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Maximal L^p-regularity of abstract evolution equations modeling closed-loop, boundary feedback control dynamics

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Abstract

We provide maximal L^p-regularity up to the level $T < \infty$ or $T = \infty$ of an abstract evolution equation in Banach space, which captures boundary closed-loop parabolic systems, defined on a bounded multidimensional domain, with finitely many boundary control vectors and finitely many boundary sensors/actuators. Illustrations given include classical parabolic equations as well as Navier-Stokes equations in $L^p(\Omega)$ or $L^q_\sigma(\Omega)$, respectively.

1. The case of boundary controls and boundary sensors/observers, [LPT.6]

Overview

The topic of maximal L^p-regularity was (apparently) first studied in the fundamental paper [Sim] (in Italian) published in 1964. In it, the author considers the generator A of a s.c. (C_0) semigroup e^{At} on the Hilbert space H and shows a definitive result in this setting: that e^{At} possesses maximal L^p-regularity up to T on the Hilbert space H if and only if it is analytic (holomorphic); with $T = \infty$ in case e^{At} is, moreover, (exponentially) uniformly stable. The sophisticated, technical proof was based (as stated in the paper's title) on the theory of singular integrals. This was truly a pioneering paper that stimulated an intense subsequent research activity, both at the abstract Banach space setting as well as at the L^p or Hölder spaces settings for the class of (parabolic) equations. At the general Banach space setting, it was established that maximal L^p-regularity of the s.c. semigroup e^{At} implies that e^{At} is analytic, but not conversely. To date known counterexamples exist in abstract Banach spaces setting, see [HNVW.2, Section 17.4.c]. Instead, the PDE-framework includes: either dynamics defined on the entire multidimensional space; or on half-spaces; or on domains exterior to multidimensional bounded domains; or else on a multidimensional domain Ω, with possibly, open-loop inhomogeneous boundary terms on $\partial\Omega$ in Triebel-Lizorkin spaces, see [DHP]. Similar results are available for Pseudodifferential setting as well. The list of significant papers will likely exceed the length permitted for this extended abstract. Thus, we must constrain ourselves to quote only a few. In contrast, the emphasis of the present extended abstract is quite different. While the setting is still at the abstract Banach space level, the modeled dynamics intend to capture closed-loop boundary feedback (parabolic) problems, with either (i) finitely many boundary controls and interior sensors/actuators [LPT.5]; or else (ii) with finitely many boundary controls and boundary sensors/actuators [LPT.6]. The assumption imposed on the two abstract models are automatically satisfied by the intended, motivating applications. These include, in addition to classical parabolic dynamics, physical important dynamics such as Navier-Stokes equations (particularly in dimension $d = 3$), Boussinesq systems, Magnetohydrodynamics (MHD) systems, etc [LPT.1, LPT.2, LPT.3, LPT.4, LPT.5, LPT.7]. Here maximal L^p-regularity is first established in the Banach space $L^q(\Omega)$, $1 < q < \infty$, or even a suitable Besov space $B^{2-2/p}_{q,p}(\Omega)$ which does not recognize boundary conditions ($1 < p < \frac{2q}{2q-1}$, $q >$ dimension d). Next, such maximal L^p regularity is exploited to obtain (well-posedness as a nonlinear semigroup and) uniform stabilization of the full nonlinear feedback model in the vicinity of an unstable equilibrium solution.

1.1. Abstract setting

The focus of the present section is the operator

$$
\begin{cases}
A_F = -A(I - GF) : Y \supset \mathcal{D}(A_F) \longrightarrow Y & \text{(1.1a)} \\
\mathcal{D}(A_F) = \{x \in Y : \ (I - GF)x \in \mathcal{D}(A)\}. & \text{(1.1b)}
\end{cases}
$$

and corresponding abstract equation

$$y_t = A_F y = -A(I - GF)y \tag{1.2}$$

under the following standing assumptions:

(**H.1**) Y is a reflexive Banach space.

