# Linear Programming based Reductions for Multiple Visit TSP and Vehicle Routing Problems

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**Abstract**

The multiple traveling salesman problem (mTSP) is an important variant of metric TSP where a set of $k$ salespeople together visit a set of $n$ cities while minimizing the total cost of the $k$ routes under a given cost metric. The mTSP problem has applications to many real-life problems such as vehicle routing. Rothkopf [14] introduced another variant of TSP called many-visits TSP (MV-TSP) where a request $r(v) \in \mathbb{Z}_+$ is given for each city $v$ and a single salesperson needs to visit each city $r(v)$ times and return to his starting point. We note that in MV-TSP the cost of loops is positive, so a TSP solution cannot be trivially extended (without an increase in cost) to a MV-TSP solution by consecutively visiting each vertex to satisfy the visit requirements. A combination of mTSP and MV-TSP, called many-visits multiple TSP (MV-mTSP) was studied by Bérczi, Mnich, and Vincze [3] where the authors give approximation algorithms for various variants of MV-mTSP.

In this work, we show a simple linear programming (LP) based reduction that converts a mTSP LP-based algorithm to an LP-based algorithm for MV-mTSP with the same approximation factor. We apply this reduction to improve or match the current best approximation factors of several variants of the MV-mTSP. Our reduction shows that the addition of visit requests $r(v)$ to mTSP does *not* make the problem harder to approximate even when $r(v)$ is exponential in the number of vertices.

To apply our reduction, we either use existing LP-based algorithms for mTSP variants or show that several existing combinatorial algorithms for mTSP variants can be interpreted as LP-based algorithms. This allows us to apply our reduction to these combinatorial algorithms while achieving improved guarantees.

## 1 Introduction

The traveling salesman problem (metric TSP) is a fundamental problem in combinatorial optimization. Given a complete graph on vertex set $V$ of size $n$ and non-negative edge costs $c_e$ for all edges $e \in E$ that satisfy the triangle inequality, the goal is to find a Hamiltonian cycle of minimum cost that visits all vertices. TSP and its variants have been at the forefront of development of algorithms, in theory as well as practice. From an approximation algorithmic perspective, Christofides [6] and

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Serdyukov [16] gave a $\frac{3}{2}$-approximation algorithm for TSP which was recently improved to roughly $\frac{3}{2} - 10^{-36}$ by Karlin, Klein, and Oveis-Gharan [11].

In this work, we aim to consider the multiple visit versions of metric TSP as well as many of its variants. In the multiple visit version of metric TSP, which we call MV-TSP, we are given a requirement $r(v) \in \mathbb{Z}_+$ for each vertex $v$ in the graph and the goal is to find a closed walk that visits each vertex exactly $r(v)$ times. We note that a TSP solution cannot be trivially extended to a MV-TSP solution since visiting a vertex $v$ twice in a row incurs a cost $c_{vv} > 0$. Simply, introducing $r(v)$ *copies* of each vertex $v \in V$ and solving the TSP instance in the corresponding semi-metric, it is easy to see that any $\rho$-approximation for metric TSP gives a $\rho$-approximation algorithm for MV-TSP. Unfortunately, this reduction is not polynomial-time since the input size is logarithmic in $\max_v r(v)$ while the algorithm takes time polynomial in $\max_v r(v)$. This raises an important question:

Is there a polynomial-time reduction that implies that a $\rho$-approximation for metric TSP gives a $\rho$-approximation for MV-TSP?

We ask the same question for variants of the metric TSP problem, in particular, for the variants inspired by the classical vehicle routing problem. An extension of metric TSP is multiple TSP, which we call mTSP, where there is a specified number of salespeople $k$ and the goal is to find $k$ cycles of minimum total cost such that each vertex is visited by some salesperson. There are several variations depending on whether the salespeople start at a fixed set of depot vertices $D$ and whether all salespeople need to be used. We refer the reader to a survey by Bektas [1] detailing different variants, applications, and several algorithms for mTSP. The many-visit version of mTSP that we call MV-mTSP is again defined similarly: we are given a graph, edge costs, a visit function $r : V \to \mathbb{Z}_+$, an integer $k$, and possibly a set of $k$ depots and the goal is to find $k$ minimum cost closed walks so that each vertex is visited $r(v)$ times. There are several variants depending on whether there are depots, if all salespeople need be used, and if the demands for each vertex can be satisfied by multiple salespeople. Approximation algorithms for many of these variants were studied recently by Bérczi, Mnich, and Vincze [3]. We give a detailed description of the different variants in Section 2.1.

## 1.1 Our Results and Contributions

Our main result is to show there is a polynomial-time reduction which given any $\rho$-approximation algorithm for metric TSP that is *linear programming based* returns a $\rho$-approximation algorithm for MV-TSP. By an LP-based algorithm, we mean any approximation algorithm whose performance guarantee is analyzed with respect to the optimum value of the classical Held-Karp LP relaxation for metric TSP.

**Theorem 1.** *If there exists a polynomial-time $\rho$-approximation algorithm for metric* TSP *whose guarantee is with respect to the Held-Karp LP relaxation (LP* (TSP)*), then there exists a polynomial-time $\rho$-approximation algorithm for* MV-TSP*.*

**Results for MV-mTSP Problems.** We also show that the above reduction also holds for various MV-mTSP variants. This allows us to either obtain improved approximation algorithms or match the best known approximation for many of these variants. Since many variants of mTSP previously only had known combinatorial algorithms, we first reinterpret these algorithms as LP-based by demonstrating that they also bound the integrality gap of the standard Held-Karp-style relaxations for these variants. For metric TSP, this part is analogous to showing that the classical Christofides $\frac{3}{2}$-approximation algorithm bounds the integrality gap of the Held-Karp relaxation to within the bound of $\frac{3}{2}$ as was done by Wolsey [18] and Shmoys and Williamson [17].

| Depot / Tour Restriction | Problem Name | Previous Work Approximation | This Work Approximation |
|---|---|---|---|
| $k$ Depots / Exactly $k$ Tours | MV-mTSP$_+$ | 3 [3] | 2, Theorem 2 |
| $k$ Depots / At Most $k$ Tours | MV-mTSP$_0$ | 2 [3] | 2, Theorem 3 |

Table 1: A comparison of results for variants of MV-mTSP. All the approximation algorithms stated in this table use our LP reduction technique.

Table 1 shows the names and constraints of the different variants of the MV-mTSP problem where we consider variants with $k$ depots and the solution can have either exactly $k$ tours (MV-mTSP$_+$) or at most $k$ tours (MV-mTSP$_0$).

In addition to the above described results, we also consider a variant of MV-mTSP called SD-MV-mTSP$_+$ where there is only one depot and we are required to find exactly $k$ tours that start at the depot and satisfy the visit requirements. We are able to improve on the previous best approximation of 3 [3] for this problem and get an approximation of $\frac{3}{2}$ by applying the following theorem to a result by Frieze [9] that gives a $\frac{3}{2}$-approximation for the single-visit version of the problem.

**Theorem 5.** *If there exists a polynomial-time $\rho$-approximation algorithm for* SD-mTSP$_+$ *whose guarantee is with respect to LP* (7)*, then there exists a polynomial-time $\rho$-approximation algorithm for* SD-MV-mTSP$_+$*.*

**Other Variants.** While the results above rely on our LP reduction technique, we also obtain new approximation results for two additional TSP variants using slightly different methods. Specifically, we improve the best known approximation for the unrestricted mTSP$_+$ problem in the single-visit setting and provide a new approximation guarantee for the single depot many-visit mTSP problem with vertex-disjoint tours.

For the unrestricted mTSP$_+$ problem, where there are no depots, we improve the previous best approximation factor of 4 (due to Bérczi, Mnich, and Vincze [3]) to 2.

**Theorem 4.** *There is a polynomial-time algorithm for the unrestricted* mTSP$_+$ *problem with an approximation factor of* 2*.*

Additionally, we consider the variant of SD-MV-mTSP$_+$ where different tours must be vertex-disjoint, meaning that each vertex is visited the required number of times by exactly one salesperson. While our LP reduction technique does not apply in this setting, we achieve the following result using ideas from Bérczi, Mnich, and Vincze [3].

**Theorem 6.** *There exists a polynomial-time algorithm for the single depot many-visit* mTSP *(*SD-MV-mTSP$_+$*) problem with vertex-disjoint tours, with an approximation factor of $\frac{7}{2}$.*

## 1.2 Related Work

Bérczi, Mnich, and Vincze [2] gave a $\frac{3}{2}$ for MV-TSP. The variant of TSP where multiple salespeople are used is usually referred to as mTSP. Frieze shows a $\frac{3}{2}$-approximation for a variant where $k$ salespeople are required to start a fixed depot vertex $v_1$. Frieze's algorithm generalizes the Christofides-Serdyukov algorithm [6, 16] for metric TSP. The mTSP problem is a relaxation of the vehicle routing problem (VRP). In VRP, a set of $k$ vehicles need to visit a set of customers with known demands while starting and ending at a fixed depot vertex. Further, the set of vehicles have a vehicle capacity which limits the total demand each vehicle can serve. If the vehicle capacity is sufficiently large so that the vehicles are not restricted by the demands then VRP is equivalent to mTSP. Thus, there are several works for VRP that apply ideas from TSP algorithms such as a paper by Christofides, Mingozzi, Toth [7] where the authors give exact VRP algorithms based on finding minimum cost trees.

A different version of mTSP is when the different salespeople are required to start from different depot vertices. Given a set of $k$ depot vertices the goal is to find at most $k$ minimum cost cycles such that each vertex contains exactly one depot and all vertices are contained in exactly one cycle. Rathinam, Sengupta, and Darbha [13] showed a 2-approximation algorithm for this problem which was then improved to $2 - \frac{1}{k}$ by Xu and Rodrigues [19]. Xu and Rodrigues [19] showed a $\frac{3}{2}$-approximation when the number of depots $k$ is constant and very recently Deppert, Kaul, and Mnich [8] showed a $\frac{3}{2}$-approximation for arbitrary $k$.

The mTSP problem with depots can be generalized further when $m$ depots are available and there are $k$ salespeople satisfying $k \leq m$. Both Malik, Rathinam, and Darbha [12] and Carnes and Shmoys [4] gave 2-approximations for this problem. Later, Xu and Rodrigues [20] gave a $(2 - 1/(2k))$-approximation. The algorithm by Xu and Rodrigues [19] can be adapted to this case to get a $\frac{3}{2}$-approximation when $m$ is constant.

Bérczi, Mnich, and Vincze [3] considered various problems that have both the constraints of mTSP and MV-TSP which are referred to as MV-mTSP. They consider 8 different variants of MV-mTSP and show equivalencies among some of the 8 variants. Additionally, they give constant factor approximations for the different variants using many ideas from previous TSP algorithms such as tree doubling.

**Organization.** We also apply this reduction to reduce different variants of MV-mTSP to mTSP, using similar techniques with some modifications. In Section **??**, we introduce a general framework that applies to multiple variants and generalizes the results shown in this section. In Section 4, we apply this framework to MV-mTSP$_0$ and MV-mTSP$_+$, while in Section 5, we apply it to SD-MV-mTSP$_+$. Finally, Sections 4.4 and 5.2 present results for unrestricted mTSP$_+$ and SD-MV-mTSP$_+$ with vertex-disjoint tours, which follow from different methods rather than our reduction technique.

For many variants that we consider, a challenge arises. Several of the existing algorithms give approximation factors as compared to the integral solution and do not compare the algorithm's solution to the cost of the linear programming relaxation. For these problems, we first formulate a Held-Karp style LP relaxation and either show that an existing algorithm has an approximation factor relative to the LP value or give a new algorithm which has a guarantee towards the LP value. For this we use characterizations of matroid intersection polytope which we apply to constrained spanning trees and related problems in Section **??**.

## 2 Preliminaries

A graph $G = (V, E)$ is defined on vertex set $V$ and edge set $E$ which we will always take to be the complete graph in this paper. For sets $A, B \subseteq V$ we denote by $E(A, B) \subseteq E$ edges $e$ such that $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$. We use $E(A)$ as a shorthand for $E(A, A)$ and $\delta(A) = E(A, V - A)$ meaning $\delta(A)$ is the set of edges with exactly one endpoint in $A$. For a single vertex $v$ we write $\delta(v) \subset E$ instead of $\delta(\{v\})$ to the set of edges that contain $v$. The degree of a vertex $v$ is denoted by $d(v)$ which is the number of edges incident to that vertex meaning $d(v) = |\delta(v)|$. We note that any loop on a vertex contribute 2 to the degree. Additionally, for a set of edges $T \subseteq E$, we use $d_T(v)$ to denote the number of edges in $T$ that contain $v$. Throughout the paper we use LPs whose variables correspond to edges of the graph and for LP variable $x \in \mathbb{R}^{|E|}$ we use $x(T) = \sum_{e \in T} x_e$ for all $T \subseteq E$.

