

# IDEALS AND STRONG AXIOMS OF DETERMINACY

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## Abstract

$\Theta$  is the least ordinal  $\alpha$  with the property that there is no surjection  $f : \mathbb{R} \rightarrow \alpha$ .  $\text{AD}_{\mathbb{R}}$  is the Axiom of Determinacy for games played on the reals. It asserts that every game of length  $\omega$  of perfect information in which players take turns to play reals is determined. An ideal  $\mathcal{I}$  on  $\omega_1$  is  $\omega_1$ -dense if the boolean algebra  $\wp(\omega_1)/\mathcal{I}$  has a dense subset of size  $\omega_1$ . We consider the theories, where CH stands for the Continuum Hypothesis,

(T<sub>1</sub>) ZFC + CH + “There is an  $\omega_1$ -dense ideal on  $\omega_1$ .”

(T<sub>2</sub>) ZF +  $\text{AD}_{\mathbb{R}}$  + “ $\Theta$  is a regular cardinal.”

The main result of this paper is that T<sub>1</sub> implies that the existence of a model of T<sub>2</sub>. Woodin, in unpublished work, showed that the consistency of T<sub>2</sub> implies the consistency of T<sub>1</sub>. We will also give a proof of this result, which, together with our main theorem, establish the equiconsistency of T<sub>1</sub> and T<sub>2</sub>.

As a consequence, this resolves part of question 12 in [35]; in particular, it shows that the theories (b) and (c) in [35, Question 12] are equiconsistent. Thus, our work completes the work that started by Woodin and Ketchersid in [5] some 25 years ago. We also establish other theorems of similar nature in this paper, showing the equiconsistency of T<sub>2</sub> and the statement that the nonstationary ideal on  $\wp_{\omega_1}(\mathbb{R})$  is strong and pseudo-homogeneous. The aforementioned results are the only known equiconsistency results at the level of  $\text{AD}_{\mathbb{R}}$  + “ $\Theta$  is a regular cardinal.”

## 1. INTRODUCTION

This paper studies the consistency of strong determinacy theories, specifically the theory

“ZF +  $\text{AD}_{\mathbb{R}}$  +  $\Theta$  is regular”

and the consistency of strong ideals on  $\omega_1$ , specifically  $\omega_1$ -dense ideals on  $\omega_1$ . The main theorems of the paper, Theorem 1.4 and Corollary 1.6, resolve a long-standing conjecture by Woodin in [35]. The work in this paper contributes to our understanding of and helps further establish the close connections between ideals and determinacy, two very seemingly different areas in set theory.

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## Some background

Famously, Ulam's investigations of the **Measure Problem**, which asks whether there is a measure on  $[0, 1]$ , led him to prove that there is no countably complete 0-1 measure, that is an *ultrafilter*, on  $\omega_1$  (e.g. [3, Chapter 10]). Ulam's theorem is often presented as showing that  $\omega_1$  is not a measurable cardinal, where we say that  $\kappa$  is a measurable cardinal if there is a  $\kappa$ -complete ultrafilter  $U$  on  $\kappa$ .

Ulam's theorem and the **Measure Problem** in general have been a source of great ideas in set theory, and one of these ideas has been the study of *ideals* that could induce nice ultrafilters on uncountable cardinals. Suppose, for example, that  $\mathcal{I} \subseteq \wp(\kappa)$  is an ideal on  $\kappa$ . Let  $\mathbb{P}_{\mathcal{I}} = \wp(\omega_1)/\mathcal{I}$  be the corresponding boolean algebra induced by  $\mathcal{I}$ . One can also think of  $\mathbb{P}_{\mathcal{I}}$  as a poset ordered by inclusion. It is not hard to see that if  $U$  is a generic object for  $\mathbb{P}_{\mathcal{I}}$ <sup>1</sup> then the function  $U^* : (\wp(\kappa))^V \rightarrow \{0, 1\}$  given by  $U^*(A) = 0 \iff A \notin U$  satisfies many of the properties of being a 0-1 measure with two major deficiencies. First  $U^*$  may not measure all subsets of  $\kappa$  that exist in  $V[U]$ , and second,  $U^*$  may not be countably complete. It is then unclear exactly in what way this approach could lead to a reasonable study of the **Measure Problem**.

The concept of ultrapower introduced the necessary formalism to eliminate the aforementioned issues. It is a well-known fact that a cardinal  $\kappa$  is a measurable cardinal if and only if there is an elementary embedding  $j : V \rightarrow M$  such that  $M$  is a transitive class of  $V$ ,  $j \neq id$ ,  $j \restriction \kappa = id$  and  $j(\kappa) > \kappa$ . If  $\kappa$  is a measurable cardinal then one obtains the  $M$  above as an ultrapower of  $V$  by a  $\kappa$ -complete ultrafilter on  $\kappa$ . The same can be done with our generic  $U$  above, and for the start one can only demand the well-foundedness of  $Ult(V, U)$ .

An ideal  $\mathcal{I}$  is called *precipitous* if whenever  $U \subseteq \mathbb{P}_{\mathcal{I}}$  is a generic ultrafilter, the generic ultrapower of  $V$  by  $U$ ,  $Ult(V, U)$ , is well-founded. This approach to the **Measure Problem** has been incredibly fruitful and has led to many great discoveries. The story has been partially told in Foreman's long manuscript [2]. The study of precipitous ideals has led to solutions of problems considered not just by set theorists but by wider mathematical community. For example, Theorem 5.42 of [2] states that the existence of a certain nice ideal implies among other things that every projective set of reals is Lebesgue measurable.

Let  $\mathcal{I}$  be an ideal on  $\omega_1$ . We write  $\mathcal{I}^+$  for the collection of  $\mathcal{I}$ -positive sets and  $\mathcal{F}_{\mathcal{I}}$  for the dual filter of  $\mathcal{I}$ .  $\mathcal{I}$  is  $\kappa$ -saturated if there is no family  $(S_i : i < \kappa)$  of sets in  $\wp(\omega_1) \setminus \mathcal{I}$  such that  $S_i \cap S_j \in \mathcal{I}$  for all  $i \neq j$ ; in other words,  $\mathcal{I}$  is  $\kappa$ -saturated if there is no antichain in  $\mathbb{P}_{\mathcal{I}}$  of size  $\kappa$ . A  $\kappa$ -complete ideal  $\mathcal{I}$  is *saturated* if it is  $\kappa^+$ -saturated. *Presaturation* is a technical weakening of saturation. An ideal  $\mathcal{I}$  on  $\omega_1$  is *presaturated* if for any  $A \in \wp(\omega_1) \setminus \mathcal{I}$ , any sequence of maximal antichains  $(\mathcal{A}_i : i < \omega)$  in  $\wp(\omega_1)/\mathcal{I}$ , there is  $B \subseteq A$  such that  $B \notin \mathcal{I}$  and such that for each  $i < \omega$ ,  $\{X \in \mathcal{A}_i : X \cap B \notin \mathcal{I}\}$  has cardinality at most  $\omega_1$ . The reader can consult Foreman's paper [2] and Woodin's book [35] for more comprehensive discussions on the topic of ideals, which is an important area of research in modern set theory.

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<sup>1</sup>I.e. intersects all dense open subsets of  $\mathbb{P}_{\mathcal{I}}$ .

Shelah, Jensen, and Steel have established the following famous theorem, which is one of the first equiconsistency results that connects ideals and large cardinals.

**Theorem 1.1.** *The following theories are equiconsistent.*

1.  $\text{ZFC} + \text{There is a pre-saturated ideal on } \omega_1.$
2.  $\text{ZFC} + \text{There is a saturated ideal on } \omega_1.$
3.  $\text{ZFC} + \text{There is a Woodin cardinal.}$

Shelah proves the consistency of 1 and 2 from the consistency of 3 by forcing techniques. Jensen and Steel prove the converse using inner model theoretic techniques, in particular core model theory. See, for example, [21, 4].

Claverie and Schindler [1] have improved the above result and shown that in fact theory 3 above is equiconsistent with the theory “there is a strong ideal on  $\omega_1$ ”.<sup>2</sup>

Below we say that the ideal  $\mathcal{I} \subseteq \wp(\omega_1)$  is  $\omega_1$ -dense if  $\mathbb{P}_{\mathcal{I}}$  has a dense set of size  $\omega_1$ .  $\omega_1$ -density is a stronger property than saturation. The consistency question of  $\omega_1$ -dense ideals has been studied extensively in the last 25 years, starting with Woodin [35] and by various other authors in [5, 14]. Unlike saturation and presaturation of ideals on  $\omega_1$ , which can be forced from a relatively mild large cardinal like a Woodin cardinal, there is no known traditional forcing construction of an  $\omega_1$ -dense ideal from a large cardinal significantly weaker than a supercompact cardinal.<sup>3</sup> The Axiom of Determinacy (AD) comes into the picture in a rather surprising and dramatic fashion via the following remarkable theorem of Woodin.

Recall that AD states that every infinite-length, two-person game of perfect information where players take turns to play integers is determined, i.e. one of the players has a winning strategy. It is worth noting that AD is equiconsistent with “ZFC + there are  $\omega$  many Woodin cardinals” and the latter theory is much weaker than a supercompact cardinal. A (stronger) variation of AD is  $\text{AD}_{\mathbb{R}}$ , which is like AD except the players are allowed to play reals. The theory “ZF +  $\text{AD}_{\mathbb{R}}$  +  $\Theta$  is regular” is strictly stronger than  $\text{ZF} + \text{AD}_{\mathbb{R}}$ ; here  $\Theta$  is the supremum of ordinals  $\alpha$  for which there is a surjection from  $\mathbb{R}$  onto  $\alpha$ . If the axiom of choice holds, then  $\Theta = \mathfrak{c}^+$ , the successor of the continuum. If AD holds, then  $\Theta$  is a limit of measurable cardinals and more. In the following theorem and this paper,  $L(\mathbb{R})$  is the minimal model of ZF that contains all the ordinals and the real numbers (see [14, Theorem 2.11.1]).

**Theorem 1.2** (Woodin). *The following theories are equiconsistent.*

1.  $\text{ZFC} + \text{“There is an } \omega_1\text{-dense ideal on } \omega_1\text{”}.$
2.  $\text{AD holds in } L(\mathbb{R}).$

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<sup>2</sup>The property of being strong is weaker than being presaturated. Strong ideals are precipitous.

<sup>3</sup>Woodin constructs  $\omega_1$ -dense ideals from an almost huge cardinal (cf. [2]). Recently, Andreas Lietz has constructed  $\omega_1$ -dense ideals from the existence of a supercompact cardinal, which is a weaker hypothesis than almost hugeness.

Woodin introduces two very important sets of techniques in the proof of the above theorem. In one direction, to show the consistency of “ZFC + There is an  $\omega_1$ -dense ideal on  $\omega_1$ ”, he develops powerful and general forcing techniques over models of determinacy, i.e.  $\mathbb{P}_{\max}$  and its cousins (cf. [35]).<sup>4</sup> To prove the other direction, *the core model induction* (CMI) technique was introduced. CMI is a general technique for obtaining lower-bound consistency by inductively proving determinacy in canonical models like  $L(\mathbb{R})$ . CMI has since then been developed further into a very powerful and versatile method for proving lower-bound consistency and equiconsistency results (see for example [14, 32, 34, 27]) from a variety of hypotheses. Part of this development is in understanding determinacy models beyond  $L(\mathbb{R})$ <sup>5</sup> and their canonical inner models of large cardinals (like the HOD).

The aforementioned theorems of Shelah, Jensen, Steel, Woodin, and others demonstrate intimate connections between different branches of set theory, namely the study of precipitous ideals and the study of models of AD. It seems that the connections that theorems like above establish are rooted in the naturalness of the constructions that produce the models of one theory given a model of another, and this naturalness – the feeling of having no barriers to naturally drift from one theory to another as if they were one and the same theory – is not fully expressed in the formal statement of the theorem, namely that the two theories are equiconsistent. We discuss this a bit more later in this section.

The main theorem of this paper, Theorem 1.4, has the same spirit as Woodin’s theorem above.

**Definition 1.3** (DI). Let DI be the conjunction of

- CH,
- there is an  $\omega_1$ -dense ideal  $\mathcal{I}$  on  $\omega_1$ .

⊢

For a sentence  $\phi$  in the language of set theory, we say that  $M$  is the minimal model of “ZF +  $\text{AD}_{\mathbb{R}}$  +  $\phi$ ” if  $M$  is a transitive model of “ZF +  $\text{AD}_{\mathbb{R}}$  +  $\phi$ ” containing all reals and ordinals, and whenever  $N$  is a transitive model of “ZF +  $\text{AD}_{\mathbb{R}}$  +  $\phi$ ” containing all reals and ordinals then  $M \subseteq N$ .

**Theorem 1.4.** ZFC + DI implies that the minimal model of ZF +  $\text{AD}_{\mathbb{R}}$  + “ $\Theta$  is a regular cardinal” exists.

The proof of [7, Theorem 6.26] explicitly establishes that the existence of divergent models of  $\text{AD}^+$  implies their common part is beyond a model of ZF +  $\text{AD}_{\mathbb{R}}$  + “ $\Theta$  is a regular cardinal”. Thus, if there is a model of ZF +  $\text{AD}_{\mathbb{R}}$  + “ $\Theta$  is a regular cardinal” then there is a minimal one. As a result of this theorem and Woodin’s unpublished work, which we will present in Section 2, we obtain the following equiconsistency result.

**Theorem 1.5.** The following theories are equiconsistent.

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<sup>4</sup>This work was partly inspired by previous work by Steel and Van Wesep [29].

<sup>5</sup>More precisely,  $\text{AD}^+$  models.  $\text{AD}^+$  is a technical strengthening of AD.

1.  $\text{ZFC} + \text{DI}$

2.  $\text{ZF} + \text{AD}_{\mathbb{R}} + “\Theta \text{ is a regular cardinal.}”$

Let  $NS_{\omega_1}$  be the nonstationary ideal on  $\omega_1$  and  $(T)$  be the theory

$$\text{ZFC} + \text{CH} + “NS_{\omega_1} \upharpoonright S \text{ is } \omega_1\text{-dense for a dense set of } S \in \wp(\omega_1) \setminus NS_{\omega_1}”.$$

Woodin, unpublished, has shown that  $\text{Con}((T))$  follows from  $\text{Con}(\text{ZF} + \text{AD}_{\mathbb{R}} + “\Theta \text{ is regular.}”)$ . This result and Theorem 1.5 immediately show

**Corollary 1.6.** *The following theories are equiconsistent.*

1.  $\text{ZF} + \text{AD}_{\mathbb{R}} + “\Theta \text{ is regular}”.$

2.  $(T)$ .

This confirms that theories (b) and (c) in [35, Question 12] are indeed equiconsistent. Below we give some more motivations for proving such theorems.

### Some definitions and more results.

For any set  $X$ , let  $\wp_{\omega_1}(X)$  be the set of countable subsets of  $X$ . Let  $\mathcal{I}$  be an ideal on  $\wp_{\omega_1}(\mathbb{R})$ . We let  $\mathcal{I}^+$  and  $\mathcal{F}_{\mathcal{I}}$  be as before and let  $\mathbb{P}_{\mathcal{I}}$  be the boolean algebra  $\wp(\wp_{\omega_1}(\mathbb{R}))/\mathcal{I}$ . Let  $\mathfrak{c}$  denote the size of the continuum.

**Definition 1.7.** An ideal  $\mathcal{I}$  on  $\omega_1$  or on  $\wp_{\omega_1}(\mathbb{R})$  is *precipitous* if whenever  $G \subseteq \mathbb{P}_{\mathcal{I}}$  is a  $V$ -generic ultrafilter, the generic ultrapower  $\text{Ult}(V, G)$  induced by  $G$  is well-founded.  $\dashv$

**Definition 1.8.** An ideal  $\mathcal{I}$  on  $\wp_{\omega_1}(\mathbb{R})$  is *strong* if

(a)  $\mathcal{I}$  is precipitous, and

(b) whenever  $G \subseteq \mathbb{P}_{\mathcal{I}}$  is  $V$ -generic, letting  $j_G : V \rightarrow \text{Ult}(V, G)$  be the ultrapower map, then  $j_G(\omega_1) = \mathfrak{c}^+$ .  $\dashv$

**Definition 1.9.** An ideal  $\mathcal{I}$  on  $\wp_{\omega_1}(\mathbb{R})$  is *pseudo-homogeneous* if for every  $\alpha \in \text{ON}$ ,  $s \in \text{ON}^{\omega}$ ,  $\lambda < \mathfrak{c}^+$ , and formula  $\theta$  in the language of set theory, letting  $G \subseteq \mathbb{P}_{\mathcal{I}}$  be a  $V$ -generic filter and  $j_G : V \rightarrow \text{Ult}(V, G)$  the corresponding ultrapower map, the truth of the statement

$$\text{Ult}(V, G) \models \theta[\alpha, j_G(s), j_G[\lambda^{\omega}]]$$

is independent of the choice of  $G$ .  $\dashv$

We obtain an equiconsistency regarding strong, pseudo-homogeneous ideals on  $\wp_{\omega_1}(\mathbb{R})$ .

**Theorem 1.10.** *The following are equiconsistent.*

1.  $\text{ZFC} + \text{“The nonstationary ideal on } \wp_{\omega_1}(\mathbb{R}) \text{ is strong and pseudo-homogeneous.”}$
2.  $\text{ZF} + \text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is a regular cardinal.”}$

## Motivations

Motivated by the success of the generic elementary embeddings induced by ideals or other similar structures, Foreman has suggested them as a possible foundational framework, and explicated his ideas in [2, Chapter 11]. As is well known, the basic foundational issue that set theory is facing is its inability to produce a single foundational framework that is accepted by all and at the same time solves all fundamental problems including the **Continuum Hypothesis**. Several successful foundational frameworks, such as **Forcing Axioms**, **Canonical Inner Models** and **Generic Embeddings**, have been proposed and developed, but they all seem to disagree on basic questions such as whether the **Continuum Hypothesis** is true or whether the universe is a ground (i.e., cannot be obtained as a non-trivial forcing extension of an inner model) and on many other such fundamental questions.

One of the main goals of CMI is to unify all of these frameworks by showing that each can be naturally interpreted in another. Given such bi-interpretations, disagreements on fundamental questions can be traced to subjective preferences in one framework over another, or preferences in one type of formalism over another.

For example, Woodin’s theorem (Theorem 1.2) and Theorem 1.5 show how to interpret natural ideas occurring in the study of generic embeddings in models of determinacy and vice versa. The reason is that, in both cases, the forcing notion used to obtain the models carrying such ideals are natural forcing notions, and in the other direction, the models of determinacy built in both cases are natural canonical models of AD. This sort of bi-interpretability demonstrates that one cannot have scientifically objective reasons for preferring generic embeddings over, say, determinacy axioms, as they are deeply interconnected: commitment to one entails commitment to the other. A bias towards a particular formalism can be justified by other more pragmatic ways, for example by insisting on the shortest or clearest or most natural possible proofs of certain desired theorems. The ideas explicated above are the motivational ideas behind proving theorems like the main theorem of this paper.<sup>6</sup> For a more detailed discussion of CMI and its role in set theory, readers may consult [11].

## The history behind the paper.

The first written presentation of CMI is Ketchersid’s PhD thesis [5], which motivated Ralf Schindler and John Steel to work on a book presenting the **Core Model Induction** (see [14]). In 2006 they organized a seminar in Berlin covering the basics of CMI. As one can see by flipping through [14], one of the main directions pursued by the community at this time was to complete Ketchersid’s project. See John Steel’s [23] for a conjecture along the same vein.

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<sup>6</sup>The authors first learned about these ideas from John Steel.

One of the main reasons this was believed to be important was that it was not known and still is not known how to force DI, clause 1 of Theorem 1.5, from conventional large cardinals that are weaker than supercompact cardinals. Woodin forced DI both over the models of  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” and from an almost huge cardinal (see [2, Chapter 7.14]). In [35], Woodin also forced  $\text{MM}^{++}(\mathfrak{c})$ , Martin’s Maximum for forcing posets of size at most the continuum, over a model of  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” (see [35, Theorem 9.40]), and just like with DI, it is not known how to force  $\text{MM}^{++}(\mathfrak{c})$  from conventional large cardinals much weaker than a supercompact cardinal. These and other results of Woodin from [35] seem to suggest that the theory  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” is in the region of supercompact cardinals, and the project of getting a model of it via CMI seemed to be equivalent to getting canonical inner models that could have supercompact cardinals in it, which has been one of the Holy Grails of set theory.

However, [7] showed that in fact the theory  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” is much weaker than a supercompact cardinal: it is weaker than a Woodin cardinal that is a limit of Woodin cardinals (see [7, Theorem 6.26]). This theorem seems to suggest the existence of a gap in our understanding of models of set theory. On the one hand, the conventional forcing and large cardinal technology that is needed to force statements such as DI or  $\text{MM}^{++}(\mathfrak{c})$  requires the complexity of a supercompact cardinal or beyond, and on the other hand, equally natural but different technologies based on [35] place the complexity far below a supercompact cardinal. This phenomenon has not yet found a proper explanation.

While [7] did show that finishing Ketchersid’s project will not lead to one of the Holy Grails of set theory, the importance of the project didn’t diminish, as it was perceived to be one of the main guiding problems for developing the CMI to a technique for producing models of  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” and beyond<sup>7</sup>. In this direction, the last chapter of the second author’s thesis [12] gave a rough outline of producing models of  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” from a strengthening of DI,<sup>8</sup> but later on a substantial error was discovered in the proof by Steel and the third author. The concept of *embeddings with condensation* introduced in [8] (see [8, Definition 11.14, Lemma 11.15]) and further developed in [32] (see [32, Definition 3.81, Lemma 3.82]) and [10] seemed good enough for correcting the aforementioned error, which is what we will do in this paper (see Theorem 6.17). However, to obtain Theorem 1.4, more substantial ideas beyond this need to be developed.

Furthermore, the fourth author, in his thesis [34], developed techniques for handling the successor stages of CMI that avoid the famous “*A*-iterability” proofs (see [14, Theorem 5.4.8] or [22, Theorem 1.46]) and various other complicated arguments originally due to Woodin. We adapt the fourth author’s arguments to our current context (see Section 5). The second, third, and fourth authors established the consistency of  $\text{ZF} + \text{AD}_{\mathbb{R}} + “\Theta$  is regular” from the aforementioned strengthening of DI in 2020-2021; this completes the project started by Ketchersid in [5]. The obvious question is how to get rid of the technical assumption used in the second author’s thesis, as

<sup>7</sup>See for example [10] for an analysis of determinacy models stronger than those of  $\text{AD}_{\mathbb{R}} + “\Theta$  is a regular cardinal” and core model induction techniques for constructing such models from strong theories like PFA.

<sup>8</sup>The strengthening is DI plus the statement: letting  $\mathcal{I}$  be an  $\omega_1$ -dense ideal  $\omega_1$ , the generic embedding induced by any generic  $G \subseteq \mathbb{P}_{\mathcal{I}}$  when restricted to the ordinals is amenable to  $V$ .



mentioned in Footnote 8. The first and fifth authors joined the ongoing work in 2022 and finished the project. The result of these collaborations is Theorem 1.4.

As mentioned above, it is a well-known unpublished theorem of Woodin that one can force DI over models of  $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is a regular cardinal”}$ . The fourth author forced some more general statements about ideals in his thesis, and we will use his argument to give a proof of this theorem of Woodin in Subsection 2.3 below. Thus, this paper presents a self-contained proof of Theorem 1.5, giving the proof of both directions in as much detail as it is possible to do in a research article. Theorem 1.5 and Theorem 1.10 are currently the only known equiconsistency results at the level of  $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is a regular cardinal”}$ .

The paper is organized as follows. In Section 2, we summarize basic facts about ideals and  $\text{AD}^+$  we need in this paper and show that DI and the existence of a strong, pseudo-homogeneous ideal on  $\wp_{\omega_1}(\mathbb{R})$  are consistent relative to  $\text{“AD}_{\mathbb{R}} + \Theta \text{ is a regular cardinal.”}$ <sup>9</sup> In Section 3, we summarize preliminaries and basic notions we need for the CMI in this paper. Section 4 outlines the proof of Theorem 1.4. Sections 5 and 6 fill in the details of the outline and complete the proof of Theorem 1.4, obtaining models of  $\text{“AD}_{\mathbb{R}} + \Theta \text{ is a regular cardinal”}$  from  $\text{ZFC} + \text{DI}$ . In Section 7, we outline the argument obtaining models of  $\text{“AD}_{\mathbb{R}} + \Theta \text{ is a regular cardinal”}$  from the assumption that the nonstationary ideal on  $\wp_{\omega_1}(\mathbb{R})$  is strong and pseudo-homogeneous. Since the argument is very similar to the argument from DI, we simply focus on the main changes, leaving the details to the reader. In the following, we will often write  $\text{“}\Theta \text{ is regular”}$  for  $\text{“}\Theta \text{ is a regular cardinal”}$ .

