Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #96, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

The commutant of divided difference operators, Klyachko's genus, and the comaj statistic

Christian Gaetz*[1], Rebecca Goldin†[2], and Allen Knutson‡[3]

[1] *Department of Mathematics, University of California, Berkeley*
[2] *Department of Mathematical Sciences, George Mason University*
[3] *Department of Mathematics, Cornell University*

Abstract. In [5, 12, 13] are studied certain operators on polynomials and power series that commute with all divided difference operators ∂_i. We introduce a second set of "martial" operators $♂_i$ that generate the full commutant, and show how a Hopf-algebraic approach naturally reproduces the operators ξ^ν from [12]. We then pause to study Klyachko's homomorphism $H^*(Fl(n)) \to H^*$(the permutahedral toric variety), and extract the part of it relevant to Schubert calculus, the "affine-linear genus". This genus is then re-obtained using Leibniz combinations of the $♂_i$. We use Nadeau-Tewari's q-analogue of Klyachko's genus to study the equidistribution of ℓ and comaj on $\binom{[n]}{k}$, generalizing known results on S_n.

Keywords: divided difference operators, Schubert calculus, comaj statistic

1 The martial operators $♂_\pi$

1.1 The ring of Schubert symbols

Given a Dynkin diagram D with Weyl group $W(D)$, define the **ring of Schubert symbols** $H(D)$ as the cohomology ring of the associated (possibly infinite-dimensional) flag variety, with the usual Schubert basis $\{\mathcal{S}_w \colon w \in W(D)\}$. The Dynkin diagrams that will interest us are primarily the semi-infinite $A_{\mathbb{Z}_+}$ and the biinfinite $A_{\mathbb{Z}}$. In these type A cases $W(D)$ is the group of finite permutations of $\mathbb{Z}_+$ or of $\mathbb{Z}$. An important difference between the two is that $H(A_{\mathbb{Z}_+})$ is generated by $\{\mathcal{S}_{r_i} \colon i \in \mathbb{Z}_+\}$, where r_i is a simple transposition, so the multiplication is entirely determined by Monk's rule, whereas $H(A_{\mathbb{Z}})$ requires additional generators $\{\mathcal{S}_{r_1 r_2 \cdots r_k}\}$ and determining its multiplication involves also the flag Pieri rule. With all that in mind we largely abandon the geometry and work with these rings symbolically.

*gaetz@berkeley.edu
†rgoldin@gmu.edu. RG was partially supported by NSF grant DMS–2152312.
‡allenk@math.cornell.edu. AK was partially supported by NSF grant DMS–2246959.

For each vertex α of D hence generator $r_\alpha \in W(D)$, we have an operator $\partial_\alpha \circlearrowright H(D)$, pronounced "partial α":

$$\partial_\alpha \mathcal{S}_\pi := \begin{cases} \mathcal{S}_{\pi r_\alpha} & \text{if } \pi r_\alpha < \pi \\ 0 & \text{if } \pi r_\alpha > \pi \end{cases}$$

from which we can well-define ∂_π for any $\pi \in W(D)$ using products.

Theorem 1 (Lascoux-Schützenberger). *There is an isomorphism $H(A_{\mathbb{Z}_+}) \to \mathbb{Z}[x_1, x_2, \ldots]$ taking the Schubert symbol $\mathcal{S}_\pi$ to its "Schubert polynomial" $S_\pi(x_1, x_2, \ldots)$. On the target ring, ∂_α acts by Newton's divided difference operation.*

Call a ring homomorphism from $H(D)$ to some other ring a **genus**,[1] making the above isomorphism the **Lascoux-Schützenberger genus**.

It was observed in [5, 13] that the operator $\nabla := \sum_i \frac{d}{dx_i}$ on the target ring has two nice properties: it commutes with each ∂_i, and its application to any Schubert polynomial is a positive combination of Schubert polynomials. Our goal in this section is to characterize operations of the first type, with an eye toward the second. To study this commutant it will be handy to work with algebra actions.

1.2 Two commuting actions of the nil Hecke algebra

Let $Nil(D)$ denote formal (potentially infinite) linear combinations of elements $\{d_\pi \colon \pi \in W(D)\}$, with a multiplication defined by $d_\pi d_\rho := \begin{cases} d_{\pi\rho} & \text{if } \ell(\pi\rho) = \ell(\pi) + \ell(\rho) \\ 0 & \text{if } \ell(\pi\rho) < \ell(\pi) + \ell(\rho). \end{cases}$ This multiplication extends to infinite sums in a well-defined way, insofar as any $w \in W(D)$ has only finitely many length-additive factorizations. Slightly abusing[2] terminology, we call this $Nil(D)$ the **nil Hecke algebra**. The association $d_\pi \mapsto \partial_\pi$ gives an action of the opposite algebra $Nil(D)^{op}$ on $H(D)$; the infinitude of the sums in $Nil(D)$ is again not problematic, because $H(D)$'s elements are finite sums of Schubert symbols.

Define $♂_\alpha$ (pronounced "martial α") by

$$♂_\alpha \mathcal{S}_\pi := \begin{cases} \mathcal{S}_{r_\alpha \pi} & \text{if } r_\alpha \pi < \pi \\ 0 & \text{if } r_\alpha \pi > \pi \end{cases}$$

We can well-define $♂_{\prod Q} := \prod_{q \in Q} ♂_q$ for each reduced word Q.

