Journal of Machine Learning Research 26 (2025) 1-51 Submitted 6/23; Revised 11/24; Published 4/25

Random Pruning Over-parameterized Neural Networks Can Improve Generalization: A Training Dynamics Analysis

Hongru Yang HY6385@UTEXAS.EDU
Department of Computer Science
The University of Texas at Austin
Austin, TX 78712, USA

Yingbin Liang LIANG.889@OSU.EDU
Department of Electrical and Computer Engineering
The Ohio State University
Columbus, OH 43210, USA

Xiaojie Guo XIAOJIE.GUO@IBM.COM
IBM Thomas J. Watson Research Center
Yorktown Heights, NY 10598, USA

Lingfei Wu LWU@ANYTIME-AI.COM
Anytime AI
White Plains, NY 10601, USA

Zhangyang Wang ATLASWANG@UTEXAS.EDU
Department of Electrical and Computer Engineering
The University of Texas at Austin
Austin, TX 78712, USA

Editor: Dan Alistarh

Abstract

It has been observed that applying pruning-at-initialization methods and training the sparse networks can sometimes yield slightly better test performance than training the original dense network. Such experimental observations are yet to be understood theoretically. This work makes the first attempt to study this phenomenon. Specifically, we identify a theoretical minimal setting and study a classification task with a one-hidden-layer neural network, which is randomly pruned according to different rates at the initialization. We show that as long as the pruning rate is below a certain threshold, the network provably exhibits good generalization performance after training. More surprisingly, the generalization bound gets better as the pruning rate mildly gets larger. To complement this positive result, we also show a negative result: there exists a large pruning rate such that while gradient descent is still able to drive the training loss toward zero, the generalization performance is no better than random guessing. This further suggests that pruning can change the feature learning process, which leads to the performance drop of the pruned neural network. To our knowledge, this is the first theory work studying how different pruning rates affect neural networks' performance, suggesting that an appropriate pruning rate might improve the neural network's generalization.

Keywords: pruning, sparsity, gradient descent, convergence, generalization

1. Introduction

Neural network pruning can be dated back to the early stage of the development of neural networks (LeCun et al., 1989). Since then, many research works have been focusing on using neural network pruning as a model compression technique, e.g. (Molchanov et al., 2019; Luo and Wu, 2017; Ye et al., 2020; Yang et al., 2021). Many of the early pruning literature focused on pruning neural networks after training to reduce inference time. On the other hand, nowadays, training models has become more and more expensive since people are training and deploying larger and larger models with tens of billions of parameters. The idea of applying pruning to reduce the cost of *training* has been catching people's attention: if we can train a sparse model which can achieve similar performance with a dense model, then the cost of training can be significantly reduced. It is not until recently that Frankle and Carbin (2018) showed a surprising phenomenon: a neural network pruned at the initialization can be trained to achieve competitive performance to the dense model. They called this phenomenon the lottery ticket hypothesis. The lottery ticket hypothesis states that there exists a sparse subnetwork inside a dense network at the random initialization stage such that when trained in isolation, it can match the test accuracy of the original dense network after training for at most the same number of iterations. On the other hand, the algorithm Frankle and Carbin (2018) proposed to find the lottery ticket requires many rounds of pruning and retraining which is computationally expensive. Many subsequent works have focused on developing new methods to reduce the cost of finding such a network at the initialization (Lee et al., 2018; Wang et al., 2019; Tanaka et al., 2020; Liu and Zenke, 2020; Chen et al., 2021b). A further investigation by Frankle et al. (2020) showed that some of these methods merely discover the layer-wise pruning ratio instead of sparsity pattern. Surprisingly, there have been empirical works showing that *random pruning* can be effective under certain cases and sometimes even produce sparse sub-networks that can perform on par as the lottery ticket sub-network (Frankle et al., 2020; Su et al., 2020; Liu et al., 2021b).

To understand the lottery ticket hypothesis, on the theory side, a line of research is focusing on finding a subnetwork inside a dense network at the random initialization such that the subnetwork can achieve good performance (Zhou et al., 2019; Ramanujan et al., 2020). In particular, Malach et al. (2020) formalized this phenomenon which they called the strong lottery ticket hypothesis: under certain assumption on the weight initialization distribution, a sufficiently overparameterized neural network at the initialization contains a subnetwork with roughly the same accuracy as the target network. Later, Pensia et al. (2020) improved the overparameterization parameters and Sreenivasan et al. (2021) showed that such a type of result holds even if the weight is binary. Unsurprisingly, as it was pointed out by Malach et al. (2020), finding such a subnetwork is computationally hard. Nonetheless, all of the analysis is from a function approximation perspective and none of the aforementioned works have considered the effect of pruning on gradient descent dynamics, let alone the neural networks' generalization. Thus, such type of analysis is far from fully explaining the success of the lottery ticket hypothesis.

Interestingly, in many empirical studies, it has been reportedly found that pruning can noticeably improve generalization in certain scenarios (Chen et al., 2021a; He et al., 2022; Jin et al., 2022). In particular, Jin et al. (2022) in their empirical work hypothesizes that pruning can (1) lead to better training (i.e., smaller training loss at the end of training) and

(2) provide additional regularization effect on the model. However, theoretical understanding of such benefit of neural network pruning is still limited. In this work, we take the first step to answer the following important open question from a theoretical perspective:

> *How does pruning fraction affect the training dynamics and the model's generalization, if the model is pruned at the initialization and trained by gradient descent?*

On the one hand, pruning methods like iterative magnitude-based pruning find sparse masks by many rounds of training and pruning and thus, the masks found by such pruning methods possess complicated relationships with the magnitude of the weights themselves, which creates theoretical hurdles for analysis. On the other hand, there have been empirical works showing that *random pruning* can be effective (Frankle et al., 2020; Su et al., 2020; Liu et al., 2021b). In this work, we identify a theoretical minimal setting where we show that different pruning rates can make the network exhibit distinct generalization behaviors *even under random pruning.* We consider a classification task where the input data consists of class-dependent sparse signal and random noise. We analyze the training dynamics of a one-hidden-layer convolutional neural network pruned at the initialization, where we offer new principled insights beyond the recent empirical observations on how pruning can improve generalization (Jin et al., 2022; He et al., 2022). Specifically, this work makes the following contributions:

- **Mild pruning.** We prove that there indeed exists a range of pruning fractions where the pruning fraction is mild and the generalization error bound gets better as the pruning fraction gets larger. In this case, the signal in the feature is well-preserved and pruning reduces the effect from noise. We provide detailed explanation in Section 3. To our knowledge, this is the first theory work studying how different pruning rates affect neural networks' performance, suggesting that mild pruning rate can improve the neural network's generalization under some setting. Further, we conduct experiments to verify our results.

- **Over pruning.** To complement the above positive result, we also show a negative result: there exists a certain range of large pruning rates such that the generalization performance of the trained network is no better than simple random guessing, although gradient descent is still able to drive the training loss toward zero. This further suggests that contrary to the common belief that the performance drop of the pruned neural network is caused by its lack of trainability or expressiveness, that can also be attributed to the change of gradient descent dynamics due to pruning.

- Technically, we develop novel analysis to bound pruning effect to weight-noise and weight-signal correlation. Further, in contrast to many previous works that considered only the binary case, our analysis handles multi-class classification with general cross-entropy loss. Here, a key technical development is a gradient upper bound for multi-class cross-entropy loss, which might be of independent interest.

Pictorially, our result is summarized in Figure 1. We point out that the neural network training we consider is in the *feature learning* regime, where the weight parameters can

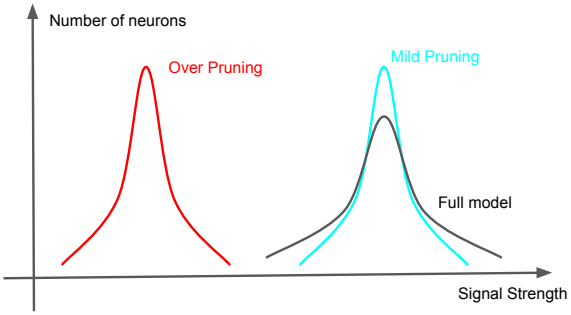


Figure 1: A pictorial demonstration of our results. The bell-shaped curves model the distribution of the signal in the features, where the mean represents the signal strength and the width of the curve indicates the variance of noise. Using the notations which we introduce later, the signal strength denotes $\langle w_{k,r} \odot m_{k,r}, \mu e_k \rangle$. Our results show that mild pruning preserves the signal strength and reduces the noise variance (and hence yields better generalization), whereas over pruning lowers signal strength albeit reducing noise variance.

go far away from their initialization. This is ***fundamentally different*** from the popular *neural tangent kernel* regime, where the neural networks essentially behave similarly to its linearization around initialization.

1.1 Related Works

The Lottery Ticket Hypothesis and Sparse Training. The discovery of the lottery ticket hypothesis (Frankle and Carbin, 2018) has inspired further investigation and applications. One line of research has focused on developing computationally efficient methods to enable sparse training: the static sparse training methods are aiming at identifying a sparse mask at the initialization stage based on different criterion such as SNIP (loss-based) (Lee et al., 2018), GraSP (gradient-based) (Wang et al., 2019), SynFlow (synaptic strength-based) (Tanaka et al., 2020), neural tangent kernel based method (Liu and Zenke, 2020) and one-shot pruning (Chen et al., 2021b). Random pruning has also been considered in static sparse training such as uniform pruning (Mariet and Sra, 2015; He et al., 2017; Gale et al., 2019; Suau et al., 2018), non-uniform pruning (Mocanu et al., 2016), expander-graph-related techniques (Prabhu et al., 2018; Kepner and Robinett, 2019) Erdös-Rényi (Mocanu et al., 2018) and Erdös-Rényi-Kernel (Evci et al., 2020). On the other hand, dynamic sparse training allows the sparse mask to be updated (Mocanu et al., 2018; Mostafa and Wang, 2019; Evci et al., 2020; Jayakumar et al., 2020; Liu et al., 2021c,d,a; Peste et al., 2021). The sparsity pattern can also be learned by using sparsity-inducing regularizer (Yang et al., 2020). Recently, He et al. (2022); Jin et al. (2022) discovered that training neural networks pruned at the initialization via iterative magnitude-based pruning can noticeably improve model's generalization when a significant portion of the training data-set labels are corrupted.

Another line of research has focused on studying pruning the neural networks at its random initialization to achieve good performance (Zhou et al., 2019; Ramanujan et al., 2020). In particular, Ramanujan et al. (2020) showed that it is possible to prune a randomly initialized wide ResNet-50 to match the performance of a ResNet-34 trained on ImageNet.

This phenomenon is named the strong lottery ticket hypothesis. Later, Malach et al. (2020) proved that under certain assumption on the initialization distribution, a target network of width d and depth l can be approximated by pruning a randomly initialized network that is of a polynomial factor (in d, l) wider and twice deeper even without any further training. However finding such a network is computationally hard, which can be shown by reducing the pruning problem to optimizing a neural network. Later, Pensia et al. (2020) improved the widening factor to being logarithmic and Sreenivasan et al. (2021) proved that with a polylogarithmic widening factor, such a result holds even if the network weight is binary. A follow-up work shows that it is possible to find a subnetwork achieving good performance at the initialization and then fine-tune (Sreenivasan et al., 2022). Our work, on the other hand, analyzes the gradient descent dynamics of a pruned neural network and its generalization after training.

Analyses of Training Neural Networks by Gradient Descent. A series of work (Allen-Zhu et al., 2019; Du et al., 2019; Lee et al., 2019; Zou et al., 2020; Zou and Gu, 2019; Ji and Telgarsky, 2019; Chen et al., 2020b; Song and Yang, 2019; Oymak and Soltanolkotabi, 2020) has proved that if a deep neural network is wide enough, then (stochastic) gradient descent provably can drive the training loss toward zero in a fast rate based on neural tangent kernel (NTK) (Jacot et al., 2018). Further, under certain assumption on the data, the learned network is able to generalize (Cao and Gu, 2019; Arora et al., 2019). However, as it is pointed out by Chizat et al. (2019), in the NTK regime, the gradient descent dynamics of the neural network essentially behaves similarly to its linearization and the learned weight is not far away from the initialization, which prohibits the network from performing any useful feature learning. In order to go beyond NTK regime, one line of research has focused on the mean field limit (Song et al., 2018; Chizat and Bach, 2018; Rotskoff and Vanden-Eijnden, 2018; Wei et al., 2019; Chen et al., 2020a; Sirignano and Spiliopoulos, 2020; Fang et al., 2021). Recently, people have started to study the neural network training dynamics in the feature learning regime where data from different class is defined by a set of class-related signals which are low rank (Allen-Zhu and Li, 2020, 2022; Cao et al., 2022; Shi et al., 2021; Telgarsky, 2022). Our work also focuses on the aforementioned feature learning regime, but for the first time characterizes the impact of pruning on the generalization performance of neural networks.

2. Preliminaries and Problem Formulation

In this section, we introduce our notation, data generation process, neural network architecture and the optimization algorithm.

Notations. We use lower case letters to denote scalars and boldface letters and symbols (e.g. $\mathbf{x}$) to denote vectors and matrices. We use $\odot$ to denote element-wise product. For an integer n, we use $[n]$ to denote the set of integers $\{1, 2, \ldots, n\}$. We use $x = O(y), x = \Omega(y), x = \Theta(y)$ to denote that there exists a constant C such that $x \leq Cy$, $x \geq Cy, x = Cy$ respectively. We use $\widetilde{O}, \widetilde{\Omega}$ and $\widetilde{\Theta}$ to hide polylogarithmic factor in these notations. Finally, we use $x = \mathrm{poly}(y)$ if $x = O(y^C)$ for some positive constant C, and $x = \mathrm{poly}\log y$ if $x = \mathrm{poly}(\log y)$.

2.1 Settings

Definition 1 (Data distribution of K classes) *Consider we are given the set of signal vectors $\{\mu \mathbf{e}_i\}_{i=1}^K$, where $\mu > 0$ denotes the strength of the signal, and $\mathbf{e}_i \in \mathbb{R}^d$ (with $d > K$) denotes the i-th standard basis vector with its i-th entry being 1 and all other coordinates being 0. Each data point $(\mathbf{x}, y)$ with $\mathbf{x} = [\mathbf{x}_1^\top, \mathbf{x}_2^\top]^\top \in \mathbb{R}^{2d}$ and $y \in [K]$ is generated from the following distribution $\mathcal{D}$:*

1. *The label y is generated from a uniform distribution over $[K]$.*
2. *A noise vector $\boldsymbol{\xi}$ is generated from the Gaussian distribution $\mathcal{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$.*
3. *With probability $1/2$, assign $\mathbf{x}_1 = \boldsymbol{\mu}_y$, $\mathbf{x}_2 = \boldsymbol{\xi}$; with probability $1/2$, assign $\mathbf{x}_2 = \boldsymbol{\mu}_y$, $\mathbf{x}_1 = \boldsymbol{\xi}$ where $\boldsymbol{\mu}_y = \mu \mathbf{e}_y$.*

The sparse signal model is motivated by the empirical observation that during the process of training neural networks, the output of each layer of ReLU is usually sparse instead of dense. This is partially due to the fact that in practice the bias term in the linear layer is used (Song et al., 2021). For samples from different classes, usually a different set of neurons fire. Our study can be seen as a formal analysis on pruning the second last layer of a deep neural network in the layer-peeled model as in Zhu et al. (2021); Zhou et al. (2022). We also point out that our assumption on the sparsity of the signal is necessary for our analysis. If we don't have this sparsity assumption and only make assumption on the ℓ_2 norm of the signal, then in the extreme case, the signal is uniformly distributed across all coordinate and the effect of pruning to the signal and the noise will be essentially the same: their ℓ_2 norm will both be reduced by a factor of $\sqrt{p}$.

Network architecture and random pruning. We consider a two-layer convolutional neural network model with polynomial ReLU activation $\sigma(z) = (\max\{0, z\})^q$, where we focus on the case when $q = 3$ [1] The network is pruned at the initialization by mask $\mathbf{M}$ where each entry in the mask $\mathbf{M}$ is generated i.i.d. from Bernoulli(p). Given the data $(\mathbf{x}, y)$, the output of the neural network can be written as $F(\mathbf{W} \odot \mathbf{M}, \mathbf{x}) = (F_1(\mathbf{W}_1 \odot \mathbf{M}_1, \mathbf{x}), F_2(\mathbf{W}_2 \odot \mathbf{M}_2, \mathbf{x}), \ldots, F_K(\mathbf{W}_K \odot \mathbf{M}_K, \mathbf{x}))$ where the j-th output is given by

$$
\begin{aligned}
F_j(\mathbf{W}_j \odot \mathbf{M}_j, \mathbf{x}) &= \sum_{r=1}^m [\sigma(\langle \mathbf{w}_{j,r} \odot \mathbf{m}_{j,r}, \mathbf{x}_1 \rangle) + \sigma(\langle \mathbf{w}_{j,r} \odot \mathbf{m}_{j,r}, \mathbf{x}_2 \rangle)] \\
&= \sum_{r=1}^m [\sigma(\langle \mathbf{w}_{j,r} \odot \mathbf{m}_{j,r}, \boldsymbol{\mu}_y \rangle) + \sigma(\langle \mathbf{w}_{j,r} \odot \mathbf{m}_{j,r}, \boldsymbol{\xi} \rangle)],
\end{aligned}
$$

where $\mathbf{m}_{j,r}$ denotes the r-th row of $\mathbf{M}_j$. The mask $\mathbf{M}$ is only sampled once at the initialization and remains fixed through the entire training process. From now on, **we use tilde over a symbol to denote its masked version, e.g., $\widetilde{\mathbf{W}} = \mathbf{W} \odot \mathbf{M}$ and $\widetilde{\mathbf{w}}_{j,r} = \mathbf{w}_{j,r} \odot \mathbf{m}_{j,r}$.**

1. We point out that as many previous works (Allen-Zhu and Li, 2020; Zou et al., 2021; Cao et al., 2022), polynomial ReLU activation can help us simplify the analysis of gradient descent, because polynomial ReLU activation can give a much larger separation of signal and noise (thus, cleaner analysis) than ReLU. Our analysis can be generalized to ReLU activation by using the arguments in (Allen-Zhu and Li, 2022).

Since we have $\boldsymbol{\mu}_j \odot \mathbf{m}_{j,r} = \mathbf{0}$, with probability $1-p$, some neurons will not receive the corresponding signal at all and will only learn noise. Therefore, for each class $j \in [K]$, we split the neurons into two sets based on whether it receives its corresponding signal or not:

$$\mathcal{S}^j_{\text{signal}} = \{r \in [m] : \boldsymbol{\mu}_j \odot \mathbf{m}_{j,r} \neq \mathbf{0}\},$$
$$\mathcal{S}^j_{\text{noise}} = \{r \in [m] : \boldsymbol{\mu}_j \odot \mathbf{m}_{j,r} = \mathbf{0}\}.$$

Gradient descent algorithm. We consider the network is trained by cross-entropy loss with softmax. We denote by $\text{logit}_i(F, \mathbf{x}) := \frac{e^{F_i(\mathbf{x})}}{\sum_{j\in[K]} e^{F_j(\mathbf{x})}}$ and the cross-entropy loss can be written as $\ell(F(\mathbf{x}, y)) = -\log \text{logit}_y(F, \mathbf{x})$. The convolutional neural network is trained by minimizing the **empirical cross-entropy loss** given by

$$L_S(\mathbf{W}) = \mathop{\mathbb{E}}_{S} \ell[F(\mathbf{W} \odot \mathbf{M}; \mathbf{x}_i, y_i)] = \frac{1}{n}\sum_{i=1}^{n} \ell[F(\mathbf{W} \odot \mathbf{M}; \mathbf{x}_i, y_i)],$$

where $S = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$ is the training data set. Similarly, we define the **generalization loss** as

$$L_{\mathcal{D}} := \mathop{\mathbb{E}}_{(\mathbf{x},y)}[\ell(F(\mathbf{W} \odot \mathbf{M}; \mathbf{x}, y))].$$

The model weights are initialized from a i.i.d. Gaussian $\mathcal{N}(0, \sigma_0^2)$. The gradient of the cross-entropy loss is given by $\ell'_{j,i} := \ell'_j(\mathbf{x}_i, y_i) = \text{logit}_j(F, \mathbf{x}_i) - \mathbb{I}(j = y_i)$.

Since

$$\begin{aligned}\nabla_{\mathbf{w}_{j,r}} L_S(\mathbf{W} \odot \mathbf{M}) &= \nabla_{\mathbf{w}_{j,r}\odot\mathbf{m}_{j,r}} L_S(\mathbf{W} \odot \mathbf{M}) \odot \mathbf{m}_{j,r} \\ &= \nabla_{\widetilde{\mathbf{w}}_{j,r}} L_S(\widetilde{\mathbf{W}}) \odot \mathbf{m}_{j,r},\end{aligned}$$

we can write the full-batch gradient descent update of the weights as

$$\begin{aligned}\widetilde{\mathbf{w}}_{j,r}^{(t+1)} &= \widetilde{\mathbf{w}}_{j,r}^{(t)} - \eta \nabla_{\widetilde{\mathbf{w}}_{j,r}} L_S(\widetilde{\mathbf{W}}) \odot \mathbf{m}_{j,r} \\ &= \widetilde{\mathbf{w}}_{j,r}^{(t)} - \frac{\eta}{n}\sum_{i=1}^{n} \ell'^{(t)}_{j,i} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right) \cdot \widetilde{\boldsymbol{\xi}}_{j,r,i} \\ &\quad - \frac{\eta}{n}\sum_{i=1}^{n} \ell'^{(t)}_{j,i} \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle\right) \boldsymbol{\mu}_{y_i} \odot \mathbf{m}_{j,r},\end{aligned}$$

for $j \in [K]$ and $r \in [m]$, where $\widetilde{\boldsymbol{\xi}}_{j,r,i} = \boldsymbol{\xi}_i \odot \mathbf{m}_{j,r}$.