We also use the notion of a matroid in this paper. A matroid $\mathcal{M}$ is defined by a ground set $E$ and a collection of independent sets $\mathcal{I} \subseteq 2^E$ satisfying three properties.

1. $\emptyset \in \mathcal{I}$.

2. If $A \in \mathcal{I}$ then $B \in \mathcal{I}$ for all $B \subseteq A$.

3. If $A, B \in \mathcal{I}$ with $|A| < |B|$, then there exists $x \in B - A$ so that $A \cup \{x\} \in \mathcal{I}$.

An independent set of maximum cardinality is called a base. We use two specific matroids in this paper: partition matroids and graphic matroids. A partition matroid is defined by a partition of the ground set $E = P_1 \dot\cup \ldots \dot\cup P_k$ each with a capacity $c_i \leq |P_i|$ and a set $S \in \mathcal{I}$ if $|S \cap P_i| \leq c_i$ for all $i = 1, \ldots, k$. A graphic matroid is defined on a graph $G$ with the set of edges as the ground set and a set $T \subseteq E$ is independent if the set of edges $T$ is acyclic in $G$. All matroids $\mathcal{M}$ have a rank function $r : 2^E \to \mathbb{Z}$ which is defined as $r(S) = \max_{A \subseteq S}\{|A| \mid A \in \mathcal{I}\}$. It is well known that the convex hull of indicator vectors of independent sets in a matroid is described by $\{\mathbf{x} \in \mathbb{R}^{|E|} \mid x \geq 0, x(S) \leq r(S) \text{ for every } S \subseteq E\}$ and for matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2)$ with rank functions $r_1, r_2$ the convex hull of the indicator vectors of common independent sets in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by $\{\mathbf{x} \in \mathbb{R}^{|E|} \mid x \geq 0, x(S) \leq \min(r_1(S), r_2(S)) \text{ for every } S \subseteq E\}$. Moreover, both the matroid and matroid intersection polytopes are TDI (totally dual integral). We refer the reader to Theorem 41.12 in Schrijver's book [15] for more details on matroids and matroid polytopes.

## 2.1 Problem Description

In this section, we formally define the problems and their feasibility requirements. We use the same names and notation as Bérczi, Mnich, and Vincze [3]. Throughout the paper, let $n$ denote the number of vertices in the input graph and let $c : V \times V \to \mathbb{R}_{\geq 0}$ be the cost function. The function $c$ is a *semi-metric*, meaning it satisfies symmetry and the triangle inequality but does not necessarily satisfy $c_{vv} = 0$. Specifically:

1. **Symmetry**: $c_{uv} = c_{vu}$ for all $u, v \in V$.

2. **Triangle Inequality**: $c_{uv} \leq c_{ux} + c_{xv}$ for all $u, v, x \in V$.

Since the triangle inequality implies $c_{vv} \leq 2c_{uv}$ for all $u, v \in V$, our algorithms may use loops to satisfy visit requirements, incurring the corresponding cost. A loop at a vertex counts as one visit and contributes twice to the vertex's degree. In many single-visit variants such as TSP, loops violate feasibility, so for most single-visit variants, we assume $c_{vv} = 0$ and that $c$ is a metric. However, there is one exception: Unrestricted mTSP$_+$, a single-visit problem that allows loops. We provide more details below.

We first describe the simpler variants:

1. **Metric TSP (**TSP**)**: Given a complete graph $G$ with vertex set $V$ and edge weights satisfying the triangle inequality, the goal is to find a minimum-cost Hamiltonian cycle.

2. **Many-Visit TSP (**MV-TSP**)**: Given a complete graph $G$ with vertex set $V$ and a visit function $r : V \to \mathbb{Z}_{\geq 1}$, the goal is to find a minimum-cost closed walk such that each vertex $v$ is visited exactly $r(v)$ times.

3. **Multiple TSP** (mTSP): Given a complete graph $G$ with vertex set $V$ and a set of $k$ depots $D \subseteq V$, the feasible solutions differ based on the following variants:

   3.1. mTSP$_+$: Find exactly $k$ cycles such that every vertex is included in exactly one cycle and each cycle contains exactly one depot.

   3.2. mTSP$_0$: Find at most $k$ cycles such that every non-depot vertex is included in exactly one cycle and each cycle contains exactly one depot.

We primarily study hybrid variants combining many-visit TSP and multiple TSP, denoted as MV-mTSP. These variants inherit the depot-based parameter from mTSP, giving rise to two problems:

1. MV-mTSP$_+$: Given a complete graph $G$ with vertex set $V$ and a subset of depots $D \subseteq V$ with $|D| = k$, find exactly $k$ closed walks such that each non-depot vertex $v$ is visited $r(v)$ times and each walk contains exactly one depot.

2. MV-mTSP$_0$: Given a complete graph $G$ with vertex set $V$ and a subset of depots $D \subseteq V$ with $|D| = k$, find at most $k$ closed walks such that each non-depot vertex $v$ is visited $r(v)$ times and each walk contains exactly one depot.

**Single Depot Variants.** These variants involve a single depot and $1 \le k \le n-1$ salespeople, with all closed walks starting at the depot vertex. Since all tours must start at the same depot, allowing at most $k$ tours is equivalent to a single-tour solution.

1. SD-mTSP$_+$: Given a complete graph and an integer $1 \le k \le n-1$, find exactly $k$ cycles of at least three vertices such that all cycles include the depot $v_1$, and every other vertex $v \neq v_1$ is in exactly one cycle. If cycles with two vertices were allowed (i.e., $v_1, v, v_1$), Frieze [9] shows a reduction to the case where cycles must have at least three vertices.

2. SD-MV-mTSP$_+$: Given a complete graph, an integer $1 \le k \le n-1$, and a visit function $r : V \setminus \{v_1\} \to \mathbb{Z}_{\ge 1}$, find exactly $k$ closed walks starting at $v_1$ such that all non-depot vertices $v$ are visited exactly $r(v)$ times.

3. SD-MV-mTSP$_+$ **with Vertex-Disjoint Tours**: Given a complete graph, an integer $k \ge 1$, and a visit function $r : V \setminus \{v_1\} \to \mathbb{Z}_{\ge 1}$, find exactly $k$ closed walks starting at $v_1$ such that all non-depot vertices $v$ are visited exactly $r(v)$ times and any two closed walks only intersect at the depot.

Bérczi, Mnich, and Vincze also consider whether depot vertices are present. If depots exist, no cycle can contain multiple depots. If depots are absent, the problem is called *unrestricted*. We analyze one unrestricted variant:

1. **Unrestricted** mTSP$_+$: Given an integer $k$, find exactly $k$ cycles spanning the graph. Loops are valid cycles in this variant.

**Representation of solution.**

## 3 Overview of Technique and MV-TSP Approximation

We now give an overview of the main technique using the example of MV-TSP and prove Theorem 1. We assume that there is a $\rho$-approximation for metric TSP which is LP-based. As previously mentioned, the run-time of the algorithm for MV-TSP needs to be polynomial in $\max_{v\in V}\log r(v)$ and $n$. A simple exponential time approximation algorithm for MV-TSP is to make $r(v)$ copies of each vertex $v$ and apply a metric TSP $\rho$-approximation algorithm to this graph. On the other hand, if $\max_{v\in V} r(v)$ was polynomial in $n$ then we would get a $\rho$-approximation polynomial-time algorithm. Our main technique is to use an LP relaxation of MV-TSP to fix certain edges in our solution (without taking a loss in the objective) and construct a new instance where the visit requirement of each vertex is polynomial. We then apply the simple reduction to TSP that we described above. We note that our reduction relies on the connection between the LP relaxations (MV-TSP) and (TSP): the LP relaxations only differ in that TSP requires every vertex has degree 2 while MV-TSP has degree $2r(v)$. As a result, our reduction is limited in that we cannot use any algorithm for TSP but only an algorithm that has a guarantee towards the LP relaxation of TSP. To illustrate our technique in more detail, let us return to the MV-TSP problem. We use the following standard Held-Karp LP relaxation for metric TSP which we call LP (TSP),

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} c_e x_e & & \text{(TSP)} \\ \text{s.t.} \quad & x(\delta(v)) = 2 & & \forall v \in V \\ & x(\delta(S)) \geq 2 & & \forall S \subset V \\ & 0 \leq x_e \leq 1 & & \forall e \in E \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} c_e x_e & & \text{(MV-TSP)} \\ \text{s.t.} \quad & x(\delta(v)) = 2r(v) & & \forall v \in V \\ & x(\delta(S)) \geq 2 & & \forall S \subset V \\ & x_e \geq 0 & & \forall e \in E \end{aligned}$$

We need the following lemma which shows that the simple reduction from MV-TSP to metric TSP is polynomial-time when the visit requests $r(v)$ are polynomial in $n$. Moreover, the reduction maintains the approximation factor of the LP relaxation based algorithm used for metric TSP. The reduction basically relies on replacing each vertex with $r(v)$ copies and then applying the LP-based algorithm.

**Lemma 3.1.** *Suppose there is a $\rho$-approximation algorithm for metric* TSP *that given an instance on a complete graph $G = (V, E(G))$ with distances $c : E(G) \to \mathbb{R}_{\geq 0}$ returns a Hamiltonian cycle $C$ such that $\sum_{e\in C} c_e \leq \rho \cdot z^\star$ where $z^\star$ is the optimum value of LP* (TSP)*. Then there exists an algorithm that given an instance of the* MV-TSP *on a complete graph $H = (V, E(H))$ with distance function $c : E(H) \to \mathbb{R}_{\geq 0}$ and requirements $r : V \to \mathbb{Z}_+$ outputs a closed walk $T : E \to \mathbb{Z}$ satisfying $\sum_{e\in E(H)} T(e)c_e \leq \rho \sum_{e\in E(H)} c_e y_e$ where $y$ is the optimal solution to LP* (MV-TSP)*. The running time of the algorithm is polynomial in $\max_{v\in V} r(v)$ and $|V|$.*

*Proof.* Given the instance on MV-TSP on graph $H$, we construct an expanded graph $H^r$ by making $r(v)$ copies of each vertex in $H$. The distance between any two copies of the same vertex $v$ is defined to be cost of the loop at $v$, i.e., $c_{vv}$ and the the distance between two copies of distinct vertices $u$ and $v$ is identical to distance between $u$ and $v$. We now apply the TSP approximation algorithm on the new instance $H^r$ to obtain a Hamiltonian cycle $C$ in the expanded graph. We can interpret this Hamiltonian cycle as a solution to MV-TSP. Observe that the cost of the solution is exactly the cost of the Hamiltonian cycle in the cost defined as above. Thus to prove the lemma, it is enough to show there exists a feasible solution to LP (TSP) on the instance $H^r$ whose cost is at most the cost of optimal solution $y$ to LP (MV-TSP) on the instance $H$.

We will convert the solution $y$ to a solution $x$ to LP (TSP) on the graph $H^r$. Let $e' = \{u_i, v_j\}$ and $e = \{u, v\}$ where $u, v$ are the original copies of $u_i, v_j$ in $V$, then we set $x_{e'} = \frac{y_e}{r(u)r(v)}$. Now we show that $x$ is a feasible solution to LP (TSP) for the graph $H^r$. For any vertex $v \in V^r$ and $i \in [r(v)]$ we have $x(\delta(v_i)) = \frac{y(\delta(v))}{r(v)} = 2$ where the first equality follows since the degree of each vertex $v$ in $y$ is distributed evenly among all $r(v)$ copies in $x$ the second equality follows by the feasibility of $y$.

