SIAM J. Discrete Math.
Vol. 38, No. 2, pp. 1269–1284

RAINBOW EVEN CYCLES*

ZICHAO DONG† AND ZIJIAN XU‡

Abstract. We prove that every family of (not necessarily distinct) even cycles $D_1,\dots,D_{\lfloor 1.2(n-1)\rfloor+1}$ on some fixed n-vertex set has a rainbow even cycle (that is, a set of edges from distinct D_i's, forming an even cycle). This resolves an open problem of Aharoni, Briggs, Holzman and Jiang. Moreover, the result is best possible for every positive integer n.

Key words. even cycle, rainbow extremal graph theory, Frankenstein graph

MSC codes. 05C35, 05C38

DOI. 10.1137/23M1564808

1. Introduction. Let $\mathcal{F}$ be a set family. A rainbow set with respect to $\mathcal{F}$ is a subset R (without repeated elements) of $\cup\mathcal{F}$ (i.e., $\bigcup_{F\in\mathcal{F}} F$) such that there exists an injection $\sigma\colon R\to\mathcal{F}$ with $r\in\sigma(r)$ for all $r\in R$. In other words, each element $r\in R$ comes from a distinct $F\in\mathcal{F}$. We think about each set in $\mathcal{F}$ as a different color class, and hence use the term "rainbow." An important remark here is that a "family" refers to a "multiset," since an element in $\cup\mathcal{F}$ may appear with more than one color.

Suppose every $F\in\mathcal{F}$ satisfies property $\mathcal{P}$. What is the minimum size of $\mathcal{F}$ such that a rainbow subset of $\cup\mathcal{F}$ satisfying $\mathcal{P}$ always exists? One famous result of this type is the colorful version of Carathéodory's theorem due to Bárány [6], which asserts that every family of $n+1$ subsets of $\mathbb{R}^n$, each containing a point p in its convex hull, has a rainbow subset whose convex hull contains p as well. Such problems are also studied in graph theory. Aharoni and Berger [1] proved that any family of $2n-1$ matchings of size n in a bipartite graph contains a rainbow matching of size n. Other results of this type on cycles and triangles can be found in [3, 9, 8].

There are studies of rainbow graphs in a different context: Given an edge-colored graph, what conditions guarantee a certain subgraph whose edges have distinct colors? Due to the relation with Latin squares, rainbow matchings have received extensive attention. See [2, 7] for recent works. As a starting point for finding colorful variants of Turán's theorem, the existence of rainbow triangles is analyzed in [4, 5]. A rainbow version of Dirac's theorem on Hamiltonian cycles can be found in [10].

Throughout the paper, a graph, without further specification, refers to a simple graph G which is a set of colored edges. Formally, G is a set of pairs $\mathsf{e}=(e,\alpha)$, where e's are distinct edges (i.e., different pairs of two distinct vertices) and α's are (not necessarily distinct) colors. For $\mathsf{e}=(uv,\alpha)\in\mathsf{G}$, where $uv\overset{\text{def}}{=}\{u,v\}$, denote $V(\mathsf{e})\overset{\text{def}}{=}\{u,v\}$, $\chi(\mathsf{e})=\alpha$. Then write $V(\mathsf{G})\overset{\text{def}}{=}\bigcup_{\mathsf{e}\in\mathsf{G}}V(\mathsf{e})$, $E(\mathsf{G})\overset{\text{def}}{=}\{V(\mathsf{e}):\mathsf{e}\in\mathsf{G}\}$, and $\chi(\mathsf{G})\overset{\text{def}}{=}\{\chi(\mathsf{e}):\mathsf{e}\in\mathsf{G}\}$ for the vertex set, the (uncolored) edge set and the color set, respectively.

*Received by the editors April 10, 2023; accepted for publication (in revised form) January 31, 2024; published electronically April 9, 2024.
https://doi.org/10.1137/23M1564808
Funding: The first author's research was supported in part through NSF grant DMS-2154063.
†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213 USA (zichaod@andrew.cmu.edu).
‡School of Mathematical Sciences, Peking University, Beijing 100871 People's Republic of China (2200010770@stu.pku.edu.cn).

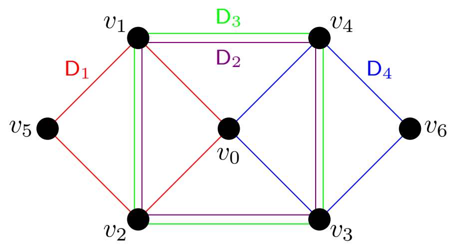


FIG. 1. *An example family $\mathcal{D}$ viewed as an edge-colored multigraph. (See electronic version for color figures.)*

Two edges $\mathsf{e}_1, \mathsf{e}_2$ are *coincident* if they are of different colors and are on the same vertex set. That is, $V(\mathsf{e}_1) = V(\mathsf{e}_2)$ yet $\chi(\mathsf{e}_1) \neq \chi(\mathsf{e}_2)$. For two graphs $\mathsf{G}_1, \mathsf{G}_2$, we call them *coincident* if there exists a bijection $\varphi\colon \mathsf{G}_1 \to \mathsf{G}_2$ such that e is coincident to $\varphi(\mathsf{e})$ for all $\mathsf{e} \in \mathsf{G}_1$. Note that coincident edges do not exist in a graph, since graphs are assumed to be simple.

This paper is devoted to the existence of a rainbow even cycle in a family of even cycles. A cycle is a graph C such that its edges $E(\mathsf{C})$, viewed as an uncolored simple graph, form a cycle. In other words, $\mathsf{C} = \{(v_1v_2, \alpha_1), \dots, (v_{\ell-1}v_\ell, \alpha_{\ell-1}), (v_\ell v_1, \alpha_\ell)\}$, where $v_1, \dots, v_\ell$ are distinct and $\ell \geq 3$ is called the *length* of C. For any $A \subseteq \{3, 4, 5, \dots\}$, an *$A$-cycle* is a cycle whose length is some number from A. For example, an *odd cycle,* a cycle of odd length, is a $\{3, 5, 7, \dots\}$-cycle. Similarly, an *even cycle,* a cycle of even length, is a $\{4, 6, 8, \dots\}$-cycle. For any integer $k \geq 3$, a k-cycle refers to a $\{k\}$-cycle.

Hereafter a family $\mathcal{F} = \{\mathsf{E}_1, \dots, \mathsf{E}_m\}$ is a family of cycles. We remark that $\mathcal{F}$ being a family implicitly implies that $\chi(\mathsf{E}_i) = \{\alpha_i\}$, while $\alpha_1, \dots, \alpha_m$ are distinct. Since each E_i is a monochromatic cycle, we view $\mathcal{F}$ as an edge-colored multigraph (i.e., a set of colored edges where coincident edges are allowed). A subgraph of $\mathcal{F}$ is then a graph E, where $\mathsf{E} \subseteq \bigcup_{i=1}^m \mathsf{E}_i$. In Figure 1, the family $\mathcal{D} = \{\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4\}$ consists of four 4-cycles on seven vertices, where $\mathsf{D}_2, \mathsf{D}_3$ are coincident. Let $\chi(\mathsf{D}_i) = \alpha_i\,(i = 1, 2, 3, 4)$. Then $\mathsf{D} \overset{\text{def}}{=} \{(v_0v_1, \alpha_1), (v_1v_2, \alpha_2), (v_2v_3, \alpha_3), (v_3v_0, \alpha_4)\}$ is a rainbow 4-cycle subgraph of $\mathcal{D}$.

We shall say that a family $\mathcal{F}$ *contains* a graph G if G is a subgraph of $\mathcal{F}$.

THEOREM 1 (see [3]). *Every family of $2\left\lceil \frac{n}{2} \right\rceil - 1$ odd cycles on n vertices contains a rainbow odd cycle.*

The tightness of Theorem 1 is witnessed by a family of $2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ many coincident odd cycles on $2\left\lceil \frac{n}{2} \right\rceil - 1$ vertices. As for even cycles, Aharoni et al. also deduced in [3] that the maximum size of a family on n vertices containing no rainbow even cycle is between roughly $\frac{6}{5}n$ and $\frac{3}{2}n$, and left the determination of the exact extremal number as an open problem. We answer this question by proving the following result.

THEOREM 2. *Every family of $\left\lfloor \frac{6(n-1)}{5} \right\rfloor + 1$ even cycles on n vertices contains a rainbow even cycle.*

The tightness of Theorem 2 for each $n \geq 4$ (no even cycle exists when $n \leq 3$) is seen as follows: The families $\mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8$ in Figure 2 are tight examples for $n = 4, 5, 6, 7, 8$, respectively. For larger n, we observe that by gluing together $\mathcal{D}_{n-5}$ (a tight example for $n - 5$) and $\mathcal{D}_6$ at exactly one vertex (edge-disjoint henceforth) the resulting family $\mathcal{D}_n$ is tight for n. We remark that the family $\mathcal{D}_6$ and the inductive argument were already presented in [3].

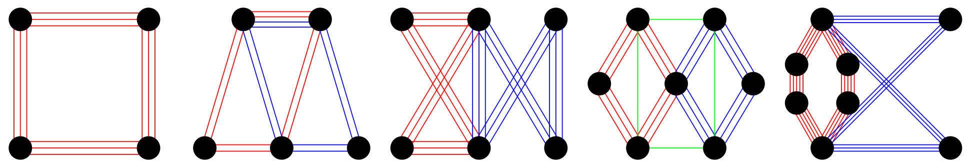


FIG. 2. *Tight examples of* Theorem 2 *for small* n.

Proof strategy. To explain the strategy of our proof, we begin with a baby version of Theorem 2 whose tightness is witnessed by, for example, a family of $n-1$ coincident Hamiltonian cycles.

THEOREM 3 (see [3, Proposition 3.2]). *Every family of n cycles on n vertices contains a rainbow cycle.*

Proof. Let $\mathcal{F}$ be such a family and F be a maximal rainbow forest subgraph of $\mathcal{F}$. Then $|\mathsf{F}| \leq n-1$, and so there is another edge e, not coincident to any edge of F, whose color does not appear in F. The maximality of F implies that e completes a rainbow cycle in the graph $\mathsf{F} \cup \{\mathsf{e}\}$. ∎

All these proofs proceed by first finding a *spanning structure* S (the rainbow forest F in the proof above) and then analyzing another edge with an absent color in S. The proof of Theorem 1 also uses a maximal rainbow forest as S. However, to prove Theorem 2 we need some new spanning structure.

It turns out that 5-cycles play a central role in the $\frac{6}{5}n$ upper bound. We thus call a cycle *long* if its length is at least 6. In particular, a rainbow $\{7, 9, 11, \ldots\}$-cycle is a long rainbow odd cycle. Then our spanning structure, which we call *Frankenstein graphs,* are (informally speaking) obtained by recursively, at single vertices, gluing together a collection of long rainbow odd cycles, rainbow trees, and another class of graphs named *bad pieces.*

We shall formally define and characterize bad pieces and Frankenstein graphs in section 2. Then section 3 is devoted to the proof of Theorem 2.

