Stochastic Processes and their Applications 176 (2024) 104429

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

Efficient integrated volatility estimation in the presence of infinite variation jumps via debiased truncated realized variations[☆,☆☆]

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ARTICLE INFO

Keywords:
Integrated volatility estimation
Itô semimartingale
High-frequency data
Truncated realized variations
Efficiency

ABSTRACT

Statistical inference for stochastic processes based on high frequency observations has been an active research area for more than two decades. One of the most well-known and widely studied problems has been the estimation of the quadratic variation of the continuous component of an Itô semimartingale with jumps. Several rate- and variance-efficient estimators have been proposed in the literature when the jump component is of bounded variation. However, to date, very few methods can deal with jumps of unbounded variation. By developing new high-order expansions of the truncated moments of a locally stable Lévy process, we propose a new rate- and variance-efficient volatility estimator for a class of Itô semimartingales whose jumps behave locally like those of a stable Lévy process with Blumenthal–Getoor index $Y \in (1, 8/5)$ (hence, of unbounded variation). The proposed method is based on a two-step debiasing procedure for the truncated realized quadratic variation of the process and can also cover the case $Y < 1$. Our Monte Carlo experiments indicate that the method outperforms other efficient alternatives in the literature in the setting covered by our theoretical framework.

1. Introduction

Statistical inference for stochastic processes based on high-frequency observations has attracted considerable attention in the literature for more than two decades. Among the many problems studied to date, arguably none has received more attention than that of the estimation of the continuous (or predictable) quadratic variation of an Itô semimartingale $X = \{X_t\}_{t\geq 0}$. Specifically, if

$$X_t := X_0 + X_t^c + X_t^j := X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + X_t^j, \quad t \in [0, T], \tag{1}$$

where $X_0 \in \mathbb{R}$, $W = \{W_t\}_{t\geq 0}$ is a Wiener process and $X^j = \{X_t^j\}_{t\geq 0}$ is a pure-jump Itô semimartingale, then our estimation target is

$$IV_T = \int_0^T \sigma_s^2 ds.$$

This quantity, also known as the *integrated volatility* or *integrated variance* of X, has many applications, especially in finance, where X typically models the log-return process of a risky asset and IV_T measures the overall uncertainty or variability inherent in X during

☆ The first author's research is partially supported by the National Science Foundation grant DMS-2413558. The second author's research is partially supported by the National Science Foundation grants: DMS-2413557, DMS-2015323.
☆☆ An earlier version of this manuscript was circulated under the title "Efficient Volatility Estimation for Lévy Processes with Jumps of Unbounded Variation". The results therein were constrained to Lévy processes, whereas here we consider a much larger class of Itô semimartingales.
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https://doi.org/10.1016/j.spa.2024.104429
Received 24 July 2023; Received in revised form 19 April 2024; Accepted 2 July 2024
Available online 8 July 2024

the time period $[0,T]$. When X is observed at evenly spaced times $0 = t_0 < t_1 < \cdots < t_n = T$, in the absence of jumps, an efficient estimator of IV_T is given by the realized quadratic variation $\widehat{IV}_T = \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2$ in the so-called high-frequency (or infill) asymptotic regime; i.e., when $n \to \infty$ and $T \equiv t_n$ is fixed. In the presence of jumps, $\widehat{IV}_T$ is no longer even consistent for IV_T, instead converging to $IV_T + \sum_{s\le T}(\Delta X_s)^2$, where $\Delta X_s := X_s - X_{s^-}$ denotes the jump at time s. To account for jumps, several estimators have been proposed, among which the most well-known are the truncated realized quadratic variation and the multipower variations. We focus on the first class, which, unlike the second, is both rate- and variance-efficient, in the Cramer–Rao lower bound sense, when jumps are of bounded variation under certain additional conditions.

The truncated realized quadratic variation (TRQV), also called truncated realized volatility, was first introduced by [1,2] and is defined as

$$\widehat{C}_n(\varepsilon) = \sum_{i=1}^n (\Delta_i^n X)^2 \mathbf{1}_{\{|\Delta_i^n X| \le \varepsilon\}}, \tag{2}$$

where $\varepsilon = \varepsilon_n > 0$ is a tuning parameter converging to 0 at a suitable rate. Above, $\Delta_i^n X := X_{t_i} - X_{t_{i-1}}$ is the ith increment of $(X_t)_{t\ge 0}$ based on evenly spaced observations $X_{t_0}, \dots, X_{t_n}$ over a fixed time interval $[0,T]$ (i.e., $t_i = ih_n$ with $h_n = T/n$). It is shown in [3] that TRQV is consistent when either the jumps have finite activity or stem from an infinite-activity Lévy process. In a semimartingale model with Lévy jumps of bounded variation, Cont and Mancini [4] showed that the TRQV admits a feasible central limit theorem (CLT), provided that $\varepsilon_n = ch_n^{\omega}$ with some $\omega \in [\frac{1}{4-Y}, \frac{1}{2})$, where $Y \in [0,1)$ denotes the Blumenthal–Getoor index. In [5], consistency was established for a general Itô semimartingale X, and a corresponding CLT is given when the jumps of X are of bounded variation. In that case, the TRQV attains the optimal rate and asymptotic variance of $\sqrt{h_n}$ and $2\int_0^T \sigma_s^4 ds$, respectively.

However, in the presence of jumps of unbounded variation, arguably the most relevant for financial applications (see, e.g., [6–8], and the results in Table 1 below), the situation is notably different, and the available literature on TRQV offers an incomplete picture. In [4], it is shown that when jumps stem from a Lévy process with stable-like small-jumps of infinite variation, the TRQV estimator $\widehat{C}_n(\varepsilon)$ converges to IV_T at a rate slower than $\sqrt{h_n}$. Further, in [9] it is shown that when the jump component J is a symmetric Y-stable Lévy process and $\varepsilon_n = h_n^{\omega}$ with $\omega \in (0,1/2)$, the decomposition $\widehat{C}_n(\varepsilon_n) - IV_T = \sqrt{h_n}Z_n + \mathcal{R}_n$ holds, where Z_n converges stably in law to $\mathcal{N}(0, 2\int_0^T \sigma_s^4 ds)$, while $\mathcal{R}_n$ is precisely of order ε_n^{2-Y} in the sense that $\mathcal{R}_n = O_P(\varepsilon_n^{2-Y})$ and $\varepsilon_n^{2-Y} = O_P(\mathcal{R}_n)$ (which decays too slowly to allow for efficiency when $Y > 1$). In [10], a smoothed version of the TRQV estimator of the form $\widehat{C}_n^{Sm}(\varepsilon) = \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2 \varphi((X_{t_i} - X_{t_{i-1}})/\varepsilon)$ is considered,[1] where $\varphi \in C^\infty$ vanishes in $\mathbb{R}\backslash(-2,2)$ and $\varphi(x) = 1$ for $x \in (-1,1)$. In that case, using the truncation level $\varepsilon_n := h_n^{\omega}$, it is shown that $\widehat{C}_n^{Sm}(\varepsilon_n) - IV_T = \sqrt{h_n}Z_n + \mathcal{R}_n$ with $\mathcal{R}_n$ such that $\varepsilon_n^{-(2-Y)}\mathcal{R}_n \to c_Y \int \varphi(u)|u|^{1-Y} du$, for a constant $c_Y \neq 0$, and still $Z_n \stackrel{st}{\to} \mathcal{N}(0, 2\int_0^T \sigma_s^4 ds)$. By taking φ such that $\int \varphi(u)|u|^{1-Y} du = 0$, a "bias-corrected" estimator was considered under the additional condition that $Y < 4/3$. Specifically, the resulting estimator is such that, for any $\tilde{\epsilon} > 0$, $\widehat{C}_n^{Sm}(\varepsilon_n) - IV_T = o_P(h_n^{1/2-\tilde{\epsilon}})$, "nearly" attaining the optimal statistical error $O_P(h_n^{1/2})$. Unfortunately, the construction of such an estimator requires knowledge or accurate estimation of the jump intensity index Y, and no feasible CLT was proved when jumps are of unbounded variation even assuming Y is known.

Apart from TQRV-based approaches, efficient estimation of IV_T when the jumps have unbounded variation is intrinsically limited in the general case. In [11], it was shown that when the jump intensity index $Y > 1$, the best possible convergence rate, in a minimax sense, over certain "bounded" classes of semimartingales, is of order $(n\log n)^{-(2-Y)/2}$. Nevertheless, in principle, a faster convergence rate may be attainable if one constrains the process X to belong to a certain semiparametric class such as when the jumps exhibit a "locally stable"-like behavior. Obviously, the fastest possible rate one can hope to achieve is $n^{-1/2}$, which coincides with the one attained by the realized quadratic variation in the continuous case and is known to be optimal in a minimax sense.

The first (and to-date, only) rate- and variance-efficient estimator of the integrated volatility known in the literature for semimartingales when $Y > 1$ was proposed by [12], under a locally-stable assumption on jumps, but with some notable additional restrictions: these results require either that the jump intensity index $Y < 3/2$, or that the "small" jumps of the process X are "symmetric".[2] Their estimator is based on locally estimating the volatility from the empirical characteristic function of the process' increments over disjoint time intervals shrinking to 0, but still containing an increasing number of observations. It requires two debiasing steps, which are simpler to explain for a Lévy process X with symmetric Y-stable jump component X^j. The first debiasing step is meant to reduce the bias introduced when attempting to estimate $\log \mathbb{E}(\cos(uX_{h_n}/\sqrt{h_n}))$ with $\log\{\frac{1}{n}\sum_{i=1}^n \cos(u\Delta_i^n X/\sqrt{h_n})\}$. The second debiasing step is aimed at eliminating the second term in the expansion $-2\log \mathbb{E}(\cos(uX_{h_n}/\sqrt{h_n}))/u^2 = \sigma^2 + 2|\gamma|^Y u^{Y-2} h_n^{1-Y/2} + O(h_n)$, which otherwise diverges when multiplied by the optimal scaling $h_n^{-1/2} = n^{1/2}$. Using an extension of this approach, Jacod and Todorov were able to apply these techniques to a more general class of Itô semimartingales in [13], even allowing any $Y < 2$, though only rate-efficient, but not variance-efficient, estimators were ultimately constructed.

On the other hand, in the special case of Lévy processes, efficiency across the full range $0 < Y < 2$ without symmetry requirements has been attained by [14], again under a locally stable assumption, via a generalized method of moments. Specifically, for some suitable smooth functions $f_1, f_2, \dots, f_m$ and a scaling factor $u_n \to \infty$, [14] proposed to search for the parameter values $\widehat{\theta} = (\widehat{\theta}_1, \dots, \widehat{\theta}_m)$ such that

$$\frac{1}{n}\sum_{i=1}^n f_j\left(u_n\Delta_i^n X\right) - \mathbb{E}_{\widehat{\theta}}\left(f_j\left(u_n\Delta_i^n \widetilde{X}\right)\right) = 0, \quad j = 1, \dots, m, \tag{3}$$

[1] The authors in [10], in fact, consider the more general estimation of $\int_0^T f(X_s)\sigma_s^2 ds$ for functions f of polynomial growth, for which IV_T is a special case.

[2] Jacod and Todorov [12] also constructed an estimator that is rate-efficient even in the presence of asymmetric jumps, but its asymptotic variance is twice as big as the optimal value $2\int_0^T \sigma_s^4 ds$ and, thus, it is not variance efficient.

Table 1
Parameter estimates for a Lévy model $\sigma W_t + X_t^j$ with semi-parametric Lévy density $(C_+\mathbf{1}_{x>0} + C_-\mathbf{1}_{x<0})q(x)|x|^{-Y-1}$ with $q(x) \xrightarrow{x\to 0} 1$ applied to stock data over a 1-year horizon at different sampling frequencies. Parameter estimates were obtained via the method [14] with $M = 1$. Intraday data was obtained from the NYSE TAQ database of 2005 trades via Wharton's WRDS system.

Stock	Freq.	$\hat{\sigma}$	$\hat{C}_+$	$\hat{C}_-$	$\hat{Y}$
INTC	1 min	0.216	0.0096	0.0075	1.43
INTC	5 sec	0.241	0.0311	0.0292	1.60
PFE	5 min	0.180	0.0232	0.0199	1.14
PFE	1 min	0.196	0.0105	0.0066	1.37
AMAT	5 sec	0.344	0.0014	0.0012	1.85
SPY	5 sec	0.103	0.0003	0.00005	1.82
AMGN	1 min	0.211	0.0032	0.0038	1.53
MOT	1 min	0.244	0.0183	0.0066	1.33

where $\widetilde{X}$ is the superposition of a Brownian motion and independent stable Lévy processes closely approximating X in a certain sense. The distribution measure $\mathbb{P}_\theta$ of $\widetilde{X}$ depends on some parameters $\theta = (\theta_1, \ldots, \theta_m)$, one of which is the volatility σ of X, and $\mathbb{E}_\theta(\cdot)$ denotes the expectation with respect to $\mathbb{P}_\theta$. Aside from the fact that this method can only be applied to Lévy processes, it also suffers from other drawbacks. First, its finite-sample performance critically depends on the chosen moment functions $f_1, \ldots, f_m$. Secondly, its implementation is computationally expensive and may lead to numerical issues since it involves solving a system of nonlinear equations (including possible non-existence of solutions to (3) over finite samples). Moreover, in addition to the required use of a numerical solver to determine the values of $\widehat{\boldsymbol{\theta}}$ in (3), the expectations appearing therein need to be numerically approximated since explicit expressions for the moments $\mathbb{E}_\theta(f_j(u_n\Delta_i^n\widetilde{X}))$ are typically not available. This fact introduces additional numerical errors that complicates its performance.

In this paper, we consider a new method to estimate the integrated volatility $IV_T = \int_0^T \sigma_s^2 ds$ of an Itô semimartingale whose jump component is given by a stochastic integral with respect to a tempered-stable-like Lévy process J of unbounded variation. To the best of our knowledge, our method, together with [12], are the only efficient methods to deal with jumps of unbounded variation for semimartingales. The idea is natural. We simply apply debiasing steps similar to those of [12] to the TRQV of [3]. To give the heuristics as to why this strategy works, consider a small-time expansion of the truncated moments $\mathbb{E}(X_{h_n}^{2k}\mathbf{1}_{\{|X_{h_n}|\le\varepsilon_n\}})$ of a Lévy process $X_t = bt + \sigma W_t + X_t^j$ in the asymptotic regime $h_n, \varepsilon_n \to 0$ with $\varepsilon_n/\sqrt{h_n} \to \infty$. Using a variety of techniques, including a change of probability measure, Fourier-based methods, and small-large jump decompositions, we show the following two expansions, for integers $k \ge 2$:

$$\mathbb{E}\left[X_{h_n}^2\mathbf{1}_{\{|X_{h_n}|\le\varepsilon_n\}}\right] = \sigma^2 h_n + c_1 h_n\varepsilon_n^{2-Y} + c_2 h_n^2\varepsilon_n^{-Y} + \text{h.o.t.},$$
$$\mathbb{E}\left[X_{h_n}^{2k}\mathbf{1}_{\{|X_{h_n}|\le\varepsilon\}}\right] = (2k-1)!!\,\sigma^{2k}h_n^k + c_3\,h_n\varepsilon_n^{2k-Y} + \text{h.o.t.},$$

for certain constants $c_1, c_2, c_3 \neq 0$ that are explicitly computed. Hereafter, h.o.t. stands for 'higher order terms'. The expansions above are the most precise of their type in the literature and are of interest in their own right. Based on the first expansion above, it is easy to see that the rescaled bias $\mathbb{E}[h_n^{-1/2}(\widehat{C}_n(\varepsilon) - \sigma^2 T)]$ satisfies

$$\mathbb{E}\left[h_n^{-\frac{1}{2}}\left(\widehat{C}_n(\varepsilon) - \sigma^2 T\right)\right] = Tc_1 h_n^{-\frac{1}{2}}\varepsilon_n^{2-Y} + Tc_2 h_n^{\frac{1}{2}}\varepsilon_n^{-Y} + \text{h.o.t.}, \tag{4}$$

which suggests the necessity of the condition $h_n^{-1/2}\varepsilon_n^{2-Y} = o(1)$ for a feasible CLT for $\widehat{C}_n(\varepsilon)$ at the rate $\sqrt{h_n}$. However, together with the (necessary) condition $\varepsilon_n/\sqrt{h_n} \to \infty$, this can happen only if $Y < 1$, and removal of the first terms in (4) is consequently necessary for efficient estimation when jumps are of unbounded variation. To that end, note that for any $\zeta > 1$,

$$\mathbb{E}\left(\widehat{C}_n(\zeta\varepsilon) - \widehat{C}_n(\varepsilon)\right) = c_1(\zeta^{2-Y} - 1)\varepsilon_n^{2-Y} + \text{h.o.t.},$$
$$\mathbb{E}\left(\widehat{C}_n(\zeta^2\varepsilon) - 2\widehat{C}_n(\zeta\varepsilon) + \widehat{C}_n(\varepsilon)\right) = c_1(\zeta^{2-Y} - 1)^2\varepsilon_n^{2-Y} + \text{h.o.t.}.$$

The above formulas motivate the "bias-corrected" estimator

$$\widehat{C}_n'(\varepsilon;\zeta) := \widehat{C}_n(\varepsilon) - \frac{\left(\widehat{C}_n(\zeta\varepsilon) - \widehat{C}_n(\varepsilon)\right)^2}{\widehat{C}_n(\zeta^2\varepsilon) - 2\widehat{C}_n(\zeta\varepsilon) + \widehat{C}_n(\varepsilon)}, \tag{5}$$

which is the essence of the debiasing procedure of [12]. As we shall see, the story is more complicated than what the simple heuristics above suggest. Our main result shows that, for the class of Itô semimartingales described above, the estimator (5) is indeed rate- and variance-efficient provided that $Y \in (1, 4/3)$. Furthermore, if $Y \in (1, 8/5)$, a second bias-correcting step will achieve both rate- and variance-efficiency (for the case $8/5 \le Y < 2$, see remark at the end of Section 3). Even though our main motivation lies in incorporating jumps of unbounded variation, we show that the debiasing steps will still achieve efficiency in the case that $Y < 1$ (though, of course, no debiasing is needed in that case because the jumps are of bounded variation).

Though our approach is natural, mathematically establishing its efficiency is highly nontrivial, starting from the new high-order expansions of the truncated moments of Lévy processes – which, beyond heuristics, ultimately play a key role in analyzing asymptotics for the debiasing technique – to the application of Jacod's stable central limit theorem for semimartingales (in particular, the verification of the asymptotic 'orthogonality' condition (2.2.40) in [15]). Our Monte Carlo experiments indicate improved performance compared to [12] (and also [14]) for the important class of CGMY Lévy processes (cf. [16]) and for a Heston stochastic volatility model with CGMY jumps in the range of values of Y covered by our theoretical framework.

If we limit ourselves to a Lévy model, our approach is more computationally efficient and numerically stable than that in [14].[3] For more general semimartingales, our procedure is simpler than that in [12] since it does not require an extra debiasing step to correct the nonlinear nature of the logarithmic transformation employed therein nor does it require a symmetrization step to deal with asymmetric jump components. Furthermore, our method does not rely on a 'localization' technique in the sense that it does not need to break the data into disjoint blocks where the integrated volatility is locally estimated. The latter step introduces an additional tuning parameter absent from our method.

Let us emphasize that our method is the first variance- and rate-efficient nonparametric method for integrated volatility free of complete symmetry assumptions on small jumps that is capable of exceeding the limit $Y < 3/2$ imposed in [12,13]. Symmetry is potentially a strong assumption for financial returns as there is a general belief that significant losses are more likely than significant gains. For instance, recently [17] examined several empirical studies from the literature and observed that the majority display negative skewness. Of course, skewness may arise from either large or small jumps, though large jump asymmetry has received most attention in empirical work. Indeed, there are few studies to date that estimate the intensity of small positive and negative jumps separately. An exception is [18], who, using MLE applied to daily S&P500 index data from 1996–2006, obtained the estimates $\hat{C}_+ = 0.7119$, $\hat{C}_- = 0.5412$, $\hat{G} = 59.94$, $\hat{M} = 59.94$, $\hat{Y}_+ = 1.0457$, and $\hat{Y}^- = 1.1521$ under a pure-jump Lévy model with Lévy density $C_+ e^{-x/G}|x|^{-Y_+-1}\mathbf{1}_{x>0} + C_- e^{-|x|/M}|x|^{-Y_--1}\mathbf{1}_{x<0}$, which points to asymmetry in small jumps. For further illustration, in Table 1 below, we fit a Lévy model $\sigma W_t + X_t^j$, with semi-parametric Lévy density $(C_+\mathbf{1}_{x>0} + C_-\mathbf{1}_{x<0})q(x)|x|^{-Y-1}$ (here, $q(x) \stackrel{x\to 0}{\longrightarrow} 1$). We use Mies' method of moments (cf. [14]) for different stocks and frequencies[4] over a 1-year period in 2005. It is clear that $\hat{C}_+$ is different from $\hat{C}_-$, sometimes by a relatively large value, indicating further evidence for asymmetry in small jump behavior. Furthermore, all values of $\hat{Y}$ are larger than 1, indicating the presence of a jump component of unbounded variation.