For any $S \subset V^r$, we need to show $x(\delta(S)) \geq 2$. Let $k$ be the number of vertices $v \in V$ such that there exists $v_i, v_j$ that are distinct copies of $v$ and $S$ contains exactly one of $v_i, v_j$. We show $x(\delta(S)) \geq 2$ for all $S \subset V^r$ by induction on $k$. If $k = 0$, then $x(\delta(S)) = y(\delta(S'))$ where $S' \subset V$ is acquired by taking the original copy of each vertex $v_j$ from $S$ implying $x(\delta(S)) \geq 2$ since $y(\delta(S')) \geq 2$ since $y$ is feasible to LP (TSP). If $k > 0$, then there exists a vertex $v \in V$ such that both $S$ and $V^r - S$ have copies of $v$. We define the following subsets of vertices based on the set $S$,

1. let $S(v)$ be the copies of $v$ in $S$ meaning $S(v) := S \cap \{v_1, \ldots, v_{r(v)}\}$
2. let $B$ be the complement of $S$ in $V^r$ meaning $B := V^r - S$
3. let $B(v)$ be the copies of $v$ not in $S$ meaning $B(v) := \{v_1, \ldots, v_{r(v)}\} - S(v)$.

First we consider the case when $B(v) = B$. We note that $S = S(v)$ and $B = B(v)$ cannot hold simultaneously since $|V| \geq 3$. WLOG, we assume that $S(v) = S$ otherwise we can switch $B$ and $S$ since $x(\delta(S)) = x(\delta(B))$. Then we have,

$$\begin{aligned} x(\delta(S)) &= x(\delta(S(v))) \\ &= |S(v)| \left(2 - x_{vv}(|S(v)| - 1)\right) \\ &\geq |S(v)| \left(2 - \frac{|S(v)| - 1}{r(v)}\right) \\ &\geq |S(v)| \left(2 - \frac{|S(v)| - 1}{|S(v)| + 1}\right) \\ &\geq 2. \end{aligned}$$

The first inequality holds since $x_{vv} = \frac{y_{vv}}{r(v)^2} \leq \frac{1}{r(v)}$ because $y(\delta(v)) = 2r(v)$ implies that $y_{vv} \leq r(v)$ since a loop contributes twice to the degree count of a vertex. The second inequality holds since $S(v)$ does not contain all copies of $v$ so $|S(v)| < r(v)$ and the third inequality holds since the function $f(x) = x(2 - \frac{x-1}{x+1})$ is an increasing function that is minimized at $x = 1$.

Now we can assume that $S(v) \subset S$ and $B \subset B(v)$. For any $v_i \in S(v)$ let $X_1 = x(E(v_i, S - S(v)))$ and $X_2 = x(E(v_i, B - B(v)))$. We note that $X_1, X_2$ do not change based on the choice of $v_i$ since all copies of $v$ are defined identically in $H^r$. Now we consider the cut $S - S(v)$ and by using the fact that $x_{vv} \geq 0$ we have,

$$\begin{aligned} x(\delta(S - S(v))) &= x(\delta(S)) + |S(v)| \left(X_1 - |B(v)| x_{vv} - X_2\right) \\ &\leq x(\delta(S)) + |S(v)| \left(X_1 - X_2\right). \end{aligned}$$

First we consider the case when $X_1 \leq X_2$ and here we have that $x(\delta(S - S(v))) \leq x(\delta(S))$ and by induction we get that $x(\delta(S - S(v))) \geq 2$ since $S - S(v)$ is a set with one less vertex $v$ that separates a pair of copies of $v$ and $S - S(v) \neq V^r$. Thus we have shown $x(\delta(S)) \geq 2$ and now we consider the case when $X_1 > X_2$. Here we have,

$$\begin{aligned} x(\delta(S+B(v))) &= x(\delta(B-B(v))) \\ &= x(\delta(B)) + |B(v)|\,(X_2 - |S(v)|x_{vv} - X_1) \\ &\le x(\delta(B)) + |B(v)|\,(X_2 - X_1) \\ &< x(\delta(B)) \\ &= x(\delta(S)). \end{aligned}$$

The first inequality follows since $x_{vv} \ge 0$ and the last inequality follows since we are in the case when $X_1 > X_2$. By induction we have $x(\delta(S+B(v))) \ge 2$ since $S+B(v)$ and is a set with one less vertex $v$ that separates a pair of copies of $v$ and $S+B(v) \ne V^r$. Thus, we have shown that $x(\delta(S)) \ge 2$.

Finally, we show that $0 \le x \le 1$. The lower bound follows immediately since $y \ge 0$, so it remains to prove that $x \le 1$. Suppose there exists $e = \{u,v\} \in E^r$ with $x_e > 1$. This implies that $x(E(u, V^r - \{u,v\})) < 1$ and $x(E(v, V^r - \{u,v\})) < 1$ since $2 = x(\delta(v)) = x(E(v, V^r - \{u,v\})) + x_e > x(E(v, V^r - \{u,v\})) + 1$ and $2 = x(\delta(u)) = x(E(u, V^r - \{u,v\})) + x_e > x(E(v, V^r - \{u,v\})) + 1$. Thus, we get that $x(\delta(\{u,v\})) = x(E(v, V^r - \{u,v\})) + x(E(u, V^r - \{u,v\})) < 2$ which is a contradiction since we showed $x(\delta(S)) \ge 2$ for all $S \subset V^r$.

Thus, we can apply the algorithm from the lemma assumption on $x$ as a solution to $H^r$ to get a Hamiltonian cycle $C$ in $H^r$ satisfying $\sum_{e \in C} c_e \le \rho c^T x$. We note that $c^T y = c^T x$ since $c^T x = \sum_{e=\{u,v\} \in E(H^r)} r(u) r(v) x_e c_e = \sum_{e \in E(H)} c_e y_e$. We now convert $C$ to a closed walk in $H$ denoted by $T : E(H) \to \mathbb{Z}$ by replacing every copy edge $\{u_i, v_j\}$ with its corresponding original edge $\{u,v\}$ in $H$. Clearly, $T$ is a closed walk in $H$. Moreover, $T$ visits every vertex $r(v)$ times in $H$ since $C$ visits each copy of $v$ one time in $H^r$. Lastly, the run-time follows since $H^r$ has at most $n \max_{v \in V} r(v)$ vertices and the algorithm in the lemma statement is a polynomial-time algorithm. □

Now we show how to use the algorithm given in Lemma 3.1 for a general instance where $r$ is not polynomially bounded. This algorithm (Algorithm (1)) solves LP (MV-TSP) and fixes edges in the solution that are integrally set and reduce the visit requests accordingly. Finally, the reduced visits are polynomial, so we can then apply Lemma 3.1. One has to carefully verify that the *reduced* linear programming solution is a feasible solution to the LP relaxation for the reduced instance which can be done by verifying the constraints carefully.

**Algorithm 1** MV-TSP Reduction Algorithm

**Input:** $G = (V,E), c : V \times V \to \mathbb{R}_{\ge 0}, r : V \to \mathbb{Z}_+$.

**Output:** an integral solution to LP (MV-TSP).

1 Solve LP (MV-TSP) to get solution $x^*$ .

2 For all edges $e$, let $\tilde{x}_e := x_e - 2k_e$ and $k_e := 0$ if $x_e \le 4$ and otherwise $k_e$ is set so that $2 \le \tilde{x}_e < 4$ and $k_e \in \mathbb{Z}$. Define a function $\tilde{r} : V \to \mathbb{Z}_+$ where $\tilde{r}(v) = r(v) - \sum_{e \in \delta(v)} k_e$.

3 Use Lemma 3.1 on the instance $G, \tilde{r}$ while setting $y$ as $\tilde{x}$ to get a closed walk $T : E \to \mathbb{Z}$.

4 Increase the number of times each edge is used in the previous step by $2k_e$ and return the resulting solution.

The next two claims will help to show that Step 3 of the Algorithm (1) runs in polynomial-time.

**Claim 3.2.** *The new visit function $\tilde{r}$ satisfies $1 \le \tilde{r}(v) \le 2n$ for all $v \in V$.*

*Proof.* For $v \in V$ we have,

$$\tilde{r}(v) = r(v) - \sum_{e\in\delta(v)} k_e$$
$$= r(v) - \sum_{e\in\delta(v)} \frac{x_e - \tilde{x}_e}{2} = \frac{1}{2}\sum_{e\in\delta(v)} \tilde{x}_e \ .$$

If all $e \in \delta(v)$ satisfy $x_e \leq 4$ then $\tilde{r}(v) = r(v) \geq 1$. Otherwise, the lower bound follows since for $x_e > 4$ we have $\tilde{x}_e \geq 2$. The upper bound follows since $\tilde{x}_e \leq 4$ for all $e \in E$. □

**Claim 3.3.** *The solution $\tilde{x}$ is feasible for LP* (MV-TSP) *with graph $G$ and $\tilde{r}$.*

*Proof.* We have $\tilde{x} \geq 0$ and $\sum_{e\in\delta(v)} \tilde{x}_e = \sum_{e\in\delta(v)} x_e - 2k_e = 2(r(v) - \sum_{e\in\delta(v)} k_e) = 2\tilde{r}(v)$. For any set $S \subset V$, if $x_e \leq 4$ for all $e \in \delta(S)$ then $\tilde{x}(\delta(S)) = x(\delta(S)) \geq 2$. Otherwise, if there is an edge $e \in \delta(S)$ such that $x_e > 4$ we get $\tilde{x}(\delta(S)) \geq \tilde{x}_e \geq 2$ by definition of $\tilde{x}$. □

Now we complete the proof of Theorem 1.

**Theorem 1.** *If there exists a polynomial-time $\rho$-approximation algorithm for metric* TSP *whose guarantee is with respect to the Held-Karp LP relaxation (LP* (TSP)*), then there exists a polynomial-time $\rho$-approximation algorithm for* MV-TSP.

*Proof.* Let $T'$ be the solution we get from the third step before we increase each edge by $2k_e$. By Claim 3.2 and Lemma 3.1 we have that $\tilde{x}$ satisfies $\sum_{e\in E} T'(e)c_e \leq \rho \sum_{e\in E} c_e\tilde{x}_e$. Let $S = \{e \in E \mid k_e > 0\}$ be the set of edges then we have that $\sum_{e\in E} T(e)c_e = \sum_{e\in E} T'(e)c_e + 2\sum_{e\in S} k_e c_e \leq \rho \sum_{e\in E} c_e\tilde{x}_e + 2\sum_{e\in S} k_e c_e \leq \rho c^T x^*$. For the run-time, we have that Step 3 runs in time polynomial in $n$ by Claim 4.6 and Lemma 4.5 and the rest of the steps are clearly polynomial-time in $n$. Finally, $T$ is a feasible integral solution to LP (MV-TSP) which follows from the following two points

1. $T'$ satisfies the cut constraints implying that $T$ also satisfies the cut constraints because $T(e) \geq T'(e)$ for all $e$

2. by definition $T$ satisfies degree constraints.

□

As a corollary of Theorem 1 we get the following by applying the work of Karlin, Klein, and Oveis-Gharan [10].

**Corollary 1.1.** *There is a polynomial time approximation algorithm for the* MV-TSP *problem with an approximation factor less than $\frac{3}{2} - 10^{-36}$.*

# 4 MV-mTSP

In this section, we present approximation algorithms for variants of mTSP. Specifically, we give a 2-approximation for MV-mTSP$_+$, MV-mTSP$_0$, and unrestricted mTSP$_+$. For MV-mTSP+ and MV-mTSP$_0$, we first provide an LP-based analysis of known algorithms for their single-visit counterparts, mTSP$_+$ and mTSP$_0$. We then show the reduction to the many-visits version, which closely follows the approach in Section 4.2 and Section 3.

### 4.1 $\mathrm{mTSP}_+$ Approximation

A 2-approximation was given by Bérczi, Mnich, and Vincze [3] for $\mathrm{mTSP}_+$. This algorithm is a simple combinatorial tree-doubling algorithm. To allow us to use this algorithm in our reduction, we first show that the tree-doubling algorithm achieves a 2-approximation relative to its LP relaxation for the single-visit case in Lemma 4.4. Given the set of depot vertices $D \subseteq V$, we get the following LP,

$$\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e x_e & & \qquad (\mathrm{mTSP}_+)\\
\text{s.t.} \quad & x(\delta(v)) = 2 & & \forall v \in V \\
& x(\delta(S \cup D)) \geq 2 & & \forall S \subset V - D \\
& x(E(D,D)) = 0 & & \\
& 0 \leq x_e \leq 2 & & \forall e \in E \ .
\end{aligned}$$

We need the notion of $D$-forest cover for the algorithm.

**Definition 1.** *A $D$-forest cover is a forest cover such that each component includes exactly one vertex from $D$ and each component has at least two vertices.*

Then we have the following algorithm.

