j, Q_j^2 g_j, q_j)$. Passing to a subsequence we obtain a Gromov–Hausdorff limit (X_∞, q_∞). By assumption, we know that $\dim X_\infty<4$. Moreover, for any fixed $R>0$, the collapsing is with curvature uniformly bounded by 4 on $B_R(q_\infty)$. So, by Proposition 3.1, X_∞ is a complete Riemannian manifold and the limit geometry of the local universal cover around q_j is not flat. On the other hand, below we will show that the limit geometry of the local universal covers is everywhere flat, which yields a contradiction. We divide into three cases.

Case (1) $\dim X_\infty=3$. By Proposition 3.7, we know that X_∞ is a special affine metric 3-manifold. In particular we know $\mathrm{Ric}^1(g_{X_\infty})\geqslant 0$. Let V be the associated positive harmonic function on X_∞. Since X_∞ is complete, by the gradient estimate (Theorem 2.8), we know that V must be a constant, which implies that X_∞ is flat, and the limit geometry of local universal covers is flat.

Case (2) $\dim X_\infty=2$. By Proposition 3.9, we know that X_∞ is a complete special Kähler 2-manifold. Lu's theorem [59] implies that X_∞ is flat, and hence the limit geometry of local universal covers is also flat.

Case (3) $\dim X_\infty=1$. By the discussion in §3.3, we know that X_∞ is an interval equipped with an affine coordinate z such that $g_\infty=V\,dz^2$ and $d\nu_\infty=V\,dz$ for a positive affine function V. Since X_∞ is complete, the interval has to be the entire set $\mathbb{R}$, but then the positivity of V implies that it must be a constant. Hence, the limit geometry of local universal covers is again flat. □

Proof of Theorem 3.21. If not, then we have a sequence (X_j^4, g_j, p_j) with

$$\int_{B_9(p_j)} |\mathrm{Rm}_{g_j}|^2 \leqslant j^{-1},$$

but $|\mathrm{Rm}_{g_j}(p_j)|\to\infty$. We choose $q_j\in\mathcal{G}_{p_j}$, and consider the rescaled sequence $(X_j, Q_j^2 g_j, q_j)$, where $Q_j\equiv|\mathrm{Rm}_{g_j}(q_j)|$. Passing to a subsequence, it converges with uniformly bounded curvature to a limit $(X_\infty, g_\infty, q_\infty)$. Lemma 3.24 implies that

$$\dim X_\infty=4 \quad \text{and} \quad |\mathrm{Rm}_{g_\infty}(q_\infty)|=1.$$

By scaling invariance, it follows that, for any fixed $R>0$, one has

$$\int_{B_R(q_j, Q_j^2 g_j)} |\mathrm{Rm}_{g_j}|^2\, d\mathrm{vol}_{g_j} \to 0,$$

so the limit metric g_∞ must be flat. This yields a contradiction. □

3.5. Perturbation to invariant hyperkähler metrics

Now we go back to the set-up at the beginning of this section. Suppose a sequence of hyperkähler manifolds (X_j^4, g_j, ν_j, p_j) converge in the measured Gromov–Hausdorff topology to a limit metric measure space $(X_\infty, d_\infty, \nu_\infty, p_\infty)$ with $d=\dim_{\mathrm{ess}}(X_\infty)<4$. The goal of this subsection is to show that, over the regular set $\mathcal{R}$, one can deform g_j to a nearby *hyperkähler* metric which exhibits local nilpotent symmetries of rank $4-d$. To prove this, we need to combine the foundational results of Cheeger–Fukaya–Gromov with a quantitative implicit function theorem argument. The following is proved in [17], and we give an explanation in Appendix A.

THEOREM 3.25. (Regular fibration) *Let $\mathcal{Q}\Subset\mathcal{R}$ be a connected compact domain with smooth boundary. Then, we can find $j_0=j_0(\mathcal{Q})>0$ and a sequence $\tau_j\to 0$ such that, for all $j\geqslant j_0$, there exists a compact connected domain $\mathcal{Q}_j\subset X_j^4$ with smooth boundary, together with a smooth fiber bundle map $F_j\colon\mathcal{Q}_j\to\mathcal{Q}$, such that the following properties hold.*

(1) *$F_j\colon\mathcal{Q}_j\to\mathcal{Q}$ is a τ_j-Gromov–Hausdorff approximation.*

(2) *For any $k\in\mathbb{Z}_+$, there exists $C_k>0$ such that, for all $j\geqslant j_0$, we have*

$$|\nabla^k F_j|\leqslant C_k. \tag{3.10}$$

(3) *There exists a uniform constant $C_0>0$ such that, for all $q\in\mathcal{Q}$ and $j\geqslant j_0$, we have*

$$|\mathrm{II}_{F_j^{-1}(q)}|\leqslant C_0,$$

where $\mathrm{II}_{F_j^{-1}(q)}$ denotes the second fundamental form of the fiber $F_j^{-1}(q)$ at $q\in\mathcal{Q}$.

(4) *F_j is an almost Riemannian submersion, in the sense that, for any vector v orthogonal to the fiber of F_j, we have*

$$(1-\tau_j)|v|_{g_j}\leqslant|dF_j(v)|_{g_\infty}\leqslant(1+\tau_j)|v|_{g_j}. \tag{3.11}$$

(5) *There are flat connections with parallel torsion on $F_j^{-1}(q)$, which depend smoothly on $q\in\mathcal{Q}$, such that each fiber of F_j is affine diffeomorphic to an infranilmanifold $\Gamma\backslash N$, where N is a simply-connected nilpotent Lie group and Γ is a cocompact subgroup of $N_L\rtimes\mathrm{Aut}(N)$, with $N_L\simeq N$ acting on N by left translation. Also, the structure group of the fibration is reduced to*

$$((\mathfrak{Z}(N)\cap\Gamma)\backslash\mathfrak{Z}(N))\rtimes\mathrm{Aut}(\Gamma)\subset\mathrm{Aff}(\Gamma\backslash N).$$

(6) *We have that $\Lambda\equiv\Gamma\cap N_L$ is normal in Γ, with $\#(\Lambda\backslash\Gamma)\leqslant w_0$ for some constant w_0 independent of i.*

This is a special case of the *nilpotent Killing structure* ($\mathcal{N}$-structure) defined in [17]. We say that a tensor field ξ on $\mathcal{Q}_j$ is $\mathcal{N}$-invariant if, for any $x\in\mathcal{Q}$, there exists a neighborhood U of x ,with $F_j^{-1}(U)\cong U\times(\Gamma\backslash N)$, such that the lift of ξ to the universal cover $U\times N$ is N_L-invariant. Below, we will construct an $\mathcal{N}$-invariant *hyperkähler* triple approximating the original hyperkähler triple $\boldsymbol{\omega}_j$. First we have

PROPOSITION 3.26. *For any sufficiently large j, there exists an $\mathcal{N}$-invariant definite triple $\boldsymbol{\omega}_j^\dagger$ on $\mathcal{Q}_j$ such that*

$$|\nabla^k_{\boldsymbol{\omega}_9 j}(\boldsymbol{\omega}_j^\dagger-\boldsymbol{\omega}_j)|_{\boldsymbol{\omega}_j}\leqslant C_k\tau_j,$$

where $\tau_j\to 0$ is given by Theorem 3.25.

Proof. The construction is via the averaging argument as in [17, §4], with the Riemannian metric replaced by the definite triple. Let $h\in N_L$ be any element and let $\tilde{v}$ and $\widetilde{w}$ be any tangent vectors on the universal cover $U\times N$ of $F_j^{-1}(U)$. Let $\widetilde{\boldsymbol{\omega}}_j$ be the lift of $\boldsymbol{\omega}_j$ on $U\times N$. Then, the function $h\mapsto\widetilde{\boldsymbol{\omega}}_j(Dh\cdot\tilde{v}, Dh\cdot\widetilde{w})$ is constant on each Λ-orbit in N_L. Since the nilpotent group N_L is unimodular, there is a canonical bi-invariant measure $\tilde{\mu}$ on N_L which descends to a unit-volume bi-invariant measure μ on $\Lambda\backslash N_L$. Therefore,

$$\widetilde{\boldsymbol{\omega}}'_j(\tilde{v},\widetilde{w})\equiv\int_{\Lambda\backslash N_L}\widetilde{\boldsymbol{\omega}}_j(Dh\cdot\tilde{v}, Dh\cdot\widetilde{w})\,d\mu$$

is N_L-invariant on $U\times N$. We denote by $\bar{\boldsymbol{\omega}}'_j$ the descending 2-form on $U\times(\Lambda\backslash N_L)$, and for any tangent vectors $\bar{v}$ and $\bar{w}$ on $U\times(\Lambda\backslash N_L)$, we define

$$\bar{\boldsymbol{\omega}}^{\dagger}_j(\bar{v},\bar{w})\equiv\frac{1}{\#(\Lambda\backslash\Gamma)}\cdot\sum_{\gamma\in\Lambda\backslash\Gamma}\bar{\boldsymbol{\omega}}'_j(D\gamma\cdot\bar{v}, D\gamma\cdot\bar{w}),$$

where $\#(\Lambda\backslash\Gamma)\leqslant w_0$ for some constant $w_0>0$ independent of j. We claim the above $(\Lambda\backslash\Gamma)$-invariant form is N_L-invariant. In fact, let $\bar{\gamma}\in\Gamma$ be any lift of $\gamma\in\Lambda\backslash\Gamma$ to Γ. Since $\Gamma\leqslant N_L\rtimes\mathrm{Aut}(N)$, for any $h\in N_L$ there is some element $\bar{h}\in N_L$ such that $\bar{\gamma}\cdot h=\bar{h}\cdot\bar{\gamma}$. Then, it is easy to verify that

$$\bar{\boldsymbol{\omega}}^{\dagger}_j(Dh\cdot\bar{v}, Dh\cdot\bar{w})=\bar{\boldsymbol{\omega}}^{\dagger}_j(\bar{v},\bar{w}).$$

Now, $\bar{\boldsymbol{\omega}}^{\dagger}_j$ descends to an $\mathcal{N}$-invariant definite triple $\boldsymbol{\omega}^{\dagger}_j$ on $\Gamma\backslash N_L$. Notice that the average of a closed form is still closed. The approximation estimate follows from [17, Proposition 4.9]. □

It is clear that the Riemannian metric $g_{\boldsymbol{\omega}^{\dagger}_j}$ determined by the definite triple $\boldsymbol{\omega}^{\dagger}_j$ is also $\mathcal{N}$-invariant. Moreover, the estimates (3.10) and (3.11) continue to hold if we replace $\boldsymbol{\omega}_j$ by $\boldsymbol{\omega}^{\dagger}_j$.

THEOREM 3.27. *For all sufficiently large j, there is an $\mathcal{N}$-invariant hyperkähler triple $\boldsymbol{\omega}^{\diamondsuit}_j$ on $\mathcal{Q}_j$ of the form $\boldsymbol{\omega}^{\diamondsuit}_j=\boldsymbol{\omega}^{\dagger}_j+dd^*(\boldsymbol{f}_j\cdot\boldsymbol{\omega}^{\dagger}_j)$, where $\boldsymbol{f}_j$ is an $\mathcal{N}$-invariant (3×3)-matrix valued function on $\mathcal{Q}_j$ satisfying that, for all $k\in\mathbb{N}$,*

$$\sup_{\mathcal{Q}_j}|\nabla^k_{\boldsymbol{\omega}^{\dagger}_j}\boldsymbol{f}_j|\to 0. \tag{3.12}$$

In particular, $\boldsymbol{\omega}^{\diamondsuit}_j$ has the same Gromov–Hausdorff collapsed limit as $\boldsymbol{\omega}_j$.

Remark 3.28. In [17] (Open Problem 1.10), Cheeger–Fukaya–Gromov asked the question that when a sufficiently collapsed Riemannian metric satisfies extra properties such as being Einstein or Kähler, whether one can perturb it to be an $\mathcal{N}$-invariant Riemannian metric in the same category. The above theorem can be viewed as giving an affirmative answer to this question in the setting of local 4-dimensional hyperkähler structures. We mention that Huang–Rong–Wang [51] made some related progress on this question of Cheeger–Fukaya–Gromov using Ricci flow.

Before proving Theorem 3.27, we make some preparations. Denote by $g_{\mathcal{Q},j}$ the quotient metric on $\mathcal{Q}$ induced by the metric $\boldsymbol{\omega}_j^\dagger$, and by H_j the mean curvature vector of the fibers of F_j. Because of the $\mathcal{N}$-invariance, we may view H_j as a vector field on the quotient $\mathcal{Q}$. Recall that we have the density function χ on $\mathcal{Q}$ for the renormalized limit measure ν_∞, as given in §2.2.

LEMMA 3.29. *On $\mathcal{Q}$, the metrics $g_{\mathcal{Q},j}$ converge smoothly to g_∞ in the Cheeger–Gromov topology, and the H_j converge smoothly to $\nabla_{g_\infty}\log\chi$.*

Proof. Given a point $q\in\mathcal{Q}$, we can find a coordinate neighborhood $\mathcal{O}$ with local coordinates $u_1,...,u_d$. Let $\widehat{\mathcal{O}}_j$ denote the universal cover of $F_j^{-1}(\mathcal{O})$ endowed with the pull-back metric $\tilde{g}_{\boldsymbol{\omega}_j^\dagger}$. The deck transformation group of $\widetilde{\mathcal{O}}_j$ is Γ. Then, by Theorem 3.25, we know $(\widetilde{\mathcal{O}}_j,\tilde{g}_{\boldsymbol{\omega}_j^\dagger},\Gamma)$ equivariantly C^k-converges to a limit $(\widetilde{\mathcal{O}}_\infty,\tilde{g}_\infty,N)$ for any $k\in\mathbb{Z}_+$. By (3.11) and (3.10), we may assume $\pi_j\circ F_j$ converges smoothly to a Riemannian submersion $F_\infty\colon\widetilde{\mathcal{O}}_\infty\to\mathcal{O}$. In particular, $u_\alpha\circ F_j$ converges smoothly to $u_\alpha\circ F_\infty$. So, for any α and β, we have

$$\lim_{i\to\infty}\langle\nabla_{\tilde{g}_{\boldsymbol{\omega}_j^\dagger}}(u_\alpha\circ F_j),\nabla_{\tilde{g}_{\boldsymbol{\omega}_j^\dagger}}(u_\beta\circ F_j)\rangle=\langle\nabla_{\tilde{g}_\infty}(u_\alpha\circ F_\infty),\nabla_{\tilde{g}_\infty}(u_\beta\circ F_\infty)\rangle.$$

It follows that the quotient metric $g_{\mathcal{Q},j}$ converges smoothly to g_∞ in the coordinates $\{u_\alpha\}$.

For the second statement, we notice that the second fundamental form Π_j of fibers of F_j can be computed in terms of the derivatives of $u_\alpha\circ F_j$. In particular, Π_j also converges smoothly to a limit Π_∞, which is the second fundamental form of the fibers of F_∞. So, the corresponding mean curvature vectors H_j converge to H_∞. It is an easy calculation that H_∞ descends to the vector field $\nabla_{g_\infty}\log\chi$ on $\mathcal{Q}$. □

Proof of Theorem 3.27. We will apply Proposition 2.12. As in §2.4, we may identify an element $\xi\in\Omega^+(\mathcal{Q}_j)\otimes\mathbb{R}^3$ with a (3×3)-matrix-valued function $\boldsymbol{f}$ on $\mathcal{Q}_j$, and ξ is $\mathcal{N}$-invariant if and only if $\boldsymbol{f}$ is $\mathcal{N}$-invariant, and hence descends to a function on $\mathcal{Q}$. We define the Banach space $\mathfrak{A}$ (resp. $\mathfrak{B}$) to be the completion of the space of $\mathcal{N}$-invariant elements in $\Omega^+(\mathcal{Q}_j)\otimes\mathbb{R}^3$ under the $C^{2,\alpha}_{g_{\mathcal{Q},j}}$ (resp. $C^{\alpha}_{g_{\mathcal{Q},j}}$) norm. Then, by Proposition 3.26, for $\eta>0$ small we know the map $\mathscr{F}\colon B_\eta(\mathbf{0})\subset\mathfrak{A}\to\mathfrak{B}$ as given by (2.5) is well defined, and $\|\mathscr{F}(\mathbf{0})\|\leqslant C\tau_j$ for some constant $C>0$.

For any $\mathcal{N}$-invariant function f on $\mathcal{Q}_j$, we have

$$\Delta_{\boldsymbol{\omega}_j^\dagger}f=\Delta_{g_{\mathcal{Q},j}}f+\langle H_j,\nabla_{g_{\mathcal{Q},j}}f\rangle.$$

As in §2.3, the Bakry–Émery Laplace operator on $(\mathcal{Q}, g_\infty, \nu_\infty)$ is given by

$$\Delta_{\nu_\infty} f \equiv \Delta_{g_\infty} f + \langle \nabla_{g_\infty} \log \chi, \nabla_{g_\infty} f \rangle,$$

where $\nu_\infty = \chi\, d\mathrm{vol}_{g_\infty}$. Let

$$\mathscr{L}(\boldsymbol{f}) \equiv \Delta_{\nu_\infty} \boldsymbol{f} \quad \text{and} \quad \mathscr{N}(\boldsymbol{f}) \equiv \mathscr{F}(\boldsymbol{f}) - \mathscr{L}(\boldsymbol{f}).$$

Then, using the above convergence and the definition of $\mathscr{F}$, it is easy to see that, for $\boldsymbol{f}, \boldsymbol{g} \in B_\eta(\mathbf{0}) \subset \mathfrak{A}$, we have

$$\|\mathscr{N}(\boldsymbol{f}) - \mathscr{N}(\boldsymbol{g})\| \leqslant (C\eta + \varepsilon_j) \|\boldsymbol{f} - \boldsymbol{g}\|$$

for some $\varepsilon_j \to 0$. On the other hand, by standard elliptic theory, there exists a bounded linear operator $\mathscr{P}: \mathfrak{B} \to \mathfrak{A}$ such that $\mathscr{L} \circ \mathscr{P}(\boldsymbol{v}) = \boldsymbol{v}$, and $\|\mathscr{P}\boldsymbol{v}\| \leqslant L\|\boldsymbol{v}\|$ for some $L>0$ and all $\boldsymbol{v} \in \mathfrak{B}$. So, for i large, we may apply Proposition 2.12 to get a solution $\boldsymbol{f}_j$ satisfying $\mathscr{F}(\boldsymbol{f}_j) = 0$ such that (3.12) holds for $k=2$. For $k \geqslant 2$, (3.12) follows from standard elliptic estimates. □

Now, we draw a few consequences of Theorem 3.27.

Corollary 3.30. (Fibers are Nil) *In the statement of Theorem 3.25, we may assume that Γ is contained in N_L, so that the collapsing fibers are nilmanifolds.*

Proof. Locally on a coordinate chart $\mathcal{O} \subset \mathcal{Q}$, we can trivialize the fibration as

$$\mathcal{O} \times (\Gamma \backslash N).$$

On the universal cover $\widetilde{\mathcal{O}}_j$ of $F_j^{-1}(\mathcal{O})$, the action of Γ preserves the hyperkähler triple $F_j^*(\boldsymbol{\omega}_j^\diamondsuit)$. It also acts by affine transformations on N. On the other hand, N_L acts transitively on the fibers of the local universal cover. Given any $\phi \in \Gamma$, we can find an element $\psi \in N_L$ such that $\psi \circ \phi$ fixes a section of F_j. By Theorem 3.27, $\boldsymbol{\omega}_j^\diamondsuit$ is $\mathcal{N}$-invariant, and hence $\psi \circ \phi$ preserves the hyperkähler triple $F_j^*(\boldsymbol{\omega}_j^\diamondsuit)$. As in the proof of Proposition 3.1, we know that the fixed point set of $\psi \circ \phi$ is either isolated or open. As it is not isolated, it follows that $\psi \circ \phi$ must be the identity, and hence $\phi \in N_L$. □

When $d=2$, by the discussion in §3.2, the limit metric g_∞ on $\mathcal{R}$ is special Kähler.

Corollary 3.31. (Local integral monodromy) *g_∞ has local integral monodromy.*

Proof. For each j, the metric $\boldsymbol{\omega}_j^\diamondsuit$ is $\mathcal{N}$-invariant. Locally consider a trivialization of the fibration $\mathcal{O} \times (\Gamma \backslash N)$, where N is the abelian group $\mathbb{R}^2$ and Γ is a lattice. Choose an integral basis $(\partial_{t_1}, \partial_{t_2})$ of Γ, then over the local universal cover as in §3.2 we may find moment maps (x_1, x_2) for the symplectic form $\omega_{i,1}$, which serve as local coordinates on $\mathcal{Q}$. These are not canonical but are unique up to $\mathbb{R}^2 \rtimes \mathrm{SL}(2;\mathbb{Z})$. This shows that the quotient metric $g^\diamondsuit_{\mathcal{Q},j}$ on $\mathcal{Q}$ is naturally a special Kähler metric with integral monodromy. Then, the conclusion follows from Lemma 3.29 and the discussion in Remark 3.18. □

4. Singularity structure I: Case $d=3$

4.1. Main results

We first state the main results of this section.

THEOREM 4.1. (Local version) *Let (X_j^4, g_j, p_j) be a sequence of hyperkähler manifolds such that $\overline{B_2(p_j)}$ is compact and*

$$(X_j^4, g_j, \nu_j, p_j) \xrightarrow{\text{mGH}} (X_\infty^3, d_\infty, \nu_\infty, p_\infty)$$

with $\dim_{\text{ess}}(X_\infty^3)=3$. Assume that the singular set $\mathcal{S}$ consists of a single point p_∞. Then, the following statements hold.

(1) *p_∞ is a conical singularity. More precisely, there exists $\delta>0$ such that the corresponding flat background geometry $(B_\delta(p_\infty)\setminus\{p_\infty\}, g_\infty^\flat)$ is isomorphic to a punctured neighborhood of the origin in $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$, and $g_\infty=Vg_\infty^\flat$ for a smooth positive harmonic function of the form $V=\sigma r^{-1}+V_0$, where $\sigma\in[0,\infty)$ and V_0 is orbifold smooth.*

(2) *If in addition*

$$\int_{B_2(p_j)} |\mathrm{Rm}_{g_j}|^2 \, d\mathrm{vol}_{g_j} \leqslant \kappa_0 \tag{4.1}$$

uniformly for some $\kappa_0>0$, then p_∞ is an orbifold singularity, i.e., the function V in statement (1) *is orbifold smooth and $\sigma=0$.*

Remark 4.2. It is not hard to see that (4.1) is equivalent to a uniform bound on the Euler characteristic. Notice that, without assuming (4.1), the constant σ does not have to vanish. As an example, consider the flat orbifold $Y_k=\mathbb{C}^2/\mathbb{Z}_{k+1}$, where $\mathbb{Z}_{k+1}\subset\mathrm{SU}(2)$ is the standard diagonal subgroup acting on $\mathbb{C}^2$. As k tends to infinity, Y_k collapses to

$$\left(\mathbb{R}^3, \frac{1}{2r} g_{\mathbb{R}^3}\right).$$

Now, let X_k be the minimal resolution of Y_k endowed with an ALE hyperkähler metric g_k such that the exceptional set has diameter comparable to k^{-1}. Choose a point p_k on the exceptional set in X_k, then (X_k, g_k, p_k) also collapses to

$$\left(\mathbb{R}^3, \frac{1}{2r} g_{\mathbb{R}^3}, 0\right).$$

Here, we have $\chi(X_k)=k+1\to\infty$. In this example, we also see an infinite bubble tree of ALE gravitational instantons. We will show that the above cannot occur under the assumption (4.1); see Proposition 7.1.

THEOREM 4.3. (Compact version) *Let g_j be a sequence of hyperkähler metrics on the* K3 *manifold $\mathcal{K}$ with* $\operatorname{diam}_{g_j}(\mathcal{K})=1$ *such that*

$$(\mathcal{K}, g_j, \nu_j) \xrightarrow{\text{mGH}} (X^3_\infty, d_\infty, \nu_\infty).$$

Then, (X^3_∞, d_∞) is isometric to a flat orbifold $\mathbb{T}^3/\mathbb{Z}_2$ and ν_∞ is a multiple of the Hausdorff measure on $\mathbb{T}^3/\mathbb{Z}_2$.

THEOREM 4.4. (Complete version) *Let (X^4_j, g_j, p_j) be a sequence of hyperkähler manifolds such that*

$$(X^4_j, g_j, \nu_j, p_j) \xrightarrow{\text{mGH}} (X^3_\infty, d_\infty, \nu_\infty, p_\infty).$$

Assume that X^3_∞ is complete non-compact and the singular set $\mathcal{S}$ is finite. Then, the following holds.

(1) *The corresponding flat background geometry of X^3_∞ is a complete flat orbifold of the form $\mathbb{R}^3/\Gamma$, where Γ is a subgroup of $\mathbb{R}^3 \rtimes \mathbb{Z}_2$. More precisely, we have the following classification* (*in terms of the asymptotic volume growth*):

(a) *Euclidean space $\mathbb{R}^3$, and its quotient $\mathbb{R}^3/\mathbb{Z}_2$;*

(b) *flat product $\mathbb{R}^2 \times S^1$, and its quotient $(\mathbb{R}^2 \times S^1)/\mathbb{Z}_2$;*

(c) *flat product $\mathbb{R} \times \mathbb{T}^2$, and its quotient $(\mathbb{R} \times \mathbb{T}^2)/\mathbb{Z}_2$.*

(2) *In case* (a), *the positive harmonic function V is of the form $\sigma r^{-1}+c$ with $\sigma \geqslant 0$ and $c \in \mathbb{R}$; in cases* (b) *and* (c), *V must be a constant.*

(3) *Assume that*

$$\int_{X^4_j} |\mathrm{Rm}_{g_j}|^2 \, d\mathrm{vol}_{g_j} \leqslant \kappa_0 \tag{4.2}$$

uniformly for some $\kappa_0 > 0$. Then, V must be a constant.

4.2. Asymptotic analysis near the singularity

Now we focus on the local situation in the setting of Theorem 4.1. The discussion in §3.1 implies that there is a special affine metric g_∞ on $B_2(p_\infty) \setminus \{p_\infty\}$. We fix a choice of the harmonic function V, and denote by $g^\flat_\infty \equiv V^{-1} g_\infty$ the flat background metric. The main goal of this subsection is to obtain a lower bound of V near p_∞ (Corollary 4.11), which gives control of the flat background geometry near p_∞ (Proposition 4.12).

We start with a simple lemma. The proof follows directly from the volume comparison theorem for the renormalized limit measure ν_∞.

LEMMA 4.5. *For any* $r\in\big(0,\frac{1}{10}\big)$, *consider the annulus* $A_{r,2r}(p_\infty)$ *centered at the singular point* p_∞. *Let* $\{x_\alpha\}_{\alpha=1}^N\subset A_{r,2r}(p_\infty)$ *be a* $\frac{1}{4}r$*-dense subset such that* $\{B_{r/20}(x_\alpha)\}_{\alpha=1}^N$ *are disjoint. Then, the following statements hold*:

(1) $A_{r,2r}(p_\infty)\subset\bigcup_{\alpha=1}^N B_{r/4}(x_\alpha)\subset A_{r/4,9r/4}(p_\infty)$;

(2) *there is a uniform constant* $N_0>0$ *independent of* r *such* $N\leqslant N_0$.

Let $\mathcal{C}_1(r), \mathcal{C}_2(r), ..., \mathcal{C}_\ell(r)$ be the connected components of the union $\bigcup_{\alpha=1}^N B_{r/4}(x_\alpha)$. Obviously, $\ell\leqslant N\leqslant N_0$. The following is a direct application of Theorem 2.8.

LEMMA 4.6. (Harnack inequality) *There is a uniform constant* $c_0>0$ *independent of* r *and the choice of the covering, such that, for any* $x,y\in A_{r,2r}(p_\infty)\cap\mathcal{C}_k(r)$ *with* $1\leqslant k\leqslant\ell$,

$$c_0^{-1}\leqslant\frac{V(x)}{V(y)}\leqslant c_0.$$

PROPOSITION 4.7. *There exists a constant* $\ell_0>0$ *such that*

$$\sup_{S_r(p_\infty)} V\geqslant \ell_0\cdot r^{3/2}$$

for all $r\in(0,1]$.