**Acknowledgments.** The work here is greatly influenced by Ketchersid’s work in his thesis [5], which in turn is greatly influenced by Woodin’s early work in the CMI. We are grateful to them for their inspiring work in this direction. We are also grateful to Woodin for his permission to include the proof of his unpublished work which shows that  $\text{Con}(\text{ZF} + \text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is a regular cardinal”})$  implies  $\text{Con}(\text{ZFC} + \text{DI})$ . The third author is grateful to the NSF for its generous support via Career Award DMS-1945592.

## 2. DENSE IDEALS AND STRONG PSEUDO-HOMOGENEOUS IDEALS FROM MODELS OF $\text{AD}_{\mathbb{R}} + \Theta \text{ IS REGULAR}$

In this section, we show the consistency of  $\text{ZFC} + \text{DI}$  and of the existence of a strong, pseudo-homogeneous ideal on  $\wp_{\omega_1}(\mathbb{R})$  from  $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular.”}$ <sup>10</sup> We first review basic facts about  $\text{AD}^+$  and ideals. In Subsection 2.3, we will give the consistency proof.

### 2.1. Basic facts about $\text{AD}^+$

We start with the definition of Woodin’s theory of  $\text{AD}^+$ . In this paper, we identify  $\mathbb{R}$  with  $\omega^\omega$ . We use  $\Theta$  to denote the sup of ordinals  $\alpha$  such that there is a surjection  $\pi : \mathbb{R} \rightarrow \alpha$ . Under AC,  $\Theta$  is

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<sup>9</sup>We adapt the proof given in the fourth author’s thesis here. We note the result that  $\text{Con}(\text{ZFC} + \text{DI})$  follows from  $\text{Con}(\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”})$  is due to Woodin.

<sup>10</sup>In fact we show the nonstationary ideal on  $\wp_{\omega_1}(\mathbb{R})$  has these properties.



just the successor cardinal of the continuum. In the context of AD, the cardinal  $\Theta$  is shown to be the supremum of  $w(A)$ <sup>11</sup> for  $A \subseteq \mathbb{R}$  (cf. [20]). The definition of  $\Theta$  relativizes to any determined pointclass  $\Gamma$  with sufficient closure properties, and we may write  $\Theta^\Gamma$  for the supremum of ordinals  $\alpha$  such that there is a surjection from  $\mathbb{R}$  onto  $\alpha$  coded by a set of reals in  $\Gamma$ .

**Definition 2.1.**  $\text{AD}^+$  is the theory  $\text{ZF} + \text{AD} + \text{DC}_{\mathbb{R}}$  plus the following two statements:

1. For every set of reals  $A$ , there are a set of ordinals  $S$  and a formula  $\varphi$  such that  $x \in A \iff L[S, x] \models \varphi[S, x]$ . The pair  $(S, \varphi)$  is called an  $\infty$ -Borel code for  $A$ .
2. For every  $\lambda < \Theta$ , every continuous  $\pi : \lambda^\omega \rightarrow \omega^\omega$ , and every set of reals  $A$ , the set  $\pi^{-1}[A]$  is determined.

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$\text{AD}^+$  is equivalent to  $\text{AD} +$  “the set of Suslin cardinals is closed below  $\Theta$ .” Another, perhaps more useful, characterization of  $\text{AD}^+$  is  $\text{AD} +$  “ $\Sigma_1$  statements reflect into the Suslin co-Suslin sets” (see [26] for the precise statement).

For  $A \subseteq \mathbb{R}$ , we let  $\theta_A$  be the supremum of all  $\alpha$  such that there is an  $OD(A)$ <sup>12</sup> surjection from  $\mathbb{R}$  onto  $\alpha$ . If  $\Gamma$  is a determined pointclass and  $A \in \Gamma$ , we write  $\Gamma \restriction A$  for the set of all  $B \in \Gamma$  that are Wadge reducible to  $A$ . If  $\alpha < \Theta^\Gamma$ , we write  $\Gamma \restriction \alpha$  for the set of all  $A \in \Gamma$  with Wadge rank strictly less than  $\alpha$ .

**Definition 2.2** ( $\text{AD}^+$ ). The **Solovay sequence** is the sequence  $\langle \theta_\alpha \mid \alpha \leq \lambda \rangle$  where

1.  $\theta_0$  is the supremum of ordinals  $\beta$  such that there is an  $OD$  surjection from  $\mathbb{R}$  onto  $\beta$ ;
2. if  $\alpha > 0$  is limit, then  $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$ ;
3. if  $\alpha = \beta + 1$  and  $\theta_\beta < \Theta$  (i.e.  $\beta < \lambda$ ), fixing a set  $A \subseteq \mathbb{R}$  of Wadge rank  $\theta_\beta$ ,  $\theta_\alpha$  is the sup of ordinals  $\gamma$  such that there is an  $OD(A)$  surjection from  $\mathbb{R}$  onto  $\gamma$ , i.e.  $\theta_\alpha = \theta_A$ .

+

Note that the definition of  $\theta_\alpha$  for  $\alpha = \beta + 1$  in Definition 2.2 does not depend on the choice of  $A$ . One can also make sense of the Solovay sequence of pointclasses that may not be constructibly closed. Such pointclasses show up in core model induction applications. The Solovay sequence  $(\theta_\alpha : \alpha < \gamma)$  of a pointclass  $\Omega$  with the property that if  $A \in \Omega$ , then  $L(A, \mathbb{R}) \models \text{AD}^+$  and  $\wp(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \Omega$  is defined as follows. First,  $\theta_0$  is the supremum of all  $\alpha$  such that there is some  $A \in \Omega$  and some  $OD^{L(A, \mathbb{R})}$  surjection  $\pi : \mathbb{R} \rightarrow \alpha$ . If  $\lambda < \gamma$  is limit, then  $\theta_\gamma = \sup_{\alpha < \lambda} \theta_\alpha$ . If  $\theta_\alpha$  has been defined and  $\alpha + 1 < \gamma$ , then letting  $A \in \Omega$  be of Wadge rank  $\theta_\alpha$ ,  $\theta_{\alpha+1}$  is the supremum of  $\beta$  such that there is some  $B \in \Omega$  and some  $OD(A)^{L(B, \mathbb{R})}$  surjection  $\pi : \mathbb{R} \rightarrow \beta$ .

<sup>11</sup> $w(A)$  is the Wadge rank of  $A$ .

<sup>12</sup> $x$  is  $OD(A)$  if there is a formula  $\phi$  and a finite set of ordinals  $s$  such that  $x$  is the unique  $y$  satisfying the formula  $\phi(y, s, A)$ .

Roughly speaking, the longer the Solovay sequence is, the stronger the associated  $\text{AD}^+$ -theory is. The minimal model of  $\text{AD}^+$  is  $L(\mathbb{R})$ , which satisfies  $\Theta = \theta_0$ . The theory  $\text{AD}^+ + \text{AD}_{\mathbb{R}}$  implies that the Solovay sequence has limit length. The theory  $\text{AD}_{\mathbb{R}} + \text{DC}$  is strictly stronger than  $\text{AD}_{\mathbb{R}}$  since by [20], DC implies  $\text{cof}(\Theta) > \omega$  whereas the minimal model<sup>13</sup> of  $\text{AD}_{\mathbb{R}}$  satisfies  $\Theta = \theta_\omega$ . The theory “ $\text{AD}_{\mathbb{R}} + \Theta$  is regular” is much stronger still, as it implies the existence of many models of  $\text{AD}_{\mathbb{R}} + \text{DC}$ . We end this section with a theorem of Woodin, which produces models with Woodin cardinals from  $\text{AD}^+$ . The theorem is important in the HOD analysis of such models.

**Theorem 2.3** (Woodin, see [6]). *Assume  $\text{AD}^+$ . Let  $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$  be the Solovay sequence. Suppose  $\alpha = 0$  or  $\alpha = \beta + 1$  for some  $\beta < \Omega$ . Then  $\text{HOD} \models \theta_\alpha$  is Woodin.*

## 2.2. Basic properties of ideals

We summarize standard facts about ideals that we will need in this paper. See for example [35] and [3] for a more detailed discussion.

Suppose  $\mathcal{I}$  is an ideal on a set  $X$ . We say that  $\mathcal{I}$  is *countably complete* if whenever  $\{A_n : n < \omega\}$  are sets in  $\mathcal{I}$  then  $\bigcup_{n < \omega} A_n \in \mathcal{I}$ . Supposing  $X$  is a cardinal (e.g.  $X = \omega_1$ ), we say  $\mathcal{I}$  is *normal* if whenever  $\{A_x : x \in X\} \subset \mathcal{I}$  then the diagonal union  $\{x \in X : \exists y \in x (x \in A_y)\} \in \mathcal{I}$ . All ideals  $\mathcal{I}$  on a cardinal considered in this paper will be assumed countably complete and normal.

Suppose  $\mathcal{I}$  is an  $\omega_1$ -dense ideal on  $\omega_1$ . The following are standard facts; see [35, Definition 6.19] and the discussion after it.

**Fact 2.4.** (i)  $\mathbb{P}_{\mathcal{I}}$  is a homogeneous forcing.<sup>14</sup>

(ii) There is a boolean isomorphism  $\pi : \mathbb{P}_{\mathcal{I}} \rightarrow \text{RO}(\text{Coll}(\omega, \omega_1))$ <sup>15</sup>. In particular,  $\mathbb{P}_{\mathcal{I}}$  is forcing equivalent to  $\text{Coll}(\omega, \omega_1)$ .

(iii) For any  $V$ -generic filter  $G \subset \text{Coll}(\omega, \omega_1)$ ,  $\pi$  induces a  $V$ -generic filter  $H \subset \mathbb{P}_{\mathcal{I}}$ , and letting  $j : V \rightarrow M =_{\text{def}} \text{Ult}(V, H) \subset V[H]$  be the associated generic ultrapower map, we have:

(a)  $j(f)(\omega_1^V) = G$  for some  $f : \omega_1 \rightarrow H_{\omega_1}$ ; in particular,  $V[H] = V[G]$ .

(b)  $j(\omega_1^V) = \omega_2^V$ .

(c)  $M$  is well-founded and  $M^\omega \subset M$  in  $V[H]$ .

Let  $\mathcal{I}$  be an  $\omega_1$ -dense ideal on  $\omega_1$ . For any  $V$ -generic  $g \subset \mathbb{P}_{\mathcal{I}} =_{\text{def}} \wp(\omega_1)/\mathcal{I}$ , let  $j_g : V \rightarrow M = \text{Ult}(V, g)$  be the associated ultrapower map. We fix a Boolean isomorphism  $\pi : \mathbb{P}_{\mathcal{I}} \rightarrow \text{RO}(\text{Coll}(\omega, \omega_1))$  as in Fact 2.4 and let  $G \subset \text{Coll}(\omega, \omega_1)$  be such that  $g$  is induced from  $G$  via  $\pi$ . When  $g$  is clear from the context, we will write  $j$  for  $j_g$ .

<sup>13</sup>From here on, whenever we talk about “models of  $\text{AD}^+$ ”, we always mean transitive models of  $\text{AD}^+$  that contain all reals and ordinals.

<sup>14</sup>A forcing  $\mathbb{P}$  is homogeneous if whenever  $p, q \in \mathbb{P}$ , there is an automorphism  $\sigma : \mathbb{P} \rightarrow \mathbb{P}$  such that  $\sigma(p)$  is compatible with  $q$ .

<sup>15</sup> $\text{RO}(\text{Coll}(\omega, \omega_1))$  is the regular open algebra of  $\text{Coll}(\omega, \omega_1)$ .

We say that a set of reals  $A$  is  $\omega_1$ -*universally Baire* (or  $\omega_1$ -UB) if there is some ordinal  $\gamma$  and a pair of trees  $T, U$  on  $\omega \times \gamma$  such that  $A = p[T] = \mathbb{R} - p[U]$  and for any forcing  $\mathbb{P}$  of size  $\leq \omega_1^V$ , for any  $V$ -generic  $h \subset \mathbb{P}$ , in  $V[h]$ ,  $p[T] = \mathbb{R} - p[U]$ . Here  $p[T] = \{x \in \mathbb{R} : \exists f \in \gamma^\omega (x, f) \in [T]\}$ .

**Lemma 2.5.** *Let  $\mathbb{P}_{\mathcal{I}}, g, G, M$  be as above. Suppose  $A \subset \mathbb{R}$  is  $\omega_1$ -UB as witnessed by trees  $(T, U)$ , then in  $V[G]$ ,  $p[T] = p[j_g(T)]$  and  $p[U] = p[j_g(U)]$ .*

*Proof.* We write  $j$  for  $j_g$ . Clearly,  $p[T] \subseteq p[j(T)]$  and  $p[U] \subseteq p[j(U)]$ . In  $M$ , equivalently in  $V[G]$ ,

$$p[j(T)] = \mathbb{R} - p[j(U)].$$

This follows from elementarity of  $j$ , the fact that in  $V$ ,  $p[T] = \mathbb{R} - p[U]$ , and property (c) of Fact 2.4.

By the fact that  $T, U$  witness  $A$  is  $\omega_1$ -UB and  $\text{Coll}(\omega, \omega_1)$  has size  $\omega_1$ , in  $V[G]$ ,  $p[T] = \mathbb{R} - p[U]$ . We must then get  $p[T] = p[j(T)]$  and  $p[U] = p[j(U)]$ .  $\square$

**Remark 2.6.** It follows easily from Lemma 2.5 that if  $(T, U), (T', U')$  witness  $A$  is  $\omega_1$ -UB then in  $V[g]$ ,  $p[T] = p[T']$  and  $p[U] = p[U']$ . Therefore, we can unambiguously write  $A_g$  for  $p[T] \cap V[g]$  as the canonical interpretation of  $A$  in  $V[g]$ .

Suppose  $X = \wp_{\omega_1}(Y)$ , where  $\wp_{\omega_1}(Y)$  is the collection of all countable subsets of  $Y$ , for some set  $Y$  (e.g.  $Y = \mathbb{R}$ ). We say  $\mathcal{I}$  is *fine* if for any  $y \in Y$ , the set  $\{\sigma \in \wp_{\omega_1}(Y) : y \notin \sigma\} \in \mathcal{I}$ . We say  $\mathcal{I}$  is *normal* if whenever  $\{A_y : y \in Y\} \subset \mathcal{I}$ , the diagonal union  $\{\sigma \in \wp_{\omega_1}(Y) : \exists y \in \sigma (\sigma \in A_y)\} \in \mathcal{I}$ .  $\mathcal{I}$  is  $|Y|$ -dense if there is a dense subset of  $\mathbb{P}_{\mathcal{I}}$  of size  $|Y|$ . All ideals on sets of the form  $\wp_{\omega_1}(Y)$  considered in this paper will be assumed countably complete, normal, and fine.

**Lemma 2.7.** *Suppose  $\mathcal{I}$  is a pseudo-homogeneous ideal on  $\wp_{\omega_1}(\mathbb{R})$ . Let  $G \subset \mathbb{P}_{\mathcal{I}}$  be  $V$ -generic and let  $j_G : V \rightarrow \text{Ult}(V, G)$  be the associated generic embedding. Then:*

- (a) *For any ordinal  $\alpha$ ,  $j_G \restriction \alpha$  does not depend on  $G$ ; in particular,  $j_G \restriction \alpha \in V$ .*
- (b) *If  $\lambda < \mathfrak{c}^+$ , then  $j_G[\lambda^\omega]$  does not depend on  $G$  and  $j_G[\lambda^\omega] \in V$ .*
- (c) *If  $A$  is a set of ordinals that is definable in  $V$  from a countable sequence of ordinals, then  $j_G(A)$  does not depend on  $G$  and  $j_G(A) \in V$ .*

*Proof.* We give the proof for (a). The other items are similar. Let  $\theta(u, v, w)$  be the formula “ $u = v(0)$ ”. Let  $\alpha$  be an ordinal. Let  $s : \omega \rightarrow \text{Ord}$  be the constant function  $s(n) = \alpha$  for all  $n \in \omega$ . For each ordinal  $\beta$  the truth of the statement  $\text{Ult}(V, G) \models \theta[\beta, j_G(s), \emptyset]$  is independent of  $G$  by pseudo-homogeneity, so the value of  $j_G(\alpha)$  is independent of  $G$ .  $\square$

### 2.3. Ideals from determinacy

We assume  $\text{AD}_{\mathbb{R}} + “\Theta$  is regular” and  $V = L(\wp(\mathbb{R}))$ . Let  $\mathbb{P}$  be a poset with the following properties:

- $\mathbb{P}$  is coded by a set of reals.

- $\mathbb{P}$  is  $\sigma$ -closed.
- $\mathbb{P}$  is homogeneous.
- $1 \Vdash_{\mathbb{P}} \mathbb{R}$  is wellorderable.
- $1 \Vdash_{\mathbb{P}} \mathfrak{c}$ -DC, dependent choices for  $\mathfrak{c}$ -sequences.

Recall that  $\mathfrak{c}$  is the size of the continuum. Examples of such  $\mathbb{P}$  are  $Coll(\omega_1, \mathbb{R})$  and  $\mathbb{P}_{\max}$ .

Let  $G \subseteq \mathbb{P}$  be  $V$ -generic and let  $H \subset Coll(\Theta, \wp(\mathbb{R}))^{V[G]}$ . Note that by the properties of  $\mathbb{P}$  and the assumption  $V = L(\wp(\mathbb{R}))$ , in  $V[G][H]$ , ZFC holds and  $\Theta = \mathfrak{c}^+$ .

**Definition 2.8.** In  $V[G][H]$  an ideal  $\mathcal{I}$  on  $\wp_{\omega_1}(\mathbb{R})$  is said to have the *ordinal covering property* with respect to  $V$  if for every function  $F : \wp_{\omega_1}(\mathbb{R}) \rightarrow \text{Ord}$  and every  $\mathcal{I}$ -positive set  $S$ , there is some  $\mathcal{I}$ -positive set  $S_0 \subseteq S$  and some  $F_0 : \wp_{\omega_1}(\mathbb{R}) \rightarrow \text{Ord}$  in  $V$  such that  $F \restriction S_0 = F_0 \restriction S_0$ .  $\dashv$

We will show that in  $V[G][H]$ , there is an ideal  $\mathcal{I}$  with the ordinal covering property with respect to  $V$ . Let  $\mu$  be the Solovay measure on  $\wp_{\omega_1}(\mathbb{R})^V$ , so  $A \in \mu$  if and only if  $A$  contains a club set in  $\wp_{\omega_1}(\mathbb{R})$ . A set  $A \subseteq \wp_{\omega_1}(\mathbb{R})$  is *club* if and only if there is a function  $F : \mathbb{R}^{<\omega} \rightarrow \mathbb{R}$  such that

$$\sigma \in A \Leftrightarrow F[\sigma^{<\omega}] \subseteq \sigma.$$

We say that  $A$  is the *club set generated by  $F$* .

The measure  $\mu$  induces an ultrapower map on the ordinals,  $j_\mu : \text{Ord} \rightarrow \text{Ord}$ . By the basic theory of  $\text{AD}^+$ ,

$$j_\mu(\omega_1) = \Theta. \tag{2.1}$$

See, for example, [34, Section 1.2] for a proof of this fact.

**Lemma 2.9.** Suppose  $V, G, H$  are as above. Suppose  $\mathcal{I}$  is an ideal on  $\wp_{\omega_1}(\mathbb{R})$  with the ordinal covering property with respect to  $V$ . Let  $K \subset \mathbb{P}_{\mathcal{I}}$  be a  $V[G][H]$ -generic filter. Then:

(a) The generic embedding  $j_K \restriction \text{Ord} = j_\mu \restriction \text{Ord}$ . In particular,  $j_K \restriction \alpha \in V[G][H]$  for every ordinal  $\alpha$  and doesn't depend on the choice of  $K$ .

(b)  $\mathcal{I}$  is strong.

*Proof.* For (a), for any  $F : \wp_{\omega_1}(\mathbb{R}) \rightarrow \text{Ord}$  in  $V[G][H]$ , the covering property gives some  $S \in K$  and  $F_0 \in V$  such that  $F \restriction S = F_0 \restriction S$ . Also,  $K \cap V = \mu$  since  $K$  is normal; this gives

$$\{F : \wp_{\omega_1}(\mathbb{R}) \rightarrow \text{Ord}\}^{V[G][H]} / K = \{F : \wp_{\omega_1}(\mathbb{R}) \rightarrow \text{Ord}\}^V / \mu$$

and  $j_K \restriction \text{Ord} = j_\mu \restriction \text{Ord}$ . Part (b) follows from (a) and (2.1).  $\square$

**Lemma 2.10.** In  $V[G][H]$ , if  $\mathcal{I}$  has the ordinal covering property relative to  $V$ , then  $\mathcal{I}$  is pseudo-homogeneous.

*Proof.* Let  $K \subset \mathbb{P}_I$  be a  $V[G][H]$ -generic filter. Let  $\alpha \in \text{Ord}$ ,  $s \in \text{Ord}^\omega$ ,  $\lambda < \mathfrak{c}^+$ , and let  $\theta$  be a formula in the language of set theory. It suffices to show that the statement  $\text{Ult}(V[G][H], K) \models \theta[\alpha, j_K(s), j_K[\lambda^\omega]]$  is independent of  $K$ . By the ordinal covering property, we can find  $F_0 \in V$  that represents  $\alpha$  in both  $\text{Ult}(V, \mu)$  and  $\text{Ult}(V[G][H], K)$ . In both ultrapowers,  $j(s)$  is represented by the constant function  $F_1(\sigma) = s$  for all  $\sigma \in \wp_{\omega_1}(\mathbb{R})$ . Fix a surjection  $\pi : \mathbb{R} \rightarrow \lambda^\omega$  in  $V$ . Then  $j_K[\lambda^\omega]$  is represented by the function  $F_2 \in V$  given by  $F_2(\sigma) = \pi[\sigma]$ . So we have  $\text{Ult}(V[G][H], K) \models \theta[\alpha, j_K(s), j_K[\lambda^\omega]]$  if and only if the set

$$S = \{\sigma : V[G][H] \models \theta[F_0(\sigma), F_1(\sigma), F_2(\sigma)]\}$$

is in  $K$ . By homogeneity of  $\mathbb{P}$ ,  $S \in V$ . But then we have  $S \in K$  if and only if  $S \in \mu$ , as desired.  $\square$

**Theorem 2.11.** *In  $V[G][H]$ , the nonstationary ideal  $\mathcal{I} = NS_{\omega_1, \mathbb{R}}$  on  $\wp_{\omega_1}(\mathbb{R})$  has the ordinal covering property with respect to  $V$ .*

To establish the covering property of  $\mathcal{I}$  in  $V[G][H]$ , or equivalently in  $V[G]$ , we will need the following lemma.

**Lemma 2.12.** *Let  $\dot{S}$  be a  $\mathbb{P}$ -name for a subset of  $\wp_{\omega_1}(\mathbb{R})$ . The following statements are equivalent for any given  $p \in \mathbb{P}$ :*

(a)  $p \Vdash \text{“}\dot{S} \text{ contains a club.”}$

(b) For a club of  $\sigma \in \wp_{\omega_1}(\mathbb{R})$ ,

$$(\dagger) \quad \forall^* g \subset \mathbb{P} \restriction \sigma \text{ containing } p \ \forall q \leq g \ q \Vdash \sigma \in \dot{S}.$$

Here  $\forall^* g$  stands for “for a comeager set of filters  $g$ ”<sup>16</sup> and  $q \leq g$  means  $\forall r \in g \ q \leq r$ .