[1] This terminology is stolen from the study of various cobordism rings of a point, e.g. the "Hirzebruch genus" and "Witten genus" are ring homomorphisms to $\mathbb{Z}$.

[2] One ordinarily considers only finite linear combinations, but we have need of certain infinite ones, and this simplifies the statement of Theorem 2.

Theorem 2. *The map $d_\pi \mapsto \male_\pi$ defines an action of $Nil(D)$ on $H(D)$ (unspoiled by the potential infinitude), commuting with the $Nil(D)^{op}$-action by the operators ∂_α. Conversely, each operator on $H(D)$ that commutes with all operators ∂_α arises as the action of a unique element of $Nil(D)$.*

In short, $Nil(D)$ and $Nil(D)^{op}$ are one another's commutants in their actions on $H(D)$.

This then characterizes the operators that commute with all ∂_i; we don't know any significance of the resulting algebra again being $Nil(D)$.

Proof. For $h \in H(D)$, let $\int h$ denote the coefficient of $\mathcal{S}_e$ in h. Each $h = \sum_\pi h_\pi \mathcal{S}_\pi$ is determined by the values

$$\int (\partial_\rho h) = \int \sum_\pi h_\pi \partial_\rho \mathcal{S}_\pi = \sum_\pi h_\pi \int \partial_\rho \mathcal{S}_\pi = \sum_\pi h_\pi \delta_{\pi,\rho} = h_\rho$$

We'll make use of the easy fact $\int \male_\pi \, \partial_\rho \, \mathcal{S}_\sigma = \begin{cases} 1 & \text{if } \sigma = \pi^{-1}\rho \text{ and } \ell(\sigma) = \ell(\pi) + \ell(\rho) \\ 0 & \text{otherwise.} \end{cases}$

Now let C be an operator on $H(D)$ commuting with all ∂_π. For each $\pi \in W(D)$, let $c_\pi := \int C(\mathcal{S}_{\pi^{-1}})$. We then confirm $C = \sum_\pi c_\pi \male_\pi$ using the determination above:

$$\begin{aligned}
\int \partial_\rho C(\mathcal{S}_\sigma) &= \int C(\partial_\rho \mathcal{S}_\sigma) = \int C\left(\mathcal{S}_{\sigma\rho^{-1}} \left[\ell(\sigma\rho^{-1}) = \ell(\sigma) - \ell(\rho)\right]\right) \\
&= \left[\ell(\sigma\rho^{-1}) = \ell(\sigma) - \ell(\rho)\right] \; c_{\rho\sigma^{-1}} \\
\int \partial_\rho \left(\sum_\pi c_\pi \male_\pi\right)(\mathcal{S}_\sigma) &= \sum_\pi c_\pi \int \male_\pi \, \partial_\rho \, \mathcal{S}_\sigma \;=\; \sum_\pi c_\pi \left[\sigma = \pi^{-1}\rho\right] \left[\ell(\sigma) = \ell(\pi) + \ell(\rho)\right] \\
&= c_{\rho\sigma^{-1}} \left[\ell(\rho\sigma^{-1}) = \ell(\sigma) - \ell(\rho)\right]
\end{aligned}$$

Here $[P] = 1$ if P is true, $[P] = 0$ if P is false, for a statement P. ☐

Example. The action of $\nabla := \sum_i \frac{d}{dx_i}$ on polynomials, pulled back to an action on $H(A_{\mathbb{Z}_+})$, is given by the operator $\sum_{n \in \mathbb{N}_+} n \male_n$. What is particularly special about ∇ is that it is a differential (i.e. satisfies the Leibniz rule), and is of degree -1.

Theorem 3. *Let $\sum_\alpha c_\alpha \male_\alpha \in Nil(D)$ be an operator of degree -1. If it is a differential (and D is simply-laced, for convenience) then each c_α is $\frac{1}{2}\sum_\beta c_\beta$ where the βs are α's neighbors in D.*

In particular if D is of finite type ADE, the only system of coefficients (c_α) is zero. If $D = A_{\mathbb{Z}_+}$, the only options are multiples of $c_i \equiv i$. If $D = A_{\mathbb{Z}}$, the space of such systems is two-dimensional, spanned by $c_i \equiv i$ and $c_i \equiv 1$.

Hence the ∇ discovered in [5] in the $A_{\mathbb{Z}_+}$ case was the only such operator available. In [12] it is explained that $\xi = \sum_{i \in \mathbb{Z}} \male_i$ is special to the back-stable situation of $A_{\mathbb{Z}}$; here we see that it is the only new option. (The result [12, Theorem 6] is very similar.)

Proof sketch. The proof amounts to applying $\sum_\alpha c_\alpha \male_\alpha$ to $(\mathcal{S}_{r_\alpha})^2 = \sum_\beta \mathcal{S}_{r_\beta r_\alpha}$ (computed using the Chevalley-Monk rule). ☐

2 Not-quite-Hopf algebras and Nenashev operators

2.1 The dual algebras

Define a pairing $Nil(D) \otimes_{\mathbb{Z}} H(D) \to \mathbb{Z}$ by

$$p \otimes s \mapsto \text{coefficient of } \mathcal{S}_e \text{ in } p(s)$$

and from there a map $Nil(D) \to H(D)^* := Hom_{\mathbb{Z}}(H(D), \mathbb{Z})$.