Proof. Suppose not, then there are a sequence of numbers $r_i\to0$ such that

$$\sup_{S_{r_i}(p_\infty)} V\leqslant r_i^{3/2}. \tag{4.3}$$

Since $\Delta_{\nu_\infty}V=0$ on $A_{r_i,1}(p_\infty)$, applying Lemma 4.6 and Theorem 2.8, we have that

$$\sup_{A_{r_i,2r_i}(p_\infty)}(|V|+r_i|\nabla_{g_\infty}V|)\leqslant Cr_i^{3/2}.$$

For any Lipchitz function ϕ with $\mathrm{Supp}(\phi)\subset A_{r_i,1}(p_\infty)$, using integration by parts,

$$\int_{A_{r_i,1}(p_\infty)}\langle\nabla_{g_\infty}V,\nabla_{g_\infty}\phi\rangle_{g_\infty}\,d\nu_\infty=0. \tag{4.4}$$

We choose a cut-off function χ_i with $\mathrm{Supp}(\chi_i)\in A_{r_i,1}(p_\infty)$, $\chi_i\equiv1$ on $A_{2r_i,1/2}(p_\infty)$, and

$$\sup_{A_{r_i,2r_i}(p_\infty)}|\nabla_{g_\infty}\chi_i|_{g_\infty}\leqslant C\cdot r_i^{-1}\quad\text{and}\quad\sup_{A_{1/2,1}(p_\infty)}|\nabla_{g_\infty}\chi_i|_{g_\infty}\leqslant C.$$

Applying $\phi\equiv\chi_i\cdot V$ to (4.4), we obtain

$$\begin{aligned}\int_{A_{2r_i,1/2}(p_\infty)}|\nabla_{g_\infty}V|^2_{g_\infty}\,d\nu_\infty&\leqslant\int_{A_{r_i,2r_i}(p_\infty)\cup A_{1/2,1}(p_\infty)}V\cdot|\nabla_{g_\infty}V|_{g_\infty}\cdot|\nabla_{g_\infty}\chi_i|_{g_\infty}\,d\nu_\infty\\&\leqslant C.\end{aligned}$$

Letting $r_i \to 0$, we find that

$$\int_{B_{1/2}(p_\infty)\setminus\{p_\infty\}} |\nabla_{g_\infty} V|^2_{g_\infty}\, d\nu_\infty < \infty \tag{4.5}$$

Now, we claim that V is a harmonic function on $B_{1/2}(p_\infty)$, in the sense of Definition 2.6. First, by [13, Theorem 5.1], given a Lipschitz function, the minimal upper gradient can be characterized by the local slope. Also, applying [10, Lemma 1.42], $\mathrm{Mod}_2(\{p_\infty\})=0$. Thus, the function $u\colon B_{1/2}(p_\infty)\to\mathbb{R}\cup\{\infty\}$ defined by setting $u(x)\equiv|\nabla_{g_\infty} V|$ for $x\neq p_\infty$ and $u(p_\infty)=\infty$, is a minimal weak upper gradient of V on $B_{1/2}(p_\infty)$. So, (4.5) implies $V\in W^{1,2}(B_{1/2}(p_\infty))$, and the Cheeger energy is given by

$$\mathrm{Ch}(V) = \int_{B_{1/2}(p_\infty)} |\nabla_{g_\infty} V|^2\, d\nu_\infty.$$

Moreover, applying similar arguments as in the proof of (4.5), one can see that (4.4) implies that

$$\int_{B_{1/2}(p_\infty)} \langle \nabla_{g_\infty} V, \nabla_{g_\infty}\phi\rangle d\nu_\infty = 0,$$

for any compactly supported Lipschitz function ϕ on $B_{1/2}(p_\infty)$. This proves the claim. Now, by Theorem 2.7, we obtain

$$\operatorname*{ess\,inf}_{B_{1/4}(p_\infty)} V \geqslant C\cdot\left(\int_{B_{1/2}(p_\infty)} V^2 d\nu_\infty\right)^{1/2} \geqslant c_0 > 0.$$

This contradicts (4.3). □

Proposition 4.8. *Any tangent cone Y at p_∞ satisfies*

$$\dim_{\mathrm{ess}}(Y) = 3.$$

Proof. We rule out the possible occurence of lower-dimensional tangent cones. Suppose that $(Y,\bar{p})$ is a tangent cone at p_∞ with $\dim_{\mathrm{ess}}(Y)\in\{1,2\}$. Then, we can find a sequence $r_j\to 0$ such that the rescaled annulus $r_j^{-1}\cdot\mathcal{A}_j$ converges to an annulus in Y, where $\mathcal{A}_j\equiv A_{r_j,2r_j}(p_\infty)$. By Proposition 4.7, there is a point $q_j\in\mathcal{A}_j$ with $V(q_j)\geqslant l_0 r_j^{3/2}$. Without loss of generality, we may assume that q_j belongs to the connected component $\mathcal{C}_1(r_j)$ in the covering constructed in Lemma 4.5. Set $\mathcal{A}_j^1\equiv\mathcal{C}_1(r_j)\cap\mathcal{A}_j$. We may also assume the rescaled space $r_j^{-1}\cdot\mathcal{A}_j^1$ converges to a connected open set in Y. By Lemma 4.6, we have $c_0^{-1}\leqslant V/V(q_j)\leqslant c_0$ uniformly on $\mathcal{A}_j^1$. This implies the flat background metric $V(q_j)g_\infty^\flat = V(q_j)V^{-1}g_\infty$ on $\mathcal{A}_j^1$ is uniformly equivalent to g_∞. Hence, the corresponding rescaled sequence of flat manifolds $(\mathcal{A}_j^1, r_j^{-2}V(q_j)g_\infty^\flat)$ also collapses to a lower-dimensional space. By the discussion in §3.1, we can find a totally geodesic torus $\mathbb{T}_j\subset(\mathcal{A}_j^1, g_\infty^\flat)$ passing q_j, whose diameter with respect to the metric $r_j^{-2}V(q_j)g_\infty^\flat$ is $\varepsilon_j\to 0$. So, the diameter of $\mathbb{T}_j$ with respect to the metric $g_\infty^\flat$ is

$$\varepsilon_j r_j V(q_j)^{-1/2}\to 0.$$

Now, choose a point $w\in A_{1,2}(p_\infty)$ and a smooth curve γ_j in $A_{r_j/2,2}$ connecting w and q_j. We can slide the torus $\mathbb{T}_j$ along γ_j, and obtain a totally geodesic torus $\mathbb{T}_j'$ (with respect to $g_\infty^\flat$) passing through w. Notice in this process that we can keep the family of flat tori along γ_j to be outside $\mathcal{A}_{j+1}$ (in particular, we do not encounter the singularity p_∞). Since the diameter of the tori is invariant along the sliding, we then obtain a sequence of totally geodesic tori contained in $A_{1,2}(p_\infty)$ with diameter going to zero. This is clearly impossible. $\square$

LEMMA 4.9. *There exists a constant $\delta_0>0$ such that, for every $r\in\left(0,\frac{1}{3}\right)$, any two points in $A_{r/2,r}(p_\infty)$ can be connected by a smooth curve $\gamma\subset A_{\delta_0\cdot r,3r}(p_\infty)$ with arc-length $|\gamma|\leqslant 10r$. In particular, $B_1(p_\infty)\setminus\{p_\infty\}$ is path-connected.*

Proof. We argue by contradiction. Suppose that there are sequences $\delta_j\to 0$, $r_j\in(0,1)$ and sequences of points $x_j, y_j\in A_{r_j/2,r_j}(p_\infty)$ such that any smooth curve γ_j connecting x_j and y_j with $|\gamma_j|\leqslant 10r_j$ satisfies $\gamma_j\cap B_{\delta_j\cdot r_j}(p_\infty)\neq\varnothing$. Choose minimizing geodesics σ_{x_j} and σ_{y_j} from p_∞ to x_j and y_j, respectively. Then, we take two points

$$\underline{x}_j\in\sigma_{x_j}\cap S_{3\delta_j\cdot r_j}(p_\infty)\quad\text{and}\quad \underline{y}_j\in\sigma_{y_j}\cap S_{3\delta_j\cdot r_j}(p_\infty).$$

By assumption, the following conditions hold:

(a) any minimizing geodesic $\bar\gamma_j$ connecting x_j and y_j must satisfy

$$\bar\gamma_j\cap B_{\delta_j\cdot r_j}(p_\infty)\neq\varnothing;$$

(b) any smooth curve $\underline{\gamma}_j$ connecting $\underline{x}_j$ and $\underline{y}_j$ with length $|\underline{\gamma}_j|\leqslant 4r_j$ must satisfy

$$\underline{\gamma}_j\cap B_{\delta_j\cdot r_j}(p)\neq\varnothing.$$

Now, we define the rescaled metric

$$\hat d_j\equiv\delta_j^{-1}\cdot r_j^{-1}\cdot d_\infty.$$

Letting $j\to\infty$ and passing to a subsequence, we obtain

$$(X_\infty^3, \hat d_j, p_\infty)\xrightarrow{\text{GH}}(\widehat Z_\infty, \hat d_\infty, \hat z_\infty), \tag{4.6}$$

where $(\widehat{Z}_\infty, \hat{d}_\infty, \hat{z}_\infty)$ is a tangent cone at $p_\infty \in X^3_\infty$, and $\underline{x}_j, \underline{y}_j$ converge to $\underline{x}_\infty, \underline{y}_\infty \in S_3(\hat{z}_\infty)$ respectively. By the discussion in §3.1, the convergence is smooth away from $\widehat{z}_\infty$.

Now, choose a sequence of minimizing geodesics $\bar{\gamma}_j$ connecting x_j and y_j. Since

$$\hat{d}_j(x_j, p_\infty) \geqslant \tfrac{1}{2}\delta_j^{-1} \quad \text{and} \quad \hat{d}_j(y_j, p_\infty) \geqslant \tfrac{1}{2}\delta_j^{-1} \tag{4.7}$$

as $j \to \infty$, it follows from the Arzelà–Ascoli lemma that by passing to a further subsequence, $\bar{\gamma}_j$ converges to a geodesic line $\bar{\gamma}_\infty \subset \widehat{Z}_\infty$. Applying Cheeger–Colding's splitting theorem, $\widehat{Z}_\infty$ is isometric to $\mathbb{R} \times W$ for a complete length space W. If W is compact, then we can slow down the rescaling slightly and obtain a tangent cone $\mathbb{R}$ at p_∞, which contradicts Proposition 4.8. So W must be non-compact. Then, it follows easily that the complement $\widehat{Z}_\infty \setminus B_2(\hat{z}_\infty)$ is path connected. In particular, we can find a smooth curve $\underline{\sigma}_\infty \subset \widehat{Z}_\infty \setminus B_2(\hat{z}_\infty)$ connecting $\underline{x}_\infty$ and $\underline{y}_\infty$. Set $\ell_0 = |\underline{\sigma}_\infty|$. Then, passing back to the sequence, for j large, we see that $\underline{x}_j$ and $\underline{y}_j$ can be connected by a smooth curve $\underline{\sigma}_j \subset A_{2\delta_j \cdot r_j, (l_0+10)\delta_j \cdot r_j}(p_\infty)$ of length $|\underline{\sigma}_j| \leqslant (\ell_0 + 10) \cdot \delta_j \cdot r_j$. This contradicts item (b). □

As an immediate consequence, we obtain an improvement of Lemma 4.6 and Proposition 4.7.

Corollary 4.10. *There exists a constant $C_0 > 0$ such that, for any $r \in (0,1)$ and $x, y \in A_{r/2,r}(p_\infty)$, we have*

$$C_0^{-1} \leqslant \frac{V(x)}{V(y)} \leqslant C_0.$$

Corollary 4.11. *There exists a constant $\ell_0 > 0$ such that*

$$\inf_{S_r(p_\infty)} V \geqslant \ell_0 \cdot r^{3/2}$$

for all $r \in (0,1]$.

The above corollary immediately implies the following.

Proposition 4.12. *The metric completion of the flat background*

$$(B_1(p_\infty) \setminus \{p_\infty\}, V^{-1} g_\infty)$$

at p_∞ is given by adding a single point.

In particular, we can identify this metric completion topologically as $B_1(p_\infty)$ itself, and we denote by $d^\flat_\infty$ the metric induced by the flat metric $g^\flat_\infty$. At this point, we encounter a non-standard *singularity removal* question.

Question 4.13. Let $\mathcal{U}$ be a connected smooth Riemannian manifold in dimension $m\geqslant 3$ with uniformly bounded sectional curvature. If the metric completion $\overline{\mathcal{U}}$ is obtained by adding *one point* p such that $\overline{\mathcal{U}}$ is locally compact, is it true that p is a Riemannian orbifold singularity?

In our setting, we are only interested in the special case when the Riemannian metric is flat. Even in this case, the above innocent looking question seems to be subtle. There is an analogous statement when $m=2$, but one needs to allow a general conical singularity. Notice that the conclusion fails if the metric completion is not locally compact; for an example in dimension 2, consider the universal cover of $\mathbb{R}^2\setminus\{0\}$ equipped with the flat metric. In the next subsection we get around this technical point in our setting, using the fact the conformal metric g_∞ is a Ricci limit space.

4.3. Proof of Theorem 4.1

By Lemma 2.2, the isometry classes $\mathcal{T}_{p_\infty}$ of all tangent cones at p_∞ is compact in $(\mathcal{M}et, d_{\mathrm{GH}})$. Let $(Y,p^*)\in\mathcal{T}_{p_\infty}$ satisfy

$$(X_\infty, r_i^{-1}d_\infty, p_\infty)\xrightarrow{\mathrm{GH}}(Y, d_Y, p^*)$$

for $r_i\to 0$. By the discussion at the end of §3.1, we know that, away from p^*, the convergence is smooth and there is a special affine metric on $Y\setminus\{p^*\}$. Notice that, by Lemma 4.9, for all $r>0$, any two points in $A_{r/2,r}(p^*)\subset Y$ can be connected by a smooth curve $\gamma\subset A_{\delta_0\cdot r,3r}(p^*)$ with arc-length $|\gamma|\leqslant 20r$. In particular, $Y\setminus\{p^*\}$ is path-connected and Y has only one end at infinity. By Proposition 4.10, the flat background $(Y\setminus\{p^*\}, g_Y^\flat)$ has a 1-point completion near p^* and is homeomorphic to Y. We always normalize the harmonic function $\widehat{V}_Y^*$ by a multiplicative constant so that

$$\sup_{S_1(p^*)}\widehat{V}_Y^*=1.$$

LEMMA 4.14. *For every $\varepsilon>0$, there exists a tangent cone (Y,p^*) such that*

$$\limsup_{R\to+\infty}\frac{|\widehat{V}_Y^*|_{L^\infty(S_R(p^*))}}{R^{3/2+\varepsilon}}\leqslant 2. \tag{4.8}$$

Proof. We argue by contradiction. Suppose the conclusion fails for some $\varepsilon_0>0$. Then for every tangent cone (Y,p^*), we can find $R_Y>1$ such that

$$\sup_{S_{R_Y}(p^*)}\widehat{V}_Y^*>2\cdot R_Y^{3/2+\varepsilon_0}. \tag{4.9}$$

Claim. *Given a tangent cone* (Y,p^*), *there exists* $\tau=\tau(Y)>0$ *such that, for any* $(W,q^*)\in B_\tau((Y,p^*))\cap\mathcal{T}_{p_\infty}$, *we have*

$$\sup_{S_{R_Y}(q^*)}\widehat{V}^*_W>\frac{3}{2}\cdot R_Y^{3/2+\varepsilon_0}.$$

Indeed, if not, then we can find a sequence of tangent cones (W_i,q_i^*) *converging to* (Y,p^*) *in the Gromov–Hausdorff topology, such that*

$$\sup_{S_{R_Y}(q_i^*)}\widehat{V}^*_{W_i}\leqslant\frac{3}{2}\cdot R_Y^{3/2+\varepsilon_0}.$$

Applying the Harnack inequality and using the convergence of special affine metrics discussed in §3.1, $\widehat{V}^*_{W_i}$ *converges uniformly away from* p^* *to* $\widehat{V}^*_Y$. *This contradicts* (4.9).

Since $\mathcal{T}_{p_\infty}$ is compact in $(\mathcal{M}et,d_{\mathrm{GH}})$, it can be covered by finite metric balls of the form $B_{\tau_\ell}((Y_\ell,p_\ell^*))$, $\ell=1,...,N$. By the claim, it follows that, for any $(Y,p^*)\in\mathcal{T}_{p_\infty}$, we have $\sup_{S_{R_{Y_\ell}}(p^*)}\widehat{V}^*_Y>(R_{Y_\ell})^{3/2+\varepsilon_0}$ for some $1\leqslant\ell\leqslant N$. Then using a simple contradiction argument, one can show that, for all $0<r\ll1$, there exists $\ell_0\in\{1,...,N\}$ with $R_0\equiv R_{Y_{\ell_0}}$ such that $\sup_{B_r(p_\infty)}V<R_0^{-3/2-\varepsilon_0}\sup_{B_{R_0\cdot r}(p_\infty)}V$. By iteration, we obtain a sequence $r_i\to0$ with $\sup_{S_{r_i}(p_\infty)}V\leqslant Cr_i^{3/2+\varepsilon_0}$, which contradicts Corollary 4.11. □

Now, we fix $\varepsilon=\frac{1}{4}$ and let (Y,d_Y,p^*) be a tangent cone given in Lemma 4.14.

Proposition 4.15. *The associated flat background geometry on* (Y,d_Y,p^*) *is complete at infinity.*

Proof. For R large, we consider the annulus $A_{R,2R}(p^*)$ in Y with respect to the metric d_Y. By Lemma 4.14, we have $\widehat{V}^*_Y\leqslant8R^{7/4}$ on $A_{R,2R}(p^*)$. Notice the flat background metric $g_Y^\flat=(\widehat{V}^*_Y)^{-1}g_Y$. Given any smooth curve $\gamma\colon[0,L]\to A_{R,2R}(p^*)$ connecting $S_R(p^*)$ and $S_{2R}(p^*)$, which is parameterized by the arc-length with respect to the metric $g_Y^\flat$, its length with respect to g_Y satisfies $L_{g_Y}(\gamma)\leqslant4LR^{7/8}$. Since $L_{g_Y}(\gamma)\geqslant R$, we see that $L\geqslant\frac{1}{4}R^{1/8}$. From this, it is easy to draw the conclusion. □

We will use the following classification result for flat ends of Riemannian manifolds.

Theorem 4.16. (Eschenburg–Schroeder [31]) *Let* Z *be a flat end in a complete Riemannian manifold* (X^n,g). *Then, there exists a compact subset* K *such that* $Z\setminus K$ *is isometric to the interior of* $(\Omega\times\mathbb{R}^k)/\Gamma$ *and one of the following three cases hold*:

(A) $\dim(\Omega)=1$, $\Omega=\mathbb{R}_+$ *and* Γ *is a Bieberbach group on* $\mathbb{R}^{n-1}$;

(B) $\dim(\Omega)=2$: Ω *is diffeomorphic to* $\mathbb{R}\times\mathbb{R}_+$ *and* Γ *is a Bieberbach group on* $\mathbb{R}\times\mathbb{R}^{n-2}$, *which preserves the Riemannian product structure of* $\Omega\times\mathbb{R}^k$;

(C) $\dim(\Omega)\geqslant3$: Ω *is the complement of a ball in* $\mathbb{R}^{n-k}$, *and* Γ *is a finite extension of a Bieberbach group on* $\mathbb{R}^k$.

PROPOSITION 4.17. *$(Y, d_Y^\flat)$ is isometric to the Euclidean space $\mathbb{R}^3$ or the flat cone $\mathbb{R}^3/\mathbb{Z}_2$, and $\widehat{V}_Y^*{=}1$.*

Proof. Propositions 4.12 and 4.15 imply that $(Y, d_Y^\flat)$ has one complete end at infinity. Since an asymptotic cone of (Y, d_Y) is itself a tangent cone at p_∞, by Proposition 4.8 the asymptotic cones of (Y, d_Y) must all be 3-dimensional, then by the discussion at the end of §3.1 we know the asymptotic cones of $(Y, d_Y^\flat)$ are also 3-dimensional. Applying Theorem 4.16 to the end of $(Y, d_Y^\flat)$, we see that we are in Case (C) and Γ is finite (the other cases have collapsed asymptotic cones). So, $(Y, d_Y^\flat)$ is isometric to either $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$ outside a compact set. Then, using the developing map and the fact that the metric singularity of $(Y, d_Y^\flat)$ consists of at most one point, we conclude that $(Y, d_Y^\flat)$ must be isometric to $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We first prove item (1). Let (Y, p^*) be a tangent cone at p_∞ whose flat background geometry $(Y, g_Y^\flat)$ is the flat cone $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$. For simplicity of notation , we will assume that $(Y, g_Y^\flat)$ is $\mathbb{R}^3$. The other case can be dealt with in the same manner. We can find $r_i \to 0$ such that $(X_\infty^3, r_i^{-1} d_\infty, p_\infty)$ converges to (Y, d_Y, p^*). As before, we also have the convergence of the corresponding flat background geometry

$$(X_\infty^3, r_i^{-1} d_\infty^\flat, p_\infty) \xrightarrow{\mathrm{GH}} (\mathbb{R}^3, g_{\mathbb{R}^3}, 0).$$

This means that the annulus $r_i^{-1} A_{r_i/2, 2r_i}(p_\infty)$ converges to the flat annulus $A_{1/2,2}(0)$ in $\mathbb{R}^3$. In particular, we can find smooth hypersurfaces $\Sigma_i {\in} A_{r_i, 2r_i}(p_\infty)$ with constant curvature 1 such that $r_i^{-1}\Sigma_i$ converges to the unit sphere in $\mathbb{R}^3$. Then, by a simple argument using the developing map, one can see that a punctured neighborhood of p_∞ in $(X_\infty^3, d_\infty^\flat)$ can be isometrically embedded in $\mathbb{R}^3$ as a punctured domain. So, the flat background geometry is smooth near p_∞. Now, V can be viewed as a positive harmonic function in a punctured domain in $\mathbb{R}^3$. The singular behavior of V then follows from the classical Bôcher's theorem. This finishes the proof of item (1) of Theorem 4.1.

The rest of this subsection is devoted to the proof of item (2). We already know that the flat background geometry on the limit X_∞^3 is a flat orbifold near p_∞, and a neighborhood of p_∞ can be identified with an open set in $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$. Moreover, the positive harmonic function V is of the form $\sigma r^{-1} {+} V_0$, where r is the radial function on $\mathbb{R}^3$, σ is a positive constant, and V_0 extends smoothly as an orbifold harmonic function. It suffices to show that $\sigma{=}0$.

Suppose that $\sigma{>}0$. Notice that item (1) implies that the tangent cone (Y, p^*) at p_∞ is unique, and $(Y, d_Y^\flat)$ can be identified with $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$. After rescaling, we may assume that $\widehat{V}_Y^* {=} \frac{1}{2r}$. Notice that Y is a metric cone over a round 2-sphere with radius $\frac{1}{2}$. We may identify the cross section of the cone with $\Sigma=\{r=\frac{1}{2}\}\subset Y$. Let B_∞ be a small tubular neighborhood of Σ. Then, we can find a domain U_i contained in X_i^4 that converges to B_∞ with uniformly bounded curvature and all its covariant derivatives. Furthermore, by Theorem 3.25, there is a smooth fibration map $F_i\colon U_i\to B_\infty$ with fibers given by smooth circles with uniformly bounded second fundamental form and all covariant derivatives. Let $\Sigma_i=F_i^{-1}(\Sigma)$. Then, Σ_i collapses to Σ along the circle bundle with uniformly bounded curvature and covariant derivatives.

Given any point $q\in B_\infty$, by assumption, we can find $q_i\in U_i$ and $\delta>0$ such that the universal cover $\widetilde{B_\delta(q_i)}$ converges smoothly to a hyperkähler limit $\widetilde{B}_\infty$, and a neighborhood of q is given by the $\mathbb{R}$ quotient of $\widetilde{B}_\infty$. As $V=\frac{1}{2r}$, the limit metric on $\widetilde{B}_\infty$ is of the form

$$\frac{1}{2r}(dr^2+r^2g_{S^2})+2r\theta^2,$$

where θ is dual to the Killing field generating the $\mathbb{R}$ action. Changing the coordinate by $r=\frac{1}{2}s^2$, one can see that this metric is flat. Moreover, the local universal covers of Σ_i converge to some subset of the level set $\{r=\frac{1}{2}\}$ in $\widetilde{B}_\infty$ which has constant curvature 1. In particular, the sectional curvature of Σ_i converges uniformly to 1. It follows from Klingenberg's estimate that the universal cover $\widetilde{\Sigma}_i$ of Σ_i has a uniform lower bound on the injectivity radius. This also implies that the universal cover $\widetilde{U}_i$ of U_i converges smoothly to a flat manifold $\widetilde{U}_\infty$, and $\widetilde{\Sigma}_i$ converges smoothly to the round sphere $\widetilde{\Sigma}_\infty\subset\widetilde{U}_\infty$. Since the two boundary components of $\widetilde{U}_\infty$ are convex, applying Sacksteder's theorem [75], $\widetilde{U}_\infty$ is isometric to a tubular neighborhood of the round sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Setting $G_i\equiv\pi_1(U_i)$, we have the following diagram:

$$\begin{tikzcd}
(\widetilde{U}_i,\tilde{g}_i,G_i) \arrow[r, "\text{eqGH}"] \arrow[d, "\pi_i"'] & (\widetilde{U}_\infty,\tilde{g}_\infty,G_\infty) \arrow[d, "\pi_\infty"] \\
(U_i,g_i) \arrow[r, "GH"] & (U_\infty,g_\infty),
\end{tikzcd} \tag{4.10}$$

where $G_\infty\leqslant\mathrm{Isom}(\widetilde{U}_\infty)$ is a closed subgroup so that $U_\infty=\widetilde{U}_\infty/G_\infty$.

For our purposes, we need to investigate more closely the above convergence. Notice that we have fixed a choice of a hyperkahler triple $\boldsymbol{\omega}_i$ on each X_i. Then, we get a triple $\widetilde{\boldsymbol{\omega}}_i$ of 2-forms on $\widetilde{U}_i$ and, passing to a further subsequence, we may assume that these converge to a hyperkähler triple $\widetilde{\boldsymbol{\omega}}_\infty$ on $\widetilde{U}_\infty$. Since the limit metric on $\widetilde{U}_\infty$ is flat, we may assume that, via the embedding $\widetilde{U}_\infty\hookrightarrow\mathbb{R}^4$, $\widetilde{\boldsymbol{\omega}}$ is given by the restriction of the standard hyperkähler triple on $\mathbb{R}^4$. Notice that the G_i action on $\widetilde{U}_i$ preserves $\widetilde{\boldsymbol{\omega}}_i$, so G_∞ preserves the triple $\widetilde{\boldsymbol{\omega}}_\infty$. If follows that G_∞ is contained in $\mathrm{SU}(2)=\mathrm{Sp}(1)$.

Now, we can restrict our attention to the smooth convergence of $\widetilde{\Sigma}_i$ to $\widetilde{\Sigma}_\infty$. Since G_∞ is a closed subgroup in the compact Lie group $\mathrm{Aut}(\widetilde{\Sigma}_\infty, \widetilde{\boldsymbol{\omega}}_\infty)=\mathrm{SU}(2)$, it follows that there is a group isomorphism $\varphi_i\colon G_i\simeq\overline{G}_i<G_\infty$ (see [60, Lemma 3.2], for instance). Moreover, for any sufficiently large i, there exists a G_i-equivariant diffeomorphism $\mathscr{F}_i\colon\widetilde{\Sigma}_i\to\widetilde{\Sigma}_\infty$ such that the following conditions hold:

(1) $\mathscr{F}_i\circ\gamma=\varphi_i(\gamma)\circ\mathscr{F}_i$ for all $\gamma\in G_i$;

(2) $\mathscr{F}_i$ is an ε_i-Gromov–Hausdorff approximation with $\varepsilon_i\to 0$;

(3) for any unit tangent vector v, one has

$$\big||d\mathscr{F}_i(v)|-1\big|\leqslant\Psi(\varepsilon_i)\quad\text{and}\quad\lim_{\varepsilon_i\to 0}\Psi(\varepsilon_i)=0. \tag{4.11}$$

A key technique in constructing the above G_i-equivariant diffeomorphism $\mathscr{F}_i$ is to use the center-of-mass technique. We refer the readers to [42] and in [73, Theorem 2.7.1] for more details.

Now, we identify $\widetilde{\Sigma}_i$ with $\widetilde{\Sigma}_\infty$, and G_i with $\overline{G}_i$ using $\mathscr{F}_i$. Consider the form $\widetilde{\omega}^1_\infty$ on $\widetilde{\Sigma}=\mathbb{S}^3$ given by $dx^1\wedge dx^2+dx^3\wedge dx^4$ in the standard coordinates. One can write down a standard contact 1-form

$$\eta_\infty=\tfrac{1}{2}((x^1dx^2-x^2dx^1)+(x^3dx^4-x^4dx^3)),$$

with $\widetilde{\omega}^1_\infty=d\eta_\infty$, and η_∞ is SU(2)-invariant. So, for i large, one can also write $\widetilde{\omega}^1_i=d\eta_i$ such that η_i converges smoothly to η_∞. Then, we can average out η_i by the group G_i to make η_i invariant under G_i, and by (4.11) we may assume that η_i still converge to η_∞ in C^0. Notice that $d\eta_i=\widetilde{\omega}^1_i$ also converges to $\widetilde{\omega}^1_\infty$ in C^0. In particular, for i large η_i is a G_i-invariant contact 1-form. Moreover, there is an obvious isotopy of G_i-invariant contact 1-forms $\eta_t=t\eta_\infty+(1-t)\eta_i$ for $t\in[0,1]$. Applying Gray's stability theorem (see [38, Theorem 2.20] or [61, pp. 135–136] for more details), we conclude that η_i and η_∞ define isomorphic contact structures on $\widetilde{\Sigma}_\infty/\overline{G}_i$. We can now apply a result in contact geometry [69], which states that the minimal symplectic filling of the space $\widetilde{\Sigma}/\overline{G}_i=\mathbb{S}^3/\overline{G}_i$ has a unique diffeomorphism type. For simplicity, we will not distinguish the notations G_i and $\overline{G}_i$. Notice that, in our setting, the subset W_i enclosed by Σ_i inside M^4_i provides a minimal symplectic filling, but, on the other hand, the minimal resolution $\widetilde{\mathbb{C}^2/G_i}$ provides another, so in particular $\chi(W_i)=\chi(\widetilde{\mathbb{C}^2/G_i})$. Now, in our setting, $|G_i|\to+\infty$, so for i large G_i is either a finite cyclic subgroup $\mathbb{Z}_{k_i+1}\leqslant\mathrm{SU}(2)$ or a binary dihedral group $2D_{2(k_i-2)}$. It then follows that

$$\chi(\widetilde{\mathbb{C}^2/G_i})=\begin{cases} k_i+1, & \text{when } G_i=\mathbb{Z}_{k_i+1},\\ k_i+1, & \text{when } G_i=2D_{2(k_i-2)},\end{cases} \tag{4.12}$$

and hence $\chi(W_i)=\chi(\widetilde{\mathbb{C}^2/G_i})\to\infty$.

