STABILITY AND BIFURCATION FOR LOGISTIC KELLER–SEGEL MODELS ON COMPACT GRAPHS

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To Fritz Gesztesy on the occasion of his 70^{th} *birthday*

Abstract. This paper concerns asymptotic stability, instability, and bifurcation of constant steady state solutions of the parabolic-parabolic and parabolic-elliptic chemotaxis models on metric graphs. We determine a threshold value $\chi^* > 0$ of the chemotaxis sensitivity parameter that separates the regimes of local asymptotic stability and instability, and, in addition, determine the parameter intervals that facilitate global asymptotic convergence of solutions with positive initial data to constant steady states. Moreover, we provide a sequence of bifurcation points for the chemotaxis sensitivity parameter that yields non-constant steady state solutions. In particular, we show that the first bifurcation point coincides with threshold value χ^* for a generic compact metric graph. Finally, we supply numerical computation of bifurcation points for several graphs.

Contents

1. Introduction

This paper is centered around the Keller–Segel model given by the following initial value problem for a system of evolution equations on a metric graph $\Gamma = (\mathcal{V}, \mathcal{E})$

$$\begin{cases} u_t = \partial_x\big(\partial_x u - \chi u \partial_x v\big) + u(a - bu), & x \in \mathcal{E}, \\ \tau v_t = \partial^2_{xx} v - v + u, & x \in \mathcal{E}, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \mathcal{E}, \end{cases} \tag{1.1}$$

where $\chi, a, b > 0, \tau \geq 0$, $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges of the graph. This pair of PDEs describes population dynamics in presence of attracting substances and stems

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from the following reaction-advection-diffusion system

$$\begin{cases}\partial_t u + \partial_x J_{u,v} - \varphi(u,v) = 0,\\ \tau\partial_t v + \partial_x J_v - \psi(u,v) = 0,\end{cases} \tag{1.2}$$

where

- $J_{u,v} = -\partial_x u + \chi\partial_x v$ is the density flux which consist of the taxis-flux term $\chi\partial_x v$ that governs the population drifts in response to attractant v and the standard flux term $-\partial_x u$ given by Fick's law,
- $J_v = -\partial_x v$ is the standard flux term describing diffusion of the chemo-attractant by Fick's law; importantly, in the case of rapid diffusion, that is, $0 < \tau \ll 1$ the second equation (1.2) can be approximated by $\partial^2_{xx} v - \psi(u,v) = 0$,
- $\varphi(u,v) = u(a - bu)$, $\psi(u,v) = -v + u$ describe the rates[1] at which the biospecies, respectively, the chemo-attractant grow or decay.

In the context of the directed movement of microorganisms in response to a chemical attractant this model is often referred to as the chemotaxis model, see [27] for illuminating discussion of chemotaxis phenomena in biomedical and social sciences. The vast mathematical literature on this model includes [3, 10, 12, 17, 18, 19, 20, 22, 23, 24, 25, 28, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45].

The two quantities central to chemotaxis models are the population density $u = u(t,x)$ and the concentration of the chemical substance $v = v(t,x)$. The classical Fick's law of diffusion combined with population drifts along the chemical gradient yield the first equation in (1.1), where $\chi > 0$ is the chemotaxis sensitivity parameter and $\chi u\partial_x v$ is the taxis-flux term. The second equation in (1.1) is the usual reaction-diffusion equation, with $\tau > 0$ corresponding to moderate diffusion rate of the chemical substances and $\tau = 0$ corresponding to rapid diffusion thereof.

In this paper, we consider (1.1) in two regimes:

(1) $\tau = 0$, in which case (1.1) is referred to as the parabolic-elliptic system,
(2) $\tau > 0$, in which case (1.1) is referred to as the parabolic-parabolic system.

Treating both regimes by different methods, we focus on local asymptotic stability of the constant steady state $(u_0, v_0) = (a/b, a/b)$ and existence of non-trivial steady states bifurcating from (u_0, v_0) in response to small variation of the chemotaxis sensitivity parameter χ. We investigate (1.1) posed on arbitrary connected compact metric graphs $\Gamma = (\mathcal{V}, \mathcal{E})$ and consider solutions $u = u(t,x)$, $v = v(t,x)$ satisfying natural Neumann–Kirchhoff vertex conditions describing continuity and preservation of flux at all vertices $\vartheta \in \mathcal{V}$, that is,

$$\begin{cases} u_e(\vartheta) = u_{e'}(\vartheta),\ v_e(\vartheta) = v_{e'}(\vartheta),\ e \sim \vartheta, e' \sim \vartheta,\ \vartheta \in \mathcal{V} \text{ (continuity at vertices)},\\ \sum\limits_{\vartheta\sim e} \partial_\nu u_e(\vartheta) = 0,\ \sum\limits_{\vartheta\sim e} \partial_\nu v_e(\vartheta) = 0,\ \vartheta \in \mathcal{V} \ \text{ (conservation of current)},\end{cases} \tag{1.3}$$

where $\partial_\nu u_e(\vartheta)$ denotes the inward normal derivative of u along the edge e at the vertex ϑ.

In [29], we have established well-posedness for general chemotaxis systems on arbitrary compact metric graphs including (1.1), (1.3) as a special case, see Theorem 2.1. In this paper, we investigate the stability, instability, and bifurcation of the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3).

[1] The logistic term can alternatively be written in the form $\varphi(u,v) = ru(c-u)$. The methods of this paper cover more general $\psi(u,v) = -\alpha v + \beta u$, $\alpha > 0$, $\beta > 0$, but we chose $\alpha = \beta = 1$ for notational convenience.

Our first result provides a threshold value $\chi^* > 0$ of the chemotaxis sensitivity parameter that separates the regimes of local asymptotic stability and instability of the constant solution (u_0, v_0).

Theorem 1.1 (Local asymptotic stability and instability). *Let Γ be a connected compact metric graph and let $\chi(\lambda)$, $\chi^* \in (0,\infty)$ be defined by*

$$\chi(\lambda) := \frac{b(\lambda - a)(1-\lambda)}{a\lambda},\ \lambda < 0,$$

and

$$\chi^* := \min\left\{\chi(\lambda) : \lambda \in \operatorname{Spec}(\Delta) \setminus \{0\}\right\},$$

where[2] $\operatorname{Spec}(\Delta)$ *is the spectrum of the Neumann–Kirchhoff Laplacian acting in* $L^2(\Gamma)$. *Then the following assertions hold for* $\tau \geq 0$.

(1) If $0 < \chi < \chi^*$ then the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) *is locally asymptotically stable.*

(2) If $\chi > \chi^*$ then the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) *is unstable.*

To prove Theorem 1.1 we compute the spectrum, via finding zeros of the perturbation determinant, of the linearization of (1.1) or, equivalently, of

$$\begin{bmatrix} 1 & 0 \\ 0 & \tau \end{bmatrix} \begin{bmatrix} \partial_t u \\ \partial_t v \end{bmatrix} = \mathcal{H}(u, v, \chi) := \begin{bmatrix} \partial_x\big(\partial_x u - \chi u \partial_x v\big) + u(a - bu) \\ \partial^2_{xx} v - v + u \end{bmatrix},$$

about the constant steady state, see Lemma 2.1.

Our next result stems from a simple observation that $\mathcal{H}(a/b, a/b, \chi) = 0$ for all $\chi \geq 0$. That is, for both parabolic-parabolic and parabolic-elliptic systems $(a/b, a/b, \chi)$ is the line of constant solutions in the space $(u, v, \chi) \in \widehat{W}^{2,2}(\Gamma) \times \widehat{W}^{2,2}(\Gamma) \times (0,\infty)$ (see (A.1) in Appendix A for the definition of $\widehat{W}^{2,2}(\Gamma)$ and other functional spaces on graphs). We show that the eigenvalues of the Neumann–Kirchhoff Laplacian give rise to a sequence $\{\chi_n\}_{n\geq 1}$ of bifurcation points that is bounded from below and that accumulates only at $+\infty$. Importantly, the first bifurcation point is precisely the threshold value χ^* where stability of the constant steady state ceases to take place.

Theorem 1.2. *Let Γ be a connected compact metric graph. Let $\lambda \in \operatorname{Spec}(\Delta)$ be a simple eigenvalue of the Neumann–Kirchhoff Laplacian on Γ, let φ be a corresponding eigenfunction and define*

$$\mathcal{D} := \left\{ u \in \widehat{W}^{2,2}(\Gamma) : \sum_{\vartheta \sim e} \partial_\nu u_e(\vartheta) = 0,\ u_e(\vartheta) = u_{e'}(\vartheta), \vartheta \sim e, e' \right\}, \tag{1.4}$$

$$\chi_\lambda := \chi(\lambda). \tag{1.5}$$

Assume, in addition, that $\chi_\lambda \neq \chi_\mu$ for $\mu \in \operatorname{Spec}(\Delta) \setminus \{\lambda\}$ then χ_λ is a bifurcation point of (1.1), (1.3). *That is, there exist $\varepsilon > 0$ and $\chi \in C^2((-\varepsilon, \varepsilon), \mathbb{R})$, $\Phi \in C^2((-\varepsilon, \varepsilon), \mathcal{D} \times \mathcal{D})$ such that $\mathcal{H}(\Phi(s), \chi(s)) = 0$ for $s \in (-\varepsilon, \varepsilon)$ and*

$$\chi(0) = \chi_\lambda, \Phi(s) = \begin{bmatrix} a/b \\ a/b \end{bmatrix} + s \begin{bmatrix} \xi \\ \eta \end{bmatrix} \varphi + o(s) \ \text{in } \widehat{W}^{2,2}(\Gamma) \times \widehat{W}^{2,2}(\Gamma) \ \text{as } s \to 0,$$

[2]see Figure 1 where an example of χ^* is provided

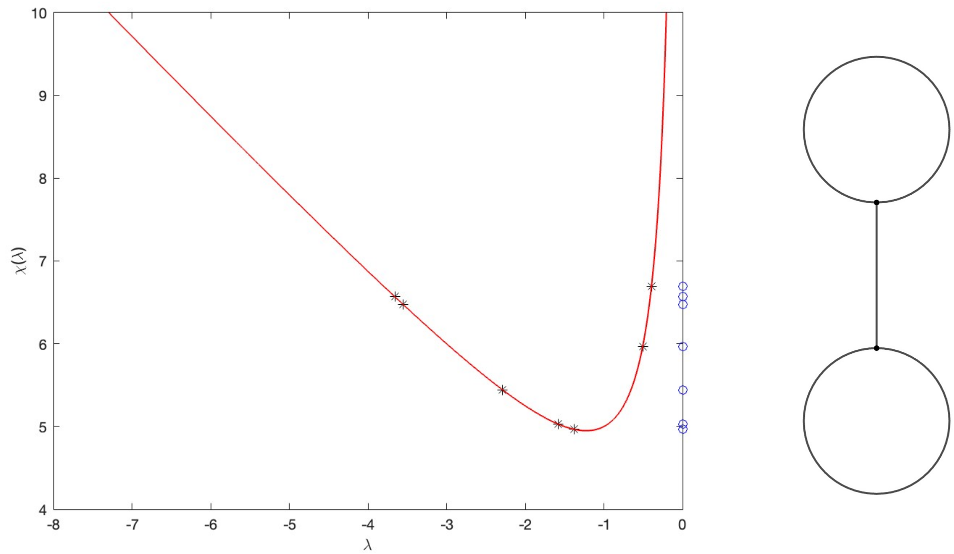

FIGURE 1. The red curve is the graph of $\chi = \chi(\lambda)$ with $a = b = 1.5$; $*$ indicates the values of χ at eigenvalues of the Kirchhoff Laplacian on a *dumbbell graph* with edge lengths $10, 5, 1$; $\circ$ indicate bifurcation points. The first bifurcation point $\chi^* \approx 4.96489$ corresponds to the $4-$th eigenvalue (we note that the first eigenvalue $\lambda = 0$ is not displayed)

where

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} \in \ker M, \quad \xi^2+\eta^2 = 1,\ \xi > 0,\ M := \begin{bmatrix}\lambda - a - \frac{\chi a}{b} & \frac{\chi a}{b}\\ 1 & \lambda - 1\end{bmatrix}.$$