2. Frankenstein graphs.

A path graph of length k is a graph of the form

$$\mathsf{P} = \{(v_0v_1, \alpha_1), (v_1v_2, \alpha_2), \ldots, (v_{k-1}v_k, \alpha_k)\},$$

where $v_0, \ldots, v_k$ are distinct. A *theta graph* is a union of 3 paths that share exactly their terminals. Formally, G is a theta graph if $\mathsf{G} = \mathsf{P}_1 \cup \mathsf{P}_2 \cup \mathsf{P}_3$, where $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$ are paths with terminals s, t and

$$V(\mathsf{P}_1) \cap V(\mathsf{P}_2) = V(\mathsf{P}_2) \cap V(\mathsf{P}_3) = V(\mathsf{P}_3) \cap V(\mathsf{P}_1) = \{s, t\},$$
$$E(\mathsf{P}_1) \cap E(\mathsf{P}_2) = E(\mathsf{P}_2) \cap E(\mathsf{P}_3) = E(\mathsf{P}_3) \cap E(\mathsf{P}_1) = \varnothing.$$

We use the name "theta" because one natural drawing of such a graph looks exactly like the Greek letter Θ. See Figure 3 below as an illustration.

Observation 4. Every rainbow theta graph has a rainbow even cycle subgraph.

Proof. Suppose $\mathsf{P}_1 \cup \mathsf{P}_2 \cup \mathsf{P}_3$ is a theta graph where $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$ are paths of common terminals. Then two of the paths, say P_1 and P_2, have lengths of the same parity, and so $\mathsf{P}_1 \cup \mathsf{P}_2$ is an even cycle. ∎

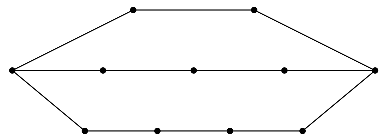


FIG. 3. *A theta graph on paths of lengths* 3, 4, 5*, respectively.*

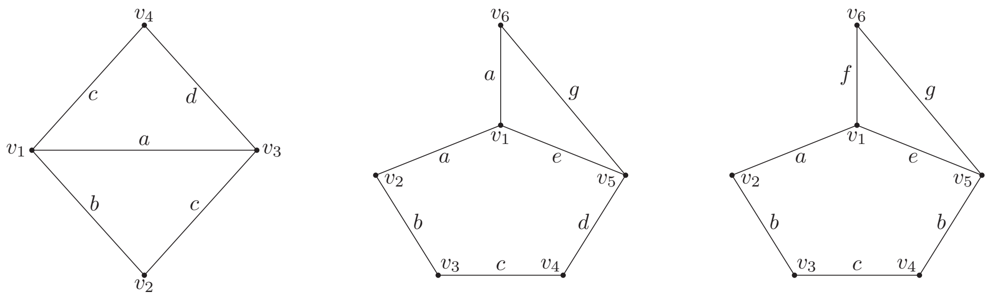


Figure 4.A **Figure 4.B** **Figure 4.C**

FIG. 4. *One example and two non-examples of bad pieces.*

We call a graph G *almost rainbow* if $|\chi(\mathsf{G})| = |\mathsf{G}| - 1$. That is, exactly two edges receive a same color, and the color of every other edge is unique. We call B a *bad piece* if B is an almost rainbow theta graph on 3 rainbow paths (sharing terminals) such that $|V(\mathsf{B})| \geq 6$.

For example, Figure 4.A is not a bad piece because it contains only 4 vertices; Figure 4.B is a bad piece on 6 vertices and 7 edges consisting of rainbow paths v_1v_5, $v_1v_2v_3v_4v_5$ and $v_1v_6v_5$; Figure 4.C is not a bad piece because $v_1v_2v_3v_4v_5$ is not rainbow (as witnessed by (v_2v_3, b) and (v_4v_5, b)).

Observation 5. If B is a bad piece, then $|V(\mathsf{B})| = |\chi(\mathsf{B})| \leq \frac{6}{5}(|V(\mathsf{B})| - 1)$.

Proof. Since B is a theta graph, we have $|V(\mathsf{B})| = |\mathsf{B}| - 1$. Notice that since B is almost rainbow, we see that $|\chi(\mathsf{B})| = |\mathsf{B}| - 1$. It follows that $n \overset{\text{def}}{=} |\chi(\mathsf{B})| = |V(\mathsf{B})| \geq 6$, and hence $\frac{|\chi(\mathsf{B})|}{|V(\mathsf{B})|-1} = \frac{n}{n-1} \leq \frac{6}{5}$. ∎

Observation 6. If B is a bad piece, then for any distinct $v_1, v_2 \in V(\mathsf{B})$, there exists in B a rainbow path subgraph whose terminals are v_1 and v_2.

Proof. Since $|\chi(\mathsf{B})| = |\mathsf{B}| - 1$, it suffices to show that v_1, v_2 are vertices of a cycle in B. Suppose B consists of three rainbow paths $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$. If v_1 and v_2 are on a same path, say P_1, then $\mathsf{P}_1 \cup \mathsf{P}_2$ is such a cycle. If v_1 and v_2 are on different paths, say P_1 and P_2, then $\mathsf{P}_1 \cup \mathsf{P}_2$ is such a cycle. ∎

Let G be a graph. We call $\mathcal{P} = \{\mathsf{G}_1, \ldots, \mathsf{G}_m\}$ a *partition* if $\mathsf{G} = \bigcup_{i=1}^m \mathsf{G}_i$ and $|V(\mathsf{G}_i) \cap V(\mathsf{G}_j)| \leq 1$, $\chi(\mathsf{G}_i) \cap \chi(\mathsf{G}_j) = \varnothing$ for every distinct $\mathsf{G}_i, \mathsf{G}_j$. We shall often abuse notation by writing $\mathcal{P}(\mathsf{G}) = \mathcal{P}$. Indeed, $\mathcal{P}(\mathsf{G})$ is not a function of G, as the partition is usually not unique. The notation emphasizes that the partition is of G. In this sense, $\mathfrak{F}$ is a *Frankenstein graph* if it admits a partition

$$\mathcal{P}(\mathfrak{F}) = \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b, \mathsf{T}_1, \ldots, \mathsf{T}_t\} \qquad (c \geq 0,\, b \geq 0,\, t \geq 0,\, c + b + t \geq 1),$$

where C's are long rainbow odd cycles, B's are bad pieces, and T's are rainbow trees, such that

(F1) $V(\mathsf{T}_p) \cap V(\mathsf{T}_q) = \varnothing$ for any distinct p, q, and
(F2) no rainbow even cycle subgraph exists in $\mathfrak{F}$.

THEOREM 7. *For any Frankenstein graph $\mathfrak{F}$ with $\mathcal{P}(\mathfrak{F}) = \{\mathsf{G}_1, \ldots, \mathsf{G}_m\}$, there exists a permutation σ on $[m]$ such that $\mathfrak{F}_i \overset{def}{=} \mathsf{G}_{\sigma(1)} \cup \cdots \cup \mathsf{G}_{\sigma(i)}$ satisfies $|V(\mathfrak{F}_i) \cap V(\mathsf{G}_{\sigma(i+1)})| \leq 1$ for each $i \in [m-1]$.*

Theorem 7 suggests the following way to think about a connected Frankenstein graph $\mathfrak{F}$: Suppose the partition of $\mathfrak{F}$ is $\mathcal{P}(\mathfrak{F}) = \{\mathsf{G}_1, \ldots, \mathsf{G}_m\}$. Then one can order the parts as $\mathfrak{F}_1 \overset{\text{def}}{=} \mathsf{G}'_1, \mathsf{G}'_2, \ldots, \mathsf{G}'_m$ and recursively glue together G'_{i+1} and the ith graph $\mathfrak{F}_i$ at some single vertex to make the $(i+1)$st graph $\mathfrak{F}_{i+1}$, such that eventually $\mathfrak{F}_m$ is exactly $\mathfrak{F}$. To prove Theorem 7, we need some preparations.

LEMMA 8. *Let C be a rainbow cycle. Assume X is a rainbow cycle or a bad piece with $\mathsf{X}, \mathsf{C} \setminus \mathsf{X}$ being color-disjoint and $E(\mathsf{C}) \setminus E(\mathsf{X}) \neq \varnothing$. If $|V(\mathsf{C}) \cap V(\mathsf{X})| \geq 2$, then $\mathsf{C} \cup \mathsf{X}$ contains a rainbow even cycle.*

Informally speaking, this technical result is helpful because it tells us that a rainbow cycle is likely to form a rainbow even cycle together with a long rainbow odd cycle or a bad piece.

Proof. Since $E(\mathsf{C}) \setminus E(\mathsf{X}) \neq \varnothing$, there exists an edge $\mathsf{e} \in \mathsf{C}$ that is not coincident to any edge of X. Starting from e and moving along C in opposite directions, we define the first vertices to meet on X as s_0, t_0, thanks to $|V(\mathsf{C}) \cap V(\mathsf{X})| \geq 2$. Then there exists a subpath P_0 (i.e., a path subgraph) of $\mathsf{C} \setminus \mathsf{X}$ satisfying $\mathsf{e} \in \mathsf{P}_0$. Here s_0, t_0 are terminals of P_0, $V(\mathsf{P}_0) \cap V(\mathsf{X}) = \{s_0, t_0\}$ and $\chi(\mathsf{P}_0) \cap \chi(\mathsf{X}) = \varnothing$.

We claim the existence of a rainbow theta subgraph in $\mathsf{X} \cup \mathsf{P}_0$, and so Observation 4 guarantees a rainbow even cycle subgraph in $\mathsf{X} \cup \mathsf{C}$.

If X is a rainbow cycle, then $\mathsf{X} \cup \mathsf{P}_0$ is a rainbow theta graph.

If X is a bad piece which consists of rainbow paths $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$ that share terminals s and t, then $\mathsf{X} \cup \mathsf{P}_0$ is almost rainbow. In fact, we can always remove a subpath containing one of the repeated-color edges on one of $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$ to get a rainbow theta graph. To be more specific, we assume without loss that the repeated color happens on P_1 and P_3. If $x, y \in V(\mathsf{P}_i)$ for some fixed $i \in [3]$, then there exists a unique subpath of P_i with terminals x and y, and we denote by $\mathsf{P}_{x,y}$ this subpath.

- If s_0 and t_0 lie on a same P_i, then one of $V(\mathsf{P}_1) \setminus \{s, t\}$ and $V(\mathsf{P}_3) \setminus \{s, t\}$ is disjoint from $V(\mathsf{P}_0)$, say $V(\mathsf{P}_1) \setminus \{s, t\}$. This implies that $(\mathsf{P}_2 \cup \mathsf{P}_3) \cup \mathsf{P}_0 \subseteq \mathsf{X} \cup \mathsf{C}$ is a rainbow theta graph.
- Otherwise, at least one of s_0 and t_0 lies on $\mathsf{P}_1 \cup \mathsf{P}_3$, say $s_0 \in V(\mathsf{P}_1)$. We further assume that the repeated-color edge (denoted by $*$) appears on P_{s,s_0} rather than P_{t,s_0} in P_1. See Figure 5.
  - If $t_0 \in V(\mathsf{P}_2)$, then by removing P_3 from $\mathsf{X} \cup \mathsf{P}_0$ we are left with a rainbow theta graph.
  - If $t_0 \in V(\mathsf{P}_3)$, then by removing P_{s,s_0} from $\mathsf{X} \cup \mathsf{P}_0$ we are left with a rainbow theta graph.