The following is well-known to the experts, if not usually expressed exactly this way (see e.g. [2], [9, §7.2]).

Theorem 4. *This map $Nil(D) \to H(D)^*$ is an isomorphism. Unfortunately the induced comultiplication $H(D) \to H(D) \otimes H(D)$ is not a ring homomorphism (example below), so the two are not thereby dual Hopf algebras. (There is an alternative statement explored in [10].)*

There is an analogue of Theorem 1 for $H(A_{\mathbb{Z}})$, taking each $\mathcal{S}_\pi$ to its **back-stable Schubert function** BS_π invented by the third author (and independently by Buch and by Lee), which were studied in [9, 12]. Define a **back-stable function** $p \in \mathbb{Z}[[\ldots, x_{-1}, x_0, x_1, x_2, \ldots]]$ to be a power series

- of finite degree, such that
- p depends only on the variables $\{x_k, k < N\}$ for some $N \gg 0$, and
- for some $M \ll 0$, p is symmetric in the variables $\{x_i, i \leq M\}$.

One way (as appears in [12]) to think of the ring of back-stable functions is as the image of the injection

$$\begin{array}{rcl} Symm \otimes_{\mathbb{Z}} \mathbb{Z}[\ldots, x_{-1}, x_0, x_1, \ldots] & \to & \mathbb{Z}[[\ldots, x_0, \ldots]] / \langle \text{elementary symmetric functions} \rangle \\ p \otimes q & \mapsto & p(\ldots, x_{-2}, x_{-1}, x_0)\, q \end{array}$$

For $\pi \in W(A_{\mathbb{Z}})$ considered as a finite permutation of $\mathbb{Z}$, and $shift_N(i) := i + N$, observe for $N \gg 0$ that $\pi[N] := shift_N(i) \circ \pi \circ shift_{-N}(i)$ is a permutation of $\mathbb{Z}$ that leaves $-\mathbb{N}$ in place, and thus has a well-defined Schubert polynomial. Define the **back-stable Schubert function**

$$BS_\pi := \lim_{N \to \infty} S_{\pi[N]}(x_{1-N}, x_{2-N}, \ldots)$$

where the limit is computed coefficient-wise (note that any single coefficient settles down to a constant value for all large enough N).

Theorem 5. *[9, Theorem 3.5] The back-stable Schubert functions lie in, and are a $\mathbb{Z}$-basis of, the ring of back-stable functions.*

In this coördinatization we can compute the comultiplication on $H(A_{\mathbb{Z}})$ and bound its failure to be a ring homomorphism. Transposing the multiplication from §1.2 of $d_\pi d_\rho$, we obtain $\Delta(BS_\sigma) = \sum\{BS_\pi \otimes BS_\rho \colon \sigma = \pi\rho,\ \ell(\sigma) = \ell(\pi) + \ell(\rho)\}$. Then, alas,

$$\begin{aligned}
\Delta(BS_{r_2}^2) = \Delta(BS_{r_1r_2} + BS_{r_3r_2}) &= (BS_{r_1r_2}\otimes 1) + (BS_{r_1}\otimes BS_{r_2}) + (1\otimes BS_{r_1r_2}) \\
&\quad + (BS_{r_3r_2}\otimes 1) + (BS_{r_3}\otimes BS_{r_2}) + (1\otimes BS_{r_3r_2}) \\
\neq \Delta(BS_{r_2})\Delta(BS_{r_2}) = (BS_{r_2}\otimes 1 + 1\otimes BS_{r_2})^2 &= (BS_{r_1r_2}\otimes 1) + (BS_{r_3r_2}\otimes 1) + (BS_{r_2}\otimes BS_{r_2}) \\
&\quad + (BS_{r_2}\otimes BS_{r_2}) + (1\otimes BS_{r_1r_2}) + (1\otimes BS_{r_3r_2})
\end{aligned}$$

Luckily $\Delta(BS_\pi BS_{\rho[N]}) = \Delta(BS_{\pi\circ(\rho[N])}) = \Delta(BS_\pi)\Delta(BS_{\rho[N]})$ for $N \gg 0$. Call this property **"separated Hopfness"**, to be used below.