Condition 2 *We consider the parameter regime described as follows: (1) Number of classes $K = O(\log d)$. (2) Total number of training samples $n = \text{poly}\log d$. (3) Dimension $d \geq C_d$ for some sufficiently large constant C_d. (4) Relationship between signal strength and noise strength: $\mu = \Theta(\sigma_n\sqrt{d}\log d) = \Theta(1)$. (5) The number of neurons in the network $m = \Omega(\text{poly}\log d)$. (6) Initialization variance: $\sigma_0 = \widetilde{\Theta}(m^{-4}n^{-1}\mu^{-1})$. (7) Learning rate: $\Omega(1/\text{poly}(d)) \leq \eta \leq \widetilde{O}(1/\mu^2)$. (8) Target training loss: $\epsilon = \Theta(1/\text{poly}(d))$.*

Conditions (1) ensures that there are not too many classes which is a mild assumption and can be satisfied by real world dataset like MNIST. Condition (2) ensure that there are enough samples in each class with high probability and at the same time not too many samples such that the noise will interfere with the neural network learning the signals. Condition (3) ensures that our setting is in high-dimensional regime. Condition (4) ensures that the full model can be trained to exhibit good generalization. Condition (5), (6) and (7) ensures that the neural network is sufficiently overparameterized and can be optimized efficiently by gradient descent. Condition (7) and (8) further ensures that training time is polynomial in d. We further discuss the practical consideration of η and ϵ to justify their condition in Theorem 23.

3. Mild Pruning

3.1 Main result

The first main result shows that there exists a threshold on the pruning fraction p such that pruning helps the neural network's generalization.

Theorem 3 (Main theorem for mild pruning, informal version of Theorem 15) *Under Condition 2, if $p \in [C_1 \frac{\log d}{m}, 1]$ for some constant C_1, then with probability at least $1 - O(d^{-1})$ over the randomness in the data, network initialization and pruning, there exists $T = \widetilde{O}(K\eta^{-1}\sigma_0^{2-q}\mu^{-q} + K^2 m^4 \mu^{-2}\eta^{-1}\epsilon^{-1})$ such that*

1. *The training loss is below ϵ: $L_S(\widetilde{\mathbf{W}}^{(T)}) \leq \epsilon$.*
2. *The generalization loss can be bounded by $L_{\mathcal{D}}(\widetilde{\mathbf{W}}^{(T)}) \leq O(K\epsilon) + \exp(-n^2/p)$.*

Theorem 3 indicates that there exists a threshold in the order of $\Theta(\frac{\log d}{m})$ such that if p is above this threshold (i.e., the fraction of the pruned weights is small), gradient descent is able to drive the training loss towards zero (as item 1 claims) and the overparameterized network achieves good testing performance (as item 2 claims): as p becomes smaller (recall that p is the probability that we keep a weight, and thus, the smaller p is, the more we prune), the generalization bound will become smaller. This implies that we can get a better generalization by pruning more. In the next subsection, we explain why pruning can help generalization, and we defer all the detailed proofs in Appendix C.

3.2 Proof Outline

Our proof establishes of the following two properties:

- First we show that after mild pruning the network is still able to learn the signal, and the magnitude of the signal in the feature is preserved.
- Then we show that given a new sample, pruning reduces the noise effect in the feature which leads to the improvement of generalization.

We first present our analysis for three stages of gradient descent: initialization, feature growing phase, and converging phase, and then establish the generalization property.

Initialization. First of all, readers might wonder why pruning can even preserve signal at all. Intuitively, a network will achieve good performance if its weights are highly correlated with the signal (i.e., their inner product is large). Two intuitive but misleading heuristics are given by the following:

- Consider a fixed neuron weight. At the random initialization, in expectation, the signal correlation with the weights is given by $\mathbb{E}_{\mathbf{w},\mathbf{m}}[|\langle \mathbf{w}\odot\mathbf{m}, \boldsymbol{\mu}\rangle|] \le p\sigma_0\mu$ and the noise correlation with the weights is given by $\mathbb{E}_{\mathbf{w},\mathbf{m},\boldsymbol{\xi}}[|\langle \mathbf{w}\odot\mathbf{m}, \boldsymbol{\xi}\rangle|] \le \sqrt{\mathbb{E}_{\mathbf{w},\mathbf{m},\boldsymbol{\xi}}[\langle \mathbf{w}\odot\mathbf{m}, \boldsymbol{\xi}\rangle^2]} = \sigma_0\sigma_n\sqrt{pd}$ by Jensen's inequality. Based on this argument, taking a sum over all the neurons, pruning will hurt weight-signal correlation more than weight-noise correlation.

- Since we are pruning with Bernoulli(p), a given neuron will not receive signal at all with probability $1-p$. Thus, there is roughly p fraction of the neurons receiving the signal and the rest $1-p$ fraction will be purely learning from noise. Even though for every neuron, roughly $\sqrt{p}$ portion of ℓ_2 mass from the noise is reduced, at the same time, pruning also creates $1-p$ fraction of neurons which do not receive signals at all and will purely output noise after training. Summing up the contributions from every neuron, the signal strength is reduced by a factor of p while the noise strength is reduced by a factor of $\sqrt{p}$. We again reach the conclusion of pruning under any rate will hurt the signal more than noise.

The above analysis shows that under any pruning rate, it seems pruning can only hurt the signal more than noise at the initialization. Such analysis would be indicative if the network training is under the *neural tangent kernel regime*, where the weight of each neuron does not travel far from its initialization so that the above analysis can still hold approximately after training. However, when the neural network training is in the *feature learning regime*, this average type analysis becomes misleading. Namely, in such a regime, the weights with large correlation with the signal at the initialization will quickly evolve into singleton neurons and those weights with small correlation will remain small. In our proof, we focus on the *featuring learning regime*, and analyze how the network weights change and what are the effect of pruning during various stages of gradient descent.

We now analyze the effect of pruning on weight-signal correlation and weight-noise correlation at the initialization. Our first lemma leverages the sparsity of our signal and shows that if the pruning is mild, then it will not hurt the maximum weight-signal correlation much at the initialization. On the other hand, the maximum weight-noise correlation is reduced by a factor of $\sqrt{p}$.

Lemma 4 (Initialization, same as Theorem 21) *With probability at least $1-2/d$, for all $i\in[n]$,*

$$\sigma_0\sigma_n\sqrt{pd} \le \max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle \le \sqrt{2\log(Kmd)}\sigma_0\sigma_n\sqrt{pd}.$$

Further, suppose $pm \ge \Omega(\log(Kd))$, with probability $1-2/d$, for all $j\in[K]$,

$$\sigma_0 \left\|\boldsymbol{\mu}_j\right\|_2 \le \max_{r\in\mathcal{S}_{\text{signal}}^j} \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle \le \sqrt{2\log(8pmKd)}\sigma_0 \left\|\boldsymbol{\mu}_j\right\|_2.$$

Given this lemma, we now prove that there exists at least one neuron that is heavily aligned with the signal after training. Similarly to previous works (Allen-Zhu and Li, 2020; Zou et al., 2021; Cao et al., 2022), the analysis is divided into two phases: feature growing phase and converging phase.

Feature Growing Phase. In this phase, the gradient of the cross-entropy is large and the weight-signal correlation grows much more quickly than weight-noise correlation thanks to the polynomial ReLU. We show that the signal strength is relatively unaffected by pruning while the noise level is reduced by a factor of $\sqrt{p}$.

Lemma 5 (Feature growing phase, informal version of Theorem 31) *Under Condition 2, there exists time T_1 such that*

1. *The max weight-signal correlation is large:* $\max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(T_1)}, \boldsymbol{\mu}_j \right\rangle \geq m^{-1/q}$ *for* $j \in [K]$.
2. *The weight-noise and cross-class weight-signal correlations are small: if* $j \neq y_i$, *then* $\max_{j,r,i} \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(T_1)}, \boldsymbol{\xi}_i \right\rangle \right| \leq O(\sigma_0 \sigma_n \sqrt{pd})$ *and* $\max_{j,r,k} \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(T_1)}, \boldsymbol{\mu}_k \right\rangle \right| \leq \widetilde{O}(\sigma_0 \mu)$.

Converging Phase. We show that gradient descent can drive the training loss toward zero while the signal in the feature is still large. An important intermediate step in our argument is the development of the following gradient upper bound for multi-class cross-entropy loss which introduces an extra factor of K in the gradient upper bound.

Lemma 6 (Gradient upper bound, informal version of Theorem 33) *Under Condition 2, we have*

$$\left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \leq O(K m^{2/q} \mu^2) L_S(\widetilde{\mathbf{W}}^{(t)}).$$

Proof Sketch To prove this upper bound, note that for a given input $(\mathbf{x}_i, y_i)$, $\ell_{y_i,i}'^{(t)} \nabla F_{y_i}(\mathbf{x}_i)$ should make major contribution to $\left\| \nabla \ell(\widetilde{\mathbf{W}}; \mathbf{x}_i, y_i) \right\|_F$. Further note that $|\ell_{y_i,i}'^{(t)}| = 1 - \mathrm{logit}_{y_i}(F; \mathbf{x}_i) = \frac{\sum_{j \neq y_i} e^{F_j(\mathbf{x}_i)}}{\sum_j e^{F_j(\mathbf{x}_i)}} \leq \frac{\sum_{j \neq y_i} e^{F_j(\mathbf{x}_i)}}{e^{F_{y_i}(\mathbf{x}_i)}}$. Now, apply the property that $F_j(\mathbf{x}_i)$ is small for $j \neq y_i$ (which we prove in the appendix), the numerator will contribute a factor of K. To bound the rest, we utilize the special property of multi-class cross-entropy loss: $|\ell_{j,i}'^{(t)}| \leq |\ell_{y_i,i}'^{(t)}| \leq \ell_i^{(t)}$. However, a naive application of this inequality will result in a factor of K^3 instead K in our bound. The trick is to further use the fact that $\sum_{j \neq y_i} |\ell_{j,i}'^{(t)}| = |\ell_{y_i,i}'^{(t)}|$. ■

Using the above gradient upper bound, we can show that the objective can be minimized.

Lemma 7 (Converging phase, informal version of Theorem 38) *Under Condition 2, there exists T_2 such that for some time $t \in [T_1, T_2]$ such that*

1. *The results from the feature growing phase (Lemma 5) hold up to constant factors.*
2. *The training loss is small* $L_S(\widetilde{\mathbf{W}}^{(t)}) \leq \epsilon$.

Notice that the weight-noise correlation still remains reduced by a factor of $\sqrt{p}$ after training. Lemma 7 proves the statement of the training loss in Theorem 3.

Generalization Analysis. Finally, we show that pruning can purify the feature by reducing the variance of the noise by a factor of p when a new sample is given. The lemma below shows that the variance of weight-noise correlation for the trained weights is reduced by a factor of p.

Lemma 8 (Same as Theorem 39) *The neural network weight $\widetilde{\mathbf{W}}^\star$ after training satisfies that*

$$\mathop{\mathbb{P}}_{\boldsymbol{\xi}}\left[\max_{j,r}\left|\langle\widetilde{\mathbf{w}}_{j,r}^\star,\boldsymbol{\xi}\rangle\right|\geq(2m)^{-\frac{2}{q}}\right]\leq 2Kme^{\left(-\frac{(2m)^{-4/q}}{O(\sigma_0^2\sigma_n^2pd)}\right)}.$$

Using this lemma, we can show that pruning yields better generalization bound (i.e., the bound on the generalization loss) claimed in Theorem 3.

4. Over Pruning

Our second result shows that there exists a relatively large pruning fraction (i.e., small p) such that the learned model yields poor generalization, although gradient descent is still able to drive the training error toward zero. The full proof is defered to Appendix D.

Theorem 9 (Main theorem for over pruning, informal version of Theorem 41) *Under Condition 2 if $p=\Theta(\frac{1}{Km\log d})$, then with probability at least $1-1/\operatorname{poly}\log d$ over the randomness in the data, network initialization and pruning, there exists $T=O(\eta^{-1}n\sigma_0^{q-2}\sigma_n^{-q}(pd)^{-q/2}+\eta^{-1}\epsilon^{-1}m^4n\sigma_n^{-2}(pd)^{-1})$ such that*

1. *The training loss is below ϵ: $L_S(\widetilde{\mathbf{W}}^{(T)})\leq\epsilon$.*
2. *The generalization loss is large: $L_{\mathcal{D}}(\widetilde{\mathbf{W}}^{(T)})\geq\Omega(\log K)$.*

Remark 10 *The above theorem indicates that in the over-pruning case, the training loss can still go to zero. However, the generalization loss of our neural network behaves no much better than random guessing, because given any sample, random guessing will assign each class with probability $1/K$, which yields a generalization loss of $\log K$. The readers might wonder why the condition for this to happen is $p=\Theta(\frac{1}{Km\log d})$ instead of $O(\frac{1}{Km\log d})$. Indeed, the generalization will still be bad if p is too small. However, now the neural network is not only unable to learn the signal but also cannot efficiently memorize the noise via gradient descent.*

Proof Outline: Now we analyze the over-pruning case. We first show that there is a good chance that the model will not capture any signal after pruning due to the sparse signal assumption and mild overparameterization of the neural network.

Lemma 11 (Over pruning initialization, same as Theorem 42) *If $m=\operatorname{poly}\log d$ and $p=\Theta(\frac{1}{Km\log d})$, with probability $1-O(1/\log d)$, for all class $j\in[K]$ we have $|\mathcal{S}_{\text{signal}}^j|=0$.*

Then, leveraging such a property, we bound the weight-signal and weight-noise properties for the feature growing and converging phases of gradient descent, as stated in the following two lemmas, respectively. Our result indicates that the training loss can still be driven toward zero by letting the neural network memorize the noise, the proof of which further exploits the fact that high dimensional Gaussian noise are nearly orthogonal.

Lemma 12 (Feature growing phase, informal version of Theorem 43) *Under Condition 2, there exists T_1 such that*

- *Some weights has large correlation with noise:* $\max_r \left\langle \widetilde{\mathbf{w}}_{y_i,r}^{(T_1)}, \boldsymbol{\xi}_i \right\rangle \geq m^{-1/q}$ *for all* $i \in [n]$.
- *The cross-class weight-noise and weight-signal correlations are small: if* $j \neq y_i$*, then* $\max_{j,r,i} \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(T_1)}, \boldsymbol{\xi}_i \right\rangle \right| = \widetilde{O}(\sigma_0 \sigma_n \sqrt{pd})$ *and* $\max_{j,r,k} \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(T_1)}, \boldsymbol{\mu}_k \right\rangle \right| \leq \widetilde{O}(\sigma_0 \mu)$.

Lemma 13 (Converging phase, informal version of Theorem 49) *Under Condition 2, there exists a time T_2 such that for some t in $[T_1, T_2]$, the results from phase 1 still holds (up to constant factors) and $L_S(\widetilde{\mathbf{W}}^{(t)}) \leq \epsilon$.*

Finally, since the above lemmas show that the network is purely memorizing the noise, this can be further utilized to show that such a network yields poor generalization performance as stated in Theorem 9.

5. Experiments

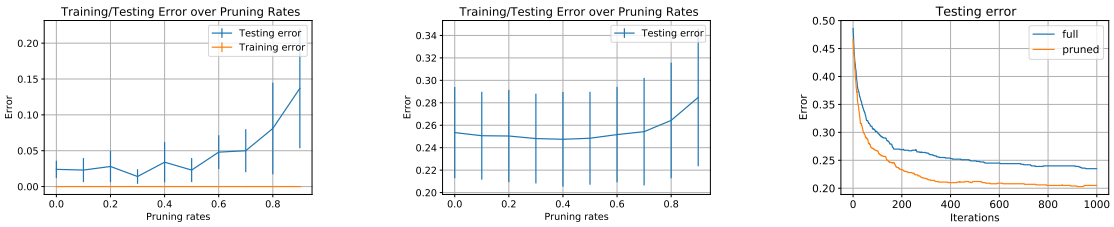
5.1 Simulations to Verify Our Results

In this section, we conduct simulations to verify our results. We conduct our experiment using binary classification task and show that our result holds for ReLU networks. Our experiment settings are the follows: we choose input to be $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] = [y\mathbf{e}_1, \boldsymbol{\xi}] \in \mathbb{R}^{800}$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{400}$, where $\boldsymbol{\xi}_i$ is sampled from a Gaussian distribution. The class labels y are $\{\pm 1\}$. We use 100 training examples and 100 testing examples. The network has width 150 and is initialized with random Gaussian distribution with variance 0.01. Then, p fraction of the weights are randomly pruned. We use the learning rate of 0.001 and train the network over 1000 iterations by gradient descent.

The observations are summarized as follows. In Figure 2a, when the noise level is $\sigma_n = 0.5$, the pruned network usually can perform at the similar level with the full model when $p \leq 0.5$ and noticably better when $p = 0.3$. When $p > 0.5$, the test error increases dramatically while the training accuracy still remains perfect. On the other hand, when the noise level becomes large $\sigma_n = 1$ (Figure 2b), the full model can no longer achieve good testing performance but mild pruning can improve the model's generalization. Note that the training accuracy in this case is still perfect (omitted in the figure). We observe that in both settings when the model test error is large, the variance is also large. However, in Figure 2b, despite the large variance, the mean curve is already smooth. In particular, Figure 2c plots the testing error over the training iterations under $p = 0.5$ pruning rate. This suggests that pruning can be beneficial even when the input noise is large.

5.2 On the Real World Dataset

To further demonstrate the mild/over pruning phenomenon, we conduct experiments on MNIST (Deng, 2012) and CIFAR-10 (Krizhevsky et al., 2009) datasets. We consider neural network architectures including MLP with 2 hidden layers of width 1024, VGG, ResNets (He et al., 2016) and wide ResNet (Zagoruyko and Komodakis, 2016). In addition to random pruning, we also add iterative-magnitude-based pruning Frankle and Carbin (2018)

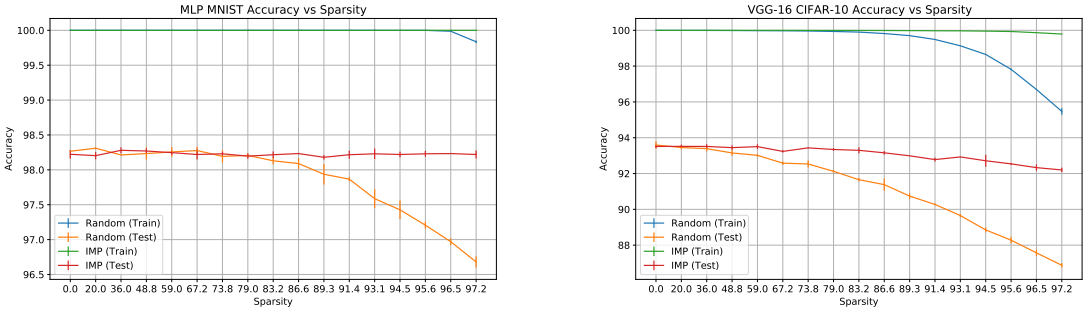


(a) (b) (c)

Figure 2: Figure (a) shows the relationship between pruning rates p and training/testing error under noise variance $\sigma_n = 0.5$. Figure (b) shows the relationship between pruning rates p and testing error under noise variance $\sigma_n = 1$. The training error is omitted since it stays effectively at zero across all pruning rates. Figure (c) shows a particular training curve under pruning rate $p = 50\%$ and noise variance $\sigma_n = 1$. Each data point is created by taking an average over 10 independent runs.

into our experiments. Both pruning methods are prune-at-initialization methods. Our implementation is based on Chen et al. (2021c).

Under the real world setting, we do not expect our theorem to hold ***exactly***. Instead, our theorem implies that (1) there exists a threshold such that the testing performance is no much worse than (or sometimes may slightly better than) its dense counter part; and (2) the training error decreases later than the testing error decreases. Our experiments on MLP (Figure 3a) and VGG-16 (Figure 3b) show that this is the case: for MLP the test accuracy is steady competitive to its dense counterpart when the sparsity is less than 79% and 36% for VGG-16. We further provide experiments on ResNet in Appendix A.2.



(a) (b)

Figure 3: Figure (a) shows the result between pruning rates p and accuracy on MLP-1024-1024 on MNIST. Figure (b) shows the result on VGG-16 on CIFAR-10. Each data point is created by taking an average over 3 independent runs.