The casework above verifies our claim, and so the proof is complete. ∎

Let $\mathfrak{F}$ be a Frankenstein graph with $\mathcal{P}(\mathfrak{F}) = \{\mathsf{G}_1, \ldots, \mathsf{G}_m\}$. To understand its structure better, we associate with it an auxiliary uncolored bipartite graph $G(\mathfrak{F}) \overset{\text{def}}{=} (V_1 \cup V_2, E)$, in which

- $V_1 \overset{\text{def}}{=} \{\mathsf{G}_1, \ldots, \mathsf{G}_m\}$, $V_2 \overset{\text{def}}{=} \{$the unique common vertex of some $\mathsf{G}_i, \mathsf{G}_j$ $(i \neq j)\}$, and
- $E \overset{\text{def}}{=} \{(\mathsf{G}, v) \in V_1 \times V_2 : v \in V(\mathsf{G})\}$.

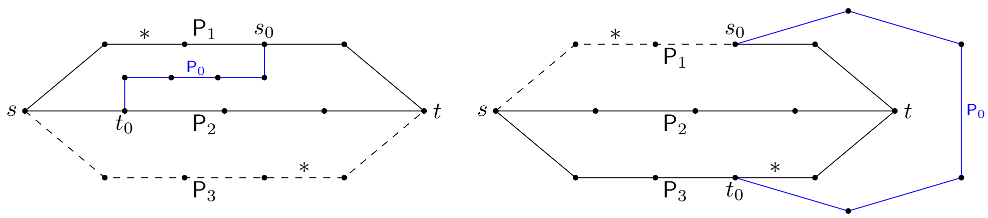


FIG. 5. *Path-removal operations where $*$ indicates the repeated color.*

LEMMA 9. *$G(\mathfrak{F})$ is acyclic for every Frankenstein graph $\mathfrak{F}$, and so is a forest.*

Proof. Assume to the contrary that $v_1\mathsf{G}_1v_2\mathsf{G}_2v_3\cdots v_k\mathsf{G}_kv_1$ presents a cycle in $G(\mathfrak{F})$, without loss of generality. Here the notation $u\mathsf{G}_i$ and G_jv refers to edges of $G(\mathfrak{F})$. From Observation 6 we deduce that there exists for each $i\in[k]$ a rainbow path P_i with terminals v_i,v_{i+1} in G_i $(v_{k+1}=v_1)$. Since different parts in $\mathfrak{F}$ are edge-disjoint and color-disjoint, $\mathsf{Q}\overset{\text{def}}{=}\mathsf{P}_1\cup\cdots\cup\mathsf{P}_k$ is a rainbow circuit, and so there exists a rainbow cycle $\mathsf{C}\subseteq\mathsf{Q}$. Since C cannot be a subgraph of any part of $\mathfrak{F}$, we can find $uv\in E(\mathsf{G}_x)\cap E(\mathsf{C})$ and $vw\in E(\mathsf{G}_y)\cap E(\mathsf{C})$, where $x\neq y$. It follows from (F1) that either G_x or G_y, say G_x, is not a rainbow tree. However, Lemma 8 then implies the existence of a rainbow even cycle subgraph in $\mathsf{C}\cup\mathsf{G}_j$, which contradicts (F2). ∎

Lemmas 8 and 9 will be applied not only in the proof of Theorem 7, but also later in many places.

Proof of Theorem 7. We induct on m. The theorem is vacuously true when $m=1$. Suppose $m\geq 2$ and let w be a leaf vertex of $G(\mathfrak{F})$. (If no leaf exists, then $E=\varnothing$ and any permutation σ satisfies the theorem.) It is easily seen from the definition that no leaf exists in V_2, and hence we assume without loss that $w=\mathsf{G}_m$. Since the partition $\{\mathsf{G}_1,\dots,\mathsf{G}_{m-1}\}$ defines a Frankenstein graph as well, the inductive hypothesis on $m-1$ implies the existence of a permutation σ on $[m-1]$ satisfying $|V(\mathfrak{F}_i)\cap V(\mathsf{G}_{\sigma(i+1)})|\leq 1$ for all $i\in[m-2]$. Then G_m being a leaf implies that $|V(\mathfrak{F}_{m-1})\cap V(\mathsf{G}_m)|\leq 1$. So, by defining $\sigma(m)\overset{\text{def}}{=}m$ to extend the definition of σ, the inductive proof is complete. ∎

The following corollaries of Theorem 7 will be useful in the proof of Theorem 2.

COROLLARY 10. *If $\mathfrak{F}$ is a Frankenstein graph, then $|\chi(\mathfrak{F})|\leq\frac{6}{5}(|V(\mathfrak{F})|-1)$.*

COROLLARY 11. *If $\mathfrak{F}$ is a Frankenstein graph with $\mathcal{P}(\mathfrak{F})=\{\mathsf{G}_1,\dots,\mathsf{G}_m\}$ and $\mathsf{C}\subseteq\mathfrak{F}$ is a cycle, then there exists $i\in[m]$ such that $\mathsf{C}\subseteq\mathsf{G}_i$.*

Proof. Write $V\overset{\text{def}}{=}V(\mathfrak{F})$. We prove Corollaries 10 and 11 by induction on m.

If $m=1$, then Corollary 11 is trivially true. To see that Corollary 10 holds, we need to check the cases when $\mathfrak{F}$ is a long rainbow odd cycle or a bad piece or a rainbow tree. Indeed, we have

$$\begin{cases}|\chi(\mathfrak{F})|=|V|<\frac{6}{5}(|V|-1) & \text{when } \mathfrak{F} \text{ is a long rainbow odd cycle (hence } |V|\geq 7),\\ |\chi(\mathfrak{F})|\leq\frac{6}{5}|(|V|-1) & \text{when } \mathfrak{F} \text{ is a bad piece (by Observation 5)},\\ |\chi(\mathfrak{F})|=|V|-1<\frac{6}{5}(|V|-1) & \text{when } \mathfrak{F} \text{ is a rainbow tree.}\end{cases}$$

Suppose $m\geq 2$ then. Assume without loss of generality that the identity $\sigma(i)\overset{\text{def}}{=}i$ satisfies Theorem 7. Then

$$|\chi(\mathfrak{F})| = |\chi(\mathfrak{F}_{m-1} \cup \mathsf{G}_m)| = |\chi(\mathfrak{F}_{m-1})| + |\chi(\mathsf{G}_m)| \le \frac{6}{5}(|V(\mathfrak{F}_{m-1})| + |V(\mathsf{G}_m)| - 2)$$
$$\le \frac{6}{5}(|V| - 1)$$

by applying the inductive hypothesis to $\mathfrak{F}_{m-1}$ and noticing that $|V(\mathfrak{F}_{m-1}) \cap V(\mathsf{G}_m)| \le 1$. Also, we have $\mathsf{C} \subseteq \mathfrak{F}_{m-1}$ or $\mathsf{C} \subseteq \mathsf{G}_m$ because the shared vertex of $\mathfrak{F}_{m-1}$ and G_m, if it exists, is a cut vertex of $\mathfrak{F}$. By applying the inductive hypothesis to $\mathfrak{F}_{m-1}$, we can find some $i \in [m]$ such that $\mathsf{C} \subseteq \mathsf{G}_i$. ∎

To prove Theorem 2, we need another technical result on Frankenstein graphs.

PROPOSITION 12. *Suppose $\mathfrak{F}$ is a Frankenstein graph and $\mathsf{P} \subseteq \mathfrak{F}$ is a path with terminals s and t. Then there exists a rainbow path $\mathsf{P}' \subseteq \mathfrak{F}$ with the same terminals s and t.*

Proof. The existence of P implies that s, t are in the same connected component of $\mathfrak{F}$. We thus assume without loss of generality that $\mathfrak{F}$ is connected. Then there exists a path in the uncolored graph $G(\mathfrak{F})$ of the form $\mathsf{G}_{i_1} v_1 \mathsf{G}_{i_2} v_2 \cdots v_{\ell-1} \mathsf{G}_{i_\ell}$ such that $s \in V(\mathsf{G}_{i_1})$, $t \in V(\mathsf{G}_{i,\ell})$ and $\ell \ge 1$. It then follows from Observation 6 that there exists a rainbow trail Q joining s and t. Obviously, any path $\mathsf{P}' \subseteq \mathsf{Q}$ with terminals s and t satisfies Proposition 12. ∎

For a Frankenstein graph $\mathfrak{F}$ given by the partition $\mathcal{P}(\mathfrak{F}) = \{\mathsf{C}_1, \dots, \mathsf{C}_c, \mathsf{B}_1, \dots, \mathsf{B}_b, \mathsf{T}_1, \dots, \mathsf{T}_t\}$, we associate with it counting parameters $c(\mathfrak{F}) \stackrel{\text{def}}{=} c$ and $b(\mathfrak{F}) \stackrel{\text{def}}{=} b$. Notice that $c(\mathfrak{F}), b(\mathfrak{F})$ depend not only on the graph $\mathfrak{F}$, but on the partition $\mathcal{P}(\mathfrak{F})$ as well. We still need another depth parameter.

For any tree T with $V(\mathsf{T}) \subset \mathbb{N}_+$, let its *root* be $r \stackrel{\text{def}}{=} \min V(\mathsf{T})$. For any vertex $v \in V(\mathsf{T})$, define its *relative depth* in T as $\text{depth}_{\mathsf{T}}(v) \stackrel{\text{def}}{=} \text{dist}_{\mathsf{T}}(r, v)$, which is the length of the unique path with terminals r and v. We henceforth define for any forest F with $V(\mathsf{F}) \subset \mathbb{N}_+$ its *total depth* as

$$\text{Depth}(\mathsf{F}) \stackrel{\text{def}}{=} \sum_{i=1}^{t} \sum_{v \in V(\mathsf{T}_i)} \text{depth}_{\mathsf{T}_i}(v),$$

where $\mathsf{T}_1, \dots, \mathsf{T}_t$ are the connected components of F. For any Frankenstein graph $\mathfrak{F}$ with $V(\mathfrak{F}) \subset \mathbb{N}_+$, we refer to its *total depth* as the total depth of its forest part, i.e., $\text{Depth}(\mathfrak{F}) \stackrel{\text{def}}{=} \text{Depth}(\mathsf{T}_1 \cup \cdots \cup \mathsf{T}_t)$.

Later in practice, we shall often construct a Frankenstein graph by a "partition"

$$\mathcal{P}(\mathfrak{F}) = \{\mathsf{C}_1, \dots, \mathsf{C}_c, \mathsf{B}_1, \dots, \mathsf{B}_b, \mathsf{F}\},$$

where C's are long rainbow odd cycles, B's are bad pieces, and $\mathsf{F} = \mathsf{T}_1 \cup \cdots \cup \mathsf{T}_t$ is the union of vertex-disjoint and color-disjoint rainbow trees, such that $\chi(\mathsf{G}_i) \cap \chi(\mathsf{G}_j) = \varnothing$ for any distinct $\mathsf{G}_i, \mathsf{G}_j \in \mathcal{P}(\mathfrak{F})$. Indeed, this $\mathcal{P}(\mathfrak{F})$ is formally not a partition since F and C_i or B_j may share more than one vertex. However, (F1) implies, up to a relabeling of the rainbow tree parts of $\mathfrak{F}$, that there is no difference between exposing the trees $\mathsf{T}_1, \dots, \mathsf{T}_t$ and exposing the forest F.

