SCAFFOLDING MOVES THAT ELICIT MODELING COMPETENCIES

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Situated within efforts to understand the complex interplay among learners, teachers, and tasks in mathematical modeling education, we examine how contingent scaffolding moves influence the modeling process. Using mixed methods, we coordinated qualitative frameworks for scaffolding and modeling competencies through their application to task-based cognitive interviews with undergraduate STEM majors. A mixed logistic regression model with participant random effect analyzed the temporally-linked frequencies of codes. The model sustains claims about the compatibility of the frameworks and predicts moves eliciting competencies.

Keywords: modeling, advanced mathematical thinking, cognition, mathematical representations

In any didactic situation, there is a triadic interaction among the learner, the teacher, and the task environment (Brousseau, 1997; Koichu & Harel, 2007). Understanding how the teacher influences the interaction between learner and task environment is a major research objective in mathematics education. In learning environments that focus on developing mathematical modeling skills, learners are assumed to enter with real-world knowledge (and therefore assumptions) that may not afford the intended mathematics (Cai et al., 2014). A number of studies have shown that educators may respond to the learners' work in ways that amount to consistent negative feedback or diminish learner autonomy in decision-making while modeling (Verschaffel et al., 2020). Additionally, support which may, on the surface, seem adaptive to error is not always contingent to a learner's in the moment needs (Wischgoll et al., 2015). For these reasons, educators have sought means for scaffolding learners' modeling processes that maintain cognitive demand, endorse and extend their autonomous ways of reasoning, and do not inadvertently teach the idea that there is a "school math" entirely distinct from "real math" (see Nunes et al., 1985; Watson, 2008). Our study is situated within the broader agenda to understand which kinds of scaffolding moves are effective in supporting modelers as they learn to construct and validate meaningful models of real-world scenarios. In particular, the aim of this study was to investigate the influence of facilitators' micro interventions on undergraduate STEM majors' modeling processes.

Literature Review

Mathematical modeling is a cognitive process.

Cognitive perspectives on mathematical modeling conceive it as a process of transforming a question about the real world into a mathematically well-posed problem (Kaiser, 2017). For example, one way the question *How rapidly will a disease spread through a community?* can be answered is by using the equation $\frac{dS}{dt} = \tau SH$ as a model of the scenario. Using an equation as a model means the modeler constructs quantitative meanings for the variables S and H which represent the number of sick people and healthy people at time t, respectively. This sub-process is known as *mathematizing*. Another important sub-process of modeling is *validating*. This is done by adopting (implied or explicit) assumptions about how the world works and evaluating the adequacy of the resulting representation against those assumptions. Assuming that having the disease does not confer immunity to it is consistent with the model in the example. In general, modelers decide which real-world conditions and assumptions are important (or not) to incorporate into their model as mathematical properties, parameters, and relationships (Schwarzkopf, 2007; Zbiek & Conner, 2006). This sub-process is known as *simplifying & structuring*. Mathematical modeling cycles (MMCs) provide a descriptive framework that organizes the cognitive sub-processes as a set of phases connecting stages of model construction (Blum & Leiß, 2007). Table 1 shows Blum and Leiß (2007)'s cycle for the stages a modeler passes through and the sub-processes that connect those stages.

Table 1 Modeling competencies from (Blum & Leiß, 2007)

Sub-Processes	Definition	Connects Stages
Understanding	Forming an initial idea about what the problem is asking for	real world → situation model
Simplifying & Structuring	Identify (un)important real-world entities and relationships	situation model → real model
Mathematizing	Represent idealized version of the real-world problem using mathematical conventions	real model → mathematical model
Working Mathematically	Analyze or solve mathematical problem	mathematical model → mathematical results
Interpreting	Re-contextualize mathematical results	mathematical results → real results
Validating	Verify results against constraints	real results → real situation