(**H.2**) $-A : Y \supset \mathcal{D}(A) \longrightarrow Y$ is the maximal dissipative generator of a C_0-contraction semigroup e^{-At} on Y, $t \geq 0$, which possesses the maximal $L^p(0,T;Y)$-regularity property up to T, either $0 < T < \infty$; or else $T = \infty$, $1 < p < \infty$; in symbols [Dore.1]

$$-A \in MReg\left(L^p(0,T;Y)\right), \quad \text{either } 0 < T < \infty; \text{ or else } T = \infty, 1 < p < \infty;$$

so that, a fortiori, the strongly continuous (s.c.) semigroup e^{At} is analytic (holomorphic) on Y. At the price (harmless for the present note) of replacing A with a suitable translation to the right ($A_k = A + k^2 I$), the fractional powers $A^\theta, 0 < \theta < 1$, of A are well-defined [Pazy].

(**H.3**) U is another Banach space and G is the ("Green") linear operator satisfying

$$G : \text{continuous } U \longrightarrow \mathcal{D}(A^{\alpha_0}) \subset Y, \text{ or } A^{\alpha_0} G \in \mathcal{L}(U;Y) \tag{1.3}$$

for some $0 < \alpha_0 < 1$.

(**H.4**) F is a linear ("feedback") operator of the form

$$Fz = \langle \gamma z, w \rangle_U \, g, \quad w, g \in U \tag{1.4}$$

where γ is a linear (trace) operator

$$\gamma : \text{continuous } \mathcal{D}(A^\sigma) \subset Y \longrightarrow U, \quad 0 < \sigma < \alpha_0 < 1 \tag{1.5}$$

so that

$$F : \text{continuous } \mathcal{D}(A^\sigma) \subset Y \longrightarrow U. \tag{1.6}$$

$\Big[$In the applications we shall take $Fz = \displaystyle\sum_{k=0}^{K} \langle \gamma z, w_k \rangle_U \, g_k, \quad w_k, g_k \in U\Big]$

Remark 1.1 F is thus unbounded as an operator on Y. For the similar problem considered in [LPT.5] in JDE, F was a bounded operator on Y. The purpose of this work is to extend to the operator (1.1) the result on maximal $L^p(0,T;Y)$-regularity of [LPT.5], $T \leq \infty$. The proof of [LPT.5] requires $F \in \mathcal{L}(Y;U)$. Thus, the proof of the present note is quite different from that in [LPT.5]. See [Las] for abstract parabolic boundary problems.

With reference to assumption (**H.3**) centered on the constant $0 < \alpha_0 < 1$, we introduce two Banach spaces, where $0 \in \rho(A)$,

$$\mathcal{E} \equiv \mathcal{D}(A^{\alpha_0}), \text{ with norm } \|x\|_{\mathcal{E}} \equiv \|x\|_{\mathcal{D}(A^{\alpha_0})} \equiv \|A^{\alpha_0} x\|_Y, \tag{1.7}$$

$$E \equiv \left[\mathcal{D}(A^{*(1-\alpha_0)})\right]' \text{ with norm } \|z\|_E \equiv \|z\|_{\left[\mathcal{D}(A^{*(1-\alpha_0)})\right]'} = \left\|A^{-(1-\alpha_0)} z\right\|_Y. \tag{1.8}$$

Accordingly we introduce the following holomorphic interpolation spaces

$$[\mathcal{E}, E]_\theta \equiv \left[\mathcal{D}\left(A^{\alpha_0}\right), \left[\mathcal{D}\left(A^{*(1-\alpha_0)}\right)\right]'\right]_\theta = \begin{cases} \mathcal{D}\left(A^{\alpha_0 - \theta}\right), \ 0 \leq \theta \leq \alpha_0, & \text{(1.9a)} \\ \left[\mathcal{D}\left(A^{*(\theta - \alpha_0)}\right)\right]', \ \alpha_0 \leq \theta \leq 1. & \text{(1.9b)} \end{cases}$$

since $\alpha_0(1-\theta) - (1-\alpha_0)\theta = \alpha_0 - \theta$, with corresponding norm

$$\|x\|_{[\mathcal{E}, E]_\theta} = \|x\|_{\mathcal{D}(A^{\alpha_0 - \theta})} = \left\|A^{\alpha_0 - \theta} x\right\|_Y, \ 0 \leq \theta \leq \alpha_0, \tag{1.10}$$

$$\|z\|_{[\mathcal{E}, E]_\theta} = \|z\|_{\left[\mathcal{D}(A^{*(\theta-\alpha_0)})\right]'} = \left\|A^{-(\theta - \alpha_0)} z\right\|_Y, \ \alpha_0 \leq \theta \leq 1. \tag{1.11}$$