3. Proof of Theorem 2. We prove Theorem 2 indirectly. Suppose $\mathcal{D} = (\mathsf{D}_1, \dots, \mathsf{D}_m)$ is a family of $m \stackrel{\text{def}}{=} \lfloor \frac{6(n-1)}{5} \rfloor + 1 > \frac{6(n-1)}{5}$ even cycles on the ambient vertex set $[n]$ without any rainbow even cycle subgraph.

Let $\mathfrak{F}_*$ be a Frankenstein subgraph of the family $\mathcal{D}$ satisfying the following maximal conditions:

(M1) The number of long rainbow odd cycles $c(\mathfrak{F}_*)$ is maximized.
(M2) The number of bad pieces $b(\mathfrak{F}_*)$ is maximized under (M1).
(M3) The number of edges $|\mathfrak{F}_*|$ is maximized under (M2).
(M4) The total depth $\mathrm{Depth}(\mathfrak{F}_*)$ is minimized under (M3).

Suppose the partition of $\mathfrak{F}_*$ is

$$\mathcal{P}(\mathfrak{F}_*) = \{\mathsf{C}_1, \dots, \mathsf{C}_c, \mathsf{B}_1, \dots, \mathsf{B}_b, \mathsf{T}_1, \dots, \mathsf{T}_t\} \quad \text{with} \quad \mathsf{F} \stackrel{\text{def}}{=} \mathsf{T}_1 \cup \cdots \cup \mathsf{T}_t,$$

where C's are long rainbow odd cycles, B's are bad pieces, and T's are vertex-disjoint rainbow trees.

3.1. Outer edges and outer cycles. Let λ be the color of the even cycle D_λ. From Corollary 10 we deduce that $|\chi(\mathfrak{F}_*)| \le \frac{6}{5}(n-1) < |\mathcal{D}|$, and hence $\Lambda \stackrel{\text{def}}{=} [m] \setminus \chi(\mathfrak{F}_*) \neq \varnothing$. Indeed, every edge of the multigraph $\mathcal{D}_\Lambda \stackrel{\text{def}}{=} \bigcup_{\lambda \in \Lambda} \mathsf{D}_\lambda$ is absent in $\mathfrak{F}_*$.

We call f in $\mathcal{D}_\Lambda$ an *outer edge* if no coincident edge of f is in $\mathfrak{F}_*$. A rainbow $\{3,5\}$-cycle containing an outer edge f in $\mathfrak{F}_* + \mathsf{f}$ is called an *outer cycle* of f. Hereafter $\mathsf{G} + \mathsf{e}$ denotes the graph generated by adding e to G (i.e., $\mathsf{G} + \mathsf{e} \stackrel{\text{def}}{=} \mathsf{G} \cup \{\mathsf{e}\}$). Moreover, whenever we write $\mathsf{G} + \mathsf{e}$, we implicitly assume that e is not coincident to any edge of G. Similarly, $\mathsf{G} - \mathsf{e}$ (assuming $\mathsf{e} \in \mathsf{G}$) refers to the graph obtained by deleting e from G (i.e., $\mathsf{G} - \mathsf{e} \stackrel{\text{def}}{=} \mathsf{G} \setminus \{\mathsf{e}\}$). Recall that a (colored) graph is a set of colored edges.

The next propositions are devoted to the existence of outer edges and outer cycles.

PROPOSITION 13. *For any $\lambda \in \Lambda$, an outer edge exists in D_λ.*

Proof. Assume for the sake of contradiction that D_λ is covered by $\mathfrak{F}_*$. That is, each $\mathsf{e} \in \mathsf{D}_\lambda$ has one coincident edge $\mathsf{e}^* \in \mathfrak{F}_*$. Indeed, this e^* is unique because no coincident edges exist in a Frankenstein graph. Define $\mathsf{D}_\lambda^* \stackrel{\text{def}}{=} \{\mathsf{e}^* : \mathsf{e} \in \mathsf{D}_\lambda\} \subseteq \mathfrak{F}_*$. Since long rainbow odd cycles and rainbow trees contain no even cycle, it follows from Corollary 11 that D_λ^* has to be contained in some bad piece B_j, and so $|\mathsf{D}_\lambda^*| - |\chi(\mathsf{D}_\lambda^*)| \in \{0,1\}$. Since no rainbow even cycle exists in $\mathcal{D}$, we obtain $|\mathsf{D}_\lambda^*| - |\chi(\mathsf{D}_\lambda^*)| = 1$. So, there exists a unique pair of distinct edges $\mathsf{e}_1^*, \mathsf{e}_2^*$ in D_λ^* such that $\chi(\mathsf{e}_1^*) = \chi(\mathsf{e}_2^*)$. Thus, $\mathsf{D}_\lambda^* - \mathsf{e}_1^* + \mathsf{e}_1$ is a rainbow even cycle in $\mathcal{D}$, a contradiction. ∎

PROPOSITION 14. *For any outer edge f, an outer cycle of f exists.*

Proof. Let $V(\mathsf{f}) \stackrel{\text{def}}{=} \{u, v\}$. Observe that u, v are in a same connected component of $\mathfrak{F}_*$, for otherwise

$$\mathcal{P}(\mathfrak{F}_* + \mathsf{f}) \stackrel{\text{def}}{=} \{\mathsf{C}_1, \dots, \mathsf{C}_c, \mathsf{B}_1, \dots, \mathsf{B}_b, \mathsf{F} + \mathsf{f}\}$$

gives another Frankenstein subgraph of $\mathcal{D}$ with one more edge than $\mathfrak{F}_*$, which contradicts (M3). It follows from Proposition 12 that f completes a rainbow (hence odd) cycle C^f in $\mathsf{F} + \mathsf{f}$.

It then suffices to disprove that C^f is long. Assume to the contrary that C^f is long. Since $\mathsf{f} \notin \mathfrak{F}_*$, from Lemma 8 we deduce that $|V(\mathsf{C}^\mathsf{f}) \cap V(\mathsf{C}_i)| \le 1$ for every $i \in [c]$. So, $\mathcal{P}(\mathfrak{F}_+) \stackrel{\text{def}}{=} \{\mathsf{C}_1, \dots, \mathsf{C}_c, \mathsf{C}^\mathsf{f}\}$ presents another Frankenstein subgraph of $\mathcal{D}$ with $c(\mathfrak{F}_+) > c(\mathfrak{F}_*)$, which contradicts (M1). ∎

For any tree T with $v \in V(\mathsf{T}) \subseteq [n]$, we define

$$\mathrm{Child}_\mathsf{T}(v) \stackrel{\text{def}}{=} \big\{ w \in V(\mathsf{T}) : vw \in E(\mathsf{T}),\ \mathrm{depth}_\mathsf{T}(w) = \mathrm{depth}_\mathsf{T}(v) + 1 \big\}.$$

The following properties characterize behaviors of outer 3-cycles.

PROPOSITION 15. *Suppose* f *is an outer edge with* $V(\mathsf{f})=\{u,v\}$*, and* C *is an outer* 3*-cycle of* f *with* $V(\mathsf{C})=\{u,v,w\}$*. Then there exists* $k\in[t]$ *such that* $u,v,w\in V(\mathsf{T}_k)$ *and* $u,v\in \mathrm{Child}_{\mathsf{T}_k}(w)$*.*

Proof. We show $u,v,w\in V(\mathsf{T}_k)$ for some k first. Since $uw,vw\in E(\mathfrak{F}_*)$, we may assume $uw\in E(\mathsf{X}_1)$ and $vw\in E(\mathsf{X}_2)$, where $\mathsf{X}_1,\mathsf{X}_2\in\mathcal{P}(\mathfrak{F}_*)$. In fact, $\mathsf{X}_\bullet\in\{\mathsf{T}_1,\dots,\mathsf{T}_t\}$ ($\bullet=1,2$), for otherwise Lemma 8 implies the existence of a rainbow even cycle in $\mathsf{C}\cup\mathsf{X}_\bullet$. It follows from (F1) that $\mathsf{X}_1=\mathsf{X}_2=\mathsf{T}_k$.

We prove $u,v\in\mathrm{Child}_{\mathsf{T}_k}(w)$ then. Suppose $\mathsf{e}_1\overset{\text{def}}{=}(uw,\alpha)$ and $\mathsf{e}_2\overset{\text{def}}{=}(vw,\beta)$ are edges in T_k. The existence of $\mathsf{e}_1,\mathsf{e}_2$ tells us that $|\mathrm{depth}_{\mathsf{T}_k}(u)-\mathrm{depth}_{\mathsf{T}_k}(v)|$ is either 0 or 2. It suffices to establish that $\mathrm{depth}_{\mathsf{T}_k}(u)=\mathrm{depth}_{\mathsf{T}_k}(v)$. If not, then assume without loss of generality that $\mathrm{depth}_{\mathsf{T}_k}(u)=\mathrm{depth}_{\mathsf{T}_k}(v)+2$. Since $\mathrm{depth}_{\mathsf{T}'_k}(u)<\mathrm{depth}_{\mathsf{T}_k}(u)$ and $\mathrm{depth}_{\mathsf{T}'_k}(x)\le\mathrm{depth}_{\mathsf{T}_k}(x)$ for all $x\in V(\mathsf{T}_k)=V(\mathsf{T}'_k)$, we deduce that $\mathsf{T}'_k\overset{\text{def}}{=}\mathsf{T}_k+\mathsf{f}-\mathsf{e}_1$ is another tree with $\mathrm{Depth}(\mathsf{T}'_k)<\mathrm{Depth}(\mathsf{T}_k)$. Then the partition

$$\mathcal{P}(\mathfrak{F}')\overset{\text{def}}{=}\mathcal{P}(\mathfrak{F}_*+\mathsf{f}-\mathsf{e}_2)=\{\mathsf{C}_1,\dots,\mathsf{C}_c,\mathsf{B}_1,\dots,\mathsf{B}_b,\mathsf{T}_1,\dots,\mathsf{T}'_k,\dots,\mathsf{T}_t\}$$

gives a Frankenstein subgraph of $\mathcal{D}$. However, this contradicts (M4) since $\mathrm{Depth}(\mathfrak{F}')<\mathrm{Depth}(\mathfrak{F}_*)$. Therefore, $\mathrm{depth}_{\mathsf{T}_k}(u)=\mathrm{depth}_{\mathsf{T}_k}(v)$, and so $u,v\in\mathrm{Child}_{\mathsf{T}_k}(w)$. ∎

PROPOSITION 16. *Suppose no outer* 5*-cycle exists. If* $\mathsf{f}=(uv,\alpha)$ *is an outer edge with outer cycle* C *on vertices* $u,v,w\in\mathsf{T}_k$ *(by Proposition* 15*), then* D_α*, the even cycle of color* α *from* $\mathcal{D}$ *containing* f*, satisfies* $V(\mathsf{D}_\alpha)\subseteq\{w\}\cup\mathrm{Child}_{\mathsf{T}_k}(w)$*. (See Figure* 6*.)*

Proof. We first show that $V(\mathsf{D}_\alpha)\subseteq V(\mathsf{T}_k)$. Define $\tau\colon\mathsf{D}_\alpha\to\mathcal{P}(\mathfrak{F}_*)$ as follows: For any edge $\mathsf{e}\in\mathsf{D}_\alpha$,

- if e is an outer edge, then $V(\mathsf{e})\subseteq V(\mathsf{T}_\ell)$ for some ℓ (by Proposition 15), and we set $\tau(\mathsf{e})\overset{\text{def}}{=}\mathsf{T}_\ell$;
- if e is coincident to $\mathsf{e}'\in\mathfrak{F}_*$, then we set $\tau(\mathsf{e})\overset{\text{def}}{=}\mathsf{X}$, where X is the part of $\mathfrak{F}_*$ that contains e'.