We recall the Chern–Gauss–Bonnet theorem on an Einstein 4-manifold (M^4,g) with boundary:

$$8\pi^2\chi(M^4)=\int_{M^4}|\mathrm{Rm}_g|^2\,d\mathrm{vol}_g+8\pi^2\int_{\partial M^4}\mathrm{TP}_\chi. \tag{4.13}$$

Denote by II and H the second fundamental form and the mean curvature of ∂M^4, respectively. Then, the above transgression of the Pfaffian is given by

$$\mathrm{TP}_\chi=\frac{1}{4\pi^2}\cdot\left(\lambda\cdot H-\mathrm{Rm}_{ikkj}\cdot\mathrm{II}_{ij}+\frac{1}{3}H^3+\frac{2}{3}\,\mathrm{Tr}(\mathrm{II}^3)-H\cdot|\,\mathrm{II}\,|^2\right)d\mathrm{vol}_{\partial M^4}, \tag{4.14}$$

where i, j and k are in the tangential direction of ∂M^4, and λ is the Einstein constant. Applying this to W_i, since the second fundamental form of Σ_i is uniformly bounded and the volume is collapsing, the boundary integral goes to zero. So, by (4.1), we obtain a uniform bound on $\chi(W_i)$. This yields a contradiction. □

Remark 4.18. In the above proof, we make use of the symplectic structure more than the Ricci-flat structure. It is possible to use signature formula on manifolds with boundary to give a proof, but we are not aware of the formula of the eta invariant on general collapsing manifolds (M_j^{2k+1},g_j). For a fixed manifold M^3 with collapsing metrics, the convergence of eta invariants is studied in [72].

4.4. Proof of Theorem 4.3

By the Chern–Gauss–Bonnet theorem,

$$\int_{\mathcal{K}}|\mathrm{Rm}_{g_j}|^2\,d\mathrm{vol}_{g_j}=192\pi^2.$$

Then, it follows from Corollary 3.22 that the singular set $\mathcal{S}$ consists of a finite number of points. The rest of the proof does not require the L^2 bound on curvature. We will only use item (1) of Theorem 4.1.

PROPOSITION 4.19. *(X^3_∞,d_∞) is isometric to a flat orbifold locally modeled on $\mathbb{R}^3/\mathbb{Z}_2$, and the limit measure ν_∞ is proportional to the Hausdorff measure.*

Proof. We consider the positive harmonic function V on $X^3_\infty\setminus\mathcal{S}$. Near each $p_\alpha\in\mathcal{S}$, we have $V=c_\alpha r_\alpha^{-1}+h_\alpha$, where $r_\alpha(x)\equiv d^\flat_\infty(x,p_\alpha)$ and $c_\alpha\in[0,\infty)$. Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ consisting of those p_α's with $c_\alpha>0$. Then, V is orbifold smooth on $X^3_\infty\setminus\mathcal{S}'$. On the other hand, $V\to\infty$ near $\mathcal{S}'$, so the minimum of V is achieved at some point in $X_\infty\setminus\mathcal{S}'$. By the strong maximum principle on the flat space, V is a constant. □

Let us review some standard facts regarding flat orbifolds. Bieberbach's theorem (see [82, Theorem 3.2.1]) states that, if $\Gamma \leqslant \mathrm{Isom}(\mathbb{R}^n)$ is a discrete and co-compact, then the lattice $\Lambda \equiv \Gamma \cap \mathbb{R}^n$ is normal in Γ with bounded index $[\Gamma:\Lambda] \leqslant w(n)$, and yields the following exact sequence:

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow H \longrightarrow 1, \tag{4.15}$$

where $H = \Gamma/\Lambda \leqslant O(n)$. We now consider a *closed* flat orbifold X^n. Applying Thurston's developability theorem (see [80, Chapter 13] or [11, Chapter III.$\mathcal{G}$]), the universal covering orbifold of X^n is isometric to $\mathbb{R}^n$, so that $X^n = \mathbb{R}^n/\Gamma$ for some discrete co-compact group $\Gamma \in \mathrm{Isom}(\mathbb{R}^n)$. Then, Biberbach's theorem implies that $X^n = \mathbb{T}^n/H$ for some finite group $H \leqslant O(n)$, where $\mathbb{T}^n \equiv \mathbb{R}^n/\Lambda$ is a flat torus.

In our setting, we can write $X^3_\infty = \mathbb{T}^3/H$ for some finite group $H \leqslant O(3)$. Let $q \in X^3_\infty$ be an orbifold point. Then, the tangent cone at q is isometric to $\mathbb{R}^3/\mathbb{Z}_2$, where the group $\mathbb{Z}_2$ is generated by the reflection

$$\begin{aligned} \iota: \mathbb{R}^3 &\longrightarrow \mathbb{R}^3, \\ x &\longmapsto -x. \end{aligned}$$

Moreover, ι induces an element in H with the fixed point q and $\det(\iota) = -1$. In particular, $H \not\subset \mathrm{SO}(3)$. Let us set $H_0 \equiv H \cap \mathrm{SO}(3)$. Then,

$$H = H_0 \cup (\iota \cdot H_0).$$

Next, we claim that any element $\gamma \in H_0$ acts freely on $\mathbb{T}^3$. If not, suppose that γ has some fixed point $x_0 \in \mathbb{T}^3$ and recall $\gamma \in H_0 \leqslant \mathrm{SO}(3)$. Then, γ fixes the rotation axis passing through x_0. However, this contradicts the assumption that X^3_∞ has only isolated singularities.

Now, since $\pi_1(\mathcal{K}) = \{1\}$, by [77] we know that $\pi_1(X^3_\infty) = \{1\}$. Then, we must have $H_0 = \{1\}$. This implies that $X^3_\infty = \mathbb{T}^3/\mathbb{Z}_2$.

4.5. Proof of Theorem 4.4

Let $(X^3_\infty, d_\infty, p_\infty)$ be given as in Theorem 4.4. By Theorem 4.1 (1), the flat background geometry has orbifold singularity near each point in $\mathcal{S}$. Fix a normalization of the harmonic function V, and let $g^\flat_\infty$ be the associated flat background metric. Near a point in $\mathcal{S}$, we have $V = \sigma r^{-1} + h$ for $\sigma \in [0, \infty)$ and h orbifold smooth. Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ consisting of points where $\sigma > 0$.

Lemma 4.20. *We have*

$$\limsup_{r\to\infty} \inf_{S_r(p_\infty)} V<\infty.$$

Proof. Otherwise, we can find an increasing sequence $r_j\to\infty$ such that

$$\inf_{S_{r_j}(p_\infty)} V\to\infty.$$

By the maximum principle for harmonic functions, we actually have $V\to\infty$ uniformly at infinity. The minimum of V is then achieved at some point in $\mathcal{S}\setminus\mathcal{S}'$. Then, by the strong maximum principle for harmonic functions, we conclude that V must be constant. This yields a contradiction. □

Proposition 4.21. *The flat background geometry $(X^3_\infty, d^\flat_\infty)$ is a complete flat orbifold.*

Proof. If (X^3_∞, g_∞) has two ends, then X^3_∞ isometrically splits off an $\mathbb{R}$. Then, it follows that X^3_∞ is smooth and $\kappa=0$. Then, $V>0$ is harmonic on the complete smooth metric measure space $(X^3_\infty, g_\infty, \nu_\infty)$, where

$$d\nu_\infty = V^{-1/2}\, d\mathrm{vol}_{g_\infty}.$$

By Theorem 2.8, V is constant.

If (X^3_∞, g_∞) has only one end, then we first claim that $\mathbb{R}$ is not an asymptotic cone at infinity. Otherwise, one can find $r_i\to\infty$ such that $A_{r_i,2r_i}(p_\infty)$ consists of two connected components. Similar to the proof of Proposition 4.8, we can find in each connected component foliations by totally geodesic flat tori with respect to $g^\flat_\infty$. Then, we can slide these tori between $A_{r_i,2r_i}(p_\infty)$ and $A_{r_{i+1},2r_{i+1}}(p_\infty)$, and we see that X^3_∞ have two ends each diffeomorphic to $\mathbb{T}^2\times\mathbb{R}_+$. This yields a contradiction.

Now, the following lemma can be proved similarly to Lemma 4.9.

Lemma 4.22. *There exists $\delta_0>0$ such that, for every sufficiently large r, any two points in $A_{r/2,r}(p_\infty)$ can be connected by a smooth curve $\gamma\subset A_{\delta_0\cdot r,3r}(p_\infty)$ with arc length $|\gamma|\leqslant 10r$.*

As in §4.2, using Lemmas 4.20 and 4.22, and the Harnack inequality for harmonic functions, we obtain that

$$\limsup_{r\to\infty} \sup_{S_r(p_\infty)} V<\infty,$$

which implies that $g^\flat_\infty$ is complete at infinity. □

By Thurston's developability theorem, we have that $(X^3_\infty, g^\flat_\infty)$ is isometric to $\mathbb{R}^3/\Gamma$ for some $\Gamma \leqslant \mathrm{Isom}(\mathbb{R}^3)$. If Γ is a free action, then the special affine structure on $(X^3_\infty, g^\flat_\infty)$ implies that X^3_∞ is isometric to $\mathbb{R}^3$, $S^1\times\mathbb{R}^2$ or $\mathbb{T}^2\times\mathbb{R}$. If Γ is not free, then there is some $\sigma\in\Gamma$ which acts as reflection at one point in $\mathbb{R}^3$. Let Γ_0 be the subgroup of Γ that preserves the orientation of $\mathbb{R}^3$. Then, $[\Gamma{:}\Gamma_0]=2$, and Γ_0 acts freely on $\mathbb{R}^3$. Otherwise, Γ_0 has an element that fixes an axis in $\mathbb{R}^3$ and the singularity of $X^3_\infty\equiv\mathbb{R}^3/\Gamma$ cannot be isolated. It follows that

$$X^3_\infty \equiv (\mathbb{R}^3/\Gamma_0)/\mathbb{Z}_2,$$

where $\mathbb{R}^3/\Gamma_0$ is isometric to $\mathbb{R}^3$, $\mathbb{R}^2\times S^1$ or $\mathbb{R}\times\mathbb{T}^2$.

Next, we classify V in the above cases. If $(X^3_\infty, g^\flat_\infty)$ is isometric to $\mathbb{R}^3$, then Bôcher's theorem implies that $V=\sigma\cdot r^{-1}+\ell$ for some constants $\sigma\geqslant 0$ and $\ell\geqslant 0$. If

$$(X^3_\infty, g^\flat_\infty) \equiv \mathbb{R}^2\times S^1,$$

then we consider the S^1-average $\overline{V}=\int_{S^1} V\,d\theta$, which is harmonic on $\mathbb{R}^2\setminus\{0^2\}$. Since the composition $\overline{V}(e^z)>0$ is harmonic on $\mathbb{C}_z$, we have that $\overline{V}$ is constant. Therefore, the harmonic function $V>0$ is smooth on $\mathbb{R}^2\times S^1$, which implies that V is constant. If

$$(X^3_\infty, g^\flat_\infty) \equiv \mathbb{R}\times\mathbb{T}^2,$$

the same average argument implies V is a positive constant. In the other cases, one can analyze the lifting of V on the $\mathbb{Z}_2$-cover and the same conclusion follows.

Finally, if (4.2) holds, then using item (2) of Theorem 4.1, V is a positive constant.

5. Singularity structure II: Case $d=2$

5.1. Main results

We first state the main results of this section.

THEOREM 5.1. (Local version) *Let (X^4_j, g_j, p_j) be a sequence of hyperkähler manifolds such that $\overline{B_2(p_j)}$ is compact and*

$$(X^4_j, g_j, \nu_j, p_j) \xrightarrow{\mathrm{mGH}} (X^2_\infty, d_\infty, \nu_\infty, p_\infty),$$

with $\dim_{\mathrm{ess}}(X^2_\infty)=2$. If $\mathcal{S}=\{p_\infty\}$, then the limit metric on X_∞ is a singular special Kähler metric in the sense of Definition 3.19, and ν_∞ is a multiple of the 2-dimensional Hausdorff measure on X^2_∞.

THEOREM 5.2. (Compact version) *Let g_j be a sequence of hyperkähler metrics on the* K3 *manifold $\mathcal{K}$, with* $\mathrm{diam}_{g_j}(\mathcal{K})=1$, *such that*

$$(\mathcal{K}, g_j, \nu_j) \xrightarrow{\mathrm{mGH}} (X^2_\infty, d_\infty, \nu_\infty),$$

with $\dim_{\mathrm{ess}}(X^2_\infty)=2$. *Then, X^2_∞ is homeomorphic to S^2, endowed with a singular special Kähler metric.*

Remark 5.3. By definition, a singular special Kähler metric has local integral monodromy around each singular point. With a uniform bound on the L^2 curvature, which is automatic in the setting of Theorem 5.2, one would expect that the limit should indeed have integral monodromy. See Conjecture 7.4.

Remark 5.4. We were informed by Shouhei Honda that using the theory of RCD spaces, one can show that the above limit space (X^2_∞, d_∞) is indeed an Alexandrov space of non-negative curvature.

THEOREM 5.5. (Complete version) *Let (X^4_j, g_j, p_j) be a sequence of hyperkähler manifolds such that*

$$(X^4_j, g_j, \nu_j, p_j) \xrightarrow{\mathrm{mGH}} (X^2_\infty, d_\infty, \nu_\infty, p_\infty).$$

Assume that (X^2_∞, d_∞) is complete non-compact and $\dim_{\mathrm{ess}}(X^2_\infty)=2$. *If $\mathcal{S}=\{p_\infty\}$, then X^2_∞ is isometric to either a flat metric cone $\mathbf{C}_\beta$ for $\beta\in\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6},1\}$, or to the flat product $\mathbb{R}\times S^1$, with the standard special Kähler structure.*

5.2. Proof of the main results

The main part of this subsection is devoted to the proof of Theorem 5.1. At the end of this section we will prove Theorems 5.2 and 5.5.

Now, assume that we are in the setup of Theorem 5.1. By §3.2, we know that $B_1(p_\infty)\setminus\{p_\infty\}$ is a special Kähler manifold. Recall that, if Σ is a smooth surface with boundary and with Gaussian curvature $K\geqslant 0$, then, by the Gauss-Bonnet theorem,

$$2\pi(2-2g(\Sigma)-n)=2\pi\chi(\Sigma)=\int_\Sigma K+\int_{\partial\Sigma} k\geqslant\int_{\partial\Sigma} k, \tag{5.1}$$

where k denotes the boundary geodesic curvature, and n is the number of boundary circles. Given a tangent cone $(Y,\bar p)$ at p_∞, suppose that it is given by the limit of $(X_\infty, r_i^{-1}d_\infty, p_\infty)$ for some $r_i\to 0$. If $\dim Y=2$, then, by the interior curvature bound discussed in §3.2, we know that $Y\setminus\{\bar p\}$ is smooth and special Kähler. If $\dim Y=1$, then

Y is either $\mathbb{R}$ or $\mathbb{R}_+$, and the collapsing is locally along a smooth circle fibration. We first claim that Y cannot be $\mathbb{R}$. Otherwise, we can choose a sequence $r_i \to 0$ such that the annulus $r_i^{-1} A_{r_i, 2r_i}$ collapses to the union of intervals $[-2,-1] \cup [1,2]$ with bounded curvature. Then, we can choose a smooth fiber C_i with uniformly bounded geodesic curvature, which is given by the union of two circles. In particular, $\int_{C_i} k \to 0$. Let Σ_i be the region bounded by C_i and C_{i+1}, whose boundary consists of four disjoint circles. Applying (5.1), we easily reach a contradiction.

So, we know that any tangent cone Y is either 2-dimensional, or it is isometric to $\mathbb{R}_+$. In both cases, we can choose a smooth circle C_i in the annulus $r_i^{-1} A_{r_i, 2r_i}$ with

$$\lim_{i \to \infty} \int_{C_i} k = c.$$

In particular, $c=0$ when $Y = \mathbb{R}_+$. Again, applying (5.1) to the region Σ_i bounded by C_i and C_{i+1}, we see that, for i large, Σ_i is diffeomorphic to a cylinder. In particular, we have shown the following result.

LEMMA 5.6. *For $\delta > 0$ small, we have that $B_\delta(p_\infty) \setminus \{p_\infty\}$ is diffeomorphic to a punctured disc in $\mathbb{R}^2$.*

Without loss of generality, we may assume that $\delta = 1$, and set

$$B = B_1(p_\infty) \quad \text{and} \quad B^* = B_1(p_\infty) \setminus \{p_\infty\}.$$

We now prove Theorem 5.1. Choose a loop σ generating $\pi_1(B^*)$, oriented so that it goes counterclockwise around p_∞ (notice that B^*, being a Riemann surface, is naturally oriented). Denote by A the monodromy of the special Kähler structure along σ. Denote by ω the Kähler form on B^*. We now divide into three cases.

Case 1. A is conjugate to Id, I_1 or I_1^{-1}. In this case, A has an invariant vector. So, we can choose a local holomorphic coordinate z such that dz is a globally defined holomorphic 1-form on B^*. Then, we have

$$\omega = \frac{\sqrt{-1}}{2} \operatorname{Im}(\tau)\, dz \wedge d\bar{z},$$

where $\operatorname{Im}(\tau)$ is positive harmonic function on B^*.

LEMMA 5.7. *For $r > 0$ small, we have $\operatorname{Im}(\tau) \geqslant C r^{3/2}$ on $S_r(p_\infty)$.*

Proof. The proof is similar to the arguments in §3.2. If the estimate does not hold, then $\operatorname{Im}(\tau)$ is a global harmonic function on B in the sense of Definition 2.6. This leads to a contradiction by the weak Harnack inequality (Theorem 2.7). □

Notice that, from the proof of Lemma 5.6, for any r small we can find a loop σ_r contained in the annulus $A_{r/2,2r}$ which is homotoptic to σ and with length bounded by Cr. Since $|dz|=1/\sqrt{\mathrm{Im}(\tau)}$, by letting $r\to 0$ it follows that $\int_\sigma dz=0$. So z is single-valued on B^*, and it extends continuously across p_∞. Adding a constant, we may assume that $z(p_\infty)=0$. Now, z defines a covering map from B^* onto a domain in $\mathbb{C}$. Suppose that the covering degree is k, then we can take $\zeta=z^{1/k}$ as a holomorphic coordinate on B^* and this embeds B^* holomorphically onto a punctured domain $\Omega^*=\Omega\setminus\{0\}$ in $\mathbb{C}$.

We may now view $\mathrm{Im}(\tau)$ as a positive harmonic function on Ω^*, so, by Bôcher's theorem, we know that

$$\mathrm{Im}(\tau)=-c\log|\zeta|+V(\zeta),$$

where $c\geqslant 0$ and V extends smoothly across zero. Then, one can directly check that the tangent cone at p_∞ is given by the flat mcetric

$$\omega_0=\frac{\sqrt{-1}}{2}k^2|\zeta|^{2k-2}\,d\zeta\wedge d\bar{\zeta}\quad\text{on }\mathbb{C}.$$

This is a cone of angle $2\pi k$. Since B is a Ricci limit space, we must have $k=1$. If $c=0$, then the metric is smooth across p_∞. If $c>0$, then, by rescaling the special holomorphic coordinate z, we may assume that $c=1$. Then, the metric is a singular special Kähler metric of type I.

Case 2. A is conjugate to $-\mathrm{Id}$, I_1^* or $(I_1^*)^{-1}$. In this case, A has an eigenvector with eigenvalue -1. Then, we can choose a local special holomorphic coordinate z such that dz transforms to $-dz$ under A.

Similar reasoning as Case 1 shows that $\mathrm{Im}(\tau)$ is a well-defined positive harmonic function on B^*, z^2 is a globally defined holomorphic function on B^* and we may assume that $z^2(p_\infty)=0$. Then, as above, $\zeta=z^{2/k}$ is a holomorphic coordinate and defines a holomorphic embedding of B^* into a punctured domain in $\mathbb{C}$. As before, we get

$$\mathrm{Im}(\tau)=-c\log|\zeta|+V(\zeta)$$

for a harmonic function V smooth at zero. As above, one can see that k must equal 1, and the tangent cone at p_∞ is $\mathbb{C}/\mathbb{Z}_2$. If $c=0$, then the singularity is of orbifold type, so the metric is a singular special Kähler metric of type III with $\beta=\frac{1}{2}$. If $c\neq 0$, then, rescaling the coordinate z, we can make $c=1$. This shows that the metric is a singular special Kähler metric of type II.

Case 3. A is elliptic or hyperbolic. Let $(Y,\bar{p})$ be a tangent cone at p_∞. It follows from (3.9) that Y must be 2-dimensional, so $Y\setminus\{\bar{p}\}$ is smooth and special Kähler, and by (3.8) we know that $Y\setminus\{\bar{p}\}$ has elliptic or hyperbolic monodromy. Again, the interior curvature bound implies that Y has quadratic curvature decay at infinity. Notice that $Y\setminus\{\bar{p}\}$ has non-negative curvature, so one can see that Y is asymptotic to a flat metric cone (C_γ, O) with angle $2\pi\gamma$ for some $\gamma\in[0,1)$, in the C^∞ Cheeger–Gromov topology.

Notice that C_γ itself is also a tangent cone at p_∞. Since C_γ is flat, by Remark 3.13 we know that the flat connection ∇ coincides with the Levi-Civita connection, and hence its monodromy around an oriented loop going counterclockwise around the singularity is given by R_γ. Now, again by (3.8), we get $\mathrm{Tr}(A)=\mathrm{Tr}(R_\gamma)$. Notice the orientation of the rotation is a conjugation invariant in $\mathrm{SL}(2;\mathbb{R})$. In particular, we must have $A=R_\gamma$, since we have chosen σ to be oriented counter-clockwise around p_∞.

The same argument shows that the monodromy of $Y\setminus\{\bar{p}\}$ is also given by R_β. Then, applying this again to the singular point $\bar{p}$ of Y, it follows that there is a tangent cone at $\bar{p}$ which is isometric to $\mathbf{C}_\beta$. This means that we can find a sequence of annuli $A_{r_j,s_j}(\bar{p})$ in Y, with $r_j\to 0$ and $s_j\to\infty$, whose boundary circles after rescaling both converge to the unit circle in $\mathbf{C}_\beta$. Applying the Gauss-Bonnet theorem on such sequences of annuli, it is easy to see that Y must be a flat metric cone, and hence is isometric to $\mathbf{C}_\beta$.

The above discussion in particular shows that there is a unique tangent cone at p_∞, which is given by $\mathbf{C}_\beta$ for some $\beta\in(0,1)$, and the original monodromy matrix satisfies $A=R_\beta$. By Corollary 3.31 and Lemma 3.17 we must have $\beta\in\left\{\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}\right\}$. In particular, A must be elliptic.

We now study the singular behavior of the limit metric near p_∞. First, we can find local special holomorphic coordinates (z,w) on B^* such that, under the monodromy along σ, we have

$$A\cdot(dz-\sqrt{-1}dw)=e^{2\pi\sqrt{-1}\beta}(dz-\sqrt{-1}\,dw).$$

This means that $d\zeta\equiv d((z-\sqrt{-1}w)^{1/\beta})$ is a well-defined global holomorphic 1-form on B^*. Since $R_\beta\neq\mathrm{Id}$, by adding some constants to (z,w), we may assume that the translation part in (3.5) vanishes. This implies that $\zeta\equiv(z-\sqrt{-1}w)^{1/\beta}$ is indeed a globally defined function on B^*. In general, τ may not be single-valued on B^*. But notice that

$$\omega_\infty=\frac{\sqrt{-1}}{2}\,\mathrm{Im}(\tau)\,dz\wedge d\bar{z}=\frac{\sqrt{-1}}{2}\,\frac{\mathrm{Im}(\tau)}{|1-\sqrt{-1}\tau|^2}\beta^2|\zeta|^{2\beta-2}\,d\zeta\wedge d\bar{\zeta}.$$

So,

$$\frac{\mathrm{Im}(\tau)}{|1-\sqrt{-1}\tau|^2}$$

is single-valued on B^*.

LEMMA 5.8. *For all $\varepsilon>0$, there exists a $C(\varepsilon)>0$ such that, for all $r\in\left(0,\frac{1}{2}\right]$, on $S_r(p_\infty)$ we have*

$$\frac{\mathrm{Im}(\tau)}{|1-\sqrt{-1}\tau|^2}\geqslant C(\varepsilon)r^\varepsilon.$$

Proof. If we suppose that this fails, then we can find a sequence $r_i \to 0$ such that

$$\sup_{A_{r_i,2r_i}(p_\infty)} \frac{\mathrm{Im}(\tau)}{|1-\sqrt{-1}\tau|^2} \geqslant (1+\delta) \inf_{A_{r_i,2r_i}(p_\infty)} \frac{\mathrm{Im}(\tau)}{|1-\sqrt{-1}\tau|^2} \tag{5.2}$$

for some $\delta>0$. Let $\widetilde{U}_i$ be the universal cover of $r_i^{-1}\cdot A_{r_i,2r_i}(p_\infty)$, endowed with the rescaled metric. Then, as $i\to\infty$ we know that $\widetilde{U}_i$ converges to the universal cover $\widetilde{U}_\infty$ of $A_{1,2}(0)\subset \mathbf{C}_\beta$, which is flat. We can find $\lambda_i>0$ such that $\sup_{\widetilde{U}_i} \lambda_i \cdot \mathrm{Im}(\tilde{\tau})=1$. Suppose that this supremum is achieved at some $q_i \in \overline{\widetilde{U}}_i$. Let $D_i = \mathrm{Re}(\lambda_i \cdot \tilde{\tau}(q_i))$. Then, by Harnack inequality, it follows that $\mathrm{Im}(\lambda_i\tilde{\tau})$ and $\mathrm{Re}(\lambda_i\tilde{\tau}-D_i)$ are locally uniformly bounded. So, passing to a subsequence, we obtain local convergence of $\lambda_i\tilde{\tau}-D_i$ to a limit $\tilde{\tau}_\infty$ on $\widetilde{U}_\infty$. The flatness of $\widetilde{U}_\infty$ implies that $\tilde{\tau}_\infty$ is a constant. This then contradicts (5.2). □

The lemma implies that

$$\left|\nabla|\zeta|^\beta\right|^2 \leqslant C(\varepsilon)^{-1} r^{-\varepsilon}.$$

In particular, ζ extends continuously across p_∞. Moreover, ζ realizes B^* as a finite cover of some punctured domain D^* in $\mathbb{C}$. So, for some $k>0$, $\zeta^{1/k}$ defines a global holomorphic coordinate on B^* which embeds B^* into $\mathbb{C}$. Now, we identify the upper half-space with the unit disk $\mathbb{D}$ via the map

$$\tau \longmapsto \xi = \frac{\tau-\sqrt{-1}}{\tau+\sqrt{-1}}. \tag{5.3}$$

Then, the monodromy transformations on ξ is given by

$$\xi \longmapsto e^{-4\pi\sqrt{-1}\beta}\xi.$$

So, $-\log|\xi|$ is well defined on B^*. By Bôcher's theorem, we have

$$-\log|\xi| = -c\log|\zeta| + v$$

for $c\geqslant 0$ and v a smooth harmonic function on B.

We claim that c cannot be zero. Otherwise, $\left|\log|\xi|\right| \leqslant C$ on B^*, which implies that τ is uniformly bounded on B^* and has definite distance away from $\sqrt{-1}$. Then, by taking limit as in the proof of the above lemma, we see that $\tilde{\tau}_\infty$ is not fixed by R_β, so is not invariant under the monodromy, which is a contradiction. So we know that $c>0$, and, as p moves to p_∞, we have $|\xi|\to 0$.

We may write

$$\omega_\infty = \frac{\sqrt{-1}}{8}(1-|\xi|^2)\beta^2|\zeta|^{2\beta-2}\, d\zeta \wedge d\bar{\zeta}.$$

Since the tangent cone at p_∞ is $\mathbf{C}_\beta$, we see that k must be 1. Without loss of generality, we may assume that $\zeta(p_\infty)=0$. We now divide into two cases:

- $\beta\in\{\frac{1}{4},\frac{3}{4}\}$. Then, ξ^2 is holomorphic across zero, and $\xi=F(\zeta)^{1/2}$ for a holomorphic function F with $F(0)=0$.
- $\beta\in\{\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}\}$. Then, ξ^3 is holomorphic across zero, and $\xi=F(\zeta)^{1/3}$ for a holomorphic function F with $F(0)=0$.

These imply ω_∞ is a singular special Kähler metric of type III. This finishes the proof of Theorem 5.1.

Proof of Theorem 5.2. By Theorem 5.1, d_∞ is a singular special Kähler metric on a compact Riemann surface. Since $\pi_1(\mathcal{K})=\{1\}$, by [77], X_∞ is simply connected which implies that X_∞ must be homeomorphic to S^2. □

Proof of Theorem 5.5. By Theorem 5.1, we know that X_∞ is endowed with a singular special Kähler metric ω. In particular, the curvature of $X_\infty\setminus\{p_\infty\}$ is positive. Then, it is easy to see that each end of X_∞ is asymptotic to a unique cone at infinity. If X_∞ has two ends, then it splits isometrically as a flat product $\mathbb{R}\times S^1$. So, we assume that X_∞ has only one end. Then, an easy application of the Gauss–Bonnet theorem implies that X_∞ is homeomorphic to $\mathbb{R}^2$. Let σ be a loop generating the fundamental group at infinity, and denote by A the monodromy matrix along σ.

Case (a). There is a 1-dimensional asymptotic cone Y. Using the Gauss–Bonnet theorem as in the beginning of this subsection, one can see that Y must be $\mathbb{R}_+$. Then, by (3.9), we know that A must be conjugate to I_1, I_1^{-1} or Id. In particular, on $X_\infty\setminus\{p_\infty\}$, there is a local special holomorphic coordinate z such that dz is globally defined, so is the positive harmonic function $\mathrm{Im}(\tau)$. Just as in the proof of Theorem 5.1, Case 1 (p. 375), the function z is indeed a global coordinate on X_∞. Then, by similar arguments as in the proof of Proposition 4.21, one can show that the flat metric

$$\omega^\flat\equiv\frac{\sqrt{-1}}{2}\,dz\wedge d\bar{z}=\mathrm{Im}(\tau)^{-1}\omega$$

is complete at infinity. So, X_∞ is biholomorphic to $\mathbb{C}$. Now, an application of Bôcher's theorem yields that $\mathrm{Im}(\tau)$ must be a constant. Hence, the metric ω itself is a flat metric on $\mathbb{C}$. This yields a contradiction.