Moreover, there exists an open set $U \subset \mathcal{D}\times\mathcal{D}\times\mathbb{R}$ *containing* $\left(\frac{a}{b}, \frac{a}{b}, \chi_\lambda\right)$ *such that*

$$\left\{(u, v, \chi) \in U : \mathcal{H}(u, v, \chi) = 0, (u, v) \neq \left(\frac{a}{b}, \frac{a}{b}\right)\right\} = \{(\Phi(s), \chi(s)) : |s| < \varepsilon, s \neq 0\}.$$

We stress that both assumptions of Theorem 1.2 hold automatically for a generic connected graph Γ that is not a circle. That is, for a given combinatorial graph $(\mathcal{V}, \mathcal{E})$ that is not a circle there exists a dense G_δ set $\mathcal{S} \subset \mathbb{R}_+^{|\mathcal{E}|}$ such that the corresponding metric graph Γ with $\{\ell_e, e \in \mathcal{E}\} \in \mathcal{S}$ satisfies the assumptions of Theorem 1.2. Indeed, the eigenvalues of the Neumann–Kirchhoff Laplacian are simple for generic graph Γ, see [13, 7]. To prove that the second condition is also generic assume that $\chi_\lambda = \chi_\mu$ for $\lambda, \mu \in \operatorname{Spec}(\Delta)$. Then stretching all edges of the graph by a factor of $t > 0$ leads to rescaling of the eigenvalues $t^2\lambda, t^2\mu$, however, the function $\nu \mapsto \chi_\nu$ is not scale-invariant. In particular, $\chi_{t^{-2}\lambda} \neq \chi_{t^{-2}\mu}$ for t near 1. Figure 1 illustrates numerically the bifurcation points for the dumbell graph, see also Section 4 for more numerical examples.

We note that in the absence of chemotaxis, that is, when $\chi = 0$, the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) is globally stable. In particular, no non-constant steady states of (1.1), (1.3) exists for $\chi = 0$. In this context, Theorem 1.2 shows that chemotaxis induces non-constant steady states via the bifurcation of the constant steady state. In the special case of Γ being a single interval the bifurcation of the constant steady state and existence of spiky solutions have been investigated, for example, in [9, 24, 31, 35, 36].

The next two theorems concern asymptotic convergence of solutions with non-trivial non-negative initial data to the constant steady state $(a/b, a/b)$ in the following regimes:

- for sufficiently small χ in the parabolic-parabolic model, see Theorem 1.3,
- for $\chi \in (0, b/2)$ in the parabolic-elliptic model, see Theorem 1.4.

Theorem 1.3 (Global stability for parabolic-parabolic model)**.** *Let Γ be a connected compact metric graph. Let $u_0 \in \widehat{C}(\overline{\Gamma})$, $v_0 \in \widehat{C}^1(\overline{\Gamma})$ be non-negative initial data $u_0 \not\equiv 0$ and let $u = u(x,t;u_0,v_0)$, $v = v(x,t;u_0,v_0)$ be a global unique positive solution of* (1.1) *with $\tau > 0$ satisfying the Neumann–Kirchhoff vertex conditions* (1.3) *and the initial condition $(u(x,0;u_0,v_0), v(x,0;u_0,v_0)) = (u_0(x), v_0(x))$* [3]*. Then there exists $C = C(\Gamma) > 0$ such that for sufficiently small $\chi > 0$ one has*

$$\lim_{t\to\infty}\left(\left\|u(t,\cdot;u_0,v_0) - \frac{a}{b}\right\|_{L^\infty(\Gamma)} + \left\|v(t,\cdot;u_0,v_0) - \frac{a}{b}\right\|_{L^\infty(\Gamma)}\right) = 0. \tag{1.6}$$

Theorem 1.4 (Global stability for parabolic-elliptic model)**.** *Let Γ be a connected compact metric graph. Let $u_0 \in \widehat{C}(\overline{\Gamma})$ be non-negative initial data $u_0 \not\equiv 0$ and let $u = u(x,t;u_0)$, $v = v(x,t;u_0)$ be a global unique positive solution of* (1.1) *with $\tau = 0$ satisfying the Neumann–Kirchhoff vertex conditions* (1.3) *and the initial condition $u(x,0;u_0) = u_0(x)$* [4]*. Then for $\chi \in (0, b/2)$ one has*

$$\lim_{t\to\infty}\left(\left\|u(t,\cdot;u_0) - \frac{a}{b}\right\|_{L^\infty(\Gamma)} + \left\|v(t,\cdot;u_0) - \frac{a}{b}\right\|_{L^\infty(\Gamma)}\right) = 0.$$

The global stability of the positive constant solution for (1.1) with $\tau > 0$ on regular convex domains Ω with Neumann boundary condition is studied in [26, 42, 44]. These works heavily rely on the following inequality

$$\frac{\partial |\nabla v|^2}{\partial \nu} \le 0, \quad x \in \partial\Omega,$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative. Such an inequality is not available in the setting of metric graphs. For parabolic-parabolic models, i.e. $\tau > 0$, we offer a new alternative approach which does apply to regular domains, see Section 3. We also note that the global stability of the positive constant solution for (1.1) with $\tau = 0$ on regular domains Ω with Neumann boundary condition have been studied in [19, 28, 31]. We adopt the approach established in [28] to prove global stability of the positive constant solution for (1.1) with $\tau = 0$.

The rest of the paper is organized as follows. In Section 2, we study the local stability, instability, and bifurcation of the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) and prove Theorems 1.1 and 1.2. In Section 3, we investigate the global stability of the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) and prove Theorems 1.3 and 1.4. We supply numerical computation of bifurcation points for several graphs in Section 4. Finally, in Appendix A, we record several facts about fractional power spaces generated by the Neumann–Kirchhoff Laplacian on compact metric graphs.

[3]cf. Theorem 2.1 (2)

[4]cf. Theorem 2.1 (1)

2. Local stability, instability, and bifurcation of constant steady states

In this section, we study the local stability, instability, and bifurcation of the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) via spectral analysis of the linearizations of (1.1), (1.3) about the steady state solution $(\frac{a}{b}, \frac{a}{b})$. We first recall a global well-posedness result from [29] in Section 2.1. We then prove Theorems 1.1 and 1.2 in Sections 2.2 and 2.3, respectively.

2.1. **Well-posedness of Keller–Segel model on graphs.** First, let us record a result concerning well-posedness of (1.1) subject to vertex conditions (1.3) in $L^p(\Gamma)$ (see (A.1) in Appendix A for the definition of $L^p(\Gamma)$ and other functional spaces on metric graphs) and regularity of solutions, in particular, their membership to the fractional power spaces $\mathcal{X}_p^\beta$ generated by the Neumann–Kirchhoff Laplacian Δ and to the space of Hölder continuous functions $\widehat{C}^\nu(\overline{\Gamma})$ (see Appendix A for definition of $\mathcal{X}_p^\beta$, $\widehat{C}^\nu(\overline{\Gamma})$).

Theorem 2.1. [29]. *Let Γ be a connected compact metric graph. Then there exists $p_0 \geq 1$ such that the following assertions hold for $p \geq p_0$.*

(1) Assume that $\tau = 0$. Then for arbitrary $u_0 \in L^p(\Gamma)$, (1.1) *has a unique global classical solution $u = u(t, x; u_0)$, $v = v(t, x; u_0)$, $t \geq 0$ satisfying Neumann–Kirchhoff vertex conditions* (1.3)*. For such a solution one has*

$$u \in C((0,\infty), \widehat{C}^\nu(\overline{\Gamma})) \cap C([0,\infty), L^p(\Gamma)) \cap C^{0,\beta}((0,\infty), \mathcal{X}_r^\beta),$$

for arbitrary $r \geq 1$, $\beta \in (0, 1/8)$, $\nu < \beta$. Moreover, if u_0 is non-negative and not equal to zero in $L^p(\Gamma)$, then $u = u(t, x; u_0) > 0$, $v = v(t, x; u_0) > 0$ for all $t > 0$, $x \in \Gamma$.

(2) Assume that $\tau > 0$ and let $(u_0, v_0) \in L^p(\Gamma) \times \widehat{W}^{1,p}(\Gamma)$. Then (1.1) *has a unique global classical solution $u = u(x, t; u_0, v_0)$, $v = v(x, t; u_0, v_0)$, $t \in [0, \infty)$ satisfying Neumann–Kirchhoff vertex conditions* (1.3)*. For such a solution one has*

$$u \in C([0,\infty), L^p(\Gamma)) \cap C((0,\infty), L^r(\Gamma)) \cap C^{0,\beta}((0,\infty), \mathcal{X}_r^\beta) \cap C^{0,\beta}((0,\infty), \widehat{C}^\nu(\overline{\Gamma})),$$
$$v \in C([0,\infty), L^p(\Gamma)) \cap C^{0,\beta}((0,\infty), \widehat{W}^{2,r}(\Gamma)) \cap C^{0,\beta}((0,\infty), \mathcal{X}_r^\beta) \cap C^{0,\beta}((0,\infty), \widehat{C}^\nu(\overline{\Gamma})),$$

for arbitrary $r \geq 1$, $\beta \in (0, 1/8)$, $\nu < \beta$. Moreover, if u_0 and v_0 are non-negative with $u_0 \neq 0$ a.e. on Γ, then $u = u(t, x; u_0, v_0) > 0$, $v = v(t, x; u_0, v_0) > 0$ for all $t > 0$, $x \in \Gamma$.

We note that well-posedness of Keller–Segel model on subset of $\mathbb{R}^n$, $n \geq 1$ has been investigated by numerous authors, see, for example, [2, 10, 28, 31, 37, 40] and references therein.

2.2. **Local asymptotic stability.** In this subsection we first discuss spectral properties of linearizations of parabolic-parabolic and parabolic-elliptic equations about the steady state solution $(a/b, a/b)$ and then prove Theorem 1.1. In particular, we show that the non-selfadjoint linearized operators have compact resolvents, hence, their spectra is discrete, and compute (in general, complex) eigenvalues in terms of the eigenvalues of Neumann-Kirchhoff Laplacian, see Lemmas 2.1 and 2.2. Then we prove the following:

- if $\chi \in (0, \chi^*)$ then all eigenvalues of the linearized operators have negative real part,
- if $\chi \in (\chi^*, \infty)$ then the linearized operators exhibit eigenvalues with positive real part.

Let us introduce the following semi-linear mappings corresponding to parabolic-parabolic and parabolic-elliptic equations respectively

$$\begin{aligned} &\mathcal{F}(u,v,\tau,\chi) : \mathcal{D}\times\mathcal{D}\times[0,\infty)\times\mathbb{R}\to L^2(\Gamma),\\ &\mathcal{F}(u,v,\tau,\chi) := \begin{bmatrix} \partial_x\big(\partial_x u-\chi u\partial_x v\big)+u(a-bu)\\ \tau^{-1}(\partial^2_{xx}v-v+u)\end{bmatrix}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &F(u,\chi) : \mathcal{D}\times\mathbb{R}\to L^2(\Gamma)\times L^2(\Gamma),\\ &F(u,\chi)u := \partial^2_{xx}u+\chi\partial_x\left(u\partial_x(\Delta-I)^{-1}u)\right)+u(a-bu), \end{aligned} \tag{2.2}$$

where $\mathcal{D}$ is as in (1.4). Let us recall, from [6, Section 3.1.1, Theorem 1.4.19], see also [4], that the spectrum of the Neumann–Kirchhoff Laplacian Δ on a compact graph is discrete[5] .