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**Algorithm 2** Tree Doubling Algorithm for $\mathrm{mTSP}_+$

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**Input:** $G = (V, E), D \subseteq V, c : V \times V \to \mathbb{R}_{\geq 0}$ with $|D| = k$

**Output:** $k$ cycles spanning the graph such that each cycle contains exactly one vertex from $D$

1 Find a min cost $D$-forest cover .

2 Double all the edges in the $D$-cover and then shortcut so that each vertex is visited exactly once and return the resulting cycles .

---

To analyze the algorithm, we need the polytope of a $D$-forest cover. We provide a description of this polytope and prove its integrality using ideas from Cerdeira [5]. For a set of vertices $D$, let $G_{/D}$ be the graph with all vertices of $D$ contracted into a single vertex $\hat{d}$ and for $S \subseteq E - E(D,D)$ let $\kappa_{G_{/D}}(S)$ be the number of components in the graph $G_{/D}$ with edges $S$.

$$\begin{aligned}
\text{minimize} \quad & \sum_{e \in E - E(D,D)} c_e x_e & & \qquad (1)\\
\text{s.t.} \quad & x(S) \geq \kappa_{G_{/D}}(\overline{S} - E(D,D)) - 1 & & \forall S \subseteq E - E(D,D) \\
& x(E(d, V - D)) \geq 1 & & \forall d \in D \\
& 0 \leq x_e \leq 1 & & \forall e \in E - E(D,D) \ .
\end{aligned}$$

**Claim 4.1.** *The value of LP* (1) *is the cost of the min cost $D$-forest cover.*

*Proof.* For a $D$-forest cover $F$ we observe that $\overline{F}$ is an edge set that satisfies $|\{\{v,d\} \in \overline{F}, v \notin D\}| < n - |D|$ for all $d \in D$ and that $F$ is spanning tree in the graph $G_{/D}$. We now define matroids $\mathcal{M}_1, \mathcal{M}_2$ on the ground set $E - E(D,D)$ with independent sets $\mathcal{I}_1, \mathcal{I}_2$ whose common independents set correspond to complements of $D$-forest covers. We define $\mathcal{M}_1$ as a partition matroid which has parts $P_d$ for each $d \in D$ that contain edges $\{\{d,v\} \mid v \notin D\}$ with capacity $n - |D| - 1$ and all other edges $e \notin E(D,D)$ go in a unique part $P_e$ with capacity 1. We define $\mathcal{M}_2$ as the dual of the graphic matroid on graph $G_{/D}$. Then the complements of $D$-forest covers are common independent sets of $\mathcal{M}_1, \mathcal{M}_2$ and the complements of all common independent sets of $\mathcal{M}_1, \mathcal{M}_2$ contain $D$-forest covers. This implies that the cost of a min cost $D$-forest cover is $c(E - E(D,D)) - \max_{F \in \mathcal{I}_2 \cap \mathcal{I}_2} c(F)$. By the characterization of the matroid intersection polytope the value of $\max_{F \in \mathcal{I}_2 \cap \mathcal{I}_2} c(F)$ is

$$\begin{aligned}
&\max \sum_{e \in E - E(D,D)} c_e x_e \\
\text{s.t.} &\sum_{e \in E(d, V-D)} x_e \leq n - |D| - 1 && \forall d \in D \\
&x(S) \leq r_{\mathcal{M}_2}(S) && \forall S \subseteq E - E(D,D) \\
&0 \leq x_e \leq 1 && \forall e \in E - E(D,D) \ .
\end{aligned}$$

Similar to Claim 5.1 we make the variable change $z_e = 1 - x_e$, add $\sum_{e \in E - E(D,D)} c_e$, negate the objective, and make it minimization to get LP (1). This follows since the following hold

1. $\sum_{e \in E - E(D,D)} c_e - \sum_{e \in E - E(D,D)} c_e x_e = \sum_{e \in E - E(D,D)} c_e z_e$
2. $\sum_{e \in E(d, V-D)} x_e \leq n - |D| - 1 \iff \sum_{e \in E(d,V-D)} z_e \geq 1$
3. $x(S) \leq |S| - \kappa_{G_{/D}}(E - E(D,D) - S) + 1 \iff z(S) \geq \kappa_{G_{/D}}(E - E(D,D) - S) - 1 \iff z(S) \geq \kappa_{G_{/D}}(\overline{S} - E(D,D)) - 1$.

Thus $x^*$ is an optimal solution if and only if $z^* = \mathbf{1} - x^*$ is an optimal solution to LP (1). ☐

Next, we show that the upper-bound $x_e \leq 1$ can be dropped and show that this gives the up-hull of the $D$-forest cover polytope. We define the up-hull LP as

$$\begin{aligned}
\text{minimize} &\sum_{e \in E - E(D,D)} c_e x_e && (2) \\
\text{s.t. } &x(S) \geq \kappa_{G_{/D}}(\overline{S} - E(D,D)) - 1 && \forall S \subseteq E - E(D,D) \\
&x(E(d, V-D)) \geq 1 && \forall d \in D \\
&x_e \geq 0 && \forall e \in E - E(D,D) \ .
\end{aligned}$$

The following claim implies that the optimal value of LP (2) is the same as the optimal value to LP (1).

**Claim 4.2.** *Let $x$ be a solution to LP (2) such that there exists $e' \in E$ where $x_{e'} > 1$. Then there exists $x'$ such that $x'$ is a feasible solution to LP (2) and $x' \leq x$ and $x'_{e'} < x_{e'}$.*

*Proof.* We will define $x'$ by keeping $x_e = x_e$ for all $e \neq e'$ and we will set $x'_{e'} = x_{e'} - \epsilon$ for some small $\epsilon > 0$. To show we can find such an $\epsilon$, it suffices to show that all constraints involving $x_{e'}$ are not tight. This follows since the two constraints in the LP are lower bounds on $x(S)$ for some set of edges $S$. If $e'$ is not adjacent to a depot vertex, then $x_e$ does not impact the second constraint, otherwise $x(E(d, V-D)) \geq x_{e'} > 1$. Let $S \subseteq E - E(D,D)$ such that $e' \in S$, we will now show that $x(S) > \kappa_{G_{/D}}(\overline{S} - E(D,D)) - 1$. Let $C_1, \ldots, C_p$ be the components in $G_{/D}$ with edges $\overline{S} - E(D,D)$, so that $\kappa_{G_{/D}}(\overline{S} - E(D,D)) = p$. Then consider the edge set $S' = S - \{e'\}$, we have that $\kappa_{G_{/D}}(\overline{S'} - E(D,D)) \geq p - 1$ since $\overline{S'} - \overline{S} = e'$ and the addition of $e'$ to $\overline{S}$ can possibly combine two components $C_i, C_j$. Thus we have, $x(S) - x_{e'} = x(S') \geq p - 2$ which implies that $x(S) > p - 1 = \kappa_{G_{/D}}(\overline{S} - E(D,D)) - 1$ since $x_{e'} > 1$. We have shown all constraints involving $x_{e'}$ are strict inequalities, so we can decrease $x_{e'}$ by some positive $\epsilon$ and still maintain feasibility in the LP. ☐

With the characterization of the $D$-forest cover, we are now ready to analyze the Algorithm (2). In the following claim, we show that the cost of min-cost $D$-forest cover as given by LP (2) is at most the cost of the optimal solution to LP (mTSP$_+$).

**Claim 4.3.** *Let $z^*$ be an optimal solution to LP* (mTSP$_+$) *and $x^*$ be an optimal solution to LP* (2) *then we have $c^T x^* \leq c^T z^*$.*

*Proof.* Let $z$ be a feasible solution to LP (mTSP$_+$), we will show that $z$ is a feasible solution to LP (2). We note that LP (2) is defined on edges $E - E(D, D)$ and $z$ is defined on edges $E$, but $z$ satisfies $z_e = 0$ for all $e \in E(D, D)$. For all $d \in D$ we have that,

$$\begin{aligned} z(E(d, V - D)) &= z(\delta(d)) - z(E(d, D)) \\ &= z(\delta(d)) = 2 > 1 \end{aligned}$$

where the second equality follows since $0 \leq z(E(d, D)) \leq z(E(D, D)) = 0$. We recall that we use the graph $G_{/D}$ in LP (2) which we get by contracting all vertices in $D$ to a single vertex. Let $\hat{d}$ be the contracted depot vertex in $G_{/D}$ and for $S \subseteq E - E(D, D)$ let $C_1, \ldots, C_p$ be the components of the sub-graph of $G_{/D}$ with edges $E - S - E(D, D)$ such that $\hat{d} \in C_1$. Then we have that

$$\begin{aligned} z(S) &\geq \sum_{i<j\leq p} z(E(C_i, C_j)) \\ &= \frac{1}{2}\left( z(\delta((C_1 - \hat{d}) \cup D)) + \sum_{i=2}^{p} z(\delta(C_i)) \right) \\ &\geq p = \kappa_{G_{/D}}(\overline{S} - E(D, D)) \\ &> \kappa_{G_{/D}}(\overline{S} - E(D, D)) - 1 \ . \end{aligned}$$

The second to last inequality follows since for $i > 1$ we have $C_i \subseteq V - D$ so $z(\delta(C_i)) = z(\delta(D \cup C_i')) \geq 2$ for some $C_i' \subseteq V - D$. ☐

The following lemma now follows straightforwardly about the LP-based guarantee for the Tree Doubling Algorithm.

**Lemma 4.4.** *Let $z^*$ be an optimal solution to linear programming relaxation for the* mTSP$_+$, *LP* (mTSP$_+$)*. Then the output of the Tree Doubling Algorithm returns a solution whose cost is at most twice the objective value of $z^*$.*

## 4.2 MV-mTSP$_+$ Approximation

Now that we have a LP based algorithm for mTSP$_+$, we are ready to give our reduction and apply the guarantee from Lemma 4.4 to get a 2-approximation for MV-mTSP$_+$. We get the following LP relaxation for MV-mTSP$_+$,

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e & & \text{(MV-mTSP}_+\text{)} \\ \text{s.t.} \quad & x(\delta(v)) = 2r(v) & & \forall v \in V - D \\ & x(\delta(v)) = 2 & & \forall v \in D \\ & x(\delta(S \cup D)) \geq 2 & & \forall S \subset V - D \\ & x(E(D, D)) = 0 & & \\ & x_e \geq 0 & & \forall e \in E \ . \end{aligned}$$

The following lemma allows us to relate the LP relaxation of the many-visit problem to its corresponding single-visit variant.

**Lemma 4.5.** *Suppose there is a $\rho$-approximation algorithm for* mTSP$_+$ *that given an instance on a complete graph $G = (V, E(G))$ with distances $c : E(G) \to \mathbb{R}_{\geq 0}$ and depot vertices $D \subset V$ returns $k := |D|$ cycles $C_1, \ldots, C_k$ such that $\sum_{i=1}^{k} \sum_{e \in C_i} c_e \leq \rho \cdot z^\star$ where $z^\star$ is the optimum value of LP* (mTSP$_+$)*. Then there exists an algorithm that given an instance of the* MV-mTSP$_+$ *on a complete graph $H = (V, E(H))$ with distance function $c : E(H) \to \mathbb{R}_{\geq 0}$, depot vertices $D \subset V$, and requirements $r : V - D \to \mathbb{Z}_+$ outputs a closed walk $T : E \to \mathbb{Z}$ satisfying $\sum_{e \in E(H)} T(e) c_e \leq \rho \sum_{e \in E(H)} c_e y_e$ where $y$ is the optimal solution to LP* (MV-mTSP$_+$)*. The running time of the algorithm is polynomial in* $\max_{v \in V} r(v)$ *and* $|V|$*.*

*Proof.* In this lemma we extend the visit function to the depot vertices and define $r(v) := 1$ for all $v \in D$. Given the instance on MV-mTSP$_+$ on graph $H$, we construct an expanded graph $H^r$ by making $r(v)$ copies of each vertex in $H$. The distance between any two copies of the same vertex $v$ is defined to be cost of the loop at $v$, i.e., $c_{vv}$ and the the distance between two copies of distinct vertices $u$ and $v$ is identical to distance between $u$ and $v$. We now apply the mTSP$_+$ approximation algorithm on the new instance $H^r$ to obtain cycles $C_1, \ldots, C_k$ in the expanded graph. We can interpret these cycles as a solution to MV-mTSP$_+$. Observe that the cost of the solution is exactly the cost of $C_1, \ldots, C_k$ in the cost defined as above. Thus to prove the lemma, it is enough to show there exists a feasible solution to LP (mTSP$_+$) on the instance $H^r$ whose cost is at most the cost of optimal solution $y$ to LP (MV-mTSP$_+$) on the instance $H$.