Many studies have investigated the sub-processes and their manifestations across grade levels and content areas (Cevikbas et al., 2021), the characteristics of tasks that evoke them (Bock et al., 2015; Maaß, 2010), and the challenges learners face in carrying them out (Galbraith & Stillman, 2006; Klock & Siller, 2020). Importantly, many studies have found modeling does not proceed linearly through the sub-processes (Ärlebäck & Bergsten, 2010; Borromeo Ferri, 2007; Czocher, 2016, 2018). Despite the low predictive power of MMC's, they remain powerful descriptive models of desirable learner engagement with modeling tasks. There are robust analytic frameworks of observational indicators for which sub-process the modeler is engaged with that are applicable across content areas and grade bands (Czocher, 2016; Maaß, 2006). Within *working mathematically*, for example, learners are seen to exhibit procedural and conceptual mathematics knowledge whereas during *simplifying* and *validating*, learners are seen to articulate and justify assumptions they make and may not draw overtly on mathematical knowledge at all. Because carrying out the sub-processes successfully is critical to constructing a viable mathematical model, they are styled as *modeling competencies* (Maaß, 2006). Modeling competencies are learning objectives in their own right and a major goal of research in modeling is understanding how a teacher, who is using modeling problems to teach mathematics (or to teach modeling), can scaffold and thereby promote learners' modeling competencies.

Scaffolding in mathematical modeling ought to be contingent.

Scaffolding a learner as they develop and validate a model has two goals. The local goal is helping the modeler arrive at a viable model for a particular problem. The global goal is promoting competencies that can be used in other problems. Both are challenging because facilitators need to focus on learners' current knowledge and understanding as it is expressed within a given sub-process of the MMC (Blum & Borromeo Ferri, 2009; Doerr, 2006; Schukajlow et al., 2015; Stender & Kaiser, 2015; Wischgoll et al., 2015). The high-level idea is that because the nature of a learner's engagement in modeling changes across modeling competencies, there are likely to be differing (and specific) moves a facilitator can make that would support each sub-process. Providing hints towards a normatively correct mathematical representation when the learner is mulling over which variables are important to include robs her of the modeling experience and does little to cultivate competencies. Investigating this conjecture calls for a view of scaffolding suitable for studying learners' productions and their relations to facilitator moves at a within-task grain size, rather than broader views that take into account classroom-level organization or cross-lesson supports (Anghileri, 2006). For these reasons, the active trend in modeling research is to adopt a Vygotskian view of scaffolding as an interactive process between a teacher and a learner that gives support to the learner as she works on a task she might not otherwise be able to accomplish (van de Pol et al., 2010, p. 274).

Building on the scaffolding means and intentions framework (van de Pol et al., 2010; van de Pol et al., 2015), Stender and Kaiser (2015) assumed that scaffolding the modeling process may be productive under three conditions: the learner has disengaged (and therefore requires motivation to re-engage in the problem), the learner asks a question, or the learner has been working unproductively for an appreciable time and does not realize it. The latter case presents the most challenging aspect of designing and evaluating scaffolding moves. Effective in-the-moment scaffolding is *contingent*, meaning that the proffered support increases facilitator control when the learner is struggling and decreases control when the learner is succeeding. In some studies, contingency is conceived as being along three-point ordinal scale (van de Pol et al., 2015). Çakmak Gürel (2023) examined the interplay between teachers' participation structures and their scaffolding methods and found that the level of support could vary according to modeling competency. These findings did not directly relate level of support to modeling competency, instead showing that scaffolding method is mediated by the teachers' preferred form of engagement in the classroom. Additionally, modeling tasks can be quite open and learners' engagement in the modeling process is idiosyncratic, based in part on their highly individual previous knowledge and experiences (Borromeo Ferri, 2006; Stillman, 2000). Thus, contingency for scaffolding modeling processes means adapting support to be responsive to the particularities of a learner's constructed knowledge and how it manifests during the modeling process, not only attending to the accuracy of learners' intermediate productions – requiring the facilitator to engage in diagnostic activities before intervening (Kaiser & Stender, 2013).