6. Further Comparison to Prior Works

We point out that previous works such as (Allen-Zhu and Li, 2020, 2022; Cao et al., 2022; Karp et al., 2021; Frei et al., 2022; Glasgow et al.) studied deep learning theory by considering

data distribution consisting of signals and noise. Our work aligns with such a line of research with the following key difference. All the above work did not study the pruning model, whereas our focus is on exploring how random pruning at initialization can improve the network's generalization after training. Our new contributions lies in characterizing how pruning neural networks at initialization will impact the training dynamics. For example, for mild pruning, Lemma 5 shows that the effect of pruning on increasing SNR can be carried through the feature growing phase. Furthermore, all the above previous work focused on binary classification setting whereas we study a multi-class classification setting, which requires further extension.

A prior work in a similar spirit to ours is (Zhang et al., 2021) whose setting assumes the labels of the input data are generated from some unknown sparse teacher network and the goal of the training is to learn a student network to recover the weights of the teacher network **given the underlying true mask of the teacher network**. Under such assumption, their conclusion is that the sparser the teacher network is, the faster convergence and better sample complexity the student network can achieve. On the other hand, our setting assumes a sparse signal structure and a binary label. However, our work doesn't have such strong assumption that the student network knows the underlying true sparse structure of the signal at all. In fact, our work **complements** Zhang et al. 2021 in a sense that our setting considers mask generated by random pruning under different pruning rate (which can hardly ever be the true mask), and we are able to show that there exists a range that the more we prune, the better generalization we can have.

7. Discussion and Future Directions

In this work, we provide theory on the generalization performance of pruned neural networks trained by gradient descent under different pruning rates in a simplified setting. Our results characterize the effect of pruning under different pruning rates: in the mild pruning case, the signal in the feature is well-preserved and the noise level is reduced which leads to improvement in the trained network's generalization; on the other hand, over pruning significantly destroys signal strength despite of also reducing the amount of noise in the feature. For **practical utility** of our work, our goal is to improve the theoretical understanding of the effectiveness of neural network pruning. We hope our work can help machine learning practitioners understand how and why pruning works. In particular, our theoretical analysis indicates that there are neurons in the neural network aligning more with the signal while other neurons aligning more with the noise. Thus, pruning the neurons aligning noise more will help improve the network's generalization. We believe that such theoretical understanding is helpful for machine learning practitioners and can lead to designing efficient and accurate pruning algorithms. We do hope our work can provide guidance on finding the optimal pruning ratio. However, currently, limited by the theoretical tools, we are unable to characterize the phase transition even for our simple data distribution.

Thus, our work is preliminary and contains many interesting future directions:

- Our work considers a simple signal-noise distribution although inspired by many recent work, still far from the real world distribution such as images. In the future, it would be interesting to consider more complicated data distribution such as the one recently studied (Ankner et al., 2022).

- Our work only considers one-hidden-layer neural networks which are although studied in the current frontier of deep learning theory, still have different learning mechanism from the deep neural networks.
- Our work studies random pruning, and we would like to study more sophisticated pruning methods such as magnitude-based pruning (Frankle and Carbin, 2018).

Acknowledgments

The work of Z. Wang was in part supported by an NSF Scale-MoDL grant (award number: 2133861) and the CAREER Award (award number: 2145346). The work of Y. Liang was supported in part by the U.S. National Science Foundation under the grants CCF-1900145, ECCS-2113860, and DMS-2134145.

Appendix A. Experiments

A.1 Experiment Details

The experiments of MLP, VGG and ResNet-32 are run on NVIDIA A5000 and ResNet-50 and ResNet-20-128 is run on 4 NIVIDIA V100s. We list the hyperparameters we used in training. All of our models are trained with SGD and the detailed settings are summarized below.

Table 1: Summary of architectures, dataset and training hyperparameters

Model	Data	Epoch	Batch Size	LR	Momentum	LR Decay, Epoch	Weight Decay
LeNet	MNIST	120	128	0.1	0	0	0
VGG	CIFAR-10	160	128	0.1	0.9	$0.1 \times [80, 120]$	0.0001
ResNets	CIFAR-10	160	128	0.1	0.9	$0.1 \times [80, 120]$	0.0001

A.2 Further Experiment Results

We plot the experiment result of ResNet-20-128 in Figure 4. This figure further verifies our results that there exists pruning rate threshold such that the testing performance of the pruned network is on par with the testing performance of the dense model while the training accuracy remains perfect.

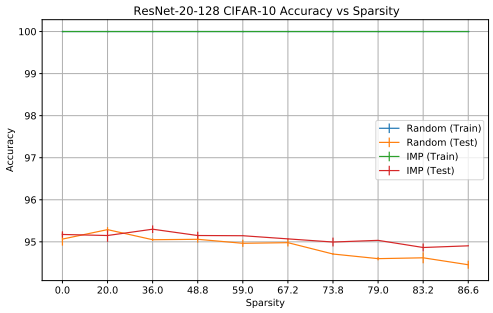
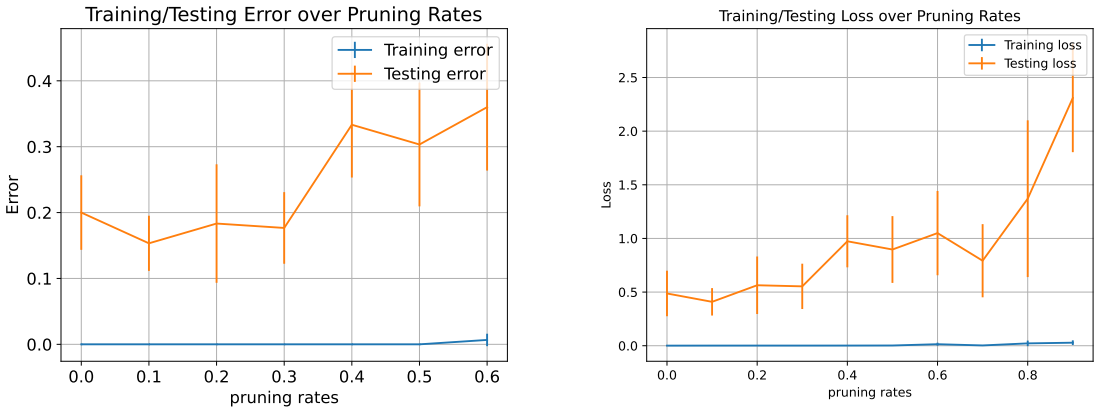


Figure 4: The figure shows the experiment results of ResNet-20-128 under various sparsity by random pruning and IMP. Each data point is averaged over 2 runs.

A.3 Further synthetic experiments

We run further experiments to verify our theory. For our experiment setting, we choose the input dimension $d = 400$, and the Gaussian noise level to be 1.5. We use 100 training samples and testing samples. The neural networks have hidden dimension 4 and the weights are initialized with standard deviation 0.01. We train the neural network with learning rate 0.1 for 5000 epoch.

Figure 5a shows mild pruning can improve generalization: 0.1 pruning rate has error noticeably smaller than the full model. On the other hand, Figure 5b shows that when the pruning rate is large, the neural network can still attain very small training loss while the testing loss is very high.



(a) (b)

Figure 5

Appendix B. Preliminary for Analysis

In this section, we introduce the following signal-noise decomposition of each neuron weight from Cao et al. (2022), and some useful properties for the terms in such a decomposition, which are useful in our analysis.

Definition 14 (signal-noise decomposition) *For each neuron weight $j \in [K]$, $r \in [m]$, there exist coefficients $\gamma_{j,r,k}^{(t)}, \zeta_{j,r,i}^{(t)}, \omega_{j,r,i}^{(t)}$ such that*

$$\widetilde{\mathbf{w}}_{j,r}^{(t)} = \widetilde{\mathbf{w}}_{j,r}^{(0)} + \sum_{k=1}^{K} \gamma_{j,r,k}^{(t)} \cdot \|\boldsymbol{\mu}_k\|_2^{-2} \cdot \boldsymbol{\mu}_k \odot \mathbf{m}_{j,r} + \sum_{i=1}^{n} \zeta_{j,r,i}^{(t)} \cdot \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^{-2} \cdot \widetilde{\boldsymbol{\xi}}_{j,r,i} + \sum_{i=1}^{n} \omega_{j,r,i}^{(t)} \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^{-2} \cdot \widetilde{\boldsymbol{\xi}}_{j,r,i},$$

where $\gamma_{j,r,j}^{(t)} \geq 0,\ \gamma_{j,r,k}^{(t)} \leq 0,\ \zeta_{j,r,i}^{(t)} \geq 0, \omega_{j,r,i}^{(t)} \leq 0$.

It is straightforward to see the following:

$$\gamma_{j,r,k}^{(0)}, \zeta_{j,r,i}^{(0)}, \omega_{j,r,i}^{(0)} = 0,$$

$$\gamma_{j,r,j}^{(t+1)} = \gamma_{j,r,j}^{(t)} - \mathbb{I}(r \in \mathcal{S}_{\text{signal}}^{j}) \frac{\eta}{n} \sum_{i=1}^{n} \ell_{j,i}^{\prime(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle\right) \|\boldsymbol{\mu}_{y_i}\|_2^2 \mathbb{I}(y_i = j),$$

$$\gamma_{j,r,k}^{(t+1)} = \gamma_{j,r,k}^{(t)} - \mathbb{I}((\mathbf{m}_{j,r})_k = 1) \frac{\eta}{n} \sum_{i=1}^{n} \ell_{j,i}^{\prime(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle\right) \|\boldsymbol{\mu}_{y_i}\|_2^2 \mathbb{I}(y_i = k),\ \forall j \neq k,$$

$$\zeta_{j,r,i}^{(t+1)} = \zeta_{j,r,i}^{(t)} - \frac{\eta}{n} \cdot \ell_{j,i}'^{(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right) \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \mathbb{I}(j = y_i),$$
$$\omega_{j,r,i}^{(t+1)} = \omega_{j,r,i}^{(t)} - \frac{\eta}{n} \cdot \ell_{j,i}'^{(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right) \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \mathbb{I}(j \neq y_i),$$

where $\{\gamma_{j,r,j}^{(t)}\}_{t=1}^T, \{\zeta_{j,r,i}^{(t)}\}_{t=1}^T$ are increasing sequences and $\{\gamma_{j,r,k}^{(t)}\}_{t=1}^T, \{\omega_{j,r,i}^{(t)}\}_{t=1}^T$ are decreasing sequences, because $-\ell_{j,i}'^{(t)} \geq 0$ when $j = y_i$, and $-\ell_{j,i}'^{(t)} \leq 0$ when $j \neq y_i$. By Lemma 18, we have $pd > n + K$, and hence the set of vectors $\{\boldsymbol{\mu}_k\}_{k=1}^K \bigcup \{\widetilde{\boldsymbol{\xi}}_i\}_{i=1}^n$ is linearly independent with probability measure 1 over the Gaussian distribution for each $j \in [K], r \in [m]$. Therefore the decomposition is unique.

Appendix C. Proof of Theorem 3

We first formally restate Theorem 3.

Theorem 15 (Formal Restatement of Theorem 3) *Under Condition 2, choose initialization variance $\sigma_0 = \widetilde{\Theta}(m^{-4}n^{-1}\mu^{-1})$ and learning rate $\eta \leq \widetilde{O}(1/\mu^2)$. For $\epsilon > 0$, if $p \geq C_1 \frac{\log d}{m}$ for some sufficiently large constant C_1, then with probability at least $1 - O(d^{-1})$ over the randomness in the data, network initialization and pruning, there exists $T = \widetilde{O}(K\eta^{-1}\sigma_0^{2-q}\mu^{-q} + K^2 m^4 \mu^{-2}\eta^{-1}\epsilon^{-1})$ such that the following holds:*

1. *The training loss is below ϵ: $L_S(\widetilde{\mathbf{W}}^{(T)}) \leq \epsilon$.*
2. *The weights of the CNN highly correlate with its corresponding class signal: $\max_r \gamma_{j,r,j}^{(T)} \geq \Omega(m^{-1/q})$ for all $j \in [K]$.*
3. *The weights of the CNN doesn't have high correlation with the signal from different classes: $\max_{j \neq k, r \in [m]} |\gamma_{j,r,k}^{(T)}| \leq \widetilde{O}(\sigma_0 \mu)$.*
4. *None of the weights is highly correlated with the noise: $\max_{j,r,i} \zeta_{j,r,i}^{(T)} = \widetilde{O}(\sigma_0 \sigma_n \sqrt{pd})$, and $\max_{j,r,i} |\omega_{j,r,i}^{(T)}| = \widetilde{O}(\sigma_0 \sigma_n \sqrt{pd})$.*

Moreover, the testing loss is upper-bounded by

$$L_{\mathcal{D}}(\widetilde{\mathbf{W}}^{(T)}) \leq O(K\epsilon) + \exp(-n^2/p).$$

The proof of Theorem 3 consists of the analysis of the pruning on the signal and noise for three stages of gradient descent: initialization, feature growing phase, and converging phase, and the establishment of the generalization property. We present these analysis in detail in the following subsections. A special note is that the constant C showing up in the following proof of each subsequent Lemmas is defined locally instead of globally, which means the constant C within each Lemma is the same but may be different across different Lemma.

C.1 Initialization

We analyze the effect of pruning on weight-signal correlation and weight-noise correlation at the initialization. We first present a few supporting lemmas, and finally provide our

main result of Theorem 21, which shows that if the pruning is mild, then it will not hurt the max weight-signal correlation much at the initialization. On the other hand, the max weight-noise correlation is reduced by a factor of $\sqrt{p}$.

Lemma 16 *Assume $n = \Omega(K^2 \log Kd)$. Then, with probability at least $1 - 1/d$,*

$$|\{i \in [n] : \ y_i = j\}| = \Theta(n/K) \quad \forall j \in [K].$$

Proof By Hoeffding's inequality, with probability at least $1 - \delta/2K$, for a fixed $j \in [K]$, we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(y_i = j) - \frac{1}{K} \right| \leq \sqrt{\frac{\log(4K/\delta)}{2n}}.$$

Therefore, as long as $n \geq 2K^2 \log(4K/\delta)$, we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(y_i = j) - \frac{1}{K} \right| \leq \frac{1}{2K}.$$

Taking a union bound over $j \in [K]$ and making $\delta = 1/d$ yield the result. ■

Lemma 17 *Assume $pm = \Omega(\log d)$ and $m = \operatorname{poly}\log d$. Then, with probability $1 - 1/d$, for all $j \in [K],\ k \in [K]$, we have $\sum_{r=1}^{m} (\mathbf{m}_{j,r})_k = \Theta(pm)$, which implies that $|\mathcal{S}^j_{\text{signal}}| = \Theta(pm)$ for all $j \in [K]$.*

Proof When $pm = \Omega(\log d)$, by multiplicative Chernoff's bound, for a given $k \in [K]$, we have

$$\mathbb{P}\left[\left| \sum_{r=1}^{m} (\mathbf{m}_{j,r})_k - pm \right| \geq 0.5pm \right] \leq 2 \exp\left\{ -\Omega\left(pm\right) \right\}.$$

Take a union bound over $j \in [K],\ k \in [K]$, we have

$$\mathbb{P}\left[\left| \sum_{r=1}^{m} (\mathbf{m}_{j,r})_k - pm \right| \geq 0.5pm,\ \forall j \in [K],\ k \in [K] \right] \leq 2K^2 \exp\left\{ -\Omega\left(pm\right) \right\} \leq 1/d.$$

■

Lemma 18 *Assume $p = 1/\operatorname{poly}\log d$. Then with probability at least $1 - 1/d$, for all $j \in [K]$, $r \in [m]$, $\sum_{i=1}^{d} (\mathbf{m}_{j,r})_i = \Theta(pd)$.*

Proof By multiplicative Chernoff's bound, we have for a given j, r

$$\mathbb{P}\left[\left| \sum_{i=1}^{d} (\mathbf{m}_{j,r})_i - pd \right| \geq 0.5pd \right] \leq 2 \exp\{ -\Omega(pd) \}.$$

Take a union bound over j, r, we have

$$\mathbb{P}\left[\left|\sum_{i=1}^{d}(\mathbf{m}_{j,r})_i - pd\right| \geq 0.5pd,\ \forall j \in [K],\ r \in [m]\right] \leq 2Km\exp\{-\Omega(pd)\} \leq 1/d,$$

where the last inequality follows from our choices of p, K, m, d. ■

Lemma 19 *Suppose $p = \Omega(1/\operatorname{poly}\log d)$, and $m, n = \operatorname{poly}\log d$. With probability at least $1 - 1/d$, we have*

$$\begin{aligned}
\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 &= \Theta(\sigma_n^2 pd),\\
\left|\left\langle\widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\xi}_{i'}\right\rangle\right| &\leq O(\sigma_n^2\sqrt{pd\log d}),\\
\left|\left\langle\boldsymbol{\mu}_k, \widetilde{\boldsymbol{\xi}}_{j,r,i}\right\rangle\right| &\leq |\left\langle\boldsymbol{\mu}, \boldsymbol{\xi}_i\right\rangle| \leq O(\sigma_n\mu\sqrt{\log d}),
\end{aligned}$$

for all $j \in \{-1, 1\},\ r \in [m],\ i, i' \in [n]$ and $i \neq i'$.

Proof From Lemma 18, we have with probability at least $1 - 1/d$,

$$\sum_{k=1}^{d}(\mathbf{m}_{j,r})_k = \Theta(pd), \quad \forall j \in [K],\ r \in [m].$$

For a set of Gaussian random variable $g_1, \ldots, g_N \sim \mathcal{N}(0, \sigma^2)$, by Bernstein's inequality, with probability at least $1 - \delta$, we have

$$\left|\sum_{i=1}^{N} g_i^2 - \sigma^2 N\right| \lesssim \sigma^2\sqrt{N\log\frac{1}{\delta}}.$$

Thus, by a union bound over j, r, i, with probability at least $1 - 1/d$, we have

$$\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 = \Theta(\sigma_n^2 pd).$$

For $i \neq i'$, again by Bernstein's bound, we have with probability at least $1 - \delta$,

$$\left|\left\langle\widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\xi}_{i'}\right\rangle\right| \leq O\left(\sigma_n^2\sqrt{pd\log\frac{Kmn}{\delta}}\right),$$

for all j, r, i. Plugging in $\delta = 1/d$ gives the result. The proof for $|\left\langle\boldsymbol{\mu}, \boldsymbol{\xi}_i\right\rangle|$ is similar. ■

Lemma 20 *Suppose we have m independent Gaussian random variables $g_1, g_2, \ldots, g_m \sim \mathcal{N}(0, \sigma^2)$. Then with probability $1 - \delta$,*

$$\max_i g_i \geq \sigma\sqrt{\log\frac{m}{\log 1/\delta}}.$$

Proof By the standard tail bound of Gaussian random variable, we have for every $x > 0$,

$$\left(\frac{\sigma}{x} - \frac{\sigma^3}{x^3}\right)\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}} \le \mathbb{P}\left[g > x\right] \le \frac{\sigma}{x}\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}}.$$

We want to pick a $x^\star$ such that

$$\begin{aligned}
&\mathbb{P}\left[\max_i g_i \le x^\star\right] = \left(\mathbb{P}\left[g_i \le x^\star\right]\right)^m = \left(1 - \mathbb{P}\left[g_i \ge x^\star\right]\right)^m \le e^{-m\,\mathbb{P}[g_i \ge x^\star]} \le \delta \\
&\Rightarrow \mathbb{P}[g_i \ge x^\star] = \Theta\left(\frac{\log(1/\delta)}{m}\right) \\
&\Rightarrow x^\star = \Theta(\sigma\sqrt{\log(m/(\log(1/\delta)\log m))}).
\end{aligned}$$

■

Lemma 21 (Formal Restatement of Theorem 4) *With probability at least* $1-2/d$*, for all* $i \in [n]$*,*

$$\sigma_0\sigma_n\sqrt{pd} \le \max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle \le \sqrt{2\log(Kmd)}\sigma_0\sigma_n\sqrt{pd}.$$

Further, suppose $pm \ge \Omega(\log(Kd))$*. Then with probability* $1 - 2/d$*, for all* $j \in [K]$*,*

$$\sigma_0 \left\|\boldsymbol{\mu}_j\right\|_2 \le \max_{r \in \mathcal{S}_{\text{signal}}^j} \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle \le \sqrt{2\log(8pmKd)}\sigma_0 \left\|\boldsymbol{\mu}_j\right\|_2.$$

Proof We first give a proof for the second inequality. From Lemma 17, we know that $|\mathcal{S}_{\text{signal}}^j| = \Theta(pm)$. The upper bound can be obtained by taking a union bound over $r \in \mathcal{S}_{\text{signal}}^j,\ j \in [K]$. To prove the lower bound, applying Lemma 20, with probability at least $1 - \delta/K$, we have for a given $j \in [K]$

$$\max_{r \in \mathcal{S}_{\text{signal}}^j} \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle \ge \sigma_0 \left\|\boldsymbol{\mu}_j\right\|_2 \sqrt{\log \frac{pm}{\log K/\delta}}.$$

Now, notice that we can control the constant in pm (by controlling the constant in the lower bound of p) such that $pm/\log(Kd) \ge e$. Thus, taking a union bound over $j \in [K]$ and setting $\delta = 1/d$ yield the result.