Finally, let us also remark that our result opens the doors to attain rate- and variance-efficient estimators free of symmetry requirements beyond the mark $8/5$ or in more general semiparametric models with successive Blumenthal–Getoor indices by considering further debiasing steps. These directions will be investigated in further work.

The rest of this paper is organized as follows. Section 2 introduces the framework and assumptions as well as some known preliminary results from the literature. Section 3 introduces the debiasing method and main results of the paper. Section 4 illustrates the performance of our method via Monte Carlo simulations and compares it to the method in [12]. The proofs of the key results are deferred to two appendix sections. The proofs of some technical lemmas and other supporting propositions are deferred to the accompanying supplemental material to this article.

2. Setting and background

In this section, we introduce the model, main assumptions, and some notation. We consider a 1-dimensional Itô semimartingale $X = (X_t)_{t\in\mathbb{R}_+}$ of the form (1), defined on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$. Since it has no impact on the value of the increments of X, for simplicity throughout we assume $X_0 = 0$. We assume the jump component X^j can be decomposed into a sum of an infinite-variation process $X^{j,\infty}$ and a finite variation process $X^{j,0}$ given, for $t \in \mathbb{R}_+$, as

$$
\begin{aligned}
X_t^j &:= X_t^{j,\infty} + X_t^{j,0} \\
&:= \int_0^t \chi_{s^-} dJ_s + \int_0^t \int \Big\{ \delta_0(s,z)\mathfrak{p}_0(ds,dz) + \delta_1(s,z)\mathfrak{p}_1(ds,dz) \Big\},
\end{aligned}
\tag{6}
$$

where $\chi = \{\chi_t\}_{t\geq 0}$ is an adapted process satisfying appropriate integrability conditions, $J := (J_t)_{t\in\mathbb{R}_+}$ is an independent pure-jump Lévy process with Lévy triplet $(b, 0, \nu)$, $\mathfrak{p}_0, \mathfrak{p}_1$, are Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}$ with intensities $\mathfrak{q}_i(ds,dz) = ds \otimes \lambda_i(dz)$, where the λ_i's are σ-finite measures on $\mathbb{R}$, and $\mathfrak{p}_1$ is assumed independent of J. The specific conditions on ν, λ_i, and on the coefficient processes δ_i, χ, and σ are given below.

The Lévy measure ν is assumed to admit a density $s : \mathbb{R}_0 \to \mathbb{R}_+$ of the form

$$
s(x) := \frac{d\nu}{dx} := \big(C_+\mathbf{1}_{(0,\infty)}(x) + C_-\mathbf{1}_{(-\infty,0)}(x)\big) q(x)\, |x|^{-1-Y}. \tag{7}
$$

Above, $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, $C_\pm > 0$, $Y \in (1,2)$, and $q : \mathbb{R}_0 \to \mathbb{R}_+$ is a bounded Borel-measurable function satisfying the following conditions:

Assumption 1.

(i) $q(x) \to 1$, as $x \to 0$;

[3] Though the method of [14] allows for simultaneous estimation of several parameters of the model (such as both σ and Y).

[4] As pointed out in [14] (see also the paragraph after (3) above), numerical issues can arise related to feasibility of the estimating equations associated with the method. We are only presenting the results when the algorithm successfully finishes and yields reasonable values.

(ii) there exist $\alpha_{\pm} \neq 0$ such that

$$\int_0^1 \left|q(x) - 1 - \alpha_+ x\right| x^{-Y-1} dx + \int_{-1}^0 \left|q(x) - 1 - \alpha_- x\right| |x|^{-Y-1} dx < \infty.$$

These processes are sometimes called "stable-like Lévy processes" and were studied in [19,20] and many other works. In simple terms, condition (i) above says that the small jumps of the Lévy process X behave like those of a Y-stable Lévy process with Lévy measure

$$\widetilde{\nu}(dx) := \left(C_+ \mathbf{1}_{(0,\infty)}(x) + C_- \mathbf{1}_{(-\infty,0)}(x)\right)|x|^{-Y-1} dx. \tag{8}$$

The condition $Y \in (1,2)$ implies that J has unbounded variation in that sense that $\sum_{i=1}^n |J_{t_i} - J_{t_{i-1}}| \to \infty$, a.s., as the partition $0 = t_0 < t_1 < \cdots < t_n = T$ is such that $\max\{t_i - t_{i-1}\} \to 0$.

As discussed in Section 1, in view of [11], the locally Y-stable aspect of Assumption 1 is crucially important for our results, and similar assumptions have been made by other authors (e.g, [12,14]). Though not completely general, the class is still relevant in applications, as many of the models proposed in the literature (especially, in finance) fall within this class. Nevertheless, from a theoretical point of view, it remains to be seen as to what the broadest assumptions may be under which one can still attain estimation efficiency.

As in [19], it will be important for our analysis to apply a density transformation technique [21, Section 6.33] to "transform" the process J into a stable Lévy process. Concretely, we can change the probability measure from $\mathbb{P}$ to another locally absolutely continuous measure $\widetilde{\mathbb{P}}$, under which W is still a standard Brownian motion independent of J, but, under $\widetilde{\mathbb{P}}$, J has Lévy triplet $(\tilde{b}, 0, \tilde{\nu})$, where $\widetilde{\nu}(dx)$ is given as in (8) and $\widetilde{b} := \bar{b} + \int_{0<|x|\le 1} x(\tilde{\nu} - \nu)(dx)$. The key assumption above is (ii), which would allow us to decompose the log-density process

$$U_t := \ln \frac{d\widetilde{\mathbb{P}}\big|_{\mathscr{F}_t}}{d\mathbb{P}\big|_{\mathscr{F}_t}},$$

as a sum of a bounded variation process and two spectrally one-sided Y-stable Lévy processes.

Finally, we give the conditions on σ and the coefficient processes b, δ_0, δ_1, and χ in (1) and (6).

Assumption 2.

(i) σ is càdlàg adapted.

(ii) The process χ is given as

$$\chi_t = \chi_0 + \int_0^t b_s^{\chi} ds + \int_0^t \Sigma_s^{\chi} dB_s.$$

(iii) The processes W, B are Brownian motions independent of $(J, \mathfrak{p}_0, \mathfrak{p}_1)$; $(\delta_1, \mathfrak{p}_1)$ is independent of $(X^c, J, \chi, \mathfrak{p}_0, \delta_0)$; J is independent of σ.

(iv) The processes Σ^{χ}, b, and b^{χ} are càdlàg adapted, and δ_0, δ_1 are predictable. There is also a sequence $\{\tau_n\}_{n\ge 1}$ of stopping times increasing to infinity, nonnegative $\lambda_i(dz)$-integrable functions H_i, and a positive sequence $\{M_n\}_{n\ge 1}$ such that

$$t \le \tau_n \implies \begin{cases} |\sigma_t| + |b_t| + |b_t^{\chi}| + |\Sigma_t^{\chi}| \le M_n, \\ |\mathbb{E}(\sigma_{t+s} - \sigma_t | \mathcal{F}_t)| + \mathbb{E}(|\sigma_{t+s} - \sigma_t|^2 | \mathcal{F}_t) \le M_n s, \\ (|\delta_0(t,z)| \wedge 1)^{r_0} \le M_n H_0(z), \\ (|\delta_1(t,z)| \wedge 1)^{r_1} \le M_n H_1(z), \end{cases}$$

for some $r_0 \in [0, \frac{Y}{2+Y})$ and $r_1 \in [0, Y/2)$.

Above, the parameters r_0, r_1 control the degree of activity in the nuisance finite-variation jump terms. Two such terms are included to allow for a broader range of finite-variation jump activity in our model setup. Note that $Y/2$ is always bigger than $Y/(2+Y)$, which shows that when the bounded variation jump component is independent from the other processes, we can incorporate a wider range of jump activity. These restrictions effectively guarantee that the bias introduced by finite variation components are negligible in comparison to leading bias terms arising from the locally-stable jumps in X.

3. Main results

In this section, we construct an efficient estimator for the integrated volatility $IV_T = \int_0^T \sigma_s^2 ds$ based on the well-studied estimator TRQV (2). All proofs are deferred to the Appendix and supplement.

Throughout, we assume the process $X = \{X_t\}_{t\ge 0}$ is sampled at n evenly spaced observations, $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$, during a fixed time interval $[0,T]$, where for $i = 0, \ldots, n$, $t_i = t_{i,n} = ih_n$ with $h_n = T/n$, and, for simplicity, assume that $T = 1$. As usual, we define the increments of a generic process $V = \{V_t\}_{t\ge 0}$ as $\Delta_i^n V := V_{t_i} - V_{t_{i-1}}$, $i = 1, \ldots, n$. We often use the shorthand notation:

$$V_i^n = V_{t_i}, \quad \mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_{t_i}].$$

As mentioned in the introduction, in the presence of jumps of unbounded variation, the TRQV estimator $\widehat{C}_n(\varepsilon)$ is not efficient since it possesses a bias that vanishes at a rate slower than $n^{-1/2}$, the rate at which the "centered" TRQV

$$\overline{C}_n(\varepsilon) = \sum_{i=1}^{n} \left\{ \left(\Delta_i^n X\right)^2 \mathbf{1}_{\{|\Delta_i^n X|\le\varepsilon\}} - \mathbb{E}_{i-1}\left[\left(\Delta_i^n X\right)^2 \mathbf{1}_{\{|\Delta_i^n X|\le\varepsilon\}}\right]\right\}, \tag{9}$$

admits a CLT. To overcome this, our idea is to apply the debiasing procedure of [12] to the TRQV. As mentioned in the introduction, our procedure is simpler than [12] since it does not require an extra debiasing step to account for the logarithmic transformation nor does it require a symmetrization step to deal with asymmetric Lévy measures. Furthermore, our method does not have to be applied in each subinterval of a partition of the time horizon, which introduces another tuning parameter.

Before constructing our estimator, we first establish the asymptotic behavior of TRQV (2) with a fully specified centering quantity $A(\varepsilon,h)$ rather than the inexplicit centering $\mathbb{E}_{i-1}\left[\left(\Delta_i^n X\right)^2 \mathbf{1}_{\{|\Delta_i^n X|\le\varepsilon\}}\right]$ of (9). It also characterizes the structure of the bias $A(\varepsilon,h)$ in the threshold parameter ε that will ultimately be exploited in our debiasing procedure. Below and throughout the rest of the paper, we use the usual notation $a_n \ll b_n$, whenever $a_n/b_n \to 0$ as $n\to\infty$.

Proposition 1. *Suppose that $Y\in(0,1)\cup(1,8/5)$ and $h_n^{\frac{3}{2(2+Y)}\wedge\frac{1}{2}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-Y}}$. Let*

$$\widetilde{Z}_n(\varepsilon) := \sqrt{n}\left(\widehat{C}_n(\varepsilon) - \int_0^1 \sigma_s^2 ds - A(\varepsilon,h)\right), \tag{10}$$

where

$$A(\varepsilon,h) := \frac{\bar{C}}{2-Y}\int_0^1 |\chi_s|^Y ds\,\varepsilon^{2-Y} - \bar{C}\frac{(Y+1)(Y+2)}{2Y}\int_0^1 |\chi_s|^Y \sigma_s^2 ds\, h\varepsilon^{-Y}, \tag{11}$$

and $\bar{C} := C_+ + C_-$. Then, as $n\to\infty$,

$$\widetilde{Z}_n(\varepsilon_n) \xrightarrow{st} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4 ds\right). \tag{12}$$

Remark 1. As expected, the statement above shows that, when $Y<1$, we have $\sqrt{n}\left(\widehat{C}_n(\varepsilon) - \int_0^1 \sigma_s^2 ds\right) \xrightarrow{st} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4 ds\right)$, whenever $h_n^{\frac{1}{2}} \ll \varepsilon_n \ll h_n^{\frac{1}{2(2-Y)}}$ and the indices r_0 and r_1 in Assumption 2 are less than Y (i.e., the constraints $r_0\in[0,\frac{Y}{2+Y})$ and $r_1\in[0,Y/2)$ are needed in Proposition 1 only in the case $Y>1$). Indeed, in this case, since the leading order bias is $O(\varepsilon^{2-Y})$, so taking $\varepsilon \ll h_n^{\frac{1}{2(2-Y)}}$ renders it asymptotically negligible.

The next theorem establishes the stable convergence (in particular, the convergence rate) of the difference $\widetilde{Z}_n(\zeta\varepsilon) - \widetilde{Z}_n(\varepsilon)$, for some $\zeta>1$, which is the second main technical result we use to deduce the efficiency of our debiased estimator.

Proposition 2. *Suppose $Y\in(0,1)\cup(1,8/5)$, and $h_n^{\frac{4}{8+Y}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-Y}}$. With the notation of Proposition 1, for arbitrary $\zeta>1$,*

$$u_n^{-1}\left(\widetilde{Z}_n(\zeta\varepsilon_n) - \widetilde{Z}_n(\varepsilon_n)\right) \xrightarrow{st} \mathcal{N}\left(0, \frac{\bar{C}}{4-Y}\int_0^1 |\chi_s|^Y ds(\zeta^{4-Y}-1)\right), \tag{13}$$

as $n\to\infty$, where $u_n := h_n^{-\frac{1}{2}}\varepsilon_n^{\frac{4-Y}{2}} \to 0$.

Remark 2. Note that (13) implies that

$$\varepsilon_n^{-Y/2}\left(\frac{\widehat{C}_n(\zeta\varepsilon) - \widehat{C}_n(\varepsilon)}{\varepsilon^{2-Y}} - \frac{\bar{C}}{2-Y}(\zeta^{2-Y}-1)\int_0^1 |\chi_s|^Y ds\right) \xrightarrow{st} \mathcal{N}\left(0, \frac{\bar{C}}{4-Y}(\zeta^{4-Y}-1)\int_0^1 |\gamma_s|^Y ds\right).$$

In particular,

$$\frac{\widehat{C}_n(\zeta\varepsilon) - \widehat{C}_n(\varepsilon)}{\varepsilon^{2-Y}} \xrightarrow{\mathbb{P}} \frac{\bar{C}}{2-Y}(\zeta^{2-Y}-1)\int_0^1 |\chi_s|^Y ds. \tag{14}$$

Expression (14) plays a role in our numerical implementation in Section 4.

We are now in a position to introduce our proposed estimator. To this end, we will exploit the structure of the bias term $A(\varepsilon,h)$ in ε. The idea is simple. Suppose that a function $f(x)$ takes the form $a+bx^\alpha$ for any $a\in\mathbb{R}$ and $\alpha,b\neq 0$. Then, it is easy to see that, for any $\zeta>1$,

$$f(x) - \frac{(f(\zeta x)-f(x))^2}{f(\zeta^2 x) - 2f(\zeta x) + f(x)} = a + bx^\alpha - \frac{b^2 x^{2\alpha}(\zeta^\alpha-1)^2}{bx^\alpha(\zeta^{2\alpha}-2\zeta^\alpha+1)} = a,$$

hence, recovering a without requiring knowledge of b and α. These heuristics suggest the following debiasing procedure to successively remove each term appearing in $A(\varepsilon,h)$. For any $\zeta_1,\zeta_2>1$, in a first step, we compute

$$\widetilde{C}'_n(\varepsilon,\zeta_1)=\widehat{C}_n(\varepsilon)-\frac{\left(\widehat{C}_n(\zeta_1\varepsilon)-\widehat{C}_n(\varepsilon)\right)^2}{\widehat{C}_n(\zeta_1^2\varepsilon)-2\widehat{C}_n(\zeta_1\varepsilon)+\widehat{C}_n(\varepsilon)}, \tag{15}$$

and, in the second step,

$$\widetilde{C}''_n(\varepsilon,\zeta_2,\zeta_1)=\widetilde{C}'_n(\varepsilon,\zeta_1)-\frac{\left(\widetilde{C}'_n(\zeta_2\varepsilon,\zeta_1)-\widetilde{C}'_n(\varepsilon,\zeta_1)\right)^2}{\widetilde{C}'_n(\zeta_2^2\varepsilon,\zeta_1)-2\widetilde{C}'_n(\zeta_2\varepsilon,\zeta_1)+\widetilde{C}'_n(\varepsilon,\zeta_1)}. \tag{16}$$

The next theorem is the main result of the paper. It establishes the rate- and variance-efficiency of the two-step debiased estimator $\widetilde{C}''_n(\varepsilon,\zeta_2,\zeta_1)$ provided $Y\in(0,1)\cup(1,8/5)$.

Theorem 1. *Suppose that $Y\in(0,1)\cup(1,8/5)$, and $h_n^{\frac{4}{8+Y}}\ll\varepsilon_n\ll h_n^{\frac{1}{4-Y}\vee\frac{2}{4+Y}}$. Then, for any fixed $\zeta_1,\zeta_2>1$, as $n\to\infty$,*

$$\sqrt{n}\left(\widetilde{C}''_n(\varepsilon,\zeta_2,\zeta_1)-\int_0^1\sigma_s^2ds\right)\xrightarrow{st}\mathcal{N}\left(0,2\int_0^1\sigma_s^4ds\right).$$

It is customary to use power thresholds of the form $\varepsilon_n=c_0h_n^{\omega}$, where $c_0>0$ and $\omega>0$ are some constants.[5] In that case, the assumption $h_n^{\frac{4}{8+Y}}\ll\varepsilon_n\ll h_n^{\frac{1}{4-Y}\vee\frac{2}{4+Y}}$ in Theorem 1 becomes

$$\left(\frac{1}{4-Y}\vee\frac{2}{4+Y}\right)<\omega<\frac{4}{8+Y}. \tag{17}$$

Remark 3. Note that, if $Y>1$ – the case where debiasing is strictly necessary for estimation efficiency – the value of $\omega=5/12$ satisfies the above constraint (17) for any value Y of the possible range $4/5<Y<8/5$ (on relaxations of these constraints, see Remark 5).

Remark 4. The case $Y=1$ is excluded from the above statements since part of our arguments rely on moment estimates for the truncated increments of Y-stable Lévy processes, whose characteristic function differs slightly when $Y=1$, though this case can be handled similarly with minor adjustments to our arguments.

Remark 5. As a consequence of the proof of Theorem 1, it follows that if $1<Y<4/3$, then only one debiasing step is needed to achieve efficiency. That is, for $1<Y<4/3$, we already have

$$\sqrt{n}\left(\widetilde{C}'_n(\varepsilon,\zeta_1)-\int_0^1\sigma_s^2ds\right)\xrightarrow{st}\mathcal{N}\left(0,2\int_0^1\sigma_s^4ds\right),$$

whenever $h_n^{\frac{1}{2Y}}\ll\varepsilon_n\ll h_n^{\frac{2}{4+Y}}$. If $4/3\le Y<8/5$, the proof of Theorem 1 shows a second debiasing step is required. These two facts suggest that further debiasing steps similar to (15)–(16) could be used to handle values of Y larger than $8/5$, or more broadly, less restrictive conditions on the jump measure of X at zero. This conjecture requires significant further analysis beyond the scope of the present paper and, hence, we leave it for future research.

4. Monte Carlo performance with CGMY jumps

In this section, we study the performance of the two-step debiasing procedure of the previous section in two settings: the case of a Lévy process with a CGMY jump component J (cf. [16]) and a Heston stochastic volatility model with again a CGMY jump component. Following [12], we also consider variants of our debiasing procedure that make use of the sign of the bias terms that lead to further improved finite-sample performance.

4.1. Constant volatility

We start by considering simulated data from the model (1) and (6), where the coefficient processes σ, b, and χ are constants, $\delta_0=\delta_1\equiv0$ (no bounded variation components), and $\{J_t\}_{t\ge0}$ is a CGMY process, independent of the Brownian motion $\{W_t\}_{t\ge0}$, with Lévy measure having a q-function, in the notation of (7), of the form:

$$q(x)=e^{-Mx}\mathbf{1}_{(0,\infty)}(x)+e^{Gx}\mathbf{1}_{(-\infty,0)}(x),$$

[5] Though, it is shown in [22] that in the case of a Lévy process X with jump component J as in Section 2, the MSE-optimal threshold ε_n^* is such that $\varepsilon_n^*\sim\sqrt{(2-Y)\sigma^2h_n\ln(1/h_n)}$, as $n\to\infty$.

and $C_+ = C_- = C$. Thus, the conditions of Assumption 1 are satisfied with $\alpha_+ = -M$ and $\alpha_- = G$. For simplicity we take $b = 0$ and $\chi = 1$, and adopt the parameter setting

$$C = 0.028, \quad G = 2.318, \quad M = 4.025, \tag{18}$$

which are similar to those used in [20],[6] and take $\sigma = 0.2$, 0.4, and $Y = 1.25$, 1.35, 1.5, 1.7, respectively. We fix $T = 1$ year and $n = 252(6.5)(60)$, which corresponds to a frequency of 1 minute (assuming 252 trading days and a 6.5-hour trading period each day).