Case (b). All asymptotic cones are 2-dimensional. In particular, they are all flat cones, and must be the unique $\mathbf{C}_\beta$ such that A is conjugate to R_β. This also implies that the tangent cone at p_∞ must also be $\mathbf{C}_\beta$. Then, the Gauss–Bonnet theorem implies that X_∞ itself is flat, ând hence must be the cone $\mathbf{C}_\beta$. □

6. Classification of gravitational instantons

6.1. Uniqueness of asymptotic cones

Let (X^4, g) be a gravitational instanton, and we fix a hyperkähler triple $\boldsymbol{\omega}$. If it is flat, then it is isometric to a flat product $\mathbb{R}^k \times \mathbb{T}^{4-k}$ with $1 \leqslant k \leqslant 3$. By Cheeger–Gromoll's splitting theorem, X^4 is isometric to a flat product $\mathbb{R} \times \mathbb{T}^3$, unless X^4 has only one end. In the following, we will always assume that X^4 is non-flat and has only one end. We will also assume that $\dim_{\mathrm{ess}}(Y) \leqslant 3$ for any asymptotic cone (Y, d_Y, p_*), since otherwise (X^4, g) is ALE, and this case has already been classified by Kronheimer [55].

Since

$$\int_{X^4} |\mathrm{Rm}_g|^2 \, d\mathrm{vol}_g < \infty,$$

Theorem 3.21 and Proposition 3.1 imply that any asymptotic cone (Y, d_Y, p_*) is smooth away from $p_* \in Y$. By Theorems 4.4 and 5.5, (Y, d_Y) is a flat space isometric to one of the following: $\mathbb{R}^3$, $\mathbb{R}^3/\mathbb{Z}_2$, $\mathbb{R}^2$, $\mathbb{R}$, $\mathbb{R}_+$, $S^1 \times \mathbb{R}^2$, $\mathbb{T}^2 \times \mathbb{R}$, $S^1 \times \mathbb{R}$, or a flat cone $\mathbf{C}_\beta$ for $\beta \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}\right\}$.

Lemma 6.1. *Any asymptotic cone is a flat metric cone.*

Proof. It suffices to rule out $S^1 \times \mathbb{R}^2$, $\mathbb{R} \times \mathbb{T}^2$ and $S^1 \times \mathbb{R}$. Notice that these spaces have moduli given by the moduli of the flat metrics on S^1 and $\mathbb{T}^2$. We first show that $S^1_{1/2} \times \mathbb{R}^2 \notin \mathcal{T}_\infty(X^4)$, where S^1_R denotes a circle of diameter R.

Suppose that one asymptotic cone Y is given by $S^1_{1/2} \times \mathbb{R}^2$. Then, we may find $r_i \to \infty$ such that $r_i^{-1} B(p, r_i)$ converges to the unit ball in $S^1_{1/2} \times \mathbb{R}^2$. On the other hand, we know that, for all $r > 0$ sufficiently large, $r^{-1} B(p, r)$ is $\varepsilon(r)$-GH close to the ball U_r of radius 1 around the vertex in an asymptotic cone Y_r, where $\lim_{r \to \infty} \varepsilon(r) = 0$. Notice that Y_r is not unique, and we simply make arbitrary choices for all r. Let $\varepsilon_0 \in \left(0, \frac{1}{100}\right)$ be small so that any asymptotic cone Y_r whose unit ball U_r is $3\varepsilon_0$-GH close to the unit ball in $S^1_R \times \mathbb{R}^2$ for some $R > \frac{1}{4}$ must be itself of the form $S^1_{R'} \times \mathbb{R}^2$ for some $R' \geqslant \frac{1}{8}$.

Now, fix $\tilde{r}$ large so that $\varepsilon(r) < \frac{1}{2}\varepsilon_0$ for $r \geqslant \tilde{r}$. For any i with $r_i \geqslant \tilde{r}$, let $s \in [r_i, r_{i+1}]$ be the smallest number such that, for all $r \in [s, r_{i+1}]$, $r^{-1} B(p, r)$ is ε_0-GH close to $S^1_R \times \mathbb{R}^2$ for some $R \geqslant \frac{1}{8}$. By assumption, we know that $s \leqslant \frac{1}{2} r_{i+1}$. We claim that $s = r_i$. Otherwise, if $s > r_i$, then for all $s' \in \left[\frac{1}{2}s, s\right]$ we have $2s' \in [s, r_{i+1}]$, so $(2s')^{-1} B(p, 2s')$ is ε_0-GH close to some ball $S^1_R \times \mathbb{R}^2$ for some $R > \frac{1}{8}$. In particular, $(s')^{-1} B(p, s')$ is $2\varepsilon_0$-GH close to $S^1_{2R} \times \mathbb{R}^2$. By assumption, it follows that the unit ball $U_{s'}$ in $Y_{s'}$ is $3\varepsilon_0$-GH close to the unit ball in $S^1_{2R} \times \mathbb{R}^2$. By our choices of ε_0, we conclude that $Y_{s'}$ is of the form $S^1_{R'} \times \mathbb{R}^2$ for $R' \geqslant 2R - 3\varepsilon_0 > \frac{1}{8}$. This contradicts the choice of s.

So, for any sufficiently large r, we may write $Y_r = S^1_{f(r)} \times \mathbb{R}^2$ for some $f(r) \in \left(\frac{1}{8}, \frac{5}{8}\right)$. Now, we claim that

$$f(2r) < \tfrac{3}{4} f(r) \quad \text{for all } r > 0, \tag{6.1}$$

so that the desired contradiction immediately arises. In fact, if (6.1) is true, then $f(r) \to 0$ as $r \to \infty$, which contradicts $f(r) > \frac{1}{8}$. To see the claim, we notice that, by assumption, $r^{-1} B(p, r)$ is $\varepsilon(r)$-GH close to the unit ball in $S^1_{f(r)} \times \mathbb{R}^2$. So, $(2r)^{-1} B(p, r)$ is $\frac{1}{2}\varepsilon(r)$-GH close to the half-ball in $S^1_{f(r)/2} \times \mathbb{R}^2$. On the other hand, $(2r)^{-1} B(p, 2r)$ is $\varepsilon(2r)$-GH close to the unit ball in Y_{2r}. It follows that $f(2r) < \frac{3}{4} f(r)$, if r is large.

Hence, we have proved that $S^1_{1/2} \times \mathbb{R}^2 \notin \mathcal{T}_\infty(X^4)$. By rescaling, $S^1_R \times \mathbb{R}^2 \notin \mathcal{T}_\infty(X^4)$ for all R. Similar arguments also show that $S^1_R \times \mathbb{R} \notin \mathcal{T}_\infty(X^4)$ for all R.

Finally, we claim that the possible unit-area flat $\mathbb{T}^2$ such that $\mathbb{R} \times \mathbb{T}^2 \in \mathcal{T}_\infty(X^4)$ must form a compact moduli. Indeed, if not, applying Lemma 2.3, one can choose a sequence of flat tori $(\mathbb{T}^2, g_j^{\text{flat}})$ with $\text{Area}_{g_j^{\text{flat}}}(\mathbb{T}^2) = 1$ and $\text{diam}_{g_j^{\text{flat}}}(\mathbb{T}^2) \to \infty$ such that, after appropriate scaling-up,

$$(\mathbb{T}^2, \tilde{g}_j^{\text{flat}}) \xrightarrow{\text{GH}} \mathbb{R} \times S^1_1.$$

It follows that rescalings of $\mathbb{R} \times \mathbb{T}^2$ converge to $\mathbb{R}^2 \times S^1_1 \in \mathcal{T}_\infty(X^4)$ in the pointed Gromov–Hausdorff sense, which is a contradiction. Then, similar arguments as above also show that $\mathbb{R} \times \mathbb{T}^2 \notin \mathcal{T}_\infty(X^4)$ for any flat torus $\mathbb{T}^2$. $\square$

PROPOSITION 6.2. *Let (X^4, g) be a gravitational instanton. Then, it has a unique asymptotic cone which is a flat metric cone $(C(W), d_C, p_*)$, where W denotes the cross-section and p_* is the cone vertex.*

Proof. By Lemma 2.3, $\mathcal{T}_\infty(X^4)$ is connected and compact. Denote by d the maximal dimension of the elements in $\mathcal{T}_\infty(X^4)$. If $d = 3$, then we choose some $Y \in \mathcal{T}_\infty(X^4)$ with $\dim_{\text{ess}}(Y) = 3$. Then, any element in a small neighborhood $\mathcal{U}$ of Y in $T_\infty(X^4)$ has dimension 3, and hence, by Lemma 6.1, it must be $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$. So, the connectedness of $\mathcal{T}_\infty(X^4)$ implies that $\mathcal{T}_\infty(X^4) = \{Y\}$. Similar arguments apply to the case $d = 2$. $\square$

In the rest of this section, we will denote by (Y, d_Y, p_*) the unique asymptotic cone of X^4, and denote $d = \dim_\infty(X^4) \equiv \dim_{\text{ess}}(Y)$. Since X has only one end, W is connected and $Y \neq \mathbb{R}$. From §3, the renormalized limit measure on Y is $\nu_Y = \chi \cdot d\text{vol}_{g_Y}$, where χ is a constant if $d > 1$, or $d = 1$ and $G_\infty = \mathbb{R}^3$; $\chi = c \cdot z^{1/2}$ if $d = 1$ and $G_\infty = \mathscr{H}_1$ (here z is the affine coordinate).

6.2. Nilpotent fibration on the end

Denote $r(x) \equiv d_g(p, x)$ and $\hat{r}(y) \equiv d_Y(p_*, y)$ for $x \in X$ and $y \in Y$. Below, we use $\tau(x)$ to denote a general function on the end of X^4 such that $\lim_{r(x) \to \infty} \tau(x) = 0$. The following is essentially due to Cheeger–Fukaya–Gromov [17]. We give an outline of the arguments in Appendix A.

THEOREM 6.3. *There exists a smooth fibration map* $F: X^4 \setminus K \to Y \setminus K'$, *where* K *and* K' *are compact such that the following properties hold.*

(1) *There are flat connections* ∇_y *with parallel torsion on the fibers* $F^{-1}(y)$ *which depend smoothly on* $y \in Y \setminus \Omega$, *such that each fiber* $(F^{-1}(y), \nabla_y)$ *is affine diffeomorphic to a nilmanifold* $\Gamma \setminus N$ *for* $\Gamma \subset N_L$, *and the structure group of the fibration is reduced to*

$$((\mathfrak{Z}(N) \cap \Gamma) \setminus \mathfrak{Z}(N)) \rtimes \mathrm{Aut}(\Gamma) \subset \mathrm{Aff}(\Gamma \setminus N).$$

(2) F *is an asymptotic Riemannian submersion in the sense that, for a tangent vector* v *at* $x \in X \setminus K$ *which is orthogonal to the fiber of* F, *we have*

$$(1-\tau(x))|v|_g \leqslant |dF_x(v)|_{g_Y} \leqslant (1+\tau(x))|v|_g \tag{6.2}$$

and, for all $k>0$, *there exists* $C_k>0$ *such that, for all* $x \in X \setminus K$,

$$|\nabla^k F(x)|_{g, g_Y} \leqslant C_k r(x)^{-k}. \tag{6.3}$$

(3) *The second fundamental form* Π *of the fibers satisfies, for all* $k \geqslant 0$,

$$\begin{cases} |\nabla^k \Pi(x)| = \tau(x) r(x)^{-1-k}, & \text{if } d=2,3, \text{ or } d=1 \text{ and } G_\infty = \mathbb{R}^3, \\ |\nabla^k \Pi(x)| \leqslant \dfrac{1}{\sqrt{3}}(1+\tau(x)) r(x)^{-1-k}, & \text{if } d=1 \text{ and } G_\infty = \mathscr{H}_1. \end{cases} \tag{6.4}$$

In our setting, applying Corollary 3.30, all the fibers are nilmanifolds. As in §3.5, we say a tensor ξ on the end of X is $\mathcal{N}$*-invariant* if its lift to the local universal covers is invariant under the full nilpotent group action of N_L.

LEMMA 6.4. *In the setting of the above theorem, there are constants* $\delta_0 \in (0,1)$ *and* $C>0$ *such that*

$$C \cdot \hat{r}(y)^{-\delta_0} \leqslant \mathrm{diam}_g(F^{-1}(y)) \leqslant C \cdot \hat{r}(y)^{\delta_0} \quad \text{for all } y \in Y \setminus K',$$

where diam_g *denotes the intrinsic diameter of the fiber.*

Proof. This is a direct consequence of the estimates on the second fundamental form (6.4). □

THEOREM 6.5. *By making* K *and* K' *larger if necessary, there exists an* $\mathcal{N}$*-invariant definite triple* $\boldsymbol{\omega}^\dagger$ *defined on* $X \setminus K$ *such that, for all* $k \in \mathbb{N}$, *we have*

$$|\nabla^k_{\boldsymbol{\omega}}(\boldsymbol{\omega}^\dagger - \boldsymbol{\omega})|_{g_{\boldsymbol{\omega}}} = O(r(x)^{-k-1+\delta_0}). \tag{6.5}$$

Proof. Following the same arguments as in the proof of Proposition 3.26, we obtain an $\mathcal{N}$-invariant definite triple $\boldsymbol{\omega}^\dagger$ on $X\setminus K$. The estimate (6.5) can be proved using the diameter growth estimate for the collapsing fibers in Lemma 6.4, as well as [17, Proposition 4.9]. □

Let $g^\dagger$ be the quotient metric on $Y\setminus K'$ induced by $\boldsymbol{\omega}^\dagger$. Then, (6.2), (6.3) and (6.5) together imply that, for all k,

$$\lim_{r\to\infty} r^k \sup_{S_r(p_*)} |\nabla^k_{g_Y}(g^\dagger - g_Y)|_{g_Y}(y) = 0. \tag{6.6}$$

An $\mathcal{N}$-invariant function f on $X\setminus K$ can be viewed as a function on $Y\setminus K'$, and we may write

$$\Delta_{\boldsymbol{\omega}^\dagger} f = \Delta_{g^\dagger} f + \langle H, \nabla_{g^\dagger} f\rangle, \tag{6.7}$$

where H denotes the mean curvature vector field of the fibers of F, viewed as a vector field on $Y\setminus K'$. By Theorem 6.3 (4) and the arguments in the proof of Lemma 3.29, we have

$$\lim_{k\to\infty} r^k \sup_{S_r(p_*)} |\nabla^k_{g_Y}(H - \nabla_Y \log\chi)| = 0 \quad \text{for all } k\in\mathbb{N}. \tag{6.8}$$

6.3. Perturbation to invariant hyperkähler metrics

For $R\gg 1$, we set

$$\mathcal{Q}_R \equiv Y\setminus B_R(p_*) \quad \text{and} \quad \mathcal{X}_R \equiv F^{-1}(\mathcal{Q}_R).$$

As in §2.4, we identify an element in $\Omega^+_{\boldsymbol{\omega}'}(\mathcal{X}_R)\otimes\mathbb{R}^3$ with a (3×3)-matrix-valued function $\boldsymbol{f}$ on $\mathcal{X}_R$, and an $\mathcal{N}$-invariant element is identified with such a function on $\mathcal{Q}_R$.

THEOREM 6.6. *Given any $\varepsilon_0\in(0,1-\delta_0)$, there exist a number $R_0>0$ and an $\mathcal{N}$-invariant hyperkähler triple $\boldsymbol{\omega}^\diamondsuit$ on $\mathcal{X}_{R_0}$ of the form $\boldsymbol{\omega}^\diamondsuit = \boldsymbol{\omega}^\dagger + dd^*(\boldsymbol{f}\cdot\boldsymbol{\omega}^\dagger)$ such that*

$$|\nabla^k_{\boldsymbol{\omega}^\dagger}\boldsymbol{f}(x)| = O(r(x)^{2-\varepsilon_0-k})$$

for all $k\in\mathbb{N}$.

In particular, we also have that, for all $k\geqslant 0$,

$$|\nabla^k_{\boldsymbol{\omega}}(\boldsymbol{\omega}^\diamondsuit - \boldsymbol{\omega})|_{g_{\boldsymbol{\omega}}} \leqslant C_k\cdot r(x)^{-k-\varepsilon_0}.$$

So, the original hyperkähler triple $\boldsymbol{\omega}$ is asymptotic to the $\mathcal{N}$-invariant triple $\boldsymbol{\omega}^\diamondsuit$.

The rest of this subsection is devoted to the proof of the above theorem. The idea is similar to the proof of Theorem 3.27. The difference is that, due to the non-compactness of $\mathcal{X}_R$, we need to work in certain weighted spaces.

Given $\delta\in\mathbb{R}$ and $k\in\mathbb{N}$, we define the following weighted (semi-)norms of an $\mathcal{N}$-invariant function f on $\mathcal{X}_R$ (or equivalently a function on $\mathcal{Q}_R$):

$$\begin{aligned}\|f\|_{C^k_\delta(\mathcal{Q}_R)} &\equiv \sum_{m=0}^{k}\sup_{r\geqslant R}\{r^{-\delta+m}\cdot\|\nabla^m_{g_Y}f\|_{C^0(A_{r,2r}(p_*))}\},\\ [f]_{C^{k,\alpha}_\delta(\mathcal{Q}_R)} &\equiv \sup_{r\geqslant R}\{r^{-\delta+k+\alpha}\cdot[f]_{C^{k,\alpha}_{g_Y}(A_{r,2r}(p_*))}\},\\ \|f\|_{C^{k,\alpha}_\delta(\mathcal{Q}_R)} &\equiv \|f\|_{C^k_\delta(\mathcal{Q}_R)}+[f]_{C^{k,\alpha}_\delta(\mathcal{Q}_R)},\end{aligned}$$

where

$$\begin{aligned}&[f]_{C^{k,\alpha}(A_{r,2r}(p_*))}\\ &\quad\equiv\sup\left\{\frac{|\nabla^k_{g_Y}f(y_1)-\nabla^k_{g_Y}f(y_2)|}{d_{g_Y}(y_1,y_2)^\alpha}: y_1,y_2\in A_{r,2r}(p_*)\text{ and } d_{g_Y}(y_1,y_2)<\operatorname{Injrad}_{g_Y}(y_1)\right\}.\end{aligned}$$

As usual, the difference in the last formula is computed in terms of the parallel transport along the minimizing geodesic. These (semi-)norms obviously extend to $\mathcal{N}$-invariant matrix-valued functions.

Now, we fix $k\geqslant 6$ and $\alpha\in(0,1)$. The following provides a suitable right inverse of the Laplace operator for us. Notice that we do not impose the boundary conditions, since we are only interested in the asymptotic behavior at infinity.

Proposition 6.7. *There exists a finite set $\Gamma\subset(0,1)$ depending only on Y such that, for all $\delta\in(0,1)\setminus\Gamma$ and all $R\geqslant 1$, one can find a bounded linear map*

$$\mathcal{S}_R: C^{k,\alpha}_{-\delta}(\mathcal{Q}_R)\longrightarrow C^{k+2,\alpha}_{-\delta+2}(\mathcal{Q}_R)$$

with the properties that

$$\Delta_{\nu_Y}\circ\mathcal{S}_R=\mathrm{Id}\quad \text{and}\quad \|\mathcal{S}_R\|\leqslant C$$

for C depending only on Y, δ, k, α (but not on R).

Proof. If $d=1$, then, in terms of the affine coordinates on $\mathbb{R}_+$ (see §3.3), we have $\Delta_{\nu_Y}=Cz^{-1}\partial_z^2$. In this case, we reduce to a simple ODE problem whose proof we omit.

We now consider the case $d\geqslant 2$. Then, ν_Y is proportional to the volume measure, and Δ_{ν_Y} is the metric Laplace operator on the flat cone $Y=C(W)$, where

$$W\equiv\begin{cases}\text{spherical space form } \mathbb{S}^2 \text{ or } \mathbb{RP}^2, & \text{if } d=3,\\ S^1_{2\pi\beta} \text{ with } \beta\in\left\{\frac12,\frac13,\frac23,\frac14,\frac34,\frac16,\frac56,1\right\}, & \text{if } d=2.\end{cases}$$

First, for any $\delta\in(0,1)$, one can construct a linear extension operator

$$E_R: C^{k,\alpha}_{-\delta}(\mathcal{Q}_R) \longrightarrow C^{k,\alpha}_{-\delta}(\mathcal{Q}_{R/2}),$$

with $\|E_R\|\leqslant C$ independent of $R\geqslant 1$. For this purpose, one can construct E_1 by the local construction in [76] or [79], then use rescaling to define E_R.

Denote by (r,Θ) the polar coordinates on Y. Let $\Sigma(W)\equiv\{\lambda_j\}_{j=0}^{\infty}$ be the spectrum (allowing multiplicities) of $-\Delta_W$ with $0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\ldots$. Let $\{\varphi_j\}_{j=0}^{\infty}$ be an orthonormal set of eigenfunctions satisfying $-\Delta_W\varphi_j=\lambda_j\cdot\varphi_j$ and $\|\varphi_j\|_{L^2(W)}=1$. Given a function $f\in C^{k,\alpha}_{-\delta}(\mathcal{Q}_R)$, we set $\tilde f\equiv E_R(f)$. Then, there is an L^2-expansion of f given by

$$\tilde f(r,\Theta)=\sum_{j=0}^{\infty} f_j(r)\varphi_j(\Theta).$$

For $j>0$, we have

$$\begin{aligned} |f_j(r)| &\equiv \left|\int_W \tilde f(r,\Theta)\varphi_j(\Theta)\right| \\ &= \left|\lambda_j^{-k}\int_W ((-\Delta_W)^k\tilde f)(r,\Theta)\varphi_j(\Theta)\right| \\ &\leqslant C(\delta)\|f\|_{C^{k,\alpha}_{-\delta}(\mathcal{Q}_R)}\lambda_j^{-k} r^{-\delta}. \end{aligned} \tag{6.9}$$

Let $u(r,\Theta)$ be a formal solution

$$u(r,\Theta)=\sum_{j=0}^{\infty} u_j(r)\varphi_j(\Theta)$$

of $\Delta_{g_Y}u=\tilde f$. Then, $u_j(r)$ satisfies

$$u_j''(r)+\frac{d-1}{r}\cdot u_j'(r)-\frac{\lambda_j}{r^2}\cdot u_j(r)=f_j(r). \tag{6.10}$$

For every $j\in\mathbb{N}$, the corresponding homogeneous ODE

$$u_j''(r)+\frac{d-1}{r}\cdot u_j'(r)-\frac{\lambda_j}{r^2}\cdot u_j(r)=0 \tag{6.11}$$

has the following fundamental solutions:

(1) when $j=0$ and $d=2$, $\mathcal{G}_0(r)\equiv\log r$ and $\mathcal{D}_0(r)\equiv 1$;

(2) when $j=0$ and $d=3$, $\mathcal{G}_0(r)\equiv 1$ and $\mathcal{D}_0(r)\equiv r^{-1}$;

(3) when $j\in\mathbb{Z}_+$, there are a growing solution $\mathcal{G}_j(r)\equiv r^{\mu_j^+}$ and a decaying solution $\mathcal{D}_j(r)\equiv r^{\mu_j^-}$; here, μ_j^+ and μ_j^- are the positive and negative roots of the following algebraic equation:

$$\mu^2+(d-2)\mu-\lambda_j=0. \tag{6.12}$$

We now set $\Gamma=\{|\mu_j^-|:j>0\}\cap(0,1)$ and $\underline{\delta}=\min\Gamma$. Let j_0 be the largest j such that $\mu_j^->-1+\underline{\delta}$. For $j=0$, we can directly integrate and define

$$u_0(r)\equiv\int_R^r s^{1-d}\left(\int_R^s t^{d-1}f_0(t)\,dt\right)ds.$$

It is easy to see that

$$|u_0(r)|\leqslant C(\delta)r^{2-\delta}\|f\|_{C^{k,\alpha}_{-\delta}(\mathcal{Q}_R)}.$$

For $1\leqslant j\leqslant j_0$, we set

$$u_j(r)=\frac{\mathcal{G}_j(r)}{\mathcal{W}_j(r)}\int_r^R\mathcal{D}_j(s)f_j(s)\,ds+\frac{\mathcal{D}_j(r)}{\mathcal{W}_j(r)}\int_R^r\mathcal{G}_j(s)f_j(s)\,ds,$$

For $j>j_0$, we set

$$u_j(r)=\frac{\mathcal{G}_j(r)}{\mathcal{W}_j(r)}\int_r^\infty\mathcal{D}_j(s)f_j(s)\,ds+\frac{\mathcal{D}_j(r)}{\mathcal{W}_j(r)}\int_R^r\mathcal{G}_j(s)f_j(s)\,ds.$$

Here, the wronskian is given by

$$\mathcal{W}_j(r)\equiv\mathcal{W}(\mathcal{G}_j(r),\mathcal{D}_j(r))=(\mu_j^+-\mu_j^-)r^{\mu_++\mu_j-1}=\sqrt{(d-2)^2+4\lambda_j}\cdot r^{1-d}.$$

It follows from (6.9) that each u_j is well defined, with

$$|u_j(r)|\leqslant C(\delta)\lambda_j^{-k}r^{2-\delta}\|f\|_{C^{k,\alpha}_{-\delta}}$$

for $\delta\in(0,1)\setminus\Gamma$. The Weyl law implies that $\lambda_j\leqslant Cj^{2/d}$. By standard elliptic estimates, $|\varphi_j|_{C^0}\leqslant C\lambda_j$ for $j\geqslant1$. Since $k\geqslant d+1$, the formal solution u converges in C^0 and, for all $r\geqslant\frac{3}{4}R$, we have

$$|u(r,\Theta)|\leqslant C(\delta)r^{2-\delta}\|f\|_{C^{k,\alpha}_{-\delta}}.$$

It is easy to check that $\Delta_{g_Y}u=\tilde{f}$ holds pointwise on $\mathcal{Q}_{3R/4}$. Using the standard interior elliptic estimates on the rescaled annulus $r^{-1}A_{r,2r}(p_*)$, we obtain the bound

$$\|u\|_{C^{k+2,\alpha}_{-\delta+2}(\mathcal{Q}_R)}\leqslant C(\delta)\|f\|_{C^{k,\alpha}_{-\delta}(\mathcal{Q}_R)}.$$

Now, we simply set $\mathcal{S}_R(f)=u$. Clearly, $\mathcal{S}_R$ is a linear operator, and the above discussion gives the uniform bound on $\|\mathcal{S}_R\|$. $\square$

We now fix $\delta_1\in(\varepsilon_0,1-\delta_0)\setminus\Gamma$. We define the Banach space $\mathfrak{A}$ to be the completion of the space of (3×3)-matrix-valued functions $\boldsymbol{f}$ on $\mathcal{Q}_R$ under the $C^{k+2,\alpha}_{-\delta_1+2}(\mathcal{Q}_R)$ norm, and define $\mathfrak{B}$ to be the completion of the same space under the $C^{k,\alpha}_{-\delta_1}(\mathcal{Q}_R)$ norm.

By Theorem 6.5, for R large, we know that the map $\mathscr{F}: B_1(\mathbf{0})\subset\mathfrak{A}\to\mathfrak{B}$ is well defined, with

$$\|\mathscr{F}(\mathbf{0})\|_{\mathfrak{B}} \leqslant CR^{-1+\delta_0+\delta_1}.$$

We let $\mathscr{L}(\boldsymbol{f})\equiv\Delta_{\nu_Y}\boldsymbol{f}$ and $\mathscr{N}(\boldsymbol{f})\equiv\mathscr{F}(\boldsymbol{f})-\mathscr{L}(\boldsymbol{f})$. Then, Proposition 6.7 provides a linear operator $\mathscr{P}: \mathfrak{B}\to\mathfrak{A}$ with $\mathscr{L}\circ\mathscr{P}=\mathrm{Id}$ and $\|\mathscr{P}\boldsymbol{v}\|_{\mathfrak{A}}\leqslant C\|\boldsymbol{v}\|_{\mathfrak{B}}$ for all $\boldsymbol{v}\in\mathfrak{B}$, where $C>0$ is a constant independent of $R\geqslant 1$.

For any $\boldsymbol{f}\in\mathfrak{A}$, we have $\Delta_{\boldsymbol{\omega}^\dagger}\boldsymbol{f}=\Delta_{g^\dagger}\boldsymbol{f}+\langle H, \nabla_{g^\dagger}\boldsymbol{f}\rangle$. Using the fact that

$$\Delta_{g^\dagger}\boldsymbol{f} = \Delta_{g_Y}\boldsymbol{f}+(g^\dagger-g_Y)*\nabla^2\boldsymbol{f}+\nabla_{g_Y}g^\dagger*\nabla_{g_Y}\boldsymbol{f}$$

and (6.8), we have

$$\|\Delta_{\boldsymbol{\omega}^\dagger}\boldsymbol{f}-\Delta_{\nu_Y}\boldsymbol{f}\|_{\mathfrak{B}} \leqslant \varepsilon(R)\|\boldsymbol{f}\|_{\mathfrak{A}}$$

for some $\varepsilon(R)\to 0$ as $R\to\infty$. Applying (2.7) and the definition of the weighted spaces, we obtain

$$\|\mathscr{N}(\boldsymbol{f})-\mathscr{N}(\boldsymbol{g})\|_{\mathfrak{B}} \leqslant (CR^{-\delta_1}+\varepsilon(R))\|\boldsymbol{f}-\boldsymbol{g}\|_{\mathfrak{A}} \quad \text{for all } \boldsymbol{f},\boldsymbol{g}\in B_1(\mathbf{0})\subset\mathfrak{A}.$$

So, we can apply Proposition 2.12 to obtain $R_0>0$ such that, for $R=R_0$, there is some $\boldsymbol{f}\in\mathfrak{A}$ that satisfies the estimate $\|\boldsymbol{f}\|_{\mathfrak{A}}\leqslant CR_0^{-1+\delta_0+\delta_1}$.