We will convert the solution $y$ to a solution $x$ to LP (mTSP$_+$) on the graph $H^r$. Let $e' = \{u_i, v_j\}$ and $e = \{u, v\}$ where $u, v$ are the original copies of $u_i, v_j$ in $V$, then we set $x_{e'} = \frac{y_e}{r(u)r(v)}$. Now we show that $x$ is a feasible solution to LP (mTSP$_+$) for the graph $H^r$. For any vertex $v \in V^r$ and $i \in [r(v)]$ we have $x(\delta(v_i)) = \frac{y(\delta(v))}{r(v)} = 2$ where the first equality follows since the degree of each vertex $v$ in $y$ is distributed evenly among all $r(v)$ copies in $x$ the second equality follows by the feasibility of $y$. For any $e \in E(D, D)$, we have $x_e = y_e = 0$ so $x(E(D, D)) = 0$. Finally $0 \leq x \leq 2$ follows since $x_e \leq 2$ if and only if $y_e \leq 2r(u)r(v)$ which follows since $y_e \leq 2\min(r(u), r(v))$.

For any $S \subset V^r - D$, we need to show $x(\delta(S \cup D)) \geq 2$. Let $k$ be the number of vertices $v \in V$ such that there exists $v_i, v_j$ that are distinct copies of $v$ and $S$ contains exactly one of $v_i, v_j$. We show $x(\delta(S \cup D)) \geq 2$ for all $S \subset V^r - D$ by induction on $k$. If $k = 0$, then $x(\delta(S \cup D)) = y(\delta(S' \cup D))$ where $S' \subset V$ is acquired by taking the original copy of each vertex $v_j$ from $S$ implying $x(\delta(S \cup D)) \geq 2$ since $y(\delta(S' \cup D)) \geq 2$ since $y$ is feasible to LP (mTSP$_+$). If $k > 0$, then there exists a vertex $v \in V - D$ such that both $S$ and $V^r - (D \cup S)$ have copies of $v$. We define the following subsets of vertices based on the set $S$,

1. let $S(v)$ be the copies of $v$ in $S$ meaning $S(v) := S \cap \{v_1, \ldots, v_{r(v)}\}$
2. let $B$ be the complement of $D \cup S$ in $V^r$ meaning $B := V^r - (S \cup D)$
3. let $B(v)$ be the copies of $v$ not in $S$ meaning $B(v) := \{v_1, \ldots, v_{r(v)}\} - S(v)$.

First we consider the case when $B(v) = B$ and we have,

$$\begin{aligned} x(\delta(S \cup D)) &= x(B(v)) \\ &= |B(v)| \left(2 - x_{vv}(|B(v)| - 1)\right) \\ &\geq |B(v)| \left(2 - \frac{|B(v)| - 1}{r(v)}\right) \\ &\geq |B(v)| \left(2 - \frac{|B(v)| - 1}{|B(v)| + 1}\right) \\ &\geq 2. \end{aligned}$$

The first inequality holds since $x_{vv} = \frac{y_{vv}}{r(v)^2} \leq \frac{1}{r(v)}$ because $y(\delta(v)) = 2r(v)$ implies that $y_{vv} \leq r(v)$ since a loop contributes twice to the degree count of a vertex. The second inequality holds since $B(v)$ does not contain all copies of $v$ so $|B(v)| < r(v)$ and the third inequality holds since the function $f(x) = x(2 - \frac{x-1}{x+1})$ is an increasing function that is minimized at $x = 1$.

Now we can assume that $B \subset B(v)$. For any $v_i \in S(v)$ let $X_1 = x(E(v_i, (S - S(v)) \cup D))$ and $X_2 = x(E(v_i, B - B(v)))$. We note that $X_1, X_2$ do not change based on the choice of $v_i$ since all copies of $v$ are defined identically in $H^r$. Now we consider the cut $S - S(v)$ and by using the fact that $x_{vv} \geq 0$ we have,

$$\begin{aligned} x(\delta((S - S(v)) \cup D)) &= x(\delta(S \cup D)) + |S(v)| \left(X_1 - |B(v)| x_{vv} - X_2\right) \\ &\leq x(\delta(S \cup D)) + |S(v)| \left(X_1 - X_2\right). \end{aligned}$$

First we consider the case when $X_1 \leq X_2$ and here we have that $x(\delta((S - S(v)) \cup D)) \leq x(\delta(S \cup D))$ and by induction we get that $x(\delta((S - S(v)) \cup D)) \geq 2$ since $S - S(v)$ is a set with one less vertex $v$ that separates a pair of copies of $v$ and $S - S(v) \subset V^r - D$. Thus we have shown $x(\delta(S \cup D)) \geq 2$ and now we consider the case when $X_1 > X_2$. Here we have,

$$\begin{aligned} x(\delta((S + B(v)) \cup D)) &= x(\delta(B - B(v))) \\ &= x(\delta(B)) + |B(v)| \left(X_2 - |S(v)| x_{vv} - X_1\right) \\ &\leq x(\delta(B)) + |B(v)| \left(X_2 - X_1\right) \\ &< x(\delta(B)) \\ &= x(\delta(S \cup D)). \end{aligned}$$

The first inequality follows since $x_{vv} \geq 0$ and the last inequality follows since we are in the case when $X_1 > X_2$. By induction we have $x(\delta((S + B(v)) \cup D)) \geq 2$ since $S + B(v)$ and is a set with one less vertex $v$ that separates a pair of copies of $v$ and $S + B(v) \subset V^r - D$. Thus, we have shown that $x(\delta(S \cup D)) \geq 2$.

Thus, we can apply the algorithm from the lemma assumption on $x$ as a solution to $H^r$ to get a cycles $C_1, \ldots, C_k$ in $H^r$ satisfying $\sum_{i=1}^{k} \sum_{e \in C_i} c_e \leq \rho c^T x$. We note that $c^T y = c^T x$ since $c^T x = \sum_{e=\{u,v\} \in E(H^r)} r(u) r(v) x_e c_e = \sum_{e \in E(H)} c_e y_e$. We now convert the cycles $C$ to closed walks in $H$ denoted by $T : E(H) \to \mathbb{Z}$ by replacing every copy edge $\{u_i, v_j\}$ with its corresponding original edge $\{u, v\}$ in $H$. Clearly, $T$ consists of $k$ closed walks in $H$. Moreover, $T$ visits every vertex $r(v)$ times in $H$ since $C$ visits each copy of $v$ one time in $H^r$. Each closed walk contains exactly one vertex from $D$ since each cycle $C_1, \ldots, C_k$ contained exactly one depot vertex. Lastly, the run-time follows since $H^r$ has at most $n \max_{v \in V} r(v)$ vertices and the algorithm in the lemma statement is a polynomial-time algorithm.

□

Using this lemma we get the following polynomial-time algorithm.

**Algorithm 3** MV-mTSP$_+$ Reduction Algorithm

**Input:** $G = (V, E), D \subseteq V, c : V \times V \to \mathbb{R}_{\geq 0}, r : V - D \to \mathbb{Z}$

**Output:** An integral solution to LP (MV-mTSP$_+$)

1 Solve LP (MV-mTSP$_+$) to get solution $x^*$ .

2 For all edges $e$ let $\tilde{x}_e := x_e - 2k_e$ and $k_e := 0$ if $x_e \leq 4$ and otherwise $k_e$ is set so that $2 \leq \tilde{x}_e < 4$ and $k_e \in \mathbb{Z}$. Define a function $\tilde{r} : V \to \mathbb{Z}$ where $\tilde{r}(v) = r(v) - \sum_{e \in \delta(v)} k_e$.

3 Use Lemma 4.5 with solution $\tilde{x}$ on instance $G, \tilde{r}$ with algorithm $\mathcal{A}$ to get $T : E \to \mathbb{Z}$.

4 Increase the number of times each edge is used in the previous step by $2k_e$ and return the resulting solution.

The next two claims will help to show that Step 3 of the algorithm runs in polynomial-time. The proof of the following claim is identical to that of Claim 3.2.

**Claim 4.6.** *The new visit function $\tilde{r}$ satisfies $1 \leq \tilde{r}(v) \leq 2n$ for all $v \in V$.*

**Claim 4.7.** *The solution $\tilde{x}$ is feasible for LP* (MV-mTSP$_+$) *with graph $G$ and $\tilde{r}$.*

*Proof.* We have $\tilde{x} \geq 0$ and $\sum_{e \in \delta(v)} \tilde{x}_e = \sum_{e \in \delta(v)} x_e - 2k_e = 2(r(v) - \sum_{e \in \delta(v)} k_e) = 2\tilde{r}(v)$. For any set $S \subset V - D$, if $x_e \leq 4$ for all $e \in \delta(S \cup D)$ then $\tilde{x}(\delta(S \cup D)) = x(\delta(S \cup D)) \geq 2$. Otherwise, if there is an edge $e \in \delta(S \cup D)$ such that $x_e > 4$ we get $\tilde{x}(\delta(S \cup D)) \geq \tilde{x}_e \geq 2$ by definition of $\tilde{x}$. Finally, we have $\tilde{x}_e = x_e = 0$ for all $e \in E(D, D)$ implying $\tilde{x}(E(D, D)) = 0$. Moreover, $T$ contains no edges between vertices in $D$ since $k_e = 0$ for all $e \in E(D, D)$ and $T'(e) = 0$ for all $e \in E(D, D)$. □

**Theorem 2.** *If there exists a polynomial-time $\rho$-approximation algorithm for* mTSP$_+$ *whose guarantee is with respect to LP* (mTSP$_+$)*, then there exists a polynomial-time $\rho$-approximation algorithm for* MV-mTSP$_+$.

*Proof.* Let $T'$ be the solution we get from the third step before we increase each edge by $2k_e$. By Claim 4.6 and Lemma 4.5 we have that $\tilde{x}$ satisfies $\sum_{e \in E} T'(e)c_e \leq \rho \sum_{e \in E} c_e \tilde{x}_e$. Let $S = \{e \in E \mid k_e > 0\}$ be the set of edges whose cost was decreased to acquire $\tilde{x}_e$ then we have that $\sum_{e \in E} T(e)c_e = \sum_{e \in E} T'(e)c_e + 2\sum_{e \in S} k_e c_e \leq \rho \sum_{e \in E} c_e \tilde{x}_e + 2\sum_{e \in S} k_e c_e \leq \rho c^T x^*$. For the runtime, we have that Step 3 runs in time polynomial in $n$ by Claim 4.6 and Lemma 4.5 and the rest of the steps are clearly polynomial-time in $n$. Finally, $T$ is a feasible solution to MV-mTSP$_+$ since $T'$ satisfies the cut constraints which implies $T$ also satisfies the cut constraints since $T(e) \geq T'(e)$ for all $e$ and by definition $T$ satisfies degree constraints. □

We get the following result by applying Lemma 4.4 and Theorem 2.

**Corollary 2.1.** *There is a polynomial-time 2-approximation algorithm for the* MV-mTSP$_+$ *problem.*

### 4.3 MV-mTSP$_0$ Approximation

In this subsection, we give a 2-approximation for the MV-mTSP$_0$ problem. Here we are given a set of depot vertices $D$ with $|D| = k$ and the goal is to find at most $k$ non-empty closed walks that each closed walk uses exactly one depot. The proofs in this subsection are very similar to those in the previous subsection for MV-mTSP$_+$. Let $G_{/D}$ be the graph with all depot vertices $D$ contracted.

We use the following LP which is very similar to LP (mTSP$_+$) and the only difference is dropping the degree constraints on $D$,

$$\begin{aligned} \text{minimize } & \sum_{e\in E} c_e x_e & & (3)\\ \text{s.t. } & x(\delta(v)) = 2 & \forall v \in V - D\\ & x(\delta(S\cup D)) \geq 2 & \forall S \subset V - D\\ & x(E(D,D)) = 0 \\ & 0 \leq x_e \leq 2 & \forall e \in E \ . \end{aligned}$$

**Algorithm 4** Tree Doubling Algorithm for mTSP$_0$

**Input:** $G = (V,E), D \subseteq V, c : V \times V \to \mathbb{R}_{\geq 0}$ with $|D| = k$

**Output:** At most $k$ cycles such that each cycle contains exactly one vertex from $D$ and each non-depot vertex is contained in a cycle.