Lemma 2.1. *The linerization of $\mathcal{F}(u,v,\tau,\chi)$ about $(a/b,a/b)$ is given by*

$$\partial_{(u,v)}\mathcal{F}\left(a/b,a/b,\tau,\chi\right) = D(a,b,\tau,\chi),$$

where $D(a,b,\tau,\chi) : L^2(\Gamma)\times L^2(\Gamma)\to L^2(\Gamma)\times L^2(\Gamma)$ is a non-selfadjoint block operator matrix given by

$$\begin{aligned} &\mathrm{dom}\left(D(a,b,\tau,\chi)\right) := \mathrm{dom}(\Delta)\times\mathrm{dom}(\Delta),\\ &D(a,b,\tau,\chi) := \begin{bmatrix}\Delta-aI_{L^2(\Gamma)} & -\frac{\chi a}{b}\Delta\\ \tau^{-1}I_{L^2(\Gamma)} & \tau^{-1}(\Delta-I_{L^2(\Gamma)})\end{bmatrix}, \end{aligned}$$

where Δ denotes the Neumann–Kirchhoff Laplacian on a compact graph Γ, a,b,τ,χ are positive constants.

Then the spectrum of $D(a,b,\tau,\chi)$ is discrete, that is, it consist of isolated eigenvalues of finite multiplicity and it is given by

$$\mathrm{Spec}\left(D(a,b,\tau,\chi)\right) = \left\{\mu\in\mathbb{C} : \det(\lambda A+B-\mu T)=0, \lambda\in\mathrm{Spec}(\Delta)\right\},$$

where

$$A := \begin{bmatrix}1 & -\frac{\chi a}{b}\\ 0 & 1\end{bmatrix}, B := \begin{bmatrix}-a & 0\\ 1 & -1\end{bmatrix}, T := \begin{bmatrix}1 & 0\\ 0 & \tau\end{bmatrix}.$$

Concretely, $\mu\in\mathrm{Spec}\left(D(a,b,\tau,\chi)\right)$ if and only if

$$\mu = \frac{-\Big(1-(1+\tau)\lambda+a\tau\Big)+\sqrt{\Big(1-(1+\tau)\lambda+a\tau\Big)^2-4\tau\Big((a-\lambda)(1-\lambda)+\chi\frac{a}{b}\lambda\Big)}}{2\tau}, \tag{2.3}$$

or

$$\mu = \frac{-\Big(1-(1+\tau)\lambda+a\tau\Big)-\sqrt{\Big(1-(1+\tau)\lambda+a\tau\Big)^2-4\tau\Big((a-\lambda)(1-\lambda)+\chi\frac{a}{b}\lambda\Big)}}{2\tau}. \tag{2.4}$$

for some eigenvalue λ of the Neumann–Kirchhoff Laplacian.

[5]in contrast to [6] we consider the positive Laplace operator Δ whose spectrum is non-positive and accumulates only at $-\infty$

Proof. In the first step we find the eigenvalues of $D = D(a, b, \tau, \chi)$, in the second step we will prove that $D - \mu$ is boundedly invertible, that is, $(D - \mu)^{-1} \in \mathcal{B}(L^2(\Gamma) \times L^2(\Gamma))$ whenever $\mu \in \mathbb{C}$ is not an eigenvalue.

Step one. Let $\Delta_2 := \Delta \oplus \Delta$ and

$$\begin{aligned}
\mathcal{A} &:= A \otimes I_{L^2(\Gamma)} = \begin{bmatrix} I_{L^2(\Gamma)} & -\frac{\chi a}{b} I_{L^2(\Gamma)} \\ 0 & I_{L^2(\Gamma)} \end{bmatrix}, \\
\mathcal{B} &:= B \otimes I_{L^2(\Gamma)} = \begin{bmatrix} -aI_{L^2(\Gamma)} & 0_{L^2(\Gamma)} \\ I_{L^2(\Gamma)} & -I_{L^2(\Gamma)} \end{bmatrix}, \\
\mathcal{T} &= T \otimes I_{L^2(\Gamma)} = \begin{bmatrix} I_{L^2(\Gamma)} & 0_{L^2(\Gamma)} \\ 0_{L^2(\Gamma)} & \tau I_{L^2(\Gamma)} \end{bmatrix}.
\end{aligned} \tag{2.5}$$

Then one has

$$D = \mathcal{T}^{-1}(\mathcal{A}\Delta_2 + \mathcal{B}),$$

and μ is an eigenvalue of D if and only if

$$\ker\left(\mathcal{A}\Delta_2 + (\mathcal{B} - \mathcal{T}\mu)\right) \neq \{0\}. \tag{2.6}$$

Since $1 \notin \operatorname{Spec}(\Delta)$, $0 \notin \operatorname{Spec}(\mathcal{A})$ one has

$$\mathcal{A}\Delta_2 + (\mathcal{B} - \mathcal{T}\mu) = \mathcal{A}(\Delta_2 - I)(I + (\Delta_2 - I)^{-1}(\mathcal{A}^{-1}(\mathcal{B} - \mathcal{T}\mu) + I)),$$

and (2.6) is equivalent to

$$\ker(I + (\Delta_2 - I)^{-1}(\mathcal{A}^{-1}(\mathcal{B} - \mathcal{T}\mu) + I)) \neq \{0\}. \tag{2.7}$$

Since $(\Delta_2 - I)^{-1}$ is a Hilbert–Schmidt operator and $(\mathcal{A}^{-1}(\mathcal{B} - \mathcal{T}\mu) + I)$ is bounded, the operator

$$V_\mu := (\Delta_2 - I)^{-1}(\mathcal{A}^{-1}(\mathcal{B} - \mathcal{T}\mu) + I)$$

is trace class. Hence, $I + V_\mu$ is boundedly invertible if and only if $\det(I + V_\mu) \neq 0$, cf.,e.g, [14, Theorem VII. 7.1]. Next, we compute this perturbation determinant explicitly. Let $P_t := \chi_{(t,\infty)}(\Delta_2)$ be the spectral projection of Δ_2 corresponding to the interval (t, ∞). Since the spectrum of Δ_2 is discrete and bounded from above, one has $\dim \operatorname{ran}(P_t) < \infty$, $t \in \mathbb{R}$ and $\lim\limits_{t\to-\infty} P_t = I_{L^2(\Gamma)\times L^2(\Gamma)}$. Then one has

$$\begin{aligned}
\det(I + V_\mu) &= \lim_{t\to-\infty} \det(I_{\operatorname{ran}P_t} + P_t(\Delta_2 - I)^{-1}(\mathcal{A}^{-1}(\mathcal{B} - \mu\mathcal{T}) + I)P_t) \\
&= \lim_{t\to-\infty} \det(I_{\operatorname{ran}P_t} + P_t(\Delta_2 - I)^{-1}P_t(\mathcal{A}^{-1}(\mathcal{B} - \mu\mathcal{T}) + I)P_t) \\
&= \lim_{t\to-\infty} \prod_{\substack{\lambda\in\operatorname{Spec}(\Delta_2) \\ \lambda>t}} \det(I_2 + (\lambda - 1)^{-1}(A^{-1}(B - \mu T) + I)) \\
&= \prod_{\lambda\in\operatorname{Spec}(\Delta_2)} \frac{\det(\lambda A + B - \mu T)}{\det(A)(\lambda - 1)^2},
\end{aligned} \tag{2.8}$$

where we used that fact that $(\mathcal{A}^{-1}(\mathcal{B} - \mu\mathcal{T}) + I)$ and P_t commute. The latter is inferred, for example, from the matrix representation of these operators with respect to spectral the decomposition

$$L^2(\Gamma) \oplus L^2(\Gamma) = \bigoplus_{\lambda\in\operatorname{Spec}(\Delta_2)} \operatorname{ran}\chi_{\{\lambda\}}(\Delta_2),$$

$$(\mathcal{A}^{-1}(\mathcal{B}-\mu\mathcal{T})+I)=\begin{bmatrix}(A^{-1}(B-\mu T)+I_2) & 0_2 & \dots\\ 0_2 & (A^{-1}(B-\mu T)+I_2) & \dots\\ \vdots & \vdots & \ddots\end{bmatrix},$$

$$P_t=\begin{bmatrix}\lambda_1 I_2 & 0_2 & \dots & 0_2 & 0_2 & \dots\\ 0_2 & \lambda_2 I_2 & \dots & 0_2 & 0_2 & \dots\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0_2 & 0_2 & \dots & \lambda_k I_2 & 0_2 & 0_2\\ 0_2 & 0_2 & \dots & 0_2 & 0_2 & \ddots\end{bmatrix},$$

where $\lambda_1\geq\dots\geq\lambda_k$ are eigenvalues of Δ_2 and λ_k is the smallest eigenvalue satisfying $\lambda_k>t$. Then (2.8) yields (2.7) which in turn shows that μ is an eigenvalue of D if and only if $\det(\lambda A+B-\mu T)=0$ for some $\lambda\in\operatorname{Spec}(\Delta)$, that is μ is as in (2.3), (2.4).

Step two. For $\mu\in\mathbb{C}$ one has

$$D-\mu=\mathcal{T}^{-1}\mathcal{A}(\Delta_2-I)(I+(\Delta_2-I)^{-1}(\mathcal{A}^{-1}(\mathcal{B}-\mathcal{T}\mu)+I)). \tag{2.9}$$

Let us pick $\mu\in\mathbb{C}$ such that $\det(\lambda A+B-\mu T)\neq 0$ for all $\lambda\in\operatorname{Spec}(\Delta_2)$[6]. Then by step one the operator in the right-hand side of (2.9) is boundedly invertible, hence,

$$(D-\mu)^{-1}=(I+(\Delta_2-I)^{-1}(\mathcal{A}^{-1}(\mathcal{B}-\mathcal{T}\mu)+I))^{-1}(\Delta_2-I)^{-1}\mathcal{A}^{-1}\mathcal{T}.$$

Since $(\Delta_2-I)^{-1}$ is compact and all other factors are bounded we infer that $(D-\mu)^{-1}$ is also compact. Therefore, the spectrum of D purely discrete and consists of eigenvalues given by (2.3), (2.4). □

Lemma 2.2. *The linearization of $F(u,\chi)$ about a/b is given by*

$$\partial_u F\,(a/b,\chi)=D(a,b,\chi),$$

where $D(a,b,\chi):L^2(\Gamma)\to L^2(\Gamma)$ is a self-adjoint operator given by

$$\begin{aligned}&\operatorname{dom}\big(D(a,b,\chi)\big):=\operatorname{dom}(\Delta),\\ &D(a,b,\chi):=\Delta-\chi\frac{a}{b}\,(\Delta-I)^{-1}-\Big(a-\chi\frac{a}{b}\Big),\end{aligned}$$

where Δ denotes the Neumann–Kirchhoff Laplacian on a compact graph Γ, a,b,χ are positive constants. Then the spectrum of $D(a,b,\chi)$ is discrete, that is, it consists of isolated eigenvalues and $\mu\in\operatorname{Spec}\big(D(a,b,\chi)\big)$ if and only if

$$\mu=\lambda-\frac{\chi a}{b(1-\lambda)}-\Big(a-\chi\frac{a}{b}\Big)$$

for some $\lambda\in\operatorname{Spec}(\Delta)$.