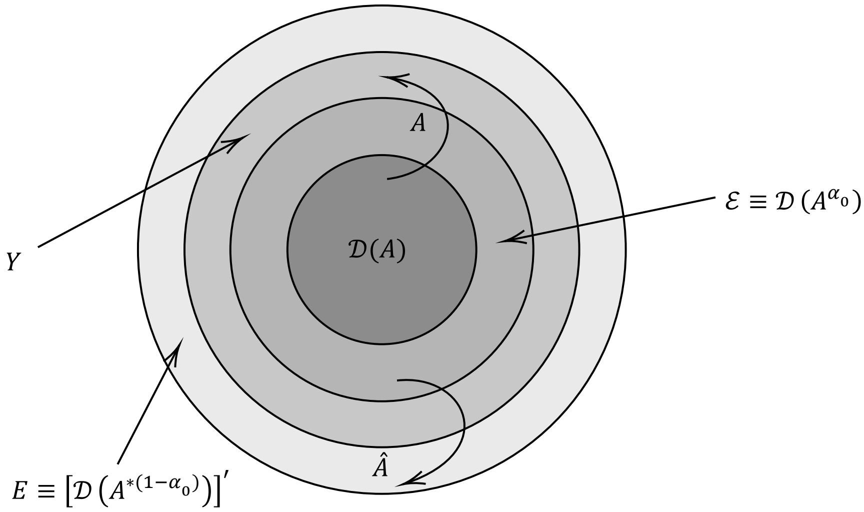

Fig 1: Symbolic illustration of the spaces and operators involved.

1.2. Main Result

Theorem 1.2 *(a) Let $0 < T < \infty$. The operator A_F in (1.1) defined on Y generates a s.c. semigroup $T_F(t)$, which is analytic on Y and, moreover, possesses the maximal $L^p(0,T;Y)$-regularity on Y, $1 < p < \infty$, $T < \infty$: the map*

$$f \to (Lf)(t) = \int_0^t e^{A_F(t-s)} f(s)ds \;\; \textit{continuous} \;\; L^p(0,T;Y) \to L^p(0,T;\mathcal{D}(A_F));$$

in symbols, [Dore.1]

$$A_F \in MReg\left(L^p(0,T;Y)\right), \quad 1 < p < \infty,\ T < \infty. \tag{1.12a}$$

(b) Let $T = \infty$. Assume further that the s.c. analytic semigroup $T_F(t)$ is uniformly stable on Y: there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| T_F(t) \right\|_{\mathcal{L}(Y)} \leq M e^{-\delta t},\ t \geq 0. \tag{1.12b}$$

Then, $T_F(t)$ possesses the maximal $L^p(0,\infty;Y)$-regularity on Y, $1 < p < \infty$, $T = \infty$; in symbols [Dore.1]

$$A_F \in MReg\left(L^p(0,\infty;Y)\right), \quad 1 < p < \infty,\ T = \infty. \tag{1.12c}$$

Actually, in the each case (a) and (b), $T_F(t)$ extends/restricts with the same properties - as s.c. analytic, uniformly stable (case (b)) semigroup, with maximal L^p-regularity ($0 < T < \infty$ in case (a), $T = \infty$ in case (b)) - on the space E in (1.8), on the space $\mathcal{E}$ in (1.7), as well as on all holomorphic interpolation spaces (1.9)-(1.11).

The proof of the present Theorem 1.2 with F unbounded as in (1.6), $F \in \mathcal{L}(\mathcal{D}(A^\sigma), U)$ given in [LPT.6], is completely different from the one in [LPT.5]. It is inspired by a proof in [LT.2] about analyticity of a specific parabolic semigroup in an Hilbert setting. It consists of three steps, (i) first, showing L^p-maximal regularity in the larger space E in (1.8); next, (ii) showing L^p-maximal regularity in the smaller space $\mathcal{E}$ in (1.7); and finally, (iii) showing L^p-maximal regularity on Y by interpolation.