Finally, applying standard elliptic estimates to the equation $\mathscr{L}(\boldsymbol{f})+\mathscr{N}(\boldsymbol{f})=0$ on the rescaled annulus $r^{-1}A_{r,2r}(p)$ as $r\to\infty$, we obtain higher-derivative estimates. This finishes the proof of Theorem 6.6.

6.4. Proof of Theorem 1.2

Let $\boldsymbol{\omega}^\diamondsuit$ be the $\mathcal{N}$-invariant hyperkähler triple constructed in Theorem 6.6, and let $g^\diamondsuit$ be the quotient metric on $\mathcal{Q}$ induced by $\boldsymbol{\omega}^\diamondsuit$. We set $\mathcal{X}\equiv\mathcal{X}_{R_0}$ and $\mathcal{Q}\equiv\mathcal{Q}_{R_0}$. We will define several families of model ends of gravitational instantons, which we will label by "AL$\mathfrak{X}$" for some letter $\mathfrak{X}\in\{\mathrm{E},\mathrm{F},\mathrm{G},\mathrm{H},\mathrm{G}^*,\mathrm{H}^*\}$. We adopt the terminology that when we say a gravitational instanton (X^4, g) is AL$\mathfrak{X}$ it means that we can smoothly identify the end of X^4 with a model end in the family AL$\mathfrak{X}$ such that

$$|\nabla_g^k(g-g_{\mathrm{model}})|_g = O(r^{-k-\varepsilon})$$

for some $\varepsilon>0$ and allfor $k\in\mathbb{N}$, where g_{model} denotes the model hyperkähler metric. By Theorem 6.6, (X^4, g) is AL$\mathfrak{X}$ if and only if $(\mathcal{X}, g_{\boldsymbol{\omega}^\diamondsuit})$ is AL$\mathfrak{X}$. To prove Theorem 1.2, we will classify the ends of the $\mathcal{N}$-invariant metric $\boldsymbol{\omega}^\diamondsuit$. Recall that we only need to consider the case when X^4 has only one end and is non-flat. Moreover, we assume that X^4 is not ALE, namely $\dim_\infty(X^4)\leqslant 3$. Theorem 1.2 will follow from Theorems 6.9, 6.13 and 6.17.

6.4.1. Case $\dim_\infty(X^4)=3$

Definition 6.8. (ALF models) ALF model ends are defined as follows.

(1) An ALF-A_k (for $k\in\mathbb{Z}$) model end is the hyperkähler metric constructed by applying the Gibbons–Hawking ansatz on $\mathbb{R}^3\setminus K$ to the positive harmonic function

$$V=\frac{k+1}{2r}+c,$$

where $c>0$.

(2) An ALF-D_k model (for $k\in\mathbb{Z}$) end is a $\mathbb{Z}_2$-quotient of an ALF-A_{2k-5} end, where the $\mathbb{Z}_2$-action covers the standard involution on $\mathbb{R}^3$.

(3) An ALF model end is an ALF-A_k or ALF-D_k model end for some $k\in\mathbb{Z}$.

THEOREM 6.9. *Any gravitational instanton* (X^4,g) *with* $\dim_\infty(X^4)=3$ *is* ALF.

Remark 6.10. ALF-A_k gravitational instantons are classified by Minerbe [64]; they are all given by multi-Taub-NUT spaces. ALF-D_k gravitational instantons are classified by Chen–Chen [21]; they are all given by the twistor space construction due to Cherkis–Hitchin–Ivanov–Kapustin–Lindström–Roček [5], [56], [52], [26], [25]. Notice that

$$k=b_2(X^4)\geqslant 0.$$

Conversely, any $k\in\mathbb{N}$ can be achieved.

Proof. We first assume that $Y=\mathbb{R}^3$. It is a standard fact that such $\boldsymbol{\omega}^\diamondsuit$ is given by the Gibbons–Hawking ansatz. Indeed, this is a special case of the discussion in §3.1. This means that the metric $(\mathcal{Q},g^\diamondsuit)$ is a special affine metric 3-manifold. Denote by $V^{-1}(x)$ the length squared of the fibers of $F^{-1}(x)$ for $x\in\mathcal{Q}$. Then, by Lemma 6.4, we know that, for all $\sigma>0$, there is $C>0$ such that $Cr^{-\sigma}\leqslant V\leqslant Cr^{\sigma}$. As in the proof of Proposition 4.15, this implies that the corresponding flat background geometry $(\mathcal{Q},g^\flat)$ is complete at infinity, and hence must be isometric to $Y\setminus K$ for some compact K. Notice that V is harmonic with respect to $g^\flat$. We consider the expansion

$$V=\sum_{j\geqslant 0}(a_j^+r^{\mu_j^+}+a_j^-r^{\mu_j^-})\varphi_j,$$

where φ_j is an L^2 orthonormal basis of Laplace eigenfunctions on the cross section of Y, with $-\Delta_{S^2}\varphi_j=\lambda_j\varphi_j$, $\lambda_j\geqslant 0$, and where $\mu_j^\pm$ are the solutions to the equation $\mu^2+\mu-\lambda_j=0$. Notice that $\lambda_0=0$ and $\lambda_j\geqslant 2$ for $j>0$. So, we have $\mu_0^+=0$, $\mu_0^-=-1$, $\mu_j^+\geqslant 1$ and $\mu_j^{-1}\leqslant -2$ for $j>0$. The growth condition on V implies that $a_j^+=0$ for all $j>0$. So, we obtain

$$V=c+\frac{l}{2r}+O(r^{-2}).$$

Here, l is the degree of the S^1 bundle $F\colon \mathcal{X}\to\mathcal{Q}$. So, we have proved that $\boldsymbol{\omega}^\diamondsuit$, and hence (X,g) is ALF-A_k for $k=l-1$. By the positive mass theorem of Minerbe [62], we know that $k\geqslant 0$.

In the case $Y\equiv\mathbb{R}^3/\mathbb{Z}_2$, by taking the $\mathbb{Z}_2$-cover outside a compact set, we may reduce to the previous case. Then, (X^4,g) is an ALF-D_k gravitational instanton. By Biquard–Minerbe [9], we have $k\geqslant 0$. □

6.4.2. Case $\dim_\infty(X^4)=2$

Definition 6.11. (ALG models) ALG model ends are defined as follows.

(1) Let $\beta\in\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6},1\}$. Let $\mathbf{C}_\beta$ be the flat cone defined in Example 3.12 with the canonical hyperkähler metric on $T^*\mathbf{C}_\beta$. Taking a lattice sub-bundle in $T^*\mathbf{C}_\beta$ which is invariant under the monodromy $\widetilde{R}_\beta$ (cf. (3.6)), the induced torus bundle gives rise to a (flat) ALG$_\beta$ model end.

(2) An ALG model end is an ALG$_\beta$ model end for some β in the above list.

Definition 6.12. (ALG* models) ALG* model ends are defined as follows.

(1) An ALG*-I_k (for $k\in\mathbb{Z}_+$) model end is obtained by applying the Gibbons–Hawking ansatz on $S^1\times\mathbb{R}^2\setminus K$ to the harmonic function $V=k\cdot\log r$, where r is the radial distance function on $\mathbb{R}^2$.

(2) An ALG*-I_k^* (for $k\in\mathbb{Z}_+$) model end is a $\mathbb{Z}_2$ quotient of an ALG*-I_{2k} model end, where $\mathbb{Z}_2$ action covers the standard involution on $\mathbb{R}^2$ and the rotation by π on S^1.

(3) An ALG* model end is an ALG*-I_k or ALG*-I_k^* model end for some $k\in\mathbb{Z}_+$.

THEOREM 6.13. *Any gravitational instanton* (X^4,g) *with* $\dim_\infty(X^4)=2$ *is either* ALG *or* ALG*.

Remark 6.14. Combining the weighted analysis developed in [23] and a direct generalization of Minerbe's positive mass theorem [62], one can conclude that ALG$_1$ and ALG*-I_k gravitational instantons do not exist. We thank Gao Chen for pointing out this. It is also proved in [23] that any ALG*-I_k gravitational instanton satisfies $1\leqslant k\leqslant 4$. On the other hand, there exist ALG$_\beta$ gravitational instantons for all $\beta\in\left\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}\right\}$, and there exist ALG*-$I_k^*$ gravitational instantons for all $k\in\{1,2,3,4\}$, which follows from the work of Hein [44]. They live on the complement of a singular fiber of finite or I_k^* monodromy on a rational elliptic surface. In [23] a partial converse to Hein's theorem was proved.

Proof. Since $\dim_\infty(X^4)=2$, $\boldsymbol{\omega}^\diamondsuit$ has local $\mathbb{T}^2$ symmetry but may have global monodromy. We divide into several subcases. Let σ be a loop generating $\pi_1(\mathcal{Q})$ which goes around the vertex $p_*\in Y$ once counterclockwise, and let $A_\sigma\in\mathrm{SL}(2;\mathbb{Z})$ be the corresponding monodromy of the $\mathbb{T}^2$ fiber. Notice that the quotient metric $g^\diamondsuit$ on $\mathcal{Q}$ is a special Kähler metric, with monodromy conjugate to A_σ.

First, assume that $A_\sigma=\widetilde{R}_\beta$ for some $\beta\in\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6},1\}$. In this case, we know that the asymptotic cone Y is given by $\mathbf{C}_\beta$. Then, by the discussion at the end of §3.5, we can find global holomorphic coordinates $\zeta=(z-\sqrt{-1}w)^{1/\beta}$ such that (z,w) is a pair of local special holomorphic coordinates. Again, τ is not single-valued in general but ξ^k ($k=2$ or $k=3$ depending on β) is single-valued. Since the asymptotic cone of $(\mathcal{Q},g^\diamondsuit)$ is $\mathbf{C}_\beta$, similar to the proof of Theorem 5.5, one can show that the flat metric

$$\sqrt{-1}\beta^2|\zeta|^{2\beta-2}\,d\zeta\wedge d\bar{\zeta}$$

is complete at infinity. It then follows that, as $\zeta\to\infty$, we have $\xi^k\to0$, so $\xi^k=\psi(\zeta^{-1})$ for a holomorphic function ψ. In particular, $\tau=\sqrt{-1}+O(|\zeta|^{-1/k})$. It then follows that the special Kähler metric $g^\diamondsuit$ is polynomially asymptotic to the standard flat cone metric in the ζ coordinate. Now, the $\mathcal{N}$-invariant metric $g_{\boldsymbol{\omega}^\diamondsuit}$ is determined by $g^\diamondsuit$ via (3.3). It follows that $g_{\boldsymbol{\omega}^\diamondsuit}$ is ALG, so is (X,g).

Next consider the case $A=I_k$ for some $k\geqslant1$. Then, we have an invariant vector of A_σ. This implies that there is a globally S^1-action on $\mathcal{X}$. In particular, $\boldsymbol{\omega}^\diamondsuit$ is given by the Gibbons–Hawking ansatz on some special affine metric 3-manifold. Similar to the case $\dim_\infty(X^4)=3$, the growth estimate on V gives a complete flat background geometry at infinity whose asymptotic cone has dimension 2. Then, the flat background geometry is itself isometric to $(S^1\times\mathbb{R}^2)\setminus K$ for some compact K. So, we can use spectral decomposition to conclude that

$$V=k\cdot\log r+c+O(r^{-\varepsilon}).$$

We may assume that $c=0$ by changing the coordinates on $\mathbb{R}^2$. In this case, we have that (X^4,g) is ALG*-I_k. , Finally, when $A=I_k^*$ for some $k\geqslant1$, we pass to a double cover and reduce to the previous case. In this case, (X,g) is ALG*-I_k^* . □

6.4.3. Case $\dim_\infty(X^4)=1$

Definition 6.15. (ALH models) An ALH model is the hyperkähler metric on the product $\mathbb{T}^3\times[0,\infty)$ for some flat $\mathbb{T}^3$.

Definition 6.16. (ALH* models) ALH* model ends are defined as follows

(1) An ALH$^*_b$ (for some $b\in\mathbb{Z}_+$) model end is the hyperkähler metric obtained by applying the Gibbons–Hawking ansatz on the product $\mathbb{T}^2\times[0,\infty)$ to the harmonic function $V=bz$ for some $b\in\mathbb{Z}_+$, where $\mathbb{T}^2$ is a flat 2-torus with area 2π, and z is the standard coordinate on $[0,\infty)$.

(2) An ALH* model end is an ALH$_b^*$ model end for some $b\in\mathbb{Z}_+$. Notice an ALH* model end is precisely a Calabi model end discussed in [46].

THEOREM 6.17. *Any gravitational instanton* (X^4, g) *with* $\dim_\infty(X^4)=1$ *is either* ALH *or* ALH*.

Proof. In this case, $\boldsymbol{\omega}^\diamondsuit$ has either a $\mathbb{T}^3$ or $\mathscr{H}_1$ symmetry. Then, it is itself an ALH or ALH* model end. Consequently, in the first case (X^4, g) is ALH, and in the second case it is ALH*. □

Remark 6.18. ALH gravitational instantons were constructed by Tian–Yau [81] and Hein [44] on the complement of a smooth fiber in a rational elliptic surface. Chen–Chen [22] proved a Torelli theorem for ALH gravitational instantons; it is also shown that ALH gravitational instantons actually have an improved exponential decay rate. ALH$_b^*$ (for $1\leqslant b\leqslant 9$) gravitational instantons have two constructions: Tian–Yau metrics [81] live on the complement of a smooth anti-canonical divisor in a weak del Pezzo surface, and Hein metrics [44] live on the complement of an I_b-fiber in a rational elliptic surface. Conversely, by Remark 6.21 below, we know that an ALH$_b^*$ gravitationl instanton must satisfy $1\leqslant b\leqslant 9$.

In the next subsection we will prove an exponential decay for ALH* gravitational instantons.

6.5. Exponential decay in the ALH* case

Let (X^4, g) be an ALH$_b^*$ gravitational instanton. As before, we fix a choice of hyperkähler triple $\boldsymbol{\omega}$. By Theorem 6.17, there exist $\varepsilon>0$ and some compact set K such that $X^4\setminus K$ is smoothly identified with an ALH$_b^*$ model end $(\mathcal{C}, \boldsymbol{\omega}_{\mathcal{C}})$, with

$$|\nabla^k_{\boldsymbol{\omega}_{\mathcal{C}}}(\boldsymbol{\omega}-\boldsymbol{\omega}_{\mathcal{C}})|_{\boldsymbol{\omega}_{\mathcal{C}}} = O(r^{-k-\varepsilon})$$

for all $k\in\mathbb{N}$, where r is the distance function with respect to $\boldsymbol{\omega}_{\mathcal{C}}$. The goal of this subsection is to prove the following.

THEOREM 6.19. *Let* $(X^4, \boldsymbol{\omega})$ *be an* ALH$_b^*$ *gravitational instanton. Then, there exist* $\delta_0>0$ *and a diffeomorphism* F *from the end of* $\mathcal{C}$ *to* X^4 *such that, for all* $k\in\mathbb{N}$,

$$|\nabla^k_{\boldsymbol{\omega}_{\mathcal{C}}}(F^*\boldsymbol{\omega}-\boldsymbol{\omega}_{\mathcal{C}})|_{\boldsymbol{\omega}_{\mathcal{C}}} \leqslant C_k\cdot e^{-\delta_0 r^{2/3}}. \tag{6.13}$$

Remark 6.20. The main interest in this theorem lies in the fact the asymptotic geometry of the Calabi model space is non-standard, which has several geometric meaningful scales. The latter is already seen in the analysis in [46]. We expect that the idea here can be applied to more general problems.

Remark 6.21. This theorem connects well with [46] and [47]. On the one hand, by construction the ALH_b^* gravitational instantons of Tian–Yau and Hein all have the exponential decay properties as stated in Theorem 6.19. Such a decay rate is a typical rate of a decaying harmonic function on the model end $\mathcal{C}$. On the other hand, under the improved decay assumption (6.13), [47] proved a partial converse to the Tian–Yau and Hein constructions in the complex-analytic sense. In particular, any ALH_b^* gravitational instanton can be compactified to a rational elliptic surface or a weak del Pezzo surface. It also implies that an ALH_b^* gravitational instanton must satisfy $1\leqslant b\leqslant 9$.

Now, we summarize the geometry of $\mathcal{C}$. Recall that $(\mathcal{C},\omega_{\mathcal{C}})$ is given by applying the Gibbons–Hawking ansatz to $V=bz$ on $\mathbb{T}^2\times[z_0,\infty)$ for some flat $\mathbb{T}^2$ with area 2π and $z_0\geqslant 10$. Then,

$$C^{-1}r^{2/3}\leqslant z\leqslant Cr^{2/3}.$$

Notice that $\mathcal{C}$ admits a natural nilpotent group action which gives rise to the $\mathcal{N}$-structure, i.e., there is a nilpotent orbit $\mathcal{N}(\boldsymbol{x})$ at every point $\boldsymbol{x}\in\mathcal{C}$. Moreover,

$$\operatorname{diam}_{g_{\mathcal{N}(\boldsymbol{x})}}(\mathcal{N}(\boldsymbol{x}))\sim r(\boldsymbol{x})^{1/3}, \quad \operatorname{Injrad}_{g_{\mathcal{C}}}(\boldsymbol{x})\sim r(\boldsymbol{x})^{-1/3} \quad \text{and} \quad \operatorname{Vol}_{g_{\mathcal{C}}}(B_r)\sim r^{4/3};$$

see [46, §2] for more details. Before proving Theorem 6.19, let us introduce a simple but useful lemma.

Lemma 6.22. *There is a triple of* 1*-forms* $\boldsymbol{\sigma}$ *such that* $\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\sigma}$ *and, for all* $k\in\mathbb{N}$ *and* $\varepsilon>0$,

$$|\nabla^k_{\boldsymbol{\omega}_{\mathcal{C}}}\boldsymbol{\sigma}|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{1/3-k+\varepsilon}). \tag{6.14}$$

Proof. Since the intrinsic diameter of the $\mathcal{N}$-orbits with respect to $\boldsymbol{\omega}$ has the order $r_{\boldsymbol{\omega}}^{1/3}$ for the $\boldsymbol{\omega}$-distance function $r_{\boldsymbol{\omega}}$, the $\mathcal{N}$-orbits in the rescaled annulus $s^{-1}A_{s,2s}(p)$ has diameter decay $\sim s^{-2/3}$. We now repeat the construction of the $\mathcal{N}$-invariant hyperkähler metric on the end of X^4. First, taking the average of $\boldsymbol{\omega}$ along the $\mathcal{N}$-orbits, we obtain a closed definite triple $\boldsymbol{\omega}^\dagger$ which is cohomologous to $\boldsymbol{\omega}$. Notice that, for any vector field ζ generating a family of diffeomorphisms $\phi_t (t\in[0,1])$, we have that

$$\phi_1^*\boldsymbol{\omega}-\boldsymbol{\omega}=\int_0^1\frac{d}{dt}\phi_t^*\boldsymbol{\omega}\,dt=d\biggl(\int_0^1\phi_t^*(\zeta\lrcorner\boldsymbol{\omega})\,dt\biggr).$$

Using this, we can write $\boldsymbol{\omega}^\dagger=\boldsymbol{\omega}+d\boldsymbol{\sigma}_1$ for some $\boldsymbol{\sigma}_1$ satisfying

$$|\nabla^k_{\boldsymbol{\omega}}\boldsymbol{\sigma}_1|_{\boldsymbol{\omega}}=O((r_{\boldsymbol{\omega}})^{1/3-k+\varepsilon})$$

for all $k\in\mathbb{N}$ and $\varepsilon>0$. Applying the construction in §6.3 to $\boldsymbol{\omega}^\dagger$ (with $\delta_0=\frac{1}{3}$), we obtain a new $\mathcal{N}$-invariant hyperkähler triple $\boldsymbol{\omega}^\diamondsuit$. By Theorem 6.6, $\boldsymbol{\omega}^\diamondsuit=\boldsymbol{\omega}^\dagger+d\boldsymbol{\sigma}_2$ with

$$|\nabla^k_{\boldsymbol{\omega}^\dagger}\boldsymbol{\sigma}_2|_{\boldsymbol{\omega}^\dagger}=O((r_{\boldsymbol{\omega}^\dagger})^{1/3-k+\varepsilon})$$

for all $k\in\mathbb{N}$ and $\varepsilon>0$. Now, in fact, $\boldsymbol{\omega}^{\diamondsuit}$ coincides with $\boldsymbol{\omega}_{\mathcal{C}}$, as both are $\mathcal{N}$-invariant hyperkähler triples which are asymptotic to each other at infinity. $\square$

Passing to a finite cover of $\mathcal{C}$, we may assume that $b=1$. We now take a closer look at the deformations of hyperkähler triples in §2.4. A triple of $\boldsymbol{\omega}_{\mathcal{C}}$-self-dual 2-forms $\boldsymbol{\theta}^+$ can be identified with a (3×3)-matrix-valued function $A_{\boldsymbol{\theta}^+}$. The 4-dimensional space of 3×3 matrices $M=(M_{ij})$ satisfying $M^T+M=\lambda\,\mathrm{Id}$ for some $\lambda\in\mathbb{R}$ is isomorphic to $\mathbb{R}^4$ via $M\mapsto\big(\frac{1}{3}\operatorname{Tr}(M), M_{23}, M_{31}, M_{12}\big)$. Define the linear operator

$$\begin{aligned} R\colon \Omega^+_{\boldsymbol{\omega}}\otimes\mathbb{R}^3 &\longrightarrow C^\infty(\mathcal{C})\otimes\mathbb{R}^4,\\ \boldsymbol{\theta}^+ &\longmapsto \tfrac{1}{2}(A_{\boldsymbol{\theta}^+}-(A_{\boldsymbol{\theta}^+})^T)+\tfrac{1}{3}\operatorname{Tr}(A_{\boldsymbol{\theta}^+})\cdot\mathrm{Id}\,. \end{aligned}$$

It follows that $R(\boldsymbol{\theta}^+)=0$ if and only if $A_{\boldsymbol{\theta}^+}$ is symmetric and trace-free. This can serve as a gauge fixing condition, due to the fact that if the triple $\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\eta}$ is hyperkähler then, by the discussion before (2.4), $R(d^+\boldsymbol{\eta})=0$ implies $d^+\boldsymbol{\eta}=\mathfrak{F}(\mathrm{tf}(-Q_{\boldsymbol{\omega}_{\mathcal{C}}}-S_{d^-\boldsymbol{\eta}}))$. The latter is elliptic in $\boldsymbol{\eta}$ when coupled with $d^*\boldsymbol{\eta}=0$ (this can always be achieved). In the proof of Theorem 6.19, we first fix the gauge such that $R(d^+\boldsymbol{\omega})=0$, and then improve the decaying order using the convexity properties of the linearized elliptic PDEs.

Step 1 (*Gauge fixing*).

On $(\mathcal{C},\boldsymbol{\omega}_{\mathcal{C}})$, we choose the complex structures J_1, J_2 and J_3 so that

$$\begin{cases} J_1\,dx=dy, & J_1\,dz=z^{-1}\theta,\\ J_2\,dy=dz, & J_2\,dx=z^{-1}\theta,\\ J_3\,dz=dx, & J_3\,dy=z^{-1}\theta, \end{cases}$$

where θ is the connection 1-form in the Gibbons–Hawking construction. It follows that ω_α is Kähler with respect to J_α, where

$$\begin{cases} \omega_1=z\,dx\wedge dy+dz\wedge\theta,\\ \omega_2=z\,dy\wedge dz+dx\wedge\theta,\\ \omega_3=z\,dz\wedge dz+dy\wedge\theta. \end{cases}$$

Without loss of generality, we may assume that $\boldsymbol{\omega}_{\mathcal{C}}\equiv(\omega_1,\omega_2,\omega_3)$. Given a vector field ξ, we have the induced infinitesimal deformation $\mathcal{L}_\xi\boldsymbol{\omega}_{\mathcal{C}}=d(\xi\lrcorner\boldsymbol{\omega}_{\mathcal{C}})$. For our purpose, we only consider vector fields generated by a 4-tuple of functions $\underline{f}\equiv(f_0,f_1,f_2,f_3)$ via

$$\xi_{\underline{f}}=\nabla f_0+\sum_{\alpha=1}^{3}J_\alpha\nabla f_\alpha.$$

Then, by straightforward computation, we obtain

$$\begin{cases} \mathcal{L}_{\xi_{\underline{f}}}\omega_\alpha\wedge\omega_\alpha=\frac{1}{2}\mathcal{L}_{\xi_{\underline{f}}}(d\mathrm{vol}_g)=2\Delta f_0\,d\mathrm{vol}_g, \quad \alpha=1,2,3,\\ \mathcal{L}_{\xi_{\underline{f}}}\omega_1\wedge\omega_2=-\mathcal{L}_{\xi_{\underline{f}}}\omega_2\wedge\omega_1=2\Delta f_3\,d\mathrm{vol}_g,\\ \mathcal{L}_{\xi_{\underline{f}}}\omega_2\wedge\omega_3=-\mathcal{L}_{\xi_{\underline{f}}}\omega_3\wedge\omega_2=2\Delta f_1\,d\mathrm{vol}_g,\\ \mathcal{L}_{\xi_{\underline{f}}}\omega_3\wedge\omega_1=-\mathcal{L}_{\xi_{\underline{f}}}\omega_1\wedge\omega_3=2\Delta f_2\,d\mathrm{vol}_g,\end{cases}$$

and, if $f_\alpha=c_\alpha z$, $\alpha=0,1,2,3$, then

$$\begin{cases} \xi_{\underline{f}}\lrcorner\omega_1=-c_0z^{-1}\theta-c_1\,dz-c_2\,dx-c_3\,dy,\\ \xi_{\underline{f}}\lrcorner\omega_2=c_0\,dy+c_1\,dx-c_2\,dz-c_3z^{-1}\theta,\\ \xi_{\underline{f}}\lrcorner\omega_3=-c_0\,dx+c_1\,dy+c_2z^{-1}\theta-c_3\,dz.\end{cases} \tag{6.15}$$

Define the linear operator

$$\begin{aligned}\mathbb{L}\colon C^\infty(\mathcal{C})\otimes\mathbb{R}^4&\longrightarrow C^\infty(\mathcal{C})\otimes\mathbb{R}^4,\\ \underline{f}\equiv(f_0,f_1,f_2,f_3)&\longmapsto R((\mathcal{L}_{\xi_{\underline{f}}}\boldsymbol{\omega}_{\mathcal{C}})^+).\end{aligned}$$

By the above calculation, we have $\mathbb{L}(\underline{f})=\Delta_{\boldsymbol{\omega}_{\mathcal{C}}}\underline{f}$. Denote by $\mathcal{Q}_w$ the region $\{z\geqslant w\}$. We adopt the definition of weighted Hölder spaces for tensors on $\mathcal{Q}_w$ given in §6.3. All the norms appearing in this subsection are taken with respect to $\boldsymbol{\omega}_{\mathcal{C}}$.

Proposition 6.23. *Given any $\delta\in(-\infty,0)\setminus\{-2\}$, $k\geqslant 20$, $\alpha\in(0,1)$, for all $w\geqslant z_0$ there exists a bounded linear map*

$$\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}\colon C^{k-2,\alpha}_\delta(\mathcal{Q}_w)\longrightarrow C^{k,\alpha}_{\delta+2}(\mathcal{Q}_w)$$

such that, for any $v\in C^{k-2,\alpha}_\delta(\mathcal{Q}_w)$, $u=\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}v$ solves $\Delta_{\boldsymbol{\omega}_{\mathcal{C}}}u=v$ and satisfies the uniform estimate

$$\|u\|_{C^{k,\alpha}_{\delta+2}(\mathcal{Q}_w)}\leqslant C(k,\alpha,\delta)\|v\|_{C^{k-2,\alpha}_\delta(\mathcal{Q}_w)}.$$

Remark 6.24. Here, the solution u is not unique. For example, the function z is always harmonic. The latter fact will be useful later.

Proof. The weighted C^0-estimate for u follows from Proposition B.3 (with $\tau=\frac{3}{2}\delta$) and the relation $C^{-1}r^{2/3}\leqslant z\leqslant Cr^{2/3}$. The higher-order estimate can be proved by standard weighted Schauder estimates. □

Now, we fix $\varepsilon>0$ sufficiently small, $k\geqslant 20$ and $\alpha\in(0,1)$.

Proposition 6.25. (Gauge fixing I) *For $w\gg 1$, there is a $C^{k-1,\alpha}$-diffeomorphism F from $\mathcal{Q}_w$ onto the end of $\mathcal{C}$ such that*

$$F^*\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\sigma}',\quad R(d^+\boldsymbol{\sigma}')=0\quad\textit{and}\quad\|\boldsymbol{\sigma}'\|_{C^{k-1,\alpha}_{1/3+\varepsilon}(\mathcal{Q}_w)}<\infty.$$

Proof. We will apply the implicit function theorem to find the desired diffeomorphism. We write $\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ has the growth in (6.14). Then, we have

$$|\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}} R(d^+\boldsymbol{\sigma})|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{-2/3-l+\delta})$$

for all $l\geqslant 0$ and $\delta>0$. We will make further improvements of decay rate of $R(d^+\boldsymbol{\sigma})$.