Proof. Let $f(t):=t-\chi ab^{-1}(1-t)^{-1}-(a-\chi ab^{-1})$, $t\leq 0$. Then $D(a,b,\chi)=f(\Delta)$ and the assertions follow from the spectral theorem combined with the fact that Δ has compact resolvent. □

We now ready to prove Theorem 1.1.

[6] such a μ exists because $\operatorname{Spec}(\Delta_2)$ is discrete

Proof of Theorem 1.1. To prove local asymptotic stability of $(\frac{a}{b}, \frac{a}{b})$ in Part (1) and the instability of $(\frac{a}{b}, \frac{a}{b})$ in Part (2), it suffices to show that the spectrum of the linearized operator is a subset of $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ if $0 < \chi < \chi^*$, and that it intersects the set $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ if $\chi > \chi^*$, cf., e.g., [16, Theorem 5.1.1]. We prove this for the cases $\tau = 0$ and $\tau > 0$ separately.

First, consider the case that $\tau = 0$. Recall (2.2) and its linearization $D(a, b, \chi)$ from Lemma 2.2. Then $\mu \in \operatorname{Spec}\big(D(a, b, \chi)\big)$ if and only if

$$\mu = \lambda - a + \chi\frac{a}{b}\left(1 - \frac{1}{1-\lambda}\right).$$

for some $\lambda \in \operatorname{Spec}(\Delta)$. Since $\lambda \leq 0$, one has $\mu = \mu(\chi)$ is a non-decreasing and vanishes at

$$\chi(\lambda) := \frac{b(\lambda - a)(1-\lambda)}{a\lambda}, \quad \lambda \in \operatorname{Spec}(\Delta).$$

Hence, $\mu < 0$ whenever $0 < \chi < \min\{\chi(\lambda) : \lambda \in \operatorname{Spec}(\Delta)\} = \chi^*$. That is

$$\operatorname{Spec}\left(D(a, b, \chi)\right) \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}, \chi \in (0, \chi^*).$$

Moreover, if $\chi > \chi^*$, then for some $\lambda \in \operatorname{Spec}(\Delta)$ one has $\mu = \lambda - a + \chi\frac{a}{b}\left(1 - \frac{1}{1-\lambda}\right) > 0$, which concludes the proof of the case $\tau = 0$.

Next, we consider the case $\tau > 0$. Recall (2.2) and its linearization $D(a, b, \tau, \chi)$ from Lemma 2.2. Then the spectrum of $D(a, b, \tau, \chi)$ is given by the following eigenvalues

$$\mu_\pm(\lambda, \chi) = \frac{-\Big(1 - (1+\tau)\lambda + a\tau\Big) \pm \sqrt{\Big(1 - (1+\tau)\lambda + a\tau\Big)^2 - 4\tau\Big((a-\lambda)(1-\lambda) + \chi\frac{a}{b}\lambda\Big)}}{2\tau}$$

for $\lambda \in \operatorname{Spec}(\Delta)$. First, let us observe that

$$1 - (1+\tau)\lambda + a\tau > 0.$$

Hence $\operatorname{Re}(\mu_-(\lambda, \chi)) < 0$ for all $\lambda \in \operatorname{Spec}(\Delta)$, $\chi > 0$.

To show $\operatorname{Re}(\mu_+(\lambda, \chi)) < 0$ for all $\lambda \in \operatorname{Spec}(\Delta)$, $\chi \in (0, \chi^*)$, we first note that

$$\mu_+(0, \chi) = \frac{-(1 + a\tau) + \sqrt{\Big(1 + a\tau\Big)^2 - 4\tau a}}{2\tau} < 0.$$

If $\lambda \in \operatorname{Spec}(\Delta) \setminus \{0\}$ then either

$$\Big(1 - (1+\tau)\lambda + a\tau\Big)^2 \geq 4\tau\Big((a-\lambda)(1-\lambda) + \chi\frac{a}{b}\lambda\Big),$$

in which case $\operatorname{Re}(\mu_+(\lambda, \chi)) < 0$, or

$$\Big(1 - (1+\tau)\lambda + a\tau\Big)^2 < 4\tau\Big((a-\lambda)(1-\lambda) + \chi\frac{a}{b}\lambda\Big),$$

in which case $\chi \mapsto \mu_+(\lambda, \chi)$ is a real-valued, non-decreasing function of χ. In the latter case, the equation $\mu_+(\lambda, \chi) = 0$ reads

$$\Big(1 - (1+\tau)\lambda + a\tau\Big) = \sqrt{\Big(1 - (1+\tau)\lambda + a\tau\Big)^2 - 4\tau\Big((a-\lambda)(1-\lambda) + \chi\frac{a}{b}\lambda\Big)}$$

and yields

$$\chi = \frac{b(\lambda - a)(1-\lambda)}{a\lambda}, \lambda \in \operatorname{Spec}(\Delta).$$

Hence, $\mathrm{Re}(\mu_+(\lambda,\chi)) = \mu_+(\lambda,\chi) < 0$ whenever $\chi \in (0,\chi^*)$ as required. To finish the proof, we note that if $\chi > \chi^*$ then there exists $\lambda \in \mathrm{Spec}(\Delta)$ such that $\mu_+(\lambda,\chi) > 0$. □

2.3. **Local bifurcation.** In this subsection, we prove Theorem 1.2 via Crandall–Rabinowitz's Theorem, cf. [8, Theorem 8.3.1].

Proof of Theorem 1.2. Our goal is to verify conditions of Crandall–Rabinowitz's Theorem as stated in [8, Theorem 8.3.1]. To that end, we first note that steady state solutions of both parabolic-parabolic and parabolic-elliptic equations stem for the same system

$$\begin{cases} \partial_x\big(\partial_x u - \chi u \partial_x v\big) + u(a-bu) = 0, \\ \partial_{xx}^2 v - v + u = 0, \end{cases}$$

or, equivalently, $\mathcal{F}(u,v,1,\chi) = 0$, cf. (2.1). The partial derivative with respect to (u,v) of this nonlinear mapping is given by

$$\begin{aligned} &L := D_{(u,v)}\mathcal{F}(u,v,1,\chi) \in \mathcal{B}(\mathcal{D}\times\mathcal{D}\times\mathbb{R}, L^2(\Gamma)\times L^2(\Gamma)), \\ &D_{(u,v)}\mathcal{F}(u,v,\chi) = L_2 + L_1 \text{ where}, \\ &L_2\begin{bmatrix} f \\ g\end{bmatrix} := \begin{bmatrix} I_{L^2(\Gamma)} & -\chi u I_{L^2(\Gamma)} \\ 0_{L^2(\Gamma)} & I_{L^2(\Gamma)}\end{bmatrix}\begin{bmatrix} \Delta & 0_{L^2(\Gamma)} \\ 0_{L^2(\Gamma)} & \Delta\end{bmatrix}, \\ &L_1\begin{bmatrix} f \\ g\end{bmatrix} := \begin{bmatrix} -\chi v' f' - \chi v'' f - \chi u' g' + af - 2buf \\ -g + f\end{bmatrix}, \end{aligned} \tag{2.10}$$

here $f,g \in \mathcal{D}$ and $\mathcal{D}$ is considered as a Banach space with $\widehat{W}^{2,2}(\Gamma)-$norm. This shows that $F \in C^2(\mathcal{D}\times\mathcal{D}\times\mathbb{R}, L^2(\Gamma)\times L^2(\Gamma))$. Next, we show that $D_{(u,v)}\mathcal{F}(u,v,1,\chi)$ is Fredholm with index zero as an operator from $\mathcal{D}\times\mathcal{D}\times\mathbb{R}$ to $L^2(\Gamma)\times L^2(\Gamma)$. Let us recall that the Neumann–Kirchhoof Laplacian $\Delta \in \mathcal{B}(\widehat{W}^{2,2}(\Gamma), L^2(\Gamma))$ is Fredholm with index zero and the first term in the right-hand side of (2.10) is Fredhlom in $L^2(\Gamma)\times L^2(\Gamma)$ with index zero. Therefore by [11, Theorem 3.16], L_2 is Fredholm with index zero as a mapping from $\widehat{W}^{2,2}(\Gamma)$ to $L^2(\Gamma)$. Next, $L_1 \in \mathcal{B}(\widehat{W}^{2,2}(\Gamma), \widehat{W}^{1,2}(\Gamma))$ and the embedding $\widehat{W}^{1,2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, therefore L_1 is compact as a mapping from $\widehat{W}^{2,2}(\Gamma)$ to $L^2(\Gamma)$. Thus by [11, Theorem 3.17] the operator $L = L_1 + L_2$ is a Fredholm with index zero.

Recalling $\mathcal{A}, \mathcal{B}, \Delta_2$ from (2.5) we obtain

$$L = D_{(u,v)}\mathcal{F}(a/b, a/b, 1, \chi) = \mathcal{A}\Delta_2 + \mathcal{B}.$$

Hence, one has

$$\ker L = \ker\left(\begin{bmatrix} \Delta & 0_{L^2(\Gamma)} \\ 0_{L^2(\Gamma)} & \Delta\end{bmatrix} + \begin{bmatrix} \left(-a+\frac{\chi a}{b}\right) I_{L^2(\Gamma)} & -\frac{\chi a}{b} I_{L^2(\Gamma)} \\ I_{L^2(\Gamma)} & -I_{L^2(\Gamma)}\end{bmatrix}\right).$$

Then $\ker L \neq \{0\}$ if and only if for some $\lambda \in \mathrm{Spec}(\Delta)$ one has

$$\det\left(\begin{bmatrix} \lambda & 0_{L^2(\Gamma)} \\ 0_{L^2(\Gamma)} & \lambda\end{bmatrix} + \begin{bmatrix} \left(-a+\frac{\chi a}{b}\right) I_{L^2(\Gamma)} & -\frac{\chi a}{b} I_{L^2(\Gamma)} \\ I_{L^2(\Gamma)} & -I_{L^2(\Gamma)}\end{bmatrix}\right) = 0,$$

that is, if and only if one has

$$\left(a - \frac{\chi a}{b} - \lambda\right)(1-\lambda) + \frac{\chi a}{b} = 0,$$

or, equivalently, the identity (1.5) holds. By assumptions, we then obtain $\dim\ker(L) = 1$ and $L\Xi = 0$, $\Xi := [\xi,\eta]^\top\varphi$.

Let us now show that the transversality condition in [8, Theorem 8.3.1] is also satisfied. That is, for $K := D^2_{(u,v),\chi}\mathcal{F}(a/b, a/b, 1, \chi)$ we show that $K[\Xi, 1]^\top \notin \operatorname{ran}(L)$. It suffices to show $\ker(L^*) = 0$. Since $L^* = (\mathcal{A}^{-1})^*(\Delta_2 + (\mathcal{B}\mathcal{A}^{-1})^*)$, we note that $\ker(L^*) \neq \{0\}$ yields a $\lambda \in \operatorname{Spec}(\Delta)$ such that

$$\det(\lambda + (BA^{-1})^*) = 0,$$

that is

$$(\lambda - a)(\lambda - 1) = 0,$$

which contradicts $\lambda \leq 0$, $a > 0$. □

3. Global stability of constant steady states

In this section, we study the global stability of the constant solution $(\frac{a}{b}, \frac{a}{b})$ of (1.1), (1.3) and prove Theorems 1.3 and 1.4. Throughout this section, C denotes a positive constant independent of a, b, χ and the solutions of (1.1), (1.3). We first establish some lemmas and then prove Theorems 1.3 and 1.4.