To address the research need for analyzing contingent support, Stender (2016) developed a framework to capture contingent interventions in learners' modeling processes that are responsive to a learner's current conceptual and (partially formed) mathematical models of the real world scenario, anticipate the specific cognitive needs of the learner, and are calibrated to provide minimal in-the-moment support such that the learner will retain control of their modeling process (excerpt in Table 2). We focus on the "B3 Codes", which classify the contingent moves. Stender and Kaiser (2015) found that requesting leaners to summarize the work they'd done thus far (code B3.1 Work Status) enabled them to continue working independently or aided the facilitator in diagnosing their work to proffer further supports, regardless of how far along the learners were in model development. Stender and Kaiser (2015) also found some expected associations between scaffolding moves and particular modeling competencies. Thus, some scaffolding moves could be competency-general while others may be capable of promoting specific competencies. Stender and Kaiser (2015) also cautioned that it was not always possible for them to determine success of an intervention because there wasn't sufficient information in the students' work. They focused on only on the few minutes before and after the facilitators' intervention into a few focal groups' work and on normative correctness of the students' models. In this study, we used task-based cognitive interviews to generate facilitator-learner interactions that could be analyzed for the extent that scaffolding moves promote modeling competencies. This maximized the amount of information available to the facilitator for informing which moves to attempt and to the analysis for examining the impact of the proffered support. We address the research question: *Which modeling competencies were more frequently elicited by which kinds of scaffolding moves?*

Table 2 Scaffolding moves (Stender & Kaiser, 2015), fitted with instances from this study.

Code Name	Description	Rule to use	Example
B3.1 Work status	Learner asked to describe current work status or what they are currently working on	Can be a direct question or implicit; Intended to orient facilitator to the learner's reasoning	Can you summarize what you have done here, so far? Can you share what you're thinking about?
B3.6 Prompt to include real-world aspects	Learner asked or encouraged to include a certain aspect	Learner asked to add an aspect to the model. Can be used to increase complexity or to draw attention to specific variable or quantity	Are there any factors that negatively influence the number of current infections?
B3.10 Request reason or explanation	Interviewer requests a reason, explanation, or justification	The reason can be about assumptions made, refer to algebra steps, or to the whole modeling process	Why did you choose multiplication here? What leads you to think that way?

Methods

We used explanatory sequential mixed methods (Creswell, 2014). We deductively coded task-based interviews according to the modeling competencies framework for participants' modeling processes and the contingent scaffolding framework for interviewer moves. The quantitative analysis used mixed logistic regression model with a participant random effect.

Data Collection

Twenty four undergraduate STEM majors at a large southern university were recruited from differential equations courses or courses listing it as a pre-requisite to participate in a set of 10 hour-long task based interviews conducted over zoom. Each participant worked on between four and eleven modeling tasks across the sessions. The modeling tasks had well-defined goals and ill-defined answers (Yeo, 2007) and were designed based on canonical problems from differential equations featuring feedback loops. We studied participants' model construction (*simplifying & structuring, mathematizing*), interpretations of models (*interpreting*), and justifications of model adequacy (*validating*) and ignored *understanding* and *working mathematically*. The tasks were given as written statements so the *understanding* competency is primarily indicated by "reading the problem statement" (Czocher, 2016), and occurs without contingent scaffolding. Additionally, many of the resulting differential equations models cannot be solved analytically, so we did not ask for their solutions (also, contingent support would be highly tailored to the mathematical content instead of participants' modeling needs).