By applying τ on D_α, we locate a closed walk $Q\subseteq G(\mathfrak{F}_*)$ as follows:

a. Put the edges of D_α on a circle $\mathcal{O}$ in order. Replace e by $\tau(\mathsf{e})$ for each $\mathsf{e}\in\mathsf{D}_\alpha$.
b. If two consecutive objects on $\mathcal{O}$ are the same, then remove one of them. Repeat.
c. If $\mathsf{G}_i,\mathsf{G}_j\in\mathcal{P}(\mathfrak{F}_*)$ are adjacent on $\mathcal{O}$, then plug in $v_{ij}\in V(\mathsf{G}_i)\cap V(\mathsf{G}_j)$ between them.

The resulting arrangement on $\mathcal{O}$ forms a closed walk $Q\subseteq G(\mathfrak{F}_*)$, where each pair of consecutive edges $v_{ij}\mathsf{G}_j$, G_jv_{jk} in Q corresponds to a path with terminals v_{ij},v_{jk} on D_α. Indeed, Q is a circuit because D_α passes through each v_{ij} exactly once. For instance, if D_α consists of $\mathsf{e}_1,\dots,\mathsf{e}_8$ in order such that

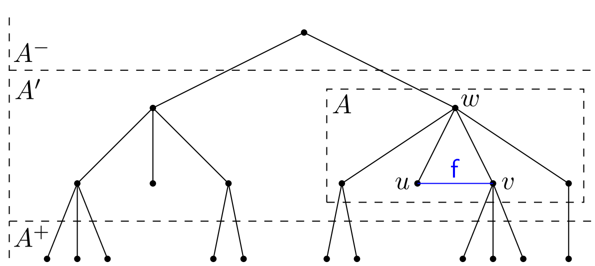
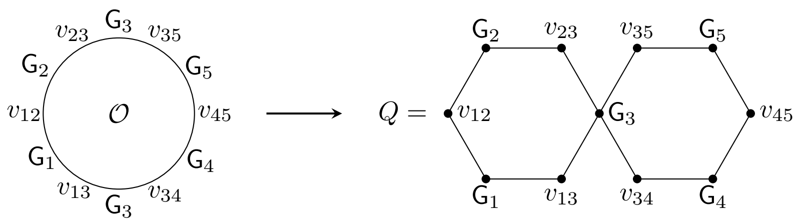


FIG. 6. $V(\mathsf{f})\subseteq\mathrm{Child}_{\mathsf{T}_k}(w)$ *implies* $V(\mathsf{D}_\alpha)\subseteq A$.

$$\tau(\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3,\mathsf{e}_4,\mathsf{e}_5,\mathsf{e}_6,\mathsf{e}_7,\mathsf{e}_8) = (\mathsf{G}_3,\mathsf{G}_3,\mathsf{G}_2,\mathsf{G}_2,\mathsf{G}_1,\mathsf{G}_3,\mathsf{G}_4,\mathsf{G}_5),$$

then the steps 1 through 3 generate



However, Lemma 9 asserts that $\mathfrak{F}_*$ is acyclic. So, Q is a single vertex, and hence $V(\mathsf{D}_\alpha) \subseteq V(\mathsf{T}_k)$.

Use abbreviations $V \overset{\text{def}}{=} V(\mathsf{T}_k)$ and $d(x) \overset{\text{def}}{=} \text{depth}_{\mathsf{T}_k}(x)$. Partition V into $A \overset{\text{def}}{=} \{w\} \cup \text{Child}_{\mathsf{T}_k}(w)$, $A^+ \overset{\text{def}}{=} \{x \in V : d(x) > d(v)\}$, $A^- \overset{\text{def}}{=} \{x \in V : d(x) < d(w)\}$, and $A' \overset{\text{def}}{=} V \setminus (A \cup A^+ \cup A^-)$. Let T_k be the uncolored copy of T_k, which is the uncolored graph on vertex set $V = V(\mathsf{T}_k)$ and edge set $E(\mathsf{T}_k)$. For all $z \in V$ and all pairs of distinct vertices $x, y \in \text{Child}_{T_k}(z)$, we add the edges xy simultaneously into $E(T_k)$ to generate a new graph $\overline{T}_k$. The vertex set of $\overline{T}_k$ is still V. Due to the absence of outer 5-cycles, from Proposition 15 we deduce that D_α, the uncolored copy of D_α, is a subgraph of $\overline{T}_k$.

Notice that any subpath of T_k with one terminal in A and the other in A' must go through A^-. It then suffices to show that $V(D_\alpha) = V(\mathsf{D}_\alpha)$ and A^- are disjoint. This breaks down to exclude the situation $d(z_+) \geq d(z_-) + 2$ for some $z_+, z_- \in V(D_\alpha)$. If such z_+, z_- exist, then D_α consists of two subpaths P_1, P_2 with terminals z_+ and z_-. Let z_i be the vertex on P_i with $d(z_i) = d(z_+) - 1$ that is nearest to z_+. The crucial observation is that z_i is the parent of z_+, which is the unique vertex in V such that $z_+ \in \text{Child}_{T_k}(z_i)$. Indeed, this follows from the fact that z_i is a cut vertex of $\overline{T}_k$ which separates z_+ from all vertices of smaller depths. However, the observation implies that $z_1 = z_2$, which is absurd. We conclude that $V(\mathsf{D}_\alpha) = V(D_\alpha) \subseteq A$, and hence the proof is complete. ∎

3.2. Finishing the proof.

LEMMA 17. *There exists a Frankenstein subgraph $\mathfrak{F}_0$ of $\mathcal{D}$ whose partition is given by*

$$\mathcal{P}(\mathfrak{F}_0) = \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b, \mathsf{F}_0\} \quad \textit{with} \quad |\mathsf{F}_0| = |\mathsf{F}|,$$

and an edge f_0 in $\mathcal{D}$ such that $\chi(\mathsf{f}_0) \notin \chi(\mathfrak{F}_0)$ and f_0 completes a rainbow 5-cycle in $\mathfrak{F}_0 + \mathsf{f}_0$.

Proof. If there is an outer 5-cycle in $\mathfrak{F}_*$, say $\mathsf{C}^{\bar{\mathsf{f}}}$ of an outer edge $\bar{\mathsf{f}}$, then $(\mathfrak{F}_0, \mathsf{f}_0) \overset{\text{def}}{=} (\mathfrak{F}_*, \bar{\mathsf{f}})$ with $\mathsf{F}_0 \overset{\text{def}}{=} \mathsf{F}$ satisfies Lemma 17. We assume no outer 5-cycle exists then. It follows from Propositions 13 and 14 that an outer edge f and its outer cycle C^{f} exist. Suppose $\mathsf{f} \overset{\text{def}}{=} (uv, \alpha)$ and D_α is the monochromatic even cycle from $\mathcal{D}$ that contains f. Assume $V(\mathsf{C}^{\mathsf{f}}) \overset{\text{def}}{=} \{u, v, w\}$. It follows from Proposition 15 that u, v, w all lie in a single rainbow tree $\mathsf{T}_k \in \mathcal{P}(\mathfrak{F}_*)$ and $u, v \in \text{Child}_{\mathsf{T}_k}(w)$. From Proposition 16 we deduce that $V(\mathsf{D}_\alpha) \subseteq \{w\} \cup \text{Child}_{\mathsf{T}_k}(w)$. Since D_α consists of at least 4 edges, at least 1 of the two adjacent edges of f on D_α is not incident to the vertex w. Assume without loss

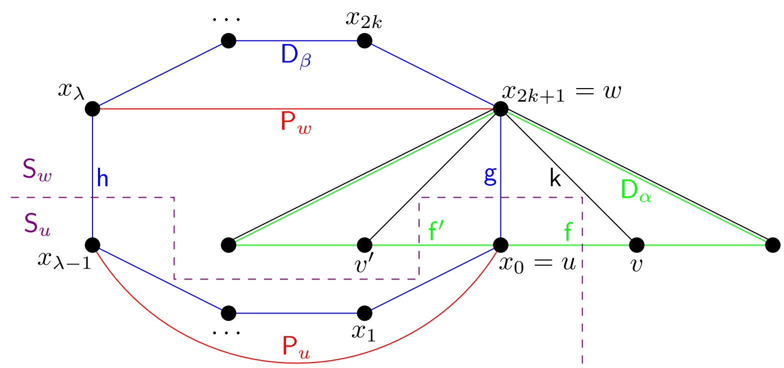


FIG. 7. *An illustration of the proof of Lemma* 17.

of generality that $\mathsf{f}' \overset{\text{def}}{=} (uv', \alpha)$ is such an edge, and hence $v' \in \text{Child}_{\mathsf{T}_k}(w)$. Observe that v and v' are symmetric despite our definition.

Let $\mathsf{g} \overset{\text{def}}{=} (uw, \beta) \in \mathfrak{F}_*$ be the edge with $V(\mathsf{g}) = \{u, w\}$. Suppose

$$\mathsf{D}_\beta = \mathsf{g} + (x_0x_1, \beta) + (x_1x_2, \beta) + \cdots + (x_{2k}x_{2k+1}, \beta) \qquad (x_0 \overset{\text{def}}{=} u,\, x_{2k+1} \overset{\text{def}}{=} w,\, k \in \mathbb{N}_+)$$

is the monochromatic even cycle from $\mathcal{D}$ containing g. From Lemma 9 we deduce that there are two connected components S_u and S_w in the graph $\mathfrak{F}_* - \mathsf{g}$ such that $u \in V(\mathsf{S}_u)$ and $w \in V(\mathsf{S}_w)$. Define λ as the smallest index such that $x_\lambda \notin V(\mathsf{S}_u)$ and write $\mathsf{h} \overset{\text{def}}{=} (x_{\lambda-1}x_\lambda, \beta)$. Then $\mathsf{h} \notin \mathfrak{F}_*$.