2.2 The Fomin-Greene–Nenashev operators ξ^ν

With these identifications, and the self-duality of the Hopf algebra *Symm* of symmetric functions, we can interpret some results of Nenashev [12]:

$$\begin{array}{ccccccc}
H(A_{\mathbb{Z}}) & \xrightarrow{\sim} & \{\text{back-stable functions}\} & \xleftarrow{\sim} & Symm \otimes_{\mathbb{Z}} \mathbb{Z}[\ldots, x_{-1}, x_0, x_1, \ldots] & \twoheadrightarrow & Symm \\
Nil(A_{\mathbb{Z}}) & & & \longleftarrow & & & Symm
\end{array}$$

The map $\twoheadrightarrow$ is the **Stanley genus**: it takes $\mathcal{S}_\pi$ to its **Stanley symmetric function** $St_\pi = \sum_\lambda a_\pi^\lambda\, Schur_\lambda$. The lower map, its transpose, takes $Schur_\lambda$ to $\sum_\pi a_\pi^\lambda\, d_\pi$. If we let this operator act on $H(A_{\mathbb{Z}})$ under the $d_\pi \mapsto \partial_\pi$ action, we get the **Fomin-Greene–Nenashev operator** $\xi^\lambda := \sum_\pi a_\pi^\lambda\, \partial_\pi$ [3, 12]. (See also the j_λ operators in the "Peterson subalgebra" defined in [9, §9.3], which are a double version of the ξ^λ.)

Let $\mathfrak{m}$ denote the kernel of the map $H(A_{\mathbb{Z}}) \twoheadrightarrow Symm$. Using the separated Hopfness and the fact that $BS_{\pi[N]} - BS_\pi \in \mathfrak{m}$, one shows that each $\Delta(pq) - \Delta(p)\Delta(q)$ (which serves as a measure of non-Hopfness) lies in $\mathfrak{m} \otimes H(A_{\mathbb{Z}}) + H(A_{\mathbb{Z}}) \otimes \mathfrak{m}$. Hence the map $H(A_{\mathbb{Z}}) \to Symm$ factors through a map of Hopf algebras. Dually, the transpose map is a Hopf map to a Hopf sub-bialgebra of $Nil(A_{\mathbb{Z}})$. In particular this Hopf map explains Nenashev's formulæ [12, §4.4]

$$\xi^\lambda \xi^\mu = \sum_\nu c_{\lambda\mu}^\nu\, \xi^\nu \qquad\qquad \xi^\nu(pq) = \sum_{\lambda,\mu} c_{\lambda\mu}^\nu\, \xi^\lambda(p)\, \xi^\mu(q)$$

2.3 Interlude (not used elsewhere): topological origin of the $\{BS_\pi\}$

The stability property underlying Lascoux-Schützenberger's definition of Schubert polynomials is the fact that each $S_\pi \in H^*(Fl(n))$ is the pullback $\iota_{n+1}^*(S_{\pi\oplus 1})$ along a map $\iota_{n+1} : Fl(n) \hookrightarrow Fl(n+1)$ taking $(E_\bullet)$ to $(F_\bullet \colon F_{i\le n} = E_i \oplus 0,\ F_{n+1} = E_n \oplus \mathbb{C})$. Chaining these together, one builds an element of the inverse limit of the cohomology rings, a ring $\mathbb{Z}[[x_1, x_2, \ldots]] / \langle \text{elementary symmetric functions } e_i \rangle$. It was then Lascoux-Schützenberger's pleasant surprise that these "inverse limit Schubert classes" lie in (and exactly span) the image of the injective ring homomorphism $\mathbb{Z}[x_1, x_2, \ldots]$ into this algebra.

This admits of a parallel story, based on a different map $\iota_{1+2n+1} : Fl(2n) \hookrightarrow Fl(2n+2)$ taking $(E_\bullet)$ to $(F_\bullet \colon F_{i \in [1,2n+1]} = \mathbb{C} \oplus E_{i-1} \oplus 0,\ F_{2n+2} = \mathbb{C} \oplus \mathbb{C}^{2n} \oplus \mathbb{C})$. Now, in order to achieve a coherent labeling (as n varies) we index the classes in $H^*(Fl(2n))$ using permutations of $[1-n, n]$ rather than of $[1, 2n]$. Once again the inverse limit is a power series ring modulo elementary symmetrics, but it is *no longer true* that the inverse limit Schubert classes are representable by polynomials; rather, they can be represented by back-stable functions. (And again, they form a basis thereof.)

One advantage of ι_{1+2n+1} is that it is equivariant w.r.t. the *duality* endomorphism of $Fl(2n)$, which takes $(E_\bullet)$ to $(E_\bullet^\perp)$, defined w.r.t. the symplectic form pairing coördinates i and $1-i$, for $i \in [1, n]$. On the level of classes, this takes $BS_\pi \mapsto BS_{w_0 \pi w_0}$ where $w_0(i) := 1 - i$. On the level of back-stable functions, it takes $x_i \mapsto -x_{1-i}$, $e_i(x_{\leq 0}) \mapsto e_i(x_{\leq 0})$.

Since this duality respects Schubert classes and the alphabet (x_i), it takes Monk's rules to Monk's rules. In particular it turns the transition formula (a specific Monk's rule)

$$BS_\pi = x_i BS_{\pi'} + \sum_{\text{certain } \pi''} BS_{\pi''} \qquad \text{into} \qquad BS_\rho = -x_j BS_{\rho'} + \sum_{\text{certain } \rho''} BS_{\rho''}$$

which implies (unstably) the cotransition formula $x_j \mathcal{S}_{\rho'} = -\cancel{\mathcal{S}_\rho} + \sum_{\text{certain } \rho''} \mathcal{S}_{\rho''}$ of [8].