The tasks were sequenced so that later tasks presented scenarios whose mathematical structures subsumed the structures of earlier tasks. The first 4 tasks included embedded scaffolding (sub-tasks) oriented towards learners' quantitative reasoning to aid them in constructing or transferring quantitative structures to the task scenario (Moore et al., 2022; Thompson, 2011). The remaining tasks did not include embedded scaffolding and featured only contingent scaffolding provided by the interviewer. Not every participant saw all tasks, depending on how "far" they got through the trajectory, which was based on their capacity to work autonomously on the tasks. A lead interviewer and a witness from the research team were present during each interview (Steffe & Thompson, 2000). The interviewer intervened in the participants' modeling process if the participant requested help, if it seemed that the participant got stuck, or to generate and test conjectures about the participants' ways of reasoning about the mathematical or real-world aspects of the task. The probing questions were designed to focus on aspects of quantities and quantitative reasoning, but overall interviewer turns were formulated so they could map to the scaffolding moves framework. The alignment between interviewer moves and the scaffolding framework was achieved through several rounds of pilot interviews and subsequent analysis, not reported here. In this paper, we consider only the tasks without embedded scaffolding to isolate the influence of contingent scaffolding. The dataset for this study comprised 51 hour-long modeling sessions.

Table 3 Summary of interview tasks

Task Name	Intended Canonical Model	n	Median Task Time
Tropical Fish	Contaminated tank	18	1:10:11
Tuberculosis	Two compartment disease transmission	17	0:47:07
Ebola	Three compartment disease transmission	11	0:53:03
Bobcats & Rabbits	Two-species predator prey	2	1:04:44
Diffusion	Fick's first law (one dimension)	2	0:20:18
Kidneys	Dialysis across a one-dimensional membrane	1	0:42:26

Data Analysis

Qualitative analysis proceeded with deductive coding procedures based on pre-defined, published codes for engagement in modeling processes (MMC codes) and contingent scaffolding moves (B3 codes). Participant engagement and interviewer moves were coded separately. To code for participant engagement, we viewed the videos in MaxQDA and assigned an MMC code if an indicator for that code could be observed in the participant's speech or writing. The MMC codes were not mutually exclusive; a participant's actions at a given time could indicate both *interpreting* and *validating*, for example. Start and stop times for the codes were determined independently of the start and stop times for other codes. Because much of the modeling process takes place in the mind of the modeler, when a modeler "really starts" to make assumptions is not accessible information. Thus, we assigned a timestamp to the earliest moment there was verbal or written evidence of an indicator for the code. To code interviewer moves, the recordings were segmented into durations of 30s and each segment was assigned each intervention code the segment evidenced. In this way, the scaffolding moves codebook produced time series corresponding to if the code is "on" or "off" during each 30s segment. Pilot studies (not reported here) adapted the scaffolding moves codebook to the research setting. We then mapped each instance of a scaffolding move to the MMC by identifying which stage of model construction the intervention referenced (situation model, real model, mathematical model, mathematical results, real results). For example, the move "Let's work on just the susceptible and infectious. And we'll pick back up the removed later" was coded as B3.23 Narrowing scope because the interviewer suggested the participant to ignore the removed population. Because the move referred to the distinct populations identified by the learner (susceptible, infectious, recovered), we inferred it to refer to the Real Model stage of the MMC. In this way, we obtained a description of the move and its modeling-stage referent.

Due to the complexity of the codebooks, total duration of the 51 sessions, and planned quantitative models, our primary concerns about reliability were the chance of missing codable segments and consistent application of the codebooks across participants and tasks. Thus, two analysts independently coded each event. To mitigate coder drift, six pairs of analysts were formed from four research team members and rotated. Pairs met regularly to reconcile codebook application and resolve disagreements based solely on code definitions. Since neither codebook was mutually exclusive, multiple codes could be added to the same data segment if warranted. Remaining disagreements were considered by the whole group and resolved by consensus.

To investigate the impact of the contingent scaffolding moves (B3 codes) on the modeling competencies (MMC phases), at each instance of a B3 and for each MMC phase, we determined if the competency was observed during the subsequent two-minute window. If the competency was observed at least once in the window, we said the B3 move was taken up by the participant. Combining the results across tasks and participants, we estimated the probability of uptake for each competency and set of B3 codes.