In contrast, the proof of [LPT.5] for $F \in \mathcal{L}(Y;U)$ was based on considering A_F^* rather than A_F. With $F \in \mathcal{L}(Y;U)$ and G satisfying $A^\gamma G \in \mathcal{L}(U;Y)$ for some γ, $0 < \gamma < 1$, the expression of A_F makes such form not directly suitable for deducing its maximal regularity on Y, as it would leave the power $A^{1-\gamma}$ on the LHS unaccounted for on Y. The form of A_F^* in [LPT.5] is more amenable to show $A_F^* \in MReg(L^p(0,T;Y^*))$ by perturbation [Dore.2, Theorem 6.2, p311], [KW.1, Remark 1, p426, for $\beta = 1$], [Weis]. Next, to show that the original A_F satisfies $A_F \in MReg(L^p(0,T;Y))$ as desired, paper [LPT.5] employs the result that on the UMD space Y, the property that $A_F \in MReg(L^p(0,T;Y))$ is equivalent to the property that the family, $\tau \in \mathcal{L}(Y)$,

$$\tau \equiv \{tR(it, A_F),\ t \in \mathbb{R}\backslash\{0\}\} \text{ be } R-\text{bounded},$$

[KW.2] where $R(\cdot, A_F)$ denotes the resolvent of A_F. And in the UMD-setting for Y, the R-boundedness property for the family τ is equivalent to the property that the corresponding dual family τ' in $\mathcal{L}(Y^*)$

$$\tau' \equiv \left\{tR(it, A_F^*),\ t \in \mathbb{R}\backslash\{0\}\right\} \text{ be } R-\text{bounded},$$

[HNVW.1, Proposition 8.4.1, p211].

2. Illustrations

For simplicity and brevity of exposition, Example # 1 (for $T < \infty$ and $T = \infty$) will be restricted to a canonical case. More general results can be given by referring to [LT.3, CV, DaV, DaG, Ves].

2.1. Case $0 < T < \infty$.

Example # 1 **The PDE model**: Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with boundary $\partial\Omega \equiv \Gamma$, assumed to be $(d-1)$-dimensional variety with Ω locally on one side of Γ, and sufficiently smooth. We consider the following canonical locally fully boundary closed loop parabolic system on Ω, with boundary control in the Neumann BC and boundary sensing (observations):

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = (\Delta - I)y(t,x) & \text{in } (0,T] \times \Omega & (2.1a) \\ y(0,x) = y_0(x) & \text{in } \Omega & (2.1b) \\ \dfrac{\partial y(t,\xi)}{\partial \nu} = f(t,\xi) \equiv \displaystyle\sum_{k=0}^{K} (\gamma y(t,\cdot), w_k(\cdot))_\Gamma g_k(\xi) & & (2.1c) \\ \qquad \equiv F y(t,\cdot) & \text{on } (0,T] \times \Gamma & (2.1d) \end{cases}$$

(a) Let

$$Y \equiv L^q(\Omega),\ 1 < q < \infty,\ A = -\Delta + I;\ Y \supset \mathcal{D}(A) \to Y \tag{2.2a}$$

$$\mathcal{D}(A) = \left\{ \varphi \in W^{2,q}(\Omega) :\ \left.\frac{\partial \varphi}{\partial \nu}\right|_\Gamma = 0 \right\}. \tag{2.2b}$$

Then $-A$ generates a s.c. contraction, analytic semigroup e^{-At}, $t \geq 0$ on $Y \equiv L^q(\Omega)$. The fractional powers A^θ, $0 < \theta < 1$, are well-defined.

(b) γ denotes any continuous operator [Trie, Wahl]

$$\gamma : \mathcal{D}(A^\sigma) \equiv W^{2\sigma,q}(\Omega) \to U \equiv L^q(\Omega),\ 2\sigma = \frac{1}{q} + \varepsilon \tag{2.3}$$

in particular the trace operator

$$\gamma\psi \equiv \psi|_\Gamma \in L^q(\Gamma),\ \psi \in W^{2\sigma,q}(\Omega). \tag{2.4}$$

Thus, the (feedback) operator F defined in (2.1d) satisfies

$$F : \mathcal{D}(A^\sigma) \equiv W^{2\sigma,q}(\Omega) \to U \equiv L^q(\Omega),\ 2\sigma = \frac{1}{q} + \varepsilon \tag{2.5}$$

as well, for all vectors $w_k \in L^{q'}(\Gamma), g \in L^q(\Gamma), \frac{1}{q} + \frac{1}{q'} = 1$, where $(\ ,\ )_\Gamma$ denotes the duality paring between $L^q(\Gamma)$ and $L^{q'}(\Gamma)$.