In a fashion similar to [12], for the threshold $\varepsilon = c_0 h^{\omega}$, we take $c_0 = \sigma_{BV}$, where

$$\sigma_{BV}^2 := \frac{\pi}{2}\sum_{i=2}^{n} |\Delta_{i-1}^n X||\Delta_i^n X|,$$

which is the standard bipower variation estimator of σ^2 first introduced by [24]. For the value of ω we take $\omega = \frac{5}{12}$, which, as mentioned above, satisfies the condition (17) for any $Y \in (1, 8/5)$. We compare the performance of the following estimators:

1. TRQV: $\hat{C}_n(\varepsilon) = \sum_{i=1}^n \left(\Delta_i^n X\right)^2 \mathbf{1}_{\{|\Delta_i^n X| \le \varepsilon\}}$;
2. 1-step debiasing estimator removing positive bias:
$$\begin{aligned} &\widetilde{C}'_{n,pb}(\varepsilon,\zeta_1,p_1) = \hat{C}_n(\varepsilon) - \eta_1\left(\hat{C}_n(\zeta_1\varepsilon) - \hat{C}_n(\varepsilon)\right), \\ &\text{where} \quad \eta_1 = \frac{\hat{C}_n(p_1\,\zeta_1\varepsilon) - \hat{C}_n(p_1\,\varepsilon)}{\hat{C}_n(p_1\,\zeta_1^2\varepsilon) - 2\hat{C}_n(p_1\,\zeta_1\varepsilon) + \hat{C}_n(p_1\,\varepsilon)} \vee 0, \end{aligned} \tag{19}$$
with $\zeta_1 = 1.45$ and $p_1 = 0.6$, which were selected to achieve favorable performance across all considered values of Y and σ. If $\widetilde{C}'_{n,pb}(\varepsilon,\zeta_1,p_1)$ is negative, we recompute η_1 with $\varepsilon = 2\varepsilon/3$. This method is inspired by [12] and is motivated by the following decomposition of the bias correction term of (15) into a product of two factors:
$$\frac{\left(\hat{C}_n(\zeta_1\varepsilon) - \hat{C}_n(\varepsilon)\right)}{\hat{C}_n(\zeta_1^2\varepsilon) - 2\hat{C}_n(\zeta_1\varepsilon) + \hat{C}_n(\varepsilon)} \times \left(\hat{C}_n(\zeta_1\varepsilon) - \hat{C}_n(\varepsilon)\right),$$
where, due to (14), the first factor estimates $(\zeta_1^{2-Y} - 1)^{-1}$, which is positive. So, we should expect $\eta_1 > 0$.
3. With $\widetilde{C}'_{n,pb}(\varepsilon) := \widetilde{C}'_{n,pb}(\varepsilon,\zeta_1,p_1)$ defined as in Step 1, the 2-step debiasing estimator removing negative bias is given by:
$$\begin{aligned} &\widetilde{C}''_{n,nb}(\varepsilon,\zeta_2,\zeta_1,p_2,p_1) = \widetilde{C}'_{n,pb}(\varepsilon) - \eta_2\left(\left(\widetilde{C}'_{n,pb}(\zeta_2\varepsilon) - \widetilde{C}'_{n,pb}(\varepsilon)\right) \vee 0\right), \\ &\text{where} \quad \eta_2 = \frac{\widetilde{C}'_{n,pb}(p_2\,\zeta_2\varepsilon,) - \widetilde{C}'_{n,pb}(p_2\,\varepsilon)}{\widetilde{C}'_{n,pb}(p_2\,\zeta_2^2\varepsilon) - 2\widetilde{C}'_{n,pb}(p_2\,\zeta_2\varepsilon) + \widetilde{C}'_{n,pb}(p_2\,\varepsilon)} \wedge 0, \end{aligned} \tag{20}$$
with $\zeta_1 = 1.2$, $\zeta_2 = 1.2$, $p_1 = 0.65$, and $p_2 = 0.75$. If it turns out that $\widetilde{C}''_{n,nb}(\varepsilon,\zeta_2,\zeta_1,p_2,p_1)$ is negative, we recompute η_2 with $\varepsilon = 2\varepsilon/3$. The reason for this adjustment is the fact that η_2 is expected to be negative since it serves as estimate of $(\zeta_2^{-Y} - 1)^{-1}$. The values of the tuning parameters $\zeta_1, \zeta_2, p_1, p_2$ were selected for 'overall favorable' performance for all considered values of Y and σ.

We further compare the simulated performance of the above estimators to the estimators proposed in [12] and [14]. Specifically, we use the Eq. (5.3) in the paper [12]:

$$\begin{aligned} &\hat{C}_{\mathrm{JT},53}(u_n,\zeta) = \hat{C}_{\mathrm{JT}}(u_n) - \eta\left(\left(\hat{C}_{\mathrm{JT}}(\zeta u_n) - \hat{C}_{\mathrm{JT}}(u_n)\right) \wedge 0\right), \\ &\text{with} \quad \eta = \frac{\hat{C}_{\mathrm{JT}}(p_0\,\zeta u_n) - \hat{C}_{\mathrm{JT}}(p_0\,u_n)}{\hat{C}_{\mathrm{JT}}(p_0\,\zeta^2 u_n) - 2\hat{C}_{\mathrm{JT}}(p_0\,\zeta u_n) + \hat{C}_{\mathrm{JT}}(p_0\,u_n)} \wedge 0, \end{aligned}$$

where $\hat{C}_{\mathrm{JT}}$ denotes their "nonsymmetrized" two-step debiased estimator (see Eq. (3.1) in therein). For the tuning parameters, we take

$$\zeta = 1.5, \quad u_n = (\ln(1/h_n))^{-1/30}/\sigma_{BV}, \quad p_0 = 0.2,$$

where the values of ζ and u_n were those suggested by [12] and the value of p_0 was chosen for favorable estimation performance. Note that, since a Lévy model has constant volatility, it is not necessary to localize the estimator and, hence, we treat the 1-year data as one block, which corresponds to taking $k_n = 252(6.5)(60)$ in the notation of [12]. For the moment estimator proposed in [14], we use the same moment functions and the parameter settings as suggested by [14]. We denote this estimator $\hat{C}_{\mathrm{M},4}$. We also examine the performance of another moment estimator, denoted $\hat{C}_{\mathrm{M},3}$, that is computed under a similar algorithm to [14] but with 3 different moment functions suggested in [22]. We remark that the moment functions used in the construction of $\hat{C}_{\mathrm{M},3}$ do not satisfy the strict

[6] Figueroa-López and Ólafsson [20] considers the asymmetric case $\nu(dx) = C_{\mathrm{sgn}(x)}\bar{q}(x)|x|^{-1-Y}\,dx$ with $C_+ = 0.015$ and $C_- = 0.041$. Here, we take $C = (C_+ + C_-)/2$ in order to simplify the simulation of the model. The parameter values of C_+, C_-, G, and M used in [20] were taken from [23], who calibrated the tempered stable model using market option prices.

Table 2
Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 0.8$ and $\sigma = 0.2$. The best tuning parameters are $\zeta_1^* = 1.4$, $\zeta_2^* = 1.35$, $p_1^* = 0.5$, $p_2^* = 0.85$; $\zeta^* = 1.2$, $p_0^* = 0.2$.

$\sigma = 0.2,\quad Y = 0.8$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.036881	0.0002	−0.0780	0.0050	9.77E−06	3.12E−03
$\widetilde{C}''^{*}_{n,nb}$	0.039902	0.0002	−0.0024	0.0047	**4.50E−08**	1.44E−04
$\widehat{C}^*_{\mathrm{JT},53}$	0.039997	0.0003	−0.0001	0.0070	7.80E−08	**1.25E−04**
$\widehat{C}_{\mathrm{M},3}$	0.038777	0.0002	−0.0306	0.0055	1.55E−06	1.23E−03
$\widehat{C}_{\mathrm{M},4}$	0.040530	0.0003	0.0132	0.0077	3.75E−07	5.08E−04
$\widetilde{C}'_{n,pb}$	0.036881	0.0002	−0.0780	0.0050	9.77E−06	3.12E−03
$\widetilde{C}''_{n,nb}$	0.038108	0.0002	−0.0473	0.0048	3.62E−06	1.90E−03
$\widehat{C}_{\mathrm{JT},53}$	0.039916	0.0010	−0.0021	0.0241	9.40E−07	1.39E−04

Table 3
Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.25$ and $\sigma = 0.2$. The best tuning parameters are $\zeta_1^* = 1.35$, $\zeta_2^* = 1.2$, $p_1^* = 0.5$, $p_2^* = 0.85$; $\zeta^* = 1.5$, $p_0^* = 0.1$.

$\sigma = 0.2,\quad Y = 1.25$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.037544	0.000205	−0.0614	0.0051	6.07E−06	2.45E−03
$\widetilde{C}''^{*}_{n,nb}$	0.039987	0.000216	−0.0003	0.0054	**4.70E−08**	**1.40E−04**
$\widehat{C}^*_{\mathrm{JT},53}$	0.040156	0.000300	0.0039	0.0075	1.14E−07	2.25E−04
$\widehat{C}_{\mathrm{M},3}$	0.039661	0.000268	−0.0085	0.0067	1.87E−07	3.28E−04
$\widehat{C}_{\mathrm{M},4}$	0.046369	0.000939	0.1592	0.0235	4.14E−05	6.37E−03
$\widetilde{C}'_{n,pb}$	0.037544	0.000205	−0.0614	0.0051	6.07E−06	2.45E−03
$\widetilde{C}''_{n,nb}$	0.040323	0.000203	0.0081	0.0051	1.45E−07	3.15E−04
$\widehat{C}_{\mathrm{JT},53}$	0.040742	0.000257	0.0185	0.0064	6.16E−07	7.60E−04

constraints imposed in [14], and therefore the asymptotic efficiency of the estimator $\widehat{C}_{\mathrm{M},3}$ has not been established. We refer to [22] for more details about the computations of $\widehat{C}_{\mathrm{M},3}$ and $\widehat{C}_{\mathrm{M},4}$.

The sample means, standard deviations (SDs), the average and SD of relative errors, the mean squared errors (MSEs), and median of absolute deviations (MADs) for each of the estimators described above are reported in Tables 2–11. In addition, we show the results corresponding to 'case-by-case favorably-tuned' versions of $\widetilde{C}''_{n,pb}$, and $\widehat{C}_{\mathrm{JT},53}$; i.e., their tuning parameters were chosen for achieving the 'best' performance for each pair (Y,σ) based on a grid search; we distinguish these estimators and their corresponding tuning parameters by the superscript *. That is, $\widetilde{C}''_{n,pb}$ is based on the choices $(\zeta_1,\zeta_2,p_1,p_2) = (1.2, 1.2, .65, .75)$ across all considered values of Y and σ. With these values of the tuning parameters, $\widetilde{C}''_{n,pb}$ exhibits generally good performance overall. However, for each given fixed pair (Y,σ), its counterpart $\widetilde{C}''^{*}_{n,pb}$ is tuned to have superior performance for those particular values of Y and σ. For instance, as shown in Table 3, when $Y = 1.25$ and $\sigma = 0.2$, the estimator $\widetilde{C}''_{n,pb}$ attains an MSE of 1.45×10^{-7}, whereas the choice of parameters $(\zeta_1^*,\zeta_2^*,p_1^*,p_2^*) = (1.35, 1.20, 0.5, 0.85)$ leads to an MSE of 4.70×10^{-8} for $\widetilde{C}''^{*}_{n,pb}$.

We provide a broad summary of our simulation results. For the estimators using the case-by-case tuned parameters $(\zeta_1^*,\zeta_2^*,p_1^*,p_2^*)$, based on both MSE and MAD, $\widetilde{C}''^{*}_{n,nb}$ outperforms every other estimator considered in each setting, except when $Y = 0.8$ when the performance between $\widetilde{C}''^{*}_{n,nb}$ and $\widehat{C}^*_{\mathrm{JT},53}$ is comparable (c.f. Table 2, top row) but $\widehat{C}^*_{\mathrm{JT},53}$ has a slight edge in both MAD and MSE when $\sigma = 0.4$ and a slight edge in MAD when $\sigma = 0.2$. Using the parameters $(\zeta_1,\zeta_2,p_1,p_2) = (1.2, 1.2, .65, .75)$, compared to the method of [14], $\widetilde{C}''_{n,nb}$ outperforms $\widehat{C}_{\mathrm{M},4}$ in all cases, as measured by MSE and MAD in Tables 3–11. When $\sigma = 0.2$ and $Y = 1.25$, or when $\sigma = 0.4$ and $Y = 1.25, 1.35, 1.5, 1.7$, $\widetilde{C}''_{n,nb}$ outperforms $\widehat{C}_{\mathrm{M},3}$ (though, as noted earlier, computation of $\widetilde{C}''_{n,nb}$ is much faster and more numerically stable than that of either $\widehat{C}_{\mathrm{M},3}$ or $\widehat{C}_{\mathrm{M},4}$). Next, compared with [12], when $\sigma = 0.2$ and $Y = 1.25, 1.35, 1.5$, or when $\sigma = 0.4$ and $Y = 1.25$ or 1.35, $\widetilde{C}''_{n,nb}$ has superior performance compared to $\widehat{C}_{\mathrm{JT},53}$ as measured by MSE and MAD. Generally, $\widetilde{C}''_{n,nb}$ significantly outperforms $\widetilde{C}'_{n,pb}$. Table 5 also shows that, though when $\sigma = 0.2$ and $Y = 1.5$, $\widetilde{C}''_{n,nb}$ has larger MSE and MAD than $\widehat{C}_n$ and $\widetilde{C}'_{n,pb}$, it still performs better than $\widehat{C}_{\mathrm{JT},53}$: the MSE and MAD of $\widetilde{C}''_{n,nb}$ in these cases are approximately 22% and 85% of those of $\widehat{C}_{\mathrm{JT},53}$, respectively. Tables 6 and 11 show that, when $Y = 1.7$, which is not covered by our theoretical framework (see Remark 5), $\widetilde{C}''_{n,nb}$ has slightly larger MSE and MAD than $\widehat{C}_{\mathrm{JT},53}$. Overall, we conclude that our debiasing procedure outperforms $\widehat{C}_{\mathrm{JT},53}$ when $Y = 1.25, 1.35, 1.5$, and outperforms $\widehat{C}_{\mathrm{M},4}$ in all parameter settings considered, and the estimation performance of $\widehat{C}_{\mathrm{JT},53}$ and $\widetilde{C}''_{n,nb}$ is comparable when $Y = 0.8$.

We also study the asymptotic approximation for the sampling distribution of $\widetilde{C}''_{n,nb}$ and $\widehat{C}_{\mathrm{JT},53}$ based on the case-by-case optimally-tuned estimators $\widetilde{C}''^{*}_{n,nb}$ and $\widehat{C}^*_{\mathrm{JT},53}$. Fig. 1 shows their normalized simulated sampling distributions, $\sqrt{n}(\hat{\sigma}-\sigma^2)/\sqrt{2\sigma^4}$, and the theoretical asymptotic normal distribution, $\mathcal{N}(0,1)$. Compared to $\widehat{C}^*_{\mathrm{JT},53}$, the simulated distribution of $\widetilde{C}''^{*}_{n,nb}$ yields a better match with the asymptotic normal distribution, especially for $Y \le 1.5$. Note that, in general, the values of $\widetilde{C}''^{*}_{n,nb}$ are much less spread out

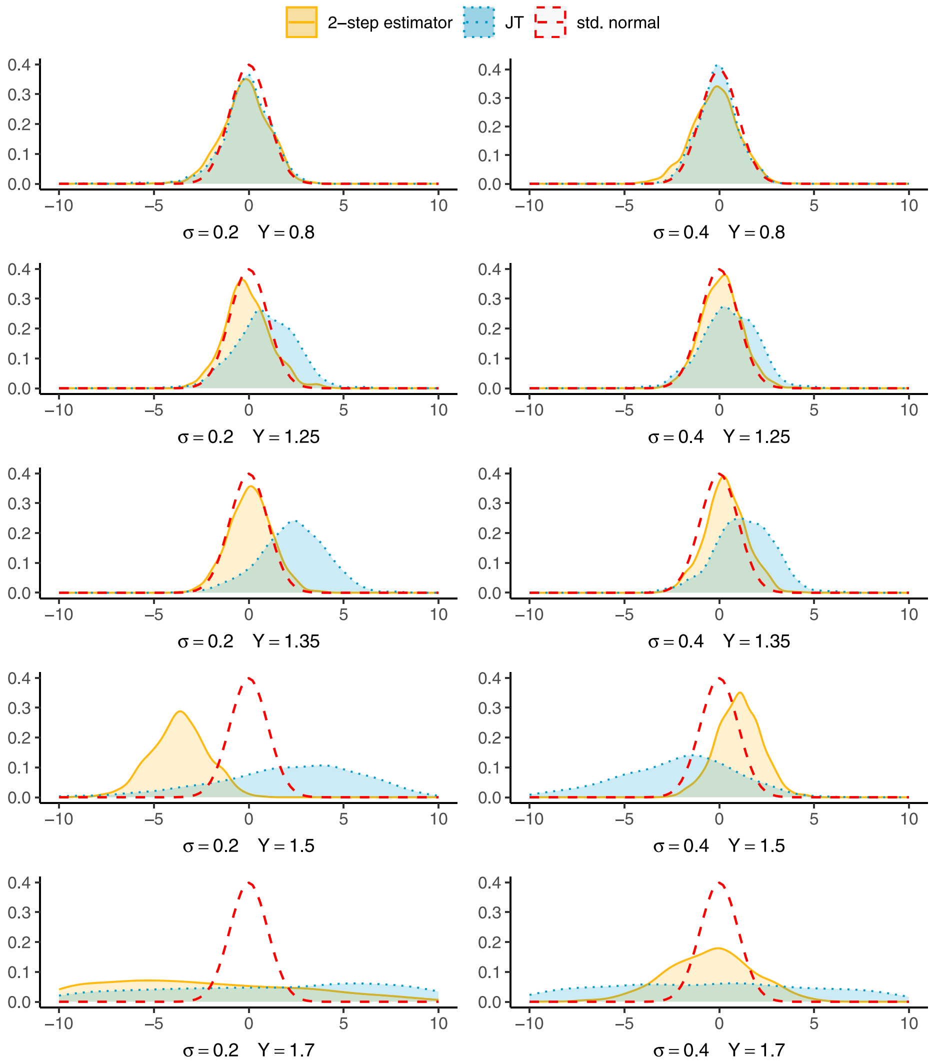

Fig. 1. Simulated and theoretical asymptotic distributions of the estimators based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The bold dashed red curve is the theoretical asymptotic distribution, $\mathcal{N}(0,1)$. The solid yellow curve is the simulated distribution of $(2\sigma^4)^{-1/2}\sqrt{n}(\widetilde{C}''^{*}_{n,nb} - \sigma^2)$. The dotted blue curve is the normalized simulated distribution of $(2\sigma^4)^{-1/2}\sqrt{n}(\widehat{C}^{*}_{\mathrm{JT}} - \sigma^2)$.

Table 4
Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.35$ and $\sigma = 0.2$. The best tuning parameters are $\zeta_1^* = 1.35$, $\zeta_2^* = 1.1$, $p_1^* = 0.6$, $p_2^* = 0.75$; $\zeta^* = 1.5$, $p_0^* = 0.1$.

	$\sigma = 0.2$, $Y = 1.35$					
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.038264	0.000203	−0.0434	0.0051	3.05E−06	1.73E−03
$\widetilde{C}''^{*}_{n,nb}$	0.040010	0.000206	0.0003	0.0052	**4.27E−08**	**1.33E−04**
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.040418	0.000316	0.0104	0.0079	2.74E−07	4.30E−04
$\widehat{C}_{\mathrm{M},3}$	0.040461	0.000277	0.0115	0.0069	2.89E−07	4.70E−04
$\widehat{C}_{\mathrm{M},4}$	0.050576	0.001121	0.2644	0.0280	1.13E−04	1.06E−02
$\widetilde{C}'_{n,pb}$	0.038264	0.000203	−0.0434	0.0051	3.05E−06	1.73E−03
$\widetilde{C}''_{n,nb}$	0.041044	0.000211	0.0261	0.0053	1.13E−06	1.05E−03
$\widehat{C}_{\mathrm{JT},53}$	0.041547	0.001111	0.0387	0.0278	3.63E−06	1.59E−03

than those of $\widehat{C}^{*}_{\mathrm{JT},53}$. Though the simulated distributions of $\widehat{C}_{\mathrm{M},3}$ and $\widehat{C}_{\mathrm{M},4}$ are not shown in Fig. 1, we can see that $\widetilde{C}''^{*}_{n,nb}$ performs much better than $\widehat{C}_{\mathrm{M},3}$ and $\widehat{C}_{\mathrm{M},4}$ as seen in Tables 3–11.