*Proof.* Fix  $p \in \mathbb{P}$ . Assume (a) holds for  $p$ . Let  $\dot{f}$  be a  $\mathbb{P}$ -name for a function from  $\mathbb{R}^{<\omega}$  into  $\mathbb{R}$  such that  $p$  forces  $\dot{S}$  to contain the club set generated by  $\dot{f}$ . By our assumption, we may assume  $\mathbb{P} \subseteq \mathbb{R}$ . To see (b), note that there is a club set of  $\sigma$  such that for all  $t \in \sigma^{<\omega}$ , the set

$$D_t = \{q \in \mathbb{P} \cap \sigma : (\exists x \in \sigma) (q \Vdash \dot{f}(t) = x)\}.$$

is dense below  $p$  in  $\mathbb{P} \cap \sigma$ . This easily gives  $(\dagger)$  for  $\sigma$  as there are countably many dense sets  $D_t$  and hence there is a comeager set of filters  $g \subset \mathbb{P} \cap \sigma$  meeting all the  $D_t$ ’s.

Assume (b) holds for  $p$ . Let

$$A = \{(q, x) : x \text{ codes } \sigma \in \wp_{\omega_1}(\mathbb{R}) \text{ and } q \Vdash \sigma \in \dot{S}\}.$$

Take  $N = L_\alpha(P_\beta(\mathbb{R}))$  satisfying  $\text{ZF}^- + \text{AD}_\mathbb{R} + \text{“}\Theta \text{ is regular”}$ , containing  $A$ , and admitting a surjection  $F : \mathbb{R} \rightarrow N$ .<sup>17</sup> Let  $B \subset \mathbb{R}$  code the first order theory of the structure  $(V_{\omega+1}, \in, A)$ . Because  $\text{AD}_\mathbb{R}$  implies that every set of reals is  $\mathbb{R}$ -universally Baire (see e.g. [34, Section 1.2],) in

<sup>16</sup>By  $\mathbb{P} \restriction \sigma$ , we mean the set of conditions in  $\mathbb{P}$  coded by a real in  $\sigma$ . Note that  $\mathbb{P} \restriction \sigma$  is countable, so the category quantifier over the set of all filters on it makes sense.

<sup>17</sup>Here  $P_\beta(\mathbb{R})$  is the set  $\{B \subset \mathbb{R} : B \text{ has Wadge rank less than } \beta\}$ . We also use the notation  $\wp(\mathbb{R}) \restriction \beta$ .

particular  $A$  and  $B$  are  $\mathbb{R}$ -universally Baire. There is then a club  $C$  of  $\sigma \in \wp_{\omega_1}(N)$  having the following properties:

- $(\dagger)$  holds for  $\sigma \cap \mathbb{R}$ .
- $\sigma \prec N$ .
- Defining  $\pi_\sigma : \sigma \rightarrow N_\sigma$  as the transitive collapse of  $\sigma$ , we have

$$(V_{\omega+1} \cap N_\sigma[h], \in, A \cap N_\sigma[h]) \prec (V_{\omega+1}, \in, A)$$

for any  $N_\sigma$ -generic filter  $h \subset \text{Coll}(\omega, \sigma \cap \mathbb{R})$ .

The last item follows from the  $\mathbb{R}$ -universal Baireness of  $B$ .

All  $\sigma \in C$  have the following property:

$$N_\sigma \models p \Vdash_{\mathbb{P} \restriction (\mathbb{R} \cap \sigma)}^g (1 \Vdash_{\text{Coll}(\omega, \mathbb{R} \cap \sigma)}^h (\forall q \leq g)((q, \sigma_h) \in \pi_\sigma(A)_{g \times h})). \quad (2.2)$$

In (2.2),  $\sigma_h$  denotes the real generically coding  $\sigma \cap \mathbb{R}$  relative to  $h$  and  $\pi_\sigma(A)_{g \times h}$  denotes the unique extension of  $\pi_\sigma(A)$  to a set of reals in  $N_\sigma[g][h]$ , which can be construed as a generic extension of  $N_\sigma$  by  $\text{Coll}(\omega, \sigma \cap \mathbb{R})$ ; the extension is given by the universal Baireness of  $A$ .

Now suppose  $G \subset \mathbb{P}$  is  $V$ -generic and  $p \in G$ . There is a club set  $D$  of  $\sigma \in C$  such that  $\sigma[G] \prec N[G]$  and  $\sigma[G] \cap V = \sigma$ . Take a  $\sigma$  in this club and  $g = G \cap \sigma$ . Note that any lower bound  $q \leq g$  forces  $\sigma \in \dot{S}$  by (2.2) and there is  $q \leq g$  in  $G$ ; so  $\sigma \cap \mathbb{R} \in \dot{S}_G$ . Therefore, the club set  $\{\sigma \cap \mathbb{R} : \sigma \in D\}$  witnesses (a).  $\square$

*Proof of Theorem 2.11.* Suppose  $p_0$  forces “ $\dot{F} : \dot{S} \rightarrow \text{Ord}$  and  $\dot{S} \subseteq \wp_{\omega_1}(\mathbb{R})$  is stationary.” Using  $(\dagger)$ , the latter part of this statement is equivalent to the following statement. For stationary many (equivalently by  $\text{AD}_{\mathbb{R}}$ , for club many) countable  $\sigma \subset \mathbb{R}$ ,

$$\exists^* g \subset \mathbb{P} \restriction \sigma \text{ containing } p_0 \exists q \leq g \ q \Vdash \sigma \in \dot{S}.$$

Under  $\text{AD}$ , a well-ordered union of meager sets is meager, so let  $F_0(\sigma)$  be the least  $\alpha$  such that

$$\exists^* g \subset \mathbb{P} \restriction \sigma \text{ containing } p_0 \exists q \leq g \ q \Vdash \dot{F}(\sigma) = \alpha.$$

By the above,  $p_0$  forces that the set of  $\sigma \in \dot{S}$  such that  $F(\sigma) = F_0(\sigma)$  is stationary.  $\square$

Theorem 2.11 and Lemmas 2.9 and 2.10 immediately give one direction of Theorem 1.10.

**Corollary 2.13.** *Con(ZF +  $\text{AD}_{\mathbb{R}}$  + “ $\Theta$  is regular”) implies Con(ZFC + “the nonstationary ideal on  $\wp_{\omega_1}(\mathbb{R})$  is strong and pseudo-homogeneous”).*

Now we proceed to prove one direction of Theorem 1.5. We show  $\text{Con}(\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”})$  implies  $\text{Con}(\text{ZFC} + \text{DI})$ . We fix objects  $V, \mathbb{P}, G, H$  as before. The following is the main theorem.

**Theorem 2.14.** *In  $V[G][H]$ , there is a  $\mathfrak{c}$ -dense ideal on  $\wp_{\omega_1}(\mathbb{R})$  with the ordinal covering property relative to  $V$ .*

We review some facts regarding generic ultrapowers by  $\text{Coll}(\omega, \mathbb{R})$ -generics. See [34] for a more detailed discussion. Let  $h \subset \text{Coll}(\omega, \mathbb{R})$  be  $V$ -generic and

$$U_h = \{A \subseteq \mathbb{R}^\omega : A \text{ is weakly comeager below some } p \in h\}.$$

Here  $A \subseteq \mathbb{R}^\omega$  is weakly comeager below a condition  $p \in \text{Coll}(\omega, \mathbb{R})$  if for a club set of  $\sigma \in \wp_{\omega_1}(\mathbb{R})$ ,  $A \cap \sigma^\omega$  is comeager below  $p$  in  $\sigma^\omega$ .<sup>18</sup>  $U_h$  is the generic ultrafilter on  $\mathbb{R}^\omega$  induced by  $h$ .  $U_h$  gives a generic embedding  $j_h : V \rightarrow \text{Ult}(V, U_h) \subset V[h]$ . Using the fact that  $\text{AD}_{\mathbb{R}} + “\Theta$  is regular” holds in  $V$ , we can prove Loś’s theorem for  $j_h$  and hence  $j_h$  is elementary. We can show that the map  $[F_0]_\mu \mapsto [F_0 \circ \text{ran}]_{U_h}$  is an isomorphism from  $\text{Ult}(\text{Ord}, \mu)$  to  $\text{Ult}(\text{Ord}, U_h)$ ,  $\mathbb{R}^{V[h]} = \mathbb{R}^{\text{Ult}(V, U_h)}$ , and  $j_\mu \upharpoonright \text{Ord} = j_{U_h} \upharpoonright \text{Ord}$ .

*Proof of Theorem 2.14.* We first prove the following claim.

**Claim 2.15.** *If  $h \subset \text{Coll}(\omega, \mathbb{R})$  is a  $V[H]$ -generic filter such that  $G \in V[h]$ , then letting  $j_h : V \rightarrow \text{Ult}(V, U_h) \subset V[h]$  denote the corresponding elementary embedding, in  $V[h][H]$ , there is an  $\text{Ult}(V, U_h)$ -generic filter  $G' \subset j_h(\mathbb{P})$  extending  $j_h “G$ .*

*Proof.* The poset  $j_h(\mathbb{P})$  is countably closed in  $\text{Ult}(V, U_h)$  and is coded by a set of reals there. In  $V[h]$ , because  $\mathbb{R} \cap V[h] = \mathbb{R} \cap \text{Ult}(V, U_h)$  the poset  $j_h(\mathbb{P})$  remains countably closed, and because  $j_h “G$  is countable there is a lower bound  $p \in j_h(\mathbb{P})$  for  $j_h “G$ .

Now note that in  $V[h]$ , there is a surjection  $f$  from  $\wp(\mathbb{R})^V$  onto  $\wp(j_h(\mathbb{P}))^{\text{Ult}(V, U_h)}$ ; this is because every subset of  $j_h(\mathbb{P})$  in  $\text{Ult}(V, U_h)$  is represented by a function  $\mathbb{R}^\omega \rightarrow \wp(\mathbb{R})$  in  $V$ , which can be coded by a set of reals in  $V$ . In  $V[G][H]$ , there is a surjection  $k$  from  $\omega_1^{V[h]} = \Theta^V$  onto  $\wp(\mathbb{R})$  whose proper initial segments are in  $V[G] \subset V[h]$ ; this follows from the fact that the forcing  $\text{Coll}(\Theta^V, \wp(\mathbb{R})^V)^{V[G]}$  is  $\mathfrak{c}^+$ -closed and  $V[G]$  satisfies  $\mathfrak{c}$ -DC. Then the surjection  $k \circ f : \omega_1^{V[h]} \rightarrow \wp(j_h(\mathbb{P}))^{\text{Ult}(V, U_h)}$  has the property that its proper initial segments are in  $V[h]$ .<sup>19</sup> Using this surjection, we recursively define a decreasing  $\omega_1$ -sequence of conditions  $(p_\alpha : \alpha < \omega_1)$  in  $j_h(\mathbb{P})$  below  $p$  whose proper initial segments are in  $V[h]$  and which generates the desired filter  $G'$ .  $\square$

By the assumptions on  $\mathbb{P}$ ,  $\mathbb{P} \times \text{Coll}(\omega, \mathbb{R})$  is forcing equivalent to  $\text{Coll}(\omega, \mathbb{R})$ ; therefore, we can find an  $h$  satisfying the hypothesis of Claim 2.15. By Claim 2.15, forcing with  $\text{Coll}(\omega, \mathbb{R})$  adds an  $\text{Ult}(V, U_h)$ -generic filter  $G' \subset j_h(\mathbb{P})$  extending  $j_h “G$ . We can then extend  $j_h$  to an elementary embedding

$$j_h^* : V[G] \rightarrow \text{Ult}(V, U_h)[G']$$

by defining  $j_h^*(\tau_G) = j_h(\tau)_{G'}$ .

Now in  $V[G][H]$ , define an ideal  $\mathcal{I}$  on  $\wp_{\omega_1}(\mathbb{R})$  by

$$S \in \mathcal{I} \iff \emptyset \Vdash_{\text{Coll}(\omega, \mathbb{R})} \check{\mathbb{R}} \notin j_h^*(\check{S}).$$

<sup>18</sup>We equip  $\sigma^\omega$  with the product of the discrete topologies on  $\sigma$ , so it is homeomorphic to the Baire space.

<sup>19</sup>We need this property for the following argument because this is the model in which  $j_h(\mathbb{P})$  is countably closed.



So  $\mathbb{P}_{\mathcal{I}}$  is isomorphic to the subalgebra  $\mathcal{B} = \{||\check{R} \in j_h^*(\check{S})|| : S \subseteq \wp_{\omega_1}(\mathbb{R})\}^{20}$  of the regular-open algebra  $\text{RO}(\text{Coll}(\omega, \mathbb{R}))$ .

$\mathcal{I}$  is fine: for any  $x \in \mathbb{R}$ , the set  $T_x = \{\sigma : x \notin \sigma\} \in \mathcal{I}$  because clearly  $\emptyset \Vdash_{\text{Coll}(\omega, \mathbb{R})} \check{R} \notin j_h^*(\check{T}_x)$ .  $\mathcal{I}$  is normal: suppose  $(S_x : x \in \mathbb{R})$  is a family of subsets of  $\wp_{\omega_1}(\mathbb{R})$  and  $S$  is the diagonal union, i.e.  $\sigma \in S$  if and only if there is some  $x \in \sigma$  such that  $\sigma \in S_x$ . Then

$$||\check{R} \in j_h^*(S)|| = ||\exists x \in \mathbb{R} (\check{R} \in j_h^*(S_x))|| = \sup_x ||\check{R} \in j_h^*(S_x)||.$$

This verifies normality of  $\mathcal{I}$  and also verifies  $\mathcal{B}$  is a  $\mathfrak{c}$ -complete subalgebra of  $\text{RO}(\text{Coll}(\omega, \mathbb{R}))$ . Since in  $V[G][H]$ ,  $\text{RO}(\text{Coll}(\omega, \mathbb{R}))$  has size  $\mathfrak{c}^+$ , has the  $\mathfrak{c}^+$ -chain condition, and is  $\mathfrak{c}$ -dense,  $\mathcal{B}$  is  $\mathfrak{c}$ -dense and is a complete subalgebra of  $\text{RO}(\text{Coll}(\omega, \mathbb{R}))$ .

We now show  $\mathcal{I}$  has the covering property relative to  $V$ . In  $V[G][H]$ , suppose  $F : S \rightarrow \text{Ord}$  where  $S \in \mathcal{I}^+$ . Note that  $F \in V[G]$ . Let  $p \in \text{Coll}(\omega, \mathbb{R})$  force “ $\check{R} \in j_h^*(S)$ ” and  $q \leq p$  force “ $j_h^*(F)(\check{R}) = \alpha$ ” for some ordinal  $\alpha$ . In  $V$ , let  $F_0 : \wp_{\omega_1}(\mathbb{R}) \rightarrow \text{Ord}$  such that  $[F_0]_\mu = \alpha$ . By the discussion above, before the proof of the theorem,

$$\emptyset \Vdash_{\text{Coll}(\omega, \mathbb{R})} [F_0]_\mu = j_h(F_0)(\check{R}) = j_h^*(F_0)(\check{R}).$$

Therefore,

$$q \Vdash_{\text{Coll}(\omega, \mathbb{R})} j_h^*(F_0)(\check{R}) = j_h^*(F)(\check{R}).$$

This means the set  $\{\sigma \in S : F(\sigma) = F_0(\sigma)\}$  is  $\mathcal{I}$ -positive. □

Now, let  $\mathbb{P}$  be such that CH holds in  $V[G][H]$ . For example, we can take  $\mathbb{P} = \text{Coll}(\omega_1, \mathbb{R})$ . So in  $V[G][H]$ ,  $\mathfrak{c} = \omega_1$  and  $\Theta^V = \omega_2$ . By Theorem 2.14, in  $V[G][H]$ , there is an  $\omega_1$ -dense ideal  $\mathcal{I}$  on  $\wp_{\omega_1}(\mathbb{R})$  that has the covering property with respect to  $V$ . Since  $|\wp_{\omega_1}(\mathbb{R})| = \omega_1$  in  $V[G][H]$ , we easily obtain an  $\omega_1$ -dense ideal on  $\omega_1$  with the ordinal covering property. This gives us one direction of Theorem 1.5.

**Corollary 2.16.** *Con(ZF + AD $_{\mathbb{R}}$  + “ $\Theta$  is regular”) implies Con(ZFC + DI).*

**Remark 2.17.** We note that the  $\omega_1$ -dense ideal constructed above has the covering property with respect to  $V$ , so in fact, it satisfies the strengthening of DI in Footnote 8, by Lemma 2.9.

### 3. PRELIMINARIES

This section, consisting of several subsections, develops some terminology and framework for the core model induction. The first subsection gives a brief summary of the theory of  $\mathcal{F}$ -premise and strategy premise developed in [16]. For a full development of these concepts, the reader should consult [16]. These concepts and notations will be used in Subsection 3.3, which defines core model induction operators, which are the operators that we construct during the course of the core model induction in this paper. Subsections 3.2 and 3.4 briefly summarize the theory of hod mice and the HOD analysis in AD $^+$  models (see [7] for a more detailed discussions of these topics). The

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<sup>20</sup> $||\check{R} \in j_h^*(\check{S})||$  is the boolean value of “ $\check{R} \in j_h^*(\check{S})$ ”.

reader who wishes to see the main argument can skip them on the first read, and go back when needed. Section 3.5 proves several important properties for reasonable hod pairs, defined in 3.32, that we need for the proof of Theorem 1.4. The key result of this section is Lemma 3.36, whose proof uses substantially Lemmata 3.34, 3.35. Lemma 3.34 appears to be a new fact in the theory of hod mice at the level of “ $\text{AD}_{\mathbb{R}} + \Theta$  is regular”. The last section reviews the technique of boolean valued comparisons for such hod pairs. Throughout this paper, we will identify a set  $A \subset HC$  with  $\text{Code}[A] \subset \mathbb{R}$ , where  $\text{Code}$  is a simple coding of elements of  $HC$  by reals.

### 3.1. $\mathcal{F}$ -premise and strategy premise

In this section, we briefly review  $\mathcal{F}$ -premise for an operator  $\mathcal{F}$ , which is an abstract generalization of first order mouse operators like  $x \mapsto x^\sharp$ , and strategy operators. These notions are developed in details in for example [14], [16]. We adapt the framework given in [16] in this section and throughout this paper.

First, the reader should review the definitions of  $\mathcal{J}$ -structures and  $\mathcal{J}$ -models over some transitive set  $a$  (with parameter  $\mathfrak{P}$ ) in [16, Section 2]. In particular, a  $\mathcal{J}$ -model over  $a$  with parameter  $\mathfrak{P}$  has the form

$$\mathcal{M} = (M; E, B, S, a, \mathfrak{P}).$$

The predicate  $E = \dot{E}^{\mathcal{M}}$  will encode the top extender; the predicate  $B = \dot{B}^{\mathcal{M}}$  will be used to code extra information such as a (partial) branch of a tree in  $M$ ; and  $S = \dot{S}^{\mathcal{M}}$  encodes a sequence  $\langle S_\alpha : \alpha \in [1, \lambda) \rangle$  of the levels of  $\mathcal{M}$ . We will omit  $\mathfrak{P}$  from the notation when it is clear from the context.

Our notion of a “ $\mathcal{J}$ -model over  $a$ ” is a bit different from the notion of “model with parameter  $a$ ” in [14] or [34, Definition 2.1.1] in that we build into our notion some fine structure and we do not have the predicate  $l$  used in [34, Definition 2.1.1]. Note that with notation in [16, Section 2], if  $\lambda$  is a successor ordinal then  $M = J(S_{\lambda-1}^{\mathcal{M}})$ , and otherwise,  $M = \bigcup_{\alpha < \lambda} |S_\alpha|$ . Also, one can recover the predicate coding the extender sequence and the predicate coding the strategy in the formulation of [34] or [14] of the levels of  $\mathcal{M}$  (not including the top predicate) from the predicate  $\dot{S}^{\mathcal{M}}$ ; so this change is mostly cosmetic. The reader should consult e.g. [16, Definitions 2.3, 2.12] for various notions concerning  $\mathcal{J}$ -models (like initial segment,  $E$ -passive/active,  $B$ -passive/active,  $\downarrow$  etc.) and the notion of an operator  $\mathcal{F}$  with domain  $D$ . In the definition of operator, the variable  $i$  should be interpreted as follows. When  $i = 0$ , we ignore history, and so  $\mathcal{P}$  is treated as a coarse object when determining  $\mathcal{F}(0, \mathcal{P})$ . When  $i = 1$  we respect the history (given it exists).

[16, Section 2] defines various abstract properties of an operator  $\mathcal{F}$ , like forgetful, historical, basic, and projecting. We refer the reader to [16, Section 2] for more details. There are two main classes of operators we have in mind: mouse operators and strategy operators. Here we give some illustrations that are useful in this paper. Strategy operators (to be explained in more detail later) are basic, and as usually defined, projecting and historical. Suppose we have an iteration strategy  $\Sigma$  and we want to build a  $\mathcal{J}$ -model  $\mathcal{N}$  (over some  $a$ ) that codes a fragment of  $\Sigma$  via its predicate

$\dot{B}$ . We feed  $\Sigma$  into  $\mathcal{N}$  by always providing  $b = \Sigma(\mathcal{T})$ , for the  $<\mathcal{N}$ -least tree  $\mathcal{T}$  for which this information is required. So given a reasonably closed level  $\mathcal{P} \triangleleft \mathcal{N}$ , the choice of which tree  $\mathcal{T}$  should be processed next will usually depend on the information regarding  $\Sigma$  already encoded in  $\mathcal{P}$  (its history). Using an operator  $\mathcal{F}$  to build  $\mathcal{N}$ , then  $\mathcal{F}(i, \mathcal{P})$  will be a structure extending  $\mathcal{P}$  and over which  $b = \Sigma(\mathcal{T})$  is encoded. The variable  $i$  should be interpreted as follows. When  $i = 1$ , we respect the history of  $\mathcal{P}$  when selecting  $\mathcal{T}$ . When  $i = 0$  we ignore history when selecting  $\mathcal{T}$ . The operator  $\mathcal{F}(X) = X^\#$  is forgetful and projecting, and not basic; here  $\mathcal{F}(X) = \mathcal{F}(0, X)$ .

**Definition 3.1.** For any  $P$  and any ordinal  $\alpha \geq 1$ , the operator  $\mathcal{J}_\alpha^m(\cdot; P)$  is defined as follows.<sup>21</sup> For  $X$  such that  $P \in \mathcal{J}_1(\hat{X})$ , let  $\mathcal{J}_\alpha^m(X; P)$  be the  $\mathcal{J}$ -model  $\mathcal{M}$  over  $X$ , with parameter  $P$ , such that  $|\mathcal{M}| = \mathcal{J}_\alpha(\hat{X})$  and for each  $\beta \in [1, \alpha]$ ,  $\mathcal{M}|\beta$  is passive. If  $P = \emptyset$  or we wish to suppress  $P$ , we just write  $\mathcal{J}_\alpha^m(\cdot)$ . +

**Definition 3.2.** Let  $\mathcal{F}$  be an operator and  $b \in C_{\mathcal{F}}$ . Let  $\mathcal{N}$  be a whole  $\mathcal{F}$ -premouse over  $b$ . A **potential continuing  $\mathcal{F}$ -premouse over  $\mathcal{N}$**  is a  $\mathcal{J}$ -model  $\mathcal{M}$  over  $\mathcal{N}$  such that  $\mathcal{M} \downarrow b$  is a potential  $\mathcal{F}$ -premouse over  $b$ .

We say that  $\mathcal{M}$  (as above) is **whole** iff  $\mathcal{M} \downarrow b$  is whole.

A **(potential) continuing  $\mathcal{F}$ -premouse** is a (potential) continuing  $\mathcal{F}$ -premouse over  $b$ , for some  $b$ . +

The definition of (potential)  $\mathcal{F}$ -premise, which generalizes the notion of (potential) premice is given in [16, Definition 2.10]. A (potential) premouse is a (potential)  $\mathcal{F}$ -premouse, where  $\mathcal{F}(x) = \mathcal{J}_1^m(x)$ .

The next couple of definitions define mouse operators. Certain first-order mouse operators (like  $x \mapsto \mathcal{M}_1^\#(x)$ ) are what we construct in the core model induction. These mouse operators will be part of the definition of core model induction operators defined in the next section.