Let $\underline{f}\equiv-\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}(R(d^+\boldsymbol{\sigma}))$, and denote by $F_t(t\geqslant 0)$ the family of diffeomorphisms generated by the vector field $\xi_{\underline{f}}$. Let $\widetilde{\boldsymbol{\omega}}\equiv F_1^*\boldsymbol{\omega}$. Then, we have

$$\widetilde{\boldsymbol{\omega}}-\boldsymbol{\omega}_{\mathcal{C}}=d\boldsymbol{\sigma}+(F_1^*\boldsymbol{\omega}_{\mathcal{C}}-\boldsymbol{\omega}_{\mathcal{C}})+d(F_1^*\boldsymbol{\sigma}-\boldsymbol{\sigma}).$$

Notice that

$$\frac{d}{dt}F_t^*\boldsymbol{\omega}_{\mathcal{C}}=F_t^*(\mathcal{L}_{\xi_{\underline{f}}}\boldsymbol{\omega}_{\mathcal{C}})=d(F_t^*(\xi_{\underline{f}}\lrcorner\boldsymbol{\omega}_{\mathcal{C}})=d(\xi_{\underline{f}}\lrcorner\boldsymbol{\omega}_{\mathcal{C}}+\boldsymbol{\beta}),$$

where $|\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}\boldsymbol{\beta}|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{-1/3-l+\delta})$ for any $l\in\mathbb{N}$, $\delta>0$. Similarly, we have

$$|\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}\mathcal{L}_{\xi_{\underline{f}}}\boldsymbol{\sigma}|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{-1/3-l+\delta}).$$

Therefore, $\widetilde{\boldsymbol{\omega}}-\boldsymbol{\omega}_{\mathcal{C}}=d\boldsymbol{\sigma}'$, where $\boldsymbol{\sigma}'=\boldsymbol{\sigma}_1'+\boldsymbol{\sigma}_2'$, $R(d^+\boldsymbol{\sigma}_1')=0$,

$$|\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}\boldsymbol{\sigma}'|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{1/3-l+\delta}) \quad\text{and}\quad |\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}\boldsymbol{\sigma}_2'|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{-1/3-l+\delta}).$$

In particular,

$$|\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}R(d^+\boldsymbol{\sigma}')|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{-4/3-l+\delta}).$$

Next, we take $\underline{h}\equiv-\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}(R(d^+\boldsymbol{\sigma}'))$, and denote by G_t the family of diffeomorphisms generated by $\xi_{\underline{h}}$. Let $\widetilde{\boldsymbol{\omega}}'\equiv G_1^*\widetilde{\boldsymbol{\omega}}$, and write $\widetilde{\boldsymbol{\omega}}'-\boldsymbol{\omega}_{\mathcal{C}}=d\boldsymbol{\sigma}''$. Similarly,

$$|\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}\boldsymbol{\sigma}''|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{1/3-l+\delta}),\quad |\nabla^l_{\boldsymbol{\omega}_{\mathcal{C}}}R(d^+\boldsymbol{\sigma}'')|_{\boldsymbol{\omega}_{\mathcal{C}}}=O(r^{-2-l+\delta}). \tag{6.16}$$

Then, we have $\underline{u}\equiv\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}(R(d^+\boldsymbol{\sigma}''))=O(r^\delta)$, and $\xi_{\underline{u}}(\boldsymbol{x})\leqslant Cr(\boldsymbol{x})^{-1+\delta}$ which is much smaller than the injectivity estimate at $\boldsymbol{x}$, which yields the asymptotics

$$\mathrm{Injrad}(\boldsymbol{x})\sim r^{-1/3}(\boldsymbol{x})\quad\text{as } r(\boldsymbol{x})\to\infty.$$

To apply the implicit function theorem on Banach spaces, we will use another way to generate diffeormorphisms from a vector field. Given a vector field ξ whose C^1 norm at each point is much smaller than the injectivity radius of $(\mathcal{C},\boldsymbol{\omega}_{\mathcal{C}})$, we define

$$\Phi_\xi(x)\equiv\exp_x(\xi(x)).$$

For some fixed $\gamma>0$, we consider the map

$$\begin{aligned}\mathscr{F}\colon C^{k,\alpha}_{\gamma}(\mathcal{Q}_w)\otimes\mathbb{R}^4 &\longrightarrow C^{k-2,\alpha}_{-2+\gamma}(\mathcal{Q}_w)\otimes\mathbb{R}^4,\\ \underline{f} &\longmapsto R((\Phi^*_{\xi_{\underline{f}}}\boldsymbol{\omega})^+-\boldsymbol{\omega}_{\mathcal{C}}).\end{aligned}$$

First, $\mathscr{F}$ is a differentiable map. Indeed, this follows from the fact that $\exp_x(V)$ is a smooth map on the tangent bundle. Set $\mathscr{L}\equiv\Delta_{\boldsymbol{\omega}_{\mathcal{C}}}$ and $\mathscr{N}\equiv\mathscr{F}-\mathscr{L}$. Then, $\mathscr{P}\equiv\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}$ is a right inverse to $\mathscr{L}$ with $\|\mathscr{P}\|\leqslant L$ for $L>0$ independent of w. For $\eta>0$ sufficiently small and independent of w, we have, for $\underline{f},\underline{g}\in B_\eta(0)$,

$$\|\mathscr{N}(\underline{f})-\mathscr{N}(\underline{g})\|_{C^{k-2,\alpha}_{-2+\gamma}(\mathcal{Q}_w)}\leqslant(3L)^{-1}\|\underline{f}-\underline{g}\|_{C^{k,\alpha}_{\gamma}(\mathcal{Q}_w)}.$$

Moreover, letting $\delta\equiv\frac{1}{2}\gamma$, by (6.16) we have

$$\|\mathscr{F}(\mathbf{0})\|_{C^{k-2,\alpha}_{-2+\gamma}(\mathcal{Q}_w)}\leqslant C_0w^{-3\gamma/4}.$$

Applying Proposition 2.12, for $w\gg1$, there exists an $\underline{f}\in C^{k,\alpha}_{\gamma}(\mathcal{Q}_w)$ with

$$R((\Phi^*_{\xi_{\underline{f}}}\boldsymbol{\omega})^+-\boldsymbol{\omega}_{\mathcal{C}})=0$$

and

$$\|\underline{f}\|_{C^{k,\alpha}_{\gamma}(\mathcal{Q}_w)}<2C_0Lw^{-3\gamma/4}.$$

As $|\xi_{\underline{f}}|\leqslant2C_0Lw^{-3\gamma/4}r^{-1+\gamma}$, we see that for $w\gg1$, $F\equiv\Phi_{\xi_{\underline{f}}}$ is a diffeomorphism from $\mathcal{Q}_w$ into $\mathcal{C}$. Then, $F^*\boldsymbol{\omega}$ satisfies the desired properties. □

In the above proposition, replacing $\boldsymbol{\omega}$ by $F^*\boldsymbol{\omega}$ and noticing that $Q_{\boldsymbol{\omega}_{\mathcal{C}}}=\mathrm{Id}$, one sees that (2.4) holds, i.e.,

$$d^+\boldsymbol{\sigma}=\mathfrak{F}(\mathrm{tf}(-S_{d^-\boldsymbol{\sigma}})).\tag{6.17}$$

By Proposition 6.23, we can solve $d^*d\boldsymbol{u}=-d^*\boldsymbol{\sigma}$ and replace $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}+d\boldsymbol{u}$, so that

$$d^*\boldsymbol{\sigma}=0,\tag{6.18}$$

and we still have the weighted estimate

$$\|\boldsymbol{\sigma}\|_{C^{k-1,\alpha}_{1/3+\varepsilon}(\mathcal{Q}_w)}<\infty.\tag{6.19}$$

Now, (6.17) and (6.18) form an elliptic system, with linearization at $\boldsymbol{\sigma}=0$ given by the Dirac operator $d^+\oplus d^*$ on $(\mathcal{C},\boldsymbol{\omega}_{\mathcal{C}})$. Notice at this point that the pointwise norm $|\boldsymbol{\sigma}|$ may still grow at infinity.

Step 2 (*$|\boldsymbol{\sigma}|$ is decaying at infinity*).

On $\mathcal{C}$, we can write

$$\boldsymbol{\sigma} = \boldsymbol{p}_0 z^{-1}\theta + \boldsymbol{p}_1\,dx + \boldsymbol{p}_2\,dy + \boldsymbol{p}_3\,dz,$$

where $\boldsymbol{p}_j$, $j=0,1,2,3$, are globally defined $\mathbb{R}^3$-valued functions on $\mathcal{C}$. Notice the pointwise norm

$$|\boldsymbol{\sigma}| = \Big(\sum_j |\boldsymbol{p}_j|^2\Big)^{1/2}.$$

The following result shows that $|\boldsymbol{\sigma}|$ is decaying at a polynomial rate at infinity.

PROPOSITION 6.26. *For all $\delta>0$, we have $\|\boldsymbol{\sigma}\|_{C^{k-1,\alpha}_{-1/3+\delta}(\mathcal{Q}_w)}<\infty$.*

Proof. We set $h(r)\equiv\|\boldsymbol{\sigma}\|_{C^{k-1,\alpha}_{-1/3+\delta}(A_{r,2r})}$, where $A_{r,2r}=\mathcal{Q}_{r^{2/3}}\setminus\mathcal{Q}_{(2r)^{2/3}}$. Then, we define

$$H(r)\equiv\frac{h(2r)}{h(r)}. \tag{6.20}$$

It suffices to prove that

$$\limsup_{r\to\infty} H(r)\leqslant 2^{-1/3}. \tag{6.21}$$

Then, the conclusion follows from an easy iteration. To prove (6.21), we use a contradiction argument. Suppose that there exists a $\delta>0$ such that

$$\limsup_{r\to\infty} H(r) > 2^{-1/3}+\delta > 2^{-1/3}.$$

Then, we can find a sequence $r_j\to\infty$ such that $H(r_j)>2^{-1/3}+\delta$. We now claim that

$$\liminf_{r\to\infty} H(r)=\infty.$$

This would imply that $|\sigma|$ is growing faster than any polynomial rate at infinity, and then we reach a contradiction with (6.19).

To prove the claim, we again use a contradiction argument. Suppose that we can find $s_j\to\infty$ such that $H\big(\frac12 s_j\big)>2^{-1/3}+\delta$, but $H(s_j)\leqslant C$ for some $C>0$. Then, we consider the sequence of rescaled spaces $(A_{s_j/2,4s_j}, s_j^{-1}g_{\boldsymbol{\omega}})$. Passing to a subsequence, they converge to the interval $\big(\frac12,4\big)$ in the asymptotic cone $\mathbb{R}_+$ of $\mathcal{C}$. The universal cover converges to a hyperkähler limit A_∞ which admits a fibration $\pi\colon A_\infty\to\big(\frac12,4\big)$ with fibers the Heisenberg algebra $\mathscr{H}_1$. Denote by $\widetilde{\boldsymbol{\sigma}}$ the lifted triple of 1-forms on the universal cover of $A_{s_j/2,4s_j}$. Let $\widetilde{\boldsymbol{\sigma}}_j=h(s_j)^{-1}\widetilde{\boldsymbol{\sigma}}$. Passing to a subsequence, we have weak $C^{k-1,\alpha}$ convergence of $\widetilde{\boldsymbol{\sigma}}_j$ to a limit $\widetilde{\boldsymbol{\sigma}}_\infty$ on A_∞. Using the fact that $\boldsymbol{\sigma}$ satisfies the elliptic system (6.17)–(6.18) and interior Schauder estimates, we may assume that $\widetilde{\boldsymbol{\sigma}}_j$ converges strongly in $C^{k-1,\alpha}$ to $\widetilde{\boldsymbol{\sigma}}_\infty$ on $\pi^{-1}(1,2)$, which is invariant under $\mathscr{H}_1$. Moreover, it satisfies the linear system $d^+\widetilde{\boldsymbol{\sigma}}_\infty = d^*\widetilde{\boldsymbol{\sigma}}_\infty = 0$. A straightforward computation shows that then we must have

$$\widetilde{\boldsymbol{\sigma}}_\infty = \boldsymbol{c}_0 z_\infty^{-1}\theta_\infty + \boldsymbol{c}_1\, dx_\infty + \boldsymbol{c}_2\, dy_\infty + \boldsymbol{c}_3\, dz_\infty,$$

where $(x_\infty, y_\infty, z_\infty, t_\infty)$ are the standard coordinates on A_∞ (as given in §3.3),

$$\theta_\infty = dt_\infty + x_\infty dy_\infty$$

and $\boldsymbol{c}_j$ are constant vectors. It follows that

$$|\widetilde{\boldsymbol{\sigma}}_\infty| = C|z_\infty|^{-1/2}.$$

This contradicts our assumption. □

Step 3 (*Decay faster than any polynomial rate*).

On $\mathcal{C}$, the $\mathcal{N}$-invariant kernel space of the Dirac operator $d^+ \oplus d^*$ acting on 1-forms is spanned by dx, dy, dz and $z^{-1}\theta$. These forms decay exactly at the rate $r^{-1/3}$. So, in order to improve the decay rate of $\boldsymbol{\sigma}$, we need to gauge out these elements. The first three forms are d-closed, so are easy to deal with; the form $z^{-1}\theta$ is *not* d-closed, and we have to invoke yet another implicit function theorem to eliminate it. For this reason, we also make use of the variation of $\boldsymbol{\omega}_{\mathcal{C}}$ induced by ξ_z (cf. (6.15)), which, by Remark 6.24, does not destroy the previous gauge fixing condition $R((\boldsymbol{\omega}-\boldsymbol{\omega}_{\mathcal{C}})^+)=0$. We denote by $S(w)$ the hypersurface $\{z=w\}\subset\mathcal{C}$, endowed with the induced Riemannian metric from $\boldsymbol{\omega}_{\mathcal{C}}$. By calculation, we have $\mathrm{Vol}(S(w)) = Cw^{1/2}$.

Proposition 6.27. (Gauge fixing II) *For $w \gg 1$, we can find a diffeomorphism F_w from $\mathcal{Q}_w$ onto the end of $\mathcal{C}$ such that $\boldsymbol{\omega}_w \equiv F_w^*\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{C}} + d\boldsymbol{\sigma}_w$, with*

$$\begin{cases} \|\boldsymbol{\sigma}_w\|_{C^{k-1,\alpha}_{-1/3+\delta}(\mathcal{Q}_w)} < \infty, \quad \textit{for all } \delta > 0, \\ R(d^+\boldsymbol{\sigma}_w) = 0 \quad (\implies d^+\boldsymbol{\sigma}_w = \mathfrak{F}(\mathrm{tf}(S_{d^-\boldsymbol{\sigma}_w}))), \\ d^*\boldsymbol{\sigma}_w = 0, \\ \displaystyle\int_{S(2^{2/3}w)} \boldsymbol{p}_j(\boldsymbol{\sigma}_w) = 0, \qquad j = 0,1,2,3. \end{cases} \tag{6.22}$$

Proof. Fix $\delta > 0$ small, and define the Banach spaces

$$\begin{aligned} \mathfrak{D} &\equiv (C^{k,\alpha}_{5/3+\delta}(\mathcal{Q}_w)\otimes\mathbb{R}^4)\oplus(C^{k,\alpha}_{5/3+\delta}(\mathcal{Q}_w)\otimes\mathbb{R}^3), \\ \mathfrak{I} &\equiv (C^{k-2,\alpha}_{-1/3+\delta}(\mathcal{Q}_w)\otimes\mathbb{R}^4)\oplus(C^{k-2,\alpha}_{-1/3+\delta}(\mathcal{Q}_w)\otimes\mathbb{R}^3)\oplus\mathbb{R}^3, \end{aligned}$$

where we fix a standard norm on $\mathbb{R}^3$. Then, we define a map $\mathscr{F}:\mathfrak{D}\to\mathfrak{J}$ by sending $(\underline{f},\boldsymbol{u})$ to

$$\bigg(R((\Phi^*_{\xi_{\underline{f}}}(\boldsymbol{\omega})-\boldsymbol{\omega}_{\mathcal{C}})^+),\, d^*(\boldsymbol{\beta}_{\xi_{\underline{f}}}+\Phi^*_{\xi_{\underline{f}}}\boldsymbol{\sigma}+d\boldsymbol{u}),\, \frac{1}{w^{5/2+3\delta/2}}\int_{S(2^{2/3}w)}\boldsymbol{p}_0(\boldsymbol{\beta}_{\xi_{\underline{f}}}+\Phi^*_{\xi_{\underline{f}}}\boldsymbol{\sigma}+d\boldsymbol{u})\bigg).$$

In the above definition, $\boldsymbol{p}_0(\boldsymbol{\alpha})$ is the $z^{-1}\theta$-component of a triple of 1-forms $\boldsymbol{\alpha}$, and given a vector field ξ, we set $\Phi_\xi(x)\equiv\exp_x(\xi(x))$ and

$$\boldsymbol{\beta}_\xi\equiv\int_0^1\Phi^*_{t\xi}\bigg(\frac{d}{dt}\Phi_{t\xi}\lrcorner\boldsymbol{\omega}_{\mathcal{C}}\bigg)\,dt.$$

Immediately, $d\boldsymbol{\beta}_\xi=\Phi^*_\xi\boldsymbol{\omega}_{\mathcal{C}}-\boldsymbol{\omega}_{\mathcal{C}}$. One can directly verify that $\mathscr{F}$ is a differentiable map and, by Proposition 6.26, $\|\mathscr{F}(0)\|\leqslant Cw^{-\delta/2-3/2}$. Moreover, the differential $d\mathscr{F}$ at zero is given by $d\mathscr{F}_0=\mathscr{L}+\mathscr{M}$, where

$$\mathscr{L}(\underline{g},\boldsymbol{v})=\bigg(\Delta_{\boldsymbol{\omega}_{\mathcal{C}}}\underline{g},\, d^*(\xi_{\underline{g}}\lrcorner\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{v}),\, \frac{1}{w^{5/2+3\delta/2}}\int_{S(2^{2/3}w)}\boldsymbol{p}_0(\xi_{\underline{g}}\lrcorner\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{v})\bigg),$$

$$\mathscr{M}(\underline{g},\boldsymbol{v})=\bigg(R((\mathcal{L}_{\xi_{\underline{g}}}(d\boldsymbol{\sigma}))^+),\, d^*(\mathcal{L}_{\xi_{\underline{g}}}\boldsymbol{\sigma}),\, \frac{1}{w^{5/2+3\delta/2}}\int_{S(2^{2/3}w)}\boldsymbol{p}_0(\mathcal{L}_{\xi_{\underline{g}}}\boldsymbol{\sigma})\bigg).$$

Notice that $\|\mathscr{M}\|$ is small when $w\gg1$. By (6.15), if $\underline{g}=z\underline{c}$, then

$$\boldsymbol{p}_0(\xi_{\underline{g}}\lrcorner\boldsymbol{\omega}_{\mathcal{C}})=\hat{\boldsymbol{c}}\equiv(-c_0,-c_3,c_2).$$

We define a right inverse $\mathscr{P}:\mathfrak{J}\longrightarrow\mathfrak{D}$ of $\mathscr{L}$ by $\mathscr{P}(\underline{h},\boldsymbol{x},\boldsymbol{q})\equiv(\underline{g},\boldsymbol{v})$, which is given as follows. Given $(\underline{h},\boldsymbol{x},\boldsymbol{q})$, we first let $\underline{g}_0=\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}(\underline{h})$ and let $\boldsymbol{v}=\Delta^{-1}_{\boldsymbol{\omega}_{\mathcal{C}}}(\boldsymbol{x}-d^*(\xi_{\underline{g}_0}\lrcorner\boldsymbol{\omega}_{\mathcal{C}}))$. Then, we let $\underline{g}=\underline{g}_0+z\underline{c}$ for a constant vector $\underline{c}$ with $c_1=0$ and, for $\alpha\neq1$, c_α is uniquely determined by

$$\hat{\boldsymbol{c}}=\frac{1}{\mathrm{Vol}(S(2^{2/3}w))}\cdot\bigg(w^{5/2+3\delta/2}\cdot\boldsymbol{q}-\int_{S(2^{2/3}w)}\boldsymbol{p}_0(\xi_{\underline{g}_0}\lrcorner\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{v})\bigg).$$

By (6.15), $d^*(\xi_{z\underline{c}}\lrcorner\boldsymbol{\omega}_{\mathcal{C}})=0$. So, it follows that $\mathscr{L}\circ\mathscr{P}=\mathrm{Id}$.

It is straightforward to check that $\|\mathscr{P}\|\leqslant C$ for $C>0$ independent of w. Moreover, one can directly estimate the non-linear term $\mathscr{N}\equiv\mathscr{F}-\mathscr{L}$. Then, as in the proof of Proposition 6.25, we can find a zero $(\underline{f},\boldsymbol{u})$ of $\mathscr{F}$ for w large, using the implicit function theorem (Theorem 2.12).

Now, we set $F_w\equiv\Phi_{\xi_{\underline{f}}}$. Then, for $w\gg1$, F_w is a diffeomorphism from $\mathcal{Q}_w$ into the end of $\mathcal{C}$. We write $F_w^*\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\sigma}'$, with $\boldsymbol{\sigma}'=\boldsymbol{\beta}_{\underline{f}}+\Phi^*_{\xi_{\underline{f}}}\boldsymbol{\sigma}+d\boldsymbol{u}$. Then, $R(d^+\boldsymbol{\sigma}')=0$ and $d^*\boldsymbol{\sigma}'=0$. By Proposition 6.26, we have $\|\boldsymbol{\sigma}_w\|_{C^{k-1,\alpha}_{-1/3+\delta}(\mathcal{Q}_w)}<\infty$ for all $\delta>0$. Also, the last condition in (6.22) is satisfied for $j=0$. Now, by adding to $\boldsymbol{\sigma}'$ the triple of 1-forms $\boldsymbol{e}_1\,dx+\boldsymbol{e}_2\,dy+\boldsymbol{e}_3\,dz$ for appropriate constant vectors $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$, we can make sure that $\boldsymbol{\sigma}'$ also satisfies the last condition in (6.22) for $j>0$. Notice that dx, dy and dz are d-closed and d^*-closed, and their norm decays at the rate $r^{-1/3}$, so the new $\boldsymbol{\sigma}$ still has the required decaying properties. □

We denote by $H_w(r)$ the function $H(r)$ given in (6.20) associated to $\boldsymbol{\sigma}_w$.

Proposition 6.28. *There exists $W_0>0$ such that, for all $w\geqslant W_0$, we have*

$$\limsup_{r\to\infty} H_w(r)=0.$$

Proof. We first claim

$$\lim_{w\to\infty} H_w(w^{3/2})=0.$$

If not, then, as in the proof of Proposition 6.26, we can take a rescaled limit and obtain a non-trivial limit $\widetilde{\boldsymbol{\sigma}}_\infty$. Now the extra normalization condition (6.22) implies that the limit must be identically zero. This yields contradiction.

Now, suppose that

$$\limsup_{r\to\infty} H_w(r)>0.$$

Then, by (6.21), we can find $w_j\to\infty$ and $r_j\geqslant w_j^{3/2}$ such that $H_{w_j}(r_j)\in\bigl(0,\frac{1}{2}\cdot 2^{-1/3}\bigr)$. But then we take rescaled limit and again get a limit $\widetilde{\boldsymbol{\sigma}}_\infty$. Still, we obtain a contradiction with the fact that $|\widetilde{\boldsymbol{\sigma}}_\infty|$ must be decaying at order $z_\infty^{-1/2}$. □

Now, we let $w=W_0$ and $\boldsymbol{\sigma}\equiv\boldsymbol{\sigma}_{W_0}$. Proposition 6.28 easily implies that, for all $m\geqslant 0$,

$$\|\boldsymbol{\sigma}\|_{C^{k-1,\alpha}_{-m}(\mathcal{Q}_{W_0})}<\infty. \tag{6.23}$$

This means that $\boldsymbol{\sigma}$ decays faster than any polynomial rate.

Step 4 (*Exponential decay*).

First, we prove an improvement of Proposition 6.27.

Lemma 6.29. (Gauge fixing III) *There exists $W_1\geqslant W_0$ such that, for all $w>W_1$, we can find a diffeomorphism F_w defined on the fixed $\mathcal{Q}_{W_1}$ such that*

$$\boldsymbol{\omega}_w\equiv F_w^*\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\sigma}_w,$$

with

$$\begin{cases} \|\boldsymbol{\sigma}_w\|_{C^{k-1,\alpha}_{-10}(\mathcal{Q}_{W_1})}\leqslant C, \\ R(d^+\boldsymbol{\sigma}_w)=0 \quad (\implies d^+\boldsymbol{\sigma}_w=\mathfrak{F}(\mathrm{tf}(S_{d^-\boldsymbol{\sigma}_w}))), \\ d^*\boldsymbol{\sigma}_w=0, \\ \displaystyle\int_{S(w+1)} \boldsymbol{p}_j(\boldsymbol{\sigma}_w)=0, \qquad j=0,1,2,3. \end{cases} \tag{6.24}$$

Furthermore, F_w is of the form Φ_ξ for $\xi=\xi_{\underline{f_w}}$, with

$$\|\underline{f_w}\|_{C^{k,\alpha}_{-9}(\mathcal{Q}_{W_1})}\leqslant C. \tag{6.25}$$

Proof. The proof is the same as Proposition 6.27. The difference here is that we can now fix the domain $\mathcal{Q}_{W_1}$. This follows from the fact that the rapid decay of $\boldsymbol{\sigma}$ guaranteed by (6.23) implies that $|\boldsymbol{p}_0(\boldsymbol{\sigma})|$ is sufficiently small on $S(W_1)$ for $w_1 \gg 1$. This also implies that the above constant C is independent of w. □

Now, we exploit a different scale of the asymptotic geometry of $\mathcal{C}$. We set

$$A_{z_1,z_2} \equiv \mathcal{Q}_{z_1} \setminus \mathcal{Q}_{z_2}$$

and

$$A^{\infty}_{z_1,z_2} \equiv \mathbb{T}^2 \times (z_1, z_2) \subset \mathbb{T}^2 \times \mathbb{R},$$

where $\mathbb{T}^2$ is the flat 2-torus involved in the definition of $\mathcal{C}$. Then, for any fixed $C>0$, as $z \to \infty$, the rescaled annulus $z^{-1/2} A_{z-C,z+C}$ (with respect to the metric $\boldsymbol{\omega}_{\mathcal{C}}$) collapses with uniformly bounded curvature to the domain $A^{\infty}_{-C,C}$ in the product cylinder $\mathbb{T}^2 \times \mathbb{R}$. We define

$$n_w(z) \equiv z^{-1} \int_{A_{z,z+1}} |\boldsymbol{\sigma}_w|^2_{g_{\mathcal{C}}} \, d\mathrm{vol}_{\mathcal{C}} \, .$$

The following arguments are well known in the study of asymptotically conical geometries (see, for example, [18], [45]), and they can easily be adapted to our setting. Let $\underline{\lambda}_1$ be the first eigenvalue of $-\Delta_{\mathbb{T}^2}$.