The proof of the first inequality is similar. ■

C.2 Supporting Properties for Entire Training Process

This subsection establishes a few properties (summarized in Theorem 24) that will be used in the analysis of feature growing phase and converging phase of gradient descent presented in the next two subsections. Define $T^\star = \eta^{-1}\,\mathrm{poly}(1/\epsilon, \mu, d^{-1}, \sigma_n^{-2}, \sigma_0^{-1}n, m, d)$. Denote

$\alpha = \Theta(\log^{1/q}(T^\star))$, $\beta = 2\max_{i,j,r,k}\left\{\left|\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_k \right\rangle\right|, \left|\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle\right|\right\}$. We need the following bound holds for our subsequent analysis.

$$4m^{1/q}\max_{j,r,i}\left\{\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i} \right\rangle, Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}, \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle, 3Cn\alpha\sqrt{\frac{\log d}{pd}}\right\} \leq 1 \tag{1}$$

Remark 22 *To see why Equation* (1) *can hold under Theorem 2, we convert everything in terms of* d*. First recall from Theorem 2 that* $m, n = \mathrm{poly}(\log d)$ *and* $\mu = \Theta(\sigma_n\sqrt{d}\log d) = \Theta(1)$*. In both mild pruning and over pruning we require* $p \geq \Omega(1/\mathrm{poly}\log d)$*. Since* $\alpha = \Theta(\log^{1/q}(T^\star))$*, if we assume* $T^\star \leq O(\mathrm{poly}(d))$ *for a moment (which we are going to justify in the next paragraph), then* $\alpha = O(\log^{1/q}(d))$*. Then if we set* d *to be large enough, we have* $4m^{1/q}Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd} \leq \frac{\mathrm{poly}\log d}{\sqrt{d}} \leq 1$*. Then, for the quantity* $4m^{1/q}\max_{j,r,i}\{\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i}\rangle, \langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i\rangle\}$*, by Theorem 4, our assumption of* $K = O(\log d)$ *in Theorem 2 and our choice of* $\sigma_0 = \widetilde{\Theta}(m^{-4}n^{-1}\mu^{-1})$ *in Theorem 3 (or Theorem 15), we can easily see that this quantity can also be made smaller than 1.*

Now, to justify that $T^\star \leq O(\mathrm{poly}(d))$*, we only need to justify that all the quantities* $T^\star$ *depend on is polynomial in* d*. First of all, based on Theorem 2,* $n, m = \mathrm{poly}\log(d)$ *and* $\mu = \Theta(\sigma_n\sqrt{d}\log d) = \Theta(1)$ *further implies* $\sigma_n^{-2} = \Theta(d\log^2 d)$*. Since Theorem 3 only requires* $\sigma_0 = \tilde{\Theta}(m^{-4}n^{-1}\mu^{-1})$*, this implies* $\sigma_0^{-1} \leq O(\mathrm{poly}\log d)$*. Hence* $\sigma_0^{-1}n = O(\mathrm{poly}\log d)$*. Together with our assumption that* $\epsilon, \eta \geq \Omega(1/\mathrm{poly}(d))$ *(which implies* $1/\epsilon, 1/\eta \leq O(\mathrm{poly}(d))$*), we have justified that all terms involved in* $T^\star$ *are at most of order* $\mathrm{poly}(d)$*. Hence* $T^\star = \mathrm{poly}(d)$*.*

Remark 23 *Here we make remark on our assumption on* ϵ *and* η *in Theorem 2.*

For our assumption on ϵ*, since the cross-entropy loss is (1) not strongly-convex and (2) achieves its infimum at infinity. In practice, the cross-entropy loss is minimized to a constant level, say 0.001. We make this assumption to avoid the pathological case where* ϵ *is exponentially small in* d *(say* $\epsilon = 2^{-d}$*) which is unrealistic. Thus, for realistic setting, we assume* $\epsilon \geq \Omega(1/\mathrm{poly}(d))$ *or* $1/\epsilon \leq O(\mathrm{poly}(d))$*.*

To deal with η*, the only restriction we have is* $\eta = O(1/\mu^2)$ *in Theorem 3 and Theorem 9. However, in practice, we don't use a learning rate that is exponentially small, say* $\eta = 2^{-d}$*. Thus, like dealing with* ϵ*, we assume* $\eta \geq \Omega(1/\mathrm{poly}(d))$ *or* $1/\eta \leq O(\mathrm{poly}\, d)$*.*

We make the above assumption to simplify analysis when analyzing the magnitude of $F_j(X)$ for $j \neq y$ given sample (X, y).

Proposition 24 *Under Theorem 2, during the training time* $t < T^\star$*, we have*

1. $\gamma_{j,r,j}^{(t)}, \zeta_{j,r,i}^{(t)} \leq \alpha$,
2. $\omega_{j,r,i}^{(t)} \geq -\beta - 6Cn\alpha\sqrt{\frac{\log d}{pd}}$.
3. $\gamma_{j,r,k}^{(t)} \geq -\beta - 2Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}$.

Notice that the lower bound has absolute value smaller than the upper bound.

Proof We use induction to prove Proposition 24.

Induction Hypothesis: Suppose Proposition 24 holds for all $t < T \leq T^\star$.

We next show that this also holds for $t = T$ via the following a few lemmas.

Lemma 25 *Under Theorem 2, for $t < T$, there exists a constant C such that*

$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)} - \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_k \right\rangle = \left(\gamma_{j,r,k}^{(t)} \pm Cn\alpha \frac{\mu\sqrt{\log d}}{\sigma_n pd} \right) \mathbb{I}((\mathbf{m}_{j,r})_k = 1),$$

$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)} - \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle = \zeta_{j,r,i}^{(t)} \pm 3Cn\alpha \sqrt{\frac{\log d}{pd}},$$

$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)} - \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle = \omega_{j,r,i}^{(t)} \pm 3Cn\alpha \sqrt{\frac{\log d}{pd}}.$$

Proof From Lemma 19, there exists a constant C such that with probability at least $1 - 1/d$,

$$\frac{\left|\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\xi}_{i'} \right\rangle\right|}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} \leq C\sqrt{\frac{\log d}{pd}},$$

$$\frac{\left|\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_k \right\rangle\right|}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} \leq C\frac{\mu\sqrt{\log d}}{\sigma_n pd},$$

$$\frac{|\left\langle \boldsymbol{\mu}_k, \boldsymbol{\xi}_i \right\rangle|}{\|\boldsymbol{\mu}_k\|_2^2} \leq C\frac{\sigma_n\sqrt{\log d}}{\mu}.$$

Using the signal-noise decomposition and assuming $(\mathbf{m}_{j,r})_k = 1$, we have

$$\begin{aligned}
&\left|\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)} - \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_k \right\rangle - \gamma_{j,r,k}^{(t)}\right| \\
&= \left|\sum_{i=1}^n \zeta_{j,r,i}^{(t)} \cdot \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^{-2} \cdot \left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_k \right\rangle + \sum_{i=1}^n \omega_{j,r,i}^{(t)} \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^{-2} \cdot \left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_k \right\rangle\right| \\
&\leq C\frac{\mu\sqrt{\log d}}{\sigma_n pd} \sum_{i=1}^n \left|\zeta_{j,r,i}^{(t)}\right| + C\frac{\mu\sqrt{\log d}}{\sigma_n pd} \sum_{i=1}^n \left|\omega_{j,r,i}^{(t)}\right| \\
&\leq 2C\frac{\mu\sqrt{\log d}}{\sigma_n pd} n\alpha.
\end{aligned}$$

where the second last inequality is by Lemma 19 and the last inequality is by induction hypothesis.

To prove the second equality, for $j = y_i$,

$$\begin{aligned}
&\left|\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)} - \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle - \zeta_{j,r,i}^{(t)}\right| \\
&= \left|\sum_{k=1}^K \gamma_{j,r,k}^{(t)} \cdot \frac{\left\langle \boldsymbol{\mu}_k, \boldsymbol{\xi}_i \right\rangle}{\|\boldsymbol{\mu}_k\|_2^2} + \sum_{i' \neq i} \zeta_{j,r,i'}^{(t)} \cdot \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\xi}_i \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i'}\right\|_2^2} + \sum_{i'=1}^n \omega_{j,r,i'}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\xi}_i \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i'}\right\|_2^2}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq C\frac{\sigma_n\sqrt{\log d}}{\mu}\sum_{k=1}^{K}|\gamma_{j,r,k}^{(t)}| + C\sqrt{\frac{\log d}{pd}}\sum_{i'\neq i}|\zeta_{j,r,i'}^{(t)}| + C\sqrt{\frac{\log d}{pd}}\sum_{i'=1}^{n}|\omega_{j,r,i'}^{(t)}| \\
&= C\frac{\sigma_n\sqrt{\log d}}{\mu}K\alpha + 2Cn\alpha\sqrt{\frac{\log d}{pd}} \\
&\leq 3Cn\alpha\sqrt{\frac{\log d}{pd}}.
\end{aligned}
$$

where the last inequality is by $n \gg K$ and $\mu = \Theta(\sigma_n\sqrt{d}\log d)$. The proof for the case of $j \neq y_i$ is similar. ■

Lemma 26 (Off-diagonal Correlation Upper Bound) *Under Theorem 2, for $t < T$, $j \neq y_i$, we have that*

$$
\begin{aligned}
\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle &\leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i} \right\rangle + Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}, \\
\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle &\leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + 3Cn\alpha\sqrt{\frac{\log d}{pd}}, \\
F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i) &\leq 1.
\end{aligned}
$$

Proof If $j \neq y_i$, then $\gamma_{j,r,k}^{(t)} \leq 0$ and we have that

$$
\begin{aligned}
\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle &\leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i} \right\rangle + \left(\gamma_{j,r,y_i}^{(t)} + Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}\right)\mathbb{I}((\mathbf{m}_{j,r})_{y_i} = 1) \\
&\leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i} \right\rangle + Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}.
\end{aligned}
$$

Further, we can obtain

$$
\begin{aligned}
\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle &\leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \omega_{j,r,i}^{(t)} + 3Cn\alpha\sqrt{\frac{\log d}{pd}} \\
&\leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + 3Cn\alpha\sqrt{\frac{\log d}{pd}}.
\end{aligned}
$$

Then, we have the following bound:

$$
\begin{aligned}
F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i) &= \sum_{r=1}^{m}[\sigma(\langle \widetilde{\mathbf{w}}_{j,r}, \boldsymbol{\mu}_{y_i} \rangle) + \sigma(\langle \widetilde{\mathbf{w}}_{j,r}, \boldsymbol{\xi}_i \rangle)] \\
&\leq m2^{q+1}\max_{j,r,i}\left\{\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i} \right\rangle, Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}, \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle, 3Cn\alpha\sqrt{\frac{\log d}{pd}}\right\}^q \\
&\leq 1.
\end{aligned}
$$

where the first inequality is by Equation (1). ■

Lemma 27 (Diagonal Correlation Upper Bound) *Under Theorem 2, for $t < T$, $j = y_i$, we have*

$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_j \right\rangle \leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle + \gamma_{j,r,j}^{(t)} + Cn\alpha \frac{\mu\sqrt{\log d}}{\sigma_n pd},$$
$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle \leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \zeta_{j,r,i}^{(t)} + 3Cn\alpha \sqrt{\frac{\log d}{pd}}.$$

If $\max\{\gamma_{j,r,j}^{(t)}, \zeta_{j,r,i}^{(t)}\} \leq m^{-1/q}$, we further have that $F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i) \leq O(1)$.

Proof The two inequalities are immediate consequences of Lemma 25. If $\max\{\gamma_{j,r,j}^{(t)}, \zeta_{j,r,i}^{(t)}\} \leq m^{-1/q}$, we have

$$\begin{aligned}
&F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i) \\
&= \sum_{r=1}^{m} [\sigma(\langle \widetilde{\mathbf{w}}_{j,r}, \boldsymbol{\mu}_j \rangle) + \sigma(\langle \widetilde{\mathbf{w}}_{j,r}, \boldsymbol{\xi}_i \rangle)] \\
&\leq 2 \cdot 3^q m \max_{j,r,i} \left\{ \gamma_{j,r}^{(t)}, \zeta_{j,r,i}^{(t)}, \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle \right|, \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle \right|, Cn\alpha \frac{\mu\sqrt{\log d}}{\sigma_n pd}, 3Cn\alpha \sqrt{\frac{\log d}{pd}} \right\}^q \leq O(1).
\end{aligned}$$

■

Lemma 28 *Under Theorem 2, for $t \leq T$, we have that*

1. $\omega_{j,r,i}^{(t)} \geq -\beta - 6Cn\alpha\sqrt{\frac{\log d}{pd}}$;
2. $\gamma_{j,r,k}^{(t)} \geq -\beta - 2Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}$.

Proof When $j = y_i$, we have $\omega_{j,r,i}^{(t)} = 0$. We only need to consider the case of $j \neq y_i$. When $\omega_{j,r,i}^{(T-1)} \leq -0.5\beta - 3Cn\alpha\sqrt{\frac{\log d}{pd}}$, by Lemma 25 we have

$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(T-1)}, \boldsymbol{\xi}_i \right\rangle \leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \omega_{j,r,i}^{(T-1)} + 3Cn\alpha\sqrt{\frac{\log d}{pd}} \leq 0.$$

Thus,

$$\begin{aligned}
\omega_{j,r,i}^{(T)} &= \omega_{j,r,i}^{(T-1)} - \frac{\eta}{n} \cdot \ell_{j,i}'^{(T-1)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(T-1)}, \boldsymbol{\xi}_i \right\rangle\right) \left\| \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\|_2^2 \mathbb{I}(j \neq y_i) \\
&= \omega_{j,r,i}^{(T-1)}
\end{aligned}$$

$$\geq -\beta - 6Cn\alpha\sqrt{\frac{\log d}{pd}}.$$

When $\omega_{j,r,i}^{(T-1)} \geq -0.5\beta - 3Cn\alpha\sqrt{\frac{\log d}{pd}}$, we have

$$\begin{aligned}
\omega_{j,r,i}^{(T)} &= \omega_{j,r,i}^{(T-1)} - \frac{\eta}{n} \cdot \ell_{j,i}'^{(T-1)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(T-1)}, \boldsymbol{\xi}_i \right\rangle\right) \left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \mathbb{I}(j \neq y_i) \\
&\geq -0.5\beta - 3Cn\alpha\sqrt{\frac{\log d}{pd}} - \frac{\eta}{n}\sigma'\left(0.5\beta + 3Cn\alpha\sqrt{\frac{\log d}{pd}}\right)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \\
&\geq -\beta - 6Cn\alpha\sqrt{\frac{\log d}{pd}},
\end{aligned}$$

where the last inequality is by setting $\eta \leq nq^{-1}\left(0.5\beta + 3Cn\alpha\sqrt{\frac{\log d}{pd}}\right)^{2-q}(C_2\sigma_n^2 d)^{-1}$ and C_2 is the constant such that $\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \leq C_2\sigma_n^2 pd$ for all j, r, i in Lemma 19.

For $\gamma_{j,r,k}^{(t)}$, the proof is similar. Consider $\mathbb{I}((\mathbf{m}_{j,r})_k) = 1$. When $\gamma_{j,r,k}^{(t)} \leq -0.5\beta - Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}$, by Lemma 25, we have

$$\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_k \right\rangle \leq \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_k \right\rangle + \gamma_{j,r,k}^{(t)} + Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd} \leq 0.$$

Hence,

$$\begin{aligned}
\gamma_{j,r,k}^{(T)} &= \gamma_{j,r,k}^{(T-1)} - \frac{\eta}{n}\sum_{i=1}^{n} \ell_{j,i}'^{(T-1)}\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(T-1)}, \boldsymbol{\mu}_k \right\rangle\right)\mu^2\mathbb{I}(y_i = k) \\
&= \gamma_{j,r,k}^{(T-1)} \\
&\geq -\beta - 2Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}.
\end{aligned}$$

When $\gamma_{j,r,k}^{(t)} \geq -0.5\beta - Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}$, we have

$$\begin{aligned}
\gamma_{j,r,k}^{(T)} &= \gamma_{j,r,k}^{(T-1)} - \frac{\eta}{n}\sum_{i=1}^{n} \ell_{j,i}'^{(T-1)}\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(T-1)}, \boldsymbol{\mu}_k \right\rangle\right)\mu^2\mathbb{I}(y_i = k) \\
&\geq -0.5\beta - Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd} - C_2\frac{\eta}{K}\sigma'\left(0.5\beta + Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd}\right)\mu^2 \\
&\geq -\beta - 2Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd},
\end{aligned}$$

where the first inequality follows from the fact that there are $\Theta(\frac{n}{K})$ samples such that $\mathbb{I}(y_i = k)$, and the last inequality follows from picking $\eta \leq K(0.5\beta + Cn\alpha\frac{\mu\sqrt{\log d}}{\sigma_n pd})^{2-q}\mu^{-2}q^{-1}C_2^{-1}$. ■

Lemma 29 *Under Theorem 2, for $t \le T$, we have $\gamma_{j,r,j}^{(t)}, \zeta_{j,r,i}^{(t)} \le \alpha$.*

Proof For $y_i \neq j$ or $r \notin \mathcal{S}_{\text{signal}}^j$, $\gamma_{j,r,j}^{(t)}, \zeta_{j,r,i}^{(t)} = 0 \le \alpha$.

If $y_i = j$, then by Lemma 26 we have

$$\left|\ell_{j,i}'^{(t)}\right| = 1 - \text{logit}_j(F; X) = \frac{\sum_{i \neq j} e^{F_i(X)}}{\sum_{i=1}^K e^{F_i(X)}} \le \frac{Ke}{e^{F_j(X)}}. \tag{2}$$

Recall that

$$\gamma_{j,r,j}^{(t+1)} = \gamma_{j,r,j}^{(t)} - \mathbb{I}(r \in \mathcal{S}_{\text{signal}}^j)\frac{\eta}{n}\sum_{i=1}^n \ell_{j,i}'^{(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i}\right\rangle\right)\left\|\boldsymbol{\mu}_{y_i}\right\|_2^2 \mathbb{I}(y_i = j),$$
$$\zeta_{j,r,i}^{(t+1)} = \zeta_{j,r,i}^{(t)} - \frac{\eta}{n} \cdot \ell_{j,i}'^{(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \mathbb{I}(j = y_i).$$

We first bound $\zeta_{j,r,i}^{(T)}$. Let $T_{j,r,i}$ be the last time $t < T$ that $\zeta_{j,r,i}^{(t)} \le 0.5\alpha$. Then we have

$$\zeta_{j,r,i}^{(T)} = \zeta_{j,r,i}^{(T_{j,r,i})} \underbrace{- \frac{\eta}{n}\ell_i'^{(T_{j,r,i})} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(T_{j,r,i})}, \boldsymbol{\xi}_i\right\rangle\right)\mathbb{I}(y_i = j)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}_{I_1}$$
$$\underbrace{- \sum_{T_{j,r,i} < t < T} \frac{\eta}{n}\ell_{j,i}'^{(T_{j,r,i})}\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\mathbb{I}(y_i = j)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}_{I_2}.$$

We bound I_1, I_2 separately. We first bound I_1 as follows.

$$|I_1| \le q\frac{\eta}{n}\left(\zeta_{j,r,i}^{(T_{j,r,i})} + 0.5\beta + 3Cn\alpha\sqrt{\frac{\log d}{pd}}\right)^{q-1} C_2\sigma_n^2 pd \le q2^q n^{-1}\eta\alpha^{q-1}C_2\sigma_n^2 pd \le 0.25\alpha,$$

where the first inequality follows from Lemma 27, the second inequality follows because $\beta \le 0.1\alpha$ and $3Cn\alpha\sqrt{\frac{\log d}{pd}} \le 0.1\alpha$, and the last inequality follows because $\eta \le n/(q2^{q+2}\alpha^{q-2}\sigma_n^2 d)$.

For $T_{j,r,i} < t < T$, by Lemma 25, we have that $\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle \ge 0.5\alpha - 0.5\beta - 3Cn\alpha\sqrt{\frac{\log d}{pd}} \ge 0.25\alpha$ and $\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle \le \alpha + 0.5\beta + 3Cn\alpha\sqrt{\frac{\log d}{pd}} \le 2\alpha$.

Now we bound I_2 as follows

$$\begin{aligned}
|I_2| &\le \sum_{T_{j,r,i} < t < T} \frac{\eta}{n} Ke \exp\{-F_j(X)\}\,\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\mathbb{I}(y_i = j)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \\
&\le \sum_{T_{j,r,i} < t < T} \frac{\eta}{n} Ke \exp\left\{-\sigma\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\right\}\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\mathbb{I}(y_i = j)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2 \\
&\le \frac{qKe\eta 2^{q-1}T^\star}{n}\exp(-\alpha^q/4^q)\alpha^{q-1}\sigma_n^2 pd \\
&\le 0.25T^\star \exp(-\alpha^q/4^q)\alpha^{q-2}\alpha
\end{aligned}$$

$$\leq 0.25\alpha,$$

where the first inequality follows from Equation (2), the second inequality follows because $F_j(X) \geq \sigma\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right)$, the fourth inequality follows by choosing $\eta \leq n/(qKe2^{q+1}\sigma_n^2 d)$, and the last inequality follows by choosing $\alpha = \Theta(\log^{1/q}(T^\star))$.

Plugging the bounds on I_1, I_2 finishes the proof for $\zeta_{j,r,i}^{(T)}$.

To prove $\gamma_{j,r,j}^{(t)} \leq \alpha$, we pick $\eta \leq 1/(qe2^{q+2}\mu^2)$ and the rest of the proof is similar. ■

Lemma 28 and Lemma 29 imply Proposition 24 holds for all $t \leq T$.