**Definition 3.3.**  $\text{Lp}^{\mathcal{F}}(a)$  for an operator  $\mathcal{F}$  denotes the stack of all countably  $\mathcal{F}$ -iterable  $\mathcal{F}$ -premise  $\mathcal{M}$  over  $a$  such that  $\mathcal{M}$  is fully sound and projects to  $a$ .<sup>22</sup>

Let  $\mathcal{N}$  be a whole  $\mathcal{F}$ -premouse over  $b$ , for  $b \in C_{\mathcal{F}}$ . Then  $\text{Lp}_+^{\mathcal{F}}(\mathcal{N})$  denotes the stack of all countably  $\mathcal{F}$ -iterable (above  $o(\mathcal{N})$ ) continuing  $\mathcal{F}$ -premise  $\mathcal{M}$  over  $\mathcal{N}$  such that  $\mathcal{M} \downarrow b$  is fully sound and projects to  $\mathcal{N}$ .<sup>23</sup>

We say that  $\mathcal{F}$  is **uniformly  $\Sigma_1$**  iff there are  $\Sigma_1$  formulas  $\varphi_1$  and  $\varphi_2$  in  $\mathcal{L}_0^-$  such that whenever  $\mathcal{M}$  is a (continuing)  $\mathcal{F}$ -premouse, then the set of whole proper segments of  $\mathcal{M}$  is defined over  $\mathcal{M}$  by  $\varphi_1$  ( $\varphi_2$ ). For such an operator  $\mathcal{F}$ , let  $\varphi_{\text{wh}}^{\mathcal{F}}$  denote the least such  $\varphi_1$ . +

<sup>21</sup>The “m” is for “model”.  $\hat{X}$  is the transitive closure of  $\{(X, \rho)\}$  where  $\rho : X \rightarrow \text{rank}(X)$  is the rank function of  $X$ .

<sup>22</sup>Countable substructures of  $\mathcal{M}$  are  $(\omega, \omega_1 + 1)$ - $\mathcal{F}$ -iterable, i.e. all iterates are  $\mathcal{F}$ -premise. See [16, Section 2] for more details on  $\mathcal{F}$ -iterability.

<sup>23</sup>Often times in this paper, when the context is clear, we will use the notation  $\text{Lp}$  for  $\text{Lp}_+$ .

**Definition 3.4** (Mouse operator). Let  $Y$  be a projecting, uniformly  $\Sigma_1$  operator. A  $Y$ -**mouse operator**  $\mathcal{F}$  with domain  $D$  is an operator with domain  $D$  such for each  $(0, X) \in D$ ,  $\mathcal{F}(0, X) \triangleleft \text{Lp}^Y(X)$ , and for each  $(1, X) \in D$ ,  $\mathcal{F}(1, X) \triangleleft \text{Lp}_+^Y(X)$ .<sup>24</sup> (So any  $Y$ -mouse operator is an operator.)

A  $Y$ -mouse operator  $\mathcal{F}$  is called **first-order** if there are formulas  $\varphi_1$  and  $\varphi_2$  in the language of  $Y$ -premise such that  $\mathcal{F}(0, X) (\mathcal{F}(1, X))$  is the first  $\mathcal{M} \triangleleft \text{Lp}^Y(X) (\text{Lp}_+^Y(X))$  satisfying  $\varphi_1$  ( $\varphi_2$ ).

A **mouse operator** is a  $\mathcal{J}_1^m$ -mouse operator.  $\dashv$

We can then define  $\mathcal{F}$ -solidity, the  $L^\mathcal{F}[\mathbb{E}]$ -construction etc. as usual (see [16] for more details). We now define the kind of condensation that mouse operators need to satisfy to ensure for example that the  $L^\mathcal{F}[\mathbb{E}]$ -construction converges. We define the coarse version of condensation (condense coarsely) here for illustrative purposes.

**Definition 3.5.** Let  $Y$  be an operator. We say that  $Y$  **condenses coarsely** iff for all  $i \in \{0, 1\}$  and  $(i, \bar{X}), (i, X) \in \text{dom}(Y)$ , and all  $\mathcal{J}$ -models  $\mathcal{M}^+$  over  $\bar{X}$ , if  $\pi : \mathcal{M}^+ \rightarrow Y_i(X)$  is fully elementary and fixes the parameters in the definition of  $Y$ , then

1. if  $i = 0$  then  $\mathcal{M}^+ \leq Y_0(\bar{X})$ ; and
2. if  $i = 1$  and  $X$  is a sound whole  $Y$ -premouse, then  $\mathcal{M}^+ \leq Y_1(\bar{X})$ .  $\dashv$

The finer version (condense finely), which is more technical, is discussed in detail in [16, Definition 2.28]; the main difference here is that in fine condensation, we weaken the elementarity requirement for embeddings (i.e. we only require  $\pi$  to be a weak  $k$ -embedding, cf [16, Definition 2.5]). In many cases, the “ $\leq$ ” above can be replaced by “ $=$ ” (cf. [16, Lemma 2.31]). The core model induction operators, which form a subclass of the  $Y$ -mouse operators, will satisfy fine condensation. [16, Lemma 2.34] shows that if  $\mathcal{F}$  is a projecting, uniformly  $\Sigma_1$  operators with fine condensation, then the  $L^\mathcal{F}[\mathbb{E}]$ -construction works out in a manner parallel to that of the standard  $L[\mathbb{E}]$ -construction.

[16, Section 3] develops the theory of  $\Sigma$ -premise, for an iteration strategy  $\Sigma$ . We first recall the operator to be used to feed in  $\Sigma$ , the  $\mathfrak{B}$ -operator defined in [16, Definition 3.1].

**Definition 3.6** ( $\mathfrak{B}(a, \mathcal{T}, b)$ ,  $b^\mathcal{N}$ ). Let  $a, \mathcal{P}$  be transitive, with  $\mathcal{P} \in \mathcal{J}_1(\hat{a})$ . Let  $\lambda > 0$  and let  $\mathcal{T}$  be an iteration tree<sup>25</sup> on  $\mathcal{P}$ , of length  $\omega\lambda$ , with  $\mathcal{T} \upharpoonright \beta \in a$  for all  $\beta \leq \omega\lambda$ . Let  $b \subseteq \omega\lambda$ . We define  $\mathcal{N} = \mathfrak{B}(a, \mathcal{T}, b)$  recursively on  $\text{lh}(\mathcal{T})$ , as the  $\mathcal{J}$ -model  $\mathcal{N}$  over  $a$  with parameter  $\mathcal{P}$ <sup>26</sup> such that:

1.  $l(\mathcal{N}) = \lambda$ ,
2. for each  $\gamma \in (0, \lambda)$ ,  $\mathcal{N} \upharpoonright \gamma = \mathfrak{B}(a, \mathcal{T} \upharpoonright \omega\gamma, [0, \omega\gamma]_\mathcal{T})$ ,
3.  $B^\mathcal{N}$  is the set of ordinals  $o(a) + \gamma$  such that  $\gamma \in b$ ,

<sup>24</sup>This restricts the usual notion defined in [14].

<sup>25</sup>We formally take an *iteration tree* to include the entire sequence  $\langle M_\alpha^\mathcal{T} \rangle_{\alpha < \text{lh}(\mathcal{T})}$  of models. So it is  $\Sigma_0(\mathcal{T}, \mathfrak{P})$  to assert that “ $\mathcal{T}$  is an iteration tree on  $\mathfrak{P}$ ”.

<sup>26</sup> $\mathcal{P} = M_0^\mathcal{T}$  is determined by  $\mathcal{T}$ .

4.  $E^{\mathcal{N}} = \emptyset$ .

We also write  $b^{\mathcal{N}} = b$ . ⊢

It is easy to see that every initial segment of  $\mathcal{N}$  is sound, so  $\mathcal{N}$  is acceptable and is indeed a  $\mathcal{J}$ -model (not just a  $\mathcal{J}$ -structure).

Using the definition of  $\mathfrak{B}$ , [16, Definition 3.4, 3.5] defines potential  $\mathcal{P}$ -strategy premice (over some set  $a$  of type  $\varphi$ ). The formula  $\varphi$  selects the “next tree”  $\mathcal{T}$  in the model that does not have its branch indexed yet. One instance of  $\varphi(\mathcal{P}, \mathcal{T})$  is, in the case  $a$  is self-wellordered, the formula “ $\mathcal{T}$  is the least tree on  $\mathcal{P}$  that doesn’t have a cofinal branch”, where least is computed with respect to the canonical well-order of the model.

In the context of a  $\Sigma$ -premouse  $\mathcal{M}$  for an iteration strategy  $\Sigma$ , if  $\mathcal{T}$  is the  $<_{\mathcal{M}}$ -least tree for which  $\mathcal{M}$  lacks instruction regarding  $\Sigma(\mathcal{T})$ , then  $\mathcal{M}$  will already have been instructed regarding  $\Sigma(\mathcal{T} \upharpoonright \alpha)$  for all  $\alpha < \text{lh}(\mathcal{T})$ . Therefore if  $\text{lh}(\mathcal{T}) > \omega$  then  $\mathfrak{B}(\mathcal{M}, \mathcal{T}, \Sigma(\mathcal{T}))$  codes redundant information (the branches already in  $\mathcal{T}$ ) before coding  $\Sigma(\mathcal{T})$ . This redundancy seems to allow one to prove slightly stronger condensation properties (e.g. fine condensation), given that  $\Sigma$  has nice condensation properties (see [16, Section 3]). It also simplifies the definition.

The original version of [28] required that when  $o(\mathcal{M}) < \eta + \text{lh}(\mathcal{T})$ , where  $\eta$  is the least such that  $\mathcal{M} \upharpoonright \eta \models \text{KP} + \varphi(\mathcal{P}, \mathcal{T})$ ,  $\dot{B}^{\mathcal{M}}$  is empty, whereas here we require that it code  $[0, o(\mathcal{M}))_{\mathcal{T}}$ , in the same way that  $\dot{B}^{\mathcal{M}}$  will have to code a new branch when  $o(\mathcal{M}) = \eta + \text{lh}(\mathcal{T})$ . Of course, letting  $\nu$  be the unique ordinal such that  $\eta + \nu = o(\mathcal{M})$ ,  $[0, \nu)_{\mathcal{T}} \in \mathcal{M}$  when  $o(\mathcal{M}) < \eta + \text{lh}(\mathcal{T})$ , so the current  $\dot{B}^{\mathcal{M}}$  seems equivalent to the original  $\dot{B}^{\mathcal{M}} = \emptyset$ . However,  $\dot{B}^{\mathcal{M}} = \emptyset$  leads to  $\Sigma_1^{\mathcal{M}}$  being too weak, with the consequence that a  $\Sigma_1$  hull of  $\mathcal{M}$  might collapse to something that is not a strategy premouse. Our current choice for  $\dot{B}^{\mathcal{M}}$  solves that problem.

**Definition 3.7** (Potential  $\Sigma$ -premouse). Let  $\Sigma$  be a (partial) iteration strategy for a transitive structure  $\mathcal{P}$ . A **potential  $\Sigma$ -premouse (over  $a$ , of type  $\varphi$ )** is a potential  $\mathcal{P}$ -strategy premouse  $\mathcal{M}$  (over  $a$ , of type  $\varphi$ ) such that  $\Sigma^{\mathcal{M}} \subseteq \Sigma$ .<sup>27</sup> ⊢

Now we define an operator that codes  $\Sigma$  via some scheme  $\varphi$ . An example of  $\varphi$  is the formula  $\varphi_{\text{all}}$  defined in Definition 3.9. This scheme will be our default scheme used in this paper.

**Definition 3.8.** Let  $\mathcal{P}$  be transitive and  $\Sigma$  a partial iteration strategy for  $\mathcal{P}$ . Let  $\varphi \in \mathcal{L}_0$ . Let  $\mathcal{F} = \mathcal{F}_{\Sigma, \varphi}$  be the operator such that:

1.  $\mathcal{F}_0(a) = \mathcal{J}_1^{\text{m}}(a; \mathcal{P})$ , for all transitive  $a$  such that  $\mathcal{P} \in \mathcal{J}_1(\hat{a})$ ;
2. Let  $\mathcal{M}$  be a sound branch-whole  $\Sigma$ -premouse of type  $\varphi$ . Let  $\lambda = l(\mathcal{M})$  and with notation as in [16, Definition 3.5], let  $\mathcal{T} = \mathcal{T}_{\lambda}$ . If  $\mathcal{T} = \emptyset$  then  $\mathcal{F}_1(\mathcal{M}) = \mathcal{J}_1^{\text{m}}(\mathcal{M}; \mathcal{P})$ . If  $\mathcal{T} \neq \emptyset$  then  $\mathcal{F}_1(\mathcal{M}) = \mathfrak{B}(\mathcal{M}, \mathcal{T}, b)$  where  $b = \Sigma(\mathcal{T})$ .

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<sup>27</sup>If  $\mathcal{M}$  is a model all of whose proper segments are potential  $\Sigma$ -premise, and the rules for potential  $\mathcal{P}$ -strategy premice require that  $B^{\mathcal{M}}$  code a  $\mathcal{T}$ -cofinal branch, but  $\Sigma(\mathcal{T})$  is not defined, then  $\mathcal{M}$  is not a potential  $\Sigma$ -premouse, whatever its predicates are.

We say that  $\mathcal{F}$  is a **strategy operator** corresponding to  $(\Sigma, \varphi)$ .  $\dashv$

The reader can consult [16, Lemma 2.29, 3.13] for more detailed discussions of mouse operators and strategy operators, particularly the proof that these operators condense coarsely and finely. The next definition defines  $\mathcal{M}_1^{\mathcal{F}, \#}$ , which is the next nontrivial operator constructed in the core model induction, given the existence of the operator  $\mathcal{F}$ .  $\mathcal{M}_1^{\mathcal{F}, \#}$  is also important in the definition of the  $\Theta$ - $g$ -organized hierarchy, discussed later in this paper.

**Definition 3.9.** Let  $a$  be transitive and let  $\mathcal{F}$  be an operator. We say that  $\mathcal{M}_1^{\mathcal{F}, \#}(a)$  **exists** iff there is a  $(0, |a|, |a| + 1)$ - $\mathcal{F}$ -iterable, non-1-small  $\mathcal{F}$ -premouse over  $a$ . We write  $\mathcal{M}_1^{\mathcal{F}, \#}(a)$  for the least such sound structure. For  $\Sigma, \mathcal{P}, a, \varphi$  as in Definition 3.8, we write  $\mathcal{M}_1^{\Sigma, \varphi, \#}(a)$  for  $\mathcal{M}_1^{\mathcal{F}_{\Sigma, \varphi}, \#}(a)$ .

Let  $\mathcal{L}_0^+$  be the language  $\mathcal{L}_0 \cup \{\dot{\prec}, \dot{\Sigma}\}$ , where  $\dot{\prec}$  is the binary relation defined by “ $\dot{a}$  is self-wellordered, with ordering  $\prec_{\dot{a}}$ , and  $\dot{\prec}$  is the canonical wellorder of the universe extending  $\prec_{\dot{a}}$ ”, and  $\dot{\Sigma}$  is the partial function defined by “ $\dot{\mathfrak{P}}$  is a transitive structure and the universe is a potential  $\dot{\mathfrak{P}}$ -strategy premouse over  $\dot{a}$  and  $\dot{\Sigma}$  is the associated partial putative iteration strategy for  $\dot{\mathfrak{P}}$ ”. Let  $\varphi_{\text{all}}(\mathcal{T})$  be the  $\mathcal{L}_0$ -formula “ $\mathcal{T}$  is the  $\dot{\prec}$ -least limit length iteration tree  $\mathcal{U}$  on  $\dot{\mathfrak{P}}$  such that  $\mathcal{U}$  is via  $\dot{\Sigma}$ , but no proper extension of  $\mathcal{U}$  is via  $\dot{\Sigma}$ ”. Then for  $\Sigma, \mathcal{P}, a$  as in Definition 3.8, we sometimes write  $\mathcal{M}_1^{\Sigma, \#}(a)$  for  $\mathcal{M}_1^{\mathcal{F}_{\Sigma, \varphi_{\text{all}}}, \#}(a)$ .

Let  $\kappa$  be a cardinal and suppose that  $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F}, \#}(a)$  exists and is  $(0, \kappa^+ + 1)$ -iterable. We write  $\Lambda_{\mathfrak{M}}$  for the unique  $(0, \kappa^+ + 1)$ -iteration strategy for  $\mathfrak{M}$  (given that  $\kappa$  is fixed).  $\dashv$

### 3.2. Hod mice

In this paper, a hod premouse  $\mathcal{P}$  is one defined as in [7]. The reader is advised to consult [7] for basic results and notations concerning hod premice and mice.

Suppose  $\mathcal{P}$  is a hod premouse and  $\gamma$  is an ordinal. By  $\mathcal{P}|_{\gamma}$ , we mean the model  $\mathcal{P}$  up to  $\gamma$ , including the top extender (if one exists); by  $\mathcal{P}||_{\gamma}$ , we mean the model  $\mathcal{P}$  up to  $\gamma$ , not including the top extender.

Let us summarize some basic first-order properties of a hod premouse  $\mathcal{P}$ . There are an ordinal  $\lambda^{\mathcal{P}}$  and sequences  $\langle (\mathcal{P}(\alpha), \Sigma_{\alpha}^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}} \rangle$  and  $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$  such that

1.  $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$  is increasing and continuous and if  $\alpha$  is a successor ordinal then  $\mathcal{P} \models$  “ $\delta_{\alpha}^{\mathcal{P}}$  is Woodin”;
2. every Woodin cardinal or limit of Woodin cardinals of  $\mathcal{P}$  is of the form  $\delta_{\alpha}^{\mathcal{P}}$  for some  $\alpha$ ;
3.  $\mathcal{P}(\alpha) \triangleleft \mathcal{P}$  is the  $\alpha$ -th layer of  $\mathcal{P}$ , and  $\delta_{\alpha}^{\mathcal{P}}$  is the largest limit cardinal of  $\mathcal{P}(\alpha)$ ;
4.  $\mathcal{P} \models$  “ $\Sigma_{\alpha}^{\mathcal{P}}$  is a  $(\omega, o(\mathcal{P}), o(\mathcal{P}))$ <sup>28</sup>-strategy for  $\mathcal{P}(\alpha)$  with hull condensation”;
5. if  $\alpha < \beta < \lambda^{\mathcal{P}}$  then  $\Sigma_{\beta}^{\mathcal{P}}$  extends  $\Sigma_{\alpha}^{\mathcal{P}}$ .

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<sup>28</sup>This just means  $\Sigma_{\alpha}^{\mathcal{P}}$  acts on all stacks of  $\omega$ -maximal, normal trees in  $\mathcal{P}$ .

We will write  $\delta^{\mathcal{P}}$  for  $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$  and  $\Sigma^{\mathcal{P}} = \oplus_{\beta < \lambda^{\mathcal{P}}} \Sigma_{\beta}^{\mathcal{P}}$ . Note that  $\mathcal{P}(0)$  is a pure extender model. Suppose  $\mathcal{P}$  and  $\mathcal{Q}$  are two hod premice. Then  $\mathcal{P} \trianglelefteq_{\text{hod}} \mathcal{Q}$  if there is  $\alpha \leq \lambda^{\mathcal{Q}}$  such that  $\mathcal{P} = \mathcal{Q}(\alpha)$ . We say then that  $\mathcal{P}$  is a *hod initial segment* of  $\mathcal{Q}$ . We say  $(\mathcal{P}, \Sigma)$  is a *hod pair* if  $\mathcal{P}$  is a hod premouse and  $\Sigma$  is a strategy for  $\mathcal{P}$  (acting on countable stacks of countable normal trees) such that  $\Sigma^{\mathcal{P}} \subseteq \Sigma$  and this fact is preserved under  $\Sigma$ -iterations. Typically, we will construct hod pairs  $(\mathcal{P}, \Sigma)$  such that  $\Sigma$  has hull condensation, branch condensation, and is  $\Gamma$ -fullness preserving for some pointclass  $\Gamma$ .

The reader should also consult [7] for the definition of  $B(\mathcal{Q}, \Sigma)$  and  $I(\mathcal{Q}, \Sigma)$ . Roughly speaking,  $B(\mathcal{Q}, \Sigma)$  is the collection of all hod pairs which are strict hod initial segments of a  $\Sigma$ -iterate of  $\mathcal{Q}$  and  $I(\mathcal{Q}, \Sigma)$  is the collection of all  $\Sigma$ -iterates of  $\mathcal{Q}$ . See [7] for the definition of  $\Gamma(\mathcal{P}, \Sigma)$ . Roughly,  $\Gamma(\mathcal{P}, \Sigma)$  is the pointclass generated by  $\Sigma$ . In the case  $\lambda^{\mathcal{P}}$  is a limit ordinal,  $\Gamma(\mathcal{P}, \Sigma)$  is the set of  $B$  such that there is some  $(\mathcal{Q}, \Lambda) \in B(\mathcal{P}, \Sigma)$ ,  $B \leq_w \Lambda$ . See [7] for the definition of  $\Gamma(\mathcal{Q}, \Sigma)$  in the case  $\lambda^{\mathcal{Q}}$  is a successor ordinal. If  $(\mathcal{P}, \Sigma)$  is a hod pair, and  $\vec{\mathcal{T}}$  is according to  $\Sigma$  with last model  $\mathcal{Q}$ , then we write  $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$  for the  $\vec{\mathcal{T}}$ -tail strategy of  $\mathcal{Q}$  induced by  $\Sigma$ , i.e.  $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}} \smallfrown \vec{\mathcal{U}})$ .

See [7] for the definition of hulls of an iteration tree/stack and [30] for a more general notion of a pseudo-hull of a stack.

**Definition 3.10.** Let  $\mathcal{P}$  be a hod premouse in the sense of [7] and  $\Sigma$  be an iteration strategy for  $\mathcal{P}$ .

- (a)  $\Sigma$  has *branch condensation* if whenever  $\vec{\mathcal{T}}, \vec{\mathcal{U}}$  are stacks according to  $\Sigma$ ,  $b = \Sigma(\vec{\mathcal{T}})$  is a non-dropping branch, and  $c$  is a cofinal, nondropping branch of  $\vec{\mathcal{U}}$  such that there is an elementary  $\sigma : \mathcal{M}_c^{\vec{\mathcal{U}}} \rightarrow \mathcal{M}_b^{\vec{\mathcal{T}}}$  with the property that  $\pi_b^{\vec{\mathcal{T}}} = \sigma \circ \pi_c^{\vec{\mathcal{U}}}$ , then  $c = \Sigma(\vec{\mathcal{U}})$ .
- (b)  $\Sigma$  has *strong hull condensation* if whenever  $\vec{\mathcal{T}}$  is according to  $\Sigma$  and  $\vec{\mathcal{U}}$  is a pseudo-hull of  $\vec{\mathcal{T}}$  then  $\vec{\mathcal{U}}$  is according to  $\Sigma$ .  $\Sigma$  has *hull condensation* if whenever  $\vec{\mathcal{T}}$  is according to  $\Sigma$  and  $\vec{\mathcal{U}}$  is a hull of  $\vec{\mathcal{T}}$  then  $\vec{\mathcal{U}}$  is according to  $\Sigma$ .

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Strong hull condensation easily implies hull condensation because every hull is a pseudo-hull. We note that strategies for hod pairs are assumed to have hull condensation, but it is not clear that hod mouse strategies constructed in [7] can have strong hull condensation. In Lemma 3.34, we show that if  $(\mathcal{P}, \Sigma)$  is a hod pair such that  $\Sigma$  has branch condensation and  $\Gamma(\mathcal{P}, \Sigma)$ -fullness preserving then  $\Sigma$  has strong hull condensation. Lemma 3.34 appears to be a new fact in hod mice theory at the level of “ $\text{AD}_{\mathbb{R}} + \Theta$  is regular.”<sup>29</sup> The lemma is used essentially in the proof of Lemma 3.36, which is a key part in the proof of Theorem 1.4.

Suppose  $(\mathcal{Q}, \Sigma)$  is a hod pair such that  $\Sigma$  has hull condensation. We say  $\mathcal{P}$  is a  $(\mathcal{Q}, \Sigma)$ -*hod premouse* if there are an ordinal  $\lambda^{\mathcal{P}}$  and sequences  $\langle (\mathcal{P}(\alpha), \Sigma_{\alpha}^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}} \rangle$  and  $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$  such that

<sup>29</sup>Lemma 3.34 should also hold for hod mice in a minimal model of LSA but we have not checked all details of this claim.