LEMMA 6.30. (Convexity lemma) *For all* $\delta \in (-\underline{\lambda}_1, \underline{\lambda}_1) \setminus \{0\}$, *there exists a*

$$W_2 = W_2(|\delta|) > W_1$$

such that, for all $w \geqslant W_1$, *if*

$$\log(n_w(W_2+1)) \geqslant \log(n_w(W_2)) + \delta,$$

then

$$\log(n_w(z+1)) \geqslant \log(n_w(z)) + \delta \quad \textit{for all } z \geqslant W_2.$$

Proof. If this fails, then we can find a sequence $w_j \geqslant W_1$ and $z_j \to \infty$ such that

$$\log n_{w_j}(z_j+1) \geqslant \log n_{w_j}(z_j) + \delta,$$

but

$$\log n_{w_j}(z_j+2) < \log n_{w_j}(z_j+1) + \delta.$$

Passing to a subsequence, we may assume that $\tilde{A}_j \equiv z_j^{-1/2} A_{z_j,z_j+3}$ collapses to the domain $A^{\infty}_{0,3}$ in $\mathbb{T}^2 \times \mathbb{R}$. We want to take the limit of $\boldsymbol{\sigma}_{w_j}$. First, set

$$\boldsymbol{\sigma}_j \equiv z_j^{-1/2} n_{w_j}(z_j+1)^{-1/2} \boldsymbol{\sigma}_{w_j}.$$

Then, the average of $|\boldsymbol{\sigma}_j|$ over $\tilde{A}_j$ is uniformly bounded. Moreover, $\boldsymbol{\sigma}_j$ satisfies the elliptic system on $\tilde{A}_j$:

$$\begin{cases} d^+\boldsymbol{\sigma}_j = n_{w_j}(z_j+1)^{1/2}\cdot z_j^{1/2}\cdot \mathfrak{F}(\mathrm{tf}(S_{d^-\boldsymbol{\sigma}_j})), \\ d^*\boldsymbol{\sigma}_j = 0. \end{cases}$$

Since $\|\boldsymbol{\sigma}_w\|_{C^{k-1,\alpha}_{-10}(\mathcal{Q}_{W_1})} < C$ for all $w \geqslant W_1$, by interior elliptic estimates for $d^+\oplus d^*$ over local universal covers one can see that

$$\|\boldsymbol{\sigma}_j\|_{C^{k-2}(z_j^{-1/2}A_{z_j+\tau, z_j+3-\tau})} \leqslant C(\tau)$$

uniformly for any $\tau > 0$ small. In particular, passing to a subsequence, we may obtain C^{k-3} convergence of $\boldsymbol{\sigma}_j$ to $\boldsymbol{\sigma}_\infty$ over local universal covers. Globally, we obtain a pair $(\boldsymbol{q}, \boldsymbol{\lambda})$ on $A^\infty_{0,3}$, where $\boldsymbol{q}$ is a vector-valued function and $\boldsymbol{\lambda}$ is a vector-valued 1-form. This is similar to the discussion of convergence of hyperkähler structures under codimension-1 collapsing in §3.1: $\boldsymbol{q}$ is given by the $\boldsymbol{\sigma}_\infty(\partial_t)$, and $\boldsymbol{\lambda}$ is given by the horizontal component of $\boldsymbol{\sigma}_\infty$. The pair $(\boldsymbol{q}, \boldsymbol{\lambda})$ satisfies

$$d\boldsymbol{q} + * d\boldsymbol{\lambda} = 0 \quad \text{and} \quad d^*\boldsymbol{\lambda} = 0.$$

In particular, both $\boldsymbol{q}$ and $\boldsymbol{\lambda}$ are harmonic. Set

$$n_\infty(z) \equiv \int_{A^\infty_{z,z+1}} (|\boldsymbol{q}|^2 + |\boldsymbol{\lambda}|^2).$$

Then, by construction and the strong interior convergence, we have

$$\log(n_\infty(1)) = 0, \quad \log(n_\infty(0)) \leqslant -\delta \quad \text{and} \quad \log(n_\infty(2)) \leqslant \delta.$$

Given a vector-valued harmonic function $\underline{u}$ on $A^\infty_{0,3}$, it is easy to see via eigenfunction expansion that

$$\|\underline{u}\|_{L^2(A^\infty_{0,1})}\cdot \|\underline{u}\|_{L^2(A^\infty_{2,3})} \geqslant \|\underline{u}\|^2_{L^2(A^\infty_{1,2})},$$

and the equality holds if and only if $\underline{u}$ is homogeneous, i.e.,

$$\underline{u} = \underline{c}e^{\lambda z}\phi_\lambda$$

for some eigenfunction ϕ_λ on $\mathbb{T}^2$. Applying this to the above limit $(\boldsymbol{q}, \boldsymbol{\lambda})$, it follows that $(\boldsymbol{q}, \boldsymbol{\lambda})$ must be homogeneous and δ must be an eigenvalue on $\mathbb{T}^2$. This contradicts our hypothesis on δ. □

LEMMA 6.31. *There exist* $\hat{\delta}\in(0,\lambda_1)$ *and* $W_3>W_1+W_2(\hat{\delta})$ *such that, for all* $w\geqslant W_1$ *and* $z\geqslant W_3$, *we must have*

$$|\log n_w(z+2)-\log n_w(z+1)|\geqslant\hat{\delta}.$$

Proof. Suppose that the conclusion fails. Then, we get a contradicting sequence $w_j\geqslant W_1$ and $z_j\to\infty$ with

$$|\log n_{w_j}(z_j+2)-\log n_{w_j}(z_j+1)|<j^{-1}.$$

Notice that, by Lemma 6.30, we know, for $j\gg1$, that

$$\log n_{w_j}(z_j+1)-\log n_{w_j}(z_j)\leqslant j^{-1}.$$

Then, we can as in the above proof pass to a subsequence and obtain a limit pair $(\boldsymbol{q},\boldsymbol{\lambda})$ on $\mathbb{T}^2\times\mathbb{R}$. This time we use the last condition in (6.24) to conclude that

$$\int_{\{z=1\}}\boldsymbol{q}=\int_{\{z=1\}}\boldsymbol{\lambda}=0.$$

Now it is easy to see, using eigenfunction expansion again, that there exists a $\delta>0$ such that, for a vector-valued harmonic function u on $A^{\infty}_{0,3}$ with $\int_{\{z=1\}}u=0$, either

$$\|u\|_{L^2(A^{\infty}_{1,2})}\geqslant e^{\delta}\|u\|_{L^2(A^{\infty}_{0,1})}\quad\text{or}\quad\|u\|_{L^2(A^{\infty}_{1,2})}\leqslant e^{-\delta}\|u\|_{L^2(A^{\infty}_{0,1})}.$$

This leads to a contradiction. □

LEMMA 6.32. *For all* $w>W_3$, *we have*

$$\log n_w(w+1)-\log n_w(w)\leqslant-\hat{\delta}.$$

Proof. If not, then, by Lemma 6.31, we must have $\log n_w(w+1)-\log n_w(w)\geqslant\hat{\delta}$ for some $w>W_3$. Now, Lemma 6.30 implies that $\log n_w(z+1)-\log n_w(z)\geqslant\hat{\delta}$ for all $z\geqslant w$. This implies that $\boldsymbol{\sigma}_w$ has exponential growth, which yields a contradiction. □

Now, given any $w>W_3$, by Lemma 6.30 again we have

$$\log n_w(z+1)-\log n_w(z)\leqslant-\hat{\delta}$$

for all $z\in[W_3+1,w]$. In particular, we must have

$$n_w(z)\leqslant Cn_w(W_3)e^{-\hat{\delta}z}.$$

Using the elliptic system satisfied by $\boldsymbol{\sigma}_w$, one can see that, passing to a subsequence $w_j\to\infty$, $\boldsymbol{\sigma}_{w_j}$ converges in C^∞_{loc} to a smooth limit $\boldsymbol{\sigma}_\infty$ with

$$\|\nabla^k_{\boldsymbol{\omega}_{\mathcal{C}}}\boldsymbol{\sigma}_\infty\|_{\boldsymbol{\omega}_{\mathcal{C}}}\leqslant C_k e^{-\delta_0 z}$$

for all $k\geqslant 0$ and $\delta\in(0,\hat{\delta})$.

Finally, by (6.25), we may also assume that F_{w_j} converges in $C^{k-2,\alpha}_{\mathrm{loc}}$ to a limit F_∞, which is again a diffomeomorphism from $\mathcal{Q}_{W_1}$ onto the end of $\mathcal{C}$, such that

$$F_\infty^*\boldsymbol{\omega}=\boldsymbol{\omega}_{\mathcal{C}}+d\boldsymbol{\sigma}_\infty.$$

Notice that both $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}_\infty$ are smooth, so F_∞ is indeed smooth. This completes the proof of Theorem 6.19.

7. Discussions and questions

7.1. Towards a bubble tree structure

Let (X^4_j,g_j,p_j) be a sequence of hyperkähler manifolds such that $\overline{B_2(p_j)}$ is compact and $(X^4_j,g_j,p_j)\xrightarrow{\mathrm{GH}}(X_\infty,d_\infty,p_\infty)$. Set $d\equiv\dim_{\mathrm{ess}}(X_\infty)$. If $d=4$, then X_∞ is a hyperkähler orbifold, and it is well known that there is a finite bubble tree structure associated to the convergence (cf. [6]). Now, we assume that $d<4$ and

$$\int_{B_2(p_j)}|\mathrm{Rm}_{g_j}|^2\,d\mathrm{vol}_{g_j}\leqslant\kappa_0,$$

uniformly for some $\kappa_0>0$. It is more involved to describe the bubble tree structure in this case. Here, we make some initial steps.

By Theorems 4.1 and 5.1, we know that there is a unique tangent cone at p_∞, which is a flat metric cone, and we denote it by (Y,d_Y,p^*). Clearly $Y\in\mathcal{B}_{p_\infty}$. Given $j\geqslant 1$ and $\lambda>0$ we denote by $X_{j,\lambda}$ the rescaled space $(X^4_j,\lambda^2 g_j,p_j)$, and by $v_{j,\lambda}$ the volume of the unit ball around p_j in $X_{j,\lambda}$. By the Bishop–Gromov volume comparison, we know that, for a fixed j, $v_{j,\lambda}$ is an increasing function of λ. So, if we rescale sufficiently large, we will get complete hyperkähaler orbifolds as bubble limits. The following result shows that there is an essentially unique scale that leads to a complete hyperkähler orbifold which is collapsing at infinity.

Proposition 7.1. (Maximal scale non-collapsing bubble) *Let $j_i\to\infty$ be any sequence. Then, passing to a subsequence, we may find $\lambda_i\to\infty$ such that the rescaled spaces X_{j_i,λ_i} converge to a complete hyperkähler orbifold (Z,d_Z,p_Z) such that*

$$\mathrm{Vol}(B_R(p_Z))=o(R^4)$$

as $R\to\infty$.

Proof. If this does not hold then, by the Bishop–Gromov volume comparison, one may find a sequence of ALE hyperkähler orbifolds $Z_k \in \mathcal{B}_{p_\infty}$ such that the structure group Γ_k at infinity satisfies $|\Gamma_k| \to \infty$. By a diagonal sequence argument, for each k we may find sequences $j_{i,k} \to \infty$ and $\lambda_{i,k} \to \infty$, and a domain $U_{i,k} \subset X_{j_{i,k},\lambda_{i,k}}$, such that $\partial U_{i,k}$ converges smoothly to the space form S^3/Γ_k. It follows from [6] that $U_{i,k}$ is diffeomorphic to an ALE gravitational instanton, with structure group Γ_k at infinity, which implies that $\chi(U_{i,k}) = c_k$ for i large. Notice that $c_k \to \infty$ as $|\Gamma_k| \to \infty$. On the other hand, by the Chern–Gauss–Bonnet theorem, we have a uniform bound on $\chi(U_{i,k})$ in terms of κ_0. This is a contradiction. □

Notice that Theorem 1.2 also holds for complete hyperkähler orbifolds with finite energy, since the proof only uses the end structure at infinity. So, the bubble limits constructed in the above proposition must be of type AL$\mathfrak{X}$, and is not ALE. This gives a rigorous explanation of the heuristic fact that non-ALE gravitational instantons are responsible for collapsing of hyperkähler manifolds.

Next, we show that the dimension of bubble limits can only increase when we zoom into a smaller scale. This is again true from the intuition.

PROPOSITION 7.2. (Dimension monotonicity) *In the setting above, we have*

$$\dim_{\mathrm{ess}}(Z) \geqslant d \quad \textit{for all } Z \in \mathcal{B}_{p_\infty}.$$

Proof. From the definition of $\mathcal{B}_{p_\infty}$, we can find $\lambda_0 > 0$, $j_0 > 0$ and $\varepsilon_j \to 0$ such that, for all $j \geqslant j_0$ and $\lambda \geqslant \lambda_0$, there is an element $Z_{j,\lambda} \in \mathcal{B}_{p_\infty}$ satisfying $d_{\mathrm{GH}}(X_{j,\lambda}, Z_{j,\lambda}) \leqslant \varepsilon_j$. Notice that a priori $Z_{j,\lambda}$ is not unique, and we simply make an arbitrary choice for each j and λ. It is clear that we can take $Z_{j,\lambda_0} = Y$ for $j \gg 1$.

The conclusion in the case $d=1$ is trivial. We will first prove the case $d=3$. Notice that every element Z in $\mathcal{B}_{p_\infty}$ with $\dim_{\mathrm{ess}} Z = 3$ belongs to the list given in Theorem 4.4, among which there are exactly two elements $Z_1 = \mathbb{R}^3$ and $Z_2 = \mathbb{R}^3/\mathbb{Z}_2$ that are metric cones. Fix $\delta > 0$ small such that any Z in $B_{2\delta}(Y) \cap \mathcal{B}_{p_\infty}$ satisfies $\dim_{\mathrm{ess}}(Z) \geqslant 3$, and

$$B_{2\delta}(Y) \cap \mathcal{B}_{p_\infty} \cap \{Z_1, Z_2\} = \{Y\}.$$

For j large, we let λ_j be the smallest λ such that $d_{\mathrm{GH}}(X_{j,\lambda}, Y) \geqslant \delta$. It is clear that

$$\liminf_{j \to \infty} \lambda_j = \infty.$$

We claim there is some $\tau > 0$ such that $v_{j,\lambda_j} \geqslant \tau$ for all j large. Given this, it follows that $v_{j,\lambda} \geqslant \tau$ for all $\lambda \geqslant \lambda_j$, and then the conclusion easily follows. To prove the claim, suppose it is not true. Then, there is a sequence $j_i \to \infty$ such that $d_{\mathrm{GH}}(X_{j_i,\lambda_{j_i}}, Y) = \delta$

but $v_{j_i,\lambda_{j_i}}\to 0$. Passing to a further subsequence, we may assume that $X_{j_i,\lambda_{j_i}}$ converges to a limit Z_∞ with $\dim_{\mathrm{ess}}(Z_\infty)=3$, and $d_{\mathrm{GH}}(Y,Z_\infty)=\delta$. So, Z_∞ is one of the spaces listed in Theorem 4.4 and $Z_\infty\notin\{Z_1,Z_2\}$. Similar to the proof of Lemma 6.1, one can show that this is impossible. For example, suppose that $Z_\infty=S^1_R\times\mathbb{R}^2$ for some $R>0$. Then, for i large, we know that $Z_{j_i,\lambda_{j_i}}$ must also be $S^1_{R_i}\times\mathbb{R}^2$ for $R_i\to R$. It follows that $Z_{j_i,\lambda_{j_i}/2}$ must be $S^1_{R_i'}\times\mathbb{R}^2$ for some $R_i'\to\frac{1}{2}R$. Then, $d_{\mathrm{GH}}(Z_{j_i,\lambda_{j_i}/2},Y)>\frac{3}{2}\delta$ for i large. This contradicts our choice of λ_{j_i}.

Now, we consider the case $d=2$. We may assume that, for j large,

$$\int_{B_2(p_j)}|\mathrm{Rm}_{g_j}|^2\,d\mathrm{vol}_{g_j}\in[l\varepsilon,(l+1)\varepsilon)$$

for some integer $l>0$, where ε is the constant given in Theorem 3.21. We will prove the conclusion by induction on l. First, consider the case $l=1$. Then, we can proceed similarly to the case $d=3$. The energy bound implies that any $Z\in\mathcal{B}_{p_\infty}$ has at most one singularity. Hence, by Theorem 5.5, any $Z\in\mathcal{B}_{p_\infty}$ with $\dim_{\mathrm{ess}}(Z)=2$ is either isometric to $\mathbf{C}_\beta$ for $\beta\in\mathbb{A}\equiv\left\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6},1\right\}$, or $S^1\times\mathbb{R}$. As above, we fix $\delta>0$ small so that any Z in $B_{2\delta}(Y)\cap\mathcal{B}_{p_\infty}$ satisfies $\dim_{\mathrm{ess}}(Z)\geqslant 2$ and

$$B_{2\delta}(Y)\cap\mathcal{B}_{p_\infty}\cap\{\mathbf{C}_\beta:\beta\in\mathbb{A}\}=\{Y\}.$$

Furthermore, we may assume that $\beta=1$ if there is some $S^1_R\times\mathbb{R}$ in $B_{2\delta}(Y)\cap\mathcal{B}_{p_\infty}$. For j large, let λ_j be the smallest λ such that $d_{\mathrm{GH}}(X_{j,\lambda},Y)\geqslant\delta$. Passing to a subsequence, we may assume that X_{j,λ_j} converges to a limit Z_∞, with

$$\dim_{\mathrm{ess}}(Z_\infty)\geqslant 2\quad\text{and}\quad d_{\mathrm{GH}}(Y,Z_\infty)=\delta.$$

We claim that any such limit Z_∞ must satisfy $\dim_{\mathrm{ess}}(Z_\infty)\geqslant 3$. If not, then there is a limit Z_∞ with $\dim_{\mathrm{ess}}(Z_\infty)=2$. By the choice of δ, it follows that $Z_\infty=S^1_R\times\mathbb{R}$ for some $R>0$ and $Y=\mathbb{R}^2$. Then, a similar reasoning as in the case $d=3$ yields a contradiction. Given the claim, then our conclusion follows from the result in the case $d=3$.

Suppose now that the conclusion holds for $l\leqslant l_0$, and consider the case $l=l_0+1$. If we run the same arguments as above, then in the end we can conclude that any limit Z_∞ either satisfies $\dim_{\mathrm{ess}}(Z_\infty)\geqslant 3$, or $\dim_{\mathrm{ess}}(Z_\infty)=2$ and Z_∞ has at least two singularities. If the latter occurs, suppose that Z_∞ is given as the limit of some subsequence $X_{j_i,\lambda_{j_i}}$, then there exists $\tau>0$ such that, for i large, we have

$$\int_{B_{\tau\lambda_{j_i}}(p_{j_i})}|\mathrm{Rm}_{g_{j_i}}|^2\,d\mathrm{vol}_{g_{j_i}}<l_0\varepsilon.$$

It follows from the induction assumption that any Gromov–Hausdorff limit of $X_{j_i,s_i\lambda_{j_i}}$ for $s_i\geqslant 1$ has dimension at least 2. Using this, one can finish the proof of the case $l=l_0+1$. □

COROLLARY 7.3. *The following statements hold.*

- *Any* $Z\in\mathcal{B}_{p_\infty}$, *with* $\dim_{\mathrm{ess}}(Z)=3$, *is isometric to* $\mathbb{R}^3$ *or* $\mathbb{R}^3/\mathbb{Z}_2$. *If* $d=3$, *then any* $Z\in\mathcal{B}_{p_\infty}$ *is isometric to either the tangent cone* Y, *or an* ALE *or* ALF *hyperkähler orbifold.*
- *Any* $Z\in\mathcal{B}_{p_\infty}$, *with* $\dim_{\mathrm{ess}}(Z)=2$ *and with a unique singularity, is isometric to* $\boldsymbol{C}_\beta$ *for some* $\beta\in\mathbb{A}$.

The first item says that the construction of Foscolo [32] essentially gives the complete picture in the case $d=3$ (modulo further development of orbifold singularities in the ALE bubbles). With more work, one expects to obtain a full bubble tree structure. The latter may also be used to prove the following. Notice that, by Remark 3.18, the statement is false without the uniform energy bound.

Conjecture 7.4. (Integral monodromy) If $d=2$, then the singular special Kähler metric has integral monodromy.

For hyperkähler metrics on the K3 manifold, it may even be possible to explicitly classify all the possible bubble trees.

7.2. Asymptotics of the period map

Let X^d_∞ be the Gromov–Hausdorff limit of a sequence of hyperkähler metrics g_j on the K3 manifold $\mathcal{K}$ with $d\equiv\dim_{\mathrm{ess}}(X^d_\infty)<4$. As before, we make a choice of a hyperkähler triple $\boldsymbol{\omega}_j$ for g_j. We can use Theorems 3.27 and 1.1 to obtain some information on the behavior of $\mathcal{P}(g_j)$ as $j\to\infty$. For example, suppose $d=3$. Then, $X_\infty=\mathbb{T}^3/\mathbb{Z}_2$. We choose $\mathcal{Q}$ to be of the form $U/\mathbb{Z}_2$, where $U\subset\mathbb{T}^3=\mathbb{R}^3/\mathbb{Z}^3$ is the complement of a small neighborhood of the finitely many points which map to the singular set $\mathcal{S}$ in X_∞ (notice that $\mathcal{S}$ contains the eight orbifold points, but it may also contain some other points). Take three disjoint geodesic circles $C_\alpha(\alpha=1,2,3)$ in U which lift to lines in $\mathbb{R}^3$ parallel to the three coordinate axes. Denote by l_α the length of C_α. For j large, we have a circle bundle $F_j\colon\mathcal{Q}_j\to\mathcal{Q}$. Denote by $E_{j,\alpha}=F_j^{-1}(C_\alpha)$ the 2-cycles in $\mathcal{Q}_j$. They span a 3-dimensional isotropic subspace of $H_2(\mathcal{K};\mathbb{Z})$. Since the hyperkähler tripe $\boldsymbol{\omega}_j^\Diamond$ given by Theorem 3.27 is $\mathcal{N}$-invariant, passing to the $\mathbb{Z}_2$-cover, the metric $g_j^\Diamond$ is given by the Gibbons–Hawking ansatz on U. That is, we may write $g_j^\Diamond=\varepsilon_j^2(V_j\cdot g_U+V_j^{-1}\theta_j^2)$, where V_j is a positive harmonic function on U with $V_j\sim\varepsilon_j^{-2}$, $d\theta_j=*dV_j$ and $\varepsilon_j\to0$. It follows that

$$\int_{E_{j,\alpha}}\omega_{j,\beta}=\int_{E_{j,\alpha}}\omega^\Diamond_{j,\beta}=\delta_{\alpha\beta}\cdot\varepsilon_j^2 l_\beta,$$

where $\varepsilon_j V_j^{-1/2}$ is the length of the S^1 orbit with respect to $g_j^{\diamondsuit}$. We also know the volume

$$\int_{X_j} \omega_{j,\alpha}^2 \sim \varepsilon_j^2.$$

A simple consequence is that this case cannot occur for collapsing *polarized* K3 surfaces. For if not, then without loss of generality we may suppose, for some $\lambda_j > 0$, that $\alpha_j = [\lambda_j \cdot \omega_{j,1}]$ is a class in $H^2(X;\mathbb{Z})$ with $\int_X \alpha_j^2 = \sigma$ independent of j. It follows that

$$\int_X \omega_{j,1}^2 = \sigma \lambda_j^{-2}.$$

So, we must have $\varepsilon_j \sim \lambda_j^{-2}$ and $\lambda_j \to \infty$. Then,

$$\int_{E_{j,1}} \alpha_j = \lambda_j \int_{E_{j,1}} \omega_{j,1} \sim \lambda_j^{-3}.$$

Since the integral is always an integer, this is impossible.

Similarly, one can treat the cases $d=2$ and $d=1$, and in each case there is some isotropic subspace of $H_2(\mathcal{K};\mathbb{Z})$ on which we know the asymptotics of the period of the hyperkähler triple. These isotropic subspaces also appear naturally in the Satake compactifications of the locally symmetric space

$$\Gamma \backslash O(3,19)/(O(3) \times O(19)).$$

One expects this to be relevant to the conjecture in [68] mentioned in the introduction. We leave it for future work.

7.3. Topological properties and the L^2-curvature energy

There are easy consequences of Theorem 1.2 which yield topological restrictions on the underlying manifolds of a gravitational instanton. Notice that any non-compact paracompact smooth manifold admits a complete Riemannian metric with quadratic curvature decay.

Corollary 7.5. *The Euler characteristic of a non-flat gravitational instanton is positive and finite.*

Proof. This follows by applying the Chern–Gauss–Bonnet theorem (4.13) to $B_r(p)$, and let r tend to infinity. Using the asymptotics at infinity, it is easy to see that the boundary term goes to zero, and hence

$$\chi(X) = \frac{1}{8\pi^2} \int_X |\mathrm{Rm}_g|^2 \, d\mathrm{vol}_g > 0. \qquad \square$$

COROLLARY 7.6. *A non-flat gravitational instanton has a vanishing first Betti number.*

Proof. By Theorem 1.2, we know that there exists $\kappa \in \left\{1, \frac{4}{3}, 2, 3\right\}$ such that

$$\mathrm{Vol}(B(p,r)) \sim Cr^{\kappa} \quad \text{as } r \to \infty.$$

If $\kappa \leqslant 2$, then the conclusion follows from a result of Anderson [3], together with the precise asymptotic description given in Theorem 1.2. If $\kappa = 3$, then X is ALF. We consider the rescaled spaces $(X^4, R^{-2}g, p)$, which collapse to the asymptotic cone of X^4 as $R \to \infty$. If $b_1(X) > 0$, then by [66] we know that the collapsing must have uniformly bounded curvature on compact sets. The latter implies that X is flat. □

As an immediate application, we consider the smooth quadric

$$Q = \{(x,y) : x^2 + y^2 = 1\}$$

in $\mathbb{C}^2$. Since $\pi_1(\mathbb{C}^2 \setminus Q) = \mathbb{Z}$ and $\chi(\mathbb{C}^2 \setminus Q) = 1$, it follows that $\mathbb{C}^2 \setminus Q$ does not support any gravitational instanton. The interest of this example lies in the fact that it admits a nowhere-vanishing holomorphic 2-form Ω, but we have shown that the Calabi–Yau equation $\omega^2 = C\Omega \wedge \bar{\Omega}$ does not have a solution which is complete at infinity and has finite energy. Notice that $\mathbb{C}^2$ admits a Ricci-flat Kähler metric ω_β with cone angle $2\pi\beta$ along Q for any $\beta \in (0,1]$, by a generalized Gibbons–Hawking ansatz [30]. It is an interesting question to understand the behavior of ω_β as $\beta \to 0$. Notice that $\mathbb{C}^2 \setminus Q$ is the same as $\mathbb{CP}^2 \setminus D$, where D is a singular elliptic curve given by the union of a line and a conic. In the case where D is smooth, it is a consequence of the result of Biquard–Guenancia in [8] that, when $\beta \to 0$, under suitable rescalings the conical Kähler–Einstein metrics on $\mathbb{CP}^2 \setminus D$ converge to the complete Calabi–Yau metric constructed by Tian–Yau in [81]. In our case, one would expect a very different picture; it is interesting to explore the connection with certain algebro-geometric "stability" notion.

It is natural to study when a complete Ricci-flat metric on an open 4-manifold has finite L^2 energy. In this regard, we make the following conjetural topological criterion.

Conjecture 7.7. (Energy finiteness conjecture) Let (X, g) be a complete Ricci-flat 4-manifold, then

$$\int_X |\mathrm{Rm}_g|^2 \, d\mathrm{vol}_g < \infty$$

if and only if X has finite topological type.

Even for exotic $\mathbb{R}^4$, we do not yet know the answer. The known infinite-energy examples of gravitational instantons constructed by Anderson–Kronheimber–LeBrun have infinite Euler characteristic; see [4].

7.4. Generalizations

The ideas and techniques developed in this paper can be likely adapted to more general settings. The first natural extension is to the case of Kähler–Einstein metrics on complex surfaces with non-positive Ricci curvature. In particular, the following question is sensible.

Problem 7.8. Classify complete Kähler–Einstein metrics with finite energy in complex dimension 2.

More generally, one can study the structure of collapsed Einstein metrics and more general canonical metrics in four dimensions, and higher-dimensional metrics of special holonomy, under suitable curvature assumptions. One interesting question arises.

Question 7.9. Do Propositions 7.1 and 7.2 hold for general Einstein metrics in all dimensions?

Over the region where the collapsing is with bounded curvature, it is possible to extend the results of this paper to show that the collapsing metric can be assumed to have genuine nilpotent symmetry. Thus, it leads to the question of understanding the geometry of dimension reduction of canonical metrics under symmetry. Notice that there has already been an extensive literature on the latter topic, mainly towards constructing examples. It seems important to systematically investigate the compactness properties of the dimension reduced equations.

Appendix A. Construction of regular fibrations

Our goal here is to outline the proof of Theorems 3.25 and 6.3. The original construction is due to Cheeger–Fukaya–Gromov in [17]. In our special case, the approach presented here is based on the harmonic splitting map of Cheeger–Colding [14], which makes it more convenient to obtain higher-regularity estimates. This observation has been used in [66] to construct bundle maps with higher regularity. In the volume-non-collapsing case, the existence of a harmonic splitting map can be also proved using the $W^{1,p}$-convergence theory of harmonic functions with respect to renormalized measure; see [2, Corollary 4.5] for more details.

Theorem A.1. (Harmonic splitting map [14]) *Given any $\varepsilon>0$ and $n\geqslant 2$, there exists some $\delta=\delta(n,\varepsilon)>0$ such that the following holds. If (M^n,g,p) is a Riemannian manifold satisfying $\mathrm{Ric}_g\geqslant-(n-1)\delta$ and $d_{\mathrm{GH}}(B_{\delta^{-1}}(p),B_{\delta^{-1}}(0^d))<\delta$, $B_{\delta^{-1}}(0^d)\subset\mathbb{R}^d$, then there exists a harmonic map*

$$\Phi=(u^{(1)},...,u^{(d)})\colon B_2(p)\longrightarrow\mathbb{R}^d$$

such that the following properties hold:

(1) $\Phi: B_2(p)\to B_2(0^d)\subset\mathbb{R}^d$ *is an* ε*-Gromov–Hausdorff approximation*;

(2) $|\nabla u^{(\alpha)}|(x)\leqslant 1+\varepsilon$ *holds for any* $x\in B_2(p)$ *and* $1\leqslant\alpha\leqslant d$;

(3) *the following estimate holds*:

$$\sum_{\alpha,\beta=1}^{d} \fint_{B_2(p)} |\langle\nabla u^{(\alpha)},\nabla u^{(\beta)}\rangle-\delta_{\alpha\beta}|\,d\mathrm{vol}_g+\sum_{\alpha=1}^{d}\fint_{B_2(p)}|\nabla^2 u^{(\alpha)}|^2\,d\mathrm{vol}_g<\varepsilon.$$

We will also need a good cut-off function with uniform derivative estimates. Here we briefly review the standard heat flow regularization, and we refer to [65, Lemma 3.1] for results on general RCD spaces.

LEMMA A.2. *Let* (X^n,g) *be a Riemannian manifold with* $\mathrm{Ric}_g\geqslant 0$. *Assume that, for any* $m\in\mathbb{N}$, *there exists a constant* $\Lambda_m>0$ *such that* $|\nabla^m\mathrm{Rm}_g|\leqslant\Lambda_m$ *uniformly on* X. *Then, for any* $p\in X$ *and* $r\in(0,1]$ *with* $\overline{B_{2r}(p)}$ *compact, there exists a cut-off function* $\psi: X^n\to[0,1]$ *which satisfies the following properties*:

(1) $\psi\equiv 1$ *on* $B_r(p)$ *and* $\psi\equiv 0$ *on* $X\setminus B_{2r}(p)$;

(2) *for any* $m\in\mathbb{Z}_+$, *there exists a constant* $C=C(m,n)>0$ *such that*

$$r^m|\nabla^m\psi|\leqslant C.$$

Proof. Without loss of generality, suppose that $r=1$. The proof below can be made purely local, but to simplify notations we assume X is complete. For any $q\in X$, we first take a cut-off function ρ defined by

$$\rho(y)=\begin{cases}1, & \text{if } y\in B_1^g(q),\\ 2-d_{g_j}(y,q), & \text{if } y\in A_{1,2}^g(q),\\ 0, & \text{if } y\in X\setminus B_2^g(q).\end{cases}$$

For $t>0$, consider the heat flow $\psi_t\equiv H_t(\rho)$ of the 1-Lipschitz cut-off function ρ. It is standard that on X we have the pointwise estimate

$$|\nabla_g\psi_t|^2+\frac{2t}{n}(\Delta_g\psi_t)^2\leqslant 1.$$

Then, for all $y\in X$, we have

$$|\psi_t(y)-\rho(y)|\leqslant\int_0^t|\Delta_g\psi_s(y)|\,ds\leqslant\sqrt{2nt}.$$

Now fix $\tau=\frac{1}{18n^2}$. Then, $\psi_\tau(y)\in\left[\frac{2}{3},1\right]$ for $y\in B_1(q)$ and $\psi_\tau(y)\in\left[0,\frac{1}{3}\right]$ for $y\in X\setminus B_2(q)$.