Induction Ends ■

C.3 Feature Growing Phase

In this subsection, we first present a supporting lemma, and then provide our main result of Theorem 31, which shows that the signal strength is relatively unaffected by pruning while the noise level is reduced by a factor of $\sqrt{p}$.

During the feature growing phase of training, the output of $F_j(X) = O(1)$ for all $j \in [K]$. Therefore, $\text{logit}_i(F, X) = O(\frac{1}{K})$ and $1 - \text{logit}_i(F, X) = \Theta(1)$ until $\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_j \right\rangle$ reaches $m^{-1/q}$.

Lemma 30 *Under the same assumption as Theorem 15, for* $T = \frac{n\eta^{-1}C_4\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{C_3(2C_1)^{q-1}[\log d]^{(q-1)/2}}$, *the following results hold:*

- $|\zeta_{j,r,i}^{(t)}| = O(\sigma_0\sigma_n\sqrt{pd})$ *for all* $j \in [K],\ r \in [m],\ i \in [n]$ *and* $t \leq T$.
- $|\omega_{j,r,i}^{(t)}| = O(\sigma_0\sigma_n\sqrt{pd})$ *for all* $j \in [K],\ r \in [m],\ i \in [n]$ *and* $t \leq T$.

Proof Define $\Psi^{(t)} = \max_{j,r,i}\{\zeta_{j,r,i}^{(t)}, |\omega_{j,r,i}^{(t)}|\}$. Then we have

$$\begin{aligned}
&\Psi^{(t+1)} \\
&\leq \Psi^{(t)} + \max_{j,r,i}\left\{\frac{\eta}{n}|\ell_{j,i}'^{(t)}| \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \sum_{k=1}^{K}\gamma_{j,r,k}^{(t)}\frac{\langle \boldsymbol{\mu}_k, \boldsymbol{\xi}_i\rangle}{\|\boldsymbol{\mu}_k\|_2^2} + \sum_{i'=1}^{n}\Psi^{(t)}\frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\xi}_i \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i'}\right\|_2^2}\right)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2\right\} \\
&\leq \Psi^{(t)} + \frac{\eta}{n}q\left(O(\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}) + K\log^{1/q}T^\star\frac{\mu\sigma_n\sqrt{\log d}}{\mu^2} + \frac{O(\sigma_n^2 pd) + nO(\sigma_n^2\sqrt{pd\log d})}{\Theta(\sigma_n^2 pd)}\Psi^{(t)}\right)^{q-1}O(\sigma_n^2 pd) \\
&\leq \Psi^{(t)} + \frac{\eta}{n}\left(O(\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}) + O(\Psi^{(t)})\right)^{q-1}O(\sigma_n^2 pd),
\end{aligned}$$

where the second inequality follows by $|\ell_{j,i}'^{(t)}|$ and applying the bounds from Lemma 19, and the last inequality follows by choosing $\frac{2K\log T^\star}{\sigma_n d\sqrt{p}} = \widetilde{O}(1/\sqrt{d}) \ll \sigma_0$. Let C_1, C_2, C_3 be the constants for the upper bound to hold in the big O notation. For any $T = \frac{n\eta^{-1}C_4\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{C_3(2C_1)^{q-1}[\log d]^{(q-1)/2}} =$

$\Theta(\frac{n\eta^{-1}\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{[\log d]^{(q-1)/2}})$, we use induction to show that

$$\Psi^{(t)} \le C_4\sigma_0\sigma_n\sqrt{pd},\ \forall t \in [T]. \tag{3}$$

Suppose that Equation (3) holds for $t \in [T']$ for $T' \le T-1$. Then

$$\begin{aligned}
\Psi^{(T'+1)} &\le \Psi^{(T')} + \frac{\eta}{n}\left(C_1\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd} + C_2C_4\sigma_0\sigma_n\sqrt{pd}\right)^{q-1} C_3\sigma_n^2 pd \\
&\le \Psi^{(T')} + \frac{\eta}{n}\left(2C_1\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}\right)^{q-1} C_3\sigma_n^2 pd \\
&\le (T'+1)\frac{\eta}{n}\left(2C_1\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}\right)^{q-1} C_3\sigma_n^2 pd \\
&\le T\frac{\eta}{n}\left(2C_1\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}\right)^{q-1} C_3\sigma_n^2 pd \\
&\le C_4\sigma_0\sigma_n\sqrt{pd},
\end{aligned}$$

where the last inequality follows by picking $T = \frac{n\eta^{-1}C_4\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{C_3(2C_1)^{q-1}[\log d]^{(q-1)/2}} = \Theta(\frac{n\eta^{-1}\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{[\log d]^{(q-1)/2}})$. Therefore, by induction, we have $\Psi^{(t)} \le C_4\sigma_0\sigma_n\sqrt{pd}$ for all $t \in [T]$. ■

Lemma 31 (Formal Restatement of Theorem 5) *Under the same assumption as Theorem 15, there exists time* $T_1 = \frac{\log 2m^{-1/q}}{\log(1+\Theta(\frac{\eta}{K})\mu^q\sigma_0^{q-2})} = O(K\eta^{-1}\sigma_0^{2-q}\mu^{-q}\log 2m^{-1/q})$ *such that*

1. $\max_r \gamma_{j,r,j}^{(T_1)} \ge m^{-1/q}$ *for* $j \in [K]$.
2. $|\zeta_{j,r,i}^{(t)}|, |\omega_{j,r,i}^{(t)}| \le O(\sigma_0\sigma_n\sqrt{pd})$ *for all* $j \in [K], r \in [m], i \in [n]$ *and* $t \le T_1$.
3. $|\gamma_{j,r,k}^{(t)}| \le O(\sigma_0\mu\,\mathrm{poly}\log d)$ *for all* $j,k \in [K],\ j \ne k,\ r \in [m]$ *and* $t \le T_1$.

Proof Consider a fixed class $j \in [K]$. Denote T_1 to be the last time for $t \in \left[0, \frac{n\eta^{-1}C_4\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{C_3(2C_1)^{q-1}[\log d]^{(q-1)/2}}\right]$ satisfying $\max_r \gamma_{j,r}^{(t)} \le m^{-1/q}$. Then for $t \le T_1$, $\max_{j,r,i}\zeta_{j,r,i}^{(t)}, |\omega_{j,r,i}^{(t)}| \le O(\sigma_0\sigma_p\sqrt{pd}) \le O(m^{-1/q})$ and $\max_{j,r}\gamma_{j,r,j}^{(t)}$. Thus, by Lemma 27, we obtain that $F_j(\widetilde{\mathbf{W}}^{(t)}, \mathbf{x}_i) \le O(1),\ \forall y_i = j$. Thus, $\ell_{j,i}'^{(t)} = \Theta(1)$. For $j \in \mathcal{S}_{\text{signal}}^j$, we have

$$\begin{aligned}
&\gamma_{j,r,j}^{(t+1)} \\
&= \gamma_{j,r,j}^{(t)} - \frac{\eta}{n}\sum_{i=1}^n \ell_{j,i}'^{(t)} \cdot \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j\right\rangle + \gamma_{j,r,j}^{(t)} + \sum_{i'=1}^n \zeta_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_j\right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} + \sum_{i'=1}^n \omega_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_j\right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}\right)\left\|\boldsymbol{\mu}_j\right\|_2^2 \mathbb{I}(y_i = j) \\
&\ge \gamma_{j,r,j}^{(t)} - \frac{\eta}{n}\sum_{i=1}^n \ell_{j,i}'^{(t)}\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j\right\rangle + \gamma_{j,r,j}^{(t)} - O(n\sigma_0\sigma_n pd\frac{\sigma_n\mu\sqrt{\log d}}{\sigma_n^2 pd})\right)\mathbb{I}(y_i = j).
\end{aligned}$$

Let $\widehat{\gamma}_{j,r,j}^{(t)} = \gamma_{j,r,j}^{(t)} + \left\langle \widehat{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle - O(n\sigma_0\sigma_n\sqrt{pd}\frac{\sigma_n\mu\sqrt{\log d}}{\sigma_n^2 pd})$ and $A^{(t)} = \max_r \widehat{\gamma}_{j,r,j}^{(t)}$. Note that by our choice of μ, we have $\frac{n\mu\sqrt{\log d}}{\sigma_n pd} = o(1)$. Since $\max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle \geq \Omega(\sigma_0\mu)$ by Lemma 21, $\max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle \geq \Omega(\sigma_0\mu) - O(n\sigma_0\sigma_n pd\frac{\sigma_n\mu\sqrt{\log d}}{\sigma_n^2 pd}) = \Omega(\sigma_0\mu)$. Then we have

$$\begin{aligned}
A^{(t+1)} &\geq A^{(t)} - \frac{\eta}{n}\sum_{i=1}^{n} \ell_{j,i}'^{(t)} \sigma'(A^{(t)})\mu^2 \mathbb{I}(y_i = j) \\
&\geq A^{(t)} + \Theta(\frac{\eta}{K})\mu^2 [A^{(t)}]^{q-1} \\
&\geq (1 + \Theta(\frac{\eta}{K}\mu^2 [A^{(t)}]^{q-2}))A^{(t)} \\
&\geq (1 + \Theta(\frac{\eta}{K}\mu^q \sigma_0^{q-2}))A^{(t)}.
\end{aligned}$$

Therefore, the sequence $A^{(t)}$ will exponentially grow and will reach $2m^{-1/q}$ within $\frac{\log 2m^{-1/q}}{\log(1+\Theta(\frac{\eta}{K})\mu^q\sigma_0^{q-2})} = O(K\eta^{-1}\sigma_0^{2-q}\mu^{-q}\log 2m^{-1/q}) \leq \Theta(\frac{n\eta^{-1}\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{[\log d]^{(q-1)/2}})$. Thus, $\max_r \gamma_{j,r}^{(t)} \geq A^{(t)} - \max_{j,r} |\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle| \geq 2m^{-1/q} - O(\sigma_0\mu) \geq 2 - m^{-1/q} = m^{-1/q}$.

Now we prove that under the same assumption as Theorem 15, for $T = O(K\eta^{-1}\sigma_0^{2-q}\mu^{-q})$, we have $|\gamma_{j,r,k}^{(t)}| \leq O(\sigma_0\mu \operatorname{poly}\log d)$ for all $r \in [m],\ j,k \in [K],\ j \neq k$ and $t \leq T$.

We show that there exists a time $T' \geq T$ such that for all $t \leq T'$, $\max_{j,r,k} |\gamma_{j,r,k}^{(t)}| \leq O(\sigma_0\mu \operatorname{poly}\log d)$. Let $T' = O(K^2\eta^{-1}\sigma_0^{2-q}\mu^{-q}\log d)$.

Define $\Phi^{(t)} = \max_{r\in[m],\ j,k\in[K],\ j\neq k}\{|\gamma_{j,r,k}^{(t)}|\}$. Since we assume $T \leq \Theta(\frac{n\eta^{-1}\sigma_0^{2-q}(\sigma_n\sqrt{pd})^{-q}}{[\log d]^{(q-1)/2}})$, by Lemma 30, we have $\zeta_{j,r,i}^{(t)}, |\omega_{j,r,i}^{(t)}| \leq O(\sigma_0\sigma_n\sqrt{pd})$.

$$\begin{aligned}
&\Phi^{(t+1)} \\
&\leq \Phi^{(t)} + \max_{j,r,k,i}\left\{\frac{\eta}{n}\sum_{i=1}^{n}\mathbb{I}(y_i = k)|\ell_{j,i}'^{(t)}|\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_k \right\rangle + \sum_{i'=1}^{n}\zeta_{j,r,i'}^{(t)}\frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\mu}_k \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i'}\right\|_2^2} + \sum_{i'=1}^{n}\omega_{j,r,i'}^{(t)}\frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\mu}_k \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i'}\right\|_2^2}\right)\mu^2\right\} \\
&\leq \Phi^{(t)} + \frac{\eta}{K}\frac{1}{K}q\left(O(\sigma_0\mu\sqrt{\log d}) + nO(\sigma_0\sigma_n\sqrt{pd})\frac{\sigma_n\mu\sqrt{\log d}}{\sigma_n^2 pd})\right)^{q-1}\mu^2 \\
&\leq \Phi^{(t)} + \frac{q\eta}{K^2}\left(O(\sigma_0\mu\sqrt{\log d})\right)^{q-1}\mu^2,
\end{aligned}$$

where the first inequality follows because $\gamma_{j,r,k}^{(t)} < 0$, the second inequality follows because there are $\Theta(n/K)$ samples from a given class k and $|\ell_{j,i}'^{(t)}| = \Theta(\frac{1}{K})$, and the last inequality follows because $\mu = \sigma_n\sqrt{d}\log d$. Now, let C be the constant such that the above holds with big O. Then, we use induction to show that $\Phi^{(t)} \leq C_2\sigma_0\mu$ for all $t \leq T$. We proceed as follows.

$$\begin{aligned}
\Phi^{(t+1)} &\leq \Phi^{(t)} + \frac{q\eta}{K^2}\left(C\sigma_0\mu\sqrt{\log d}\right)^{q-1}\mu^2 \\
&\leq T\frac{q\eta}{K^2}\left(C\sigma_0\mu\sqrt{\log d}\right)^{q-1}\mu^2
\end{aligned}$$

$$\leq C_2\sigma_0\mu\text{poly}\log d,$$

where the last inequality follows by picking $T = \frac{C_2K^2\eta^{-1}\sigma_0^{2-q}\mu^{-q}\sqrt{\log d}}{C^{q-1}} = O(K^2\eta^{-1}\sigma_0^{2-q}\mu^{-q}\log d)$. ■

C.4 Converging Phase

In this subsection, we show that gradient descent can drive the training loss toward zero while the signal in the feature is still large. An important intermediate step in our argument is the development of the following gradient upper bound for multi-class cross-entropy loss.

In this phase, we are going to show that

- $\max_r \gamma_{j,r,j}^{(t)} \geq m^{1/q}$ for all $j \in [K]$.
- $\max_{j\neq k, r\in[m]} |\gamma_{j,r,k}^{(t)}| \leq \beta_1$ where $\beta_1 = \widetilde{O}(\sigma_0\mu)$.
- $\max_{j,r,i}\{\zeta_{j,r,i}^{(t)}, |\omega_{j,r,i}^{(t)}|\} \leq \beta_2$ where $\beta_2 = O(\sigma_0\sigma_n\sqrt{pd})$

Define $\mathbf{W}^\star$ as follows:

$$\mathbf{w}_{j,r}^\star = \mathbf{w}_{j,r}^{(0)} + \Theta(m\log(1/\epsilon))\frac{\boldsymbol{\mu}_j}{\mu^2}.$$

Lemma 32 *Based on the result from the feature growing phase,*

$$\left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star\right\|_F^2 \leq O(Km^3\log^2(1/\epsilon)\mu^{-2}).$$

Proof We first compute

$$\begin{aligned}
&\left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^{(0)}\right\|_F^2 \\
&= \sum_{j=1}^K\sum_{r=1}^m \left\| \gamma_{j,r,j}^{(T_1)}\frac{\boldsymbol{\mu}_j\odot\mathbf{m}_{j,r}}{\mu^2} + \sum_{k\neq j}\gamma_{j,r,k}^{(T_1)}\frac{\boldsymbol{\mu}_k\odot\mathbf{m}_{j,r}}{\mu^2} + \sum_i \zeta_{j,r,i}^{(T_1)}\frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} + \sum_i \omega_{j,r,i}^{(T_1)}\frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}\right\|_2^2 \\
&\leq \sum_j\sum_r \left(\gamma_{j,r,j}^{(T_1)}\frac{1}{\mu} + \sum_{k\neq j}\gamma_{j,r,k}^{(T_1)}\frac{1}{\mu} + \sum_i \zeta_{j,r,i}^{(T_1)}\frac{1}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2} + \sum_i \omega_{j,r,i}^{(T_1)}\frac{1}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2}\right)^2 \\
&\leq \sum_j\sum_r \left(\widetilde{O}(\frac{1}{\mu}) + K\widetilde{O}(\sigma_0) + n\widetilde{O}(\sigma_0)\right)^2 \\
&\leq \sum_j\sum_r \widetilde{O}(\frac{1}{\mu^2}) \\
&= \widetilde{O}(Km\frac{1}{\mu^2}),
\end{aligned}$$

where the first inequality follows from triangle inequality, the second inequality follows from Lemma 31, and the last inequality follows from our choice of σ_0. On the other hand,

$$\left\|\widetilde{\mathbf{W}}^{(0)} - \widetilde{\mathbf{W}}^{\star}\right\|_F^2 = \sum_{j,r} m^2 \log^2(1\epsilon)\frac{1}{\mu^2} = O(Km^3\log^2(1/\epsilon)\frac{1}{\mu^2}).$$

Thus, we obtain

$$\left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^{\star}\right\|_F^2 \leq 4\left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^{(0)}\right\|_F^2 + 4\left\|\widetilde{\mathbf{W}}^{(0)} - \widetilde{\mathbf{W}}^{\star}\right\|_F^2 \leq O(Km^3\log^2(1/\epsilon)\frac{1}{\mu^2}).$$

■

Lemma 33 (Gradient upper bound, formal version of Theorem 6) *Under Theorem 2, for $t \leq T^{\star}$, there exists constant $C = O(Km^{2/q}\max\{\mu^2, \sigma_n^2 pd\})$ such that*

$$\left\|\nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M}\right\|_F^2 \leq C L_S(\widetilde{\mathbf{W}}^{(t)}).$$

Proof We need to prove that $|\ell'^{(t)}_{y_i,i}|\left\|\nabla F(\widetilde{\mathbf{W}}^{(t)}, \mathbf{x}_i) \odot \mathbf{M}\right\|_F^2 \leq C$. Assume $y_i \neq j$. Then we obtain

$$\begin{aligned}
\left\|\nabla F_j(\widetilde{\mathbf{W}}_j, \mathbf{x}_i) \odot \mathbf{M}\right\|_F &\leq \sum_r \left\|\sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i}\right\rangle\right)\boldsymbol{\mu}_{y_i} + \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\widetilde{\boldsymbol{\xi}}_i\right\|_2 \\
&\leq \sum_r \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i}\right\rangle\right)\left\|\boldsymbol{\mu}_{y_i}\right\|_2 + \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i\right\rangle\right)\left\|\widetilde{\boldsymbol{\xi}}_i\right\|_2 \\
&\leq m^{1/q}\left[F_j(\widetilde{\mathbf{W}}_j, \mathbf{x}_i)\right]^{(q-1)/q}\max\{\mu, C\sigma_n\sqrt{pd}\} \\
&\leq m^{1/q}\max\{\mu, C\sigma_n\sqrt{pd}\},
\end{aligned}$$

where the first and second inequality follow from triangle inequality, the third inequality follows from Hölder's inequality, and the last inequality follows from Lemma 26. Similarly, on the other hand, if $y_i = j$, then

$$\left\|\nabla F_{y_i}(\widetilde{\mathbf{W}}) \odot \mathbf{M}\right\|_F \leq m^{1/q}\left[F_{y_i}(\widetilde{\mathbf{W}}_{y_i}, \mathbf{x}_i)\right]^{(q-1)/q}\max\{\mu, C\sigma_n\sqrt{pd}\}.$$

Therefore,

$$\begin{aligned}
\sum_{j\neq y_i}|\ell'^{(t)}_{j,i}|\left\|\nabla F_j(\widetilde{\mathbf{W}}_j, \mathbf{x}_i) \odot \mathbf{M}_j\right\|_F^2 &\leq \sum_{j\neq y_i}|\ell'^{(t)}_{j,i}|m^{2/q}O(\max\{\mu^2, \sigma_n^2 pd\}) \\
&= |\ell'^{(t)}_{y_i,i}|m^{2/q}O(\max\{\mu^2, \sigma_n^2 pd\}) \\
&\leq Ke\exp\{-F_{y_i}(\mathbf{x}_i)\}m^{2/q}O(\max\{\mu^2, \sigma_n^2 pd\}),
\end{aligned}$$

and

$$|\ell'^{(t)}_{y_i,i}|\left\|\nabla F_{y_i}(\widetilde{\mathbf{W}}_{y_i}, \mathbf{x}_i) \odot \mathbf{M}_{y_i}\right\|_F^2$$

$$\leq Ke\exp\{-F_{y_i}(\mathbf{x}_i)\}m^{2/q}\left[F_{y_i}(\widetilde{\mathbf{W}}_{y_i},\mathbf{x}_i)\right]^{2(q-1)/q}O(\max\{\mu^2,\sigma_n^2pd\},$$

where the inequality follows from Equation (2). Thus,

$$\begin{aligned}
&\sum_{j=1}^{K}|\ell_{j,i}'^{(t)}|^2\left\|\nabla F_j(\widetilde{\mathbf{W}}_j,\mathbf{x}_i)\odot\mathbf{M}_j\right\|_F^2\\
&\leq|\ell_{y_i,i}'^{(t)}|\sum_{j=1}^{K}|\ell_{j,i}'^{(t)}|\left\|\nabla F_j(\widetilde{\mathbf{W}}_j,\mathbf{x}_i)\odot\mathbf{M}_j\right\|_F^2\\
&\leq|\ell_{y_i,i}'^{(t)}|Ke\exp\{-F_{y_i}(\mathbf{x}_i)\}m^{2/q}O(\max\{\mu^2,\sigma_n^2pd\}\left(\left[F_{y_i}(\widetilde{\mathbf{W}}_{y_i},\mathbf{x}_i)\right]^{(q-1)/q}+1\right)\\
&\leq|\ell_{y_i,i}'^{(t)}|O(Km^{2/q}\max\{\mu^2,\sigma_n^2pd\}),
\end{aligned}\tag{4}$$

where the first inequality follows because $|\ell_{j,i}'^{(t)}|\leq|\ell_{y_i,i}'^{(t)}|$, and the last inequality uses the fact that $\exp\{-x\}(1+x^{(q-1)/q})=O(1)$ for all $x\geq 0$.