Lemma 3.1. *Assume the setting of Theorem* 1.3. *Then for* $T > 0$ *one has*

$$\limsup_{t\to\infty} \int_\Gamma u(t; u_0, v_0)dx \leq \frac{a|\Gamma|}{b}, \tag{3.1}$$

$$\limsup_{t\to\infty} \int_\Gamma v(t; u_0, v_0)dx \leq \frac{a|\Gamma|}{b}. \tag{3.2}$$

Proof. Integrating both sides of the first equation in (1.1) over Γ we obtain

$$\frac{d}{dt}\int_\Gamma u(t; u_0, v_0)dx = a\int_\Gamma u(t; u_0, v_0)dx - b\int_\Gamma u^2(t; u_0, v_0)dx.$$

Combining this with

$$\int_\Gamma u^2(t; u_0, v_0)dx \geq \frac{1}{|\Gamma|}\left(\int_\Gamma u(t; u_0, v_0)dx\right)^2,$$

we arrive at

$$\frac{d}{dt}\int_\Gamma u(t; u_0, v_0)dx \leq a\int_\Gamma u(t; u_0, v_0) - \frac{b}{|\Gamma|}\left(\int_\Gamma u(t; u_0, v_0)\right)^2 dx.$$

Then, since $f(t) = |\Gamma|ab^{-1}$ solves the differential equation $f' = af - b(|\Gamma|)^{-1}f^2$, the comparison principle yields

$$\limsup_{t\to\infty} \int_\Gamma u(t; u_0, v_0)dx \leq \frac{a|\Gamma|}{b},$$

as asserted in (3.1).

To prove (3.3) we integrate the second equation in (1.1) to obtain,

$$\frac{d}{dt}\int_\Gamma v(t; u_0, v_0)dx \leq \int_\Gamma u(t; u_0, v_0)dx - \int_\Gamma v(t; u_0, v_0)dx,$$

which together with (3.1) yields (3.2). □

Lemma 3.2. *Assume the setting of Theorem* 1.3. *Then there exists a constant* $C = C(\Gamma) > 0$ *such that*

$$\limsup_{t\to\infty} \|v(t)\|_{\widehat{C}^1(\overline{\Gamma})} \leq C\frac{a}{b}. \tag{3.3}$$

Proof. Let us fix $\beta \in \left(\frac{1}{2}, 1\right)$ and $q > 1$ satisfying $2\beta - q^{-1} > 1$. The by Theorem A.1 one has

$$\mathfrak{X}_q^\beta \hookrightarrow \widehat{C}^1(\overline{\Gamma}). \tag{3.4}$$

By the Duhamel principle, the second equation in (1.1) yields

$$v(t) = e^{(\Delta-\omega)\frac{(t-t_0)}{\tau}} v(t_0) + \frac{1}{\tau}\int_{t_0}^{t} e^{(\Delta-\omega)\frac{t-s}{\tau}} w(s)ds,$$

where

$$v(t) = v(t; u_0, v_0),\ w(s) := u(s) + \sigma_0 v(s),\ \omega := \sigma_0 + 1,$$

and $\sigma_0 > 0$ as in Theorem A.2. Then for some $\delta \in (0, \sigma_0)$ one obtains

$$\begin{aligned}
\|v(t)\|_{\widehat{C}^1(\overline{\Gamma})} &\underset{(3.4)}{\leq} C\|v(t)\|_{\mathfrak{X}_q^\beta} \leq C\|e^{(\Delta-\omega)\frac{t-t_0}{\tau}}(\omega-\Delta)^\beta v(t_0)\|_{L^q(\Gamma)} \\
&\qquad + C\int_{t_0}^{t} \|(\omega-\Delta)^\beta e^{(\Delta-\omega)\frac{t-s}{2\tau}} e^{(\Delta-I)\frac{t-s}{2\tau}} w(s)\|_{L^q(\Gamma)} ds \\
&\leq C\|e^{(\Delta-\omega)\frac{t-t_0}{\tau}}(\omega-\Delta)^\beta v(t_0)\|_{L^q(\Gamma)} \\
&\qquad + C\int_{t_0}^{t} \left(\frac{t-s}{2\tau}\right)^{-\beta} e^{-\frac{\delta(t-s)}{4\tau}} \|e^{(\Delta-\omega)\frac{t-s}{2\tau}} w(s)\|_{L^q(\Gamma)} ds \\
&\leq C\|e^{(\Delta-\omega)\frac{t-t_0}{\tau}}(\omega-\Delta)^\beta v(t_0)\|_{L^q(\Gamma)} \\
&\qquad + C\int_{t_0}^{t} \left(\frac{t-s}{2\tau}\right)^{-\beta-\frac{1}{2}(1-\frac{1}{q})} e^{-\frac{\delta(t-s)}{2\tau}} \|w(s)\|_{L^1(\Gamma)} ds \\
&\leq Ce^{\frac{\delta(t_0-t)}{2\tau}}\|(\omega-\Delta)^\beta v(t_0)\|_{L^q(\Gamma)} + C\sup_{t\in[t_0,\infty)}\|w(t)\|_{L^1(\Gamma)}\int_{t_0}^{t}\left(\frac{t-s}{2\tau}\right)^{-\beta-\frac{1}{2}(1-\frac{1}{q})} e^{-\frac{\delta(t-s)}{2\tau}} ds.
\end{aligned}$$

This implies (3.3) by choosing sufficiently large t_0 such that $\sup_{t\in[t_0,\infty)}\|w(t)\|_{L^1(\Gamma)} \leq 2|\Gamma|\omega\frac{a}{b}$ and then letting $t \to \infty$. □

In order to estimate the L^2-norm of the logistic term $u(a - bu)$ in the proof of Lemma 3.5 below, we provide an upper bound of the L^4-norm of u next.

Lemma 3.3. *Assume the setting of Theorem* 1.3. *Then one has*

$$\limsup_{t\to\infty}\|u(t)\|_{L^4(\Gamma)} \leq C\left(\frac{a}{b} + \frac{a^2\chi^2}{b^3}\right). \tag{3.5}$$

Proof. Multiplying both sides of the first equation in (1.1) by u^3 and integrating over Γ we obtain

$$\int_\Gamma u_t u^3 dx = \int_\Gamma u^3\partial_x(\partial_x u - \chi u\partial_x v)dx + \int_\Gamma u^4(a - bu)dx.$$

We note that

$$\begin{aligned}
\int_\Gamma u^3 u_{xx} dx &= -\int_\Gamma (u^3)_x u_x dx + \sum_{\theta\in\mathcal{V}}\sum_{e\sim\theta} u_e^3(\theta)\partial_\nu u_e(\theta) \\
&= -\int_\Gamma (u^3)_x u_x dx + \sum_{\theta\in\mathcal{V}} u^3(\theta)\sum_{e\sim\theta}\partial_\nu u_e(\theta) = -\int_\Gamma (u^3)_x u_x dx = -3\int_\Gamma (uu_x)^2 dx,
\end{aligned}$$

where we used the fact that u satisfies the Neumann-Kirchhoff vertex conditions. Similarly, one has

$$\int_\Gamma u^3\partial_x(uv_x)dx = -\int_\Gamma (u^3)_x uv_x dx + \sum_{\theta\in\mathcal{V}}\sum_{e\sim\theta} u_e^4(\theta)\partial_\nu v_e(\theta) = -\int_\Gamma (u^3)_x uv_x dx.$$

Therefore, we have

$$\frac{1}{4}\frac{d}{dt}\int_\Gamma u^4 dx = -3\int_\Gamma (uu_x)^2 dx + 3\chi\int_\Gamma u^3u_xv_x dx + \int_\Gamma u^4(a-bu)dx. \tag{3.6}$$

We note that Young's inequality with exponents $2,2$ yields

$$\chi\int_\Gamma u^3u_xv_xdx = \int_\Gamma (uu_x)(\chi u^2v_x)dx \le \frac{\chi^2}{4}\int_\Gamma u^4|v_x|^2dx + \int_\Gamma (uu_x)^2dx. \tag{3.7}$$

By Hölder's inequality we have

$$\int_\Gamma u^5dx \ge \frac{1}{|\Gamma|^{1/5}}\left(\int_\Gamma u^4dx\right)^{5/4}.$$

Combining these inequalities with (3.6) we obtain

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Gamma u^4 dx &= -3\int_\Gamma (uu_x)^2 dx + 3\chi\int_\Gamma u^3u_xv_x dx + \int_\Gamma u^4(a-bu)dx\\ &\underset{(3.7)}{\le} \frac{3\chi^2}{4}\int_\Gamma u^4|v_x|^2dx + a\int_\Gamma u^4dx - b\int_\Gamma u^5dx\\ &\le \frac{3\chi^2}{4}\|v\|^2_{\widehat{C}^1(\overline{\Gamma})}\int_\Gamma u^4dx + a\int_\Gamma u^4dx - \frac{b}{|\Gamma|^{\frac15}}\Big(\int_\Gamma u^4dx\Big)^{\frac54}\\ &= \Big(\frac{3\chi^2}{4}\|v\|^2_{\widehat{C}^1(\overline{\Gamma})} + a - \frac{b}{|\Gamma|^{\frac15}}\big(\int_\Gamma u^4dx\big)^{\frac14}\Big)\int_\Gamma u^4dx.\end{aligned}$$

Denoting $y(t) = \int_\Gamma u^4dx$, this together with (3.3) yield

$$y' \le \left(\frac{\chi^2a^2}{b^2} - a - \frac{b}{|\Gamma|^{\frac15}}y^{1/4}\right)y.$$

Hence, there exists $C>0$ such that

$$\limsup_{t\to\infty} y(t) \le C\left(\frac{a}{b} + \frac{a^2\chi^2}{b^3}\right)^4,$$

which gives (3.5). □

Lemma 3.4. *Assume the setting of Theorem* 1.3. *Then for arbitrary* $\gamma\in(0,1/2)$ *one has*

$$\limsup_{t\to\infty}\|(\omega-\Delta)^\gamma u(t)\|_{L^2(\Gamma)} \le p(a,b,\chi), \tag{3.8}$$

where $\omega=\sigma_0+1$ *and* $\sigma_0>0$ *is as in Theorem* A.2, *and* $\chi\mapsto p(a,b,\chi)$ *is a polynomial of degree* 4 *with coefficients dependent only on* a,b,Γ.