(c) We introduce the Neumann (Green) map [LT.4]

$$Gg \equiv \varphi \iff \left\{ (\Delta - I)\varphi \equiv 0 \text{ in } \Omega,\ \frac{\partial \varphi}{\partial \nu} = g \text{ on } \Gamma \right\} \tag{2.6a}$$

$$G : U \equiv L^q(\Omega) \to W^{1+1/q,q}(\Omega) \subset \mathcal{D}(A^{\alpha_0}),\ \alpha_0 = \frac{1}{2} + \frac{1}{2q} - \varepsilon. \tag{2.6b}$$

(d) We observe from (2.3) and (2.6b) that

$$\sigma = \frac{1}{2q} + \frac{\varepsilon}{2} < \alpha_0 = \frac{1}{2} + \frac{1}{2q} - \varepsilon \tag{2.7}$$

The abstract model. As is well known, we can rewrite (2.1a) as

$$y_t = (\Delta - I)y = (\Delta - 1)(y - Gf),\ \text{since } (\Delta - 1)(Gf) \equiv 0 \text{ in } \Omega \tag{2.8}$$

by (2.6a), recalling f in (2.1c). Moreover

$$\frac{\partial(y - Gf)}{\partial\nu} = \frac{\partial y}{\partial\nu} - \frac{\partial(Gf)}{\partial\nu} = f - f \equiv 0 \text{ on } \Gamma \tag{2.9}$$

and so $(y - Gf)$ satisfies the boundary conditions of the operator A in (2.2b). In conclusion, recalling $f = Fy$ from (2.1d) we can rewrite (2.8) as

$$y_t = -A(I - GF)y = A_F y \tag{2.10}$$

which is the abstract model on $Y \equiv L^q(\Omega)$ of the original PDE feedback model (2.1a-d). We now verify that the abstract model (2.10) for (2.1a-d) satifies all abstract assumptions of Section 1.

(**H.1**)is satisfied since $Y \equiv L^q(\Omega)$, $1 < q < \infty$ is reflexive Banach space. (**H.2**)is satisfied since the operator $-A$ in (2.2a) is the maximal dissipative generator of a C_0-contraction semigroup e^{-At} on Y, $t \geq 0$, which possesses the maximal $L^p(0,T;Y)$-regularity property, $0 < T < \infty$, $1 < p < \infty$. (**H.3**)is satisfied since $U = L^q(\Gamma)$ is a Banach space $A^{\alpha_0}G \in \mathcal{L}(U;Y)$ from (2.6b), $\alpha_0 < 1$. (**H.4**)is satisfied by (2.5).
In conclusion: Problem (2.1a-d) satisfies all assumptions of Theorem 1.2, for $0 < T < \infty$, and hence $A_F \in MReg(L^p(0,T;L^q(\Omega)))$, $1 < p < \infty$, $1 < p < \infty$, $T < \infty$ with $A_F = -A(I - GF)$ in (2.10). This conclusion is true for all $w_k \in L^{q'}(\Gamma)$, $g_k \in L^q(\Gamma)$. Below we shall consider the case $T = \infty$.

Example # 2: We return to [LT.6, LPT.2] and consider the linearized Navier-Stokes problem over a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega \equiv \Gamma$ (after translation by the equilibrium solution, see [LPT.2, Eq (1.28)]])

$$\begin{cases} w_t - \nu_o \Delta w + L_e(w) + \nabla\chi = 0 & \text{in } Q \quad (2.11a)\\ \operatorname{div} w = 0 & \text{in } Q \quad (2.11b)\\ w \equiv v \equiv \sum_{k=0}^{K} \langle \gamma w, p_k \rangle_\Gamma \, g_k \equiv Fw & \text{on } \Sigma \quad (2.11c)\\ w(0,x) = w_0(x) & \text{on } \Omega \quad (2.11d) \end{cases}$$