1 Find a min cost spanning tree $T^*$ in the graph $G_{/D}$ which is formed by contracting $D$ .

2 Double all the edges in $T^*$ and then shortcut so that each vertex is visited exactly once and return the resulting cycles .

We use the following LP to characterize the up-hull of spanning trees in $G_{/D}$. The proof of integrality of the LP follows by the same argument as the one used in Claim 4.2.

$$\begin{aligned} \text{minimize } & \sum_{e\in E-E(D,D)} c_e x_e & & (4)\\ \text{s.t. } & x(S) \geq \kappa_{G_{/D}}(\overline{S} - E(D,D)) - 1 & \forall S \subseteq E - E(D,D)\\ & x_e \geq 0 & \forall e \in E - E(D,D) \ . \end{aligned}$$

The proof of the following claim is identical to the proof of Claim 4.3.

**Claim 4.8.** *Let $z^*$ be an optimal solution to LP (3) and $x^*$ be an optimal solution to LP (4) then we have $c^T x^* \leq c^T z^*$.*

The following lemma now follows easily using the properties of the Tree Doubling Algorithm and the triangle inequality.

**Lemma 4.9.** *Let $z^*$ be an optimal solution to linear programming relaxation for the* mTSP$_0$, *LP (3). Then the output of the Tree Doubling Algorithm returns a solution whose cost is at most twice the objective value of $z^*$.*

Next we give the LP relaxation for the multi-visit version. The LP is given by,

$$\begin{aligned} \text{minimize } & \sum_{e\in E} c_e x_e & & (5)\\ \text{s.t. } & x(\delta(v)) = 2r(v) & \forall v \in V - D\\ & x(\delta(S\cup D)) \geq 2 & \forall S \subset V - D\\ & x(E(D,D)) = 0 \\ & 0 \leq x_e \leq 2 & \forall e \in E \end{aligned}$$

The proof of the theorem follows almost exactly applying the ideas from Theorem 1. We note that the only difference here is the absence of a degree constraint on the depot vertices in LP (5), compared to LP (MV-mTSP$_+$). However, this difference does not affect the proofs.

**Theorem 3.** *If there exists a polynomial-time $\rho$-approximation algorithm for* mTSP$_0$ *whose guarantee is with respect to LP* (3)*, then there exists a polynomial-time $\rho$-approximation algorithm for* MV-mTSP$_0$.

We get the following result by applying Lemma 4.9 and Theorem 3.

**Corollary 3.1.** *There is a polynomial-time* 2*-approximation algorithm for the* MV-mTSP$_0$ *problem.*

### 4.4 Approximation for Unrestricted Variant

In this subsection, we show there is a 2-approximation for unrestricted mTSP$_+$. We note that we allow using loops for the single-visit version here. Our algorithm is the following.

---

**Algorithm 5** Unrestricted mTSP$_+$

**Input:** $G, k \in \mathbb{Z}, 1 \leq k \leq n$

**Output:** $k$ cycles that cover all vertices in the graph

1 Add a new vertex $d$ that has all edges to vertices of $G$ to get a new graph $G'$. Extend the cost function of the graph by setting $c_{dv} = \frac{c_{vv}}{2}$ .

2 Find a tree of minimum cost that has degree $k$ on the vertex $d$ .

3 Remove the vertex $d$ and all edges incident to it. Among the remaining $k$ components, if the component is a singleton, then add a loop in that component. Otherwise, double the edges of the tree in the remaining components and shortcut so that each component is a cycle. Return the resulting $k$ cycles.

---

We now give the LP formulation of the tree found in step 1 of Algortihm (5). We denote by $E' = E \cup \{\{d, v\} \mid v \in V\}$ as the edge set of $G'$ where $d$ is a new dummy vertex that we added to the graph. We extend the cost function by setting $c_{dv} = \frac{c_{vv}}{2}$ for all $v \in V$ and get the following LP,

$$\begin{aligned} \text{minimize} \; & \sum_{e \in E'} c_e z_e & & \quad (6) \\ \text{s.t.} \; & z(E(S)) \leq |S| - 1 & & \forall S \subseteq V \cup \{d\} \\ & z(\delta(d)) = k & & \\ & z(E) = n - 1 & & \\ & 0 \leq z_e \leq 1 & & \forall e \in E' \; . \end{aligned}$$

**Claim 4.10.** *Let $T_k^*$ be a minimum cost spanning tree among all spanning trees in $G$ that have degree $k$ on vertex $v_1$. Then we have that the indicator vector $T_k^*$ is an optimal solution to LP* (6)*.*

*Proof.* Let $G'$ denote the extended graph with the new vertex $d$, $\mathcal{M}_1$ be the graphic matroid on $G'$ and $\mathcal{M}_2$ be a partition matroid with one part containing all edges incident to $d$ with capacity $k$ and another part containing the remaining edges with capacity $n - k$. Then $T_k^*$ is an optimal common base in $\mathcal{M}_1, \mathcal{M}_2$ and the polytope of common bases is give by the constraints of LP (6) □

Our LP relaxation for unrestricted $\text{mTSP}_+$ is LP (6). We will show the value of this LP lower bounds the optimal value of our problem. For the previous problems this property was implicitly true as we showed a fractional solution $x$ was in the up-hull of the tree polytope.

**Theorem 4.** *There is a polynomial-time algorithm for the unrestricted* $\text{mTSP}_+$ *problem with an approximation factor of* 2.

*Proof.* Let $M_1, \ldots, M_k$ be the output of Algorithm (5) and $z^*$ be an optimal solution to LP (6). We will show that $\sum_{i=1}^{k} c(M_i) \leq 2c^T z^*$. Let $T^*$ be the tree from the second step of Algorithm (5). First, we show that $\sum_{i=1}^{k} c(M_i) \leq 2c(T^*)$. If we acquire $M_i$ by adding a loop to a singleton vertex $v$ then the cost of $M_i$ is $c_{vv}$ while the cost of the edge adjacent to $M_i$ in $T^*$ is $\frac{c_{vv}}{2}$ which is twice the cost of the edge in the algorithm's output. Now we consider when $M_i$ is a non-singleton component, let $\{d, v\}$ be the edge in $T^*$ such that $v \in M_i$ and let $u \in V$ such that $\{u, v\}$ is an edge in $T^*$. Then we have that $c_{uv} + \frac{c_{vv}}{2} \leq 2c_{uv}$. For any other edge $e \in M_i$, the algorithm pays at most $2c_e$ while the cost in $T^*$ is $c_e$. Then summing up the cost of all cycles and applying these bounds gives $\sum_{i=1}^{k} c(M_i) \leq 2c(T^*)$. Then the proof is concluded by observing $c(T^*) = c^T z^*$ since $T^*$ is an optimal solution to LP (6) by Claim 4.10. □

# 5 Single Depot Many-Visit mTSP

In this section, we give a $\frac{3}{2}$-approximation for the SD-MV-$\text{mTSP}_+$ problem and a $\frac{7}{2}$-approximation for the SD-MV-$\text{mTSP}_+$ with vertex disjoint tours problem.

## 5.1 Approximation for Non Vertex Disjoint Tours

First, we give an LP analysis for Frieze's [9] SD-mTSP algorithm since Frieze shows that this algorithm achieves a $\frac{3}{2}$-approximation relative to the integral optimal solution. We note that unlike Frieze we allow a cycle to contain 2 vertices because it is necessary for our LP analysis.

**Algorithm 6** Single Depot mTSP

**Input:** $G = (V = \{v_1, \ldots, v_n\}), c : V \times V \to \mathbb{R}_{\geq 0}, k \in \mathbb{N}$

**Output:** $k$ cycles that contain $v_1$ that span the graph and vertices not equal to $v_1$ are visited exactly once

Find a min cost multi-set of edges $T^*$ that satisfies (1) if $T' \subseteq T^*$ where $T'$ contains exactly one copy of each edge in $T^*_{2k}$, then $T'$ is a spanning tree of $G$ (2) Each edge $e \in \delta(v_1)$ appears at most $k$ times in $T^*$ and (3) Each edge $e \notin \delta(v_1)$ appears at most once in $T^*$

Find a min cost perfect matching $M^*$ on the vertices with odd degree in $T^*$.

Add $M^*$ to $T^*$ which is now an Eulerian graph. Let $w_1 = v_1, \ldots, w_s = v_1$ be the Eulerian tour and let $U$ be the neighbors of $v_1$ in $T^*$. Delete a node $w_i$ in the sequence if

1. $w_i$ has appeared before and $w_i \neq v_1$ or
2. $w_i \in U$ and $v_1 \notin \{w_{i-1}, w_{i+1}\}$.

Return the sequence obtained after short cutting.

We use the following LP for SD-mTSP. We note that the LP does not exactly fit the LP in the general framework (LP ($\text{mTSP}_+$)) since there is a different constraint on the degree of vertex $v_1$. We still use the general framework in this section, but we show that each part of the framework still holds with the additional degree constraint.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e x_e & & (7)\\
\text{s.t.} \quad & x(\delta(v)) = 2 & \forall v \in V - v_1 &\\
& x(\delta(v_1)) = 2k & &\\
& x(\delta(S)) \geq 2 & \forall S \subset V &\\
& x_e \geq 0 & \forall e \in E &
\end{aligned}
$$

We now characterize the polytope of the tree $T^*$ found in the first step of the algorithm. We first characterize connected graphs that have fixed degree $2k$ on vertex $v_1 \in V$ and define $\kappa(S)$ as the number of components in the graph $(V, S)$ for all $S \subseteq E$.. Then we get the following LP,

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e z_e & & (8)\\
\text{s.t.} \quad & z(S) \geq \kappa(\overline{S}) - 1 & \forall S \subseteq E &\\
& z(\delta(v_1)) = 2k & &\\
& 0 \leq z_e \leq 1 & \forall e \in E \ . &
\end{aligned}
$$

**Claim 5.1.** *Let $G$ be a connected graph where vertex $v_1$ in $G$ has degree at least $2k$ and $T^*_{2k} \subseteq E$ be a minimum cost spanning tree among all spanning trees in $G$ that have degree $2k$ on vertex $v_1$. Then we have that the indicator vector of $T^*_{2k}$ is an optimal solution to LP (8).*

*Proof.* First, we define matroid $\mathcal{M}_1$ as the dual of the graphic matroid on graph $G$. Next, we define $\mathcal{M}_2$ as a partition matroid with parts $P_1, P_2, \dots P_{|E|}$ where $P_1$ contains edges incident to $v_1$ and has capacity $d(v_1) - 2k$, and the remaining edges go in a unique $P_j$ with capacity 1. Thus, common independent sets of $\mathcal{M}_1, \mathcal{M}_2$ are sets $R \subseteq E$ such that $R$ has at most $d(v_1) - 2k$ edges incident to $v_1$ and $E - R$ contains a spanning tree. We now show that there is an optimal solution such that $E - R$ is a spanning tree. If not, then $E - R$ contains a cycle $C$. If the cycle does not have any edges incident to $v_1$ then for any $e \in C$ we have $R + e$ is feasible and $c(R + e) \geq c(R)$. Otherwise, $C$ has two edges incident to $v_1$ and there must be an edge $e \in C$ such that $e \cap v_1 = \emptyset$ and again we get that $R + e$ is a feasible solution and $c(R + e) \geq c(R)$.

By the matroid intersection theorem, the polytope given by constraints $\{x \geq 0 \mid x(S) \leq r_{\mathcal{M}_i}(S), \forall i \in [2] \text{ and } S \subseteq E\}$ is totally-dual integral and therefore integral. Turning an inequality to equality in a TDI system maintains the TDI property, so we can restrict our polytope to common independent sets of $\mathcal{M}_1, \mathcal{M}_2$ that have degree exactly $d(v_1) - 2k$ on $v_1$. Then we observe that $E - T^*_{2k}$ is an optimal solution to $\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} c(I)$ and is also an optimal solution to the following LP

$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} c_e x_e & & (9)\\
\text{s.t.} \quad & x(\delta(v_1)) = d(v_1) - 2k & &\\
& x(S) \leq r_{\mathcal{M}_1}(S) & \forall S \subseteq E &\\
& 0 \leq x_e \leq 1 & \forall e \in E \ . &
\end{aligned}
$$

Here we have that $r_{\mathcal{M}_1}(S) = |S| - \kappa(\overline{S}) + 1$ since $\mathcal{M}_1$ is the dual matroid to the graphic matroid. If we negate the objective of LP (9), change it to a minimization problem, add $\sum_{e \in e} c_e$ to the objective, and make the variable change $z_e = 1 - x_e$ we get LP (8). This is true since the following hold for all $S \subseteq E$

1. $\sum_{e\in E} c_e - \sum_{e\in E} c_e x_e = \sum_{e\in e} c_e z_e$
2. $x(\delta(v_1)) = d(v_1) - 2k \iff 2k = d(v_1) - x(\delta(v_1)) \iff 2k = z(\delta(v_1))$
3. $x(S) \leq |S| - \kappa(E - S) + 1 \iff \kappa(\overline{S}) - 1 \leq |S| - x(S) \iff \kappa(\overline{S}) - 1 \leq z(S)$.