Table 5

Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.5$ and $\sigma = 0.2$. The best tuning parameters are $\zeta_1^* = 1.45$, $\zeta_2^* = 1.3$, $p_1^* = 0.1$, $p_2^* = 0.2$; $\zeta^* = 1.5$, $p_0^* = 0.1$.

$\sigma = 0.2$, $Y = 1.5$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.041326	0.000223	0.0332	0.0056	1.81E−06	1.33E−03
$\widetilde{C}''^{*}_{n,nb}$	0.039333	0.000259	−0.0167	0.0065	**5.12E−07**	6.58E−04
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.040371	0.000740	0.0093	0.0185	6.85E−07	**6.20E−04**
$\widehat{C}_{\mathrm{M},3}$	0.044022	0.000311	0.1006	0.0078	1.63E−05	4.02E−03
$\widehat{C}_{\mathrm{M},4}$	0.063113	0.001598	0.5778	0.0400	5.37E−04	2.30E−02
$\widetilde{C}'_{n,pb}$	0.041326	0.000223	0.0332	0.0056	1.81E−06	1.33E−03
$\widetilde{C}''_{n,nb}$	0.044331	0.000232	0.1083	0.0058	1.88E−05	4.33E−03
$\widehat{C}_{\mathrm{JT},53}$	0.040092	0.009267	0.0023	0.2317	8.59E−05	5.12E−03

Table 6

Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.7$ and $\sigma = 0.2$. The best tuning parameters are $\zeta_1^* = 1.3$, $\zeta_2^* = 1.3$, $p_1^* = 0.4$, $p_2^* = 0.8$; $\zeta^* = 2$, $p_0^* = 0.4$. The parameters for $\widetilde{C}'_{n,pb}$ and $\widetilde{C}''_{n,nb}$ are $\zeta_1 = 1.2$, $\zeta_2 = 1.2$, $p_1 = 0.6$, and $p_2 = 0.85$.

$\sigma = 0.2$, $Y = 1.7$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.063772	0.000359	0.5943	0.0090	5.65E−04	2.38E−02
$\widetilde{C}''^{*}_{n,nb}$	0.038238	0.001842	−0.0441	0.0461	**6.50E−06**	**1.91E−03**
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.039833	0.006706	−0.0042	0.1677	4.50E−05	4.38E−03
$\widehat{C}_{\mathrm{M},3}$	0.069247	0.000469	0.7312	0.0117	8.56E−04	2.93E−02
$\widehat{C}_{\mathrm{M},4}$	0.110533	0.002272	1.7633	0.0568	4.98E−03	7.05E−02
$\widetilde{C}'_{n,pb}$	0.063772	0.000359	0.5943	0.0090	5.65E−04	2.38E−02
$\widetilde{C}''_{n,nb}$	0.069421	0.001187	0.7355	0.0297	8.67E−04	2.93E−02
$\widehat{C}_{\mathrm{JT},53}$	0.066487	0.000488	0.6622	0.0122	7.02E−04	2.65E−02

Table 7

Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.25$ and $\sigma = 0.2$. The best tuning parameters are $\zeta_1^* = 1.4$, $\zeta_2^* = 1.35$, $p_1^* = 0.5$, $p_2^* = 0.85$; $\zeta^* = 1.5$, $p_0^* = 0.1$.

$\sigma = 0.4$, $Y = 0.8$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.147467	0.0008	−0.0783	0.0049	1.58E−04	1.26E−02
$\widetilde{C}''^{*}_{n,nb}$	0.159698	0.0008	−0.0019	0.0047	6.66E−07	5.17E−04
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.159963	0.0007	−0.0002	0.0047	**5.61E−07**	**4.75E−04**
$\widehat{C}_{\mathrm{M},3}$	0.160908	0.0008	0.0057	0.0051	1.49E−06	9.39E−04
$\widehat{C}_{\mathrm{M},4}$	0.160531	0.0008	0.0033	0.0049	9.00E−07	6.34E−04
$\widetilde{C}'_{n,pb}$	0.147467	0.0008	−0.0783	0.0049	1.58E−04	1.26E−02
$\widetilde{C}''_{n,nb}$	0.152373	0.0007	−0.0477	0.0047	5.87E−05	7.61E−03
$\widehat{C}_{\mathrm{JT},53}$	0.159312	0.0038	−0.0043	0.0235	1.47E−05	5.73E−04

Table 8

Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.25$ and $\sigma = 0.4$. The best tuning parameters are $\zeta_1^* = 1.35$, $\zeta_2^* = 1.3$, $p_1^* = 0.5$, $p_2^* = 0.85$; $\zeta^* = 1.5$, $p_0^* = 0.2$.

$\sigma = 0.4$, $Y = 1.25$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.148523	0.000775	−0.0717	0.0048	1.32E−04	1.14E−02
$\widetilde{C}''^{*}_{n,nb}$	0.160102	0.000754	0.0006	0.0047	**5.79E−07**	**5.24E−04**
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.160376	0.001062	0.0023	0.0066	1.27E−06	7.70E−04
$\widehat{C}_{\mathrm{M},3}$	0.162777	0.000825	0.0174	0.0052	8.39E−06	2.78E−03
$\widehat{C}_{\mathrm{M},4}$	0.166381	0.001159	0.0399	0.0072	4.21E−05	6.32E−03
$\widetilde{C}'_{n,pb}$	0.148523	0.000775	−0.0717	0.0048	1.32E−04	1.14E−02
$\widetilde{C}''_{n,nb}$	0.159879	0.000785	−0.0008	0.0049	6.31E−07	5.34E−04
$\widehat{C}_{\mathrm{JT},53}$	0.160376	0.001062	0.0023	0.0066	1.27E−06	7.70E−04

4.2. Stochastic volatility

In this section, we apply our two-step debiasing procedure to estimate the daily integrated variance under a stochastic volatility model with a CGMY jump component and compare it with the estimator of Jacod and Todorov [12].

Table 9
Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.35$ and $\sigma = 0.4$. The best tuning parameters are $\zeta_1^* = 1.2$, $\zeta_2^* = 1.2$, $p_1^* = 0.6$, $p_2^* = 0.85$; $\zeta^* = 1.5$, $p_0^* = 0.2$.

$\sigma = 0.4$, $Y = 1.35$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.149582	0.000788	−0.0651	0.0049	1.09E−04	1.04E−02
$\widetilde{C}''^{*}_{n,nb}$	0.160284	0.000757	0.0018	0.0047	**6.53E−07**	**5.65E−04**
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.160950	0.001125	0.0059	0.0070	2.17E−06	1.09E−03
$\widehat{C}_{\mathrm{M},3}$	0.164366	0.000860	0.0273	0.0054	1.98E−05	4.38E−03
$\widehat{C}_{\mathrm{M},4}$	0.170625	0.001292	0.0664	0.0081	1.15E−04	1.06E−02
$\widetilde{C}'_{n,pb}$	0.149582	0.000788	−0.0651	0.0049	1.09E−04	1.04E−02
$\widetilde{C}''_{n,nb}$	0.160937	0.000758	0.0059	0.0047	1.45E−06	9.54E−04
$\widehat{C}_{\mathrm{JT},53}$	0.160950	0.001125	0.0059	0.0070	2.17E−06	1.09E−03

Table 10
Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.5$ and $\sigma = 0.4$. The best tuning parameters are $\zeta_1^* = 1.35$, $\zeta_2^* = 1.1$, $p_1^* = 0.5$, $p_2^* = 0.9$; $\zeta^* = 1.4$, $p_0^* = 0.2$.

$\sigma = 0.4$, $Y = 1.5$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.153623	0.000811	−0.0399	0.0051	4.13E−05	6.39E−03
$\widetilde{C}''^{*}_{n,nb}$	0.160721	0.000826	0.0045	0.0052	**1.20E−06**	**8.13E−04**
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.158146	0.002499	−0.0116	0.0156	9.68E−06	1.89E−03
$\widehat{C}_{\mathrm{M},3}$	0.168253	0.000981	0.0516	0.0061	6.91E−05	8.24E−03
$\widehat{C}_{\mathrm{M},4}$	0.183132	0.001785	0.1446	0.0112	5.38E−04	2.31E−02
$\widetilde{C}'_{n,pb}$	0.153623	0.000811	−0.0399	0.0051	4.13E−05	6.39E−03
$\widetilde{C}''_{n,nb}$	0.165112	0.000800	0.0319	0.0050	2.68E−05	5.13E−03
$\widehat{C}_{\mathrm{JT},53}$	0.163082	0.001305	0.0193	0.0082	1.12E−05	3.08E−03

Table 11
Estimation based on simulated 1-minute observations of 1000 paths over a 1 year time horizon. The parameters are $Y = 1.7$ and $\sigma = 0.4$. The best tuning parameters are $\zeta_1^* = 1.2$, $\zeta_2^* = 1.35$, $p_1^* = 0.2$, $p_2^* = 0.3$; $\zeta^* = 1.5$, $p_0^* = 0.2$.

$\sigma = 0.4$, $Y = 1.7$						
	Sample Mean	Sample SD	Mean of RE	SD of RE	MSE	MAD
$\widehat{C}_n$	0.178820	0.000946	0.1176	0.0059	3.55E−04	1.88E−02
$\widetilde{C}''^{*}_{n,nb}$	0.159715	0.001594	−0.0018	0.0100	**2.62E−06**	**1.09E−03**
$\widehat{C}^{*}_{\mathrm{JT},53}$	0.151628	0.013172	−0.0523	0.0823	2.44E−04	6.87E−03
$\widehat{C}_{\mathrm{M},3}$	0.192848	0.001222	0.2053	0.0076	1.08E−03	3.29E−02
$\widehat{C}_{\mathrm{M},4}$	0.230606	0.002466	0.4413	0.0154	4.99E−03	7.05E−02
$\widetilde{C}'_{n,pb}$	0.178820	0.000946	0.1176	0.0059	3.55E−04	1.88E−02
$\widetilde{C}''_{n,nb}$	0.192674	0.000972	0.2042	0.0061	1.07E−03	3.27E−02
$\widehat{C}_{\mathrm{JT},53}$	0.151628	0.013172	−0.0523	0.0823	2.44E−04	6.87E−03

Specifically, we consider the following Heston model:

$$X_t = 1 + \int_0^t \sqrt{V_s}\, dW_s + J_t, \quad V_t = \theta + \int_0^t \kappa\big(\theta - V_s\big)\, ds + \xi \int_0^t \sqrt{V_s}\, dB_s,$$

where $\{W_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are two correlated standard Brownian motions with correlation ρ and $\{J_t\}_{t\geq 0}$ is a CGMY Lévy process independent of $\{W_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$. The parameters are set as

$$\kappa = 5, \quad \xi = 0.5, \quad \theta = 0.16, \quad \rho = -0.5.$$

The values of κ, ξ, and ρ above are borrowed from [25]. The CGMY parameters are the same as those in the previous section.

We consider 1-min observations over a one-year (252 days) time horizon with 6.5 trading hours per day. We break each path into 252 blocks (one for each day) and estimate the integrated volatility $IV = \int_t^{t+1/252} V_s ds$ for each day ($t = 0, 1/252, \ldots, 251/252$). As suggested and used in [12], to improve the stability of the estimates, the estimated bias terms in (19) and (20) are split into two components each: $\big(\widehat{C}_n(\zeta_1\varepsilon) - \widehat{C}_n(\varepsilon)\big)$ and $\big(\widetilde{C}'_{n,pb}(\zeta_2\varepsilon, \zeta_1, p_1) - \widetilde{C}'_{n,pb}(\varepsilon, \zeta_1, p_1)\big)$. These are computed using the data in each day, and the factors η_1, η_2, which only depend on Y, are computed using the data during the whole time horizon. In practice, one would precompute η_1 and η_2 using historical data over 1 year and use those values to compute the daily integrated volatility afterward. The precise formulas for our estimators are described below:

Table 12
The MSE and MADs for $\overline{C}''_{n,nb}$ and $\overline{C}_{\text{JT},53}$. The results are based on simulated 1-minute observations of 1000 paths over a one-year time horizon with $Y = 0.8, 1.25, 1.35, 1.5, 1.7$. The parameters for debiasing method are $\zeta_1 = 1.2$, $\zeta_2 = 1.2$, $p_1 = 0.65$, and $p_2 = 0.75$ in all cases. The smallest average MSE and smallest average MAD is displayed in bold in each row.

Estimation performance in a Heston model								
Y	Method		Day 2	Day 63	Day 126	Day 189	Day 252	Average
0.8	$\overline{C}''_{n,nb}$	MSE	2.38E−09	2.77E−09	2.64E−09	2.64E−09	2.39E−09	**2.57E−09**
		MAD	2.86E−05	2.92E−05	2.90E−05	3.02E−05	2.82E−05	**2.93E−05**
	$\overline{C}_{\text{JT},53}$	MSE	7.89E−09	8.30E−09	8.09E−09	8.98E−09	8.17E−09	8.68E−09
		MAD	3.34E−05	3.51E−05	3.23E−05	3.38E−05	3.34E−05	3.40E−05
1.25	$\overline{C}''_{n,nb}$	MSE	2.50E−09	2.37E−09	2.48E−09	2.57E−09	2.45E−09	**2.45E−09**
		MAD	2.90E−05	2.93E−05	2.98E−05	2.91E−05	2.89E−05	**2.88E−05**
	$\overline{C}_{\text{JT},53}$	MSE	5.80E−09	5.49E−09	5.26E−09	5.21E−09	5.79E−09	5.52E−09
		MAD	3.49E−05	3.43E−05	3.45E−05	3.26E−05	3.44E−05	3.40E−05
1.35	$\overline{C}''_{n,nb}$	MSE	2.52E−09	2.40E−09	2.55E−09	2.55E−09	2.45E−09	**2.47E−09**
		MAD	2.83E−05	2.93E−05	2.97E−05	2.91E−05	2.81E−05	**2.90E−05**
	$\overline{C}_{\text{JT},53}$	MSE	7.63E−09	7.31E−09	6.85E−09	6.50E−09	7.55E−09	7.25E−09
		MAD	3.63E−05	3.55E−05	3.51E−05	3.29E−05	3.53E−05	3.46E−05
1.5	$\overline{C}''_{n,nb}$	MSE	3.05E−09	3.12E−09	3.10E−09	3.19E−09	3.01E−09	**3.07E−09**
		MAD	3.25E−05	3.48E−05	3.37E−05	3.30E−05	3.22E−05	**3.31E−05**
	$\overline{C}_{\text{JT},53}$	MSE	6.25E−09	6.27E−09	5.76E−09	5.62E−09	6.21E−09	5.98E−09
		MAD	3.65E−05	3.76E−05	3.85E−05	3.53E−05	3.60E−05	3.66E−05
1.7	$\overline{C}''_{n,nb}$	MSE	2.08E−08	2.14E−08	2.04E−08	2.13E−08	2.09E−08	2.07E−08
		MAD	1.23E−04	1.28E−04	1.26E−04	1.30E−04	1.24E−04	1.26E−04
	$\overline{C}_{\text{JT},53}$	MSE	1.89E−08	1.86E−08	1.82E−08	1.75E−08	1.88E−08	**1.82E−08**
		MAD	1.06E−04	1.07E−04	1.05E−04	1.04E−04	1.06E−04	**1.05E−04**

1. 1-step debiasing estimator removing positive bias:
$$\overline{C}'_{n,pb}(\varepsilon,\zeta_1)_t = \widehat{C}_n(\varepsilon)_t - \eta_1\left(\widehat{C}_n(\zeta_1\varepsilon)_t - \widehat{C}_n(\varepsilon)_t\right),$$
$$\eta_1 = \frac{\sum_{i=0}^{251}\left(\widehat{C}_n(p_1\,\zeta_1\varepsilon)_{\frac{i}{252}} - \widehat{C}_n(p_1\,\varepsilon)_{\frac{i}{252}}\right)}{\sum_{i=0}^{251}\left(\widehat{C}_n(p_1\,\zeta_1^2\varepsilon)_{\frac{i}{252}} - 2\widehat{C}_n(p_1\,\zeta_1\varepsilon)_{\frac{i}{252}} + \widehat{C}_n(p_1\,\varepsilon)_{\frac{i}{252}}\right)} \vee 0,$$
2. With the estimator $\overline{C}'_{n,pb}(\varepsilon)_t := \overline{C}'_{n,pb}(\varepsilon,\zeta_1)_t$ defined in Step 1 above, the 2-step debiasing estimator removing negative bias is given by:
$$\overline{C}''_{n,nb}(\varepsilon,\zeta_2,\zeta_1)_t = \overline{C}'_{n,pb}(\varepsilon)_t - \eta_2\left(\left(\overline{C}'_{n,pb}(\zeta_2\varepsilon)_t - \overline{C}'_{n,pb}(\varepsilon)_t\right)\vee 0\right),$$
$$\eta_2 = \frac{\sum_{i=0}^{251}\left(\overline{C}'_{n,pb}(p_2\,\zeta_2\varepsilon)_{\frac{i}{252}} - \overline{C}'_{n,pb}(p_2\,\varepsilon)_{\frac{i}{252}}\right)}{\sum_{i=0}^{251}\left(\overline{C}'_{n,pb}(p_2\,\zeta_2^2\varepsilon)_{\frac{i}{252}} - 2\overline{C}'_{n,pb}(p_2\,\zeta_2\varepsilon)_{\frac{i}{252}} + \overline{C}'_{n,pb}(p_2\,\varepsilon)_{\frac{i}{252}}\right)} \wedge 0,$$
with the same parameters used in the previous subsection $\zeta_1 = 1.2$, $\zeta_2 = 1.2$, $p_1 = 0.65$, and $p_2 = 0.75$.

For the estimator of [12], we use Eq. (5.3) therein with tuning parameter $k_n = 130$ (number of observation in each block), $\xi = 1.5$, and $u_n = (-\ln h_n)^{-1/30}/\sqrt{BV}$ (these values were suggested in [12]). Here, BV is the bipower variation of the previous day. The resulting estimator is denoted as $\overline{C}_{\text{JT},53}$. To assess the accuracy of the different methods, we compute the Median Absolute Deviation (MAD) around the true value, $IV_t = \int_t^{t+1/252} V_s ds$, and the MSE, i.e. the sample mean of $(\widehat{IV}_t - IV_t)^2$, for 5 arbitrarily chosen days over 1000 simulation paths.

The results are shown in Table 12. The last column of Table 12 shows the sample means of the MSE and MAD over the 252 days for the different estimators. When $Y = 0.8, 1.25, 1.35, 1.5$, all MSEs and MADs of $\overline{C}''_{n,nb}$ are smaller than those of $\overline{C}_{\text{JT},53}$, with the MSE of $\overline{C}''_{n,nb}$ typically half of that of $\overline{C}_{\text{JT},53}$ or smaller. For the case $Y = 1.7$ (outside the scope of our theoretical framework), the MSE and MAD of $\overline{C}''_{n,nb}$ is slightly larger than those of $\overline{C}_{\text{JT},53}$.

This behavior can also be observed in Fig. 2, which shows the true daily integrated volatility (dashed red line) for one fixed simulated path compared with the estimates corresponding to $\overline{C}''_{n,nb}$ (solid black line) and $\overline{C}_{\text{JT},53}$ in [12] (dotted blue line). From the figure, we conclude that for this specific stochastic volatility model, our debiasing method achieves significant improvement when $Y \le 1.5$. For $Y = 1.7$, both estimators $\overline{C}''_{n,nb}$ and $\overline{C}_{\text{JT},53}$ are very close and significantly overestimate the true daily integrated volatility for this simulated path. This changes from path to path, though $\overline{C}''_{n,nb}$ and $\overline{C}_{\text{JT},53}$ are typically close when $Y = 1.7$.