We claim that $x_\lambda \in \mathsf{S}_w$. If not, then h cannot complete any cycle in $\widehat{\mathsf{F}} \overset{\text{def}}{=} \mathsf{F}+\mathsf{f}-\mathsf{g}+\mathsf{h}$, and so $\widehat{\mathsf{F}}$ is a rainbow forest. Observe that the trees in $\widehat{\mathsf{F}}$ containing f or h (they are possibly the same) share at most one vertex with any of $\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b$. This implies that the partition

$$\mathcal{P}(\mathfrak{F}_* + \mathsf{f} - \mathsf{g} + \mathsf{h}) \overset{\text{def}}{=} \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b, \widehat{\mathsf{F}}\}$$

presents another Frankenstein subgraph of $\mathcal{D}$ on $|\mathfrak{F}_*| + 1$ edges, which contradicts (M3).

By Proposition 12, we can find a rainbow path $\mathsf{P}_u \subseteq \mathsf{S}_u$ with terminals $u, x_{\lambda-1}$ and a rainbow path $\mathsf{P}_w \subseteq \mathsf{S}_w$ with terminals w, x_λ. Here we allow P_u to be empty if $x_0 = x_{\lambda-1}$, and allow P_w to be empty if $x_{2k+1} = x_\lambda$. Note that $\mathsf{P}_u = \mathsf{P}_w = \varnothing$ cannot happen, since $|\mathsf{D}_\beta| \geq 4$. Assume further that the length of P_w is minimized, and so $v \notin V(\mathsf{P}_w)$ or $v' \notin V(\mathsf{P}_w)$, say $v \notin V(\mathsf{P}_w)$. Thus, $\widetilde{\mathsf{C}} \overset{\text{def}}{=} \mathsf{f} + \mathsf{P}_u + \mathsf{h} + \mathsf{P}_w + \mathsf{k}$ is a rainbow odd cycle with $|\widetilde{\mathsf{C}}| \geq 5$. Here k denotes the edge of T_k with $V(\mathsf{k}) = \{v, w\}$.

Since $V(\mathsf{T}_k + \mathsf{f} - \mathsf{g}) = V(\mathsf{T}_k)$, we can define another Frankenstein subgraph $\mathfrak{F}_0 \overset{\text{def}}{=} \mathfrak{F}_* + \mathsf{f} - \mathsf{g}$ by

$$\mathcal{P}(\mathfrak{F}_0) \overset{\text{def}}{=} \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b, \mathsf{T}_1, \ldots, \mathsf{T}_k + \mathsf{f} - \mathsf{g}, \ldots, \mathsf{T}_t\}.$$

We claim that $\mathsf{f}_0 \overset{\text{def}}{=} \mathsf{h}$ is as desired. It suffices to show that $\widetilde{\mathsf{C}}$ is a rainbow 5-cycle. Since $\mathsf{h} \in \widetilde{\mathsf{C}}$ and $\beta \notin \chi(\mathfrak{F}_0)$, Lemma 8 tells us that $|V(\widetilde{\mathsf{C}}) \cap V(\mathsf{C}_i)| \leq 1\, (\forall i \in [c])$. Then $\mathcal{P}(\mathfrak{F}') \overset{\text{def}}{=} \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \widetilde{\mathsf{C}}\}$ gives another Frankenstein subgraph of $\mathcal{D}$ with $c(\mathfrak{F}') > c$ if $\widetilde{\mathsf{C}}$ is long, which contradicts (M1). Thus, $|\widetilde{\mathsf{C}}| \geq 5$ implies that $\widetilde{\mathsf{C}}$ is a rainbow 5-cycle in $\mathfrak{F}_0 + \mathsf{h}$. The proof of Lemma 17 is complete. ∎

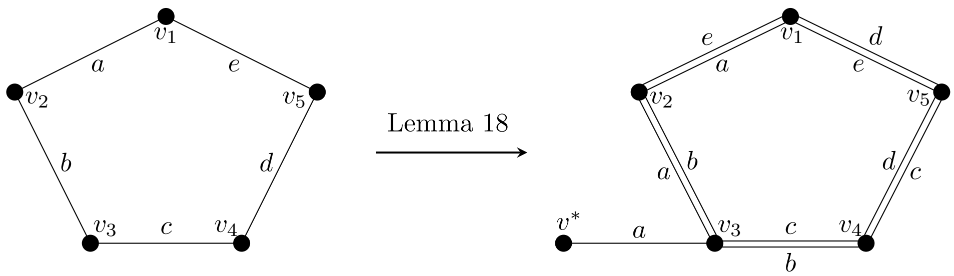


FIG. 8. *The "growth" of a rainbow 5-cycle.*

LEMMA 18. *Suppose the rainbow 5-cycle found in Lemma 17 is $\widetilde{\mathsf{C}} \overset{def}{=} \{(v_iv_{i+1}, \alpha_i) : i \in [5]\}$, with the convention $v_{\ell+5} = v_\ell$. Write $\mathsf{e}_i \overset{def}{=} (v_iv_{i+1}, \alpha_i)$. Then there exists a shifting parameter $j \in \{0,1,2,3,4\}$, a set of five edges $\mathsf{e}'_i \overset{def}{=} (v_iv_{i+1}, \alpha_{i+j})$ from $\mathcal{D}$, a vertex $v^* \in [n] \setminus \{v_1,\dots,v_5\}$, and an index $k \in [5]$, such that at least one of the edges (v^*v_k, α_{k+j-1}) and (v^*v_k, α_{k+j}) appears in $\mathcal{D}$.*

Informally speaking, Lemma 18 is dedicated to "grow" one more edge from the 5-cycle guaranteed by Lemma 17. That is, after a possible cyclic shift of the colors on $\widetilde{\mathsf{C}}$, we would like to find out another edge on one of the monochromatic even cycles in $\mathcal{D}$ "leaving" $\widetilde{\mathsf{C}}$ (i.e., incident to $v^* \notin \{v_1,\dots,v_5\}$). Such a configuration will then help us to locate another bad piece in $\mathcal{D}$, which contradicts (M2). For ease of notation, we write $(a,b,c,d,e) \overset{\text{def}}{=} (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ in the coming example and in the proof of Lemma 18. Figure 8 illustrates one possible output of Lemma 18 in which $(j,k) = (4,3)$.

Proof of Lemma 18. Assume without loss of generality that $\mathsf{e}_i \in \mathsf{D}_i \in \mathcal{D}$. Suppose e_i^+ and e_i^- are the edges in D_i satisfying $V(\mathsf{e}_i) \cap V(\mathsf{e}_i^+) = \{v_{i+1}\}$ and $V(\mathsf{e}_i) \cap V(\mathsf{e}_i^-) = \{v_i\}$, respectively.

Write $V \overset{\text{def}}{=} \{v_1,\dots,v_5\}$ for brevity. If there exists $v \in V(\mathsf{e}_i^\bullet) \setminus V$ for some $i \in [5]$ and $\bullet \in \{+,-\}$, say $i = 1$ and $\bullet = +$, then by choosing $v^* \overset{\text{def}}{=} v$ and $(j,k) = (0,1)$ the proof is done.

We thus assume that $V(\mathsf{e}_i^\bullet) \subseteq V$ for any $i \in [5]$ and $\bullet \in \{+,-\}$, and claim that this is impossible. To see this, we prove by contradiction. The following observation is quite useful.

FACT. $V(\mathsf{e}_i^-) = \{v_{i-1}, v_i\}$ *or* $\{v_i, v_{i+3}\}$*, and* $V(\mathsf{e}_i^+) = \{v_{i+1}, v_{i+2}\}$ *or* $\{v_{i+1}, v_{i+3}\}$.

Proof of fact. Let $V(\mathsf{e}_i^-) \overset{\text{def}}{=} \{v_{i-1}, v'\}$. Then $v' \in \{v_{i-1}, v_{i+2}, v_{i+3}\}$ is forced. However, $v' \neq v_{i+2}$, for otherwise $\mathsf{e}_i^-, \mathsf{e}_{i+2}, \mathsf{e}_{i+3}, \mathsf{e}_{i+4}$ form a rainbow 4-cycle in $\mathcal{D}$. The $V(\mathsf{e}_i^+)$ case is similar. ◻

If e_i^+ and e_{i+1} are coincident for all $i \in [5]$, then we cyclically shift the vertices via increasing j by 1 (note that the shift cannot happen indefinitely since the cycles D_i are even). This does not change the situation, and so we may assume without loss of generality that $V(\mathsf{e}_1^+) \neq \{v_2, v_3\}$. It follows from the above fact that $V(\mathsf{e}_1^+) = \{v_2, v_4\}$, which forces $V(\mathsf{e}_1^-) = \{v_1, v_5\}$, as shown in Figure 9.A.

We next look at $\mathsf{e}_2^\pm$. If $V(\mathsf{e}_2^-) = \{v_1, v_2\}$, then $\mathsf{e}_2^-, \mathsf{e}_1^+, \mathsf{e}_4, \mathsf{e}_5$ form a rainbow even cycle, a contradiction. So, $V(\mathsf{e}_2^-) = \{v_2, v_5\}$, and hence $V(\mathsf{e}_2^+) = \{v_3, v_4\}$ by the fact, as shown in Figure 9.B.

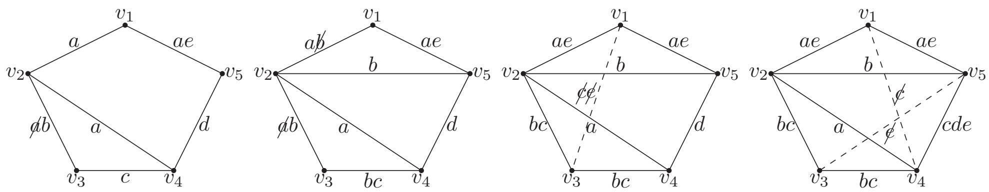


FIG. 9. *An illustration of the proof of Lemma* 18.

We turn to $\mathsf{e}_3^{\pm}$ and $\mathsf{e}_5^{\pm}$ then. At this moment, we have a configuration that is symmetric in (a,e) and (b,c) (as seen in Figure 9.B). If $V(\mathsf{e}_3^-)=\{v_1,v_3\}$, then $\mathsf{e}_3^-,\mathsf{e}_2^+,\mathsf{e}_4,\mathsf{e}_5$ form a rainbow even cycle, a contradiction. So, the above fact implies $V(\mathsf{e}_3^-)=\{v_2,v_3\}$. By symmetry, $V(\mathsf{e}_5^+)=\{v_1,v_2\}$. We thus arrive at Figure 9.C. If $V(\mathsf{e}_3^+)=\{v_1,v_4\}$, then $\mathsf{e}_1,\mathsf{e}_2^-,\mathsf{e}_4,\mathsf{e}_3^+$ form a rainbow 4-cycle, which is impossible. It then follows from the fact and symmetry that $V(\mathsf{e}_3^+)=V(\mathsf{e}_5^-)=\{v_4,v_5\}$, as illustrated in Figure 9.D.