The gradient norm can be bounded by

$$\begin{aligned}
\left\|\nabla L_S(\widetilde{\mathbf{W}}^{(t)})\right\|_F^2&\leq\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\nabla L(\widetilde{\mathbf{W}}^{(t)},\mathbf{x}_i)\right\|_F\right)^2\\
&=\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{\sum_{j=1}^{K}|\ell_{j,i}'^{(t)}|^2\left\|\nabla F_j(\widetilde{\mathbf{W}}_j^{(t)},\mathbf{x}_i)\right\|_F^2}\right)^2\\
&\leq\left(\frac{1}{n}\sum_{i=1}^{n}|\ell_{y_i,i}'^{(t)}|\left\|\nabla F(\widetilde{\mathbf{W}}_j^{(t)},\mathbf{x}_i)\right\|_F\right)^2\\
&\leq\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{|\ell_{y_i,i}'^{(t)}|O(Km^{2/q}\max\{\mu^2,\sigma_n^2d\})}\right)^2\\
&\leq O(Km^{2/q}\max\{\mu^2,\sigma_n^2d\})\frac{1}{n}\sum_{i=1}^{n}|\ell_{y_i,i}'^{(t)}|\\
&\leq O(Km^{2/q}\max\{\mu^2,\sigma_n^2d\})L_S(\widetilde{\mathbf{W}}^{(t)}),
\end{aligned}$$

where the first inequality uses triangle inequality, the second inequality follows because $|\ell_{j,i}'^{(t)}|\leq|\ell_{y_i,i}'^{(t)}|$, the third inequality uses the bound (4), the fourth inequality uses Jensen's inequality and the last inequality follows because $|\ell_{y_i,i}'^{(t)}|\leq\ell_i^{(t)}$. ■

Lemma 34 *For $T_1\leq t\leq T^\star$, we have for all $j\neq y_i$,*

$$\left\langle\nabla F_{y_i}(\widetilde{\mathbf{W}}_{y_i}^{(t)},\mathbf{x}_i),\widetilde{\mathbf{W}}_{y_i}^{\star}\right\rangle-\left\langle\nabla F_j(\widetilde{\mathbf{W}}_j^{(t)},\mathbf{x}_i),\widetilde{\mathbf{W}}_j^{\star}\right\rangle\geq q\log\frac{2qK}{\epsilon}.$$

Proof [Proof of Lemma 34] The proof of this lemma depends on the next two lemmas.

Lemma 35 *For $T_1 \le t \le T^\star$ and $j = y_i$, we have $\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \ge \Theta(m^{1/q}\log(1/\epsilon))$.*

Proof By Lemma 31, we have

$$\begin{aligned}
\max_r \left\{ \left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_j \right\rangle \right\} &= \max_r \left\{ \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_j \right\rangle + \gamma_{j,r,j}^{(t)} + \sum_{i=1}^n \zeta_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_j \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\|_2^2} + \sum_{i=1}^n \omega_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_j \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\|_2^2} \right\} \\
&\ge m^{-1/q} - O(\sigma_0 \mu \sqrt{\log d}) - O(n\sigma_0\sigma_n\sqrt{d}\frac{\mu\sqrt{\log d}}{\sigma_n p d}) \\
&\ge \Theta(m^{-1/q}),
\end{aligned}$$

where the last inequality follows by picking $\sigma_0 \le O(m^{-1}n^{-1}\mu^{-1}(\log d)^{-1/2})$. On the other hand,

$$\left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle \right| \le \left| \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle \right| + |\omega_{j,r,i}^{(t)}| + |\zeta_{j,r,i}^{(t)}| + O(n\sqrt{\frac{\log d}{pd}}\alpha) + O(n\frac{\mu\sqrt{\log d}}{\sigma_n p d}\alpha) \le O(1), \quad (5)$$

where the first inequality follows from Lemma 25 and the second inequality follows from Equation (1) and Theorem 24. Therefore,

$$\begin{aligned}
&\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \\
&= \sum_r \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_j \right\rangle \right) \left\langle \boldsymbol{\mu}_j, \widetilde{\mathbf{w}}_{j,r}^\star \right\rangle + \sum_r \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle \right) \left\langle \boldsymbol{\xi}_i, \widetilde{\mathbf{w}}_{j,r}^\star \right\rangle \\
&\ge \sum_r \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_j \right\rangle \right) \Theta(m\log(1/\epsilon)) - \sum_r O(\sigma_0\sigma_n\sqrt{pd\log d} + \frac{\sigma_n\sqrt{\log d}}{\mu} m\log(1/\epsilon)) \\
&\ge \Theta(m^{1/q}\log(1/\epsilon)) - O(m\sigma_0\sigma_n\sqrt{pd\log d} + \frac{\sigma_n\sqrt{\log d}}{\mu} m^2\log(1/\epsilon)) \\
&\ge \Theta(m^{1/q}\log(1/\epsilon)),
\end{aligned}$$

where the last inequality follows because $m\sigma_0\sigma_n\sqrt{pd\log d} = o(1)$ and $\frac{\sigma_n\sqrt{\log d}}{\mu} m^2 = o(1)$ by our choices of μ, σ_0. ■

Lemma 36 *For $T_1 \le t \le T$ and $j \ne y_i$, we have $\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \le O(1)$.*

Proof First we have

$$\begin{aligned}
\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle &= \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\mu}_{y_i} \right\rangle + \gamma_{j,r,y_i}^{(t)} + \sum_{i=1}^n \zeta_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_j \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\|_2^2} + \sum_{i=1}^n \omega_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\mu}_j \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\|_2^2} \\
&\le O(\sigma_0\mu\sqrt{\log d} + \sigma_0\mu\,\mathrm{poly}\log d + n\sigma_0\sigma_n\sqrt{pd}\frac{\sigma_n\mu\sqrt{\log d}}{\sigma_n^2 p d}) \\
&\le O(1), \quad (6)
\end{aligned}$$

where the first inequality follows from Lemma 21, Lemma 19 and Lemma 31, and the last inequality follows from our choices of σ_0, μ. Then, we have

$$\begin{aligned}
&\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \\
&= \sum_r \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_{y_i} \right\rangle\right) \langle \boldsymbol{\mu}_{y_i}, \widetilde{\mathbf{w}}_{j,r}^\star \rangle + \sum_r \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right) \langle \boldsymbol{\xi}_i, \widetilde{\mathbf{w}}_{j,r}^\star \rangle \\
&\le mO(\sigma_0 \mu \sqrt{\log d}) + mO(\sigma_0 \sigma_n \sqrt{d \log d} + m \log(1/\epsilon) \frac{\sigma_n \sqrt{\log d}}{\mu}) \\
&\le O(1),
\end{aligned}$$

where the second inequality follows from Equation (6) and Equation (5), and the last inequality follows from our choices of μ, σ_0. ■

Applying the lower bound and upper bound from Lemma 35 and Lemma 36, we have

$$\begin{aligned}
&\left\langle \nabla F_{y_i}(\widetilde{\mathbf{W}}_{y_i}^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_{y_i}^\star \right\rangle - \left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \\
&\ge \Theta(m^{1/q} \log(1/\epsilon)) - O(1) \\
&\ge q \log \frac{2qK}{\epsilon}.
\end{aligned}$$

■

Lemma 37 *Under the same assumption as Theorem 15, we have*

$$\left\| \widetilde{\mathbf{W}}^{(t)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 - \left\| \widetilde{\mathbf{W}}^{(t+1)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 \ge 5\eta L_S(\widetilde{\mathbf{W}}^{(t)}) - \eta\epsilon.$$

Proof To simplify our notation, we define $\widehat{F}_j^{(t)}(\mathbf{x}_i) = \left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle$.

We use the fact that the network is q-homogeneous.

$$\begin{aligned}
&\left\| \widetilde{\mathbf{W}}^{(t)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 - \left\| \widetilde{\mathbf{W}}^{(t+1)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 \\
&= 2\eta \left\langle \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M}, \widetilde{\mathbf{W}}^{(t)} - \widetilde{\mathbf{W}}^\star \right\rangle - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&= \frac{2\eta}{n} \sum_{i=1}^n \sum_{j=1}^K \ell_{j,i}'^{(t)} \left[q F_j(\widetilde{\mathbf{W}}_j^{(t)}; \mathbf{x}_i, y_i) - \left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&\ge \frac{2q\eta}{n} \sum_{i=1}^n \left[\log(1 + \sum_{j=1}^K e^{F_j - F_{y_i}}) - \log(1 + \sum_{j=1}^K e^{(\hat{F}_j - \hat{F}_{y_i})/q}) \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&\ge \frac{2q\eta}{n} \sum_{i=1}^n \left[\ell(\widetilde{\mathbf{W}}^{(t)}; \mathbf{x}_i, y_i) - \log(1 + K e^{-\log(2qK/\epsilon)}) \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2
\end{aligned}$$

$$\geq \frac{2q\eta}{n}\sum_{i=1}^{n}\left[\ell(\widetilde{\mathbf{W}}^{(t)};\mathbf{x}_i,y_i)-\frac{\epsilon}{2q}\right]-\eta^2\left\|\nabla L_S(\widetilde{\mathbf{W}}^{(t)})\odot\mathbf{M}\right\|_F^2$$
$$\geq C\eta L_S(\widetilde{\mathbf{W}}^{(t)})-\eta\epsilon,$$

where the first inequality follows from the convexity of the cross-entropy loss with softmax, the second inequality follows from Lemma 34, the third inequality follows because $\log(1+x)\leq x$, and the last inequality follows from Lemma 33 for some constant C. ■

Lemma 38 (Formal Restatement of Theorem 7) *Under the same assumption as Theorem 15, choose* $T_2 = T_1 + \frac{\|\widetilde{\mathbf{W}}^{(T_1)}-\widetilde{\mathbf{W}}^\star\|_F^2}{2\eta\epsilon} = T_1 + \widetilde{O}(Km^3\log^2(1/\epsilon)\mu^{-2})$. *Then for any time* t *during this stage, we have* $\max_r \gamma_{j,r,j}^{(t)} \geq m^{1/q}$ *for all* $j\in[K]$, $\max_{j,r,i}\{|\zeta_{j,r,i}^{(t)}|,|\omega_{j,r,i}^{(t)}|\}\leq 2\beta_1$, $\max_{j\neq k,r\in[m]}\{|\gamma_{j,r,k}^{(t)}|\}\leq 2\beta_2$, *and*

$$\frac{1}{t-T_1}\sum_{s=T_1}^{t}L_S(\widetilde{\mathbf{W}}^{(s)})\leq\frac{\left\|\widetilde{\mathbf{W}}^{(T_1)}-\widetilde{\mathbf{W}}^\star\right\|_F^2}{C\eta(t-T_1)}+\frac{\epsilon}{C}.$$

Proof From Theorem 31, we have $max_r\gamma_{j,r,j}^{(T_1)}\geq m^{1/q}$ and since $\gamma^{(t)}$ is an increasing sequence over t, we have $max_r\gamma_{j,r,j}^{(t)}\geq m^{1/q}$ for all $t\in[T_1,T_2]$. We have

$$\left\|\widetilde{\mathbf{W}}^{(s)}-\widetilde{\mathbf{W}}^\star\right\|_F^2-\left\|\widetilde{\mathbf{W}}^{(s+1)}-\widetilde{\mathbf{W}}^\star\right\|_F^2\geq C\eta L_S(\widetilde{\mathbf{W}}^{(s)})-\eta\epsilon.$$

Taking a telescopic sum from T_1 to t yields

$$\sum_{s=T_1}^{t}L_S(\widetilde{\mathbf{W}}^{(s)})\leq\frac{\left\|\widetilde{\mathbf{W}}^{(T_1)}-\widetilde{\mathbf{W}}^\star\right\|_F^2+\eta\epsilon(t-T_1)}{C\eta}.$$

Combining Lemma 32, we have

$$\sum_{s=T_1}^{t}L_S(\widetilde{\mathbf{W}}^{(s)})\leq O(\eta^{-1}\left\|\widetilde{\mathbf{W}}^{(T_1)}-\widetilde{\mathbf{W}}^\star\right\|_F^2)=O(\eta^{-1}Km^3\log^2(1/\epsilon)\mu^{-2}).\tag{7}$$

Define $\Psi^{(t)}=\max_{j,r,i}\{\zeta_{j,r,i}^{(t)},|\omega_{j,r,i}^{(t)}|\}$ and $\Phi^{(t)}=\max_{j\neq k,r\in[m]}|\gamma_{j,r,k}^{(t)}|$ and $\beta_2=\widetilde{O}(\sigma_0\mu)$. Now we use induction to prove $\Psi^{(t)}\leq 2\beta_1$ and $\Phi^{(t)}\leq 2\beta_2$. Suppose the result holds for time $t\leq t'$. Then

$$\Psi^{(t+1)}$$
$$\leq\Psi^{(t)}+\max_{j,r,i}\left\{\frac{\eta}{n}|\ell_{j,i}'^{(t)}|\cdot\sigma'\left(\left\langle\widetilde{\mathbf{w}}_{j,r}^{(0)},\boldsymbol{\xi}_i\right\rangle+\sum_{k=1}^{K}\gamma_{j,r,k}^{(t)}\frac{\langle\boldsymbol{\mu}_k,\boldsymbol{\xi}_i\rangle}{\|\boldsymbol{\mu}_k\|_2^2}+\sum_{i'=1}^{n}\Psi^{(t)}\frac{\left\langle\widetilde{\boldsymbol{\xi}}_{j,r,i'},\boldsymbol{\xi}_i\right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i'}\right\|_2^2}\right)\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2\right\}$$

$$\leq \Psi^{(t)} + \frac{\eta}{n} q \max_i |\ell'^{(t)}_{y_i,i}| \Bigg(O(\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}) + K\log^{1/q} T^\star \frac{\mu\sigma_n\sqrt{\log d}}{\mu^2} + \frac{O(\sigma_n^2 pd) + nO(\sigma_n^2\sqrt{pd\log d})}{\Theta(\sigma_n^2 pd)} \Psi^{(t)} \Bigg)^{q-1} O(\sigma_n^2 pd)$$

$$\leq \Psi^{(t)} + \frac{\eta}{n}\sum_{i=1}^n |\ell'^{(t)}_{y_i,i}| \left(O(\sqrt{\log d}\sigma_0\sigma_n\sqrt{pd}) + O(\Psi^{(t)}) \right)^{q-1} O(\sigma_n^2 pd),$$

where the second inequality follows by $|\ell'^{(t)}_{j,i}| \leq |\ell'^{(t)}_{y_i,i}|$ and applying the bounds from Lemma 19, and the last inequality follows by choosing $\frac{K\log^{1/q} T^\star}{\sqrt{d}} = \widetilde{O}(\frac{1}{\sqrt{d}}) \ll \sigma_0\sigma_n\sqrt{pd}$. Unrolling the recursion by taking a sum from T_1 to t' we have

$$\begin{aligned}
\Psi^{(t'+1)} &\overset{(i)}{\leq} \Psi^{(T_1)} + \frac{\eta}{n}\sum_{s=T_1}^{t'}\sum_{i=1}^n |\ell'^{(s)}_{y_i,i}| O(\sigma_n^2 pd\,\mathrm{poly}\log d)\beta_1^{q-1} \\
&\overset{(ii)}{\leq} \Psi^{(T_1)} + \frac{\eta}{n} O(\sigma_n^2 pd\,\mathrm{poly}\log d)\beta_1^{q-1}\sum_{s=T_1}^{t'}\sum_{i=1}^n \ell_i^{(s)} \\
&= \Psi^{(T_1)} + \frac{\eta}{n} O(\sigma_n^2 pd\,\mathrm{poly}\log d)\beta_1^{q-1}\sum_{s=T_1}^{t'} L_S(\widetilde{\mathbf{W}}^{(s)}) \\
&\overset{(iii)}{\leq} \Psi^{(T_1)} + \frac{1}{n} O(Km^3\mu^{-2}\sigma_n^2 pd\,\mathrm{poly}\log d)\beta_1^{q-1} \\
&\overset{(iv)}{\leq} \beta_1 + \widetilde{O}(Km^3)\beta_1^{q-1} \\
&\overset{(v)}{\leq} 2\beta_1,
\end{aligned}$$

where (i) follows from induction hypothesis $\Psi^{(t)} \leq 2\beta_1$, (ii) follows from the property of cross-entropy loss with softmax $|\ell'_{j,i}| \leq |\ell'_{y_i,i}| \leq \ell_i$, (iii) follows from Equation (7), (iv) follows from our choice of μ, n, K, and (v) follows because $\widetilde{O}(Km^3){\beta_1}^{q-2} \leq \widetilde{O}(Km^3\sigma_0\sigma_n\sqrt{pd}) \leq 1$. Therefore, by induction $\Psi^{(t)} \leq 2\beta_1$ holds for time $t \leq t'+1$.

On the other hand,

$$\begin{aligned}
&\Phi^{(t'+1)} \\
&\overset{(i)}{\leq} \Phi^{(t)} + \max_{j,r,k,i} \left\{ \frac{\eta}{n}\sum_{i=1}^n \mathbb{I}(y_i = k)|\ell'^{(t)}_{j,i}|\sigma' \left(\left\langle \widetilde{\mathbf{w}}^{(0)}_{j,r}, \boldsymbol{\mu}_k \right\rangle + \sum_{i'=1}^n \zeta^{(t)}_{j,r,i'} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\mu}_k \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i'} \right\|_2^2} + \sum_{i'=1}^n \omega^{(t)}_{j,r,i'} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\mu}_k \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i'} \right\|_2^2} \right) \mu^2 \right\} \\
&\overset{(ii)}{\leq} \Phi^{(t)} + \Theta(\frac{\eta}{K}) \max_{j,i} |\ell'^{(t)}_{j,i}| \left(O(\sigma_0\mu\sqrt{\log d}) + nO(\sigma_0\sigma_n\sqrt{pd})\frac{\sigma_n\mu\sqrt{\log d}}{\sigma_n^2 pd}) \right)^{q-1} \mu^2 \\
&\overset{(iii)}{\leq} \Phi^{(T_1)} + \Theta(\frac{\eta}{K})\mu^2 \sum_{s=T_1}^t \sum_{i=1}^n \ell_i^{(s)} \left(O(\sigma_0\mu\sqrt{\log d}) \right)^{q-1} \\
&\overset{(iv)}{\leq} \beta_2 + O(m^3)\beta_2^{q-1}
\end{aligned}$$

$\overset{(v)}{\leq} 2\beta_2,$

where (i) follows because $\gamma_{j,r,k}^{(t)} \leq 0$, (ii) follows from Lemma 21 and Lemma 19, (iii) follows because $\max_{j,i} |\ell_{j,i}'^{(t)}| \leq \max_i |\ell_{y_i,i}'^{(t)}| \leq \max_i \ell_i^{(t)} \leq \sum_i \ell_i^{(t)}$, (iv) follows from Equation (7), and (v) follows because $O(m^3)\beta_2^{q-2} \leq \widetilde{O}(m^3\sigma_0\mu) \leq 1$. ■

C.5 Generalization Analysis

In this subsection, we show that pruning can purify the feature by reducing the variance of the noise by a factor of p when a new sample is given.

Now the network has parameter

$$\widetilde{\mathbf{w}}_{j,r}^\star = \widetilde{\mathbf{w}}_{j,r}^{(0)} + \sum_{k=1}^K \gamma_{j,r,k}^\star \frac{\boldsymbol{\mu}_k \odot \mathbf{m}_{j,r}}{\mu^2} + \sum_{i=1}^n \zeta_{j,r,i}^\star \frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} + \sum_{i=1}^n \omega_{j,r,i}^\star \frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}.$$

We have $\left\|\widetilde{\mathbf{w}}_{j,r}^\star\right\|_2 = O(\sigma_0\sqrt{pd} + \mu^{-1}\log^{1/q}(T^\star) + K\sigma_0 \operatorname{poly}\log d + n\sigma_0\sigma_n\sqrt{pd}\frac{1}{\sigma_n\sqrt{pd}}) = O(\sigma_0\sqrt{pd})$.