1.  $\langle \delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$  is increasing and continuous and if  $\alpha$  is a successor ordinal then  $\mathcal{P} \models \delta_\alpha^{\mathcal{P}}$  is Woodin;
2. every Woodin cardinal or limit of Woodin cardinals of  $\mathcal{P}$  is of the form  $\delta_\alpha^{\mathcal{P}}$  for some  $\alpha$ ;
3.  $\mathcal{P}(0) = \text{Lp}_\omega^\Sigma(\mathcal{P}|\delta_0)^{\mathcal{P}}$  (so  $\mathcal{P}(0)$  is a  $\Sigma$ -premouse built over  $\mathcal{Q}$ ); for  $\alpha < \lambda^{\mathcal{P}}$ ,  $\mathcal{P}(\alpha + 1) = (\text{Lp}_\omega^{\Sigma \oplus \Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$ ; for limit  $\alpha \leq \lambda^{\mathcal{P}}$ ,  $\mathcal{P}(\alpha) = (\text{Lp}_\omega^{\Sigma \oplus \Lambda}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$ , where  $\Lambda = \oplus_{\beta < \alpha} \Sigma_\beta^{\mathcal{P}}$ ;
4.  $\mathcal{P} \models \text{"}\Sigma \cap \mathcal{P} \text{ is a } (\omega, o(\mathcal{P}), o(\mathcal{P}))\text{-strategy for } \mathcal{Q} \text{ with hull condensation"}$ ;
5.  $\mathcal{P} \models \text{"}\Sigma_\alpha^{\mathcal{P}} \text{ is a } (\omega, o(\mathcal{P}), o(\mathcal{P}))\text{-strategy for } \mathcal{P}(\alpha) \text{ with hull condensation"}$ ;
6. if  $\alpha < \beta < \lambda^{\mathcal{P}}$  then  $\Sigma_\beta^{\mathcal{P}}$  extends  $\Sigma_\alpha^{\mathcal{P}}$ .

Inside  $\mathcal{P}$ , the strategies  $\Sigma_\alpha^{\mathcal{P}}$  act on stacks above  $\mathcal{Q}$  and every  $\Sigma_\alpha^{\mathcal{P}}$  iterate is a  $\Sigma$ -premouse. Again, we write  $\delta^{\mathcal{P}}$  for  $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$  and  $\Sigma^{\mathcal{P}} = \oplus_{\beta < \lambda^{\mathcal{P}}} \Sigma_\beta^{\mathcal{P}}$ . We say  $(\mathcal{P}, \Lambda)$  is a  $(\mathcal{Q}, \Sigma)$ -hod pair if  $\mathcal{P}$  is a  $(\mathcal{Q}, \Sigma)$ -hod premouse and  $\Lambda$  is a strategy for  $\mathcal{P}$  such that  $\Sigma^{\mathcal{P}} \subseteq \Lambda$  and this fact is preserved under  $\Lambda$ -iterations.

Suppose  $(\mathcal{R}, \Lambda)$  is a hod pair and  $\Gamma$  is a nice pointclass. We say that  $\Lambda$  is  $\Gamma$ - $\mathcal{Q}$ -structure guided if whenever  $\mathcal{T}$  is according to  $\Lambda$  and short, then  $\Lambda(\mathcal{T}) = b$  is such that  $\mathcal{Q}(b, \mathcal{T})$  exists and the phalanx  $\Phi(\mathcal{T} \hat{\ } b)^{30}$  is  $(\omega_1, \omega_1)$ -iterable with unique strategy in  $\Gamma$ . We show in essence that the branch  $b$  must be unique in Lemma 3.33. We also note that if  $\delta(\mathcal{T})$  is a cutpoint of  $\mathcal{Q}(b, \mathcal{T})$  then the phalanx iterability condition reduces to the iterability of  $\mathcal{Q}(b, \mathcal{T})$  above  $\delta(\mathcal{T})$ .

Suppose  $\mathcal{P}$  is  $\Sigma$ -premouse,  $\delta$  is a cardinal of  $\mathcal{P}$ , and  $A \subseteq \mathbb{R}$  is  $OD_\Sigma$ . We say  $\mathcal{P}$  *weakly term captures*  $A$  at  $\delta$  if for each  $n < \omega$  there is a term relation  $\tau \in \mathcal{P}^{Coll(\omega, (\delta^{+n})^{\mathcal{P}})}$  such that for comeager many  $\mathcal{P}$ -generics  $g \subseteq Coll(\omega, (\delta^{+n})^{\mathcal{P}})$ , we have  $\tau_g = \mathcal{P}[g] \cap A$ . We say  $\mathcal{P}$  *term captures*  $A$  if the equality holds for all generics. Given a  $\Sigma$ -suitable  $\mathcal{P}$  and an  $OD_\Sigma$  set of reals  $A$ , we let  $\tau_{A, \delta, n}^{\mathcal{P}}$  be the standard name for a set of reals in  $\mathcal{P}^{Coll(\omega, (\delta^{+n})^{\mathcal{P}})}$  witnessing the fact that  $\mathcal{P}$  weakly captures  $A$ . When  $\delta$  is clear from the context, we omit it from the notation and just write  $\tau_{A, n}^{\mathcal{P}}$ . Let

$$\gamma_A^{\mathcal{P}} = \sup(\delta \cap \text{Hull}_1^{\mathcal{P}}(\{\tau_{A, n}^{\mathcal{P}} : n < \omega\})).$$

We then let

$$f_A(\mathcal{P}) = \langle \tau_{A, n}^{\mathcal{P}} : n < \omega \rangle. \quad (3.1)$$

Suppose  $(\mathcal{R}, \Lambda)$  is a hod pair and  $\lambda^{\mathcal{R}} = \alpha + 1$  for some  $\alpha \geq 0$ , where  $\lambda^{\mathcal{R}}$  is the order type of the set  $\{\delta : \delta \text{ is either a Woodin cardinal or a limit of Woodin cardinals in } \mathcal{R}\}$ ; we will write  $\delta_\alpha^{\mathcal{R}}$  for the  $\alpha$ -th member of this set. Recall the notations  $\delta^{\mathcal{R}}$ ,  $(\mathcal{R}^-, \Lambda_{\mathcal{R}^-})$ ,  $\mathbb{B}(\mathcal{R}^-, \Lambda_{\mathcal{R}^-})$  from [7].<sup>31</sup> [7, Lemma 5.19] gives that  $\text{AD}^+$  implies there is some tail  $(\mathcal{S}, \Psi)$  of  $(\mathcal{R}, \Lambda)$  and some  $\vec{B} = \{B_i : i < \omega\}$  that *strongly guides*  $\Psi$ . This means that

- $\Lambda$  is  $\Gamma$ - $\mathcal{Q}$ -structure guided, where  $\Gamma = \Gamma(\mathcal{R}, \Lambda)$ .

<sup>30</sup>This is the set of models in the tree  $\mathcal{T} \hat{\ } b$  along with the exchange ordinals.

<sup>31</sup> $\mathcal{R}^- = \mathcal{R}(\alpha - 1)$  and  $\Lambda_{\mathcal{R}^-}$  is just  $\Lambda_{\mathcal{R}(\alpha - 1)}$ . In the case  $\alpha = 0$ ,  $(\mathcal{R}^-, \Lambda_{\mathcal{R}^-}) = (\emptyset, \emptyset)$ .  $\delta^{\mathcal{R}}$  is the largest Woodin cardinal of  $\mathcal{R}$ .



- There are terms  $(\tau_{i,k}^{\mathcal{S}} = \tau_{B_i,k}^{\mathcal{S}} : i, k < \omega, \tau_{i,k}^{\mathcal{S}} \in \mathcal{S}^{Coll(\omega, (\delta^{+k})^{\mathcal{S}})})$  for the  $B_i$ 's such that whenever  $l : \mathcal{S} \rightarrow \mathcal{Q}$  is an iteration map by  $\Psi$  of a maximal tree, then for each  $i, k < \omega$ ,  $l(\tau_{i,k}^{\mathcal{S}}) = \tau_{B_i,k}^{\mathcal{Q}}$ ,  $\sup\{\gamma_{B_i}^{\mathcal{Q}} : i < \omega\} = \delta^{\mathcal{S}}$ , the branch  $b$  giving rise to the embedding  $l$  is the unique branch whose branch embedding moves the terms for  $B_i$ 's correctly, and whenever  $\vec{\mathcal{T}}$  is according to  $\Psi$  with branch embedding  $\pi$ ,  $\vec{\mathcal{U}}$  is according to  $\Psi$ , and suppose  $b$  is a cofinal branch of  $\vec{\mathcal{U}}$  such that there is an elementary map  $\sigma : \mathcal{M}_{\vec{\mathcal{U}}} \rightarrow \mathcal{M}^{\vec{\mathcal{T}}}$  such that  $\sigma \circ \pi_b^{\vec{\mathcal{U}}} = \pi^{\vec{\mathcal{T}}}$ , then for each  $i, k$ ,

$$\sigma^{-1}(\tau_{B_i,k}^{\mathcal{M}^{\vec{\mathcal{T}}}}) = \tau_{B_i,k}^{\mathcal{M}_{\vec{\mathcal{U}}}} = \pi_b^{\vec{\mathcal{U}}}(\tau_{B_i,k}^{\mathcal{S}}).$$

When we don't want to specify the  $B_i$ 's or the particular  $B_i$ 's are not important to specify, we simply say  $\Psi$  is strongly guided. The above notion of strongly guided can be defined in an obvious way for  $(\mathcal{R}, \Lambda)$ , where  $\lambda^{\mathcal{R}} = \alpha + n$  for some  $n < \omega$ . We omit details and refer the reader to [7] for a full discussion. The next section will elaborate more on this topic in the context of the HOD analysis.

In the following, if  $\Sigma$  is a strategy and  $\Gamma$  is a pointclass,  $\mathcal{N} \triangleleft \text{Lp}^{\Sigma, \Gamma}(x)$  if  $\mathcal{N}$  is sound, projecting to  $x$  and whenever  $\pi : \mathcal{M} \rightarrow \mathcal{N}$  is elementary and  $\mathcal{M}$  is countable, transitive then  $\mathcal{M}$  has a unique strategy  $\Lambda$  witnessing  $\mathcal{M}$  is a  $\Sigma^{\pi}$ -mouse over  $\pi^{-1}(x)$  and  $\Lambda \in \Gamma$ .

**Definition 3.11** ( $\Gamma$ -Fullness preservation). Suppose  $(\mathcal{P}, \Sigma)$  is a hod pair such that  $\mathcal{P} \in HC$  and  $\Gamma$  is a nice pointclass. We say  $\Sigma$  is  $\Gamma$ -fullness preserving if  $\Sigma$  is  $\Gamma$ - $\mathcal{Q}$ -structure guided and the following holds for all  $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{P}, \Sigma)$ .

- (1) For all limit  $\alpha < \lambda^{\mathcal{Q}}$ , letting  $\mathcal{R} = \mathcal{Q}(\alpha)$ , then

$$\mathcal{R} = \text{Lp}_{\omega}^{\oplus_{\beta < \alpha} \Sigma_{\mathcal{R}(\beta)}, \vec{\mathcal{T}}, \Gamma}(\mathcal{R} | \delta^{\mathcal{R}}).$$

- (2) For all successor  $\alpha < \lambda^{\mathcal{Q}}$ , letting  $\mathcal{R} = \mathcal{Q}(\alpha)$  and  $\beta = \alpha - 1$ ,

$$\mathcal{R} = \text{Lp}_{\omega}^{\Sigma_{\mathcal{R}(\beta)}, \vec{\mathcal{T}}, \Gamma}(\mathcal{R} | \delta^{\mathcal{R}}).$$

- (3) If  $\eta$  is a cardinal strong cutpoint of  $\mathcal{Q}$ , letting  $\alpha$  be the largest such that  $\mathcal{Q}(\alpha) \triangleleft \mathcal{Q} | \eta$  and  $\mathcal{R} = \mathcal{Q}(\alpha)$ , then

$$\mathcal{Q} | (\eta^+)^{\mathcal{Q}} = \text{Lp}^{\Gamma, \Sigma_{\mathcal{R}}, \vec{\mathcal{T}}}(\mathcal{Q} | \eta).$$

- (4) Furthermore, letting for  $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ ,

$$U_{\mathcal{Q}(\alpha), \Sigma} = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ codes a countable set } a \text{ and } y \text{ codes a sound } \Sigma_{\mathcal{Q}(\alpha)}\text{-mouse } \mathcal{M} \text{ over } a \text{ whose unique strategy is in } \Gamma \text{ such that } \rho(\mathcal{M}) = a\},$$

and

$W_{Q(\alpha),\Sigma} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U_{Q(\alpha),\Sigma} \text{ and } z \text{ codes an iteration tree on the mouse } \mathcal{M} \text{ coded by } y\},$

then whenever  $(\vec{U}, \mathcal{R}) \in I(Q(\alpha+1), \Sigma_{Q(\alpha+1), \vec{T}})$  such that  $\vec{U}$  only uses extenders with critical points above  $\delta_\alpha^Q$  and its images along branch embeddings of  $\vec{U}$ , we have

$$\pi^{\vec{U}}(f_A(Q)) = f_A(\mathcal{R}),$$

where  $A = U_{Q(\alpha),\Sigma} \oplus W_{Q(\alpha),\Sigma}$  and  $f_A$  is defined in (3.1) above.

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**Remark 3.12.** In [7], clauses (1)–(3) comprise the definition of fullness preservation of  $\Sigma$ ; if in addition, clause (4) holds for  $\Sigma$ , then  $\Sigma$  is said to be super fullness preserving (with respect to  $\Gamma$ ). We simplify the terminology by combining these two notions into one definition. For  $Q$  satisfying (1)–(3), we say that  $Q$  is *full in  $\Gamma$*  or  *$\Gamma$ -full*. For  $\mathcal{R}$  as in (1), we say that  $\mathcal{R}$  is full with respect to  $\oplus_{\beta < \alpha} \Sigma_{\mathcal{R}(\beta), \vec{T}}$ -mice in  $\Gamma$ ; a similar statement can be made for the mice in (2), (3). If  $\Gamma$  is  $\wp(\mathbb{R})$ , we simply omit  $\Gamma$  from our notation and say that  $\Sigma$  is fullness preserving etc.

Under  $\text{AD}^+$  and the hypothesis that there are no models of  $\text{AD}_{\mathbb{R}} + “\Theta \text{ is regular},”$  [7] constructs hod pairs that are fullness preserving and have branch condensation (see [7] for a full discussion of these notions). Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of  $\text{AD}^+$  models. Under  $\text{AD}^+$ , for hod pairs  $(\mathcal{M}_\Sigma, \Sigma)$ , if  $\Sigma$  is a fullness preserving strategy with branch condensation and  $\vec{T}$  is a stack on  $\mathcal{M}_\Sigma$  with last model  $\mathcal{N}$ , then  $\Sigma_{\mathcal{N}, \vec{T}}$  is independent of  $\vec{T}$ . Therefore, later on we will omit the subscript  $\vec{T}$  from  $\Sigma_{\mathcal{N}, \vec{T}}$  whenever  $\Sigma$  is a strategy with branch condensation and  $\mathcal{M}_\Sigma$  is a hod mouse. In a core model induction, at the moment  $(\mathcal{M}_\Sigma, \Sigma)$  is constructed we don’t quite have an  $\text{AD}^+$ -model  $M$  such that  $(\mathcal{M}_\Sigma, \Sigma) \in M$ , but we do know that every  $(\mathcal{R}, \Lambda) \in B(\mathcal{M}_\Sigma, \Sigma)$  belongs to such a model. We then can show (using our hypothesis) that  $(\mathcal{M}_\Sigma, \Sigma)$  belongs to an  $\text{AD}^+$ -model.

We briefly review definitions and notations related to the analysis of stacks in [7, Section 6.2]; see [7, Section 6.2] for a more detailed discussion. These notions will be useful in Section 6. Suppose  $\mathcal{P}$  is a hod premouse and  $\vec{T}$  is a stack on  $\mathcal{P}$ . Let  $\mathcal{S}$  be a model that appears in  $\vec{T}$ . By  $\vec{T}_{\leq \mathcal{S}}$  we mean the part of  $\vec{T}$  up to and including  $\mathcal{S}$  (according to the tree order of  $\vec{T}$ ), we define  $\vec{T}_{\geq \mathcal{S}}, \vec{T}_{< \mathcal{S}}, \vec{T}_{> \mathcal{S}}$  similarly. We let  $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$  be the normal components of  $\vec{T}$ , i.e.  $\mathcal{M}_0 = \mathcal{P}$ ,  $\mathcal{T}_\alpha$  is a normal tree on  $\mathcal{M}_\alpha$ , and  $\mathcal{M}_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}$ . We say  $\mathcal{R}$  is a *terminal node* of  $\vec{T}$  if for some  $\alpha, \beta$ ,  $\mathcal{R} = \mathcal{M}_\beta^{\mathcal{T}_\alpha}$  and  $\pi_{0,\beta}^{\mathcal{T}_\alpha}$  is defined. We say  $\mathcal{R}$  is a *non-trivial terminal node* of  $\vec{T}$  if letting  $(\alpha, \beta)$  witness that  $\mathcal{R}$  is a terminal node of  $\vec{T}$ , the extender  $E_\beta^{\mathcal{T}_\alpha}$  is applied to  $\mathcal{R}$  in the tree  $\mathcal{T}_\alpha$  to obtain the model  $\mathcal{M}_{\beta+1}^{\mathcal{T}_\alpha}$ . We write  $tn(\vec{T})$  for the set of terminal nodes of  $\vec{T}$  and  $ntn(\vec{T})$  for the set of non-trivial terminal nodes of  $\vec{T}$ . If  $\mathcal{R}$  is a non-trivial terminal node then  $\xi^{\vec{T}, \mathcal{R}}$  is the least  $\xi$  such that  $E_\beta^{\mathcal{T}_\alpha} \in \mathcal{R}(\xi+1)$ .

For  $Q, \mathcal{R} \in tn(\vec{T})$ , we write  $Q \prec^{\vec{T}} \mathcal{R}$  if the  $Q$ -to- $\mathcal{R}$  iteration embedding in  $\vec{T}$  exists, and we write  $\pi_{Q, \mathcal{R}}^{\vec{T}}$  for this embedding. We write  $Q \prec^{\vec{T}, s} \mathcal{R}$  if letting  $\vec{U}$  be the part of  $\vec{T}$  between  $Q$  and  $\mathcal{R}$ , then  $\vec{U}$  is an iteration on  $Q$ . We write  $\vec{T}_{Q, \mathcal{R}}$  for  $\vec{U}$ .

Let  $C \subseteq \text{tn}(\vec{T})$ . We say  $C$  is *linear* (*strongly linear* respectively) if  $C$  is linearly ordered by  $\prec^{\vec{T}}$  ( $\prec^{\vec{T},s}$  respectively). We say  $C$  is *closed* if  $C$  is strongly linear and whenever  $\alpha$  is a limit point of  $C$ , then letting  $\mathcal{R}$  be the direct limit of  $C \restriction \alpha$  (under the iteration embeddings), we have  $\mathcal{R} \in C$ . We say  $C$  is *cofinal* if for every  $\mathcal{S} \in \vec{T}$ , there are  $\mathcal{Q}, \mathcal{R} \in C$  such that  $\mathcal{Q} \prec^{\vec{T},s} \mathcal{R}$  and  $\mathcal{S}$  is in  $\vec{T}_{\mathcal{Q},\mathcal{R}}$ . Note that if  $\vec{T}$  doesn't have a last model, but there is a strongly closed and cofinal  $C \subseteq \text{tn}(\vec{T})$ , then  $C$  uniquely determines a cofinal branch of  $\vec{T}$ . If such a  $C$  doesn't exist, then  $\eta$  is a successor ordinal, say  $\eta = \alpha + 1$ . Let  $\mathcal{U} = \vec{T}_\alpha$  and  $D = \{\mathcal{S} \in \text{tn}(\mathcal{U}) : \mathcal{U}_{\geq \mathcal{S}} \text{ is a tree on } \mathcal{S}\}$ . In this case  $D$  has a  $\prec^{\vec{T},s}$ -largest element and we write  $\mathcal{S}_{\vec{T}}$  for this element. Then  $\vec{T}_{\mathcal{S}_{\vec{T}}}$  is a normal tree based on  $\mathcal{S}_{\vec{T}}(\beta + 1)$  and above  $\delta_\beta^{\mathcal{S}_{\vec{T}}}$  for some  $\beta < \lambda^{\mathcal{S}_{\vec{T}}}$ .

### 3.3. Core model induction operators

In this section, our main goal is to introduce the main concepts that one uses in the core model induction through the hierarchy  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \Sigma \restriction \text{HC})$ <sup>32</sup> <sup>33</sup>. Here  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \Sigma \restriction \text{HC})$  is the union of all sound,  $\Theta$ - $g$ -organized  $\Sigma$ -premise  $\mathcal{M}$  over  $(\mathbb{R}, \Sigma \restriction \text{HC})$  such that  $\rho_\omega(\mathcal{M}) = \mathbb{R}$  and whenever  $\pi : \mathcal{M}^* \rightarrow \mathcal{M}$  is sufficiently elementary and  $\mathcal{M}^*$  is countable and transitive, then  $\mathcal{M}^*$  has a unique  $(\omega_1 + 1)$ - $\Sigma$ -iteration strategy  $\Lambda$ .<sup>34</sup> See [16] for a precise definition of  $g$ -organized  $\Sigma$ -premise,  $\Theta$ - $g$ -organized  $\Sigma$ -premise,  $\text{Lp}^{\mathcal{G}\Sigma}(x)$ ,  $\text{Lp}_+^{\mathcal{G}\Sigma}(x)$  and other related concepts. When we write  $\text{Lp}^{\mathcal{G}\Sigma}$  or  $\text{Lp}_+^{\mathcal{G}\Sigma}$ , we refer to the hierarchy of  $g$ -organized  $\Sigma$ -mice; when we write  $\text{Lp}^{\mathcal{G}\Sigma}$  or  $\text{Lp}_+^{\mathcal{G}\Sigma}$ , we refer to the hierarchy of  $\Theta$ - $g$ -organized  $\Sigma$ -mice. The  $g$ -organized hierarchy of  $\Sigma$ -mice is considered (instead of the traditional “least branch” hierarchy of  $\Sigma$ -mice) because the  $S$ -constructions (cf. [15], where they are called  $P$ -constructions) work out nicely for this hierarchy.<sup>35</sup> The  $\Theta$ - $g$ -organized hierarchy, which is a slight modification of the  $g$ -organized hierarchy, is considered because the scales analysis under optimal hypotheses can be carried out in  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \Sigma \restriction \text{HC})$  in much the same manner as the scales analysis in  $\text{Lp}(\mathbb{R})$ .<sup>36</sup> For the purpose of this paper, it will not be important to go into the detailed definitions of these hierarchies. Whenever it makes sense to define  $\text{Lp}^\Sigma(x)$  and  $\text{Lp}^{\mathcal{G}\Sigma}(x)$ , [16] shows that  $\wp(x) \cap \text{Lp}^\Sigma(x) = \wp(x) \cap \text{Lp}^{\mathcal{G}\Sigma}(x)$  (and similarly for  $\text{Lp}^{\mathcal{G}\Sigma}(x)$ ); also in the case it is not clear how to make sense of  $\text{Lp}^\Sigma(x)$  (say for instance when  $x = \mathbb{R}$ ), it still makes sense to define  $\text{Lp}^{\mathcal{G}\Sigma}(x)$  and  $\text{Lp}^{\mathcal{G}\Sigma}(x)$  and in that case, [16] shows that  $\wp(x) \cap \text{Lp}^{\mathcal{G}\Sigma}(x) = \wp(x) \cap \text{Lp}^{\mathcal{G}\Sigma}(x)$ .

In core model induction applications, we often have a pair  $(\mathcal{P}, \Sigma)$  where  $\mathcal{P}$  is a hod premouse (cf. Section 3.2) and  $\Sigma$  is  $\mathcal{P}$ 's strategy with branch condensation and is fullness preserving (relative to mice with strategies in some pointclass) or  $\mathcal{P}$  is a sound (hybrid) premouse projecting to some

<sup>32</sup>An equivalent way to define this is to first fix a canonical coding function  $\text{Code} : \text{HC} \rightarrow \mathbb{R}$  and consider  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \text{Code}(\Sigma \restriction \text{HC}))$ .

<sup>33</sup>Instead of feeding  $\Sigma$  into the hierarchy, which is not at all clear how it could be done in a canonical way, we feed in  $\Lambda$ , the canonical strategy of  $\mathcal{M}_1^{\Sigma, \#}$ , into the hierarchy. Roughly speaking, the trees according to  $\Lambda$  that we feed into  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \text{Code}(\Sigma \restriction \text{HC}))$  are those making the local HOD of  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \text{Code}(\Sigma \restriction \text{HC}))|_\alpha$  generically generic, for appropriately chosen ordinals  $\alpha$ . See [16].