Proof. By Duhamel's principle, the first equation in (1.1) yields

$$\begin{aligned} u(t) = e^{(\Delta-\omega)(t-t_0)}u(t_0) - \chi\int_{t_0}^{t} e^{(\Delta-\omega)(t-s)}\partial_x\left(u(s)\partial_x v(s)\right)ds \\ + \int_{t_0}^{t} e^{(\Delta-\omega)(t-s)}u(s)(a+\omega-bu(s))ds. \end{aligned} \tag{3.9}$$

Let us note that the following inequalities hold for some $\delta\in(0,\sigma_0)$

$$\begin{aligned} &\|(\omega-\Delta)^{\gamma}e^{(\Delta-\omega)\frac{t-s}{2}}e^{(\Delta-\omega)\frac{t-s}{2}}\partial_x(u(s)\partial_x v(s))\|_{L^2(\Gamma)} \\ &\underset{(A.9)}{\leq} C(t-s)^{-\gamma}e^{-\frac{\delta(t-s)}{2}}\|e^{(\Delta-\omega)\frac{t-s}{2}}\partial_x(u(s)\partial_x v(s))\|_{L^2(\Gamma)} \\ &\underset{(A.10)}{\leq} C(t-s)^{-\gamma-1/2}e^{-\delta(t-s)}\|u(s)\partial_x v(s)\|_{L^2(\Gamma)} \\ &\leq C(t-s)^{-\gamma-1/2}e^{-\delta(t-s)}\sup_{r\geq t_0}\|\partial_x v(r)\|_{L^\infty(\Gamma)}\|u(s)\|_{L^2(\Gamma)}, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\|(\omega-\Delta)^{\gamma}e^{(\Delta-\omega)(t-s)}u(s)(a+\omega-bu(s))\|_{L^2(\Gamma)} \\ &\underset{(A.9)}{\leq} C(t-s)^{-\gamma}e^{-\delta(t-s)}\|(a+\omega)u(s)-bu^2(s)\|_{L^2(\Gamma)} \\ &\leq C(t-s)^{-\gamma}e^{-\delta(t-s)}\left((a+\omega)\|u(s)\|_{L^2(\Gamma)}+b\|u(s)\|^2_{L^4(\Gamma)}\right), \end{aligned} \tag{3.11}$$

and

$$\|u(s)\|_{L^2(\Gamma)} \leq |\Gamma|^{\frac14}\|u(s)\|_{L^4(\Gamma)}. \tag{3.12}$$

Combining (3.9), (3.10), (3.11), (3.12) we obtain

$$\begin{aligned} &\|(\omega-\Delta)^{\gamma}u(t)\|_{L^2(\Gamma)} \\ &\leq Ce^{-\delta(t-t_0)}\|(\omega-\Delta)^{\gamma}u(t_0)\|_{L^2(\Gamma)} \\ &\quad +C\chi\int_{t_0}^{t}(t-s)^{-\gamma-1/2}e^{-\delta(t-s)}\sup_{r\geq t_0}\|\partial_x v(r)\|_{L^\infty(\Gamma)}\|u(s)\|_{L^2(\Gamma)}ds \\ &\quad +C\int_{t_0}^{t}(t-s)^{-\gamma}e^{-\delta(t-s)}\left((a+\omega)\|u(s)\|_{L^2(\Gamma)}+b\|u(s)\|^2_{L^4(\Gamma)}\right)ds \\ &\leq Ce^{-\delta(t-t_0)}\|(\omega-\Delta)^{\gamma}u(t_0)\|_{L^2(\Gamma)} \\ &\quad +C\chi\sup_{r\geq t_0}\|\partial_x v(r)\|_{L^\infty(\Gamma)}\sup_{r\geq t_0}\|u(r)\|_{L^4(\Gamma)}\int_{t_0}^{t}(t-s)^{-\gamma-1/2}e^{-\delta(t-t_0)}ds \qquad (3.13) \\ &\quad +C\left((a+\omega)\sup_{r\geq t_0}\|u(r)\|_{L^4(\Gamma)}+b\sup_{r\geq t_0}\|u(r)\|^2_{L^4(\Gamma)}\right)\int_{t_0}^{t}(t-s)^{-\gamma}e^{-\delta(t-t_0)}ds \qquad (3.14) \end{aligned}$$

This together with (3.3) and (3.5) implies (3.8) (choosing t_0 sufficiently large). □

Remark 3.1. The polynomial $\chi\mapsto p(a,b,\chi)$ can be written out explicitly using (3.3), (3.5) in (3.13), (3.14).

Lemma 3.5. *Assume the setting of Theorem* 1.3. *Then one has*

$$\limsup_{t\to\infty}\|\Delta v(t)\|_{L^\infty(\Gamma)} \leq \kappa(a,b,\chi), \tag{3.15}$$

where $\chi \mapsto \kappa(a,b,\chi)$ is a polynomial of degree 4 with coefficients dependent only on a,b,Γ.

Proof. The second equation in (1.1) together with Duhamel's principle yields

$$v(t) = e^{(\Delta-\omega)\frac{(t-t_0)}{\tau}}v(t_0) + \frac{1}{\tau}\int_{t_0}^{t} e^{(\Delta-\omega)\frac{t-s}{\tau}}w(s)ds,$$

where

$$v(t) = v(t;u_0,v_0),\ w(s) := u(s) + \sigma_0 v(s),\ \omega := \sigma_0 + 1,$$

and $\sigma_0 > 0$ as in Theorem A.2. Hence,

$$(\omega-\Delta)v(t) = e^{(\Delta-\omega)\frac{t-t_0}{\tau}}(\omega-\Delta)v(t_0) + \frac{1}{\tau}\int_{t_0}^{t}(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}w(s)ds.$$

Let us estimate $L^\infty(\Gamma)$ norm of each term above. To that end, we first note the following embedding

$$\mathfrak{X}_2^\alpha \hookrightarrow \widehat{C}^\nu(\overline{\Gamma}),\ \frac{1}{4} < \alpha < 1,\ 0 < \nu < 2\alpha - \frac{1}{2}.$$

Choose $\frac{1}{4} < \alpha < \gamma < \frac{1}{2}$. For some $\delta \in (0,\sigma_0)$ one has

$$\begin{aligned}
&\int_{t_0}^{t} \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}u(s)\|_{L^\infty(\Gamma)}ds \\
&\qquad \le \int_{t_0}^{t} \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}u(s)\|_{C^\nu(\overline{\Gamma})}ds \\
&\qquad \le C\int_{t_0}^{t} \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}u(s)\|_{\mathfrak{X}_2^\alpha}ds \\
&\qquad \le C\int_{t_0}^{t} \|(\omega-\Delta)^{1+\alpha}e^{(\Delta-\omega)\frac{t-s}{\tau}}u(s)\|_{L^2(\Gamma)}ds \\
&\qquad \le C\int_{t_0}^{t} \|(\omega-\Delta)^{1+\alpha-\gamma}e^{(\Delta-\omega)\frac{t-s}{\tau}}(I-\Delta)^\gamma u(s)\|_{L^2(\Gamma)}ds \\
&\qquad \le C\int_{t_0}^{t} (t-s)^{-(1+\alpha-\gamma)}e^{-\frac{\delta(t-s)}{\tau}}\|(\omega-\Delta)^\gamma u(s)\|_{L^2(\Gamma)}ds.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{t} \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}v(s)\|_{L^\infty(\Gamma)}ds \\
&\qquad \le \int_{t_0}^{t} \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}v(s)\|_{C^\nu(\overline{\Gamma})}ds \\
&\qquad \le C\int_{t_0}^{t} \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-s}{\tau}}v(s)\|_{\mathfrak{X}_2^\alpha}ds \\
&\qquad \le C\int_{t_0}^{t} \|(\omega-\Delta)^{\alpha}e^{(\Delta-\omega)\frac{t-s}{\tau}}\omega v(s)\|_{L^2(\Gamma)}ds \\
&\qquad\ \ + C\int_{t_0}^{t} \|(\omega-\Delta)^{\alpha}e^{(\Delta-\omega)\frac{t-s}{2\tau}}e^{(\Delta-\omega)\frac{t-s}{2\tau}}\partial_x(\partial_x v(s))\|_{L^2(\Gamma)}ds
\end{aligned}$$

$$\leq C\int_{t_0}^{t}(t-s)^{-(\frac{1}{2}+\alpha)}e^{-\frac{\delta(t-s)}{\tau}}\|v(s)\|_{\widehat{C}^1(\overline{\Gamma})}ds$$

Combining these inequalities with

$$\begin{aligned}\|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-t_0}{\tau}}v(t_0)\|_{L^\infty(\Gamma)} &\leq \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-t_0}{\tau}}v(t_0)\|_{C^\nu(\overline{\Gamma})}\\ &\leq \|(\omega-\Delta)e^{(\Delta-\omega)\frac{t-t_0}{\tau}}v(t_0)\|_{\mathfrak{X}_2^\alpha}\\ &\leq Ce^{\frac{\delta(t_0-t)}{\tau}}\|(\omega-\Delta)^{1+\alpha}v(t_0)\|_{L^2(\Gamma)},\end{aligned}$$

we obtain

$$\begin{aligned}\|(\omega-\Delta)v(t)\|_{L^\infty(\Gamma)} \leq\ & Ce^{\frac{\delta(t_0-t)}{2\tau}}\|(\omega-\Delta)^{1+\alpha}v(t_0)\|_{L^2(\Gamma)}\\ &+C\int_{t_0}^{t}(t-s)^{-(1+\alpha-\gamma)}e^{-\frac{\delta(t-s)}{\tau}}\|(\omega-\Delta)^{\gamma}u(s)\|_{L^2(\Gamma)}ds\\ &+C\int_{t_0}^{t}(t-s)^{-(\frac{1}{2}+\alpha)}e^{-\frac{\delta(t-s)}{\tau}}\|v(s)\|_{\widehat{C}^1(\overline{\Gamma})}ds.\end{aligned}$$

This inequality together with (3.3) and (3.8) yields (3.15). □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Let us observe that for arbitrary $\varepsilon>0$, Lemma 3.5 yields a $\chi-$independent constant such that for arbitrary $x\in\overline{\Gamma}$, $t>0$ and $\chi>0$ one has

$$-(\kappa(a,b,\chi)+\varepsilon)\leq v_{xx}(t,x)\leq\kappa(a,b,\chi)+\varepsilon,\quad \text{for } t\gg 1. \tag{3.16}$$

Employing (3.16) we obtain the following inequalities

$$\begin{aligned}u_t&=u_{xx}-\chi(uv_x)_x+u(a-bu)\\ &=u_{xx}-\chi u_xv_x-\chi uv_{xx}+u(a-bu)\\ &\geq u_{xx}-\chi u_xv_x-(\kappa(a,b,\chi)+\varepsilon)u+u(a-bu),\quad \text{for } t\gg 1,\end{aligned}$$

and

$$\begin{aligned}u_t&=u_{xx}-\chi(u_xv)_x+u(a-bu)\\ &=u_{xx}-\chi u_xv_x-\chi uv_{xx}+u(a-bu)\\ &\leq u_{xx}-\chi u_xv_x+(\kappa(a,b,\chi)+\varepsilon)u+u(a-bu),\quad \text{for } t\gg 1,\end{aligned}$$

where $\kappa(a,b,\chi)$ is as in Lemma 3.5. Therefore the partial differential equation

$$\varphi_t=\varphi_{xx}-\chi\varphi_xv_x+\chi(\kappa(a,b,\chi)+\varepsilon)\varphi+\varphi(a-b\varphi),$$

exhibits the subsolution u and a constant solution

$$\overline{u}_\varepsilon:=\frac{a+\chi(\kappa(a,b,\chi)+\varepsilon)}{b},$$

and, similarly u is a supersolution of

$$\varphi_t=\varphi_{xx}-\chi\varphi_xv_x-\chi(\kappa(a,b,\chi)+\varepsilon)\varphi+\varphi(a-b\varphi),$$

while a constant solution is given by

$$\underline{u}_\varepsilon:=\frac{a-\chi(\kappa(a,b,\chi)+\varepsilon)}{b}.$$

Therefore, one obtains

$$\underline{u}_\varepsilon \le u(t,x) \le \overline{u}_\varepsilon,\ \ x\in\Gamma, t>0. \tag{3.17}$$