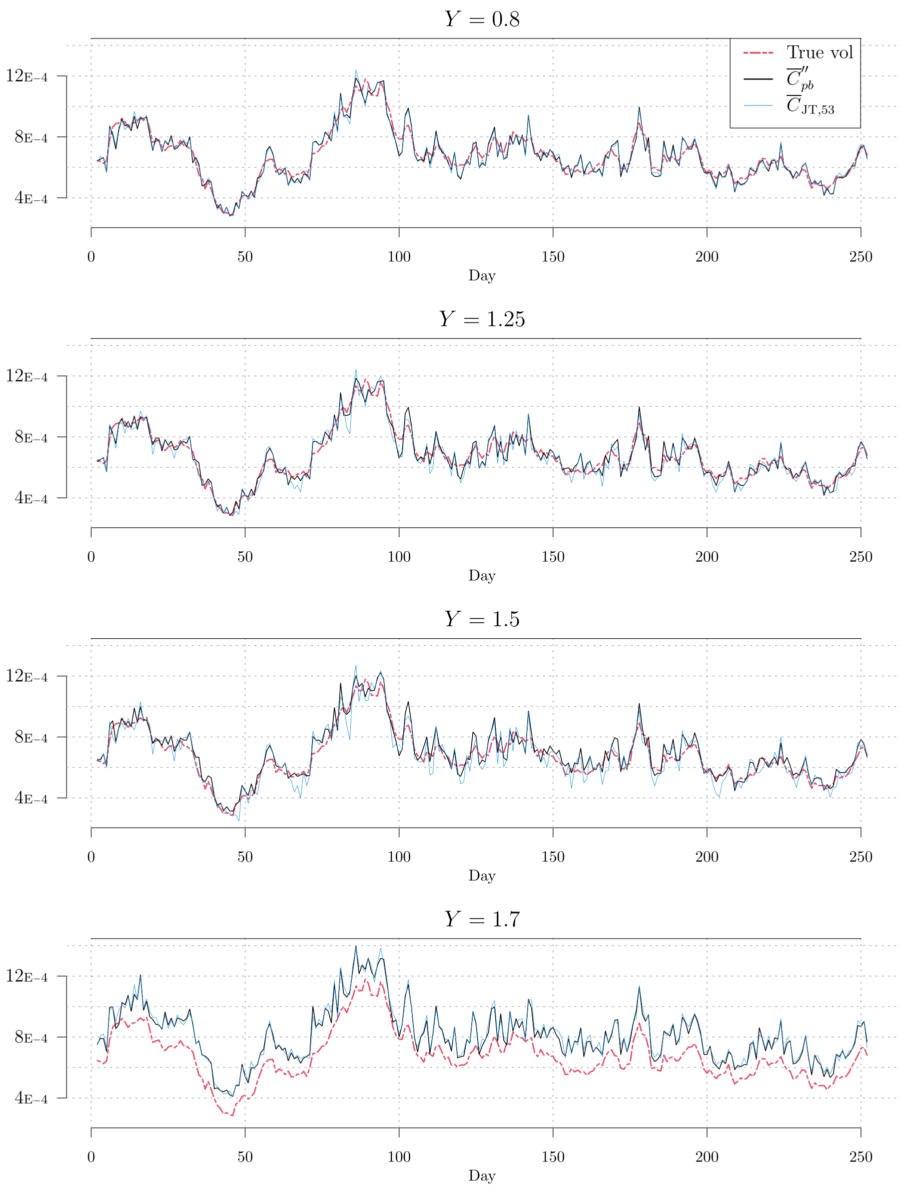

Fig. 2. Plots of daily integrated volatility estimates $\overline{C}''_{n,nb}(\varepsilon,\zeta_2,\zeta_1,p_2,p_1)_{(\frac{t}{252},\frac{t+1}{252}]}$. Except for a change in the jump index Y of the CGMY component, the same path of X_t is for each plot. Above are the plots for $Y = 0.8, 1.25, 1.5, 1.7$ ($Y = 1.35$ is similar to $Y = 1.25$ and was removed to save space). The dashed red line corresponds to the true daily integrated volatility, while the solid black (respectively, blue) line corresponds to the daily estimates using our debiasing estimator $\overline{C}''_{n,nb}$ (respectively, the estimator given in expression (5.3) in [12]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are truly grateful to the Editor, Associate Editor, and two anonymous referees for their numerous suggestions that helped to significantly improve the original manuscript.

Appendix A. Proofs of the main results

Throughout all appendices, we routinely make use of the following fact: under the assumption $h_n^{4/(8+Y)} \ll \varepsilon_n$ made throughout Section 3, it holds $\exp\left(-\frac{\varepsilon_n^2}{2\sigma^2 h_n}\right) \ll h_n^s$ for any $s > 0$. For notational simplicity, we also often omit the subscript n in h_n and ε_n. We denote by C or K generic constants independent of n that may be different from line to line. In all proofs, based on Assumption 2, by a standard localization argument, we may assume without restriction that $|\sigma|, |b|, |b^\chi|, |\chi|, |\Sigma^\chi|$, are almost surely bounded by a nonrandom constant, and $(|\delta_i(t,z)| \wedge 1)^{r_i} \leq K H_i(z)$, $i = 0, 1$. Further, we have the following estimates valid for all $s, t > 0$:

$$\mathbb{E}\left(|\chi_{t+s} - \chi_t|^p \middle| \mathcal{F}_t\right) \leq K s^{p/2}, \quad p \geq 1; \quad \mathbb{E}\left(|\sigma_{t+s} - \sigma_t|^2 \middle| \mathcal{F}_t\right) \leq K s.$$

In the proofs below, we will show that we can neglect the finite variation jump component $X^{j,0}$ and prove the results for the Itô semimartingale

$$X' := X - X^{j,0} = X_t^c + X_t^{j,\infty};$$

i.e., the process consisting of the continuous component $X_t^c = \int_0^t b_s ds + \int_0^t \sigma_s dW_s$ and the infinite variation jump component $X_t^{j,\infty} = \int_0^t \chi_{s-} dJ_s$.

Proof of Proposition 1. Note that $\widetilde{Z}_n(\varepsilon)$ in (10) can be decomposed as follows:

$$\begin{aligned}
\widetilde{Z}_n(\varepsilon) &= \sqrt{n} \sum_{i=1}^n \left(\left(\Delta_i^n X\right)^2 \mathbf{1}_{\{|\Delta_i^n X| \leq \varepsilon\}} - \left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}} \right) \\
&\quad + \sqrt{n} \sum_{i=1}^n \left(\left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}} - \mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right] \right) \\
&\quad + \sqrt{n} \sum_{i=1}^n \left(\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right] - \sigma_{t_{i-1}}^2 h - \widehat{A}_i(\varepsilon, h) h \right) \\
&\quad + \sqrt{n} \left(\sum_{i=1}^n \left(\sigma_{t_{i-1}}^2 h + \widehat{A}_i(\varepsilon, h) h \right) - \int_0^1 \sigma_s^2 ds - A(\varepsilon, h) \right) \\
&=: T_0 + T_1 + T_2 + T_3,
\end{aligned}$$

where $\widehat{A}_i(\varepsilon, h)$ is defined as in (B.4). Lemma 5 implies $T_0 = o_P(1)$, and Lemma 2 directly implies that $T_2 = o_P(1)$. We also have $T_3 = o_P(1)$, which follows from the fact that

$$\begin{aligned}
&\sum_{i=1}^n \sigma_{t_{i-1}}^2 h - \int_0^1 \sigma_s^2 ds = o_P(n^{-1/2}), \\
&\varepsilon^{2-Y} \sum_{i=1}^n |\chi_{t_{i-1}}|^Y h - \varepsilon^{2-Y} \int_0^1 |\chi_s|^Y ds = o_P(n^{-1/2}), \\
\text{and} \quad &h\varepsilon^{-Y} \sum_{i=1}^n |\chi_{t_{i-1}}|^Y \sigma_{t_{i-1}}^2 h - h\varepsilon^{-Y} \int_0^1 |\chi_s|^Y \sigma_s^2 ds = o_P(n^{-1/2}).
\end{aligned}$$

The above is established in Lemma 5 of the accompanying supplemental material to this article for completeness.

We now show $T_1 \xrightarrow{st} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4 ds\right)$ by applying the martingale difference CLT (Theorem 2.2.15 of [15]), which will complete the proof. Define

$$\xi_i^n := \sqrt{n} \left(\left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}} - \mathbb{E}_{i-1}\left(\left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}} \right) \right).$$

We first need to show that $V_n := \sum_{i=1}^n \mathbb{E}_{i-1}\left[(\xi_i^n)^2\right] \xrightarrow{\mathbb{P}} 2\int_0^1 \sigma_s^4 ds$. The left-hand side can be written as

$$V_n = n \sum_{i=1}^n \left(\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^4 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right] - \left[\mathbb{E}_{i-1}\left(\Delta_i^n X'\right)^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right]^2 \right).$$

Lemmas 2 and 4 imply that

$$\begin{aligned}
\mathbb{E}_{i-1}\left[(\Delta_i^n X')^2 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right] &= \sigma_{t_{i-1}}^2 h + O_P(h\varepsilon^{2-Y}), \\
\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^4 \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right] &= 3\sigma_{t_{i-1}}^4 h^2 + O_P(h\varepsilon^{4-Y}).
\end{aligned}$$

Since, due to our conditions on ε, $h\varepsilon^{4-Y} \ll h^2$, we clearly have

$$V_n = n \sum_{i=1}^n \left(3\sigma_{t_{i-1}}^4 h^2 + o_P(h^2) - \left(\sigma_{t_{i-1}}^2 h + O_P(h\varepsilon^{2-Y})\right)^2 \right) \xrightarrow{\mathbb{P}} 2\int_0^1 \sigma_s^4 ds.$$

Next, we show that $\sum_{i=1}^{n}\mathbb{E}_{i-1}\left[(\xi_i^n)^4\right]\xrightarrow{\mathbb{P}}0$. We have

$$\mathbb{E}_{i-1}\left[(\xi_i^n)^4\right] \le n^2 K\Big(\mathbb{E}_{i-1}[\left(\Delta_i^n X'\right)^8\mathbf{1}_{\{|\Delta_i^n X'|\le\varepsilon\}}]+\left[\mathbb{E}_{i-1}\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{|\Delta_i^n X'|\le\varepsilon\}}\right]^4\Big).$$

Lemma 4 implies that,

$$\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^{2k}\mathbf{1}_{\{|\Delta_i^n X'|\le\varepsilon\}}\right]=O_P(h^k)+O_P(h\varepsilon^{2k-Y}),$$

for $k\ge 1$. Then, $\sum_{i=1}^{n}\mathbb{E}_{i-1}\left[(\xi_i^n)^4\right]=n^3\left(O_P(h^4)+O_P(h\varepsilon^{8-Y})\right)=o_P(1)$, since our assumption $\varepsilon\ll h^{\frac{1}{4-Y}}$ implies that $\varepsilon\ll h^{\frac{1}{4-Y/2}}$, which is equivalent to $n^3h\varepsilon^{8-Y}\ll 1$. It remains to check the condition (2.2.40) in [15]:

$$\sum_{i=1}^{n}\mathbb{E}_{i-1}\left[\xi_i^n(\Delta_i^n M)\right]\xrightarrow{\mathbb{P}}0, \tag{A.1}$$

when $M=W$ or M is a square-integrable martingale orthogonal to W. This is proved in Lemma 6 of the accompanying supplemental material to this article. □

Proof of Proposition 2. We consider the decomposition:

$$\begin{aligned}
&u_n^{-1}\left(\widetilde{Z}_n(\zeta\varepsilon_n)-\widetilde{Z}_n(\varepsilon_n)\right)\\
&=\varepsilon_n^{\frac{Y-4}{2}}\sum_{i=1}^{n}\left(\left(\Delta_i^n X\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X|\le\zeta\varepsilon\}}-\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}\right)\\
&+\varepsilon_n^{\frac{Y-4}{2}}\sum_{i=1}^{n}\left(\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}-\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}\right]\right)\\
&+\varepsilon_n^{\frac{Y-4}{2}}\sum_{i=1}^{n}\left(\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}\right]-\widehat{A}_i(\zeta\varepsilon,h)h+\widehat{A}_i(\varepsilon,h)h\right)\\
&+\varepsilon_n^{\frac{Y-4}{2}}\left(\sum_{i=1}^{n}\left(\widehat{A}_i(\zeta\varepsilon,h)-\widehat{A}_i(\varepsilon,h)\right)h-A(\zeta\varepsilon,h)+A(\varepsilon,h)\right)\\
&=: T_0+T_1+T_2+T_3,
\end{aligned}$$

where $\widehat{A}_i(\varepsilon,h)$ is defined as in (B.4). From Lemma 5 we have

$$T_0=n\varepsilon^{(Y-4)/2}o_P(h\varepsilon^{2-Y/2})=o_P(1).$$

Lemma 3 shows that $T_2=o_P(1)$. To show that $T_3=o_P(1)$, Lemma 5 of the supplement implies

$$\sum_{i=1}^{n}\left(\widehat{A}_i(\zeta\varepsilon,h)-\widehat{A}_i(\varepsilon,h)\right)h-A(\zeta\varepsilon,h)+A(\varepsilon,h)=o_P(h^{\frac{1}{2}}\varepsilon^{2-Y})+o_P(h^{\frac{3}{2}}\varepsilon^{-Y}),$$

and each of the terms above is $o_P(\varepsilon^{\frac{4-Y}{2}})$ since $h^{\frac{1}{2}}\varepsilon^{2-Y}\gg h^{\frac{3}{2}}\varepsilon^{-Y}$ and $h^{\frac{1}{2}}\varepsilon^{2-Y}\ll\varepsilon^{\frac{4-Y}{2}}$, which is implied from our assumption $h^{\frac{4}{8+Y}}\ll\varepsilon$ since $\frac{1}{Y}>\frac{4}{8+Y}$. It remains to show $T_1\xrightarrow{st}\mathcal{N}\left(0,2\int_0^1\sigma_s^4ds\right)$, for which we shall again use Theorem 2.2.15 in [15]. Define

$$\tilde{\xi}_i^n:=\varepsilon^{\frac{Y-4}{2}}\left(\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}-\mathbb{E}_{i-1}\left(\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}\right)\right).$$

We first need to compute $V_n:=\sum_{i=1}^{n}\mathbb{E}_{i-1}\left[(\tilde{\xi}_i^n)^2\right]$. We clearly have

$$\begin{aligned}
V_n=\varepsilon^{Y-4}\sum_{i=1}^{n}\Big(&\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^4\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}\right]\\
&-\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}\right]^2\Big).
\end{aligned}$$

Lemmas 2 and 4 imply that

$$\begin{aligned}
\mathbb{E}_{i-1}(\Delta_i^n X')^2\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}&=O_P(h\varepsilon^{2-Y}),\\
\mathbb{E}_{i-1}\left(\Delta_i^n X'\right)^4\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\le\zeta\varepsilon\}}&=\frac{\bar{C}|\chi_{t_{i-1}}|^Y}{4-Y}h\varepsilon^{4-Y}(\zeta^{4-Y}-1)+o_P(h\varepsilon^{4-Y}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_n&=\varepsilon^{Y-4}\sum_{i=1}^{n}\left(\frac{\bar{C}}{4-Y}|\chi_{t_{i-1}}|^Y h\varepsilon^{4-Y}(\zeta^{4-Y}-1)+o_P(h\varepsilon^{4-Y})+O_P(h\varepsilon^{2-Y})^2\right)\\
&\xrightarrow{\mathbb{P}}\frac{\bar{C}}{4-Y}(\zeta^{4-Y}-1)\int_0^1|\chi_s|^Y ds,
\end{aligned}$$

since $\varepsilon^{Y-4}n(h\varepsilon^{2-Y})^2 = h\varepsilon^{-Y} \ll 1$. Next, we show $\sum_{i=1}^n \mathbb{E}_{i-1}(\tilde{\xi}_i^n)^4 \xrightarrow{\mathbb{P}} 0$. We have

$$\mathbb{E}_{i-1}(\tilde{\xi}_i^n)^4 \leq K\varepsilon^{2Y-8}\left(\mathbb{E}_{i-1}[(\Delta_i^n X')^8 \mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\leq\zeta\varepsilon\}}] + \left[\mathbb{E}_{i-1}(\Delta_i^n X')^2 \mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\leq\zeta\varepsilon\}}\right]^4\right).$$

Lemma 4 implies that for $k \geq 1$,

$$\mathbb{E}_{i-1}\left(\Delta_i^n X'\right)^{2k} \mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\leq\zeta\varepsilon\}} = O_P(h\varepsilon^{2k-Y}),$$

and thus, $\sum_{i=1}^n \mathbb{E}_{i-1}(\tilde{\xi}_i^n)^4 = \varepsilon^{2Y-8}O_P(\varepsilon^{8-Y}) = o_P(1)$. It remains to check the condition (2.2.40) in [15]:

$$\sum_{i=1}^n \mathbb{E}_{i-1}\left[\tilde{\xi}_i^n(\Delta_i^n M)\right] \xrightarrow{\mathbb{P}} 0, \tag{A.2}$$

when $M = W$ or M is a square-integrable martingale orthogonal to W. The proof is much more involved and technical than that of (A.1) and is given in Lemma 7 of the accompanying supplemental material to this article. □

Proof of Theorem 1. Recall the notation of Propositions 1 and 2. In addition, we set up the following notation:

$$a_1(\varepsilon) := \frac{\bar{C}}{2-Y}\int_0^1 |\chi_s|^Y ds\varepsilon^{2-Y} =: \kappa_1\varepsilon^{2-Y},$$
$$a_2(\varepsilon) := a_2(\varepsilon,h) := -\bar{C}\frac{(Y+1)(Y+2)}{2Y}\int_0^1 |\chi_s|^Y\sigma_s^2 dsh\varepsilon^{-Y} =: \kappa_2 h\varepsilon^{-Y},$$
$$\Phi_n := u_n^{-1}\left(\widetilde{Z}_n(\zeta_1\varepsilon) - \widetilde{Z}_n(\varepsilon)\right) = O_P(1),$$
$$\Psi_n := u_n^{-1}\left(\widetilde{Z}_n(\zeta_1^2\varepsilon) - 2\widetilde{Z}_n(\zeta_1\varepsilon) + \widetilde{Z}_n(\varepsilon)\right) = O_P(1),$$

where the stochastic boundedness of Φ_n and Ψ_n is a consequence of Proposition 2. The proof is obtained in two steps.

Step 1. We first analyze the behavior of $\widetilde{C}_n'(\varepsilon,\zeta_1) = \widehat{C}_n(\varepsilon) - \widehat{a}_1(\varepsilon)$, where

$$\widehat{a}_1(\varepsilon) := \frac{\left(\widehat{C}_n(\zeta_1\varepsilon) - \widehat{C}_n(\varepsilon)\right)^2}{\widehat{C}_n(\zeta_1^2\varepsilon) - 2\widehat{C}_n(\zeta_1\varepsilon) + \widehat{C}_n(\varepsilon)}. \tag{A.3}$$

If we let $\eta_1(\zeta) = \zeta^{2-Y} - 1$ and $\eta_2(\zeta) = \zeta^{-Y} - 1$, then, for $i = 1,2$, we have

$$a_i(\zeta_1\varepsilon) - a_i(\varepsilon) = \eta_i(\zeta_1)a_i(\varepsilon), \quad a_i(\zeta_1^2\varepsilon) - 2a_i(\zeta_1\varepsilon) + a_i(\varepsilon) = \eta_i^2(\zeta_1)a_i(\varepsilon).$$

For simplicity, we often omit the variable ζ_1 on $\eta_i(\zeta_1)$. Also, note that, by definition, $\widehat{C}_n(\varepsilon) = \sqrt{h}\widetilde{Z}_n(\varepsilon) + \int_0^1 \sigma_s^2 ds + A(h,\varepsilon)$ and $A(\varepsilon,h) = a_1(\varepsilon) + a_2(\varepsilon,h)$. Therefore, we may write

$$\widehat{a}_1(\varepsilon) = \frac{(\eta_1 a_1(\varepsilon) + \eta_2 a_2(\varepsilon) + \sqrt{h}u_n\Phi_n)^2}{\eta_1^2 a_1(\varepsilon) + \eta_2^2 a_2(\varepsilon) + \sqrt{h}u_n\Psi_n}. \tag{A.4}$$