3 Relation to Klyachko's genus

3.1 Klyachko's ideal and its prime factors

Let $T \leq GL_n(\mathbb{C})$ denote the group of diagonal matrices, and $TV_{perm} \subseteq Fl(n)$ be the **permutahedral toric variety** obtained as the closure of a generic T-orbit on the flag manifold $Fl(n)$. This subvariety arises as a Hessenberg variety (see e.g. [1]) and is of key importance in [6, 11].

The inclusion $\iota \colon TV_{perm} \hookrightarrow Fl(n)$ induces a map backwards on cohomology, which is neither injective nor surjective. Klyachko [7] presented its image $im(\iota^*)$ (with rational coefficients), and a formula for ι^* evaluated on Schubert symbols:

$$\begin{aligned} H^*(Fl(n);\mathbb{Q}) &\to im(\iota^*) \cong \mathbb{Q}[k_0, \ldots, k_n] \Big/ \left\langle \begin{matrix} k_i(-k_{i-1} + 2k_i - k_{i+1}) = 0, \ \ 1 \leq i \leq n-1 \\ k_0 = k_n = 0 \end{matrix} \right\rangle \\ \mathcal{S}_\pi &\mapsto \frac{1}{\ell(\pi)!} \sum_{Q \in RW(\pi)} \prod_{q \in Q} k_q \quad \text{where } RW(\pi) \text{ is the set of reduced words} \end{aligned}$$

Taking forward- and back-stable limits, while leaving behind geometry, we get the **Klyachko genus** $H(A_{\mathbb{Z}}) \to \mathbb{Q}[\ldots,k_{-1},k_0,k_1,\ldots] \Big/ \langle\, k_i(-k_{i-1}+2k_i-k_{i+1}) = 0\ \forall i \in \mathbb{Z}\,\rangle$ whose map on Schubert symbols is given by the same formula. We use this to recover a result of Nenashev, foreshadowing some results in §5:

Theorem 6. *[12, Proposition 3 and discussion after] Let $RW(\pi)$ denote the set of reduced words for π. There must* **exist** *(but the proof doesn't find one) a "rectification" map*

$$\{\textit{shuffles of any word in } RW(\pi) \textit{ with any word in } RW(\rho)\} \to \coprod_\sigma RW(\sigma)$$

whose fiber over any reduced word for σ has size $c^\sigma_{\pi\rho}$, the coefficient from $\mathcal{S}_\pi\mathcal{S}_\rho = \sum_\sigma c^\sigma_{\pi\rho}\,\mathcal{S}_\sigma$.

Proof. Apply the Klyachko genus to that last equation, then set all $k_i = 1$, obtaining

$$\frac{1}{\ell(\pi)!}\sum_{P\in RW(\pi)}\prod_P 1 \quad \frac{1}{\ell(\rho)!}\sum_{R\in RW(\rho)}\prod_R 1 = \sum_\sigma c^\sigma_{\pi\rho}\,\frac{1}{\ell(\sigma)!}\sum_{S\in RW(\sigma)}\prod_S 1$$

Since $c^\sigma_{\pi\rho} = 0$ unless $\ell(\sigma) = \ell(\pi)+\ell(\rho)$, we can restrict to those σ. Multiplying through:

$$\#RW(\pi)\ \#RW(\rho)\binom{\ell(\pi)+\ell(\rho)}{\ell(\pi)} = \sum_\sigma c^\sigma_{\pi\rho}\,\#RW(\sigma)$$

Let $C^\sigma_{\pi\rho}$ be a set with cardinality $c^\sigma_{\pi\rho}$ (and wouldn't you like to know one?). Then we can interpret the above as

$$\#\{\text{shuffles of any word in } RW(\pi) \text{ with any word in } RW(\rho)\} \;=\; \#\coprod_\sigma (C^\sigma_{\pi\rho}\times RW(\sigma))$$

Hence there exists a bijection; compose it with the projection to $\coprod_\sigma RW(\sigma)$. ☐

We can further simplify the target of this genus by modding out by each of the minimal prime ideals that contain the Klyachko ideal. We get ahold of these using the Nullstellensatz,[3] i.e. by looking at the components of the solution set to Klyachko's equations.

Proposition 1. *Consider $\mathbb{Z}$-ary tuples $(k_i)_{i\in\mathbb{Z}}$ of complex numbers satisfying the Klyachko equalities. This set is the (nondisjoint) union of the following countable set of 2-planes:*

- *For $a,b\in\mathbb{C}$, let $k_m = am+b$.*
- *For $i\le j$ each in $\mathbb{Z}$, and $x,y\in\mathbb{C}$ a pair of "slopes", let* $k_m = \begin{cases} x(m-i) & \textit{if } k\le i \\ 0 & \textit{if } k\in[i,j] \\ y(m-j) & \textit{if } k\ge j.\end{cases}$

[3]This isn't quite fair, in that we are working in infinite dimensions, but we won't worry about it. All we're really trying to do here is choose, for each i, which factor of $k_i(-k_{i-1}+2k_i-k_{i+1})$ to mod out.

After completing this work, we learned of a very similar calculation in [11, §3.4], so we omit the proof of proposition 1 (obtainable as a sort of $q \to 1$ limit of theirs).