Next, let us introduce

$$U(t,x) := u(t,x) - \frac{a}{b},\quad V(t,x) := v(t,x) - \frac{a}{b}.$$

Then we have

$$U_t = U_{xx} - \chi(u_x V)_x - buU, \tag{3.18}$$

and

$$\tau V_t = V_{xx} - V + U. \tag{3.19}$$

Since U satisfies the Neumann–Kirchhoff vertex conditions, multiplying (3.18) and integrating by parts as in the proof of Lemma 3.1 we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Gamma U^2 dx &= -\int_\Gamma |U_x|^2 dx + \chi\int_\Gamma uU_xV_x dx - b\int_\Gamma uU^2 dx\\ &\le \frac{\chi^2}{4}\int_\Gamma u^2|V_x|^2 dx - b\int_\Gamma uU^2 dx\\ &\le \frac{\chi^2}{4}\bar{u}_\varepsilon^2\int_\Gamma |V_x|^2 dx - b\underline{u}_\varepsilon\int_\Gamma U^2 dx\end{aligned} \tag{3.20}$$

we used (3.17) and

$$\chi\int_\Gamma uU_xV_x dx = \int_\Gamma \chi V_x u U_x dx \le \int_\Gamma |U_x|^2 dx + \frac{\chi^2}{4}\int_\Gamma u^2|V_x|^2 dx$$

which, in turn, follows from Young's inequality. Similarly, multiplying (3.19) by V and integrating by parts yields

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\int_\Gamma V^2 dx &= -\int_\Gamma |V_x|^2 - \int_\Gamma V^2 dx + \int_\Gamma UV dx\\ &\le -\int_\Gamma |V_x|^2 dx - \frac{1}{2}\int_\Gamma V^2 dx + \frac{1}{2}\int_\Gamma U^2 dx,\end{aligned} \tag{3.21}$$

where in the last step we used Young's inequality. Hence, combining (3.20), (3.21) we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Gamma U^2 dx + \frac{\tau}{2}\frac{\chi^2}{4}\overline{u}_\varepsilon^2\int_\Gamma V^2 dx\\ &\le -\frac{1}{2}\frac{\chi^2}{4}\overline{u}_\varepsilon^2\int_\Gamma V^2 dx - \left(b\underline{u}_\varepsilon - \frac{\chi^2}{4}\overline{u}_\varepsilon^2\right)\int_\Gamma U^2 dx.\end{aligned} \tag{3.22}$$

Provided χ is sufficiently small we have

$$b\underline{u}_\epsilon - \frac{\chi^2}{4}\overline{u}_\epsilon^2 > 0,$$

which together with (3.22) yield

$$\lim_{t\to\infty}\int_\Gamma (U^2+V^2)dx = 0. \tag{3.23}$$

Let us now switch to the proof of (1.6). Assume that

$$\limsup_{t\to\infty}\left(\left\|u-\frac{a}{b}\right\|_{L^\infty(\Gamma)}+\left\|v-\frac{a}{v}\right\|_{L^\infty(\Gamma)}\right)>0,$$

Then for some $\epsilon_0>0$, $t_n\to\infty$ and $x_n\in\Gamma$, $n\in\mathbb{N}$ one has

$$\left|u(t_n,x_n)-\frac{a}{b}\right|+\left|v(t_n,x_n)-\frac{a}{b}\right|\geq\epsilon_0.$$

Combining this inequality with the uniform continuity of u and v yields a $\delta_0>0$ such that

$$\left|u(t_n,x)-\frac{a}{b}\right|+\left|v(t_n,x)-\frac{a}{b}\right|\geq\frac{\epsilon_0}{2},\ n\in\mathbb{N},x\in\Gamma,|x-x_n|\leq\delta_0.$$

This implies that

$$\liminf_{n\to\infty}\int_\Gamma(U^2(t_n,x)+V^2(t_n,x))>0,$$

which contradicts (3.23). □

Finally, we prove Theorem 1.4.

Proof of Theorem 1.4. Let us define

$$\overline{u}=\limsup_{t\to\infty}\left(\sup_{x\in\Gamma}u(x,t)\right)\quad\text{and}\quad\underline{u}=\liminf_{t\to\infty}\left(\inf_{x\in\Gamma}u(x,t)\right),$$

then for $\varepsilon>0$ there exists $t_\varepsilon>0$ such that

$$\underline{u}-\varepsilon\leq\inf_{x\in\Gamma}u(x,t)\leq u(x,t)\leq\sup_{x\in\Gamma}u(x,t)\leq\overline{u}+\varepsilon,\qquad t\geq t_\varepsilon.$$

Let us note that $v_{xx}(x,t)-v(x,t)+u(x,t)=0$ together with the comparison principle for elliptic equations yield

$$\underline{u}_\epsilon:=\underline{u}-\epsilon\leq v(x,t)\leq\overline{u}+\epsilon:=\overline{u}_\epsilon,\quad x\in\Gamma,t\geq t_\epsilon.\tag{3.24}$$

Hence, using the first equation in (1.1) we obtain

$$u_t\leq u_{xx}-\chi u_xv_x-\chi u(\underline{u}^\epsilon-u)+u(a-bu).\tag{3.25}$$

Consider the initial value problem for the following logistic equation

$$\begin{cases}\overline{w}_t&=\chi\overline{w}(\overline{w}-\underline{u}^\epsilon)+\overline{w}(a-b\overline{w})\\ &=-(b-\chi)\overline{w}^2+(a-\chi\underline{u}^\epsilon)\overline{w},\ t\geq t_\epsilon,\\ \overline{w}(t_\epsilon)&=\max\limits_{x\in\Gamma}u(x,t_\varepsilon).\end{cases}\tag{3.26}$$

Combining (3.25), (3.26) and the comparison principle for parabolic equations one obtains

$$u(x,t)\leq\overline{w}(t),x\in\Gamma,t\geq t_\varepsilon.\tag{3.27}$$

Moreover, the logistic equation (3.26) yields

$$\overline{w}(t)\to\frac{(a-\chi\underline{u}^\epsilon)_+}{b-\chi}\quad\text{as}\quad t\to\infty.$$

Then employing (3.27) one infers

$$\overline{u}\leq\frac{(a-\chi(\underline{u}-\varepsilon))_+}{b-\chi},\quad\forall\,\varepsilon>0.$$

Hence, one has

$$\overline{u} \le \frac{(a - \chi \underline{u})_+}{b - \chi} \tag{3.28}$$

By a similar argument, using

$$u_t \ge u_{xx} - \chi u_x v_x - \chi u(\overline{u}^\epsilon - u) + u(a - bu),$$

one obtains

$$\underline{u} \ge \frac{a - \chi \overline{u}}{b - \chi}. \tag{3.29}$$

Note that $a - \chi \underline{u} > 0$, for otherwise, by (3.28), we have $\overline{u} = 0$ and then $\underline{u} = 0$. But by (3.29), we have $\underline{u} \ge \frac{a}{b-\chi} > 0$, which yields a contradiction. Hence, one has $a - \chi \underline{u} > 0$. Then, employing (3.29), (3.28) we obtain

$$a - \chi \overline{u} + (b - \chi)\overline{u} \le (b - \chi)\underline{u} + (a - \chi \underline{u}),$$
$$(b - 2\chi)\overline{u} \le (b - 2\chi)\underline{u}.$$

Recalling $b > 2\chi$ and $\overline{u} \ge \underline{u}$ we obtain $\overline{u} = \underline{u}$. This identity together with (3.28) (resp. (3.29)) gives $\overline{u} \le \frac{a}{b}$ (resp. $\overline{u} \ge \frac{a}{b}$). Hence $\overline{u} = \underline{u} = \frac{a}{b}$, which in turn shows

$$\lim_{t \to \infty} \left\| u(t, \cdot; u_0) - \frac{a}{b} \right\|_{L^\infty(\Gamma)} = 0.$$

Similar inequality for v follows from (3.24). □

4. Numerical illustrations

In this section we provide numerical computation of bifurcation points for the following graphs:

- the dumbbell graph, see Figure 1, whose eigenvalues $\lambda = k^2$ can be determined from the secular equation

$$\sin\frac{\ell_1 k}{2} \sin\frac{\ell_3 k}{2}\left[\left(4\sin\frac{\ell_1 k}{2}\sin\frac{\ell_3 k}{2} - \cos\frac{\ell_1 k}{2}\cos\frac{\ell_3 k}{2}\right)\sin\frac{\ell_2 k}{2} - 2\cos\frac{\ell_2 k}{2}\sin\frac{(\ell_1 + \ell_3)k}{2}\right] = 0,$$

- the tadpole graph, see Figure 2, whose eigenvalues $\lambda = k^2$ can be determined from the secular equation

$$2\cos(\ell_1 k)\cos(\ell_2 k) - \sin(\ell_1 k)\sin(\ell_2 k) = 0,$$

- the figure 8 graph, see Figure 3, whose eigenvalues $\lambda = k^2$ can be determined from the secular equation

$$\sin\left(\frac{\ell_1 k}{2}\right)\sin\left(\frac{\ell_2 k}{2}\right)\sin\left(\frac{(\ell_1 + \ell_2)k}{2}\right) = 0,$$

- the 3-star graph, see Figure 4, whose eigenvalues $\lambda = k^2$ can be determined from the secular equation

$$\sin(\ell_1 k)\cos(\ell_2 k)\cos(\ell_3 k) + \cos(\ell_1 k)\sin(\ell_2 k)\sin(\ell_3 k) +$$
$$+ \sin(\ell_1 k)\sin(\ell_2 k)\cos(\ell_3 k) = 0.$$

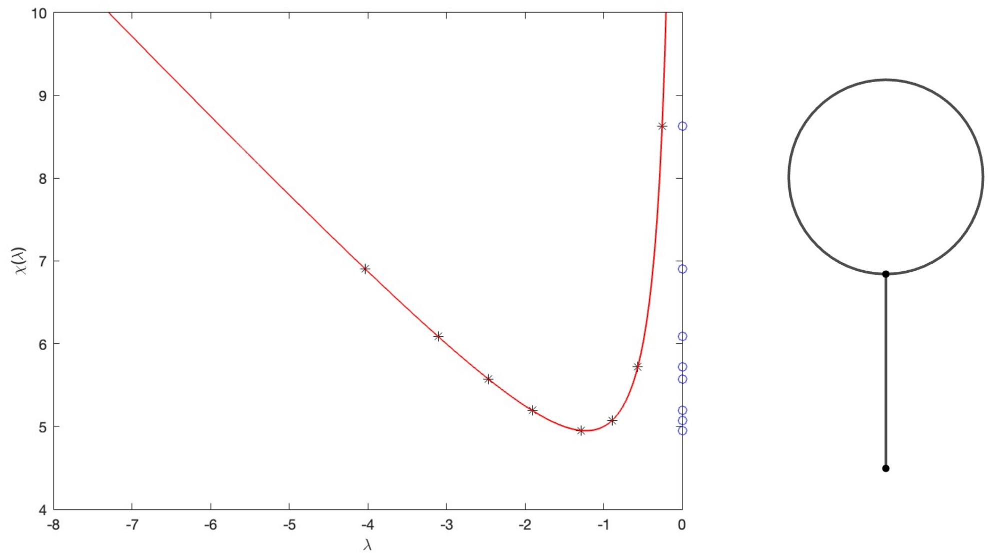

FIGURE 2. The red curve is the graph of $\chi = \chi(\lambda)$ with $a = b = 1.5$; $*$ indicates the values of χ at eigenvalues of the Kirchhoff Laplacian on a *tadpole graph* with edge lengths $10, 5$; $\circ$ indicate bifurcation points. The first bifurcation point $\chi^* \approx 4.95279$ corresponds to the $5-$th eigenvalue (we note that the first eigenvalue $\lambda = 0$ is not displayed).