As seen in Table 4, the number of instances observed varied considerably by competency. As expected, *understanding* (233) and *working mathematically* (261) competencies were rarely elicited, and so were excluded from analysis. However, *validating* was observed nearly twice as often as *interpreting*. To account for variation, we estimated a base probability under the null or no effect model where the null assumption is that MMC codes are uniformly distributed across the sessions. Under the null model, we let X_k be the number of instances of MMC code k in a given two-minute window. Then X_k follows a binomial distribution with size equal to the total number of instances of code k and probability equal to 2 divided by the total combined time of the sessions. The base probability is $p_{b_k} = P(X_k > 0)$. We then normalized the probability of uptake, by computing an odds ratio:

$$\text{OR} = \frac{(p/(1-p)}{(p_{b_k}/(1-p_{b_k})}.$$

If the uptake probability equals the expected value under the null model, then $OR = 1$, and indicates the contingent scaffolding move is not associated with an increased uptake of competency k. On the other hand, $OR > 1$ indicates that the contingent scaffolding move promotes competency k.

Initial investigations indicated that the proportion of uptake depended on the modeling stages referred to. Hence, we used the analysis of B3 code instances in terms of model construction stage to collapse to four broad categories: Real (scaffolding move refers to situation and/or real model), Math (scaffolding move refers to mathematical model), Both (scaffolding move refers to both Real and Math), and Neither (scaffolding move refers to neither Math nor Real). Only 15 instances of intervention codes referring only to math result, 14 referring only to real result, and 9 referring to both were observed. We excluded the low counts. Instances of B3.10 Request Reason or Explanation could fall into distinct secondary categories, depending on the stage referred to by the specific move at that time in the interview.

Results and Interpretations

Figure 1 shows variation in the odds of uptake of each competency following interviewer moves referring to the stages of model construction. To fully characterize the differences observed in Figure 1, for each competency we fit a mixed logistic regression model with a participant random effect, u_i, to account for dependence (Agresti, 2012):

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1\text{Real} + \beta_2\text{Math} + \beta_3\text{Both} + u_i$$

Results of the models are shown in the Relationship with Stage Referred to column of Table 4. "$A < B$" indicates that the Odds of uptake for that competency is significantly less for stage A than B. "$A = B$" indicates no significant difference.

Figure 1 Odds ratio of uptake of modeling competency by stage of scaffolding move referred to.

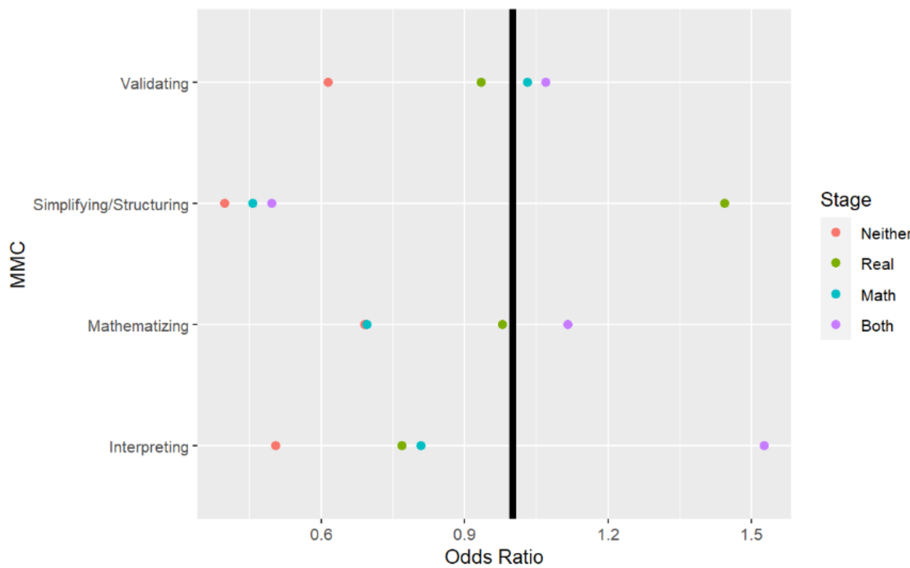

Table 4 Summary of competencies, relationship with stages referred to, and log odds for three of the contingent scaffolding moves from Table 2