If $x^*$ is an optimal solution to LP (9) then $z^* = \mathbf{1} - x^*$ is an optimal solution to LP (8), so $T^*_{2k}$ is an optimal solution to LP (8) since $E - T^*_{2k}$ is an optimal solution to LP (9). □

Now we use Claim 5.1 to characterize a multi-set of edges with slightly different properties than the tree in LP (5.1).

$$\begin{aligned} \text{minimize } & \sum_{e\in E} c_e z_e && && (10)\\ \text{s.t. } & z(S) \geq \kappa(\overline{S}) - 1 && \forall S \subseteq E \\ & z(\delta(v_1)) = 2k \\ & 0 \leq z_e \leq 1 && \forall e \notin \delta(v_1) \\ & 0 \leq z_e \leq k && \forall e \in \delta(v_1) \end{aligned}$$

**Claim 5.2.** *Let $T^*_{2k}$ be a minimum cost multi-set of edges satisfying the following three properties.*

1. *Each edge $e \in \delta(v_1)$ appears at most $k$-times in $T^*_{2k}$.*
2. *Each edge $e \notin \delta(v_1)$ appears at most 1-time in $T^*_{2k}$.*
3. *If $T' \subseteq T^*_{2k}$ where $T'$ contains exactly one copy of each edge in $T^*_{2k}$, then $T'$ is a spanning tree of $G$.*

*Then we have that the indicator vector of $T^*_{2k}$ is a optimal solution to LP (10).*

*Proof.* First, we construct a graph $G^k_{v_1}$ where all vertices $v \neq v_1$ have $k$ copies $v_1, \ldots, v_k$ and there is an edge between vertices $u_1, v_1$ in the new graph if $\{u, v\}$ is an edge in the original $G$. Moreover, for each $v \in V$ the set of copies $v_1, \ldots, v_k$ contains a path $v_1, \ldots, v_k$ where the edges in the path have cost 0. Then we observe there is a one to one correspondence (that maintains costs) between spanning trees in $G^k_{v_1}$ with degree $2k$ on vertex $v_1$ and multi-sets that satisfy the three properties described in the claim. Then the proof of the claim follows by applying Claim 5.1. □

With the characteriation of the $T^*$ we now show the cost of $T^*$ is at most the cost of the LP relaxation for the problem.

**Lemma 5.3.** *Let $x^*$ be an optimal solution to LP (7). Then we have,*

$$c(T^*) \leq c^T x^* \ .$$

*Proof.* We show that any solution $x$ to LP (7) is feasible for LP (10) which will conclude the proof. The solution $x$ clearly satisfies $x(\delta(v_1)) = 2k$. Let $S \subset E$ and $C_1, \ldots, C_m$ be the connected components of the vertex set $V$ in the graph $(V, \overline{S})$. Then we have that,

$$\begin{aligned} x(S) &\geq \sum_{i<j} x(E(C_i, C_j)) = \frac{1}{2}\sum_{i=1}^{m} x(\delta(C_i)) \\ &\geq m = \kappa(\overline{S}) > \kappa(\overline{S}) - 1 \ . \end{aligned}$$

The first inequality follows since $E(C_i, C_j) \subseteq S$ since $C_1, \ldots, C_m$ are components in the graph with edges $\overline{S}$ and the second inequality follows since $x$ is feasible for LP (7). If $e = \{u, v\} \notin \delta(v_1)$ then we that $x(\delta(\{u, v\})) = x(\delta(u)) + x(\delta(v)) - 2x_e = 4 - 2x_e \geq 2$ which implies $x_e \leq 1$. Similarly, for $e = \{v_1, u\} \in \delta(v_1)$ we have $x(\delta(\{v_1, u\})) = x(\delta(u)) + x(\delta(v_1)) - 2x_e = 2 + 2k - 2x_e \geq 2$ which implies $x_e \leq k$. □

We can show the cost of the matching is at most 1/2 the cost of the LP optimum.

**Claim 5.4.** *Let $x^*$ be an optimal solution to LP (7). Then we have $c(M^*) \leq \frac{c^T x^*}{2}$.*

*Proof.* Let $S$ be the set of odd degree vertices in $T*$ then $M^*$ is a min-cost $S$-join in $G$. The polytope for the up-hull of $S$-joins is given by $\{x \geq 0 \mid x(\delta(P)) \geq 1, \forall P \text{ such that } |P \cap S| \text{ is odd}\}$. Then the claim follows since $x^*/2$ is a feasible solution for the $S$-join polytope. □

Then the above two lemmas imply the following.

**Lemma 5.5.** *Let $x^*$ be an optimal solution to LP (7). Algorithm (6) returns a solution $C$ satisfying $\sum_{e \in C} c_e \leq \frac{3}{2} c^T x^*$.*

*Proof.* This follows since by the triangle inequality $\sum_{e \in C} c_e \leq c(M^*) + c(T^*) \leq \frac{3}{2} c^T x^*$. □

Now we are ready to get a $\frac{3}{2}$ algorithm for the many-visit variant. We need the following to characterize solutions to the problem.

**Lemma 5.6.** *Given a connected graph with edge set $T$ such that $d_T(v_1) = 2k$ and $d_T(v) = 2r(v)$, we can decompose the edges of $T$ into $k$ closed walks containing $v_1$.*

*Proof.* The graph $G$ is Eulerian, so there exists an Eulerian $C$ circuit starting at $v_1$ and the circuit is given by a sequence of vertices $w_1 = v_1, \ldots, w_k = v_1$. Let $w_1, w_2, \ldots, w_j$ be a prefix of the sequence such that $j$ is the smallest index greater than 1 such that $w_j = v_1$. We will use $w_1, \ldots, w_j$ as the first closed walk. Next we reduce the graph by deleting all edges used by the first closed walk and then by removing any isolated vertices. We now show this graph is still Eulerian. Clearly, all vertices have even degree since we removed an even number of edges from each vertex. The graph remains connected since the Eulerian circuit $C$ will not use any of the removed vertices or edges in the graph. Thus, we can inductively repeat this process to get $k$ closed walks containing $v_1$ so that each vertex $v$ is visited a total of $r(v)$ times. □

We use following LP for the many-visit version of the problem.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e x_e && (11)\\
\text{s.t.} \quad & x(\delta(v_1)) = 2k \\
& x(\delta(v)) = 2r(v) && \forall v \in V \\
& x(\delta(S)) \geq 2 && \forall S \subset V \\
& x_e \geq 0 && \forall e \in E
\end{aligned}
$$

The proof of the next claim is nearly identical to that of Lemma 4.5 and the only difference comes from the fact that the depot vertex $v_1$ has degree $2k$ while in Lemma 4.5 all vertices have degree 2.

**Lemma 5.7.** *Suppose there is a $\rho$-approximation algorithm for* SD-mTSP *that given an instance on a complete graph $G = (V, E(G))$, depot vertex $v_1$, integer $k$, and distances $c : E(G) \to \mathbb{R}_{\geq 0}$ returns $k$ cycles $C_1, \ldots, C_k$ that contain vertex $v_1$ such that $\sum_{i=1}^{k} \sum_{e \in C_i} c_e \leq \rho \cdot z^\star$ where $z^\star$ is the optimum value of LP (7). Then there exists an algorithm that given an instance of the* SD-MV-mTSP *on a complete graph $H = (V, E(H))$, depot vertex $v_1 \in V$, and integer $k$, with distance function $c : E(H) \to \mathbb{R}_{\geq 0}$ and requirements $r : V \to \mathbb{Z}_+$ outputs a solution to* SD-MV-mTSP, *$T : E(H) \to \mathbb{Z}$ satisfying $\sum_{e \in E(H)} T(e) c_e \leq \rho \sum_{e \in E(H)} c_e y_e$ where $y$ is the optimal solution to LP (11). The running time of the algorithm is polynomial in $\max_{v \in V} r(v)$ and $|V|$.*

*Proof.* In this lemma, we extend the visit function to the depot vertex and define $r(d) := 1$. We construct the graph $H^r$ identically as in Lemma 4.5: $H^r$ has $r(v)$ copies of each vertex $v$, and we extend the cost function by defining the distance between copy vertices to be the same as the distance between their original counterparts in $H$. We now apply the SD-mTSP approximation algorithm on the new instance $H^r$ to obtain cycles $C_1, \ldots, C_k$ in the expanded graph. We can interpret these cycles as a solution to SD-MV-mTSP. Observe that the cost of the solution is exactly the cost of $C_1, \ldots, C_k$ in the cost defined as above. Thus to prove the lemma, it is enough to show there exists a feasible solution to LP (7) on the instance $H^r$ whose cost is at most the cost of optimal solution $y$ to LP (11) on the instance $H$.

We will convert the solution $y$ to a solution $x$ to LP (7) on the graph $H^r$. Let $e' = \{u_i, v_j\}$ and $e = \{u, v\}$ where $u, v$ are the original copies of $u_i, v_j$ in $V$, then we set $x_{e'} = \frac{y_e}{r(u)r(v)}$. Now we show that $x$ is a feasible solution to LP (7) for the graph $H^r$. For any vertex $v \in V^r$ and $i \in [r(v)]$ we have $x(\delta(v_i)) = \frac{y(\delta(v))}{r(v)}$ where the first equality follows since the degree of each vertex $v$ in $y$ is distributed evenly among all $r(v)$ copies in $x$. Thus, by the feasbility of $y$ we have that $x(\delta(v_1)) = 2k$ and for all $v \in V^r - V_1$ we have $x(\delta(v)) = 2$. Finally $x \geq 0$ holds since $y \geq 0$.

For any $S \subset V^r$, we need to show $x(\delta(S)) \geq 2$. Let $z$ be the number of vertices $v \in V$ such that there exists $v_i, v_j$ that are distinct copies of $v$ and $S$ contains exactly one of $v_i, v_j$. We show $x(\delta(S)) \geq 2$ for all $S \subset V^r$ by induction on $z$. If $z = 0$, then $x(\delta(S)) = y(\delta(S'))$ where $S' \subset V$ is acquired by taking the original copy of each vertex $v_j$ from $S$ implying $x(\delta(S)) \geq 2$ since $y(\delta(S')) \geq 2$ since $y$ is feasible to LP (mTSP$_+$). If $z > 0$, then there exists a vertex $v \in V$ such that both $S$ and $V^r - S$ have copies of $v$. We note that $v \neq v_1$ since $r(v_1) = 1$ so $H^r$ only has one copy of $v_1$ implying $S$ cannot separate copies of $v_1$. We define the following subsets of vertices based on the set $S$,

1. let $S(v)$ be the copies of $v$ in $S$ meaning $S(v) := S \cap \{v_1, \ldots, v_{r(v)}\}$
2. let $B$ be the complement of $S$ in $V^r$ meaning $B := V^r - S$
3. let $B(v)$ be the copies of $v$ not in $S$ meaning $B(v) := \{v_1, \ldots, v_{r(v)}\} - S(v)$.

First we consider the case when $B(v) = B$. We note that $S = S(v)$ and $B = B(v)$ cannot hold simultaneously since $|V| \geq 2$. WLOG, we assume that $S(v) = S$ otherwise we can switch $B$ and $S$ since $x(\delta(S)) = x(\delta(B))$. Then we have,

$$\begin{aligned} x(\delta(S)) &= x(\delta(S(v))) \\ &= |S(v)| \left(2 - x_{vv}(|S(v)| - 1)\right) \\ &\geq |S(v)| \left(2 - \frac{|S(v)| - 1}{r(v)}\right) \\ &\geq |S(v)| \left(2 - \frac{|S(v)| - 1}{|S(v)| + 1}\right) \\ &\geq 2. \end{aligned}$$

The first inequality holds since $x_{vv} = \frac{y_{vv}}{r(v)^2} \leq \frac{1}{r(v)}$ because $y(\delta(v)) = 2r(v)$ implies that $y_{vv} \leq r(v)$ since a loop contributes twice to the degree count of a vertex. The second inequality holds since $S(v)$ does not contain all copies of $v$ so $|S(v)| < r(v)$ and the third inequality holds since the function $f(x) = x(2 - \frac{x-1}{x+1})$ is an increasing function that is minimized at $x = 1$.