APPENDIX A. AUXILIARY RESULTS

In this section we record several facts about fractional power spaces $\mathfrak{X}_p^\alpha$, $\alpha \in (0,1)$, $1 \le p < \infty$ generated by the Neumann–Kirchhoff Laplacian on compact metric graphs on $\Gamma = (\mathcal{V}, \mathcal{E})$. Let

$$L^p(\Gamma) := \bigoplus_{e\in\mathcal{E}} L^p(e) \quad \text{and} \quad \widehat{W}^{s,p}(\Gamma) := \bigoplus_{e\in\mathcal{E}} W^{s,p}(e). \tag{A.1}$$

By definition, see, e.g., [16, Section 1.3], $\mathfrak{X}_p^\alpha := \operatorname{dom}((I_{L^p(\Gamma)} - \Delta)^\alpha)$ is equipped with the graph norm of $(I_{L^p(\Gamma)} - \Delta)^\alpha$. Throughout this paper we used bounded embeddings of $\mathfrak{X}_p^\alpha$ into various function spaces. Such embeddings are well known in the case of classical domains $\Omega \subset \mathbb{R}^n$, cf. [16, Chapter 1]. To the best of our knowledge, these type of embedding for metric graphs, although expected, have not appeared in print. For completeness of exposition we present them in Theorem A.1 below. Let us first introduce some notation. The edge-wise direct sum of Banach spaces of functions will be denoted by $\widehat{\ }$, in particular, we write

$$\widehat{C}_0^\infty(\Gamma) := \bigoplus_{e\in\mathcal{E}} C_0^\infty(e),\ \widehat{C}^\nu(\overline{\Gamma}) := \bigoplus_{e\in\mathcal{E}} \widehat{C}^\nu(\overline{e}), \tag{A.2}$$

where $\nu > 0$ and $C^\nu(\overline{e})$ denotes the usual space of Hölder continuous functions defined on the closed interval $\overline{e}$ endowed with the norm

$$\|u\|_{C^\nu(\overline{e})} = \sum_{\alpha\in\mathbb{N}_0, \alpha\le[\nu]} \sup_{x\in\overline{e}} |u^{(\alpha)}(x)| + \sup_{x,y\in\overline{e}, x\ne y} \frac{|u^{([\nu])}(x) - u^{([\nu])}(y)|}{|x-y|^{\nu-[\nu]}}.$$

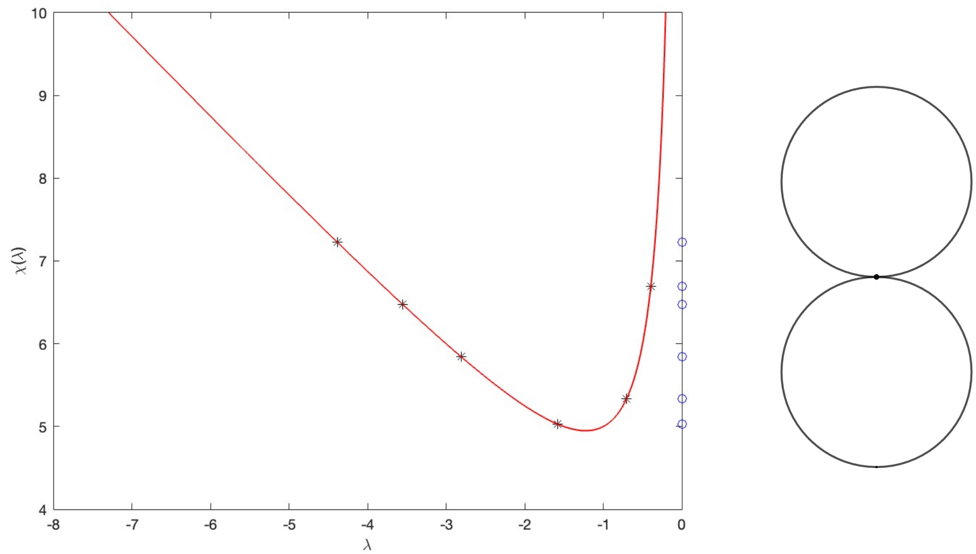

FIGURE 3. The red curve is the graph of $\chi = \chi(\lambda)$ with $a = b = 1.5$; $*$ indicates the values of χ at eigenvalues of the Kirchhoff Laplacian on a *figure* 8 *graph* with edge lengths $10, 5$; $\circ$ indicate bifurcation points. The first bifurcation point $\chi^* \approx 5.01774$ corresponds to the 4$-$th eigenvalue (we note that the first eigenvalue $\lambda = 0$ is not displayed).

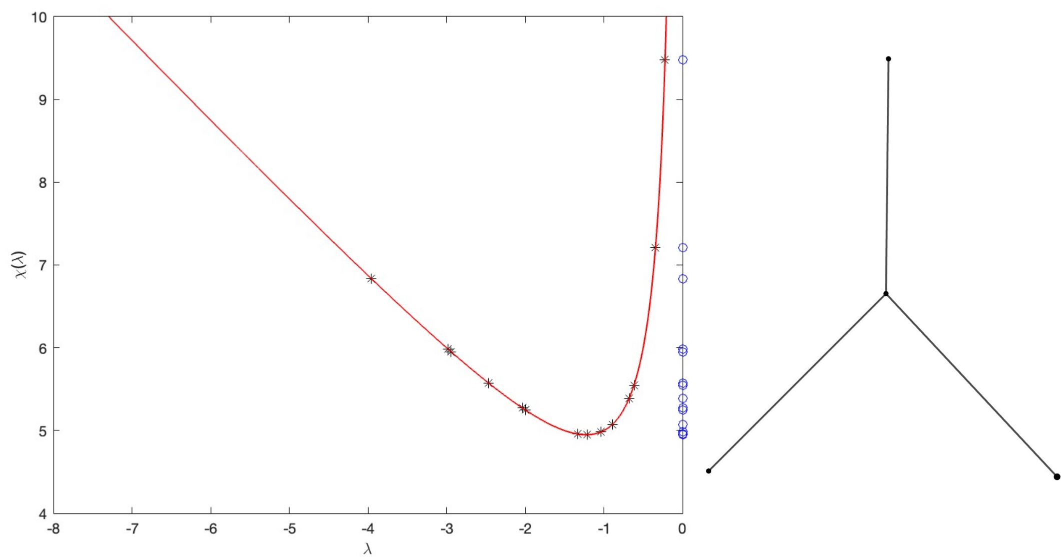

FIGURE 4. The red curve is the graph of $\chi = \chi(\lambda)$ with $a = b = 1.5$; $*$ indicates the values of χ at eigenvalues of the Kirchhoff Laplacian on a *3 star graph* with edge lengths $10, 5, 1$; $\circ$ indicate bifurcation points. The first bifurcation point $\chi^* \approx 4.94967$ corresponds to the 8$-$th eigenvalue (we note that the first eigenvalue $\lambda = 0$ is not displayed).

Let us note that the edge-wise direct sums introduced in (A.2) induce no vertex conditions as oppose to the space of continuous functions on the closure $\overline{\Gamma}$ of the graph

$$C(\overline{\Gamma}) = \{u \in \widehat{C}(\overline{\Gamma}) : u \text{ is continuous at the vertices of } \Gamma\}.$$

In the following theorem $(L^p(\gamma), \widehat{W}^{2,p}(\Gamma))_{\theta,q}$ denotes the interoplation space between $L^p(\Gamma)$ and $\widehat{W}^{2,p}(\Gamma)$ via the K-method, where $0 < \theta < 1$ and $1 \le q < \infty$ (see [32, Section 1.3.2] for definition).

Theorem A.1. *Suppose that $1 \le p < \infty$. Then one has*

(1) For any $q \ge p$ and $s - \frac{1}{p} > t - \frac{1}{q}$, there holds

$$\widehat{W}^{s,p}(\Gamma) \hookrightarrow \widehat{W}^{t,q}(\Gamma), \tag{A.3}$$

$$\widehat{W}^{s,p}(\Gamma) \hookrightarrow \widehat{C}^{r}(\overline{\Gamma}) \quad r < s - \frac{1}{p}, \tag{A.4}$$

and for $s \in (0,1) \setminus \{\frac{1}{2}\}$ one has

$$(L^p(\Gamma), \widehat{W}^{2,p}(\Gamma))_{s,p} = \widehat{W}^{2s,p}(\Gamma). \tag{A.5}$$

(2) One has

$$(\widehat{L}^p(\Gamma), X_p^\alpha)_{\theta,p} = (\widehat{L}^p(\Gamma), \mathcal{D}(A_p))_{\alpha\theta,p}, \quad 0 < \theta < 1, \tag{A.6}$$

$$X_p^\alpha \hookrightarrow \widehat{W}^{2\alpha\theta,p}(\Gamma), \quad 0 < \theta < 1, \tag{A.7}$$

and

$$X_p^\alpha \hookrightarrow \widehat{C}^{\nu}(\overline{\Gamma}), \quad 0 < \nu < 2\alpha - \frac{1}{p}. \tag{A.8}$$

Proof. (1) (A.3) and (A.4) follow from [1, Theorem 11.5], and (A.5) follows from [1, Theorem 11.6]. (2) First, (A.6) follows from [32, (unnumbered) Theorem on page 101].

To prove (A.7), for a given $0 < \nu < 2\alpha - \frac{1}{p}$, let us choose $\theta \in (0,1)$ such that $\frac{\alpha}{2}\theta \ne 1$ and $2\alpha\theta - \frac{1}{p} > \nu$. Then one has

$$\mathcal{X}_p^\alpha \subset (L^p(\Gamma), \mathcal{X}_p^\alpha)_{\theta,p} = (L^p(\Gamma), \mathcal{D}(A))_{\alpha\theta,p} \underset{(A.6)}{\subset} ((L^p(\Gamma), \widehat{W}^{2,p}(\Gamma))_{\alpha\theta,p} \underset{(A.5)}{=} \widehat{W}^{2\alpha\theta,p}(\Gamma).$$

The embedding (A.8) follows from (A.4) and (A.7). ☐

Proposition A.1. [15, Theorem 1.4.3]. *Let $\sigma > 0$, $\alpha \in [0,1)$, $p \in [1,\infty)$. Then there exists $C > 0$ such that for arbitrary $t > 0$ and $u \in L^p(\Gamma)$, $v \in \mathcal{X}^\alpha$ one has*

$$\begin{aligned} &\|(\sigma - \Delta)^\alpha e^{(\Delta-\sigma)t} u\|_{L^p(\Gamma)} \le C t^{-\alpha} e^{-\frac{\sigma}{2}t} \|u\|_{L^p(\Gamma)}, \\ &\|(e^{(\Delta-\sigma)t} - I)v\|_{L^p(\Gamma)} \le C t^\alpha \|v\|_{\mathcal{X}_p^\alpha}. \end{aligned} \tag{A.9}$$

Theorem A.2. [29]. *Let Γ be a compact metric graph, $1 \le p < \infty$, $q \in [p,\infty)$, then there exists $\sigma_0 = \sigma_0(\Gamma) > 0$ such that the following assertion holds.*

For every $\sigma > \sigma_0$ there exists $\delta \in (0,\sigma_0)$ and $C = C(\sigma_0, \delta, p, q, \Gamma) > 0$ such that for all $t > 0$ one has

$$\begin{aligned} &||e^{(\Delta-\sigma)t}||_{\mathcal{B}(L^p(\Gamma),L^q(\Gamma))} \le C e^{-\delta t} t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}, \\ &\|e^{(\Delta-\sigma)t}\partial_x\|_{\mathcal{B}(L^p(\Gamma),L^q(\Gamma))} \le C e^{-t\delta} t^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, \\ &\|\partial_x e^{(\Delta-\sigma)t} u\|_{\mathcal{B}(L^p(\Gamma),L^q(\Gamma))} \le C e^{-t\delta} t^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}. \end{aligned} \tag{A.10}$$

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