By expanding the squares in the numerator and using the notation

$$\widetilde{a}_1(\varepsilon) := a_1(\varepsilon) + \widetilde{\eta}_2 a_2(\varepsilon) := a_1(\varepsilon) + \frac{2\eta_1\eta_2 - \eta_2^2}{\eta_1^2}a_2(\varepsilon),$$

we may express $\widehat{a}_1(\varepsilon)$ as

$$\begin{aligned}
\widehat{a}_1(\varepsilon) &= \widetilde{a}_1(\varepsilon) + \frac{\eta_2^2(1-\widetilde{\eta}_2)a_2^2(\varepsilon) + \sqrt{h}\left[2u_n\Phi_n(\eta_1 a_1(\varepsilon) + \eta_2 a_2(\varepsilon)) - \widetilde{a}_1(\varepsilon)u_n\Psi_n\right] + hu_n^2\Phi_n^2}{\eta_1^2 a_1(\varepsilon) + \eta_2^2 a_2(\varepsilon) + \sqrt{h}u_n\Psi_n} \\
&= \widetilde{a}_1(\varepsilon) + \sqrt{h}\times\frac{O(h^{3/2}\varepsilon^{-2Y}) + O_P(u_n\varepsilon^{2-Y}) + O_P(\sqrt{h}u_n^2)}{\eta_1^2 a_1(\varepsilon) + o(a_1(\varepsilon)) + O_P(\sqrt{h}u_n)} \\
&= \widetilde{a}_1(\varepsilon) + \sqrt{h}\times\frac{O(h^{3/2}\varepsilon^{-2-Y}) + O_P(u_n) + O_P(h^{-1/2}\varepsilon^2)}{\eta_1^2\kappa_1 + o(1) + O_P(\varepsilon^{Y/2})} \\
&= \widetilde{a}_1(\varepsilon) + \sqrt{h}\times O_P(u_n),
\end{aligned} \tag{A.5}$$

where in the last equality we use our assumption $h_n^{\frac{4}{8+Y}} \ll \varepsilon_n$ to conclude that $h^{3/2}\varepsilon^{-2-Y} \ll u_n$. Then, we see that $\widetilde{C}_n'(\varepsilon,\zeta_1) = \widehat{C}_n(\varepsilon) - \widehat{a}_1(\varepsilon)$ is given by

$$\widetilde{C}_n'(\varepsilon,\zeta_1) = \sqrt{h}\widetilde{Z}_n(\varepsilon) + \int_0^1 \sigma_s^2 ds + A(h,\varepsilon) - \widetilde{a}_1(\varepsilon) + O_P(h^{1/2}u_n)$$

$$
\begin{aligned}
&= \sqrt{h}\widetilde{Z}_n(\varepsilon) \\
&\quad + \int_0^1 \sigma_s^2 ds + a_1(\varepsilon) + a_2(\varepsilon,h) - [a_1(\varepsilon) + \widetilde{\eta}_2 a_2(\varepsilon)] + O_P(h^{1/2}u_n) \\
&= \sqrt{h}\widetilde{Z}_n(\varepsilon) + \int_0^1 \sigma_s^2 ds + a_2'(\varepsilon) + O_P(h^{1/2}u_n),
\end{aligned} \tag{A.6}
$$

where $a_2'(\varepsilon) = (1-\widetilde{\eta}_2)a_2(\varepsilon)$. So,

$$
\widetilde{Z}_n'(\varepsilon) := \sqrt{n}\left(\widetilde{C}_n'(\varepsilon,\zeta_1) - \int_0^1 \sigma_s^2 ds - a_2'(\varepsilon)\right) = \widetilde{Z}_n(\varepsilon) + O_P(u_n), \tag{A.7}
$$

where the $O_P(u_n)$ term is a consequence of expression (A.6). Then, by Proposition 2,

$$
\widetilde{Z}_n'(\varepsilon) = \widetilde{Z}_n(\varepsilon) + O_P(u_n) \xrightarrow{st} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4 ds\right), \tag{A.8}
$$

since $u_n \to 0$ by our Assumption $\varepsilon_n \ll h_n^{\frac{1}{4-Y}}$. Note that if $\varepsilon_n \gg h_n^{\frac{1}{2Y}}$, then $\sqrt{n}a_2' \ll 1$ and in place of (A.5) we have $\widetilde{a}_1(\varepsilon) + \sqrt{h} \times o_P(1)$, from which we conclude that

$$
\sqrt{n}\left(\widetilde{C}_n'(\varepsilon,\zeta_1) - \int_0^1 \sigma_s^2 ds\right) \xrightarrow{st} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4 ds\right).
$$

Step 2. Now we analyze $\widetilde{C}_n''(\varepsilon,\zeta_2,\zeta_1) = \widetilde{C}_n'(\varepsilon,\zeta_1) - \widehat{a}_2'(\varepsilon,\zeta_1,\zeta_2)$, where

$$
\widehat{a}_2'(\varepsilon,\zeta_1,\zeta_2) := \frac{\left(\widetilde{C}_n'(\zeta_2\varepsilon,\zeta_1) - \widetilde{C}_n'(\varepsilon,\zeta_1)\right)^2}{\widetilde{C}_n'(\zeta_2^2\varepsilon,\zeta_1) - 2\widetilde{C}_n'(\zeta_2\varepsilon,\zeta_1) + \widetilde{C}_n'(\varepsilon,\zeta_1)}.
$$

For simplicity, we omit the dependence on ζ_1 and ζ_2 in $\widetilde{C}_n'(\varepsilon,\zeta_1), C_n''(\varepsilon,\zeta_1,\zeta_2)$, etc. First, analogous to Φ_n, Ψ_n defined in Step 1, we define

$$
\begin{aligned}
\Phi_n' &:= u_n^{-1}\left(\widetilde{Z}_n'(\zeta_2\varepsilon) - \widetilde{Z}'(\varepsilon)\right) = u_n^{-1}\left(\widetilde{Z}_n(\zeta\varepsilon) - \widetilde{Z}_n(\varepsilon) + O_P(u_n)\right) = O_P(1), \\
\Psi_n' &:= u_n^{-1}\left(\widetilde{Z}_n'(\zeta_2^2\varepsilon) - 2\widetilde{Z}_n'(\zeta_2\varepsilon) + \widetilde{Z}_n'(\varepsilon)\right) = O_P(1),
\end{aligned}
$$

where the stochastic boundedness of Φ_n', Ψ_n' follows from (A.7) and (13). Now, by definition (A.7), $\widetilde{C}_n'(\varepsilon) = \sqrt{h}\widetilde{Z}_n'(\varepsilon) + \int_0^1 \sigma_s^2 ds + a_2'(\varepsilon)$. Also, with the notation $\eta_2'(\zeta) = \zeta^{-Y} - 1$, the term $a_2'(\varepsilon) = (1-\widetilde{\eta}_2)a_2(\varepsilon) =: \kappa_2' h\varepsilon^{-Y}$ satisfies

$$
\begin{aligned}
a_2'(\zeta_2\varepsilon) - a_2'(\varepsilon) &= \eta_2'(\zeta_2)a_2'(\varepsilon), \\
a_2'(\zeta_2^2\varepsilon) - 2a_2'(\zeta_2\varepsilon) + a_2'(\varepsilon) &= (\eta_2')^2(\zeta_2)a_2'(h,\varepsilon).
\end{aligned}
$$

Therefore, we may express

$$
\begin{aligned}
\widehat{a}_2'(\varepsilon) &= \frac{(\eta_2' a_2'(\varepsilon) + \sqrt{h}u_n\Phi_n')^2}{(\eta_2')^2 a_2'(\varepsilon) + \sqrt{h}u_n\Psi_n'} \\
&= a_2'(\varepsilon) + \frac{\sqrt{h}a_2'(\varepsilon)u_n(2\Phi_n' - \Psi_n') + hu_n^2(\Phi_n')^2}{(\eta_2')^2 a_2'(\varepsilon) + \sqrt{h}u_n\Psi_n'} \\
&= a_2'(\varepsilon) + \sqrt{h} \times \frac{O_P(u_n) + O_P(h^{-3/2}\varepsilon^4)}{(\eta_2')^2\kappa_2' + o_P(1)} \\
&= a_2'(\varepsilon) + \sqrt{h} \times O_P(u_n),
\end{aligned}
$$

where, in the last equality we used that $\varepsilon_n \ll h_n^{2/(4+Y)}$ to conclude that

$$
\begin{aligned}
(a_2')^{-1}\sqrt{h}u_n &= O_P(h^{-1}\varepsilon^Y\varepsilon^{\frac{4-Y}{2}}) = O_P(h^{-1}\varepsilon^{\frac{4+Y}{2}}) = o_P(1), \\
(a_2')^{-1}h^{1/2}u_n^2 &= h^{-3/2}\varepsilon^4 \ll u_n = h_n^{-\frac{1}{2}}\varepsilon^{\frac{4-Y}{2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\sqrt{n}\left(\widetilde{C}_n''(\varepsilon,\zeta_2,\zeta_1) - \int_0^1 \sigma_s^2 ds\right) \\
&= h^{-1/2}\left(\widetilde{C}_n'(\varepsilon,\zeta_1) - \int_0^1 \sigma_s^2 ds - \widehat{a}_2'(\varepsilon,\zeta_1,\zeta_2)\right)
\end{aligned}
$$

$$= h^{-1/2}\left(\widetilde{C}_n'(\varepsilon,\zeta_1) - \int_0^1 \sigma_s^2 ds - a_2'(\varepsilon)\right) + O(u_n)$$
$$= \widetilde{Z}_n'(\varepsilon) + O_P(u_n) \xrightarrow{st} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4 ds\right),$$

where the third and fourth limit follow from (A.7) and (A.8), respectively. □

Appendix B. Asymptotic expansions for truncated moments

In this section, we provide high-order asymptotic expansions for the truncated moments of the Itô semimartingale X. As in Appendix A, we denote C or K generic constants that may be different from line to line.

To simplify some proofs, we now lay out some additional notation related to the process J. Let N be the Poisson jump measure of J and let $\bar{N}$ be its compensated measure. Observe that due to condition (ii) in Assumption 1, there exists $\delta_0 \in (0,1)$ such that $q(x) > 0$ for all $|x| \leq \delta_0$. Next, let $\check{J}$ be a pure-jump Lévy process independent of J with triplet $(0,0,\check{\nu})$, where $\check{\nu}(dx) = e^{-|x|^p}(C_+\mathbf{1}_{(0,\infty)}(x) + C_-\mathbf{1}_{(-\infty,0)}(x))\mathbf{1}_{|x|>\delta_0}|x|^{-1-Y}dx$, for a fixed $p < 1 \wedge Y$, and define the Lévy process

$$J_t^\infty = \Big(\bar{b} + \int_{\delta_0<|x|\leq 1} x\nu(dx)\Big)t + \int_0^t\int_{|x|\leq\delta_0} x\bar{N}(ds,dx) + \check{J}_t. \tag{B.1}$$

In other words, J^∞ has Lévy measure $\nu(dx)\mathbf{1}_{\{|x|\leq\delta_0\}} + \check{\nu}(dx)\mathbf{1}_{\{|x|>\delta_0\}}$, and, in particular, J_t^∞ satisfies all the conditions of Assumption 2 of the accompanying supplemental material to this article and, thus, we can apply the asymptotics of the truncated moments established therein. Next, we write

$$J_t^0 := J_t - J_t^\infty = \int_0^t\int_{|x|>\delta_0} xN(ds,dx) - \check{J}_t, \tag{B.2}$$

and observe J^0 has finite jump activity.

As an intermediate step, we first establish estimates for the truncated moments of the process

$$X_t' := \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \chi_s dJ_s^\infty. \tag{B.3}$$

In a subsequent step, we show the same estimates also hold for X up to asymptotically negligible terms (Lemma 5, below). Note that

$$X_t - X_t' = X_t^{j,0} + \int_0^t \chi_s dJ_s^0.$$

In other words, the process X' includes the continuous component of X and the infinite variation component $\int_0^t \chi_s dJ_s^\infty$, and $X - X'$ contains only finite variation terms.

Lemma 2. *Let*

$$\widehat{A}_i(\varepsilon,h) = \frac{\bar{C}|\chi_{t_{i-1}}|^Y}{2-Y}\varepsilon^{2-Y} - \bar{C}\frac{(Y+1)(Y+2)}{2Y}\sigma_{t_{i-1}}^2|\chi_{t_{i-1}}|^Y\, h\varepsilon^{-Y}. \tag{B.4}$$

Suppose that $Y \in (0,1)\cup(1,\frac{8}{5})$ *and* $h_n^{\frac{3}{2(2+Y)}\wedge\frac{1}{2}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-s}}$, *for some* $s \in (0,4)$. *Then, for any* $i = 1,\ldots,n$,

$$\mathbb{E}_{i-1}\left[(\Delta_i^n X')^2\mathbf{1}_{\{|\Delta_i^n X'|\leq\varepsilon\}}\right] = \sigma_{t_{i-1}}^2 h + \widehat{A}_i(\varepsilon,h)h + o_P(h^{3/2}). \tag{B.5}$$

Proof. We use the following notation

$$\begin{aligned}
&x_i := b_{t_{i-1}}h + \sigma_{t_{i-1}}\Delta_i^n W + \chi_{t_{i-1}}\Delta_i^n J^\infty =: x_{i,1} + x_{i,2} + x_{i,3},\\
&\mathcal{E}_i := \Delta_i^n X' - x_i, \quad \mathcal{E}_{i,1} = \int_{t_{i-1}}^{t_i} b_s ds - b_{t_{i-1}}h,\\
&\mathcal{E}_{i,2} = \int_{t_{i-1}}^{t_i}\sigma_s dW_s - \sigma_{t_{i-1}}\Delta_i W, \qquad \mathcal{E}_{i,3} = \int_{t_{i-1}}^{t_i}\chi_s dJ_s^\infty - \chi_{t_{i-1}}\Delta_i^n J^\infty.
\end{aligned} \tag{B.6}$$

The following estimates, established in Lemma 3 of the accompanying supplemental material to this article, are often used:

$$\mathbb{E}_{i-1}\left[|\mathcal{E}_{i,\ell}|^p\right] \leq C\begin{cases} h^p, & \ell = 1,\ p>0\\ h^{\frac{(2\wedge p)+p}{2}}, & \ell=2,\ p>0\\ h^{1+\frac{p}{2}} & \ell = 3,\ p\in[1,\infty)\cap(Y,\infty).\end{cases} \tag{B.7}$$

In particular $\mathbb{E}_{i-1}\left[|\mathcal{E}_i|^p\right] \leq Ch^{1+\frac{p}{2}}$ for all $p \geq 2$. By our expansion for the truncated second-order moment of Lévy process given in Proposition 2 of the accompanying supplemental material to this article, we can easily see that

$$\mathbb{E}_{i-1}\left[x_i^2\mathbf{1}_{\{|x_i|\leq\varepsilon\}}\right] = \sigma_{t_{i-1}}^2 h + \widehat{A}_i(\varepsilon,h)h + o_P(h^{\frac{3}{2}}),$$

because the higher-order terms $h^3\varepsilon^{-Y-2}$, $h^2\varepsilon^{2-2Y}$, and $h\varepsilon^{2-\bar{\delta}}$ ($\bar{\delta}>0$ arbitrary) are smaller than $h^{\frac{3}{2}}$ due to the restrictions $Y<8/5$ and $h_n^{\frac{3}{2(2+Y)}\wedge\frac{1}{2}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-s}}$. Therefore, for (B.5) to hold, it suffices that

$$\mathcal{R}_i := \mathbb{E}_{i-1}\left[(\Delta_i^n X')^2\mathbf{1}_{\{|\Delta_i^n X'|\le\varepsilon\}}\right] - \mathbb{E}_{i-1}\left[x_i^2\mathbf{1}_{\{|x_i|\le\varepsilon\}}\right] = o_P(h^{3/2}). \tag{B.8}$$

Clearly,

$$\begin{aligned}|\mathcal{R}_i| &\le 2\mathbb{E}_{i-1}\left[x_i^2\mathbf{1}_{\{|x_i+\mathcal{E}_i|\le\varepsilon<|x_i|\}}\right] + 2\mathbb{E}_{i-1}\left[\mathcal{E}_i^2\mathbf{1}_{\{|x_i+\mathcal{E}_i|\le\varepsilon<|x_i|\}}\right] \\ &\quad + 2\left|\mathbb{E}_{i-1}\left[x_i\mathcal{E}_i\mathbf{1}_{\{|x_i+\mathcal{E}_i|\le\varepsilon,|x_i|\le\varepsilon\}}\right]\right| \\ &\quad + \mathbb{E}_{i-1}\left[x_i^2\mathbf{1}_{\{|x_i|\le\varepsilon<|x_i+\mathcal{E}_i|\}}\right] + \mathbb{E}_{i-1}\left[\mathcal{E}_i^2\mathbf{1}_{\{|x_i+\mathcal{E}_i|\le\varepsilon,|x_i|\le\varepsilon\}}\right] = \sum_{\ell=1}^5 \mathcal{R}_{i,\ell}.\end{aligned}$$

The terms $\mathcal{R}_{i,2}$ and $\mathcal{R}_{i,5}$ are straightforward since $\mathcal{R}_{i,\ell}\le\mathbb{E}_{i-1}\left[\mathcal{E}_i^2\right]\le Ch^2 = o_P(h^{3/2})$. For the term $\mathcal{R}_{i,3}$, expanding the product $x_i\mathcal{E}_i$, we get 9 terms of the form $A_{\ell,\ell'} := \left|\mathbb{E}_{i-1}\left[\mathcal{E}_{i,\ell}x_{i,\ell'}\mathbf{1}_{\{|x_i+\mathcal{E}_i|\le\varepsilon,|x_i|\le\varepsilon\}}\right]\right|$ (one for each pair $\ell,\ell'\in\{1,2,3\}$). For terms with $x_{i,1}$, since $|x_{i,1}|\le Kh$ clearly

$$A_{\ell,1}\le Ch\mathbb{E}_{i-1}\left[|\mathcal{E}_{i,\ell}|\right]\le Ch\mathbb{E}_{i-1}\left[|\mathcal{E}_{i,\ell}|^2\right]^{\frac{1}{2}}\le Ch^2.$$

For terms involving $x_{i,3}$, Lemma 12 of the accompanying supplemental material to this article implies

$$\begin{aligned}A_{\ell,3} &\le C\mathbb{E}_{i-1}\left[|\Delta_i^n J^\infty||\mathcal{E}_{i,\ell}|\mathbf{1}_{\{|x_i|\le\varepsilon\}}\right] \\ &\le C\mathbb{E}_{i-1}\left[|\Delta_i^n J^\infty|^2|\mathbf{1}_{\{|x_i|\le\varepsilon\}}\right]^{\frac{1}{2}}\mathbb{E}_{i-1}\left[\mathcal{E}_{i,\ell}^2|\right]^{\frac{1}{2}}\le Ch^{\frac{1}{2}}\varepsilon^{\frac{2-Y}{2}}h = o_P(h^{3/2}).\end{aligned}$$

The terms involving $x_{i,2}$ are more delicate. We start with

$$A_{\ell,2}\le C\left|\mathbb{E}_{i-1}\left[\Delta_i^n W\mathcal{E}_{i,\ell}\right]\right| + C\mathbb{E}_{i-1}\left[|\Delta_i^n W||\mathcal{E}_{i,\ell}|\mathbf{1}_{\{|x_i+\mathcal{E}_i|>\varepsilon\ \text{ or }\ |x_i|>\varepsilon\}}\right]. \tag{B.9}$$

Clearly, $\mathbb{E}_{i-1}\left[\Delta_i^n W\mathcal{E}_{i,\ell}\right]=0$ for $\ell=1,3$. For $\ell=2$, $\mathbb{E}_{i-1}\left[\Delta_i^n W\mathcal{E}_{i,2}\right]=\mathbb{E}_{i-1}\left[\int_{t_{i-1}}^{t_i}(\sigma_s-\sigma_{t_{i-1}})ds\right]=O_P(h^2)$. For the second term in (B.9) on the event $\{|x_i+\mathcal{E}_i|>\varepsilon\ \text{ or }\ |x_i|>\varepsilon\}$, we have that $|x_{i,\ell}|>\varepsilon/4$ for at least one ℓ or $|\mathcal{E}_i|>\varepsilon/4$. The case $|x_{i,1}|>\varepsilon/4$ is eventually impossible for n large enough (since b is bounded), while both cases $|x_{i,2}|>\varepsilon/4$ and $|\mathcal{E}_i|>\varepsilon/4$ are straightforward to handle using the Markov's and Hölder's inequalities. For instance, for any $m\ge 1$,

$$\begin{aligned}&\mathbb{E}_{i-1}\left[|\Delta_i^n W||\mathcal{E}_{i,\ell}|\mathbf{1}_{\{|\mathcal{E}_i|>\varepsilon/4\}}\right] \\ &\quad\le\frac{C}{\varepsilon^m}\mathbb{E}_{i-1}\left[|\Delta_i^n W||\mathcal{E}_{i,\ell}||\mathcal{E}_i|^m\right] \\ &\quad\le\frac{C}{\varepsilon^m}\mathbb{E}_{i-1}\left[|\Delta_i^n W|^3\right]^{\frac{1}{3}}\mathbb{E}_{i-1}\left[|\mathcal{E}_{i,\ell}|^3\right]^{\frac{1}{3}}\mathbb{E}_{i-1}\left[|\mathcal{E}_i|^{3m}\right]^{\frac{1}{3}} \\ &\quad\le\frac{C}{\varepsilon^m}h^{\frac{1}{2}}h^{\frac{5}{6}}h^{\frac{1}{3}+\frac{m}{2}} = Ch^{\frac{1}{2}}h^{\frac{5}{6}}h^{\frac{1}{3}}\left(\frac{h}{\varepsilon^2}\right)^{\frac{m}{2}}.\end{aligned} \tag{B.10}$$

So, by picking $m=1$, we can make this term $o_P(h^{3/2})$. The remaining term is when $|x_{i,3}|>\varepsilon/4$. In that case, for any $p,q>2$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, applying Lemma 6,

$$\begin{aligned}&\mathbb{E}_{i-1}\left[|\Delta_i^n W||\mathcal{E}_{i,\ell}|\mathbf{1}_{\{|x_{i,3}|>\varepsilon/4\}}\right] \\ &\le C\mathbb{E}_{i-1}\left[|\Delta_i^n W|^p\right]^{\frac{1}{p}}\mathbb{P}_{i-1}\left[|\Delta_i^n J^\infty|>\frac{\varepsilon}{4}\right]^{\frac{1}{q}}\mathbb{E}_{i-1}\left[|\mathcal{E}_{i,\ell}|^2\right]^{\frac{1}{2}} \\ &\le Ch^{\frac{3}{2}}(h\varepsilon^{-Y})^{\frac{1}{q}}\ll h^{\frac{3}{2}},\end{aligned}$$

since, by our assumption on ε, we have $h^{\frac{1}{Y}}\ll\varepsilon$. We then conclude that $\mathcal{R}_{i,3}=o_P(h^{\frac{3}{2}})$.