<sup>34</sup>This means whenever  $\mathcal{T}$  is an iteration tree according to  $\Lambda$  with last model  $\mathcal{N}$ , then  $\mathcal{N}$  is a  $\Sigma$ -premouse.

<sup>35</sup>It is not clear how one can perform  $S$ -constructions over the least branch hierarchy.

<sup>36</sup>[16] generalizes Steel's scales analysis in [25, 24] to  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \Sigma \restriction \text{HC})$  for various classes of nice strategies  $\Sigma$ . It is not clear that one can carry out the full scales analysis for the hierarchy  $\text{Lp}^{\mathcal{G}\Sigma}(\mathbb{R}, \Sigma \restriction \text{HC})$ .

countable set  $a$  and  $\Sigma$  is the unique (normal)  $(\omega_1 + 1)$ -strategy for  $\mathcal{P}$ . Let  $\mathcal{F}$  be the operator corresponding to  $\Sigma$  (using the formula  $\varphi_{\text{all}}$  in Definition 3.9) and suppose  $\mathcal{M}_1^{\mathcal{F},\#}$  exists. Then [16, Lemma 4.8] shows that  $\mathcal{F}$  condenses finely and  $\mathcal{M}_1^{\mathcal{F},\#}$  generically interprets  $\mathcal{F}$ . The core model induction in this paper will give us that  $\mathcal{F} \upharpoonright \mathbb{R}$  is self-scaled (defined below) and  $\mathcal{M}_1^{\mathcal{F},\#}$  exists. In the following, we will write  $\mathcal{M}_1^{\Sigma,\#}$  for  $\mathcal{M}_1^{\mathcal{F},\#}$ . To be descriptive, we will sometimes write “ $\Sigma$ -premouse” to mean “ $\mathcal{F}$ -premouse”,  $\text{Lp}^{\Sigma}$  for  $\text{Lp}^{\mathcal{F}}$  etc. These properties of  $\Sigma$  and the existence of  $\mathcal{M}_1^{\Sigma,\#}$  allows us to define  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  and analyze its scales pattern as done in [16].

$g$ -organization is a slight variation of the reorganized hod mice hierarchy in [7, Section 3.7]. We will use the reorganized hierarchy of hod premice in [7, Section 3.7] in this paper. The purpose of using the reorganized (or  $g$ -organized) hierarchy is to ensure the  $S$ -constructions (cf. [7, Section 3.8]) work out nicely for hod mice and for  $\Sigma$ -mice as mentioned above. In the paragraph below, we briefly remark on how the  $S$ -constructions work for the  $g$ -organized hierarchy and for the  $\Theta$ - $g$ -hierarchy.

Suppose  $\mathcal{F}$  is a nice operator (with parameter  $\mathfrak{P}$ )<sup>37</sup> and suppose  $\mathcal{M}$  is a  $\mathcal{G}$ -mouse (over some transitive  $a$ ), where  $\mathcal{G}$  is either  ${}^g\mathcal{F}$  or  ${}^G\mathcal{F}$ . Suppose  $\delta$  is a cutpoint of  $\mathcal{M}$  and suppose  $\mathcal{N}$  is a transitive structure such that  $\delta \subseteq \mathcal{N} \subseteq \mathcal{M}|\delta$  and  $\mathfrak{P} \in \mathcal{N}$ . Suppose  $\mathbb{P} \in \mathcal{J}_\omega[\mathcal{N}]$  is such that  $\mathcal{M}|\delta$  is  $\mathbb{P}$ -generic over  $\mathcal{J}_\omega[\mathcal{N}]$  and suppose whenever  $\mathcal{Q}$  is a  $\mathcal{G}$ -mouse over  $\mathcal{N}$  such that  $H_\delta^\mathcal{Q} = \mathcal{N}$  then  $\mathcal{M}|\delta$  is  $\mathbb{P}$ -generic over  $\mathcal{Q}$ . Then the  $S$ -constructions (or  $P$ -constructions) from [15] give a  $\mathcal{G}$ -mouse  $\mathcal{R}$  over  $\mathcal{N}$  such that  $\mathcal{R}[\mathcal{M}|\delta] = \mathcal{M}$ . The  $S$ -constructions give the sequence  $(\mathcal{R}_\alpha : \delta < \alpha \leq \lambda)$  of  $\mathcal{G}$ -premise over  $\mathcal{N}$ , where

- (i)  $\mathcal{R}_{\delta+1} = \mathcal{J}_\omega^m(\mathcal{N})$ ;
- (ii) if  $\alpha$  is limit then let  $\mathcal{R}_\alpha^* = \bigcup_{\beta < \alpha} \mathcal{R}_\beta$ . If  $\mathcal{M}|\alpha$  is passive, then let  $\mathcal{R}_\alpha = \mathcal{R}_\alpha^*$ . So  $\mathcal{R}_\alpha$  is passive. If  $B^{\mathcal{M}|\alpha} \neq \emptyset$ , then let  $\mathcal{R}_\alpha = (|\mathcal{R}_\alpha^*|; \emptyset, B^{\mathcal{M}|\alpha}, \bigcup_{\beta < \alpha} S^{\mathcal{R}_\beta}, \mathcal{N}, \mathfrak{P})$ . Suppose  $E^{\mathcal{M}|\alpha} \neq \emptyset$ ; let  $E^* = E^{\mathcal{M}|\alpha} \cap |\mathcal{R}_\alpha^*|$ , then we let  $\mathcal{R}_\alpha = (|\mathcal{R}_\alpha^*|; E^*, \emptyset, \bigcup_{\beta < \alpha} S^{\mathcal{R}_\beta}, \mathcal{N}, \mathfrak{P})$ . By the hypothesis, we have  $\mathcal{R}_\alpha[\mathcal{M}|\delta] = \mathcal{M}|\alpha$ .
- (iii) Suppose we have already constructed  $\mathcal{R}_\alpha$  and (by the hypothesis) maintain that  $\mathcal{R}_\alpha[\mathcal{M}|\delta] = \mathcal{M}|\alpha$ . Then  $\mathcal{R}_{\alpha+1} = \mathcal{J}_\omega^m(\mathcal{R}_\alpha)$ .
- (iv)  $\lambda$  is such that  $\mathcal{R}_\lambda[\mathcal{M}|\delta] = \mathcal{M}$ . We set  $\mathcal{R}_\lambda = \mathcal{R}$ .

We note that the full constructions from [15] do not require that  $\delta$  is a cutpoint of  $\mathcal{M}$  but we don't need the full power of the  $S$ -constructions in our paper. Also, the fact that  $\mathcal{M}$  is  $g$ -organized (or  $\Theta$ - $g$ -organized) is important for our constructions above because it allows us to get past levels  $\mathcal{M}|\alpha$  for which  $B^{\mathcal{M}|\alpha} \neq \emptyset$ . Because of this fact, in this paper, hod mice are reorganized into the  $g$ -organized hierarchy, that is if  $\mathcal{P}$  is a hod mouse then  $\mathcal{P}(\alpha + 1)$  is a  $g$ -organized  $\Sigma_{\mathcal{P}(\alpha)}$ -premouse for all  $\alpha < \lambda^{\mathcal{P}}$ . The  $S$ -constructions are also important in many other contexts. One such context

<sup>37</sup>Nice is defined in [16, Definition 3.8]. Roughly speaking, these are operators that condense well and determine themselves on generic extensions. CMI operators defined in this section are nice.

is the local HOD analysis of levels of  $\text{Lp}^{\mathcal{F}}(\mathbb{R}, \mathcal{F} \upharpoonright \mathbb{R})$ , which features in the scales analysis of  $\text{Lp}^{\mathcal{F}}(\mathbb{R}, \mathcal{F} \upharpoonright \mathbb{R})$  (cf. [16]).

In the following, a transitive structure  $N$  is *closed* under an operator  $\mathcal{F}$  if whenever  $x \in \text{dom}(\mathcal{F}) \cap N$ , then  $\mathcal{F}(x) \in N$ . We are now in a position to introduce the core model induction operators that we will need in this paper. These are particular kinds of mouse operators (in the sense of [18, Example 3.41]) that are constructed during the course of the core model induction. These operators can be shown to satisfy the sort of condensation described in [18, Section 3] (e.g. condense coarsely, cf. Definition 3.5, and condense finely, cf. [16, Definition 2.28]), relativize well, and determine themselves on generic extensions.

**Definition 3.13** (relativizes well). Let  $\mathcal{F}$  be an a  $Y$ -mouse operator for some operator  $Y$ .<sup>38</sup> We say that  $\mathcal{F}$  *relativizes well* if there is a formula  $\phi(x, y, z)$  such that for any  $a, b \in \text{dom}(\mathcal{F})$  such that  $a \in L_1(b)$ , whenever  $N$  is a transitive model of  $\text{ZFC}^-$  such that  $N$  is closed under  $Y$  and  $a, b, \mathcal{F}(b) \in N$ , then  $\mathcal{F}(a) \in N$  and is the unique  $x \in N$  such that  $N \models \phi[x, a, \mathcal{F}(b)]$ .  $\dashv$

**Definition 3.14** (determines itself on generic extensions). Suppose  $\mathcal{F}$  is an operator. We say that  $\mathcal{F}$  *determines itself on generic extensions* if there is a formula  $\phi(x, y, z)$  and a parameter  $c \in \text{HC}$  such that for any countable transitive structure  $N$  of  $\text{ZFC}^-$  such that  $N$  contains  $c$  and is closed under  $\mathcal{F}$ , for any generic extension  $N[g]$  of  $N$  in  $V$ ,  $\mathcal{F} \cap N[g] \in N[g]$  and is definable over  $N[g]$  via  $(\phi, c)$ , i.e. for any  $e \in N[g] \cap \text{dom}(\mathcal{F})$ ,  $\mathcal{F}(e) = d$  if and only if  $d$  is the unique  $d' \in N[g]$  such that  $N[g] \models \phi[c, d', e]$ .  $\dashv$

**Definition 3.15.** Let  $\Gamma$  be an inductive-like pointclass.<sup>39</sup> For  $x \in \mathbb{R}$ ,  $C_\Gamma(x)$  denotes the set of all  $y \in \mathbb{R}$  such that for some ordinal  $\gamma < \omega_1$ ,  $y$  (as a subset of  $\omega$ ) is  $\Delta_\Gamma(\{\gamma, x\})$ , where  $\Delta_\Gamma$  is the intersection of  $\Gamma$  with its dual pointclass.

Let  $x \in \text{HC}$  be transitive and let  $f : \omega \rightarrow x$  be a surjection. Then  $c_f \in \mathbb{R}$  denotes the code for  $(x, \in)$  determined by  $f$ . And  $C_\Gamma(x)$  denotes the set of all  $y \in \text{HC} \cap \wp(x)$  such that for all surjections  $f : \omega \rightarrow x$  we have  $f^{-1}(y) \in C_\Gamma(c_f)$ .  $\dashv$

We say that  $\vec{A}$  is a self-justifying-system (sjs) if for any  $A \in \text{rng}(\vec{A})$ ,  $\neg A \in \text{rng}(\vec{A})$  and there is a scale  $\varphi$  on  $A$  such that the set of prewellorderings associated with  $\varphi$  is a subset of  $\text{rng}(\vec{A})$ . A set  $Y \subseteq \mathbb{R}$  is *self-scaled* if there are scales on  $Y$  and  $\mathbb{R} \setminus Y$  which are projective in  $Y$ .

In the following,  $\eta$  is a *strong cutpoint* of an  $\mathcal{F}$ -premouse  $\mathcal{N}$  (for some operator  $\mathcal{F}$ ) if there is no extender  $E$  on the sequence of  $\mathcal{N}$  such that  $\text{crt}(E) \leq \eta \leq \text{lh}(E)$ .  $\eta$  is a *cutpoint* of an  $\mathcal{F}$ -premouse  $\mathcal{N}$  if there is no extender  $E$  on the sequence of  $\mathcal{N}$  such that  $\text{crt}(E) < \eta \leq \text{lh}(E)$ .

**Definition 3.16.** Let  $\mathcal{F}$  be a nice operator,  $\Gamma$  be an inductive-like pointclass, and  $t \in \text{HC}$  such that for some  $a$ ,  $\mathcal{M}_1^{\mathcal{F}, \sharp}(a), \mathfrak{P} \in J_1(t)$ . Let  $1 \leq k < \omega$ . A premouse  $\mathcal{N}$  over  $t$  is  $\mathcal{F}$ - $\Gamma$ - $k$ -suitable (or just  $k$ -suitable if  $\Gamma$  and  $\mathcal{F}$  are clear from the context) iff there is a strictly increasing sequence  $\langle \delta_i \rangle_{i < k}$  such that

<sup>38</sup> $Y$  may be the rud operator, in which case  $\mathcal{F}$  is just a mouse operator in the usual sense.

<sup>39</sup> $\Gamma$  is inductive-like if it is  $\omega$ -parametrized, closed under real quantification, recursive substitutions, and has the scale property.

1.  $\forall \delta \in \mathcal{N}$ ,  $\mathcal{N} \models \text{"}\delta \text{ is Woodin"}$  if and only if  $\exists i < k$  ( $\delta = \delta_i$ ).
2.  $o(\mathcal{N}) = \sup_{i < \omega} (\delta_{k-1}^{+i})^{\mathcal{N}}$ .
3. If  $\mathcal{N}|\eta$  is a strong cutpoint of  $\mathcal{N}$  then  $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \text{Lp}_+^{\mathcal{F}, \Gamma}(\mathcal{N}|\eta)$ .<sup>40</sup>
4. Let  $\xi < o(\mathcal{N})$ , where  $\mathcal{N} \models \text{"}\xi \text{ is not Woodin"}$ . Then  $C_\Gamma(\mathcal{N}|\xi) \models \text{"}\xi \text{ is not Woodin"}$ .

We write  $\delta_i^{\mathcal{N}} = \delta_i$ ; also let  $\delta_{-1}^{\mathcal{N}} = 0$  and  $\delta_k^{\mathcal{N}} = o(\mathcal{N})$ .<sup>41</sup>

If  $\mathcal{N}$  is 1-suitable, we simply say  $\mathcal{N}$  is suitable, and we write  $\delta^{\mathcal{N}}$  for  $\delta_0^{\mathcal{N}}$ . Additionally, we also write  $\mathcal{N}$  is  $\Sigma$ -suitable or  $\Sigma$ - $\Gamma$ -suitable if we want to remind the reader that  $\mathcal{N}$  is an  $\mathcal{F}$ -premouse.

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Let  $\mathcal{N}$  be 1- $\Sigma$ -suitable and let  $\xi \in o(\mathcal{N})$  be a limit ordinal such that  $\mathcal{N} \models \text{"}\xi \text{ isn't Woodin"}$ . Let  $Q \triangleleft \mathcal{N}$  be the  $Q$ -structure for  $\xi$ . Let  $\alpha$  be such that  $\xi = o(\mathcal{N}|\alpha)$ . If  $\xi$  is a strong cutpoint of  $\mathcal{N}$  then  $Q \triangleleft \text{Lp}_+^{\mathcal{F}, \Gamma}(\mathcal{N}|\xi)$  by clause 3 of the definition. Assume now that  $\mathcal{N}$  is reasonably iterable. If  $\xi$  is a strong cutpoint of  $Q$ , our mouse capturing hypothesis (namely Mouse Capturing with respect to  $\Sigma$ ,  $\text{MC}(\Sigma)$ , which is what we show during the course of the core model induction) combined with clause 4 gives that  $Q \triangleleft \text{Lp}_+^{\mathcal{F}, \Gamma}(\mathcal{N}|\xi)$ . If  $\xi$  is an  $\mathcal{N}$ -cardinal then indeed  $\xi$  is a strong cutpoint of  $Q$ , since  $\mathcal{N}$  has only finitely many Woodins. If  $\xi$  is not a strong cutpoint of  $Q$ , then by definition, we do not have  $Q \triangleleft \text{Lp}_+^{\mathcal{F}, \Gamma}(\mathcal{N}|\xi)$ . However, using  $*$ -translation (see [23]), one can find a level of  $\text{Lp}_+^{\mathcal{F}, \Gamma}(\mathcal{N}|\xi)$  which corresponds to  $Q$  (and this level is in  $C_\Gamma(\mathcal{N}|\xi)$ ).

If  $\mathcal{F}$  is a nice operator (in the sense of [16], see Footnote 37) and  $\Sigma$  is an iteration strategy for a  $\mathcal{F}$ - $\Gamma$ -1-suitable premouse  $\mathcal{P}$  such that  $\Sigma$  has branch condensation and is  $\Gamma$ -fullness preserving (for some pointclass  $\Gamma$ ), then we say that  $(\mathcal{P}, \Sigma)$  is a  $\mathcal{F}$ - $\Gamma$ -suitable pair or just  $\Gamma$ -suitable pair or just suitable pair if the pointclass and/or the operator  $\mathcal{F}$  is clear from the context.

**Definition 3.17** (Core model induction operators). Suppose  $(\mathcal{P}, \Sigma)$  is a  $\mathcal{G}$ - $\Omega^*$ -suitable pair for some nice operator  $\mathcal{G}$  or a hod pair such that  $\Sigma$  has branch condensation and is  $\Omega^*$ -fullness preserving for some inductive-like  $\Omega^*$ . Let  $\mathcal{F} = \mathcal{F}_{\Sigma, \varphi_{\text{all}}}$  as defined in Definition 3.9. Assume  $\text{Code}(\Sigma)$  is self-scaled. We say  $J$  is a  $\Sigma$ -core model induction operator or just a  $\Sigma$ -cmi operator if one of the following holds:

1.  $J$  is a nice  $\mathcal{F}$ -mouse operator (or  $g$ -organized  $\mathcal{F}$ -mouse operator) defined on a cone of HC above some  $a \in \text{HC}$ . Furthermore,  $J$  condenses finely, relativizes well and determines itself on generic extensions.
2. For some  $\alpha \in \text{OR}$  such that  $\alpha$  ends either a weak or a strong gap in the sense of [24] and [16], letting  $M = \text{Lp}^{\mathcal{G}, \Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|\alpha$  and  $\Gamma = (\Sigma_1)^M$ ,  $M \models \text{AD}^+ + \text{MC}(\Sigma)$ .<sup>42</sup> For some

<sup>40</sup>  $\mathcal{N} \triangleleft \text{Lp}_+^{\mathcal{F}, \Gamma}(\mathcal{N}|\eta)$  iff  $\mathcal{N} \triangleleft \text{Lp}_+^{\mathcal{F}}(\mathcal{N}|\eta)$  and whenever  $\mathcal{N}^*$  is countable transitive and embeddable into  $\mathcal{N}$ ,  $\mathcal{N}^*$  has a unique iteration strategy above the preimage of  $\eta$  in  $\Gamma$ .

<sup>41</sup> We could also define a suitable premouse  $\mathcal{N}$  as a  $\Theta$ - $g$ -organized  $\mathcal{F}$ -premouse and all the results that follow in this paper will be unaffected.

<sup>42</sup>  $\text{MC}(\Sigma)$  stands for Mouse Capturing relative to  $\Sigma$  which says that for  $x, y \in \mathbb{R}$ ,  $x$  is  $\text{OD}(\Sigma, y)$  (or equivalently  $x$  is  $\text{OD}(\Sigma, y)$ ) iff  $x$  is in some  $g$ -organized  $\Sigma$ -mouse over  $y$ .  $\text{SMC}$  is the statement that for every hod pair  $(\mathcal{P}, \Sigma)$  such that  $\Sigma$  is fullness preserving and has branch condensation,  $\text{MC}(\Sigma)$  holds.

transitive  $b \in \text{HC}$  and some 1-suitable<sup>43</sup>  $\mathcal{F}$ -premouse  $\mathcal{Q}$  over  $b$ ,  $J = \Lambda$ , where  $\Lambda$  is an  $(\omega_1, \omega_1)$ -iteration strategy for  $\mathcal{Q}$  which is  $\Gamma$ -fullness preserving, has branch condensation and is guided by some self-justifying-system (sjs)  $\vec{A} = (A_i : i < \omega)$ <sup>44</sup> such that for some real  $x$ , for each  $i$ ,  $A_i \in \text{OD}_{b, \Sigma, x}^M$  and  $\vec{A}$  seals the gap that ends at  $\alpha$ .

When  $\Sigma$  is clear from the context or that we don't want to specify  $\Sigma$ , we simply say  $J$  is a cmi operator.  $\dashv$

**Remark 3.18.** Let  $\Gamma, M$  be as in clause 2 above. The (lightface) envelope of  $\Gamma$  is defined as:  $A \in \text{Env}(\Gamma)$  iff for every countable  $\sigma \subset \mathbb{R}$  there is some  $A'$  such that  $A'$  is  $\Delta_1$ -definable over  $M$  from ordinal parameters and  $A \cap \sigma = A' \cap \sigma$ . For a real  $x$ , we define  $\text{Env}(\Gamma(x))$  similarly: here  $\Gamma(x) = \Sigma_1(x)^M$  and  $A \in \text{Env}(\Gamma(x))$  iff for every countable  $\sigma \subset \mathbb{R}$  there is some  $A'$  that is  $\Delta_1(x)$ -definable over  $M$  from ordinal parameters such that  $A \cap \sigma = A' \cap \sigma$ . We now let  $\text{Env}(\Gamma) = \bigcup_{x \in \mathbb{R}} \text{Env}(\Gamma(x))$ . Note that  $\text{Env}(\Gamma) = \wp(\mathbb{R})^M$  if  $\alpha$  ends a weak gap and  $\text{Env}(\Gamma) = \wp(\mathbb{R})^{\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|(\alpha+1)}$  if  $\alpha$  ends a strong gap.

In clause 2 above, we say  $\vec{A}$  seals the gap that ends at  $\alpha$  if letting  $\Gamma$  be defined as above, then  $\vec{A}$  is Wadge cofinal in  $\text{Env}(\Gamma)$ . We also say  $\vec{A}$  seals  $\text{Env}(\Gamma)$ .

The following definitions are obvious generalizations of those defined in [14]. For example, see [14, Definition 3.2.1] for the definition of a coarse  $(k, U)$ -Woodin mouse. We let  $\Sigma, \mathcal{F}$  be as in Definition 3.17.

**Definition 3.19.** We say that the coarse mouse witness condition  $W_\gamma^{*, \text{G}\Sigma}$  holds if, whenever  $U \subseteq \mathbb{R}$  and both  $U$  and its complement have scales in  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Omega \upharpoonright \text{HC})|_\gamma$ , then for all  $k < \omega$  and  $x \in \mathbb{R}$  there is a coarse  $(k, U)$ -Woodin mouse  $M$  containing  $x$  and closed under the strategy  $\Lambda$  of  $\mathcal{M}_1^{\Sigma, \#}$  with an  $(\omega_1 + 1)$ -iteration strategy whose restriction to  $\text{HC}$  is in  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma$ .<sup>45</sup>  $\dashv$

**Remark 3.20.** By the proof of [14, Lemma 3.3.5],  $W_\gamma^{*, \text{G}\Sigma}$  implies  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma \models \text{AD}^+$ .

**Definition 3.21.** An ordinal  $\gamma$  is a *critical ordinal* in  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  if there is some  $U \subseteq \mathbb{R}$  such that  $U$  and  $\mathbb{R} \setminus U$  have scales in  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|(\gamma + 1)$  but not in  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma$ . In other words,  $\gamma$  is critical in  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  just in case  $W_{\gamma+1}^{*, \text{G}\Sigma}$  does not follow trivially from  $W_\gamma^{*, \text{G}\Sigma}$ .  $\dashv$

To any  $\Sigma_1$  formula  $\theta(v)$  in the language of  $\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  we associate formulae  $\theta_k(v)$  for  $k \in \omega$ , such that  $\theta_k$  is  $\Sigma_k$ , and for any  $\gamma$  and any real  $x$ ,

$$\text{Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|(\gamma + 1) \models \theta[x] \iff \exists k < \omega \text{ Lp}^{\text{G}\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma \models \theta_k[x].$$

**Definition 3.22.** Suppose  $\theta(v)$  is a  $\Sigma_1$  formula (in the language of set theory expanded by a name for  $\mathbb{R}$  and a predicate for  $\text{G}\Sigma$ ), and  $z$  is a real; then a  $\langle \theta, z \rangle$ -prewitness is an  $\omega$ -sound  $g$ -organized  $\Sigma$ -premouse  $N$  over  $z$  in which there are  $\delta_0 < \dots < \delta_9$ ,  $S$ , and  $T$  such that  $N$  satisfies the formulae expressing

<sup>43</sup>Recall this means  $\mathcal{F}$ - $\Gamma$ -1-suitable.