whose abstract version is given by

$$\frac{dw}{dt} = \mathcal{A}_q w - \mathcal{A}_q D\left(\sum_{k=1}^{K} \langle \gamma w, p_k \rangle_\Gamma \, g_k\right) \tag{2.12a}$$

$$= \mathcal{A}_q w - \mathcal{A}_q DFw = \mathcal{A}_q(I - DF) \equiv \mathbb{A}_{F,q} w. \tag{2.12b}$$

see [LPT.2, Eq (4.3)] with $m \equiv 0$ and a modified boundary control v. We have

$$Y \equiv L^q_\sigma(\Omega),\ q \geq 2,\ \mathcal{A}_q = -(\nu_o A_q + A_{o,q}),\ \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \quad \text{[LPT.2, Eq (2.16)]} \tag{2.13}$$

$$A_q z = -P_q \Delta z,\ \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \quad \text{[LPT.2, Eq (2.14)]} \tag{2.14}$$

$$L_e(z) = (y_e \,.\, \nabla)z + (z \,.\, \nabla)y_e \quad \text{[LPT.2, Eq (1.9)]} \tag{2.15}$$

$$A_{o,q} z = P_q L_e(z) = P_q[(y_e \,.\, \nabla)z + (z \,.\, \nabla)y_e], \tag{2.16a}$$

$$\mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \subset L^q_\sigma(\Omega). \quad \text{[LPT.2, Eq (2.15)]} \tag{2.16b}$$

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega) \quad \text{(Helmholtz direct sum decomposition)} \tag{2.17}$$

$$L^q_\sigma(\Omega) = \{g \in L^q(\Omega) : \div g = 0;\ g \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad \text{[Ga]} \tag{2.18}$$

the solenoidal space. It is verified in [LPT.6] that all the assumptions of Theorem 1.2 are satisfied for the feedback operator $\mathbb{A}_{F,q} = \mathcal{A}_q(I - DF)$ in (2.12b) on $Y \equiv L^q_\sigma(\Omega)$. In particular, (**H.2**) is verified since the operator $\mathcal{A}_q$ in (2.13) has maximal L^p-regularity on $Y \equiv L^q_\sigma(\Omega)$, for $0 < T < \infty$. (**H.3**) is satisfied with

$$U \equiv U_q \equiv \{g \in L^q(\Gamma) :\ g \cdot \nu = 0 \text{ on } \Gamma\} \tag{2.19}$$

$$D : U_q \to W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}(A_q^{1/2q - \varepsilon}) \tag{2.20a}$$

$$\text{or } A_q^{1/2q - \varepsilon} D \in \mathcal{L}(U_q, L^q_\sigma(\Omega)), \quad \sigma_0 = \frac{1}{2q} - \varepsilon \tag{2.20b}$$

(**H.4**) is satisfied by taking

$$\gamma : \text{ continuous } \mathcal{D}(A_q^{\sigma}) \subset Y \equiv L^q_\sigma(\Omega) \to U \text{ with } 0 < \sigma < \sigma_0 = \frac{1}{2q} - \varepsilon \tag{2.21}$$

so that then

$$F : \text{ continuous } \mathcal{D}(A_q^{\sigma}) \subset Y \to U \tag{2.22}$$

as well. Then we take $p_k \in L^{q'}(\Gamma)$ and $g \in L^q(\Gamma)$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then all the assumptions of Theorem 1.2 are satisfied for the feedback operator $\mathbb{A}_{F,q} = \mathcal{A}_q(I - DF)$ in (2.12b). See [Sol.1, Sol.2, Sol.3, Sol.4] for open-loop problems.