It remains to analyze $\mathcal{R}_{i,1}$ and $\mathcal{R}_{i,4}$. The proof is similar in both cases and we only give the details for the second case to save space. For some $\delta=\delta_n\to 0$ ($0<\delta<\varepsilon$), whose precise asymptotic behavior will be determined below, we consider the decomposition:

$$\mathcal{R}_{i,4}\le\mathbb{E}_{i-1}\left[x_i^2\mathbf{1}_{\{\varepsilon-\delta<|x_i|\le\varepsilon\}}\right]+\mathbb{E}_{i-1}\left[x_i^2\mathbf{1}_{\{|x_i|\le\varepsilon-\delta,\varepsilon\le|x_i+\mathcal{E}_i|\}}\right]=:\mathcal{D}_{i,1}+\mathcal{D}_{i,2}.$$

By the expansion in the Proposition 3 of the accompanying supplemental material to this article and our assumptions, we have

$$\begin{aligned}\mathcal{D}_{i,1}&=Ch[\varepsilon^{2-Y}-(\varepsilon-\delta)^{2-Y}]+C'h^2[\varepsilon^{-Y}-(\varepsilon-\delta)^{-Y}]+o_P(h^{\frac{3}{2}}) \\ &=Ch\varepsilon^{1-Y}\delta+o_P(h\varepsilon^{1-Y}\delta)+o_P(h^{\frac{3}{2}}).\end{aligned} \tag{B.11}$$

Therefore, to obtain $\mathcal{D}_{i,1}=o_P(h^{\frac{3}{2}})$, we require

$$\delta\ll h^{\frac{1}{2}}\varepsilon^{Y-1}. \tag{B.12}$$

For $\mathcal{D}_{i,2}$, note that $|x_i| \le \varepsilon - \delta$ and $\varepsilon \le |x_i + \mathcal{E}_i|$ imply that $|\mathcal{E}_i| > \delta$. Also, $\varepsilon \le |x_i + \mathcal{E}_i|$ implies that $|x_{i,\ell}| > \varepsilon/4$ for at least one ℓ or $|\mathcal{E}_i| > \varepsilon/4$. The case $|x_{i,1}| > \varepsilon/4$ is eventually impossible for n large enough, while both cases $|x_{i,2}| > \varepsilon/4$ and $|\mathcal{E}_i| > \varepsilon/4$ are again straightforward to handle using Markov's and Hölder's inequalities as in (B.10). Therefore, we need only to consider the case when $|x_{i,3}| > \varepsilon/4$ and $\max\{|x_{i,1}|, |x_{i,2}|, |\mathcal{E}_i|\} \le \varepsilon/4$. In particular, since $|x_i| = |x_{i,1} + x_{i,2} + x_{i,3} + \mathcal{E}_i| < \varepsilon$, we have $\varepsilon/4 < |x_{i,3}| \le C\varepsilon$ for some C. Then, we are left to analyze the following term:

$$\mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\le C\varepsilon, |\mathcal{E}_i|>\delta\}}\right] \le C\sum_{\ell=1}^{3} \mathbb{E}_{i-1}\left[x_{i,\ell}^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\le C\varepsilon, |\mathcal{E}_i|>\delta\}}\right] =: \sum_{\ell=1}^{3} \mathcal{V}_{i,\ell}.$$

Clearly, $\mathcal{V}_{i,1} = O_P(h^2) = o_P(h^{3/2})$. For $\mathcal{V}_{i,2}$, by Hölder's inequality, for any $p, q > 1$, $r \ge 2$, such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, recalling (B.7) and Lemma 6 below,

$$\begin{aligned}\mathcal{V}_{i,2} &\le C\frac{1}{\delta}\mathbb{E}_{i-1}\left[(\Delta_i^n W)^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\}} |\mathcal{E}_i|\right] \\ &\le C\frac{1}{\delta}\mathbb{E}_{i-1}\left[(\Delta_i^n W)^{2p}\right]^{\frac{1}{p}} \mathbb{P}_{i-1}\left[\frac{\varepsilon}{4} < |x_{i,3}|\right]^{\frac{1}{q}} \mathbb{E}_{i-1}\left[|\mathcal{E}_i|^r\right]^{\frac{1}{r}} \\ &\le C\frac{1}{\delta} h (h\varepsilon^{-Y})^{\frac{1}{q}} (h^{1+\frac{r}{2}})^{\frac{1}{r}} = \frac{C}{\delta} h^{\frac{5}{2}-\frac{1}{p}} \varepsilon^{-\frac{Y}{q}}.\end{aligned}$$

If $Y < 1$, we take $q \to \infty$ and $p = r = 2$, to conclude that we only need $\delta \gg h^{1/2}$ for $\mathcal{V}_{i,2} = o_P(h^{3/2})$ to hold. This is consistent with (B.12) (meaning they can be met simultaneously for at least one choice of the sequence δ), since $Y < 1$.

Now we consider the case $Y \in (1, 8/5)$. Clearly

$$\mathcal{V}_{i,2} \le C\frac{1}{\delta}\mathbb{E}_{i-1}\left[(\Delta_i^n W)^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\}} (|\mathcal{E}_{i,1}| + |\mathcal{E}_{i,2}| + |\mathcal{E}_{i,3}|)\right].$$

Note $\mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\}} \le \mathbf{1}_{\{K\varepsilon<|\Delta_i^n J^\infty|\}}$. Thus,

$$\begin{aligned}&\mathbb{E}_{i-1}\left[(\Delta_i^n W)^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\}} |\mathcal{E}_{i,1}|\right] \\ &\quad\le h\mathbb{E}_{i-1}(\Delta_i^n W)^2 \mathbb{P}\left(K\varepsilon < |\Delta_i^n J^\infty|\right) \\ &\quad\le C h^3 \varepsilon^{-Y}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}_{i-1}\left[(\Delta_i^n W)^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\}} |\mathcal{E}_{i,2}|\right] \\ &\quad\le \left(\mathbb{E}_{i-1}(\Delta_i^n W)^4 \mathbb{E}_{i-1}|\mathcal{E}_{i,2}|^2\right)^{1/2} \mathbb{P}\left(K\varepsilon < |\Delta_i^n J^\infty|\right) \\ &\quad\le C h^3 \varepsilon^{-Y}.\end{aligned}$$

For $\ell = 3$, with $p, q > 1$, $Y < r < 2$, such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, from (B.7) we have

$$\mathbb{E}_{i-1}\left[(\Delta_i^n W)^2 \mathbf{1}_{\{\frac{\varepsilon}{4}<|x_{i,3}|\}} |\mathcal{E}_{i,3}|\right] \le C h (h\varepsilon^{-Y})^{\frac{1}{q}} (h^{1+\frac{r}{2}})^{\frac{1}{r}} = C h^{\frac{5}{2}-\frac{1}{p}} \varepsilon^{-\frac{Y}{q}}.$$

Then, taking r close to Y, p large, and q close to $Y/(Y-1)$, we obtain, for some $s', s'' > 0$ that can be made arbitrarily small,

$$\mathcal{V}_{i,2} \le \frac{C}{\delta}\left(h^{\frac{5}{2}-s'} \varepsilon^{1-Y-s''} + h^3 \varepsilon^{-Y}\right) = O\left(\delta^{-1} h^{\frac{5}{2}-s'} \varepsilon^{1-Y-s''}\right),$$

where we used $h^{\frac{1}{2}} \ll \varepsilon$ to conclude that $h^3\varepsilon^{-Y} \ll h^{\frac{5}{2}}\varepsilon^{1-Y} \ll h^{\frac{5}{2}-s'}\varepsilon^{1-Y-s''}$. Thus, for $\mathcal{V}_{i,2} = o_P(h^{3/2})$ to hold, it suffices that for some appropriately small $s', s'' > 0$,

$$h^{1-s'} \varepsilon^{1-Y-s''} \ll \delta. \tag{B.13}$$

The conditions (B.12) and (B.13) are consistent, since

$$h^{1-s'} \varepsilon^{1-Y-s''} \ll h^{\frac{1}{2}} \varepsilon^{Y-1} \iff h^{\frac{1-2s'}{4Y-4+2s''}} \ll \varepsilon, \tag{B.14}$$

which is implied by our condition $\varepsilon \gg h^{\frac{3}{2(2+Y)}}$ when provided s', s'' are both chosen small enough, since $\frac{3}{2(2+Y)} < \frac{1}{4-4Y}$ when $Y < 8/5$.

It remains to analyze $\mathcal{V}_{i,3}$. As a consequence of Lemma 4 in the accompanying supplemental material to this article,[7] provided that the condition $\delta \gg h^{1/2}\varepsilon^{Y/2}$ holds, we have

$$\mathbb{E}\mathcal{V}_{i,3} \le C\mathbb{E}[(\Delta_i^n J^\infty)^2 \mathbf{1}_{\{|\Delta_i^n J^\infty|\le\varepsilon\}} \mathbf{1}_{\{|\mathcal{E}_i|>\delta_n\}}] = O(h^2\varepsilon^{2-2Y}) = o(h^{3/2}),$$

[7] This estimate is shaper than what can be obtained by applying simply Hölder's inequality and, hence, require some special handling.

where the second equality follows from $\varepsilon \gg h^{\frac{3}{2(2+Y)}}$. The condition $\delta \gg h^{1/2}\varepsilon^{Y/2}$ can be met under (B.12) since $h^{1/2}\varepsilon^{Y/2} \ll h^{1/2}\varepsilon^{Y-1}$. Thus, (B.8) holds and this concludes the proof. □

Lemma 3. *Let $\widehat{A}_i(\varepsilon, h)$ be as in (B.4) and suppose that $Y \in (0,1) \cup (1, \frac{8}{5})$ and $h_n^{\frac{3}{2(2+Y)}\wedge\frac{1}{2}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-Y}}$. Then, for any $i = 1, \dots, n$ and $\zeta > 1$,*

$$\begin{aligned}
&\mathbb{E}_{i-1}\left[(\Delta_i^n X')^2 \mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\leq\zeta\varepsilon\}}\right] \\
&\quad = \widehat{A}_i(\zeta\varepsilon, h)h - \widehat{A}_i(\varepsilon, h)h + o_P\left(h\varepsilon_n^{\frac{4-Y}{2}}\right).
\end{aligned} \tag{B.15}$$

Proof. We use the same notation as in (B.6). From the expansion in Proposition 2 of the accompanying supplemental material to this article, we have that

$$\mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i|\leq\zeta\varepsilon\}}\right] = \widehat{A}_i(\zeta\varepsilon, h)h - \widehat{A}_i(\varepsilon, h)h + o_P\left(h\varepsilon_n^{\frac{4-Y}{2}}\right),$$

since all higher-order terms $h^3\varepsilon^{-Y-2}$, $h^2\varepsilon^{2-2Y}$, $h^{\frac{3}{2}}\varepsilon^{1-\frac{Y}{2}}$, and $h\varepsilon^{2-\bar{\delta}}$ are all $o(h\varepsilon^{\frac{4-Y}{2}})$, for $\bar{\delta}$ small enough. Therefore, it suffices to show that

$$\begin{aligned}
\overline{\mathcal{R}}_i &:= \mathbb{E}_{i-1}\left[(\Delta_i^n X')^2 \mathbf{1}_{\{\varepsilon\leq|\Delta_i^n X'|\leq\zeta\varepsilon\}}\right] - \mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon\leq|x_i|\leq\zeta\varepsilon\}}\right] \\
&= o_P\left(h\varepsilon_n^{\frac{4-Y}{2}}\right).
\end{aligned} \tag{B.16}$$

We have the decomposition:

$$\begin{aligned}
|\overline{\mathcal{R}}_i| &\leq 2\mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i+\mathcal{E}_i|\leq\zeta\varepsilon, |x_i|>\zeta\varepsilon\}}\right] + 2\mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i+\mathcal{E}_i|\leq\zeta\varepsilon, |x_i|<\varepsilon\}}\right] \\
&\quad + 4\mathbb{E}_{i-1}\left[\mathcal{E}_i^2\right] + \mathbb{E}_{i-1}\left[|x_i\mathcal{E}_i| \mathbf{1}_{\{\varepsilon<|x_i+\mathcal{E}_i|\leq\zeta\varepsilon, \varepsilon<|x_i|\leq\zeta\varepsilon\}}\right] \\
&\quad + \mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i|\leq\zeta\varepsilon, |x_i+\mathcal{E}_i|>\zeta\varepsilon\}}\right] + 2\mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i|\leq\zeta\varepsilon, |x_i+\mathcal{E}_i|<\varepsilon\}}\right] \\
&=: \sum_{\ell=1}^{6} \overline{\mathcal{R}}_{i,\ell}.
\end{aligned}$$

The term $\overline{\mathcal{R}}_{i,3}$ is clearly $O_P(h^2)$ and hence, $o_P(h\varepsilon^{(4-Y)/2})$. For $\overline{\mathcal{R}}_{i,4}$, by Cauchy's inequality, the expansion (2.7) in the Proposition 2 of the accompanying supplemental material to this article, and (B.7),

$$\begin{aligned}
\overline{\mathcal{R}}_{i,4} \leq 2\mathbb{E}_{i-1}\left[|x_i\mathcal{E}_i| \mathbf{1}_{\{\varepsilon<|x_i|\leq\zeta\varepsilon\}}\right] &\leq C\mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i|\leq\zeta\varepsilon\}}\right]^{\frac{1}{2}} \mathbb{E}_{i-1}\left[|\mathcal{E}_i|^2\right]^{\frac{1}{2}} \\
&\leq C\left(h\varepsilon^{2-Y}\right)^{\frac{1}{2}}(h^2)^{\frac{1}{2}} = h^{\frac{3}{2}}\varepsilon^{\frac{2-Y}{2}},
\end{aligned}$$

which is $o_P(h\varepsilon^{(4-Y)/2})$ since $h^{\frac{1}{2}} \ll \varepsilon$. The proofs of the remaining terms are similar. We give only one of those for simplicity. Consider $\overline{\mathcal{R}}_{i,1}$. We decompose it as

$$\begin{aligned}
(1/2)\overline{\mathcal{R}}_{i,1} &\leq \mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\zeta\varepsilon<|x_i|\leq\zeta\varepsilon+\delta\}}\right] + \mathbb{E}_{i-1}\left[x_i^2 \mathbf{1}_{\{\varepsilon<|x_i+\mathcal{E}_i|\leq\zeta\varepsilon, |x_i|>\zeta\varepsilon+\delta\}}\right] \\
&=: \overline{\mathcal{D}}_{i,1} + \overline{\mathcal{D}}_{i,2}.
\end{aligned}$$

By the expansion (2.7) in Proposition 2.2 and our assumptions, we have

$$\begin{aligned}
\overline{\mathcal{D}}_{i,1} &= Ch[(\zeta\varepsilon)^{2-Y} - (\zeta\varepsilon-\delta)^{2-Y}] + C'h^2[(\zeta\varepsilon)^{-Y} - (\zeta\varepsilon-\delta)^{-Y}] + o_P\left(h\varepsilon^{\frac{4-Y}{2}}\right) \\
&= Ch\varepsilon^{1-Y}\delta + o_P(h\varepsilon^{1-Y}\delta) + o_P\left(h\varepsilon^{\frac{4-Y}{2}}\right).
\end{aligned}$$

Therefore, to obtain $\overline{\mathcal{D}}_{i,1} = o_P(h\varepsilon^{(4-Y)/2})$, we require

$$\delta \ll \varepsilon^{1+\frac{Y}{2}}. \tag{B.17}$$

For $\overline{\mathcal{D}}_{i,2}$, we follow the same analysis as in the proof of Lemma 2. Indeed, the arguments following expression (B.12) show that under the condition

$$\delta \gg h^{1/2}\varepsilon^{Y/2}, \tag{B.18}$$

we have

$$\overline{\mathcal{D}}_{i,2} = O_P\left(\delta^{-1}h^{\frac{5}{2}}\varepsilon^{1-Y}\right) + O_P(h^2\varepsilon^{2-2Y}) + o_P\left(h\varepsilon_n^{\frac{4-Y}{2}}\right). \tag{B.19}$$

Observe the condition (B.18) is consistent with (B.17) since $h^{1/2}\varepsilon^{Y/2} \ll \varepsilon^{1+Y/2} \iff h^{1/2} \ll \varepsilon$. For the first term on the right-hand side of (B.19) to be $o_P(h\varepsilon^{(4-Y)/2})$, we require $h^{\frac{3}{2}}\varepsilon^{-1-\frac{Y}{2}} \ll \delta$, which is consistent with the condition (B.17) since $h^{\frac{3}{2}}\varepsilon^{-1-\frac{Y}{2}} \ll \varepsilon^{1+\frac{Y}{2}}$ under $\varepsilon \gg h^{\frac{3}{2(2+Y)}}$. Therefore taking any $\delta \to 0$ such that $(h^{1/2}\varepsilon \ \vee \ h^{\frac{3}{2}}\varepsilon^{-1-\frac{Y}{2}}) \ll \delta \ll \varepsilon^{1+Y/2}$, we obtain $\overline{\mathcal{D}}_{i,2} = o_P\left(h\varepsilon_n^{\frac{4-Y}{2}}\right)$, which establishes (B.16) and completes the proof. □

Lemma 4. *Suppose that $\sqrt{h} \ll \varepsilon \ll h^{\frac{1}{4-Y}}$ and $Y \in (0,1) \cup (1, 8/5)$. Then, for any $k \geq 2$,*

$$\mathbb{E}_{i-1}\left[\left(\Delta_i^n X'\right)^{2k} \mathbf{1}_{\{|\Delta_i^n X'| \leq \varepsilon\}}\right] \tag{B.20}$$
$$= (2k-1)!!\,\sigma_{t_{i-1}}^{2k} h^k + \frac{\bar{C}|\chi_{t_{i-1}}|^Y}{2k-Y} h\varepsilon^{2k-Y} + o_P\left(h\varepsilon^{2k-Y}\right) + O_P\left(h^{k+\frac{1}{2}}\right).$$

Proof. We use the same notation as in (B.6). The proof is similar to that of Lemma 2. From the expansion of Proposition 3 in the accompanying supplemental material to this article, under our assumptions, we have that

$$\mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{|x_i| \leq \varepsilon\}}\right] = d_1\sigma_{t_{i-1}}^{2k} h^k + d_2 h\varepsilon^{2k-Y} + o_P\left(h\varepsilon^{2k-Y}\right), \tag{B.21}$$

where $d_1 = (2k-1)!!$ and $d_2 = \frac{\bar{C}}{2k-Y}|\chi_{t_{i-1}}|^Y$. Therefore, for (B.20) to hold, it suffices to show that

$$\widetilde{\mathcal{R}}_i := \mathbb{E}_{i-1}\left[(\Delta_i^n X)^{2k}\mathbf{1}_{\{|\Delta_i^n X| \leq \varepsilon\}}\right] - \mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{|x_i| \leq \varepsilon\}}\right]$$
$$= o_P\left(h\varepsilon^{2k-Y}\right). \tag{B.22}$$

Consider the decomposition:

$$|\widetilde{\mathcal{R}}_i| \leq C\mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{|x_i+\mathcal{E}_i| \leq \varepsilon < |x_i|\}}\right] + C\mathbb{E}_{i-1}\left[\mathcal{E}_i^{2k}\mathbf{1}_{\{|x_i+\mathcal{E}_i| \leq \varepsilon < |x_i|\}}\right]$$
$$+ \sum_{\ell=0}^{2k-1}\binom{2k}{\ell}\mathbb{E}_{i-1}\left[\left|x_i^{\ell}\mathcal{E}_i^{2k-\ell}\right|\mathbf{1}_{\{|x_i+\mathcal{E}_i| \leq \varepsilon, |x_i| \leq \varepsilon\}}\right]$$
$$+ \mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{|x_i| \leq \varepsilon < |x_i+\mathcal{E}_i|\}}\right] = \sum_{m=1}^{4}\widetilde{\mathcal{R}}_{i,m}.$$

By (B.7), the term $\widetilde{\mathcal{R}}_{i,2} = O_P(h^{1+k})$ and, thus, is $o_P(h\varepsilon^{2k-Y})$. In light of (B.21), the ℓ–th summand appearing in $\widetilde{\mathcal{R}}_{i,3}$ is bounded by a constant times

$$\mathbb{E}_{i-1}\left[\left|x_i^{\ell}\mathcal{E}_i^{2k-\ell}\right|\mathbf{1}_{\{|x_i| \leq \varepsilon\}}\right] \leq \mathbb{E}_{i-1}\left[|x_i|^{2\ell}\mathbf{1}_{\{|x_i| \leq \varepsilon\}}\right]^{\frac{1}{2}}\mathbb{E}_{i-1}\left[|\mathcal{E}_i|^{4k-2\ell}\right]^{\frac{1}{2}}$$
$$\leq \left(O_P(h^{\frac{\ell}{2}}) + O_P(h^{\frac{1}{2}}\varepsilon^{\frac{2\ell-Y}{2}})\right)(h^{1+2k-\ell})^{\frac{1}{2}}$$
$$= O_P\left(h^{k+\frac{1}{2}}\right) + O_P\left(h^{1+k-\frac{\ell}{2}}\varepsilon^{\ell-\frac{Y}{2}}\right).$$

The second term above is $o_P(h\varepsilon^{2k-Y})$ when $\varepsilon \gg \sqrt{h}$.