Next, we choose a smooth cut-off function $h\colon [0,1]\to[0,1]$ which satisfies

$$h(s)=\begin{cases} 1, & \text{if } \frac{2}{3}\leqslant s\leqslant 1,\\ 0, & \text{if } 0\leqslant s\leqslant \frac{1}{3},\end{cases}$$

and set $\psi=h\circ\psi_\tau$. Since ψ_t solves the heat equation, the higher-order derivative estimate of ψ follows from the standard parabolic estimate. □

Proof of Theorem 3.25. We adopt the notation in the setting of Theorem 3.25. By a simple rescaling, we may assume that $\mathrm{Injrad}_{g_\infty}(q)\geqslant 10$ for any $q\in\mathcal{Q}$. To begin with, we fixe $\varepsilon>0$ sufficiently small, and define the rescaled Riemannian metric $h_\varepsilon\equiv\varepsilon^{-4}\cdot g_\infty$ on $\mathcal{R}$. Throughout the proof, we will denote by $\tau(\varepsilon)$ a general function of ε satisfying $\lim_{\varepsilon\to 0}\tau(\varepsilon)=0$. Now, for any $q\in\mathcal{Q}$, there is a harmonic coordinate system

$$\varpi_q\equiv(w_1,\dots,w_d)\colon B_5^{h_\varepsilon}(q)\longrightarrow B_5(0^d)$$

such that

(i) $\Delta_{h_\varepsilon} w_\alpha=0$ for any $1\leqslant\alpha\leqslant d$,

(ii) $|h_{\varepsilon,\alpha\beta}-\delta_{\alpha\beta}|_{C^0(B_4(q))}+|\partial_{w_\gamma}h_{\varepsilon,\alpha\beta}|_{C^0(B_4(q))}\leqslant\tau(\varepsilon)$,

where $h_{\varepsilon,\alpha\beta}\equiv h_\varepsilon(\nabla_{h_\varepsilon}w_\alpha,\nabla_{h_\varepsilon}w_\beta)$. Moreover, we have

$$d_{\mathrm{GH}}(B^{h_\varepsilon}_{\varepsilon^{-1}}(q),B_{\varepsilon^{-1}}(0^d))<\tau(\varepsilon),$$

where $0^d\in\mathbb{R}^d$.

In the following, we will also work with the rescaled metrics $h_i\equiv\varepsilon^{-1}\cdot g_i$ on X_i^4. Unless otherwise specified, the metric balls below will be measured in terms of h_i and h_ε, respectively.

We will prove the theorem in three steps. In the first step, using the harmonic splitting map, we will construct local fiber bundle maps over every ball in $\mathcal{Q}$ which looks like a ball in $\mathbb{R}^d$. The second step is to glue the local fiber bundle maps by the well-behaved partition of unity. In the last step, we will show the desired estimates and identify the topology of the collapsing fibers.

Step 1 (Construction of local fiber bundles).

Let $\{\underline{q}_\ell\}_{\ell=1}^N$ be a subset of $\mathcal{Q}$ such that $\mathcal{Q}\subset\bigcup_{\ell=1}^N B_1(q_\ell)\subset\mathcal{R}$ and, for all $1\leqslant\ell,\ell'\leqslant N$ with $\ell\neq\ell'$, we have $d_{h_\varepsilon}(\underline{q}_\ell,\underline{q}_{\ell'})>\frac{1}{2}$. For every $1\leqslant\ell\leqslant N$, let $q_{i,\ell}\in X_i^4$ be such that

$$(B_{\varepsilon^{-1}}(q_{i,\ell}),h_i)\xrightarrow{\mathrm{GH}}(B_{\varepsilon^{-1}}(\underline{q}_\ell),h_\varepsilon).$$

Then, for any sufficiently large i, we have

$$d_{\mathrm{GH}}(B_{\varepsilon^{-1}}(q_{i,\ell}),B_{\varepsilon^{-1}}(0^d))<2\tau(\varepsilon).$$

Then, there exists a harmonic map

$$\Phi^*_{i,\ell}=(u^{(1)}_{i,\ell},...,u^{(k)}_{i,\ell})\colon B_3(q_{i,\ell})\longrightarrow\mathbb{R}^d$$

which satisfies the following integral estimates:

$$\sum_{\alpha,\beta=1}^{d}\fint_{B_3(q_{i,\ell})}|h_i(\nabla_{h_i}u^{(\alpha)}_{i,\ell},\nabla_{h_i}u^{(\beta)}_{i,\ell})-\delta_{\alpha\beta}|+\sum_{\alpha=1}^{d}\fint_{B_3(q_{i,\ell})}|\nabla^2u^{(\alpha)}_{i,\ell}|^2_{h_i}\leqslant\tau(\varepsilon).$$

Since $(B_4(q_{i,\ell}),h_i)$ is collapsing with uniformly bounded geometry, the above integral estimate can be strengthened to the following pointwise estimate on $B_2(q_{i,\ell})$:

$$\sum_{\alpha,\beta=1}^{k}|g_i(\nabla u^{(\alpha)}_{i,\ell},\nabla u^{(\beta)}_{i,\ell})-\delta_{\alpha\beta}|+\sum_{\alpha=1}^{k}|\nabla^2u^{(\alpha)}_{i,\ell}|^2\leqslant\tau(\varepsilon).$$

This implies that, for every $1\leqslant\ell\leqslant N$, the composition

$$\Phi_{i,\ell}\equiv(\varpi_\ell)^{-1}\circ\Phi^*_{i,\ell}\colon B_2(q_{i,\ell})\longrightarrow B_2(\underline{q}_\ell)$$

is a fiber bundle map, where the diffeomorphism $\varpi_\ell\colon B_2(\underline{q}_\ell)\to B_2(0^d)$ is given by the harmonic coordinate system at $\underline{q}_\ell$. Moreover, $\Phi_{i,l}$ is a $\tau(\varepsilon)$-Gromov–Hausdorff approximation, for all i large.

Step 2 (Gluing local bundle maps).

Let us take domains with smooth boundary $\mathcal{Q}_i\subset\bigcup_{\ell=1}^N B_2(q_{i,\ell})$ such that

$$(\mathcal{Q}_i,h_i)\xrightarrow{\text{GH}}(\mathcal{Q},h_\varepsilon).$$

We will glue the above local harmonic maps to obtain a fiber bundle map $F_i\colon\mathcal{Q}_i\to\mathcal{Q}$.

For every $1\leqslant\ell\leqslant N$, let ψ_ℓ be the good cut-off function in Lemma A.2 such that

$$\psi_\ell(y)=\begin{cases}1, & \text{if } y\in B_1(q_{i,\ell}),\\ 0, & \text{if } y\in X^4_i\setminus B_2(q_{i,\ell}),\end{cases}$$

and, for all $m\in\mathbb{Z}_+$, $|\nabla^m\psi_\ell|\leqslant C_m$ holds everywhere on M^4_i. Then, we take the partition of unity subordinate to the cover $\{B_2(q_{i,\ell})\}_{\ell=1}^N$ of $\mathcal{Q}_i$ given by

$$\phi_\ell\equiv\psi_\ell\biggl(\sum_{\ell=1}^N\psi_\ell\biggr)^{-1}.$$

It follows from volume comparison that the multiplicity in the above cover is bounded by some absolute constant $Q_0 > 0$. We set

$$\mathcal{B}_i \equiv \bigcup_{\ell=1}^{N} B_2(q_{i,\ell}) \quad \text{and} \quad \mathcal{B}_\infty \equiv \bigcup_{\ell=1}^{N} B_2(\underline{q}_\ell).$$

For any $1 \leqslant \ell \leqslant N$, we define

$$\mathfrak{d}_\varepsilon(\underline{x}, \underline{y}) \equiv \sum_{\alpha=1}^{d} |w_\alpha(\underline{x}) - w_\alpha(\underline{y})|^2,$$

which is determined by the harmonic coordinate system $(w_1, ..., w_d)$ on $B_2(q_\ell)$. It follows from the estimates on the harmonic coordinates that $|\mathfrak{d}_\varepsilon - d_{h_\varepsilon}^2| \leqslant \tau(\varepsilon)$ holds on $B_2(\underline{q}_\ell)$. Then, let us define the energy function $\mathfrak{E}: \mathcal{B}_i \times \mathcal{B}_\infty \to (0, \infty)$ by

$$\mathfrak{E}(q_i, q_\infty) \equiv \frac{1}{2} \sum_{\ell=1}^{N} \phi_\ell(q_i) \cdot \mathfrak{d}_\varepsilon(\Phi_{i,\ell}(q_i), q_\infty).$$

By convexity, for any $q_i \in \mathcal{B}_i$, the function $\mathfrak{E}(q_i, \cdot): \mathcal{B}_\infty \to [0, \infty)$ has a unique minimum $\mathfrak{z}(q_i)$. It is straightforward to verify that, for any $q_i \in B_2(q_{i,\ell})$,

$$d_{h_\varepsilon}(\mathfrak{z}(q_i), \Phi_{i,\ell}(q_i)) < \tau(\varepsilon),$$

and

$$|h_i(\nabla_{h_i}(w_\alpha \circ \mathfrak{z}), \nabla_{h_i}(w_\beta \circ \mathfrak{z})) - h_i(\nabla_{h_i} u_{i,\ell}^{(\alpha)}, \nabla_{h_i} u_{i,\ell}^{(\beta)})| \leqslant \tau(\varepsilon).$$

Then, we define the map

$$\begin{aligned} F_i: \mathcal{B}_i &\longrightarrow \mathcal{B}_\infty, \\ q_i &\longmapsto \mathfrak{z}(q_i). \end{aligned}$$

Combining the above estimates on the harmonic splitting maps, harmonic coordinates, as well as the good cut-off functions, we conclude that F_i is non-degenerate, and hence it is a fiber bundle map. For fixed $\varepsilon > 0$, the mapping $F_i: \mathcal{B}_i \to \mathcal{B}_\infty$ converges to a diffeomorphism $F_\infty: \mathcal{B}_\infty \to \mathcal{B}_\infty$ as $i \to \infty$. It is straightforward to check that $F_\infty^{-1} \circ F_i$ is a δ_i-Gromov–Hausdorff approximation with $\lim_{i \to \infty} \delta_i = 0$.

Step 3 (Proof of the higher-order regularity estimates).

In this step, we will rescale everything back to the original metrics g_i and g_∞, respectively. Notice that the uniform estimates for the higher derivatives of the good cut-off functions (constructed in Lemma A.2) hold in our case, and the higher-order estimates for the splitting maps $\Phi_{i,\ell}$ and the harmonic coordinates on $\mathcal{Q}$ hold as well. Then, we obtain the pointwise estimate on the second fundamental form in item (2) and the higher-order estimate $\nabla^k F_i$ in item (3). We skip the details.

We will prove item (4) by contradiction. Assume that there exist a sequence $\eta_i\to 0$, a constant $\tau_0>0$, and a sequence of bundle maps $F_i: \mathcal{Q}_i\to\mathcal{Q}$ which are η_i-Gromov–Hausdorff approximations such that, for all sufficiently large i,

$$\left|\frac{|dF_i(v)|_{g_\infty}}{|v|_{g_i}}-1\right|\geqslant\tau_0 \tag{A.1}$$

holds for a sequence of vectors $v_i\in T_{x_i}\mathcal{Q}_i$ orthogonal to the fiber of F_i. We assume $|v|_{g_i}=1$. We take the universal cover of $B_{r_0}(x_i)$ for some sufficiently small constant $r_0>0$, which gives the following equivariant C^k-convergence for any $k\in\mathbb{Z}_+$:

$$\begin{tikzcd}
(\widetilde{B_{r_0}(x_i)}, \tilde{g}_i, \Gamma_i, \tilde{x}_i) \arrow[r, "C^k"] \arrow[d, "\pi_i"'] & (\widetilde{B}_\infty, \tilde{g}_\infty, \Gamma_\infty, \tilde{x}_\infty) \arrow[d, "\pi_\infty"] \\
(B_{r_0}(x_i), g_i) \arrow[r, "\mathrm{GH}"] & (B_{r_0}(x_\infty), g_\infty),
\end{tikzcd}$$

where

$$\pi_i: (\widetilde{B_{r_0}(x_i)}, \tilde{x}_i)\longrightarrow (B_{r_0}(x_i), x_i)$$

is the Riemannian universal cover with $\pi_i(\tilde{x}_i)=x_i$, $\Gamma_i\equiv\pi_1(B_{r_0}(x_i))$ and Γ_∞ is a closed subgroup in $\mathrm{Isom}_{\tilde{g}_\infty}(\widetilde{B}_\infty)$. The above diagram of equivariant convergence implies that $\pi_\infty: \widetilde{B}_\infty\to B_{r_0}(x_\infty)=\widetilde{B}_\infty/\Gamma_\infty$ is a Riemannian submersion. Let $\widetilde{F}_i\equiv F_i\circ\pi_i$ and let $\tilde{v}_i$ be the lift of v_i to $\tilde{x}_i$. Then, the C^k convergence implies that $\widetilde{F}_i$ converges to π_∞, and $\tilde{v}_i$ converges to a limiting vector $\tilde{v}_\infty$ with $|d\pi_\infty(v_\infty)|_{g_\infty}=1$. This contradicts (A.1), which completes the proof of item (4).

Based on the Gromov–Hausdorff estimate in item (1) and the second fundamental form estimate in item (3), we can conclude that all the fibers of F_i are almost flat manifolds in the sense that

$$\operatorname{diam}(F_i^{-1}(q))^2\cdot|\sec_{F_i^{-1}(q)}|<\delta_i \quad \text{for all } q\in\mathcal{Q},$$

where $\delta_i\to 0$ as $i\to\infty$. If ε is chosen sufficiently small, then items (5) and (6) follow from Gromov and Ruh's theorems on the almost flat manifolds and Fukaya's fibration theorem; see [40], [74], [36], [17]. □

Proof of Theorem 6.3. By Theorem 3.25, there exists a fiber bundle map

$$F_j\colon A^g_{2^j,2^{j+2}}(p) \longrightarrow A^{d_Y}_{2^j,2^{j+2}}(p^*)$$

for any sufficiently large integer $j \geqslant j_0$ such that

$$F_j\colon A^{g_j}_{1,4}(p) \longrightarrow A_{1,4}(p^*)$$

is a δ_j-Gromov–Hausdorff approximation, with $\lim_{j\to\infty}\delta_j=0$. Here, $A^{g_j}_{1,4}(p)$ is the scale down of $A^g_{2^j,2^{j+2}}(p)$ by the factor 2^{-j}.

Now, we glue the above fiber bundle maps over the annuli, and thus obtain a global fiber bundle map

$$F\colon X^4\setminus B_{R_0}(p) \longrightarrow Y^d\setminus B_{R_0}(p^*).$$

The procedure is well known, once we have the higher-derivative estimates on the local bundle maps. We outline the arguments. Denote by Σ the cross-section of the flat cone Y. Now, let us consider the two adjacent annuli $A^g_{2^j,2^{j+2}}(p)$ and $A^g_{2^{j+1},2^{j+3}}(p)$. Then, there exists some isometry $\rho_j\in\mathrm{Isom}(\Sigma)$ such that $|F_{j+1}-\rho_j\circ F_j|<\delta_j$ holds on the intersection $A_{2^{j+1},2^{j+2}}(p)$, where $\lim_{j\to\infty}\delta_j=0$. Moreover, the higher-order regularity estimates in Theorem 3.25 implies that the above approximation can be improved to the C^k sense. Then there exists a self-diffeomorphism $\sigma_j\colon A_{2^{j+1},2^{j+2}}(p)\to A_{2^{j+1},2^{j+2}}(p)$, which is close to the identity map, such that $F_{j+1}=\rho_j\circ F_j\circ\sigma_j$. One can choose $\sigma_j\colon F_j^{-1}(q)\to F_{j+1}^{-1}(q)$ as the normal projection from the fiber. It is indeed a diffeomorphism, since the normal injectivity radius of each fiber has a uniform lower bound. We refer the readers to [17, Proposition A2.2] for more details. Using the good cut-off function

$$\chi_j(y)=\begin{cases} 1, & \text{if } y\in A_{2^{j+5/4},2^{j+7/4}}(p),\\ 0, & \text{if } y\in X^4\setminus A_{2^{j+1},2^{j+2}}(p),\end{cases}$$

given in Lemma A.2, we can construct a modified fibration

$$\widehat{F}_j\colon A^g_{2^j,2^{j+3}}(p) \longrightarrow A^{d_Y}_{2^j,2^{j+3}}(p^*),$$

which satisfies the properties in Theorem 3.25. Inductively, we finally obtain a global fiber bundle map

$$F\colon X^4\setminus B_{R_0}(p) \longrightarrow Y^d\setminus B_{R_0}(p^*),$$

which satisfies items (1) and (2).

The estimate on the second fundamental form in item (3) depends on the special limiting geometry in the hyperkähler setting for the sequence $A^{g_j}_{1,4}$. There are three cases to analyze.

- $d=3$. The limiting universal cover is flat, and the limit of the fibers are given by totally geodesic lines $\mathbb{R}$. Since we have convergence of the second fundamental form (see Lemma 3.29), we get the conclusion.

• $d=2$. The limiting geometry is again flat and the proof is similar to above.

• $d=1$. The limiting geometry is either the flat or nilpotent geometry. In the first case, the proof is the same as above; in the second case, we make use of Lemma 3.20, and one can compute explicitly the relation between the distance function and the coordinate function z. □

Appendix B. Poisson's equation on the Calabi model space

Let $(\mathcal{C}, \boldsymbol{\omega}_{\mathcal{C}})$ be a Calabi model space. We identify $\mathcal{C}$ differentiably with the product space $[2,\infty)\times\mathfrak{N}^3$, where we use the moment coordinate z and $\mathfrak{N}^3$ is a nilmanifold. We first recall the separation of variables arguments in [46]; see [46, §4] for more details. Denote $\mathfrak{N}^3_z\equiv\{z\}\times\mathfrak{N}^3$, and let $\Lambda\equiv\{\Lambda_k\}_{k=0}^{\infty}$ be the spectrum of $-\Delta_{h_0}$ on the fixed slice $\mathfrak{N}^3_{z_0}$. Then, we have $\Lambda_k=(2z_0)^{-1}\cdot\lambda_k+2z_0\cdot j_k^2$, with $\lambda_k\geqslant j_k$ and $j_k\in\mathbb{Z}_{\geqslant 0}$. Given a continuous function u on $\mathcal{C}$, we can write the L^2 expansion

$$u(z,\boldsymbol{y})=\sum_{k=0}^{\infty}u_k(z)\cdot\varphi_k(\boldsymbol{y}),\quad \boldsymbol{y}\in\mathfrak{N}^3_{z_0},$$

where $-\Delta_{h_0}\varphi_k=\Lambda_k\varphi_k$. The equation $\Delta_{\boldsymbol{\omega}_{\mathcal{C}}}u=v$ is equivalent to the fact that, for all k,

$$\frac{d^2u_k(z)}{dz^2}-(j_k^2z^2+\lambda_k)u_k(z)=v_k(z)\cdot z,\quad z\geqslant 1, \tag{B.1}$$

where $v_k(z)$ is the corresponding coefficient in the expansion of v. The corresponding homogeneous equation has two explicit fundamental solutions $\mathcal{F}_k(z)$ and $\mathcal{U}_k(z)$:

(1) if $j_k=\lambda_k=0$, then

$$\mathcal{F}_k(z)=z\quad\text{and}\quad\mathcal{U}_k(z)=1;$$

(2) if $j_k=0$, $\lambda_k>0$, then

$$\mathcal{F}_k(z)=e^{\sqrt{\lambda_k}\cdot z}\quad\text{and}\quad\mathcal{U}_k(z)=e^{-\sqrt{\lambda_k}\cdot z};$$

(3) if $j_k>0$, then

$$\mathcal{F}_k(z)=e^{-j_k\cdot z^2/2}H_{-h-1}(-\sqrt{j_k}\cdot z)\quad\text{and}\quad\mathcal{U}_k(z)=e^{-j_k\cdot z^2/2}H_{-h-1}(\sqrt{j_k}\cdot z),$$

where h satisfies $\lambda_k=(2h+1)j_k$, and

$$H_{-h-1}(y)\equiv\int_0^{\infty}e^{-t^2-2ty}t^h\,dt.$$

When $j_k>0$, we set $y\equiv\sqrt{j_k}\cdot z$ and define

$$F_k(t)\equiv -t^2+2ty+h\log t \quad \text{and} \quad U_k(t)\equiv -t^2-2ty+h\log t.$$

Let t_k and s_k be the unique positive critical points of $F_k(t)$ and $U_k(t)$, respectively:

$$t_k=\frac{y}{2}+\sqrt{\frac{h^2}{2}+\frac{y^2}{4}} \quad \text{and} \quad s_k=-\frac{y}{2}+\sqrt{\frac{h^2}{2}+\frac{y^2}{4}},$$

and define

$$\widehat{F}_k(z)\equiv -\frac{j_k z^2}{2}+F_k(t_k(z)) \quad \text{and} \quad \widehat{U}_k(z)\equiv -\frac{j_k z^2}{2}+U_k(s_k(z)).$$

The following two results are taken from in [46, Lemmas 4.6 and 4.7].

Lemma B.1. *The following uniform estimates hold*:

$$\mathcal{F}_k(z)\leqslant(1+\sqrt{\pi}\,)\widehat{F}_k(z) \quad \textit{and} \quad \mathcal{U}_k(z)\leqslant(1+\sqrt{\pi}\,)\widehat{U}_k(z).$$

Lemma B.2. *The following statements hold*:

(1) *$\widehat{F}_k(z)$ is increasing for $z\geqslant 1$ and $\widehat{U}_k(z)$ is decreasing for $z\geqslant 1$.*

(2) *There exists a uniform constant $C_0>0$ independent of k such that*

$$0<\mathcal{W}_k(z)^{-1}\cdot(e^{\widehat{F}_k(z)+\widehat{U}_k(z)})\leqslant C_0,$$

where

$$\mathcal{W}_k(z)\equiv\mathcal{F}_k'(z)\mathcal{U}_k(z)-\mathcal{F}_k(z)\mathcal{U}_k'(z)$$

is the Wronskian of $\mathcal{F}_k$ and $\mathcal{U}_k$.

We set $\mathcal{Q}_w\equiv\{x\in\mathcal{C}:z(\boldsymbol{x})\geqslant w\}$, and fix any $\tau\in(-\infty,0)\setminus\{-3\}$. The following result is used in the proof of Proposition 6.23.

Proposition B.3. *There exists a constant $C>0$ such that, if $v\in C^5(\mathcal{Q}_w)$ for some $w>2$ satisfies*

$$\sum_{\ell=0}^{5}(z(\boldsymbol{x}))^{3\ell/2}\cdot|\nabla^{\ell}_{g_{\mathcal{C}}}v(\boldsymbol{x})|_{g_{\mathcal{C}}}\leqslant\mathfrak{b}\cdot(z(\boldsymbol{x}))^{\tau} \quad \textit{for all } \boldsymbol{x}\in\mathcal{Q}_w,$$

then $\Delta_{\boldsymbol{\omega}_{\mathcal{C}}}u=v$ has a solution $u\in C^6(\mathcal{Q}_w)$ satisfying

$$|u(\boldsymbol{x})|\leqslant C\cdot\mathfrak{b}\cdot(z(\boldsymbol{x}))^{3+\tau} \quad \textit{for all } \boldsymbol{x}\in\mathcal{Q}_w. \tag{B.2}$$

The proof depends on the estimates of solutions to the non-homogeneous equation (B.1). We write the expansion of v as

$$v(z,\boldsymbol{y}) = \sum_{k=0}^{\infty} v_k(z)\cdot\varphi_k(\boldsymbol{y}).$$

In the case $j_k=0$ and $\lambda_k=0$, we set

$$u_k(z) \equiv \int_w^z \left(\int_w^t v_k(s)\,ds\right)dt. \tag{B.3}$$

In the case $j_k=0$ and $\lambda_k\neq 0$, we set

$$u_k(z) \equiv \frac{1}{2\sqrt{\lambda_k}}\left(e^{-\sqrt{\lambda_k}z}\cdot\int_w^z e^{\sqrt{\lambda_k}t}\cdot v_k(t)\cdot t\,dt + e^{\sqrt{\lambda_k}z}\cdot\int_z^\infty e^{-\sqrt{\lambda_k}t}\cdot v_k(t)\cdot t\,dt\right). \tag{B.4}$$

In the case $j_k\in\mathbb{Z}_+$, we set

$$u_k(z) \equiv \frac{\mathcal{U}_k(z)}{\mathcal{W}_k(z)}\int_w^z \mathcal{F}_k(t)\cdot v_k(t)\cdot t\,dt + \frac{\mathcal{F}_k(z)}{\mathcal{W}_k(z)}\int_z^\infty \mathcal{U}_k(t)\cdot v_k(t)\cdot t\,dt. \tag{B.5}$$

Lemma B.4. *There exists a constant $C>0$ independent of k such that, for all $w>2$, any solution given by* (B.3)–(B.5) *satisfies*

$$\sup_{z\geqslant w}|u_k(z)|z^{-2-\tau} \leqslant C\sup_{z\geqslant w}|v_k(z)|z^{-\tau}.$$

Proof. For (B.3), this is immediate. Below, we only treat the solution given by (B.5). The case for (B.4) can be dealt with in a similar fashion. Set $\mathfrak{B}_k = \sup_{z\geqslant w}|v_k(z)|z^{-\tau}$. We first estimate the second term in (B.5). Applying Lemma B.1, we have

$$\int_z^\infty \mathcal{U}_k(t)\cdot v_k(t)\cdot t\,dt \leqslant C\cdot\mathfrak{B}_k\cdot\int_0^\infty e^{\widehat{U}_k(u+z)+(1+\tau)\log(u+z)}\,du.$$

We set $\widetilde{U}_k(u)\equiv\widehat{U}_k(u+z)+(1+\tau)\log(u+z)$. By a simple computation, if $z>1$, then we have $\widetilde{U}_k'(0) = -j_k\cdot z+(1+\tau)z^{-1}<0$ and $\widetilde{U}_k''(u)<0$ for all $u>0$. Therefore,

$$\widetilde{U}_k(u) \leqslant \widetilde{U}_k(0)+\widetilde{U}_k'(0)\cdot u \quad \text{for all } u\geqslant 0.$$

So, it follows that

$$\int_z^\infty \mathcal{U}_k(t)\cdot v_k(t)\cdot t\,dt \leqslant C\cdot\mathfrak{B}_k\cdot e^{\widetilde{U}_k(0)}\cdot\int_0^\infty e^{\widetilde{U}_k'(0)\cdot u}\,dt \leqslant C\cdot\mathfrak{B}_k\cdot e^{\widehat{U}_k(z)}\cdot z^{2+\tau}.$$

Therefore, combining the above estimate and Lemma B.2 (2), we get

$$\frac{\mathcal{F}_k(z)}{\mathcal{W}_k(z)}\cdot\int_z^\infty \mathcal{U}_k(t)\cdot v_k(t)\cdot t\,dt \leqslant C\cdot\mathfrak{B}_k\cdot\frac{e^{\widehat{F}_k(z)+\widehat{U}_k(z)}}{\mathcal{W}_k(z)}\cdot z^{2+\tau} \leqslant C\cdot\mathfrak{B}_k\cdot z^{2+\tau}.$$

For the first term of (B.5), we apply the uniform estimate in Lemma B.1, the monotonicity of $\widehat{F}_k(z)$ in Lemma B.2, as well as Lemma B.2 (2), to obtain

$$\frac{\mathcal{U}_k(z)}{\mathcal{W}_k(z)}\cdot\int_w^z \mathcal{F}_k(t)\cdot v_k(t)\cdot t\,dt \leqslant C\cdot\mathfrak{B}_k\cdot z^{2+\tau}.$$

Adding up the above two terms, we obtain the conclusion. □

Proof of Proposition B.3. First, consider the case $\Lambda_k>0$. By the same computations as in the proof of [46, Lemma 4.9], there is some constant $C>0$, independent of $k\in\mathbb{Z}_+$, such that

$$\begin{aligned}
&|v_k(z)|\\
&\leqslant C\cdot(\Lambda_k+1)^{-2}\cdot\mathrm{Vol}_{h_0}(\mathfrak{N}^3_{z_0})^{1/2}\cdot\|(-\tau_{h_0})^2 v\|_{C^0(\mathfrak{N}^3_{z_0})}\\
&\leqslant C\cdot(\Lambda_k+1)^{-2}\cdot\mathrm{Vol}_{h_0}(\mathfrak{N}^3_{z_0})^{1/2}\cdot(z^2\|\nabla^4 v\|_{C^0(\mathfrak{N}^3_z)}+z^{3/2}\|\nabla^3 v\|_{C^0(\mathfrak{N}^3_z)}+z\|\nabla^2 v\|_{C^0(\mathfrak{N}^3_z)})\\
&\leqslant C\cdot(\Lambda_k+1)^{-2}\cdot z(\boldsymbol{x})^\tau.
\end{aligned}$$

It is easy to see that the same estimate also holds when $\Lambda_k=0$.

Now, consider the formal solution

$$u(\boldsymbol{x})=\sum_{k=0}^{\infty}u_k(z)\cdot\varphi_k(\boldsymbol{y}),$$

where $u_k(z)$ is given by (B.3)–(B.5). By Lemma B.4 and Weyl's law, we see that $u(\boldsymbol{x})$ is convergent and satisfies

$$|u(\boldsymbol{x})|\leqslant C\cdot z(\boldsymbol{x})^{3+\tau}+C\cdot\left(\sum_{k=2}^{\infty}\frac{1}{k^{4/3}}\right)\cdot(z(\boldsymbol{x}))^{3+\tau}\leqslant C\cdot(z(\boldsymbol{x}))^{3+\tau},$$

where $C>0$ is independent of $x\in\mathcal{Q}_w$, and $w>2$. □

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