Finally, we focus on $\mathsf{e}_4^{\pm}$. Indeed, we have $V(\mathsf{e}_4^-)=\{v_2,v_4\}$ or $\{v_3,v_4\}$ by the fact. Figure 9.D shows that the former case generates a rainbow 4-cycle on $\mathsf{e}_1,\mathsf{e}_4^-,\mathsf{e}_3^+,\mathsf{e}_5$, while the latter generates a rainbow 4-cycle on $\mathsf{e}_2^-,\mathsf{e}_3^-,\mathsf{e}_4^-,\mathsf{e}_5^-$. We thus obtain the desired contradiction. ∎

Assume $\mathfrak{F}_0$ and f_0 satisfy Lemma 17. Let F_0 be the forest part of $\mathfrak{F}_0$. That is,

$$\mathcal{P}(\mathfrak{F}_0)=\{\mathsf{C}_1,\dots,\mathsf{C}_c,\mathsf{B}_1,\dots,\mathsf{B}_b,\mathsf{F}_0\}\quad\text{with}\quad |\mathsf{F}_0|=|\mathsf{F}|.$$

From Lemma 18 we can find a subgraph of $\mathcal{D}$ on six vertices $v_1,\dots,v_5$ and v_*. After some possible renaming of vertices, edges, and colors, we assume this subgraph consists of the ingredients below:

- $\widetilde{\mathsf{C}}\overset{\text{def}}{=}\{\mathsf{e}_i=(v_iv_{i+1},\alpha_i): i\in[5]\}$ is the rainbow 5-cycle in $\mathfrak{F}_0+\mathsf{f}_0$ located by Lemma 17, and
- $\mathsf{p}\overset{\text{def}}{=}(v^*v_1,\alpha_1)$ is a pendant edge of color α_1 on vertices v^* and v_1 located by Lemma 18.

We first claim that $\widetilde{\mathsf{C}}-\mathsf{f}_0\subseteq\mathsf{F}_0$. Since $\mathsf{f}_0\notin\mathfrak{F}_0$ and $\chi(\mathsf{f}_0)\notin\chi(\mathfrak{F}_0)$, it follows from Lemma 8 that $\widetilde{\mathsf{C}}$ is edge-disjoint from $\mathsf{C}_1,\dots,\mathsf{C}_c$ and $\mathsf{B}_1,\dots,\mathsf{B}_b$. In particular, $\widetilde{\mathsf{C}}-\mathsf{f}_0\subseteq\mathsf{F}_0$.

We then claim that $\mathsf{p}\notin\mathfrak{F}_0$. If $\mathsf{f}_0=\mathsf{e}_1$, then $\chi(\mathsf{p})=\chi(\mathsf{f}_0)\notin\chi(\mathfrak{F}_0)$ follows from the choice of f_0 in Lemma 17, and so $\mathsf{p}\notin\mathfrak{F}_0$. If $\mathsf{f}_0\in\{\mathsf{e}_2,\mathsf{e}_3,\mathsf{e}_4,\mathsf{e}_5\}$, then $\mathsf{e}_1\in\mathsf{F}_0$. This implies $\mathsf{p}\notin\mathsf{F}_0$ since F_0 is rainbow, and $\mathsf{p}\notin\mathsf{C}_i$, $\mathsf{p}\notin\mathsf{B}_j$ since $\mathsf{C}_i,\mathsf{B}_j$ are color-disjoint from F_0. We conclude that $\mathsf{p}\notin\mathfrak{F}_0$.

Let $\widetilde{\mathsf{C}}-\mathsf{f}_0$ be a subgraph of $\mathsf{T}_k\in\mathcal{P}(\mathfrak{F}_0)$. Set $\mathfrak{F}_0'\overset{\text{def}}{=}\mathfrak{F}_0+\mathsf{f}_0-\mathsf{e}_5$ and $\mathsf{F}_0'\overset{\text{def}}{=}\mathsf{F}_0+\mathsf{f}_0-\mathsf{e}_5$. Note that F_0' differs from F_0 only at T_k', the rainbow tree from $\mathcal{P}(\mathfrak{F}_0')$ containing $\widetilde{\mathsf{C}}-\mathsf{e}_5$. Since $V(\mathsf{T}_k')=V(\mathsf{T}_k)$,

$$\mathcal{P}(\mathfrak{F}_0')\overset{\text{def}}{=}\{\mathsf{C}_1,\dots,\mathsf{C}_c,\mathsf{B}_1,\dots,\mathsf{B}_b,\mathsf{F}_0'\}$$

shows that $\mathfrak{F}_0'$ is a Frankenstein subgraph of $\mathcal{D}$ with $|\mathsf{F}_0'|=|\mathsf{F}_0|=|\mathsf{F}|$. We remark that $\mathsf{p}\notin\mathfrak{F}_0'$.

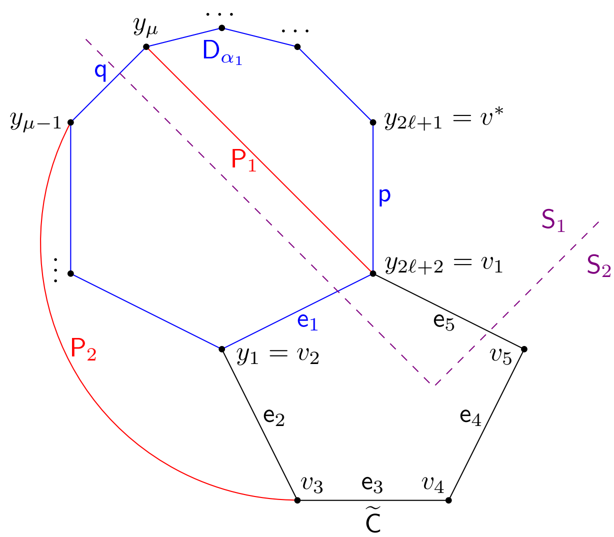


FIG. 10. *An illustration of the proof of Theorem* 2.

Let D_{α_1} be the monochromatic even cycle from $\mathcal{D}$ that contains e_1 and p. Then

$$\mathsf{D}_{\alpha_1} \stackrel{\text{def}}{=} \mathsf{e}_1 + (y_1y_2, \alpha_1) + \cdots + (y_{2\ell+1}y_{2\ell+2}, \alpha_1)$$
$$\left(y_1 \stackrel{\text{def}}{=} v_2,\, y_{2\ell+1} \stackrel{\text{def}}{=} v^*,\, y_{2\ell+2} \stackrel{\text{def}}{=} v_1,\, \ell \in \mathbb{N}_+\right).$$

By Lemma 9, there are two connected components S_1 and S_2 of $\mathfrak{F}_0' - \mathsf{e}_1$ such that $v_1 \in V(\mathsf{S}_1)$ and $v_2 \in V(\mathsf{S}_2)$. Define μ as the smallest index with $y_\mu \notin V(\mathsf{S}_2)$. For reasons similar to "$x_{\lambda-1} \in \mathsf{S}_u$ and $x_\lambda \in \mathsf{S}_w$" in the proof of Lemma 17, we have that $y_{\mu-1} \in \mathsf{S}_2$ and $y_\mu \in \mathsf{S}_1$. Indeed, if $y_\mu \notin \mathsf{S}_1$, then

$$\mathcal{P}(\mathfrak{F}_0' + \mathsf{e}_5 - \mathsf{e}_1 + \mathsf{q}) \stackrel{\text{def}}{=} \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b, \mathsf{F}_0' + \mathsf{e}_5 - \mathsf{e}_1 + \mathsf{q}\},$$

where $\mathsf{q} \stackrel{\text{def}}{=} (y_{\mu-1}y_\mu, \alpha_1)$, is another Frankenstein subgraph on $|\mathfrak{F}_*| + 1$ edges, which contradicts (M3).

According to Proposition 12, we can find a rainbow path $\mathsf{P}_1 \subseteq \mathsf{S}_1$ with terminals y_μ and v_1. We can also find a rainbow path $\mathsf{P}_2 \subseteq \mathsf{S}_2$ whose terminals are $y_{\mu-1}$ and some $v_t \in \{v_2, v_3, v_4, v_5\}$. Assume further that P_2 is of minimum length. Note that $\mathsf{P}_1 = \varnothing$ if $v_1 = y_\mu$, and $\mathsf{P}_2 = \varnothing$ if $v_t = y_{\mu-1}$.

CLAIM. $\mathsf{P}_1, \mathsf{P}_2, \widetilde{\mathsf{C}}$ *are pairwise color-disjoint.*

Proof of claim. Recall that $\widetilde{\mathsf{C}} - \mathsf{e}_5 \subseteq \mathsf{T}_k'$. If $\mathsf{P}_1, \mathsf{P}_2$ intersect some same part $\mathsf{G} \in \mathcal{P}(\mathfrak{F}_0')$, then it follows from the definitions of $\mathsf{S}_1, \mathsf{S}_2$ that $\mathsf{G} = \mathsf{T}_k'$. Since T_k' is rainbow, and different parts in $\mathcal{P}(\mathfrak{F}_0')$ are color-disjoint, we conclude that $\chi(\mathsf{P}_1), \chi(\mathsf{P}_2), \chi(\widetilde{\mathsf{C}})$ are pairwise disjoint. ∎

Decompose $\widetilde{\mathsf{C}}$ into two rainbow paths $\widetilde{\mathsf{P}}_1, \widetilde{\mathsf{P}}_2$ with terminals v_1 and v_t such that $\mathsf{e}_1 \in \widetilde{\mathsf{P}}_1$. For instance, in Figure 10 we have $v_t = v_3$, $\widetilde{\mathsf{P}}_1 = \mathsf{e}_1 + \mathsf{e}_2$ and $\widetilde{\mathsf{P}}_2 = \mathsf{e}_3 + \mathsf{e}_4 + \mathsf{e}_5$. Set $\widetilde{\mathsf{P}} \stackrel{\text{def}}{=} \mathsf{P}_1 \cup \mathsf{P}_2 \cup \{\mathsf{q}\}$, and define $\widetilde{\mathsf{B}} \stackrel{\text{def}}{=} \widetilde{\mathsf{P}} \cup \widetilde{\mathsf{P}}_1 \cup \widetilde{\mathsf{P}}_2$. We are going to verify that $\widetilde{\mathsf{B}}$ is a bad piece in three steps.

First, we show that $\widetilde{\mathsf{B}} = \widetilde{\mathsf{P}} \cup \widetilde{\mathsf{P}}_1 \cup \widetilde{\mathsf{P}}_2$ is a theta graph with common terminals v_1 and v_t. Let $\widetilde{C}, P_1, P_2, q$ be the uncolored copies of $\widetilde{\mathsf{C}}, \mathsf{P}_1, \mathsf{P}_2, \mathsf{q}$, respectively. It suffices to show that $\widetilde{C}, P_1, P_2, \{q\}$ are pairwise disjoint. The definitions of $\mathsf{S}_1, \mathsf{S}_2$ indicate $q \notin P_1$, $q \notin P_2$ and $P_1 \cap P_2 = \varnothing$, $P_1 \cap \widetilde{C} = \varnothing$. The minimum-length assumption on P_2 implies $P_2 \cap \widetilde{C} = \varnothing$. To see that $q \notin \widetilde{C}$, we argue indirectly. If $q \in \widetilde{C}$, then $V(\mathsf{q}) \subseteq V(\widetilde{\mathsf{C}})$ and $y_\mu = v_1$. This implies $\mathsf{q} = \mathsf{p}$, hence $v^* \in \{v_1, \ldots, v_5\}$, a contradiction.