Each component defines a quotient of the Klyachko ring, namely

$$\mathbb{Q}[\ldots, k_{-1}, k_0, k_1, \ldots] \Big/ \langle -k_{m-1} + 2k_m - k_{m+1} = 0 \ \forall m \in \mathbb{Z} \rangle$$

$$\forall i \leq j, \qquad \mathbb{Q}[\ldots, k_{-1}, k_0, k_1, \ldots] \Big/ \left\langle \begin{array}{cc} k_m = 0 & \forall m \in [i,j] \\ -k_{m-1} + 2k_m - k_{m+1} = 0 & \forall m \notin [i,j] \end{array} \right\rangle$$

Call the map of $H(A_{\mathbb{Z}})$ to the first quotient the **affine-linear genus**.

There is a slight subtlety in that the Klyachko ideal is not radical, and as such, the map from the Klyachko ring to the direct sum of these quotients is not injective. We will return to this minor matter below.

3.2 Dropping the other genera

The other components (besides the one giving the affine-linear genus) are useless, in the following senses. Say $k_m = 0$ for some m; then there are three situations.

1. *Some reduced word for a permutation π uses the letter m.* Then *all* reduced words do, with the effect that $\mathcal{S}_\pi \mapsto 0$ in the quotient ring.

2. *Each reduced word for π uses some letters $> m$ and some $< m$.* Then $\pi = \pi_{<m}\pi_{>m}$ where each uses only letters $< m$, $> m$ respectively. In this case $\mathcal{S}_\pi = \mathcal{S}_{\pi_{<m}}\mathcal{S}_{\pi_{>m}}$.

3. *Each reduced word for π only uses letters on one side of m.* At this point there is nothing to be gained by setting $k_m = 0$; we could work with just the affine-linear genus.

Our principal interest in genera is to study **Schubert calculus**, the structure constants $c_{\pi\rho}^{\sigma}$ of the multiplication of Schubert symbols. That is hard to do if the symbols map to zero (situation #1), silly to do directly if the symbols are are themselves products (situation #2), and in situation #3 might as well be done using the affine-linear genus. As such, at this point we cast aside the Klyachko genus in favor of the affine-linear genus γ:

$$\gamma : \ H(A_{\mathbb{Z}}) \ \to \ \mathbb{Q}[a,b], \qquad \qquad \mathcal{S}_\pi \mapsto \frac{1}{\ell(\pi)!} \sum_{P \in RW(\pi)} \prod_{i \in P} (ai + b)$$

The assiduous reader might be guessing now that the information lost when passing from the Klyachko ideal to its radical is similarly negligible for Schubert calculus purposes. And indeed: if we factor the Klyachko ideal as an intersection of primary instead of prime components, we run into the ideals

$$\forall i \leq j, \qquad \mathbb{Q}[\ldots, k_{-1}, k_0, k_1, \ldots] \Big/ \left\langle \begin{array}{cc} k_m^2 = 0 & \forall m \in [i+1, j-1] \\ k_i = k_j = 0 & \\ -k_{m-1} + 2k_m - k_{m+1} = 0 & \forall m \notin [i,j] \end{array} \right\rangle$$

These would let us study π, ρ, σ whose reduced words use only the letters in the range $[i+1, j-1]$, and each at most once. This is an extremely limited case.

4 The affine-linear genus γ from the martial derivations

Recall the derivations

$$\nabla = \sum_m m \male_m \qquad\qquad \xi = \sum_m \male_m$$

Being derivations, they exponentiate to automorphisms of $\mathbb{Q} \otimes_{\mathbb{Z}} H(A_{\mathbb{Z}})$ (where the $\mathbb{Q}$ is necessitated by the denominators in the exponential series).

Theorem 7. *The following triangle commutes:*

$$\begin{array}{ccc} & H(A_{\mathbb{Z}}) & \\ e^{a\nabla + b\xi} \swarrow & & \searrow \gamma \\ \mathbb{Q}[a,b] \otimes_{\mathbb{Z}} H(A_{\mathbb{Z}}) & \twoheadrightarrow & \mathbb{Q}[a,b] \\ \mathcal{S}_\pi & \mapsto & \delta_{\pi,e} \end{array}$$

Proof. The proof is not conceptual; we compute both sides and compare. Indeed, we find the statement intriguing exactly because we know of no geometric reason the two maps should be related.

$$e^{a\nabla + b\xi} \cdot \mathcal{S}_\pi \;=\; \sum_n \frac{1}{n!}(a\nabla + b\xi)^n \cdot \mathcal{S}_\pi \;\mapsto\; \frac{(a\nabla + b\xi)^{\ell(\pi)} \cdot \mathcal{S}_\pi}{\ell(\pi)!} \;=\; \frac{\left(\sum_i (ai+b)\male_i\right)^{\ell(\pi)} \cdot \mathcal{S}_\pi}{\ell(\pi)!}$$

Expanding $\left(\sum_i (ai+b)\male_i\right)^{\ell(\pi)}$, the nonvanishing terms correspond to reduced words of length $\ell(\pi)$, and only those that multiply to π^{-1} survive application to $\mathcal{S}_\pi$. ◻

In particular the proof of Theorem 6 essentially amounts to applying $\exp(\xi)$. (Oddly, the original proof in [12] is closer to an application of $\exp(\nabla)$.)