Now we can assume that $S(v) \subset S$ and $B \subset B(v)$. For any $v_i \in S(v)$ let $X_1 = x(E(v_i, S - S(v)))$ and $X_2 = x(E(v_i, B - B(v)))$. We note that $X_1, X_2$ do not change based on the choice of $v_i$ since all copies of $v$ are defined identically in $H^r$. Now we consider the cut $S - S(v)$ and by using the fact that $x_{vv} \geq 0$ we have,

$$\begin{aligned} x(\delta(S - S(v))) &= x(\delta(S)) + |S(v)|\left(X_1 - |B(v)|x_{vv} - X_2\right) \\ &\leq x(\delta(S)) + |S(v)|\left(X_1 - X_2\right). \end{aligned}$$

First we consider the case when $X_1 \leq X_2$ and here we have that $x(\delta(S - S(v))) \leq x(\delta(S))$ and by induction we get that $x(\delta(S - S(v))) \geq 2$ since $S - S(v)$ is a set with one less vertex $v$ that separates a pair of copies of $v$ and $S - S(v) \neq V^r$. Thus we have shown $x(\delta(S)) \geq 2$ and now we consider the case when $X_1 > X_2$. Here we have,

$$\begin{aligned} x(\delta(S + B(v))) &= x(\delta(B - B(v))) \\ &= x(\delta(B)) + |B(v)|\left(X_2 - |S(v)|x_{vv} - X_1\right) \\ &\leq x(\delta(B)) + |B(v)|\left(X_2 - X_1\right) \\ &< x(\delta(B)) \\ &= x(\delta(S)). \end{aligned}$$

The first inequality follows since $x_{vv} \geq 0$ and the last inequality follows since we are in the case when $X_1 > X_2$. By induction we have $x(\delta(S + B(v))) \geq 2$ since $S + B(v)$ and is a set with one less vertex $v$ that separates a pair of copies of $v$ and $S + B(v) \neq V^r$. Thus, we have shown that $x(\delta(S)) \geq 2$.

Thus, we can apply the algorithm from the lemma assumption on $x$ as a solution to $H^r$ to get $k$ cycles $C_1, \ldots, C_k$ in $H^r$ satisfying $\sum_{e \in C} c_e \leq \rho c^T x$. We note that $c^T y = c^T x$ since $c^T x = \sum_{e=\{u,v\} \in E(H^r)} r(u) r(v) x_e c_e = \sum_{e \in E(H)} c_e y_e$. We now convert the $k$ cycles to $k$ closed walks in $H$ denoted by $T : E(H) \to \mathbb{Z}$ by replacing every copy edge $\{u_i, v_j\}$ with its corresponding original edge $\{u, v\}$ in $H$. Clearly, $T$ is consists of $k$ closed walks in $H$. Moreover, $T$ visits every vertex $r(v)$ times in $H$ since $C_1, \ldots, C_k$ visits each copy of $v$ one time in $H^r$. Lastly, the run-time follows since $H^r$ has at most $n \max_{v \in V} r(v)$ vertices and the algorithm in the lemma statement is a polynomial-time algorithm. □

Similar to the previous sections, we get the following reduction algortihm.

**Algorithm 7** Many-Visit TSP Single Depot

**Input:** $G = (V = \{v_1, \ldots, v_n\}), c : V \times V \to \mathbb{R}_{\geq 0}, m \in \mathbb{N}, r : V - v_1 \to \mathbb{Z}$

**Output:** $k$ tours that contain $v_1$ such that vertices $v \neq v_1$ are visited $r(v)$ times

1 Solve LP (11) to get solution $x^*$ .

2 For all edges $e$, let $\tilde{x}_e = x_e - 2k_e$ such that $k_e = 0$ if $x_e \leq 4$ and otherwise $k_e$ is set so that $2 \leq \tilde{x}_e < 4$ and $k_e \in \mathbb{Z}$. Define a function $\tilde{r} : V - v_1 \to \mathbb{Z}$ where $\tilde{r}(v) = r(v) - \sum_{e \in \delta(v)} k_e$ and $\tilde{k} = \frac{1}{2}\tilde{x}(\delta(v_1))$.

3 Use Lemma 5.7 with solution $\tilde{x}$ on instance $G, \tilde{r}, \tilde{k}$ .

4 Increase the number of times each edge is used in the previous step by $2k_e$ and return the resulting solution.

As in the previous section we show the following lemmas and claims to show this algorithm gets a $\rho$-approximation.

The proof of this claim is identical to Claim 4.6.

**Claim 5.8.** *For all $v \in V - v_1$ we have $1 \leq \tilde{r}(v) \leq 2n$ and $\tilde{k} \geq 1$.*

*Proof.* The proof of Claim 4.6 shows $1 \leq \tilde{r}(v) \leq 2n$ for all $v \in V - v_1$. Now we show $\tilde{k} \geq 1$. If we have that if $x_e \leq 4$ for all $e \in \delta(v_1)$ then $\tilde{k} = k \geq 1$ otherwise if there exists $e$ such that $x_e > 4$ then $\tilde{k} = \frac{1}{2}\tilde{x}(\delta(v_1)) \geq \frac{\tilde{x}_e}{2} > 2$. □

**Claim 5.9.** *The solution $\tilde{x}$ is a feasible solution for LP (11) with graph $G$ and $\tilde{r}, \tilde{k}$.*

*Proof.* By definition, we have $\tilde{x}(\delta(v_1)) = 2\tilde{k}$ and the rest claim follows from the proof of Claim 4.7. □

Then we get the following theorem whose proof is identical to the proof of Theorem 2.

**Theorem 5.** *If there exists a polynomial-time $\rho$-approximation algorithm for* SD-mTSP$_+$ *whose guarantee is with respect to LP (7), then there exists a polynomial-time $\rho$-approximation algorithm for* SD-MV-mTSP$_+$.

This gives the following corollary which we get by using the analysis of the Frieze algorithm we showed at the beginning of the section.

**Corollary 5.1.** *There is an approximation algorithm for the* SD-MV-mTSP *problem with an approximation factor of $\frac{3}{2}$.*

### 5.2 Approximation for Vertex Disjoint Tours

Here we show an algorithm for the vertex disjoint variant that achieves a 7/2-approximation. We note that this result does not follow the general framework and follows from a simple use of the single-visit algorithm. The idea for this algorithm is from Bérczi, Mnich, and Vincze [3] which is that we can first find a mTSP solution that visits all vertices once. Then we add loops to the different tours to satisfy the visit requirements while maintaining the vertex disjoint property.

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**Algorithm 8** Many-Visit TSP Single Depot Vertex Disjoint

**Input:** $G = (V = \{v_1, \ldots, v_n\}), c : V \times V \to \mathbb{R}_{\geq 0}, k \in \mathbb{N}, r : V - v_1 \to \mathbb{Z}$

**Output:** $k$ closed walks that contain $v_1$ such that vertices $v \neq v_1$ are visited $r(v)$ times, closed walks are disjoint outside of $v_1$

Use Algorithm (6) to get $k$-cycles containing $v_1$ so that all vertices are visited once.

Add $r(v) - 1$ loops to all vertices $v \neq v_1$ to the solution from the previous step.

---

We need the following claim.

**Claim 5.10.** *Let* OPT *be the value of the optimum solution. Then we have $\sum_{v \in V - v_1} r(v)c(vv) \leq$ 2OPT.*

*Proof.* Let $T : E \to \mathbb{Z}$ be an optimal solution. Then we have

$$\begin{aligned}
\mathrm{OPT} &= \frac{1}{2}\sum_{e\in\delta(v_1)} c_e T(e) + \frac{1}{2}\sum_{v\in V-v_1}\sum_{e\in\delta(v)} T(e)c_e \\
&\geq \frac{1}{2}\sum_{e\in\delta(v_1)} c_e T(e) + \frac{1}{2}\sum_{v\in V-v_1} 2r(v)\min_{e\in\delta(v)} c_e \\
&\geq \frac{1}{2}\sum_{e\in\delta(v_1)} c_e T(e) + \frac{1}{2}\sum_{v\in V-v_1} r(v)c_{vv} \\
&\geq \frac{1}{2}\sum_{v\in V-v_1} r(v)c_{vv} \enspace .
\end{aligned}$$

The first inequality follows since for all $v \in V - v_1$ we have $\sum_{e\in\delta(v_1)} T(e) = 2r(v)$ and the second inequality follows by the triangle inequality since for any edge $c \in \delta(v)$ we have $c_{vv} \leq 2c_e$. □

Then showing the following claim will imply that we get a $\frac{7}{2}$-approximation.

**Claim 5.11.** *Let $c_1, \ldots, c_k$ be the $k$ cycles returned in the first step of the algorithm. Then we have that $\sum_{i=1}^{k} c(c_i) \leq \frac{3}{2}\mathrm{OPT}$.*

*Proof.* Let $p_1, \ldots, p_m$ be an optimal solution with value OPT. We have that any two cycles $p_i, p_j$ only intersect at the depot vertex $v_1$ since we are in the vertex disjoint tours setting. For each $p_i$ we can shortcut to get cycle $r_i$, so that all vertices in $p_i$ are visited once and by the triangle inequality we have that $c(r_i) \leq c(p_i)$ implying $\sum_{i=1}^{k} c(r_i) \leq \mathrm{OPT}$. Thus, we get that $\sum_{i=1}^{k} c(c_i) \leq \frac{3}{2}\sum_{i=1}^{k} c(r_i) \leq \mathrm{OPT}$ where the first inequality follows since Algorithm (6) is a $\frac{3}{2}$-approximation. □

Thus, the above two claims imply the following theorem.

**Theorem 6.** *There exists a polynomial-time algorithm for the single depot many-visit* mTSP *(*SD-MV-mTSP$_+$*) problem with vertex-disjoint tours, with an approximation factor of $\frac{7}{2}$.*

*Proof.* Let $c_1, \ldots, c_k$ be the cycle acquired in the first step of the algorithm. The cycles $c_1, \ldots, c_k$ only intersect at the depot vertex $v_1$ since they are a feasible solution to SD-mTSP. Adding the loops to the cycles keeps this property so Algorithm (8) outputs a feasible solution. Finally, the cost of the solution is $\sum_{i=1}^{k} c(c_i) + \sum_{v\in V-v_1}(r(v)-1)c_{vv} \leq \frac{7}{2}\mathrm{OPT}$ where the last inequality follows by Claim 5.11 and Claim 5.10. The run-time follows immediately since both steps of the algorithm are polynomial-time. □

## 6 Further Directions

In this paper, we gave a reduction from various many-visit TSP problems and their respective single-visit versions. Our reduction relies on the connection between the LP relaxations of many-visit variants and their respective single-visit variants. There are two open questions that follow naturally.

**Get a $\frac{3}{2}$-approximation for** MV-mTSP$_0$**.** For the MV-mTSP$_0$ problem, we are given $k$ depots and the visit function $r$ and the goal is to find at most $k$ closed walks so that all non-depot vertices $v$ are visited $r(v)$ times and each closed walk contains exactly one depot. Recently, Deppert, Kaul, and Mnich [8] showed that LP (3) for mTSP$_0$ has an integrality gap of 2 and gave a $\frac{3}{2}$-approximation for mTSP$_0$ This means we cannot apply our reduction to MV-mTSP$_0$ by using LP (3) for mTSP$_0$

and get an approximation better than 3/2. One direction is to get a reduction from MV-mTSP$_0$ to mTSP$_0$ that does not use LPs.

**Apply the reduction to the unrestricted MV-**mTSP$_+$**.** In Section 4 we gave a 2-approximation for the unrestricted mTSP$_+$ problem and the approximation factor was with respect to the value of LP (6). We recall that LP (6) was not a LP relaxation where the characteristic vectors of integral solutions to the problem are feasible, but instead it was the convex hull of specfici trees that all integral solutions contain. We are not able to apply our reduction technique as the LP does not follow the structure of the LP described in the general framework. In particular, it is difficult to find a feasible solution $\tilde{x}$ for the reduced visit function $\tilde{r}$. Either finding a different LP relaxation or finding a different way to apply the reduction to LP (6) would improve the approximation factor of unrestricted MV-mTSP$_+$ from 4 to 2.

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