Lemma 39 (Same as Theorem 8) *With probability at least* $1 - 2Km\exp\left(-\frac{(2m)^{-4/q}}{O(\sigma_0^2\sigma_n^2 pd)}\right)$,

$$\max_{j,r} \left|\left\langle \widetilde{\mathbf{w}}_{j,r}^\star, \boldsymbol{\xi}\right\rangle\right| \leq (2m)^{-2/q}.$$

Proof Since $\left\langle \widetilde{\mathbf{w}}_{j,r}^\star, \boldsymbol{\xi}\right\rangle$ follows a Gaussian distribution with variance $O(\sigma_0^2\sigma_n^2 pd)$, we have

$$\mathbb{P}\left[\left|\left\langle \widetilde{\mathbf{w}}_{j,r}^\star, \boldsymbol{\xi}\right\rangle\right| \geq (2m)^{-2/q}\right] \leq 2\exp\left(-\frac{(2m)^{-4/q}}{O(\sigma_0^2\sigma_n^2 pd)}\right).$$

Applying a union bound over $j \in [K], r \in [m]$ gives the result. ■

Theorem 40 (Formal Restatement of Generalization Part of Theorem 3) *Under the same assumptions as Theorem 15, within* $\widetilde{O}(K\eta^{-1}\sigma_0^{2-q}\mu^{-q} + K^2m^4\mu^{-2}\eta^{-1}\epsilon^{-1})$ *iterations, we can find* $\widetilde{\mathbf{W}}^\star$ *such that*

- $L_S(\widetilde{\mathbf{W}}^\star) \leq \epsilon$.
- $L_{\mathcal{D}} \leq O(K\epsilon) + \exp(-n^2/p)$.

Proof Let $\mathcal{E}$ be the event that Lemma 39 holds. Then, we can divide $L_{\mathcal{D}}(\widetilde{\mathbf{W}}^\star)$ into two parts:

$$\mathbb{E}[\ell(F(\widetilde{\mathbf{W}}^\star, \mathbf{x}))] = \underbrace{\mathbb{E}[\mathbb{I}(\mathcal{E})\ell(F(\widetilde{\mathbf{W}}^\star, \mathbf{x}))]}_{I_1} + \underbrace{\mathbb{E}[\mathbb{I}(\mathcal{E}^c)\ell(F(\widetilde{\mathbf{W}}^\star, \mathbf{x}))]}_{I_2}.$$

Since $L_S(\widetilde{\mathbf{W}}^\star) \le \epsilon$, for each class $j \in [K]$ there must exist one training sample $(\mathbf{x}_i, y_i) \in S$ with $y_i = j$ such that $\ell(F(\widetilde{\mathbf{W}}^\star, \mathbf{x}_i)) \le K\epsilon \le 1$ by pigeonhole principle. This implies that $\sum_{j' \neq j} \exp(F_{j'}(\mathbf{x}_i) - F_j(\mathbf{x}_i)) \le 2K\epsilon$. Conditioning on the event $\mathcal{E}$, by Lemma 39, we have

$$\begin{aligned}
|F_j(\widetilde{\mathbf{W}}^\star, \mathbf{x}) - F_j(\widetilde{\mathbf{W}}^\star, \mathbf{x}_i)| &\le \sum_r \sigma(\langle \widetilde{\mathbf{w}}_{j,r}^\star, \boldsymbol{\xi}_i \rangle) + \sum_r \sigma(\langle \widetilde{\mathbf{w}}_{j,r}^\star, \boldsymbol{\xi} \rangle) \\
&\le \sum_r (2m)^{-1} + \sum_r (2m)^{-1} \\
&\le 1.
\end{aligned}$$

Thus, we have $\exp(F_{j'}(\mathbf{x}) - F_j(\mathbf{x})) \le 2K\epsilon e^2 = O(K\epsilon)$. Next we bound the term I_2.

$$\begin{aligned}
\ell(F(\widetilde{\mathbf{W}}^\star, \mathbf{x})) &= \log\left(1 + \sum_{j' \neq y} \exp(F_{j'}(\mathbf{x}) - F_y(\mathbf{x}))\right) \\
&\le \log\left(1 + \sum_{j' \neq y} \exp(F_{j'}(\mathbf{x}))\right) \\
&\le \sum_{j' \neq y} \log(1 + \exp(F_{j'}(\mathbf{x}))) \\
&\le K + \sum_{j' \neq y} F_{j'}(\mathbf{x}) \\
&= K + \sum_{j' \neq y} \sigma(\langle \widetilde{\mathbf{w}}_{j',r}^\star, \boldsymbol{\mu}_y \rangle) + \sigma(\langle \widetilde{\mathbf{w}}_{j',r}^\star, \boldsymbol{\xi} \rangle) \\
&\le K + Km(O(\sigma_0 \mu \sqrt{\log d}))^q + \widetilde{O}(m(\sigma_0 \sigma_n \sqrt{d})^q) \, \|\boldsymbol{\xi}/\sigma_n\|_2^q \\
&\le 2K + \|\boldsymbol{\xi}/\sigma_n\|_2^q, \qquad (8)
\end{aligned}$$

where the first inequality follows because $F_y(\mathbf{x}) \ge 0$, the second and third inequalities follow from the property of log function, and the last inequality follows from our choice of $\sigma_0 \le \widetilde{O}(m^{-4} n^{-1} \sigma_n^{-1} d^{-1/2})$. We further have

$$\begin{aligned}
I_2 &\le \sqrt{\mathbb{E}[\mathbb{I}(\mathcal{E})]} \sqrt{\mathbb{E}[\ell(F(\widetilde{\mathbf{W}}^\star, \mathbf{x}))^2]} \\
&\le \sqrt{\mathbb{P}(\mathcal{E}^c)} \sqrt{4K^2 + \mathbb{E}\, \|\boldsymbol{\xi}/\sigma_n\|_2^{2q}} \\
&\le \exp(-C m^{-2/q} \sigma_0^{-2} \sigma_n^{-2} p^{-1} d^{-1} + \log(d)) \\
&\le \exp(-n^2/p),
\end{aligned}$$

where the first inequality follows from Cauchy-Schwarz inequality, the second inequality follows from Equation (8), the third inequality follows from Lemma 39, and the last inequality follows because $\sigma_0 \le \widetilde{O}(m^{-4} n^{-1} \sigma_n^{-1} d^{-1/2})$.

■

Appendix D. Proof of Theorem 9

In this section, we show that there exists a relatively large pruning fraction (i.e., small p) such that while gradient descent is still able to drive the training error toward zero, the learned model yields poor generalization. We first provide a formal restatement of Theorem 9.

Theorem 41 (Formal Version of Theorem 9) *Under Condition 2, choose initialization variance $\sigma_0 = \widetilde{\Theta}(m^{-4}n^{-1}\mu^{-1})$ and learning rate $\eta \leq \widetilde{O}(1/\mu^2)$. For $\epsilon > 0$, if $p = \Theta(\frac{1}{Km\log d})$, then with probability at least $1 - 1/\log(d)$, there exists $T = O(\eta^{-1}n\sigma_0^{q-2}\sigma_n^{-q}(pd)^{-q/2} + \eta^{-1}\epsilon^{-1}m^4n\sigma_n^{-2}(pd)^{-1})$ such that the following holds:*

1. *The training loss is below ϵ: $L_S(\widetilde{\mathbf{W}}^{(T)}) \leq \epsilon$.*
2. *The model weight doesn't learn any of its corresponding signal at all: $\gamma_{j,r,j}^{(t)} = 0$ for all $j \in [K],\ r \in [m]$.*
3. *The model weights is highly correlated with the noise: $\max_{r\in[m]} \zeta_{j,r,i}^{(T)} \geq \Omega(m^{-1/q})$ if $y_i = j$.*

Moreover, the testing loss is large:

$$L_{\mathcal{D}}(\widetilde{\mathbf{W}}^{(T)}) \geq \Omega(\log K).$$

The proof of Theorem 9 consists of the analysis of the over-pruning for three stages of gradient descent: initialization, feature growing phase, and converging phase, and the establishment of the generalization property. We present these analysis in detail in the following subsections.

D.1 Initialization

Lemma 42 *When $m = \operatorname{poly}\log d$ and $p = \Theta(\frac{1}{Km\log d})$, with probability $1 - O(1/\log d)$, for all class $j \in [K]$ we have $|\mathcal{S}_{\text{signal}}^j| = 0$.*

Proof First, the probability that a given class j receives no signal is $(1-p)^m$. We use the inequality that

$$1 + t \geq \exp\{O(t)\} \quad \forall t \in (-1/4, 1/4).$$

Then the probability that $|\mathcal{S}_{\text{signal}}^j| = 0,\ \forall j \in [K]$ is given by

$$(1-p)^{Km} \geq \exp\{-O(pKm)\} \geq 1 - O\left(\frac{1}{\log d}\right).$$

■

D.2 Feature Growing Phase

Lemma 43 (Formal Restatement of Theorem 12) *Under the same assumption as Theorem 41, there exists $T_1 < T^\star$ such that $T_1 = O(\eta^{-1} n \sigma_0^{q-2} \sigma_n^{-q} (pd)^{-q/2})$ and we have*

- $\max_r \zeta_{y_i,r,i} \geq m^{-1/q}$ *for all* $i \in [n]$.
- $\max_{j,r,i} |\omega_{j,r,i}^{(t)}| = \widetilde{O}(\sigma_0 \sigma_n \sqrt{pd})$.
- $\max_{j,r,k} |\gamma_{j,r,k}^{(t)}| \leq \widetilde{O}(\sigma_0 \mu)$.

Proof First of all, recall that from Definition 14 we have for $j = y_i$

$$\begin{aligned}
&\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle \\
&= \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \zeta_{j,r,i}^{(t)} + \sum_{k \neq j} \gamma_{j,r,k}^{(t)} \frac{\left\langle \boldsymbol{\mu}_k, \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\rangle}{\mu^2} + \sum_{i' \neq i} \zeta_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\xi}_i \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i'} \right\|_2^2} + \sum_{i'=1}^{n} \omega_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i'}, \boldsymbol{\xi}_i \right\rangle}{\left\| \widetilde{\boldsymbol{\xi}}_{j,r,i'} \right\|_2^2}.
\end{aligned}$$

Let

$$B_i^{(t)} = \max_{j=y_i,r} \left\{ \zeta_{j,r,i}^{(t)} + \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle - O(n \log^{1/q} T^\star \sqrt{\frac{\log d}{pd}}) - O(n \sigma_0 \sigma_n \sqrt{pd} \sqrt{\frac{\log d}{pd}}) \right\}.$$

Since $\max_{j=y_i,r} \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle \geq \Omega(\sigma_0 \sigma_n \sqrt{pd})$, we have

$$B_i^{(0)} \geq \Omega(\sigma_0 \sigma_n \sqrt{pd}) - O(n \log^{1/q} T^\star \sqrt{\frac{\log d}{pd}}) - O(n \sigma_0 \sigma_n \sqrt{pd} \sqrt{\frac{\log d}{pd}}) \geq \Omega(\sigma_0 \sigma_n \sqrt{pd}).$$

Let T_i to be the last time that $\zeta_{j,r,i}^{(t)} \leq m^{-1/q}$. We can compute the growth of $B_i^{(t)}$ as

$$\begin{aligned}
B_i^{(t+1)} &\geq B_i^{(t)} + \Theta(\frac{\eta \sigma_n^2 pd}{n}) [B_i^{(t)}]^{q-1} \\
&\geq B_i^{(t)} + \Theta(\frac{\eta \sigma_n^2 pd}{n}) [B_i^{(0)}]^{q-2} B_i^{(t)} \\
&\geq \left(1 + \Theta \left(\frac{\eta \sigma_0^{q-2} \sigma_n^q p^{q/2} d^{q/2}}{n} \right) \right) B_i^{(t)}.
\end{aligned}$$

Therefore, $B_i^{(t)}$ will reach $2m^{-1/q}$ within $\widetilde{O}(\eta^{-1} n \sigma_0^{q-2} \sigma_n^{-q} (pd)^{-q/2})$ iterations.

On the other hand, by Proposition 24, we have $|\omega_{j,r,i}^{(t)}| \leq \beta + 6Cn\alpha \sqrt{\frac{\log d}{pd}} = O(\sigma_0 \sigma_n \sqrt{pd \log d})$. ■

D.3 Converging Phase

From the first stage we know that

$$\widetilde{\mathbf{w}}_{j,r}^{(T_1)} = \widetilde{\mathbf{w}}_{j,r}^{(0)} + + \sum_{k\neq j} \gamma_{j,r,k}^{(t)} \frac{\boldsymbol{\mu}_k \odot \mathbf{m}_{j,r}}{\mu^2} + \sum_{i=1}^{n} \zeta_{j,r,i}^{(T_1)} \frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} + \sum_{i=1}^{n} \omega_{j,r,i}^{(T_1)} \frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}.$$

Now we define $\widetilde{\mathbf{W}}^\star$ as follows:

$$\widetilde{\mathbf{w}}_{j,r}^\star = \widetilde{\mathbf{w}}_{j,r}^{(0)} + \Theta(m\log(1/\epsilon)) \left[\sum_{i=1}^{n} \mathbb{I}(j=y_i) \frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}\right].$$

Lemma 44 *Based on the result from feature growing phase,*

$$\left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star\right\|_F \leq O(m^2 n^{1/2}\log(1/\epsilon)\sigma_n^{-1}(pd)^{-1/2}).$$

Proof We derive the following bound:

$$\begin{aligned}
&\left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star\right\|_F \\
&\leq \left\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^{(0)}\right\|_F + \left\|\widetilde{\mathbf{W}}^{(0)} - \widetilde{\mathbf{W}}^\star\right\|_F \\
&\leq \sum_{j,r}\left(\left\|\sum_{k\neq j}\gamma_{j,r,k}^{(t)}\frac{\boldsymbol{\mu}_k}{\mu^2}\right\|_2 + \left\|\sum_{i=1}^{n}\zeta_{j,r,i}^{(T_1)}\frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}\right\|_2 + \left\|\sum_{i=1}^{n}\omega_{j,r,i}^{(T_1)}\frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}\right\|_2\right) \\
&\quad + \Theta(m^2 n^{1/2}\log(1/\epsilon)\sigma_n^{-1}(pd)^{-1/2}) \\
&\leq Km(O(\sqrt{K}\sigma_0) + O(n^{1/2}\sigma_n^{-1}(pd)^{-1/2}\log^{1/q}T^\star)) + \widetilde{O}(m^2 n^{1/2}\log(1/\epsilon)\sigma_n^{-1}(pd)^{-1/2}) \\
&\leq \widetilde{O}(m^2 n^{1/2}\log(1/\epsilon)\sigma_n^{-1}(pd)^{-1/2}),
\end{aligned}$$

where the first inequality follows from triangle inequality, the second inequality follows from the expression of $\mathbf{W}^{(T_1)}, \mathbf{W}^\star$, and the third inequality follows from Lemma 19 and the fact that $\zeta_{j,r,i}^{(t)} > 0$ if and only if $j = y_i$. ■

Lemma 45 *For $T_1 \leq t \leq T^\star$, we have*

$$\left\langle \nabla F_{y_i}(\widetilde{\mathbf{W}}_{y_i}, \mathbf{x}_i), \widetilde{\mathbf{W}}_{y_i}^\star \right\rangle - \left\langle \nabla F_j(\widetilde{\mathbf{W}}_j, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \geq q\log\frac{2qK}{\epsilon}.$$

Lemma 46 *For $T_1 \leq t \leq T^\star$ and $j = y_i$, we have*

$$\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \geq \Theta(m^{1/q}\log(1/\epsilon)).$$

Proof By Lemma 19, we have $\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \widetilde{\mathbf{w}}_{j,r}^{\star} \right\rangle = \Theta(m\log(1/\epsilon))$ and by Lemma 43 for $j = y_i$, $\max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle \geq \max_r \zeta_{j,r,i} - \max_r \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle - O(n\sqrt{\frac{\log d}{d}}\alpha) \geq \Theta(m^{-1/q})$. Then we have

$$\begin{aligned}\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^{\star} \right\rangle &= \sum_{r=1}^{m} \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right) \left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \widetilde{\mathbf{w}}_{j,r}^{\star} \right\rangle \\ &\geq \Theta(m^{1/q}\log(1/\epsilon)).\end{aligned}$$

■

Lemma 47 *For $T_1 \leq t \leq T^{\star}$ and $j \neq y_i$, we have*

$$\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^{\star} \right\rangle \leq O(1).$$

Proof We first compute $\left\langle \widetilde{\mathbf{w}}_{j,r}^{\star}, \boldsymbol{\xi}_i \right\rangle = \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \Theta(m\log(1/\epsilon)) \sum_{i=1}^{n} \mathbb{I}(j = y_i) \frac{\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\xi}_i \rangle}{\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\|_2^2} = O(\sigma_0\sigma_n\sqrt{pd\log d})$. Further,

$$\begin{aligned}&\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle \\ &= \left\langle \widetilde{\mathbf{w}}_{j,r}^{(0)}, \boldsymbol{\xi}_i \right\rangle + \sum_{k\neq j} \gamma_{j,r,k}^{(t)} \frac{\left\langle \boldsymbol{\mu}_k, \widetilde{\boldsymbol{\xi}}_{j,r,i} \right\rangle}{\mu^2} + \sum_{i=1}^{n} \zeta_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\xi}_i \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} + \sum_{i=1}^{n} \omega_{j,r,i}^{(t)} \frac{\left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \boldsymbol{\xi}_i \right\rangle}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} \\ &\leq O(\sigma_0\sigma_n\sqrt{pd\log d}),\end{aligned}$$

where the inequality follows from Lemma 19 and Lemma 29. Thus, we have

$$\begin{aligned}\left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^{\star} \right\rangle &= \sum_{r=1}^{m} \sigma'\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}_i \right\rangle\right) \left\langle \widetilde{\boldsymbol{\xi}}_{j,r,i}, \widetilde{\mathbf{w}}_{j,r}^{\star} \right\rangle \\ &\leq mO\left(\sigma_0\sigma_n\sqrt{pd\log d}\right)^q \\ &\leq O(1),\end{aligned}$$

where the last inequality follows from our choice of $\sigma_0 \leq \widetilde{O}(m^{-1/q}\mu^{-1})$. ■

Lemma 48 *Under the same assumption as Theorem 41, we have*

$$\left\|\mathbf{W}^{(t)} - \mathbf{W}^{\star}\right\|_F^2 - \left\|\mathbf{W}^{(t+1)} - \mathbf{W}^{\star}\right\|_F^2 \geq C\eta L_S(\widetilde{\mathbf{W}}^{(t)}) - \eta\epsilon.$$

Proof To simplify our notation, we define $\widehat{F}_j^{(t)}(\mathbf{x}_i) = \left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^{\star} \right\rangle$. The proof is exactly the same as the proof of Lemma 37.

$$\left\|\widetilde{\mathbf{W}}^{(t)} - \widetilde{\mathbf{W}}^{\star}\right\|_F^2 - \left\|\widetilde{\mathbf{W}}^{(t+1)} - \widetilde{\mathbf{W}}^{\star}\right\|_F^2$$

$$
\begin{aligned}
&= 2\eta \left\langle \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M}, \widetilde{\mathbf{W}}^{(t)} - \widetilde{\mathbf{W}}^\star \right\rangle - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&= \frac{2\eta}{n} \sum_{i=1}^n \sum_{j=1}^K \ell_{j,i}'^{(t)} \left[q F_j(\widetilde{\mathbf{W}}_j^{(t)}; \mathbf{x}_i, y_i) - \left\langle \nabla F_j(\widetilde{\mathbf{W}}_j^{(t)}, \mathbf{x}_i), \widetilde{\mathbf{W}}_j^\star \right\rangle \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&\geq \frac{2q\eta}{n} \sum_{i=1}^n \left[\log(1 + \sum_{j=1}^K e^{F_j - F_{y_i}}) - \log(1 + \sum_{j=1}^K e^{(\hat{F}_j - \hat{F}_{y_i})/q}) \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&\geq \frac{2q\eta}{n} \sum_{i=1}^n \left[\ell(\widetilde{\mathbf{W}}^{(t)}; \mathbf{x}_i, y_i) - \log(1 + K e^{-\log(2qK/\epsilon)}) \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&\geq \frac{2q\eta}{n} \sum_{i=1}^n \left[\ell(\widetilde{\mathbf{W}}^{(t)}; \mathbf{x}_i, y_i) - \frac{\epsilon}{2q} \right] - \eta^2 \left\| \nabla L_S(\widetilde{\mathbf{W}}^{(t)}) \odot \mathbf{M} \right\|_F^2 \\
&\geq C\eta L_S(\widetilde{\mathbf{W}}^{(t)}) - \eta\epsilon,
\end{aligned}
$$

where the first inequality follows from the convexity of the cross-entropy loss with softmax, the second inequality follows from Lemma 34, the third inequality follows because $\log(1+x) \leq x$, and the last inequality follows from Lemma 33 for some constant $C > 0$. ■

Lemma 49 (Formal Restatement of Theorem 13) *Under the same assumption as Theorem 41, choose* $T_2 = T_1 + \left\lceil \frac{\|\widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star\|_F^2}{2\eta\epsilon} \right\rceil = T_1 + \widetilde{O}(\eta^{-1}\epsilon^{-1}m^4 n\sigma_n^{-2}(pd)^{-1})$. *Then for any time* t *during this stage we have* $\max_{j,r} |\omega_{j,r,i}^{(t)}| = O(\sigma_0\sqrt{pd})$ *and*

$$
\frac{1}{t - T_1} \sum_{s=T_1}^t L_S(\widetilde{\mathbf{W}}^{(s)}) \leq \frac{\left\| \widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star \right\|_F^2}{C\eta(t - T_1)} + \frac{\epsilon}{C}.
$$