Second, we prove that $\widetilde{\mathsf{P}}$, $\widetilde{\mathsf{P}}_1$, $\widetilde{\mathsf{P}}_2$ are all rainbow, and that $\widetilde{\mathsf{B}}$ is almost rainbow. Indeed, $\widetilde{\mathsf{P}}_1, \widetilde{\mathsf{P}}_2$ are rainbow because $\widetilde{\mathsf{C}}$ is rainbow. Since $\chi(\mathsf{q}) = \chi(\mathsf{e}_1) = \alpha_1$ and $\mathsf{e}_1 \in \widetilde{\mathsf{C}}$, the claim then implies that $\widetilde{\mathsf{P}}$ is rainbow and $\widetilde{\mathsf{B}}$ is almost rainbow.

Third, we check that $|\widetilde{\mathsf{B}}| \geq 7$. Since $\mathsf{q} \in \widetilde{\mathsf{B}}$, $\widetilde{\mathsf{C}} \subseteq \widetilde{\mathsf{B}}$, and $\mathsf{q} \notin \widetilde{\mathsf{C}}$, we obtain $|\widetilde{\mathsf{B}}| \geq 6$. If $|\widetilde{\mathsf{B}}| = 6$, then $V(\mathsf{q}) \subseteq \{v_1, \ldots, v_5\}$ and hence $y_{\mu-1} = v_t$, $y_\mu = v_1$, which contradicts $v^* \notin \{v_1, \ldots, v_5\}$. So, $|\widetilde{\mathsf{B}}| \geq 7$.

If $|V(\widetilde{\mathsf{B}}) \cap V(\mathsf{X})| \leq 1$ for all $\mathsf{X} \in \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b\}$, then the partition

$$\mathcal{P}(\widetilde{\mathfrak{F}}) \stackrel{\text{def}}{=} \{\mathsf{C}_1, \ldots, \mathsf{C}_c, \mathsf{B}_1, \ldots, \mathsf{B}_b, \widetilde{\mathsf{B}}\}$$

exposes a Frankenstein subgraph of $\mathcal{D}$ with $c(\widetilde{\mathfrak{F}}) = c$ and $b(\widetilde{\mathfrak{F}}) > b$, which contradicts (M2). So, there exists $\mathsf{X}_0 = \mathsf{C}_i$ or B_j such that $|V(\widetilde{\mathsf{B}}) \cap V(\mathsf{X}_0)| \geq 2$.

Consider the rainbow cycle $\widehat{\mathsf{C}} \stackrel{\text{def}}{=} \widetilde{\mathsf{P}} \cup \widetilde{\mathsf{P}}_2$. We claim that $\chi(\mathsf{X}_0) \cap \chi(\widehat{\mathsf{C}} \setminus \mathsf{X}_0) = \varnothing$. To see this, we begin by noticing that $\widehat{\mathsf{C}}$ is a disjoint union $\mathsf{P}_1 \cup \mathsf{P}_2 \cup \widetilde{\mathsf{P}}_2 \cup \{\mathsf{q}\}$. The claim is then verified by

- $\chi(\mathsf{X}_0) \cap \chi(\mathsf{P}_1 \cup \mathsf{P}_2 \setminus \mathsf{X}_0) = \varnothing$ follows from $\mathsf{X}_0 \in \mathcal{P}(\mathfrak{F}'_0)$ and $\mathsf{P}_1 \cup \mathsf{P}_2 \subseteq \mathfrak{F}'_0$;
- $\chi(\mathsf{X}_0) \cap \chi(\widetilde{\mathsf{P}}_2 \setminus \mathsf{X}_0) = \varnothing$ since $\mathsf{X}_0 \in \mathcal{P}(\mathfrak{F}_0)$, $\chi(\mathsf{f}_0) \notin \chi(\mathfrak{F}_0)$, and $\widetilde{\mathsf{P}}_2 \subseteq \mathfrak{F}_0 + \mathsf{f}_0$;
- $\chi(\mathsf{q}) \notin \chi(\mathsf{X}_0)$ because $\chi(\mathsf{q}) = \chi(\mathsf{e}_1) \in \chi(\mathsf{T}'_k)$ and $\mathsf{T}'_k, \mathsf{X}_0$ are color-disjoint.

It follows from $y_{\mu-1} \in \mathsf{S}_2$ and $y_\mu \in \mathsf{S}_1$ that $\mathsf{q} \notin \mathfrak{F}'_0$, and hence $\mathsf{q} \notin \mathsf{X}_0$. Since $\mathsf{q} \in \widetilde{P} \subseteq \widehat{\mathsf{C}}$, we locate an edge $\mathsf{q} \in \widehat{\mathsf{C}} \setminus \mathsf{X}_0$. Lemma 8 then tells us that $|V(\widehat{\mathsf{C}}) \cap V(\mathsf{X}_0)| \leq 1$, for otherwise a rainbow even cycle appears in $\mathcal{D}$. Similarly, it follows from $\chi(\mathsf{X}_0) \cap \chi(\widetilde{\mathsf{C}} \setminus \mathsf{X}_0) = \varnothing$, $\mathsf{e}_1 \in \widetilde{\mathsf{C}} \setminus \mathsf{X}_0$, and Lemma 8 that $|V(\widetilde{\mathsf{C}}) \cap V(\mathsf{X}_0)| \leq 1$. We thus obtain $|V(\widetilde{\mathsf{B}}) \cap V(\mathsf{X}_0)| = 2$ and $\widetilde{\mathsf{B}} \cap \mathsf{X}_0 = \varnothing$ by noticing $\widetilde{\mathsf{B}} = \widehat{\mathsf{C}} \cup \widetilde{\mathsf{C}}$.

Suppose $V(\widetilde{\mathsf{B}}) \cap V(\mathsf{X}_0) \stackrel{\text{def}}{=} \{u, u_1\}$ with $u \in V(\widetilde{\mathsf{P}}) \setminus V(\widetilde{\mathsf{P}}_2)$ and $u_1 \in V(\widetilde{\mathsf{P}}_1) \setminus V(\widetilde{\mathsf{P}}_2)$. Denote by P_{u,v_t} the subpath of $\widetilde{\mathsf{P}}$ with terminals u, v_t, and by P_{u_1,v_t} the subpath of $\widetilde{\mathsf{P}}_1$ with terminals u_1, v_t. Write $\widehat{\mathsf{P}} \stackrel{\text{def}}{=} \mathsf{P}_{u,v_t} \cup \mathsf{P}_{u_1,v_t}$. Then $\widehat{\mathsf{P}}$ is a rainbow path because $\widehat{\mathsf{P}} \subseteq \widetilde{\mathsf{B}}$ and $\mathsf{e}_1 \notin \widehat{\mathsf{P}}$. Since $\widetilde{\mathsf{B}} \cap \mathsf{X}_0 = \varnothing$, from Lemma 8 we deduce that $\widehat{\mathsf{P}} \cup \mathsf{X}_0$ contains a rainbow even cycle, a contradiction.

The proof of Theorem 2 is complete.

4. Concluding remarks. Write $\langle n \rangle \stackrel{\text{def}}{=} \{3, 4, \ldots, n\}$. For any positive integer n and any $A \subseteq \langle n \rangle$, let $f(n, A)$ be the minimum positive integer N such that a rainbow A-cycle is guaranteed in every family of N many A-cycles. It then follows from Theorems 1 to 3 that

$$f(n, A) = \begin{cases} n & \text{when } A = \langle n \rangle, \\ 2\left\lceil \frac{n}{2} \right\rceil - 1 & \text{when } A = \langle n \rangle \cap (2\mathbb{Z} + 1), \\ \left\lfloor \frac{6(n-1)}{5} \right\rfloor + 1 & \text{when } A = \langle n \rangle \cap 2\mathbb{Z}. \end{cases}$$

We were unable to determine $f(n, A)$ when $A = \langle n \rangle \cap (a\mathbb{Z} + b)$ in general. Another nice problem is to estimate $f(n, \{k\})$. It was proved independently by Győri [9] and Goorevitch and Holzman [8] that $f(n, \{3\}) \approx \frac{n^2}{8}$. In particular, the value of $f(n, \{n\})$ concerning Hamiltonian cycles seems mysterious.

Acknowledgments. The first author is grateful to Boris Bukh, Ting-Wei Chao, and Zilin Jiang for fruitful discussions. Part of this work was done after his graduation from Carnegie Mellon University in May 2023. As a postdoctoral researcher, the first author would like to thank

- Alfréd Rényi Institute of Mathematics (Budapest, Hungary) for hosting from Sep. to Dec. 2023 (supported by ERC grant 882971, "GeoScape," and the Erdős Center), and
- Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS, Daejeon, South Korea) for hosting since Jan. 2024 (supported by IBS-R029-C4).

The second author would like to thank Peking University for a preadmission in his tenth grade, and to thank Beijing National Day School (high school) for allowing him to skip all regular classes in the academic year 2021–2022. These privileges resulted in plenty of free time to study all kinds of exciting new mathematics, especially to work on this problem on rainbow even cycles.

Both authors thank two anonymous referees for their valuable feedback on earlier versions of this paper.

REFERENCES

[1] R. Aharoni and E. Berger, *Rainbow matchings in r-partite r-graphs*, Electron. J. Combin., 16 (2009), 119, http://www.combinatorics.org/Volume_16/Abstracts/v16i1r119.html.

[2] R. Aharoni, E. Berger, M. Chudnovsky, and S. Zerbib, *Rainbow paths and large rainbow matchings*, Electron. J. Combin., 29 (2022), 1.10, https://doi.org/10.37236/10173.

[3] R. Aharoni, J. Briggs, R. Holzman, and Z. Jiang, *Rainbow odd cycles*, SIAM J. Discrete Math., 35 (2021), pp. 2293–2303, https://doi.org/10.1137/20M1380557.

[4] R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, and R. Šámal, *A rainbow version of Mantel's theorem*, Adv. Combin., (2020), 2, https://doi.org/10.19086/aic.12043.

[5] R. Aharoni, M. DeVos, and R. Holzman, *Rainbow triangles and the Caccetta-Häggkvist conjecture*, J. Graph Theory, 92 (2019), pp. 347–360, https://doi.org/10.1002/jgt.22457.

[6] I. Bárány, *A generalization of Carathéodory's theorem*, Discrete Math., 40 (1982), pp. 141–152, https://doi.org/10.1016/0012-365X(82)90115-7.

[7] D. M. Correia, A. Pokrovskiy, and B. Sudakov, *Short proofs of rainbow matchings results*, Int. Math. Res. Not. IMRN, 2023 (2023), pp. 12441–12476, https://doi.org/10.1093/imrn/rnac180.

[8] I. Goorevitch and R. Holzman, *Rainbow Triangles in Families of Triangles*, preprint, https://arxiv.org/abs/2209.15493, 2022.

[9] E. Győri, *Triangle-free hypergraphs*, Combin. Probab. Comput., 15 (2006), pp. 185–191, https://doi.org/10.1017/S0963548305007108.

[10] F. Joos and J. Kim, *On a rainbow version of Dirac's theorem*, Bull. Lond. Math. Soc., 52 (2020), pp. 498–504, https://doi.org/10.1112/blms.12343.