It remains to analyze $\widetilde{\mathcal{R}}_{i,1}$ and $\widetilde{\mathcal{R}}_{i,4}$. The proof is similar in both cases and we only give the details for the second case to save space. For some $\delta \to 0$ ($0 < \delta < \varepsilon$), whose precise asymptotic behavior will be determined below, we consider the decomposition:

$$\widetilde{\mathcal{R}}_{i,4} \leq \mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{\varepsilon-\delta < |x_i| \leq \varepsilon\}}\right] + \mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{|x_i| \leq \varepsilon-\delta, \varepsilon \leq |x_i+\mathcal{E}_i|\}}\right] =: \widetilde{\mathcal{D}}_{i,1} + \widetilde{\mathcal{D}}_{i,2}.$$

By the expansion (13) of Proposition 3 in the accompanying supplemental material to this article, we have

$$\widetilde{\mathcal{D}}_{i,1} = Ch[\varepsilon^{2k-Y} - (\varepsilon-\delta)^{2k-Y}] + o_P\left(h\varepsilon^{2k-Y}\right)$$
$$= O_P(h\varepsilon^{2k-1-Y}\delta) + o_P\left(h\varepsilon^{2k-Y}\right).$$

Thus, to obtain $\widetilde{\mathcal{D}}_{i,1} = o_P\left(h\varepsilon^{2k-Y}\right)$, we require

$$\delta \ll \varepsilon. \tag{B.23}$$

As in the proof of Lemma 2, when dealing with $\mathcal{D}_{i,2}$, it suffices to analyze the term:

$$\mathbb{E}_{i-1}\left[x_i^{2k}\mathbf{1}_{\{\frac{\varepsilon}{4} < |x_{i,3}| | \leq C\varepsilon, |\mathcal{E}_i| > \delta\}}\right] \leq C\sum_{\ell=1}^{3}\mathbb{E}_{i-1}\left[x_{i,\ell}^{2k}\mathbf{1}_{\{\frac{\varepsilon}{4} < |x_{i,3}| | \leq C\varepsilon, |\mathcal{E}_i| > \delta\}}\right]$$
$$=: \sum_{\ell=1}^{3}\widetilde{\mathcal{V}}_{i,\ell}.$$

Clearly, $\widetilde{\mathcal{V}}_{i,1} = O_P(h^{2k}) = o_P(h\varepsilon^{2k-Y})$. For $\widetilde{\mathcal{V}}_{i,2}$, by Hölder's inequality, for any $p, q > 1$ and $r > Y$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$:

$$\widetilde{\mathcal{V}}_{i,2} \leq C\frac{1}{\delta}\mathbb{E}_{i-1}\left[(\Delta_i^n W)^{2k}\mathbf{1}_{\{\frac{\varepsilon}{4} < |x_{i,3}|\}}|\mathcal{E}_i|\right]$$

$$\leq C\frac{1}{\delta}\mathbb{E}_{i-1}\left[(\Delta_i^n W)^{2kp}\right]^{\frac{1}{p}}\mathbb{P}_{i-1}\left[\frac{\varepsilon}{4}<|x_{i,3}|\right]^{\frac{1}{q}}\mathbb{E}_{i-1}\left[|\mathcal{E}_i|^r\right]^{\frac{1}{r}}$$
$$\leq C\frac{1}{\delta}h^k(h\varepsilon^{-Y})^{\frac{1}{q}}(h^{1+\frac{r}{2}})^{\frac{1}{r}}=\frac{C}{\delta}h^{k+\frac{3}{2}-\frac{1}{p}}\varepsilon^{-\frac{Y}{q}}.$$

For the above to be smaller than $h\varepsilon^{2k-Y}$, we need $\delta \gg h^{k+\frac{1}{2}-\frac{1}{p}}\varepsilon^{Y-2k-\frac{Y}{q}}$. For δ to be consistent with (B.23), we need that $h^{k+\frac{1}{2}-\frac{1}{p}}\varepsilon^{Y-2k-\frac{Y}{q}-1} \ll 1$. This is always possible if we take q close to 1, $p \nearrow \infty$, and $r \nearrow \infty$ because $h^{k+\frac{1}{2}}\varepsilon^{-2k-1} \ll 1$ under our condition $\varepsilon \gg \sqrt{h}$.

It remains to analyze $\widetilde{\mathcal{V}}_{i,3}$. Under the additional constraint

$$h^{\frac{1}{2}}\varepsilon^{Y/2} \ll \delta, \tag{B.24}$$

Lemma 4 in the accompanying supplemental material to this article implies

$$\widetilde{\mathcal{V}}_{i,3} \leq \mathbb{E}_{i-1}\left[(\Delta_i^n J^\infty)^{2k}\mathbf{1}_{\{|\Delta_i^n J^\infty|\leq C\varepsilon\}}\mathbf{1}_{\{|\mathcal{E}_i|>\delta_n\}}\right]=O(h^2\varepsilon^{2k-2Y}),$$

which implies $\widetilde{\mathcal{V}}_{i,3}=o_P(h\varepsilon^{2k-Y})$ since $h^2\varepsilon^{2k-2Y} \ll h\varepsilon^{2k-Y}$ under our condition $\varepsilon \gg h^{1/2}$. The conditions (B.23) and (B.24) are consistent since $h^{\frac{1}{2}}\varepsilon^{Y/2} \ll \varepsilon$, under our conditions. Thus, we conclude that (B.22) holds, which completes the proof. □

Lemma 5. *Provided $h_n^{\frac{1}{2}-s} \ll \varepsilon_n \ll h_n^{\frac{1}{4-Y}}$ for some $s \in (0, 1/2)$, and $r_0 \vee r_1 \in (0, Y \wedge 1)$, for any $\alpha \geq 1$,*

$$\left|\left(\Delta_i^n X\right)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon\}}-(\Delta_i^n X')^{2k}\mathbf{1}_{\{|\Delta_i^n X'|\leq\varepsilon\}}\right|$$
$$=O_P(h\varepsilon^{2k-r_1})+O_P(h\varepsilon^{2k-\alpha r_0})+O_P(h^2\varepsilon^{2k-Y-r_1\alpha})+O_P(h\varepsilon^{2k-Y-1+\alpha}). \tag{B.25}$$

In particular, when $h_n^{\frac{4}{8+Y}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-Y}}$, with r_0, r_1 as in Assumption 2,

$$\left|\left(\Delta_i^n X\right)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon\}}-(\Delta_i^n X')^{2k}\mathbf{1}_{\{|\Delta_i^n X'|\leq\varepsilon\}}\right|=o_P(h\varepsilon^{2-Y/2}), \tag{B.26}$$

and the estimates (B.5), (B.15) and (B.20) hold with X in place of X'.

Proof. Let $\mathcal{D}_{2k} := \left|\left(\Delta_i^n X\right)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon\}}-(\Delta_i^n X')^{2k}\mathbf{1}_{\{|\Delta_i^n X'|\leq\varepsilon\}}\right|$ and $r=r_0\vee r_1$. Let us recall the definition of J^0 and X' in (B.2) and (B.3), respectively. Write $V_t = X_t - X'_t = X_t^{j,0} + \int_0^t \chi_s dJ_s^0$, and set

$$X_t^{j,0}=\int_0^t\int\delta_0(s,z)\mathfrak{p}_0(ds,dz)+\int_0^t\int\delta_1(s,z)\mathfrak{p}_1(ds,dz)=:Y_t^0+Y_t^1.$$

For any fixed integer $k \geq 1$, we have

$$\begin{aligned}\mathcal{D}_{2k}&=\left|(\Delta_i^n X)^{2k}-(\Delta_i^n X')^{2k}\right|\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon,\ |\Delta_i^n X'|\leq\varepsilon\}}\\&\quad+(\Delta_i^n X)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon,\ |\Delta_i^n X'|>\varepsilon\}}\\&\quad+(\Delta_i^n X')^{2k}\mathbf{1}_{\{|\Delta_i^n X|>\varepsilon,\ |\Delta_i^n X'|\leq\varepsilon\}}\\&=:T_1+T_2+T_3.\end{aligned}$$

A Taylor expansion of $(x'+v)^{2k}$ at $v=0$ gives $|(x'+v)^{2k}-(x')^{2k}| \leq K|v|\left(|x'|^{2k-1}+|v|^{2k-1}\right)$, so

$$T_1 \leq K\left(|\Delta_i^n V|^{2k}+|\Delta_i^n V||\Delta_i^n X'|^{2k-1}\right)\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon,\ |\Delta_i^n X'|\leq\varepsilon\}}. \tag{B.27}$$

Note that Corollary 2.1.9 in [15] implies that for each $p \geq 1$,

$$\mathbb{E}_{i-1}\left(\frac{|\Delta_i^n Y^m|}{\varepsilon}\wedge 1\right)^p \leq Kh\varepsilon^{-r_m}, \quad m=0,1. \tag{B.28}$$

Also, since J^0 is compound Poisson, writing $N^0(ds,dx)$ for the jump measure of J^0,

$$\mathbb{E}_{i-1}\left(\left|\frac{1}{\varepsilon}\int_{t_{i-1}}^{t_i}\chi_s dJ_s^0\right|\wedge 1\right)^p \leq \mathbb{P}\left(N^0([t_{i-1},t_i],\mathbb{R})>0\right) \leq Kh.$$

Thus we obtain $\mathbb{E}_{i-1}\left(\left|\Delta_i^n V\right|^p\mathbf{1}_{\{|\Delta_i^n V|\leq\varepsilon\}}\right) \leq Kh\varepsilon^{p-r}$. Therefore, since $|x+v| \leq \varepsilon$ and $|x| \leq \varepsilon$ imply $|v| \leq 2\varepsilon$, from (B.27), we have

$$\begin{aligned}\mathbb{E}_{i-1}T_1&\leq K\left(\mathbb{E}_{i-1}(\Delta_i^n V)^{2k}\mathbf{1}_{\{|\Delta_i^n V|\leq 2\varepsilon\}}+\varepsilon^{2k-1}\mathbb{E}_{i-1}|\Delta_i^n V|\mathbf{1}_{\{|\Delta_i^n V|\leq 2\varepsilon\}}\right)\\&\leq Kh\varepsilon^{2k-r}.\end{aligned}$$

On T_2, write

$$\begin{aligned}T_2&=(\Delta_i^n X)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon,\ |\Delta_i^n X'|>\varepsilon+\varepsilon^\alpha\}}+(\Delta_i^n X)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon<|\Delta_i^n X'|\leq\varepsilon+\varepsilon^\alpha\}}\\&=:T_2'+T_2'',\end{aligned} \tag{B.29}$$

where $\alpha \geq 1$ is to be later chosen. Using that $|x' + v| \leq \varepsilon$ and $|x'| > \varepsilon + \varepsilon^{\alpha}$ imply $|v| > \varepsilon^{\alpha}$, we get

$$\begin{aligned}\mathbb{E}_{i-1}T_2' &\leq K\varepsilon^{2k}\mathbb{P}_{i-1}\left(|\Delta_i^n V| > \varepsilon^{\alpha}, |\Delta_i^n X'| > \varepsilon + \varepsilon^{\alpha}\right)\\ &\leq K\varepsilon^{2k}\Big(\ \mathbb{P}_{i-1}\left(|\Delta_i^n(V - Y^1)| > \varepsilon^{\alpha}/2\right)\\ &\qquad + \mathbb{P}_{i-1}\left(|\Delta_i^n Y^1| > \varepsilon^{\alpha}/2\right)\mathbb{P}\left(|\Delta_i^n X'| > \varepsilon + \varepsilon^{\alpha}\right)\Big)\\ &\leq K\varepsilon^{2k}\left(\varepsilon^{-\alpha r_0}h + h^2\varepsilon^{-Y-r_1\alpha}\right),\end{aligned}$$

where we used that $\mathbb{P}(|\Delta X'| > \varepsilon) \leq Kh\varepsilon^{-Y}$ as a consequence of Lemma 6 and that $\mathbb{P}_{i-1}\left(|\Delta_i^n Y^1| > u\right) \leq \mathbb{E}_{i-1}\left(\frac{|\Delta_i^n Y^1|}{u} \wedge 1\right) \leq hu^{-r_1}$ per (B.28).

On the other hand, for T_2'', since $|x' + v| \leq \varepsilon$ and $|x'| \leq \varepsilon + \varepsilon^{\alpha}$ give $|v| \leq 2\varepsilon + \varepsilon^{\alpha}$,

$$\begin{aligned}\mathbb{E}_{i-1}T_2'' &= \mathbb{E}_{i-1}(\Delta_i^n X)^{2k}\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon<|\Delta_i^n X'|\leq\varepsilon+\varepsilon^{\alpha}\}}\\ &\leq K\mathbb{E}_{i-1}\left[\left((\Delta_i^n X')^{2k} + (\Delta_i^n V)^{2k}\right)\mathbf{1}_{\{|\Delta_i^n X|\leq\varepsilon<|\Delta_i^n X'|\leq\varepsilon+\varepsilon^{\alpha}\}}\right]\\ &\leq \mathbb{E}_{i-1}(\Delta_i^n X')^{2k}\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\leq\varepsilon+\varepsilon^{\alpha}\}} + O_P(\varepsilon^{2k-r}h).\end{aligned}$$

Since $\alpha \geq 1$, arguing as in (B.11), by applying Lemma 4, we obtain

$$\begin{aligned}\mathbb{E}_{i-1}(\Delta_i^n X')^{2k}\mathbf{1}_{\{\varepsilon<|\Delta_i^n X'|\leq\varepsilon+\varepsilon^{\alpha}\}} &= O_P\left(h\left((\varepsilon+\varepsilon^{\alpha})^{2k-Y} - \varepsilon^{2k-Y}\right)\right)\\ &= O_P(h\varepsilon^{2k-1-Y+\alpha}).\end{aligned}$$

Now, turning to T_3, write

$$\begin{aligned}T_3 &= (\Delta_i^n X')^{2k}\mathbf{1}_{\{|\Delta_i^n X|>\varepsilon,\ |\Delta_i^n X'|\leq\varepsilon-\varepsilon^{\alpha}\}} + (\Delta_i^n X')^2\mathbf{1}_{\{\varepsilon-\varepsilon^{\alpha}<|\Delta_i^n X'|\leq\varepsilon\}}\\ &= T_3' + T_3''.\end{aligned}$$

Since $|x' + v| > \varepsilon$ and $|x'| \leq \varepsilon - \varepsilon^{\alpha}$ imply $|v| > \varepsilon^{\alpha}$, the same arguments for T_2' and T_2'' apply to T_3', T_3'', giving (B.25).

To obtain (B.26) under $h_n^{\frac{4}{8+Y}} \ll \varepsilon_n \ll h_n^{\frac{1}{4-Y}}$, consider first $k = 1$. In this case, we may simultaneously satisfy each of $\varepsilon^{2-\alpha r_0}h \ll h\varepsilon^{2-\frac{Y}{2}}$, ($\iff \alpha < \frac{Y}{2r_0}$), $h^2\varepsilon^{2-Y-r_1\alpha} \ll h\varepsilon^{2-\frac{Y}{2}}$ ($\iff h^{\frac{2}{Y+2\alpha r_1}} \ll \varepsilon \impliedby \varepsilon \gg h^{\frac{4}{8+Y}}$ if $\alpha < \frac{16-2Y}{8r_1}$), and $h\varepsilon^{1-Y+\alpha} \ll h\varepsilon^{2-\frac{Y}{2}}$ ($\iff \alpha > 1 + Y/2$) provided α is chosen such that

$$1 + \frac{Y}{2} < \alpha < \left(\frac{Y}{2r_0}\right) \wedge \left(\frac{16-2Y}{8r_1}\right),$$

which is always possible under the restrictions on r_0, r_1 in Assumption 2. Since $r_1 \leq Y/2$, we also have $\varepsilon^{2-r_1} \ll \varepsilon^{2-\frac{Y}{2}}$, concluding (B.26) for $k = 1$. When $k \geq 2$, $\varepsilon^{2k-\alpha r_0}h \ll h\varepsilon^{2-\frac{Y}{2}}$, $h^2\varepsilon^{2k-Y-r_1\alpha} \ll h\varepsilon^{2-\frac{Y}{2}}$, and $h\varepsilon^{2k-1-Y+\alpha} \ll h\varepsilon^{2-\frac{Y}{2}}$ all hold by taking $\alpha = 1$.

Finally, since $\varepsilon \ll h^{\frac{1}{4-Y}}$ implies $h\varepsilon^{2-Y/2} \ll h^{3/2}$, expression (B.26) implies each of (B.5), (B.15) and (B.20) hold with X in place of X'. □

Lemma 6. *Suppose $0 < Y < 2$, and $\varepsilon \to 0$ with $\varepsilon \gg h^{1/Y}$. For all $p > 1 \vee Y$, the following estimates hold:*

$$\mathbb{E}\left(\frac{|\Delta_i^n J|}{\varepsilon} \wedge 1\right)^p \leq Kh\varepsilon^{-Y}, \quad \mathbb{E}_{i-1}\left(\frac{|\Delta_i^n X|}{\varepsilon} \wedge 1\right)^p \leq K\left(h\varepsilon^{-Y} + h^{p/2}\varepsilon^{-p}\right). \tag{B.30}$$

In particular, with $V = J$ or $V = X$, $\mathbb{P}_{i-1}\left[|\Delta_i^n V| > \varepsilon\right] \leq h\varepsilon^{-Y}$.

Proof. We only give the proof for $|\Delta_i^n X|$ since the proof for other term is similar. It suffices to establish the bound for $\int_{t_{i-1}}^{t_i} b_t dt$, $\int_{t_{i-1}}^{t_i} \sigma_t dW_t$ and $\Delta_i^n X^j$ separately. It holds immediately for $\int_{t_{i-1}}^{t_i} b_t dt$, since $\varepsilon^{-p}\left|\int_{t_{i-1}}^{t_i} b_t dt\right|^p \leq K\varepsilon^{-p}h^p \ll \varepsilon^{-p}h^{p/2}$. for the second term, Burkholder–Davis–Gundy inequality gives $\varepsilon^{-p}\mathbb{E}_{i-1}\left|\int_{t_{i-1}}^{t_i} \sigma_t dW_t\right|^p \leq K_p(h^{1/2}\varepsilon^{-1})^p$. For $\Delta_i^n X^j$, we first consider $\Delta_i^n X^{j,\infty}$ as in (6). Applying Corollary 2.1.9(a)[8] in, we obtain

$$\mathbb{E}_{i-1}\left(\frac{|\Delta_i^n X^{j,\infty}|}{\varepsilon} \wedge 1\right)^p \leq Kh\varepsilon^{-Y}.$$

Applying Corollary 2.1.9(c) in [15], we have

$$\mathbb{E}_{i-1}\left(\frac{|\Delta_i^n X^{j,0}|}{\varepsilon} \wedge 1\right)^p \leq Kh\varepsilon^{-r},$$

[8] Strictly speaking, this corollary assumes $\varepsilon = h^q$ for some $q \in (0, 1/Y)$, though a straightforward adaptation shows it holds provided $h^{1/Y} \ll \varepsilon \ll 1$.

where $r = r_0 \vee r_1$, which gives

$$\mathbb{E}_{i-1}\left(\frac{|\Delta_i^n X^j|}{\varepsilon} \wedge 1\right)^p \leq K h \varepsilon^{-Y}.$$

The remaining statements follow since

$$\mathbb{P}_{i-1}\left[|\Delta_i^n V| > \varepsilon\right] \leq \mathbb{E}_{i-1}\left(\frac{|\Delta_i^n V|}{\varepsilon} \wedge 1\right)^p,$$

for every $p > 0$, and $h^{p/2}\varepsilon^{-p} \ll h\varepsilon^{-Y}$ for p large enough. This completes the proof. □

Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spa.2024.104429.

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