There is a fascinating q-**Klyachko genus** introduced in [11, §3.4]:

$$\begin{aligned} \gamma_q : \; H(A_{\mathbb{Z}}) \;&\to\; \mathbb{Q}(q)[\alpha, \beta] \\ \mathcal{S}_\pi \;&\mapsto\; \frac{1}{\ell(\pi)^{q}_{\bullet}} \sum_{Q:\, \prod Q = \pi} q^{\mathrm{comaj}(Q)} \prod_{i \in Q} \left(\alpha q^i + \beta\right) \end{aligned}$$

Here $m^{q}_{\bullet}$ is the q-torial $\prod_{j=1}^m [j]_q$, and $\mathrm{comaj}(Q)$ is the sum of the positions of the ascents. We looked for a long time for a q-analogue of Theorem 7, to no avail: it would provide an automorphism of $H(A_{\mathbb{Z}})(q)[\alpha, \beta]$ whose $\ell = 0$ part is the q-Klyachko genus.

5 Rectification and the q-statistic

We pursue a q-analogue of (Nenashev's) Theorem 6. Applying Nadeau-Tewari's q-Klyachko genus to $\mathcal{S}_\pi \mathcal{S}_\rho = \sum_\sigma c^\sigma_{\pi\rho} \mathcal{S}_\sigma$ we get

$$\begin{aligned} &\frac{1}{\ell(\pi)^q_\bullet} \sum_{P\in RW(\pi)} q^{\mathrm{comaj}(P)} \prod_{i\in P} (\alpha q^i + \beta) \quad \frac{1}{\ell(\rho)^q_\bullet} \sum_{R\in RW(\rho)} q^{\mathrm{comaj}(Q)} \prod_{i\in R} (\alpha q^i + \beta) \\ =\ & \sum_\sigma c^\sigma_{\pi\rho} \frac{1}{\ell(\sigma)^q_\bullet} \sum_{S\in RW(\sigma)} q^{\mathrm{comaj}(S)} \prod_{i\in S} (\alpha q^i + \beta) \end{aligned}$$

Multiplying through, we get

$$\binom{\ell(\pi)+\ell(\rho)}{\ell(\pi)}_q \sum_{\substack{P\in RW(\pi)\\ R\in RW(\rho)}} q^{\substack{\mathrm{comaj}(P)\\ +\mathrm{comaj}(R)}} \prod_{i\in P\coprod R} (\alpha q^i + \beta) \quad = \quad \sum_\sigma c^\sigma_{\pi\rho} \sum_{S\in RW(\sigma)} q^{\mathrm{comaj}(S)} \prod_{i\in S} (\alpha q^i + \beta)$$

Let's interpret both sides at $\alpha = \beta = q = 1$, again using a mystery set $C^\sigma_{\pi\rho}$ with cardinality $c^\sigma_{\pi\rho}$. Define a **barred word** for π as a reduced word in which some letters are overlined, e.g. $12\overline{1}$ for (13). Then the left side of the above equation counts pairs (P, R) of barred words, shuffled together, where the barring indicates "use the αq^i term" rather than the β term. Meanwhile, the right side counts pairs (τ, S) where S is a barred word for some σ, and τ is in $C^\sigma_{\pi\rho}$.

Theorem 8. *Define the q-statistic of a barred word as the sum of the locations of the ascents, plus the sum of the barred letters.*

Define the q-statistic of a shuffle $ш$ of a pair (P, R) of barred words as the sum of the two q-statistics, plus the number of inversions in the shuffle (letters in R leftward of letters in P).

Then there **exists** *(but the proof doesn't find one) a "rectification" map*

$$\{\textit{shuffles of pairs } (P,R) \textit{ of barred words for } \pi,\rho\} \quad \to \quad \coprod_\sigma \{\textit{barred words for } \sigma\}$$

preserving the number of bars and the q-statistic, whose fiber over each word for σ is of size $c^\sigma_{\pi\rho}$.

We note that the affine-linear genus doesn't let one produce such a combinatorial result, insofar as the factors $ai + b$ can involve $i < 0$ (in the back-stable setting of $A_\mathbb{Z}$).

Example. These examples get large very quickly, so we restrict to the fully barred case. Let $\pi = \rho = 12463578$, chosen to give a $c^\sigma_{\pi\rho} > 1$ (and chosen stably enough that the terms in the product don't move $-\mathbb{N}$). Each of π and ρ have two reduced words (354 and 534, comajs 1 and 2, each of total 12), and there are $\binom{6}{3}$ ways to shuffle, for a total of $2 \cdot 2 \cdot \binom{6}{3} = 80$; the resulting q-statistics range from $26 = 1 + 12 + 1 + 12 + 0$

to $37 = 2 + 12 + 2 + 12 + 3 \cdot 3$. There are 7 terms $\mathcal{S}_\sigma$ in the product $\mathcal{S}_\pi \mathcal{S}_\rho$ (one with coefficient 2) with various numbers of reduced words.

q-statistic:	26	27	28	29	30	31	32	33	34	35	36	37	
	1	3	5	8	11	12	12	11	8	5	3	1	total = 80
σ													
23561478	1	1	2	1	2	1	1						
14562378		1		1	1	1		1					
13572468			1	2	2	3	3	2	2	1			
13572468			1	2	2	3	3	2	2	1			again
23471568		1	1	2	2	2	1	1					
13482567					1	1	2	2	2	1	1		
12673458					1		1	1	1		1		
12583467						1	1	2	1	2	1	1	

In the line with "total = 80", we count the number of fully barred shuffles with given q-statistic. In each of the lower lines, we put σ on the left, and on the right we list the number of fully barred words for it with given q-statistic. Then the theorem asserts that each number atop a column is the sum of the numbers below. There is a silly rotational near-symmetry tracing to the fact that π and ρ are Grassmannian permutations for self-conjugate partitions.