Competency	Count	Relationship Stage Referred to	B3.1	B3.6	B3.10
Simplifying & Structuring	1218	Neither = Math = Both < Real	-0.289	0.027	-0.100
Mathematizing	929	Neither = Math < Real = Both	0.391	0.943	-.656
Interpreting	787	Neither< Real = Math < Both	0.208	0.408	0.192
Validating	1428	Neither< Real = Math = Both	0.552	-0.139	-0.101

As expected, contingent scaffolding moves referring to the Real stages were much more likely to elicit *simplifying/structuring* than those referring to other stages. Specifically, the odds are 1.4 times expectations under the null model. Scaffolding moves classified as Real and Both had the greatest odds of eliciting *mathematizing*, which is sensible because the mathematizing competency bridges thinking about real-world conditions and assumptions to reasoning about mathematical properties and parameters (Zbiek & Conner, 2006) it also, according to the MMC, ought to follow chronologically from thinking about real-world conditions and assumptions. Additionally, the odds of eliciting the *interpreting* competency are greatest when a move refers to Both (Math and Real) stages of model construction. This makes sense theoretically because *interpreting* competency, like *mathematizing*, bridges real-world and mathematical knowledge.

Finally, a wide range of scaffolding moves can elicit *validating* – as long as the move refers to either Real or Math or Both -- corroborating claims in previous work that validating arises throughout the modeling process in response to multiple knowledge sources (Czocher, 2018; Ishibashi & Uegatani, 2022).

Due to space constraints, we discuss only three of the 48 B3 Codes. The log odds (Table 4) show increased rates of *mathematizing, interpreting,* and *validating* followed a request for the participant to summarize their current work (B3.1 Work Status), relative to the base rates of occurrence for these competencies, while it is not an effective move for eliciting *simplifying & structuring*. Because *mathematizing* and *interpreting* are likely to follow B3.6 Prompt to Include Aspects (of the real world), moves which primarily refer to Real stages of model construction, it seems that prompting learners to attend to their ideas about how the world works leads to competencies associated with increasing model complexity and scope. We are not surprised to see a positive association between B3.10 Request Reason or Explanation and *interpreting*, because asking a learner their rationale for a modeling decision would often necessitate interpreting situationally relevant meanings. However, its negative association with *validating*, which shares aspects of justifying and explaining (Czocher et al., 2018) was surprising. We had anticipated that B3.10 would increase elicitation of *validating*.

Finally, there were several scaffolding moves for which the odds of eliciting any of the four focal competencies were less than the base rate predicted by the null model. The move 3.11p Math Procedure was one such example. In contrast, the move B3.14 Suggestion for Action, Related to Content (focus on directing attention to variables and quantities) elicited all four focal competencies more than expected. Thus, it seems attending to the role of quantities and quantitative reasoning promotes modeling competencies.

Conclusions

We conclude that the logistic regression model adequately captures expected relationships between contingent scaffolding moves, their referents relative to the MMC, and elicitation of modeling competencies. We view it as an initial model capable of sustaining claims about (a) the compatibility of the analytic frameworks and (b) predicting which moves are capable of eliciting which modeling competencies. Importantly, the regression model quantifies variation and differences across competencies, scaffolding moves, and the likelihoods of their interactions. This is a promising advance for work seeking to understand the impact of modeling-forward learning environments on learners' modeling competencies. The approach retains the nuance of the critical aspects of contingent scaffolding, as articulated by the scaffolding moves framework, while offering a vision of the larger cross-participant and cross-task patterns. One limitation is that presently, it is unclear the extent to which the random effect model adequately accounts for the person-dependence of each competency. Future iterations would improve on this uncertainty. In the end, the holy grail is a model capable of informing facilitators which contingent scaffolding moves are most and least likely to promote which competencies so they may focus on developing powerful moves. Due to the large number of codes, we clumped them according to the stage they referred to. Future iterations can examine individual moves to understand which perform similarly with respect to competency elicitation and distill move types into strategies.

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