<sup>44</sup> $\vec{A} = (A_i : i < \omega)$  is a self-justifying system if for any  $A_i$ , there is some  $k < \omega$  such that  $A_k = \neg A_i$  and some function  $f : \omega \rightarrow \omega$  such that  $(A_{f(n)} : n < \omega)$  codes a scale on  $A_i$ .

<sup>45</sup>We demand the strategy has the property that iterates of  $M$  according to the strategy are closed under  $\Lambda$ .



- (a) ZFC,
- (b)  $\delta_0, \dots, \delta_9$  are Woodin,
- (c)  $S$  and  $T$  are trees on some  $\omega \times \eta$  which are absolutely complementing in  $V^{\text{Col}(\omega, \delta_9)}$ , and
- (d) For some  $k < \omega$ ,  $p[T]$  is the  $\Sigma_{k+3}$ -theory (in the language with names for each real and predicate for  ${}^\Sigma\Sigma$ ) of  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma$ , where  $\gamma$  is least such that  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma \models \theta_k[z]$ .

If  $N$  is also  $(\omega, \omega_1, \omega_1 + 1)$ -iterable (as a  $g$ -organized  $\Sigma$ -mouse), then we call it a  $\langle \theta, z \rangle$ -witness.  $\dashv$

**Definition 3.23.** We say that the fine mouse witness condition  $W_\gamma^{g\Sigma}$  holds if whenever  $\theta(v)$  is a  $\Sigma_1$  formula (in the language  $\mathcal{L}^+$  of  $g$ -organized  $\Sigma$ -premise (cf. [16])),  $z$  is a real, and  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma \models \theta[z]$ , then there is a  $\langle \theta, z \rangle$ -witness  $\mathcal{N}$  whose  ${}^g\Sigma$ -iteration strategy, when restricted to countable trees on  $\mathcal{N}$ , is in  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma$ .  $\dashv$

**Lemma 3.24.**  $W_\gamma^{*,g\Sigma}$  implies  $W_\gamma^{g\Sigma}$  for limit  $\gamma$ .

The proof of the above lemma is a straightforward adaptation of that of [14, Lemma 3.5.4]. One main point is the use of the  $g$ -organization:  $g$ -organized  $\Sigma$ -mice behave well with respect to generic extensions in the sense that if  $\mathcal{P}$  is a  $g$ -organized  $\Omega$ -mouse and  $h$  is set generic over  $\mathcal{P}$  then  $\mathcal{P}[h]$  can be rearranged to a  $g$ -organized  $\Sigma$ -mouse over  $h$ .

**Remark 3.25.** In light of the discussion above, the core model induction (through  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$ ) inductively shows  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})|_\gamma \models \text{AD}^+$  by showing that  $W_\gamma^{*,g\Sigma}$  holds for critical ordinals  $\gamma$ . This, in turn, is done by constructing appropriate  $\Sigma$ -cmi operators “capturing” the theory of those levels (as specified in Definitions 3.19 and 3.23).

Finally, as in [33], the maximal model of  $\Theta = \theta_\Sigma$  is  $\text{sLp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$ , an initial segment (possibly strict) of  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$ .

**Definition 3.26.** We define  $\text{sLp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  to be the union of those  $\mathcal{M} \triangleleft \text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  such that whenever  $\pi : \mathcal{M}^* \rightarrow \mathcal{M}$  is elementary,  $\mathcal{P} \in \pi^{-1}(\text{HC})$ , and  $\mathcal{M}^*$  is countable and transitive, then  $\mathcal{M}^*$  is  $\mathcal{F}$ -( $\omega_1 + 1$ )-iterable with unique strategy  $\Lambda$  such that  $\Lambda \upharpoonright \text{HC} \in \mathcal{M}$ .  $\dashv$

In Section 5, we will outline the core model induction, showing that  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC}) \models \text{AD}^+ + \text{MC}(\Sigma)$ <sup>46</sup> for sufficiently nice  $\Sigma$ . We note that by [9], if  $M$  is a model of  $\text{AD}^+ + \text{MC}(\Sigma)$  satisfying  $\Theta = \Theta_\Sigma$  and  $V = L(\wp(\mathbb{R}))$ , then  $M$  satisfies that every set of reals  $A$  belongs to  $\text{sLp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$ . So in fact, in the situation of this paper,

$$\text{sLp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC}) = \text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC}).$$

For notational simplicity, from now on, we denote  $\text{Lp}^{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$  by  $\text{Lp}^{\Sigma}(\mathbb{R})$ .

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<sup>46</sup> $\text{MC}(\Sigma)$  states that if  $x, y \in \mathbb{R}$  and  $x \in \text{OD}(y, \Sigma)$ , then there is a  $\mathcal{F}$ -mouse  $\mathcal{M}$  over  $y$  such that  $\mathcal{M}$  is sound,  $\rho_\omega(\mathcal{M}) = \omega$ , and  $x \in \mathcal{M}$ .

### 3.4. HOD and $\text{HOD}_\Sigma$ under $\text{AD}^+$

Suppose  $\Sigma$  is an iteration strategy of some hod mouse  $\mathcal{Q}$  and suppose  $\Sigma$  is fullness preserving (see [7]) and has branch condensation. Assume further that  $V = L(\wp(\mathbb{R}))$  and  $\text{MC}(\Sigma)$  holds and  $\Theta = \theta_\Sigma$ . We outline the analysis of HOD and  $\text{HOD}_\Sigma$  in [7].

**Definition 3.27** ( $S(\Gamma, \Sigma)$  and  $F(\Gamma, \Sigma)$ ). Suppose  $\Gamma$  is a pointclass. Let  $S(\Gamma, \Sigma) = \{\mathcal{Q} : \mathcal{Q} \text{ is } \Sigma\text{-}\Gamma\text{-suitable}^{47}\}$ . Also, we let  $F(\Gamma, \Sigma)$  be the set of functions  $f$  such that  $\text{dom}(f) = S(\Gamma, \Sigma)$  and for each  $\mathcal{P} \in S(\Gamma, \Sigma)$ ,  $f(\mathcal{P}) \subseteq \mathcal{P}$  and  $f(\mathcal{P})$  is amenable to  $\mathcal{P}$ , i.e., for every  $X \in \mathcal{P}$ ,  $X \cap f(\mathcal{P}) \in \mathcal{P}$ .  $\dashv$

We let  $\Gamma = \wp(\mathbb{R})$  and for the duration of this subsection, we drop  $\Gamma$  from our notation whenever it is unambiguous to do so. Thus, a  $\Sigma$ -suitable premouse is a  $\Sigma$ - $\Gamma$ -suitable premouse etc. We remark that by [9],

$$V = L(\text{Lp}^\Sigma(\mathbb{R})).$$

Also, we allow for the case  $(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$ , in which case  $V = L(\text{Lp}(\mathbb{R}))$  and  $\text{HOD}_\Sigma = \text{HOD}$ . The following lemma is essentially due to Woodin and the proof for mice can be found in [14].

**Lemma 3.28.** *Suppose  $\mathcal{P}$  is  $\Sigma$ -suitable and  $A \subseteq \mathbb{R}$  is  $OD_\Sigma$ . Then  $\mathcal{P}$  weakly term captures  $A$ . Moreover, there is a  $\Sigma$ -suitable  $\mathcal{Q}$  which term captures  $A$ .*

The following lemma is one of the most fundamental lemmas used to compute HOD and it is originally due to Woodin. Again, the proof can be found in [14]. See also [14, Section 4.1] for detailed discussions of related standard notions like (strong)  $f$ -iterability and  $f$ -quasi-iterability.

**Theorem 3.29.** *For each  $f \in F(\Gamma, \Sigma)$  such that  $f \in OD_\Sigma$ , there is a  $\Sigma$ -suitable premouse  $\mathcal{P}$  which is strongly  $f$ -iterable.*

To save some ink, in what follows, we will sometimes say  $A$ -iterable instead of  $f_A$ -iterable and similarly for other notions. Also, we will use  $A$  in our subscripts instead of  $f_A$ . See Equation (3.1) for the definition of  $f_A$ .

Given  $\mathcal{P} \in S(\Gamma, \Sigma)$  and  $f \in F(\Gamma, \Sigma)$  we let  $f_n(\mathcal{P}) = f(\mathcal{P}) \cap \mathcal{P} \setminus ((\delta^\mathcal{P})^{+n})^\mathcal{P}$ . Then  $f(\mathcal{P}) = \bigcup_{n < \omega} f_n(\mathcal{P})$ . We also let

$$\gamma_f^\mathcal{P} = \sup(\delta^\mathcal{P} \cap \text{Hull}_1^\mathcal{P}(\{f_n(\mathcal{P}) : n < \omega\})).$$

Notice that

$$\gamma_f^\mathcal{P} = \delta^\mathcal{P} \cap \text{Hull}_1^\mathcal{P}(\gamma_f^\mathcal{P} \cup \{f_n(\mathcal{P}) : n < \omega\}).$$

We then let

$$H_f^\mathcal{P} = \text{Hull}_1^\mathcal{P}(\gamma_f^\mathcal{P} \cup \{f_n(\mathcal{P}) : n < \omega\}).$$

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<sup>47</sup>See Definition 3.16

If  $\mathcal{P} \in S(\Gamma, \Sigma)$ ,  $f \in F_{\Sigma, od}$ , and  $i : \mathcal{P} \rightarrow \mathcal{Q}$  is an embedding, then we let  $i(f(\mathcal{P})) = \bigcup_{n < \omega} i(f_n(\mathcal{P}))$ .

The following are the next block of definitions that routinely generalize into our context: (1)  $(f, \Sigma)$ -iterability, (2)  $\vec{b} = \langle b_k : k < m \rangle$  witnesses  $(f, \Sigma)$ -iterability for  $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ , and (3) strong  $(f, \Sigma)$ -iterability.

If  $\mathcal{P}$  is strongly  $(f, \Sigma)$ -iterable and  $\vec{\mathcal{T}}$  is a  $(\Gamma, \Sigma)$ -correctly guided finite stack on  $\mathcal{P}$  with last model  $\mathcal{R}$  (cf. [7, Section 4.1]) then we let

$$\pi_{\mathcal{P}, \mathcal{R}, f}^{\Sigma} : H_f^{\mathcal{P}} \rightarrow H_f^{\mathcal{R}}$$

be the embedding given by any  $\vec{b}$  which witnesses the  $(f, \Sigma)$ -iterability of  $\vec{\mathcal{T}}$ , i.e., fixing  $\vec{b}$  which witnesses  $f$ -iterability for  $\vec{\mathcal{T}}$ ,

$$\pi_{\mathcal{P}, \mathcal{R}, f}^{\Sigma} = \pi_{\vec{\mathcal{T}}, \vec{b}} \upharpoonright H_f^{\mathcal{P}}.$$

Clearly,  $\pi_{\mathcal{P}, \mathcal{R}, f}^{\Sigma}$  is independent of  $\vec{\mathcal{T}}$  and  $\vec{b}$ . Here we keep  $\Sigma$  in our notation for  $\pi_{\mathcal{P}, \mathcal{R}, f}^{\Sigma}$  because it depends on a  $(\Gamma, \Sigma)$ -correct iteration. It is conceivable that  $\mathcal{R}$  might also be a  $(\Gamma, \Lambda)$ -correct iterate of  $\mathcal{P}$  for another  $\Lambda$ , in which case  $\pi_{\mathcal{P}, \mathcal{R}, f}^{\Sigma}$  might be different from  $\pi_{\mathcal{P}, \mathcal{R}, f}^{\Lambda}$ . However, the point is that these embeddings agree on  $H_f^{\mathcal{P}}$ .

Given a finite sequence of functions  $\vec{f} = \langle f_i : i < n \rangle$  in  $F_{\Sigma, od}$ , we let  $\oplus_{i < n} f_i \in F_{\Sigma, od}$  be the function given by  $(\oplus_{i < n} f_i)(\mathcal{P}) = \langle f_i(\mathcal{P}) : i < n \rangle$ . We set  $\oplus \vec{f} = \oplus_{i < n} f_i$ .

We let  $F = F(\Gamma, \Sigma) \cap OD_{\Sigma}$  and

$$\mathcal{I}_{F, \Sigma} = \{(\mathcal{P}, \vec{f}) : \mathcal{P} \in S(\Gamma, \Sigma), \vec{f} \in F^{<\omega} \text{ and } \mathcal{P} \text{ is strongly } \oplus \vec{f}\text{-iterable}\}$$

and

$$\mathcal{F}_{F, \Sigma} = \{H_f^{\mathcal{P}} : (\mathcal{P}, f) \in \mathcal{I}_{F, \Sigma}\}.$$

We then define  $\preceq_{F, \Sigma}$  on  $\mathcal{I}_{F, \Sigma}$  by letting  $(\mathcal{P}, \vec{f}) \preceq_{F, \Sigma} (\mathcal{Q}, \vec{g})$  iff  $\mathcal{Q}$  is a  $\Sigma$ -correct iterate of  $\mathcal{P}$  and  $\vec{f} \subseteq \vec{g}$ . Given  $(\mathcal{P}, \vec{f}) \preceq_{F, \Sigma} (\mathcal{Q}, \vec{g})$ , we have

$$\pi_{\mathcal{P}, \mathcal{Q}, \vec{f}}^{\Sigma} : H_{\oplus \vec{f}}^{\mathcal{P}} \rightarrow H_{\oplus \vec{f}}^{\mathcal{Q}}$$

Notice that  $\preceq_{F, \Sigma}$  is directed. Let then  $\mathcal{M}_{\infty, F, \Sigma}$  be the direct limit of  $(\mathcal{F}_{F, \Sigma}, \preceq_{F, \Sigma})$  under the maps  $\pi_{\mathcal{P}, \mathcal{Q}, \vec{f}}^{\Sigma}$ . Given  $(\mathcal{P}, \vec{f}) \in \mathcal{I}_{F, \Sigma}$ , we let  $\pi_{\mathcal{P}, \vec{f}, \infty}^{\Sigma} : H_{\oplus \vec{f}}^{\mathcal{P}} \rightarrow \mathcal{M}_{\infty, F, \Sigma}$  be the direct limit embedding. Let

$$\mathcal{M}_{\infty} = \mathcal{M}_{\infty, F, \Sigma}.$$

**Theorem 3.30** (Woodin, [14]).  $\delta^{\mathcal{M}_{\infty}} = \Theta$ ,  $\mathcal{M}_{\infty} \in \text{HOD}_{\Sigma}$ , and

$$\mathcal{M}_{\infty} \upharpoonright \Theta = (V_{\Theta}^{\text{HOD}_{\Sigma}}, \vec{E}^{\mathcal{M}_{\infty} \upharpoonright \Theta}, S^{\mathcal{M}_{\infty}}, \in),$$

where  $S^{\mathcal{M}_{\infty}}$  is the predicate of  $\mathcal{M}_{\infty}$  describing  $\Sigma$ .

**Remark 3.31.** In some of the arguments below, for convenience, we actually use the “one cardinal” version of suitability. More precisely, for  $(\mathcal{P}, f) \in \mathcal{I}_{F, \Sigma}$  we consider direct limits of  $(\hat{\mathcal{P}}, \hat{f})$  where  $\delta = \delta^{\mathcal{P}}$ ,  $\hat{\mathcal{P}} = \mathcal{P}|(\delta^+)^{\mathcal{P}}$ , and  $\hat{f} = \vec{f}(\mathcal{P}) \cap \mathcal{P}|(\delta^+)^{\mathcal{P}}$ . We define  $\gamma_{\hat{f}}^{\hat{\mathcal{P}}} = \sup(\delta^{\mathcal{P}} \cap \text{Hull}_1^{\mathcal{P}}(\{f_0(\mathcal{P})\}))$  etc. We let  $\hat{\mathcal{M}}_{\infty}$  be the direct limit of such pairs  $(\hat{\mathcal{P}}, \hat{f})$ . Then it is easy to see also that  $\hat{\mathcal{M}}_{\infty}|\Theta = (V_{\Theta}^{\text{HOD}_{\Sigma}}, \vec{E}^{\mathcal{M}_{\infty}|\Theta}, S^{\mathcal{M}_{\infty}}, \in)$ .

Finally, if  $a \in H_{\omega_1}$  is self-wellordered then we could define  $\mathcal{M}_{\infty}(a)$  by working with  $\Sigma$ -suitable premece over  $a$ . Everything we have said about  $\Sigma$ -suitable premece can also be said about  $\Sigma$ -suitable premece over  $a$ , and in particular the equivalent of Theorem 3.30 can be proven using  $\text{HOD}_{(\Sigma, a) \cup \{a\}}$  instead of  $\text{HOD}_{\Sigma}$  and  $\mathcal{M}_{\infty}(a)$  instead of  $\mathcal{M}_{\infty}$ .

[7] computes  $\text{HOD}$  (up to  $\Theta$ ) in models of  $(V = L(\wp(\mathbb{R}))) + \text{SMC} + \text{AD}_{\mathbb{R}}$  below  $\text{AD}_{\mathbb{R}} + “\Theta$  is regular” by exhibiting a hod premouse  $\mathcal{M}_{\infty}$  satisfying

1.  $\mathcal{M}_{\infty} \in \text{HOD}$ .
2.  $\mathcal{M}_{\infty}$  is a hod premouse.
3.  $\mathcal{M}_{\infty}|\Theta = (V_{\Theta}^{\text{HOD}}, \vec{E}^{\mathcal{M}_{\infty}|\Theta}, S^{\mathcal{M}_{\infty}}, \in)$ , where  $S^{\mathcal{M}_{\infty}|\Theta}$  is the predicate for strategies of hod initial segments of  $\mathcal{M}_{\infty}|\Theta$ .

Here **SMC** is Strong Mouse Capturing, which is the statement that for any  $x, y \in \mathbb{R}$ , if  $x \in OD_{y, \Sigma}$  where  $(\mathcal{P}, \Sigma)$  is a hod pair such that  $\Sigma$  has branch condensation and is fullness preserving, then  $x$  is in a  $\Sigma$ -mouse  $\mathcal{M}$  over  $y$ . We call  $\mathcal{M}_{\infty}$  the *hod limit*. Here  $\mathcal{M}_{\infty} = \bigcup_{(\mathcal{Q}, \Lambda)} \mathcal{M}_{\infty}(\mathcal{Q}, \Lambda)$ , where  $(\mathcal{Q}, \Lambda)$  is a hod pair with branch condensation and is fullness preserving and  $\mathcal{M}_{\infty}(\mathcal{Q}, \Lambda)$  is the direct limit of all (non-dropping)  $\Lambda$ -iterates of  $\mathcal{Q}$ . The reader can consult [7] for more details on this computation.

### 3.5. Strategies with strong hull condensation pulls back

**Definition 3.32.** We say a hod pair  $(\mathcal{P}, \Sigma)$  *reasonable* if it has the following additional properties:

- $\Sigma$  has branch condensation.
- $\Sigma$  is  $\Gamma(\mathcal{P}, \Sigma)$ -fullness preserving.

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We will show that properties listed above for  $\Sigma$  hold for hold mice constructed in this paper. For the next several proofs, the reader is advised to review [7] for basic properties and terminologies of hod pair strategies. See also [16, Lemma 3.18] for a similar argument.

**Lemma 3.33.** *Suppose  $(\mathcal{R}, \Lambda)$  is a reasonable hod pair. Let  $\Gamma = \Gamma(\mathcal{R}, \Lambda)$ . Suppose  $\vec{\mathcal{U}}$  is according to  $\Lambda$  with the following properties:*

- $\vec{\mathcal{U}} = \vec{\mathcal{U}}_0 \hat{\cap} \vec{\mathcal{U}}_1$ , where  $\vec{\mathcal{U}}_0 = \mathcal{W}^{\frown} d$  and  $d = \Lambda(\mathcal{W})$ ,

- letting  $\mathcal{S} = \mathcal{M}_d^{\mathcal{W}}$ , there is  $\beta < \lambda^{\mathcal{S}}$  such that the set of generators used in  $\vec{U}_0 \restriction \alpha(\vec{U}_0) \subset (\delta_\beta^{\mathcal{S}})^{<\omega}$ ,
- $\vec{U}_1$  is based on  $\mathcal{S}(\beta + 1)$ , has the last normal component, and is above  $\delta_\beta^{\mathcal{S}}$ ,
- suppose  $b$  is a cofinal well-founded branch such that  $\mathcal{Q}(b, \vec{U}_1)$  exists and the phalanx  $\Phi(\vec{U}_1 \restriction b)$  is iterable in  $\Gamma$ .

Then  $b = \Lambda_{\vec{U}_0, \mathcal{S}}(\vec{U}_1)$ .

*Proof.* Let  $\Sigma = \Lambda_{\vec{U}_0, \mathcal{S}} \restriction \mathcal{S}(\beta)$ . Let  $c = \Lambda_{\vec{U}_0, \mathcal{S}}(\vec{U}_1)$ . We want to show  $b = c$ . There are two cases.

Suppose  $\delta =_{\text{def}} \delta(\vec{U}_1)$  is a cutpoint of  $\mathcal{Q}(b, \vec{U}_1)$ .<sup>48</sup> This means that

$$\mathcal{Q}(b, \vec{U}_1) \triangleleft \text{Lp}_+^{\Sigma, \Gamma}(\mathcal{M}(\vec{U}_1)).$$

Recall  $\mathcal{M}(\vec{U}_1)$  is the common part model of  $\vec{U}_1$ , or more precisely, the common part model of the last normal component of  $\vec{U}_1$ . This follows from the fact that  $\mathcal{Q}(b, \vec{U}_1)$  must be iterable in  $\Gamma$  for trees above  $\delta$ . But by  $\Gamma$ -fullness preservation of  $\Lambda$ ,  $\mathcal{Q}(c, \vec{U}_1)$  exists, and  $\mathcal{Q}(c, \vec{U}_1) \triangleleft \text{Lp}_+^{\Sigma, \Gamma}(\mathcal{M}(\vec{U}_1))$ . Therefore,  $\mathcal{Q}(c, \vec{U}_1) = \mathcal{Q}(b, \vec{U}_1)$ . So  $b = c$ .

Suppose now  $\delta$  is not a cutpoint of  $\mathcal{Q}(b, \vec{U}_1)$ . Let  $E$  be the least extender on the  $\mathcal{Q}(b, \vec{U}_1)$ -sequence with the property that  $\text{crt}(E) < \delta(\vec{U}_1) < \text{lh}(E)$ . Let  $\mathcal{U}' = \vec{U}_1 \restriction E$ . Let  $\lambda + 1 = \text{lh}(\mathcal{U}')$ ,  $\kappa = \text{crt}(E)$ ,  $\xi = \mathcal{U}' - \text{pred}(\lambda)$ . Then it is easy to see that there is a  $\gamma < o(\mathcal{M}_\xi^{\mathcal{U}'})$  such that

$$\mathcal{M}_\infty^{\mathcal{U}'} = \mathcal{M}_\lambda^{\mathcal{U}'} = \text{Ult}_n(\mathcal{M}_\xi^{\mathcal{U}'} \restriction \gamma, E),$$

where  $n$  is least such that  $\rho_{n+1}(\mathcal{M}_\xi^{\mathcal{U}'}) \leq \kappa$ .<sup>49</sup> By the minimality of  $E$ , we also have

$$\mathcal{M}_\xi^{\mathcal{U}'} \restriction \gamma \models \text{“}\kappa \text{ is a measurable limit of cutpoint Woodin cardinals”}$$

and

$$\mathcal{M}_\infty^{\mathcal{U}'} \models \text{“}\delta \text{ is a cutpoint Woodin cardinal”}.$$

This implies  $\mathcal{M}_\infty^{\mathcal{U}'} \triangleleft \text{Lp}_+^{\Sigma, \Gamma}(\mathcal{M}(\mathcal{U}_1))$ . Furthermore,  $\rho_{n+1}(\mathcal{M}_\infty^{\mathcal{U}'}) < \delta(\vec{U}_1)$  and the above properties of  $\kappa, \delta$  imply that both  $b, c$  drop;<sup>50</sup> so  $\mathcal{Q}(\vec{U}_1, c)$  exists. Suppose  $\mathcal{Q}(\vec{U}_1, b) \neq \mathcal{Q}(\vec{U}_1, c)$ . We claim that letting  $\mathcal{Y}, \mathcal{Z}$  be the padded trees extending  $\vec{U}_1 \restriction b, \vec{U}_1 \restriction c$  respectively, that are the results of comparing the phalanxes  $\Phi(\vec{U}_1 \restriction b), \Phi(\vec{U}_1 \restriction c)$ , then for every  $\alpha \geq \lambda$ ,  $[0, \alpha]_{\mathcal{Y}}, [0, \alpha]_{\mathcal{Z}}$  both drop. This gives a standard contradiction.<sup>51</sup>

Suppose not. Let  $\alpha \geq \lambda$  be least such that either  $F = E_\alpha^{\mathcal{Y}}$  or  $F = E_\alpha^{\mathcal{Z}}$  has critical point  $< \delta$  and hence overlaps  $\delta$ , i.e.  $\text{crt}(F) < \delta < \text{lh}(F)$ . Then  $[0, \alpha']_{\mathcal{Y}}$  and  $[0, \alpha']_{\mathcal{Z}}$  both drop for  $\alpha' \in [\lambda, \alpha]$  by

<sup>48</sup>Technically, this is the  $\mathcal{Q}$ -structure for the last normal component of  $\vec{U}_1$ , but we abuse notation here.