6 Equidistribution of inversion number vs. comaj on $\binom{[n]}{m}$

Let $J \subseteq \mathbb{Z}$ be a set of n numbers, no two adjacent. Then the product $\prod_{j \in J} r_j$ is well-defined i.e. is independent of the order; indeed, the reduced words for $\prod_{j \in J} r_j$ are in correspondence with permutations of J. The same holds when multiplying subsets of J.

Fix $K \subseteq J$ and let $\rho = \prod_K r_k$, $\pi = \prod_{J \setminus K} r_j$. Then $\mathcal{S}_\pi \mathcal{S}_\rho = \mathcal{S}_{\pi\rho}$, and Theorem 8 (again in the fully barred case) predicts a bijection

$$\{\text{insertions of reduced words } R \text{ for } \rho \text{ into reduced words } P \text{ for } \pi\} \quad \to \quad RW(\pi\rho)$$

such that $\left[\text{comaj}(P) + \text{comaj}(R) + \text{the inversion number of the shuffle}\right]$ matches comaj of the resulting word ш. Note that the obvious map (just insert R where the shuffle suggests) does *not* correspond these two statistics!

If we break J not into two subsets, but all the way down into individual letters, this recovers the equidistribution on S_n of the statistics ℓ and comaj (or maj); see e.g. [14, Proposition 1.4.6].

This hints at a stronger result: that for any two strings P, R such that PR has no repeats, on the set {shuffles ш} the distributions of the statistic $\text{comaj}(P) + \text{comaj}(R) + \ell(ш)$ and the statistic $\text{comaj}(ш)$ match. (Theorem 8 only guarantees this *after summing* over all $P \in RW(\pi), R \in RW(\rho)$.) And indeed, this stronger claim holds [4].

Acknowledgements

We thank Marcelo Aguiar, Hugh Dennin, Yibo Gao, Thomas Lam, Philippe Nadeau, Gleb Nenashev, Mario Sanchez, Thomas Bååth Sjöblom, and David E Speyer.

References

[1] H. Abe and T. Horiguchi. "A survey of recent developments on Hessenberg varieties". *Schubert Calculus and Its Applications in Combinatorics and Representation Theory: Guangzhou, China, November 2017* (2020), pp. 251–279.

[2] A. Berenstein and E. Richmond. "Littlewood–Richardson coefficients for reflection groups". *Advances in Mathematics* **284** (2015), pp. 54–111.

[3] S. Fomin and C. Greene. "Noncommutative Schur functions and their applications". *Discrete Mathematics* **193**.1-3 (1998), pp. 179–200.

[4] A. M. Garsia and I. Gessel. "Permutation statistics and partitions". *Adv. in Math.* **31**.3 (1979), pp. 288–305. DOI.

[5] Z. Hamaker, O. Pechenik, D. E. Speyer, and A. Weigandt. "Derivatives of Schubert polynomials and proof of a determinant conjecture of Stanley". *Alg. Comb.* **3**.2 (2020), pp. 301–307.

[6] J. Huh and E. Katz. "Log-concavity of characteristic polynomials and the Bergman fan of matroids". *Mathematische Annalen* **354** (2012), pp. 1103–1116.

[7] A. A. Klyachko. "Orbits of a maximal torus on a flag space". *Functional analysis and its applications* **19**.1 (1985), pp. 65–66.

[8] A. Knutson. "Schubert polynomials, pipe dreams, equivariant classes, and a co-transition formula". *Facets of algebraic geometry. Vol. II*. Vol. 473. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2022, pp. 63–83.

[9] T. Lam, S. J. Lee, and M. Shimozono. "Back stable Schubert calculus". *Compositio Mathematica* **157**.5 (2021), pp. 883–962.

[10] M. Lanini, R. Xiong, and K. Zainoulline. "Structure algebras, Hopf algebroids and oriented cohomology of a group" (2023). arXiv:2303.02409.

[11] P. Nadeau and V. Tewari. "A q-deformation of an algebra of Klyachko and Macdonald's reduced word formula". 2021. arXiv:2106.03828.

[12] G. Nenashev. "Differential operators on Schur and Schubert polynomials". arXiv:2005.08329.

[13] O. Pechenik and A. Weigandt. "A dual Littlewood-Richardson rule and extensions". 2022. arXiv:2202.11185.

[14] R. P. Stanley. "Enumerative Combinatorics Volume 1 second edition". *Cambridge Studies in Advanced Mathematics* (2011).