2.2. Case $T = \infty$

We return to Example # 1, except that, to make the problem more significant, we replace (2.1a) by the canonical equation

$$\frac{\partial y(t,x)}{\partial t} = (\Delta + k^2) y(t,x) \tag{2.23}$$

k^2 large, while keeping Eqts (2.1b-c). Thus, for $f \equiv 0$, the corresponding free dynamics operator

$$A\varphi = (\Delta + k^2)\varphi, \quad Y \equiv L^q(\Omega) \supset \mathcal{D}(A) = \left\{ \varphi \in W^{2,q}(\Omega), \ \frac{\partial \varphi}{\partial \nu}\bigg|_\Gamma = 0 \right\} \tag{2.24}$$

is the generator of a s.c. analytic semigroup on Y which is unstable and possesses maximal $L^p(0,T;Y)$-regularity, $T < \infty$. We take the boundary vectors $g_k \in L^q(\Gamma)$ to be linearly independent. According to Theorem 1.1(b), or the basis of the analysis of Example # 1 (k^2 rather than -1 is irrelevant), we only need to verify the additional assumption that, for suitable vectors $w_k \in L^{q'}(\Gamma)$, $g_k \in L^q(\Gamma)$, the semigroup $T_F(t) = e^{A_F t}$, $A_F = -A(I - GF)$ in (2.10), is exponentially stable

$$\left\| e^{A_F t} \right\|_{\mathcal{L}(Y)} \equiv \left\| T_F(t) \right\|_{\mathcal{L}(Y)} \le M e^{-\delta t}, \quad t \ge 0, \ \delta > 0, \ Y \equiv L^q(\Omega). \tag{2.25}$$

This statement amounts to saying that the original boundary homogeneous problem (2.23), (2.1a-d) which with $f \equiv 0$ is unstable (i.e. it has finitely many unstable eigenvalues on $\mathcal{C}^+ = \{\lambda \in \mathcal{C} : \operatorname{Re} \lambda \ge 0\}$) can be uniformly stabilized by a finite dimensional feedback control $f(t,\xi) =$ RHS of (2.1c), with suitable boundary vectors $g_k \in L^q(\Gamma)$ and boundary sensors $w_k \in L^{q'}(\Gamma)$. This problem was originally studied in early 1980s, see [LT.2, LT.3, Tr.1, Tr.2, Tr.3] and references therein. The vectors w_k have to be chosen to satisfy the algebraic condition

$$\text{rank } W_k = \ell_k = \text{ algebraic/geometric multiplicity of the unstable eigenvalue } \lambda_k \text{ of the self-adjoint operator } A \text{ in (2.24)} \tag{2.26}$$

where

$$W_k = \begin{bmatrix} \langle w_1, \Phi_{k1} \rangle_\Gamma, \langle w_1, \Phi_{k2} \rangle_\Gamma & \cdots & \langle w_1, \Phi_{k\ell_k} \rangle_\Gamma \\ \langle w_2, \Phi_{k1} \rangle_\Gamma, \langle w_2, \Phi_{k2} \rangle_\Gamma & \cdots & \langle w_2, \Phi_{k\ell_k} \rangle_\Gamma \\ \vdots & & \vdots \\ \langle w_K, \Phi_{k1} \rangle_\Gamma, \langle w_K, \Phi_{k2} \rangle_\Gamma & \cdots & \langle w_K, \Phi_{k\ell_k} \rangle_\Gamma \end{bmatrix} \tag{2.27}$$

$\langle \cdot, \cdot \rangle_\Gamma$ duality pair, where $\{\Phi_{k1}, \dots, \Phi_{k\ell_k}\}$ are the normalized eigenvectors in Y of the unstable eigenvalues λ_k of the operator A in (2.24). Condition (2.27) can always be satisfied by infinite choices of the vectors $w_1, \dots, w_K$, since for every λ_k, the Dirichlet traces $\{\Phi_{k1}|_\Gamma, \Phi_{k2}|_\Gamma, \dots, \Phi_{k\ell_k}|_\Gamma\}$ are linearly independent [Tr.4, Tr.5, Tr.6].

It is known [LT.3] that if Ω is either a d-sphere or a d-parallelepiped, is it always possible to select boundary vectors g_k, $k = 1, \dots, k$ such that the exponential decay (2.25) holds true [LT.3] and hence Theorem 1.2 holds true for $T = \infty$ for these special geometries. For other geometries, $d \ge 2$, technical conditions are available which cannot be recalled here for brevity of exposition. We refer to the reference [LT.5] Moreover, if $d = 1$, the uniform stability (2.25) is impossible if A has at least 3 unstable eigenvalues, with only to boundary vectors g_1, g_2 at $x = 0$, or $x = 1$ with $\Omega = (0,1)$ [LT.3]. See [LT.1] for Dirichlet boundary feedback problems.

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