Proof We have

$$
\left\| \widetilde{\mathbf{W}}^{(s)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 - \left\| \widetilde{\mathbf{W}}^{(s+1)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 \geq C\eta L_S(\widetilde{\mathbf{W}}^{(s)}) - \eta\epsilon.
$$

Taking a telescopic sum from T_1 to t yields

$$
\sum_{s=T_1}^t L_S(\widetilde{\mathbf{W}}^{(s)}) \leq \frac{\left\| \widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star \right\|_F^2 + \eta\epsilon(t - T_1)}{C\eta}.
$$

Combining Lemma 44, we have

$$
\sum_{s=T_1}^t L_S(\widetilde{\mathbf{W}}^{(s)}) \leq O(\eta^{-1} \left\| \widetilde{\mathbf{W}}^{(T_1)} - \widetilde{\mathbf{W}}^\star \right\|_F^2) = \widetilde{O}(\eta^{-1}m^4 n\sigma_n^{-2}(pd)^{-1}).
$$

■

D.4 Generalization Analysis

Theorem 50 (Formal Version of the Generalization Result of Theorem 9) *Under the same assumption as Theorem 41, within $O(\eta^{-1}n\sigma_0^{q-2}\sigma_n^{-q}(pd)^{-q/2}+\eta^{-1}\epsilon^{-1}m^4n\sigma_n^{-2}(pd)^{-1})$ iterations, we can find $\widetilde{\mathbf{W}}^{(T)}$ such that $L_S(\widetilde{\mathbf{W}}^{(T)}) \le \epsilon$, and $L_{\mathcal{D}}(\widetilde{\mathbf{W}}^{(t)}) \ge \Omega(\log K)$.*

Proof First of all, from Theorem 49 we know there exists $t \in [T_1, T_2]$ such that $L_S(\widetilde{\mathbf{W}}^{(T)}) \le \epsilon$. Then, we can bound

$$\begin{aligned}
\left\|\widetilde{\mathbf{w}}_{j,r}^{(t)}\right\|_2 &= \left\|\widetilde{\mathbf{w}}_{j,r}^{(0)} + \sum_{k\neq j}\gamma_{j,r,k}^{(t)}\frac{\boldsymbol{\mu}_k}{\mu^2} + \sum_{i=1}^{n}\zeta_{j,r,i}^{(t)}\frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2} + \sum_{i=1}^{n}\omega_{j,r,i}^{(t)}\frac{\widetilde{\boldsymbol{\xi}}_{j,r,i}}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2^2}\right\|_2 \\
&\le \left\|\widetilde{\mathbf{w}}_{j,r}^{(0)}\right\|_2 + \sum_{k\neq j}|\gamma_{j,r,k}^{(t)}|\frac{1}{\mu} + \sum_{i=1}^{n}\zeta_{j,r,i}^{(t)}\frac{1}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2} + \sum_{i=1}^{n}|\omega_{j,r,i}^{(t)}|\frac{1}{\left\|\widetilde{\boldsymbol{\xi}}_{j,r,i}\right\|_2} \\
&\le O(\sigma_0\sqrt{d}) + \widetilde{O}(n\sigma_n^{-1}(pd)^{-1/2}).
\end{aligned}$$

Consider a new example $(\mathbf{x}, y)$. Taking a union bound over r, with probability at least $1 - d^{-1}$, we have

$$\left|\left\langle \mathbf{w}_{y,r}^{(t)}, \boldsymbol{\xi}\right\rangle\right| = \widetilde{O}(\sigma_0\sigma_n\sqrt{d} + n(pd)^{-1/2}),$$

for all $r \in [m]$. Then,

$$\begin{aligned}
F_y(\mathbf{x}) &= \sum_{r=1}^{m}\sigma\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\mu}_y\right\rangle\right) + \sigma\left(\left\langle \widetilde{\mathbf{w}}_{j,r}^{(t)}, \boldsymbol{\xi}\right\rangle\right) \\
&\le m\max_r\left|\left\langle \mathbf{w}_{y,r}^{(t)}, \boldsymbol{\xi}\right\rangle\right|^q \\
&\le m\widetilde{O}(\sigma_0^q\sigma_n^q d^{q/2} + n^q(pd)^{-q/2}) \\
&\le 1,
\end{aligned}$$

where the last inequality follows because $\sigma_0 \le \widetilde{O}(m^{-1/q}\mu^{-1})$ and $d \ge \widetilde{\Omega}(m^{2/q}n^2)$. Thus, with probability at least $1 - 1/d$,

$$\ell(F(\widetilde{\mathbf{W}}^{(t)}; \mathbf{x})) \ge \log(1 + (K-1)e^{-1}).$$

■

References

Zeyuan Allen-Zhu and Yuanzhi Li. Towards understanding ensemble, knowledge distillation and self-distillation in deep learning. ***arXiv preprint arXiv:2012.09816***, 2020.

Zeyuan Allen-Zhu and Yuanzhi Li. Feature purification: How adversarial training performs robust deep learning. In ***2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)***, pages 977–988. IEEE, 2022.

Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via over-parameterization. In ***International Conference on Machine Learning***, pages 242–252. PMLR, 2019.

Zachary Ankner, Alex Renda, Gintare Karolina Dziugaite, Jonathan Frankle, and Tian Jin. The effect of data dimensionality on neural network prunability. ***arXiv preprint arXiv:2212.00291***, 2022.

Sanjeev Arora, Simon Du, Wei Hu, Zhiyuan Li, and Ruosong Wang. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. In ***International Conference on Machine Learning***, pages 322–332. PMLR, 2019.

Yuan Cao and Quanquan Gu. Generalization bounds of stochastic gradient descent for wide and deep neural networks. ***Advances in neural information processing systems***, 32, 2019.

Yuan Cao, Zixiang Chen, Mikhail Belkin, and Quanquan Gu. Benign overfitting in two-layer convolutional neural networks. ***arXiv preprint arXiv:2202.06526***, 2022.

Tianlong Chen, Zhenyu Zhang, Santosh Balachandra, Haoyu Ma, Zehao Wang, Zhangyang Wang, et al. Sparsity winning twice: Better robust generalization from more efficient training. In ***International Conference on Learning Representations***, 2021a.

Tianyi Chen, Bo Ji, Tianyu Ding, Biyi Fang, Guanyi Wang, Zhihui Zhu, Luming Liang, Yixin Shi, Sheng Yi, and Xiao Tu. Only train once: A one-shot neural network training and pruning framework. ***Advances in Neural Information Processing Systems***, 34, 2021b.

Xiaohan Chen, Yu Cheng, Shuohang Wang, Zhe Gan, Jingjing Liu, and Zhangyang Wang. The elastic lottery ticket hypothesis. ***Github Repository, MIT License***, 2021c.

Zixiang Chen, Yuan Cao, Quanquan Gu, and Tong Zhang. A generalized neural tangent kernel analysis for two-layer neural networks. ***Advances in Neural Information Processing Systems***, 33:13363–13373, 2020a.

Zixiang Chen, Yuan Cao, Difan Zou, and Quanquan Gu. How much over-parameterization is sufficient to learn deep relu networks? In ***International Conference on Learning Representations***, 2020b.

Lenaic Chizat and Francis Bach. On the global convergence of gradient descent for over-parameterized models using optimal transport. ***Advances in neural information processing systems***, 31, 2018.

Lenaic Chizat, Edouard Oyallon, and Francis Bach. On lazy training in differentiable programming. ***Advances in Neural Information Processing Systems***, 32, 2019.

Li Deng. The mnist database of handwritten digit images for machine learning research [best of the web]. ***IEEE signal processing magazine***, 29(6):141–142, 2012.

Simon Du, Jason Lee, Haochuan Li, Liwei Wang, and Xiyu Zhai. Gradient descent finds global minima of deep neural networks. In ***International conference on machine learning***, pages 1675–1685. PMLR, 2019.

Utku Evci, Trevor Gale, Jacob Menick, Pablo Samuel Castro, and Erich Elsen. Rigging the lottery: Making all tickets winners. In ***International Conference on Machine Learning***, pages 2943–2952. PMLR, 2020.

Cong Fang, Jason Lee, Pengkun Yang, and Tong Zhang. Modeling from features: a mean-field framework for over-parameterized deep neural networks. In ***Conference on learning theory***, pages 1887–1936. PMLR, 2021.

Jonathan Frankle and Michael Carbin. The lottery ticket hypothesis: Finding sparse, trainable neural networks. In ***International Conference on Learning Representations***, 2018.

Jonathan Frankle, Gintare Karolina Dziugaite, Daniel Roy, and Michael Carbin. Pruning neural networks at initialization: Why are we missing the mark? In ***International Conference on Learning Representations***, 2020.

Spencer Frei, Niladri S Chatterji, and Peter L Bartlett. Benign overfitting without linearity: Neural network classifiers trained by gradient descent for noisy linear data. ***arXiv preprint arXiv:2202.05928***, 2022.

Trevor Gale, Erich Elsen, and Sara Hooker. The state of sparsity in deep neural networks. ***arXiv preprint arXiv:1902.09574***, 2019.

Margalit Glasgow, Colin Wei, Mary Wootters, and Tengyu Ma. Max-margin works while large margin fails: Generalization without uniform convergence. In ***The Eleventh International Conference on Learning Representations***.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In ***Proceedings of the IEEE conference on computer vision and pattern recognition***, pages 770–778, 2016.

Yihui He, Xiangyu Zhang, and Jian Sun. Channel pruning for accelerating very deep neural networks. In ***Proceedings of the IEEE international conference on computer vision***, pages 1389–1397, 2017.

Zheng He, Zeke Xie, Quanzhi Zhu, and Zengchang Qin. Sparse double descent: Where network pruning aggravates overfitting. In ***International Conference on Machine Learning***, pages 8635–8659. PMLR, 2022.

Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks. ***Advances in neural information processing systems***, 31, 2018.

Siddhant Jayakumar, Razvan Pascanu, Jack Rae, Simon Osindero, and Erich Elsen. Top-kast: Top-k always sparse training. ***Advances in Neural Information Processing Systems***, 33:20744–20754, 2020.

Ziwei Ji and Matus Telgarsky. Polylogarithmic width suffices for gradient descent to achieve arbitrarily small test error with shallow relu networks. In ***International Conference on Learning Representations***, 2019.

Tian Jin, Michael Carbin, Daniel M Roy, Jonathan Frankle, and Gintare Karolina Dziugaite. Pruning's effect on generalization through the lens of training and regularization. In ***Advances in Neural Information Processing Systems***, 2022.

Stefani Karp, Ezra Winston, Yuanzhi Li, and Aarti Singh. Local signal adaptivity: Provable feature learning in neural networks beyond kernels. ***Advances in Neural Information Processing Systems***, 34:24883–24897, 2021.

Jeremy Kepner and Ryan Robinett. Radix-net: Structured sparse matrices for deep neural networks. In ***2019 IEEE International Parallel and Distributed Processing Symposium Workshops (IPDPSW)***, pages 268–274. IEEE, 2019.

Alex Krizhevsky, Geoffrey Hinton, et al. Learning multiple layers of features from tiny images. 2009.

Yann LeCun, John Denker, and Sara Solla. Optimal brain damage. ***Advances in neural information processing systems***, 2, 1989.

Jaehoon Lee, Lechao Xiao, Samuel Schoenholz, Yasaman Bahri, Roman Novak, Jascha Sohl-Dickstein, and Jeffrey Pennington. Wide neural networks of any depth evolve as linear models under gradient descent. ***Advances in neural information processing systems***, 32, 2019.

Namhoon Lee, Thalaiyasingam Ajanthan, and Philip Torr. Snip: Single-shot network pruning based on connection sensitivity. In ***International Conference on Learning Representations***, 2018.

Shiwei Liu, Tianlong Chen, Xiaohan Chen, Zahra Atashgahi, Lu Yin, Huanyu Kou, Li Shen, Mykola Pechenizkiy, Zhangyang Wang, and Decebal Constantin Mocanu. Sparse training via boosting pruning plasticity with neuroregeneration. ***Advances in Neural Information Processing Systems***, 34, 2021a.

Shiwei Liu, Tianlong Chen, Xiaohan Chen, Li Shen, Decebal Constantin Mocanu, Zhangyang Wang, and Mykola Pechenizkiy. The unreasonable effectiveness of random pruning: Return of the most naive baseline for sparse training. In ***International Conference on Learning Representations***, 2021b.

Shiwei Liu, Decebal Constantin Mocanu, Amarsagar Reddy Ramapuram Matavalam, Yulong Pei, and Mykola Pechenizkiy. Sparse evolutionary deep learning with over one million artificial neurons on commodity hardware. ***Neural Computing and Applications***, 33 (7):2589–2604, 2021c.

Shiwei Liu, Lu Yin, Decebal Constantin Mocanu, and Mykola Pechenizkiy. Do we actually need dense over-parameterization? in-time over-parameterization in sparse training. In ***International Conference on Machine Learning***, pages 6989–7000. PMLR, 2021d.

Tianlin Liu and Friedemann Zenke. Finding trainable sparse networks through neural tangent transfer. In ***International Conference on Machine Learning***, pages 6336–6347. PMLR, 2020.

Jian-Hao Luo and Jianxin Wu. An entropy-based pruning method for cnn compression. ***arXiv preprint arXiv:1706.05791***, 2017.

Eran Malach, Gilad Yehudai, Shai Shalev-Schwartz, and Ohad Shamir. Proving the lottery ticket hypothesis: Pruning is all you need. In ***International Conference on Machine Learning***, pages 6682–6691. PMLR, 2020.

Zelda Mariet and Suvrit Sra. Diversity networks: Neural network compression using determinantal point processes. ***arXiv preprint arXiv:1511.05077***, 2015.

Decebal Constantin Mocanu, Elena Mocanu, Phuong H Nguyen, Madeleine Gibescu, and Antonio Liotta. A topological insight into restricted boltzmann machines. ***Machine Learning***, 104(2):243–270, 2016.

Decebal Constantin Mocanu, Elena Mocanu, Peter Stone, Phuong H Nguyen, Madeleine Gibescu, and Antonio Liotta. Scalable training of artificial neural networks with adaptive sparse connectivity inspired by network science. ***Nature communications***, 9(1):1–12, 2018.

P Molchanov, S Tyree, T Karras, T Aila, and J Kautz. Pruning convolutional neural networks for resource efficient inference. In ***5th International Conference on Learning Representations, ICLR 2017-Conference Track Proceedings***, 2019.

Hesham Mostafa and Xin Wang. Parameter efficient training of deep convolutional neural networks by dynamic sparse reparameterization. In ***International Conference on Machine Learning***, pages 4646–4655. PMLR, 2019.

Samet Oymak and Mahdi Soltanolkotabi. Toward moderate overparameterization: Global convergence guarantees for training shallow neural networks. ***IEEE Journal on Selected Areas in Information Theory***, 1(1):84–105, 2020.

Ankit Pensia, Shashank Rajput, Alliot Nagle, Harit Vishwakarma, and Dimitris Papailiopoulos. Optimal lottery tickets via subset sum: Logarithmic over-parameterization is sufficient. ***Advances in Neural Information Processing Systems***, 33:2599–2610, 2020.

Alexandra Peste, Eugenia Iofinova, Adrian Vladu, and Dan Alistarh. Ac/dc: Alternating compressed/decompressed training of deep neural networks. ***Advances in Neural Information Processing Systems***, 34, 2021.

Ameya Prabhu, Girish Varma, and Anoop Namboodiri. Deep expander networks: Efficient deep networks from graph theory. In ***Proceedings of the European Conference on Computer Vision (ECCV)***, pages 20–35, 2018.

Vivek Ramanujan, Mitchell Wortsman, Aniruddha Kembhavi, Ali Farhadi, and Mohammad Rastegari. What's hidden in a randomly weighted neural network? In ***Proceedings of the IEEE CVF Conference on Computer Vision and Pattern Recognition***, pages 11893–11902, 2020.

Grant M Rotskoff and Eric Vanden-Eijnden. Neural networks as interacting particle systems: Asymptotic convexity of the loss landscape and universal scaling of the approximation error. ***stat***, 1050:22, 2018.

Zhenmei Shi, Junyi Wei, and Yingyu Liang. A theoretical analysis on feature learning in neural networks: Emergence from inputs and advantage over fixed features. In ***International Conference on Learning Representations***, 2021.

Justin Sirignano and Konstantinos Spiliopoulos. Mean field analysis of neural networks: A law of large numbers. ***SIAM Journal on Applied Mathematics***, 80(2):725–752, 2020.

Mei Song, Andrea Montanari, and P Nguyen. A mean field view of the landscape of two-layers neural networks. ***Proceedings of the National Academy of Sciences***, 115(33): E7665–E7671, 2018.

Zhao Song and Xin Yang. Quadratic suffices for over-parametrization via matrix chernoff bound. ***arXiv preprint arXiv:1906.03593***, 2019.

Zhao Song, Shuo Yang, and Ruizhe Zhang. Does preprocessing help training over-parameterized neural networks? ***Advances in Neural Information Processing Systems***, 34, 2021.

Kartik Sreenivasan, Shashank Rajput, Jy-yong Sohn, and Dimitris Papailiopoulos. Finding everything within random binary networks. ***arXiv preprint arXiv:2110.08996***, 2021.

Kartik Sreenivasan, Jy-yong Sohn, Liu Yang, Matthew Grinde, Alliot Nagle, Hongyi Wang, Kangwook Lee, and Dimitris Papailiopoulos. Rare gems: Finding lottery tickets at initialization. ***arXiv preprint arXiv:2202.12002***, 2022.

Jingtong Su, Yihang Chen, Tianle Cai, Tianhao Wu, Ruiqi Gao, Liwei Wang, and Jason D Lee. Sanity-checking pruning methods: Random tickets can win the jackpot. ***Advances in Neural Information Processing Systems***, 33:20390–20401, 2020.

Xavier Suau, Luca Zappella, and Nicholas Apostoloff. Network compression using correlation analysis of layer responses. 2018.

Hidenori Tanaka, Daniel Kunin, Daniel L Yamins, and Surya Ganguli. Pruning neural networks without any data by iteratively conserving synaptic flow. ***Advances in Neural Information Processing Systems***, 33:6377–6389, 2020.

Matus Telgarsky. Feature selection with gradient descent on two-layer networks in low-rotation regimes. ***arXiv preprint arXiv:2208.02789***, 2022.

Chaoqi Wang, Guodong Zhang, and Roger Grosse. Picking winning tickets before training by preserving gradient flow. In ***International Conference on Learning Representations***, 2019.

Colin Wei, Jason D Lee, Qiang Liu, and Tengyu Ma. Regularization matters: Generalization and optimization of neural nets vs their induced kernel. ***Advances in Neural Information Processing Systems***, 32, 2019.

Huanrui Yang, Wei Wen, and Hai Li. Deephoyer: Learning sparser neural network with differentiable scale-invariant sparsity measures. In ***International Conference on Learning Representations***, 2020. URL `https://openreview.net/forum?id=rylBK34FDS`.

Qing Yang, Jiachen Mao, Zuoguan Wang, and "Helen" Li Hai. Dynamic regularization on activation sparsity for neural network efficiency improvement. ***ACM Journal on Emerging Technologies in Computing Systems (JETC)***, 17(4):1–16, 2021.

Mao Ye, Chengyue Gong, Lizhen Nie, Denny Zhou, Adam Klivans, and Qiang Liu. Good subnetworks provably exist: Pruning via greedy forward selection. In ***International Conference on Machine Learning***, pages 10820–10830. PMLR, 2020.

Sergey Zagoruyko and Nikos Komodakis. Wide residual networks. In ***British Machine Vision Conference 2016***. British Machine Vision Association, 2016.

Shuai Zhang, Meng Wang, Sijia Liu, Pin-Yu Chen, and Jinjun Xiong. Why lottery ticket wins? a theoretical perspective of sample complexity on sparse neural networks. ***Advances in Neural Information Processing Systems***, 34:2707–2720, 2021.

Hattie Zhou, Janice Lan, Rosanne Liu, and Jason Yosinski. Deconstructing lottery tickets: Zeros, signs, and the supermask. ***Advances in neural information processing systems***, 32, 2019.

Jinxin Zhou, Xiao Li, Tianyu Ding, Chong You, Qing Qu, and Zhihui Zhu. On the optimization landscape of neural collapse under mse loss: Global optimality with unconstrained features. ***arXiv preprint arXiv:2203.01238***, 2022.

Zhihui Zhu, Tianyu Ding, Jinxin Zhou, Xiao Li, Chong You, Jeremias Sulam, and Qing Qu. A geometric analysis of neural collapse with unconstrained features. ***Advances in Neural Information Processing Systems***, 34, 2021.

Difan Zou and Quanquan Gu. An improved analysis of training over-parameterized deep neural networks. ***Advances in neural information processing systems***, 32, 2019.

Difan Zou, Yuan Cao, Dongruo Zhou, and Quanquan Gu. Gradient descent optimizes over-parameterized deep relu networks. ***Machine Learning***, 109(3):467–492, 2020.

Difan Zou, Yuan Cao, Yuanzhi Li, and Quanquan Gu. Understanding the generalization of adam in learning neural networks with proper regularization. ***arXiv preprint arXiv:2108.11371***, 2021.