<sup>49</sup>This situation is what Sargsyan calls a “fatal drop” in [7]. See also [16] for an alternative treatment and more details of such a situation.

<sup>50</sup>This is because Woodin cardinals are cutpoints of hod mice  $\mathcal{P}$  we consider and there are no measurable  $\kappa$  such that  $\mathcal{P} \restriction \kappa = \bigcup_{\alpha < \kappa} \mathcal{P}(\alpha)$  (i.e. below “ $\text{AD}_{\mathbb{R}} + \Theta$  is measurable”).

<sup>51</sup>The fact that the last branches of  $\mathcal{Y}, \mathcal{Z}$  drop give that some pairs of extenders in  $\mathcal{Y}, \mathcal{Z}$  must be compatible. This contradicts the fact that  $\mathcal{Y}, \mathcal{Z}$  are comparison trees. We note that the phalanx  $\Phi(\vec{U}_1 \restriction c)$  is iterable by the strategy induced by  $\Lambda$  and  $\Phi(\vec{U} \restriction b)$  has an iteration strategy in  $\Gamma$  by hypothesis.

the above argument. Note that  $\delta$  is Woodin in  $M^{\mathcal{Y}} \parallel \text{lh}(F)$  and if there is any  $F'$  on the sequence of  $M^{\mathcal{Y}} \parallel \text{lh}(F)$  that overlaps  $\delta$ , then  $[0, \beta]_{\mathcal{Y}}, [0, \beta]_{\mathcal{Z}}$  both drop for all  $\beta > \alpha$  by the smallness assumption on our hod mice.

So suppose  $F$  is the least extender overlapping  $\delta$ , and so  $\alpha = \lambda$ . Let  $\kappa' = \text{crt}(F)$  and  $\epsilon$  be the least ordinal less than the length of the last normal component of  $\vec{\mathcal{U}}_1$  such that  $F$  is applied to some  $\mathcal{Q} \trianglelefteq \mathcal{M}_{\epsilon}^{\vec{\mathcal{U}}_1}$  according to the rules of normal trees. Then  $\mathcal{Y} \restriction [\epsilon, \text{lh}(\mathcal{Y}))$  and  $\mathcal{Z} \restriction [\epsilon, \text{lh}(\mathcal{Z}))$  are equivalent to above- $\kappa'$ , normal trees on  $\mathcal{Q}$ . If  $\mathcal{Q} \triangleleft \mathcal{M}_{\epsilon}^{\vec{\mathcal{U}}_1}$ , we are done. Otherwise,  $[0, \epsilon]_{\vec{\mathcal{U}}_1}$  must drop because our hod mice are below “ $\text{AD}_{\mathbb{R}} + \Theta$  is measurable” and  $\kappa'$  is an inaccessible limit of Woodin cardinals.

So  $\mathcal{Q}(\vec{\mathcal{U}}_1, b) = \mathcal{Q}(\vec{\mathcal{U}}_1, c)$  and hence  $b = c$ .

□

**Lemma 3.34.** *Suppose  $(\mathcal{R}, \Lambda)$  is a reasonable hod pair, then  $\Lambda$  has strong hull condensation.*

*Proof.* Suppose  $\vec{\mathcal{T}}$  is according to  $\Lambda$  and  $\vec{\mathcal{U}}$  is a pseudo-hull of  $\vec{\mathcal{T}}$ . We assume for ease of notations in the following argument that  $\vec{\mathcal{U}} = \vec{\mathcal{U}}_0 \hat{\ } \vec{\mathcal{U}}_1$  and letting  $\mathcal{Q}$  be the last model of  $\vec{\mathcal{U}}_0$ , then there is an ordinal  $\beta$  such that:

- $\alpha(\vec{\mathcal{U}}_0)$ , the set of generators used in  $\vec{\mathcal{U}}_0$ , is contained in  $(\delta_{\beta}^{\mathcal{Q}})^{<\omega}$ .
- $\vec{\mathcal{U}}_1$  is based on  $\mathcal{Q}(\beta + 1)$  and is above  $\delta_{\beta}^{\mathcal{Q}}$ .
- $\vec{\mathcal{U}}_0$  is non-dropping and is according to  $\Lambda$ .

This is indeed the main case; the proof of other cases is similar and we will leave that to the reader.

In this case, we also have that  $\vec{\mathcal{T}} = \vec{\mathcal{T}}_0 \hat{\ } \vec{\mathcal{T}}_1$ , where  $\vec{\mathcal{T}}_0$  has last model  $\mathcal{S}$  and the embedding  $\varphi : \mathcal{Q} \rightarrow \mathcal{S}$  is the natural map given by the fact that  $\vec{\mathcal{U}}$  is a pseudo-hull of  $\vec{\mathcal{T}}$ . Hence, we have  $\varphi \circ i_0 = j_0$  where  $i_0$  is the iteration map given by  $\vec{\mathcal{U}}_0$  and  $j_0$  is the iteration map given by  $\vec{\mathcal{T}}_0$ . We also have that  $\mathcal{S}$  is such that  $\vec{\mathcal{T}}_1$  is above  $\mathcal{S}(\varphi(\beta))$ . Let

$$b^* = \Lambda_{\vec{\mathcal{T}}_0, \mathcal{S}}(\vec{\mathcal{T}}_1)$$

and

$$\mathcal{S}^* = \mathcal{M}_{b^*}^{\vec{\mathcal{T}}_1}.$$

Suppose the following holds.

$$\Lambda_{\vec{\mathcal{T}}_0, \mathcal{S}}^{\varphi} \restriction \mathcal{Q}(\beta) = \Lambda_{\vec{\mathcal{U}}_0, \mathcal{Q}} \restriction \mathcal{Q}(\beta). \quad (3.2)$$

Call the strategy in (3.2)  $\Sigma$ . Let  $\Psi = \Lambda_{\vec{\mathcal{T}}_0, \mathcal{S}}^{\varphi}$ ,  $b = \Psi(\vec{\mathcal{U}}_1)$  and  $c = \Lambda_{\vec{\mathcal{U}}_0, \mathcal{Q}}(\vec{\mathcal{U}}_1)$ , we then show that

$$b = c.$$

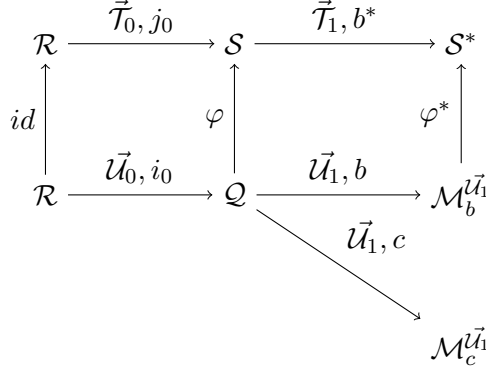


Figure 1: Strong hull condensation.

There are two cases. Suppose  $b$  does not drop. Then there is a map  $\varphi^* : \mathcal{M}_b^{\vec{U}_1} \rightarrow \mathcal{S}^*$  given by the fact that  $\vec{U}$  is a pseudo-hull of  $\vec{T}$ . We then have that  $\pi^{\vec{T}} = \varphi^* \circ \pi_b^{\vec{U}_1} \circ i_0$ . Applying branch condensation, we have that  $b = c$ . See Figure 1.

Suppose  $b$  drops, then  $\mathcal{Q}(b, \vec{U}_1)$  exists and the phalanx  $\Phi(\vec{U}_1 \hat{\wedge} b)$  is iterable (above  $\delta_\beta^{\mathcal{Q}}$ ) in  $\Gamma$  because it is embeddable into the phalanx  $\Phi(\vec{T}_1 \hat{\wedge} b^*)$  (cf. [17, Lemma 4.20]).<sup>52</sup> Lemma 3.33 then implies that  $b = c$ .

Now we prove equation (3.2). Suppose not. Let  $\vec{W} = \vec{W}_0 \hat{\wedge} \vec{W}_1$  be a minimal disagreement stack on  $\mathcal{Q}(\beta)$ . More precisely,  $\vec{W}_0$  is a nondropping stack on  $\mathcal{Q}(\beta)$  according to both  $\Lambda_{\vec{T}_0, \mathcal{S}}^\varphi \upharpoonright \mathcal{Q}(\beta) =_{\text{def}} \Lambda^1$  and  $\Lambda_{\vec{U}_0, \mathcal{Q}}^1 \upharpoonright \mathcal{Q}(\beta) =_{\text{def}} \Lambda^2$  with last model  $\mathcal{P}^*$ ,  $\alpha(\vec{W}_0) \subseteq (\delta_\gamma^{\mathcal{P}^*})^{<\omega}$  and  $\vec{W}_1$  is on  $[\delta_\gamma^{\mathcal{P}^*}, \delta_{\gamma+1}^{\mathcal{P}^*})$  for some  $\gamma$ , and  $\Lambda_{\vec{W}_0, \mathcal{P}^*}^1(\vec{W}_1) \neq \Lambda_{\vec{W}_0, \mathcal{P}^*}^2(\vec{W}_1)$ . Let  $\psi : \mathcal{P}^* \rightarrow \mathcal{R}^*$  be the copy map from  $\vec{W}_0$  to  $\varphi\vec{W}_0$ . Let  $b_i = \Lambda_{\vec{W}_0, \mathcal{P}^*}^i(\vec{W}_1)$  for  $i = 1, 2$ .

Again there are two cases just like above. If  $b_1$  drops, then the same argument as above gives us  $b_1 = b_2$ . Now suppose  $b_1$  does not drop. Let  $\tau : \mathcal{M}_{b_1}^{\vec{W}_1} \rightarrow \mathcal{R}^{**}$  obtained by copying  $\vec{W}_1 \hat{\wedge} b_1$  to  $\psi\vec{W}_1 \hat{\wedge} b_1$ . Note that

$$\tau \circ \pi_{b_1}^{\vec{W}_1} \circ \pi^{\vec{W}_0} \circ \pi^{\vec{U}_0} = \pi_{b_1}^{\psi\vec{W}_1} \circ \pi^{\varphi\vec{W}_0} \circ \pi^{\vec{T}_1}.$$

By branch condensation, again,  $b_1 = \Lambda_{\vec{U}_0 \hat{\wedge} \vec{W}_0}^1(\vec{W}_1)$ . Therefore,  $b_1 = b_2$ . Contradiction. This shows Equation (3.2) holds and hence completes the proof of the lemma.  $\square$

The following lemma will be used in Lemma 3.36. Lemma 3.36 also uses Lemma 3.34 in an essential way. Lemma 3.36 may also be of independent interest and is used in an essential way in the proof of several theorems, including 5.5, 5.12.

**Lemma 3.35.** *Assume CH+ there is an  $\omega_1$ -dense ideal  $\mathcal{I}$  on  $\omega_1$ . Let  $g \subseteq \mathbb{P}_{\mathcal{I}}$  be  $V$ -generic and  $j = j_g : V \rightarrow M$  be the corresponding generic embedding. Suppose  $(\mathcal{R}, \Lambda)$  is a reasonable hod*

<sup>52</sup>In the case  $\delta(\vec{U}_1)$  is a cutpoint of  $\mathcal{Q}(b, \vec{U}_1)$ , as mentioned above, we simply have that  $\mathcal{Q}(b, \vec{U}_1)$  is iterable above  $\delta(\vec{U}_1)$  as a  $\Sigma$ -mouse via a unique iteration strategy in  $\Gamma$ .



pair where  $|\mathcal{R}|^V \leq \omega_1$  and  $\Lambda$  is an  $(\omega_2, \omega_2)$ -strategy. Suppose  $A \subseteq \omega_1^V$  and  $A$  codes  $H_{\omega_1}^V$ . Then in  $L_{\omega_2^V}^\Lambda[A][g]$ , there is no largest cardinal.

*Proof.* First, suppose  $\pi : \mathcal{P} \rightarrow \mathcal{R}$  is elementary and  $\mathcal{P}$  is countable. Let  $\Psi = \Lambda^\pi$ .  $\Lambda$  has hull condensation, and hence  $\Psi$  has hull condensation (see [7] for a proof that hull condensation “pulls back”). We first claim that for any  $x \in HC$  containing  $\mathcal{P}$ ,

$$L_{\omega_2^V}^\Psi[x] = j(L_{\omega_1^V}^\Psi[x]). \quad (*)$$

Suppose not. Then let  $\mathcal{T}$  be a tree in  $L_{\omega_2^V}^\Psi[x] \cap j(L_{\omega_1^V}^\Psi[x])$  such that  $\Psi(\mathcal{T}) \neq j(\Psi)(\mathcal{T})$ . Let  $\mathcal{T}$  be the least such (in the constructibility order of the models). Now the following are easy to see:

- (a)  $\mathcal{T} \in V$ .
- (b)  $j(\pi) = j \circ \pi$ .
- (c)  $j(\Psi) = j(\Lambda^\pi) = j(\Lambda)^{j \circ \pi}$ .

So

$$j(\Psi)(\mathcal{T}) = j(\Lambda)^{j \circ \pi}(\mathcal{T}) = j(\Lambda)(j \circ \pi \mathcal{T}) = \Lambda(\pi \mathcal{T}) = \Psi(\mathcal{T}). \quad (3.3)$$

The first equality follows from (c). The second and last equalities follow from definitions. To see the third equality, first note that by (a),  $\pi \mathcal{T} \in V$  and therefore,  $j \circ \pi \mathcal{T}$  is a hull of  $j(\pi \mathcal{T})$ . Since  $j(\pi \mathcal{T})$  is according to  $j(\Lambda)$ , so is  $j \circ \pi \mathcal{T}$  by hull condensation of  $j(\Lambda)$ .<sup>53</sup> Now let  $b = \Lambda(\pi \mathcal{T})$ , then  $j(b) = j(\Lambda)(j(\pi \mathcal{T}))$ . Since  $j \circ \pi \mathcal{T} \cap b$  is a hull of  $j(\pi \mathcal{T}) \cap j(b)$ , by hull condensation of  $j(\Lambda)$ ,  $b = j(\Lambda)(j \circ \pi \mathcal{T})$  as desired. The last equality follows from the definition of  $\Lambda, \Psi$ .

(3.3) contradicts the assumption that  $\Psi(\mathcal{T}) \neq j(\Psi)(\mathcal{T})$ . So  $(*)$  holds.  $(*)$  implies that there is no  $\alpha < \omega_1^V$  such that

$$L_{\omega_2^V}^\Psi[x] \models \alpha^+ = \omega_1^V.$$

This is because otherwise, in  $j(L_{\omega_2^V}^\Psi[x]) \models \alpha^+ = j(\omega_1^V) = \omega_2^V$ . This implies then that

$$j(L_{\omega_1^V}^\Psi[x]) \models “\omega_1^V \text{ is not a cardinal}”.$$

On the other hand,

$$L_{\omega_2^V}^\Psi[x] \models “\omega_1^V \text{ is a cardinal}.”$$

$(*)$  then immediately gives a contradiction.

Now let  $A \subseteq \omega_1^V$  and  $A$  codes  $H_{\omega_1}^V$ . Let  $X = L_{\omega_2^V}^\Lambda[A][g]$ . To see that there is no largest cardinal in  $X$ . It is enough to show there is no largest cardinal in  $L_{\omega_2^V}^\Lambda[A]$ .<sup>54</sup> The argument above (showing Equations (3.3) hold) shows that

$$\Lambda = j(\Lambda)^j \upharpoonright V,$$

<sup>53</sup>We note that  $j \circ \pi \mathcal{T}$  is countable in  $V[g]$  and therefore is in  $M$ .

<sup>54</sup>We use that  $\mathbb{P}_\mathcal{I}$  is forcing equivalent to  $Coll(\omega, \omega_1)$ .

and

$$L_{\omega_2^V}^\Lambda[A] = L_{\omega_2^V}^{j(\Lambda)^j}[A]. \quad (3.4)$$

Now,  $\mathcal{R}, A \in HC^M$ , and  $j \restriction \mathcal{R} : \mathcal{R} \rightarrow j(\mathcal{R})$  is elementary in  $M$  (because  $M$  is closed under countable sequences in  $V[g]$ ), so the argument above, applied in  $M$ , shows that

$$\omega_1^M = \omega_2^V \text{ is not a successor cardinal in } L_{\omega_2^V}^{j(\Lambda)^j}[A] \quad (**)$$

(\*\*) and Equation (3.4) imply that there is no largest cardinal in  $L_{\omega_2^V}^\Lambda[A]$ .  $\square$

**Lemma 3.36.** *Assume CH+ there is an  $\omega_1$ -dense ideal  $\mathcal{I}$  on  $\omega_1$ . Suppose  $(\mathcal{R}, \Lambda)$  is a reasonable hod pair such that  $|\mathcal{R}|^V \leq \omega_1$  and  $\Lambda$  is an  $\omega_2$ -iteration strategy for  $\mathcal{R}$ . Let  $g \subseteq \mathbb{P}_{\mathcal{I}}$  be  $V$ -generic and  $j = j_g : V \rightarrow M$  be the corresponding generic embedding. Then  $\Lambda = j(\Lambda)^j$ .*

*Proof.* By Lemma 3.34,  $\Lambda$  has strong hull condensation. By strong hull condensation and [17, Theorem 7.3], there is a unique extension of  $\Lambda$  in  $V[g]$ . Hence we identify  $\Lambda$  with its canonical extension in  $V[g]$ . First let  $\mathcal{T} \in V$  be according to  $\Lambda$ . Then  $j\mathcal{T}$  is a pseudo-hull (in fact, a hull) of  $j(\mathcal{T})$  and  $j(\mathcal{T})$  is according to  $j(\Lambda)$ , so  $j\mathcal{T}$  is according to  $j(\Lambda)$  by strong hull condensation of  $j(\Lambda)$ . But then  $\mathcal{T}$  is by  $j(\Lambda)^j$ .

Suppose  $\mathcal{T} \in M$  is according to  $\Lambda$ . Then there is a  $\mathcal{U} \in V$  according to  $\Lambda$  such that  $\mathcal{T}$  is a pseudo-hull of  $\mathcal{U}$  (see [17, Theorem 7.3]); we note that to apply [17, Theorem 7.3] to get the existence of  $\mathcal{U}$ , we need to work inside  $X = L_{\omega_2^V}^\Lambda[tr.cl.(\{\dot{\mathcal{T}}\} \cup H_{\omega_1}^V)][g]$ , where  $\dot{\mathcal{T}} \in H_{\omega_2^V}^V$  is a  $Coll(\omega, \omega_1)$ -name of  $\mathcal{T}$ . For [17, Theorem 7.3] to apply, we need that  $\omega_2^V > (\omega_1^V)^{+,X}$ . This follows from Lemma 3.35.

This means  $j\mathcal{T}$  is a pseudo-hull of  $j\mathcal{U}$ <sup>55</sup> and  $j\mathcal{U}$  is by  $j(\Lambda)$  by the argument above. By strong hull condensation of  $j(\Lambda)$ ,  $j\mathcal{T}$  is by  $j(\Lambda)$ . Therefore,  $\mathcal{T}$  is by  $j(\Lambda)^j$ .  $\square$

### 3.6. Boolean-valued comparison and ZFC comparison of hod pairs

Suppose  $(\mathcal{P}, \Sigma)$  is a reasonable hod pair such that  $\Sigma$  is  $\omega_1$ -UB. Suppose  $p \in Coll(\omega, \omega_1^V)$  and  $G \subset Coll(\omega, \omega_1^V)$  is  $V$ -generic and  $p \in G$ ; let  $g \subseteq \mathbb{P}_{\mathcal{I}}$  be the corresponding generic induced by  $G, \pi$  and  $j_g : V \rightarrow M$  be the corresponding generic embedding. Suppose  $\Gamma \in V[G]$  is an inductive-like pointclass.<sup>56</sup> For each  $q \leq p$ , let  $G_p = G - G \restriction dom(q) \cup q$  be the “finite variation” of  $G$  induced by  $q$ . Note that  $V[G] = V[G_q]$  for all  $q \leq p$ ; for each  $q$ , let  $g_q \subseteq \mathbb{P}_{\mathcal{I}}$  be the corresponding generic induced by  $G_q, \pi$  and  $j_{g_q} : V \rightarrow M_q$  be the corresponding generic embedding. In  $V[G]$ , suppose  $(\mathcal{P}_q, \Lambda_q)$  is a (countable)  $\Sigma$ - $\Gamma$ -suitable mouse with  $\Lambda_q$  being a  $(\omega_1, \omega_1 + 1)$ ,  $\Gamma$ -fullness preserving strategy for  $\mathcal{P}_q$  and  $\Lambda_q$  is strongly guided by a sjs  $\mathcal{A}_q$  that seals  $\underline{Env}(\Gamma)$  (see Remark 3.18). Then Woodin’s Boolean comparison theorem ([14]) gives us that we can compare  $\{(\mathcal{P}_q, \Lambda_q) : q \leq p\}$  in  $V[G]$  and the comparison results in a pair  $(\mathcal{R}, \Lambda)$  such that  $\mathcal{R} \in V$ ,  $|\mathcal{R}|^V \leq \omega_1$ ,  $\Lambda \restriction H_{\omega_2^V}^V \in V$ . Furthermore,  $\Lambda$  is the tail of all the  $\Lambda_q$ ’s via the iteration trees that appear in the comparison.

<sup>55</sup>This fact can be easily verified, by chasing through the definition of pseudo-hull. See [30]. Furthermore, [17, Theorem 7.3] gives that if  $\mathcal{T}$  is nondropping, then so is  $\mathcal{U}$ .

<sup>56</sup>In cases of interest,  $\Gamma$  is typically of the form  $j_g(\Gamma^*)$  for some inductive-like  $\Gamma^* \in V$ .

In our present context,<sup>57</sup> we only know  $\Lambda_q$  is an  $(\omega_1, \omega_1)$ -iteration strategy in  $V[G]$  for each  $q$ . However, we can still conclude the comparison above terminates in less than  $\omega_1^{V[G]} = \omega_2^V$  many steps. This is because by  $\Sigma_1$ -reflection (inside the pointclass  $\Gamma$ ), we have that for every  $q \leq p$ , there is a countable tree  $\mathcal{T}_q$  such that:

- $\mathcal{T}_q$  is correctly guided, i.e. whenever  $\alpha < lh(\mathcal{T}_q)$  is limit, then  $\mathcal{Q}(\mathcal{T}_q \restriction \alpha)$  exists and  $\mathcal{Q}(\mathcal{T}_q \restriction \alpha) \triangleleft \text{Lp}_+^{\Sigma, \Gamma}(\mathcal{M}(\mathcal{T}_q \restriction \alpha))$ .<sup>58</sup>
- $\mathcal{T}_q$  is maximal and has last model  $M_q = \text{Lp}_+^{\Sigma, \Gamma}(\mathcal{M}(\mathcal{T}_q))$ .
- $\{\mathcal{T}_q : q \leq p\}$  are obtained by the least-extender disagreement process.

Futhermore, for any  $q \neq r$  such that  $q, r \leq p$ ,

- $M_q = M_r$ .

The tree  $\mathcal{T}_q$ 's above are precisely the trees occurred during the Boolean comparison process. This is possible because  $\mathcal{T}_q$  is countable; that  $\mathcal{T}_q$  is countable is a consequence of the fact that  $\omega_1$  is measurable in  $\Gamma$ . Therefore, the comparison process succeeds and results in  $(\mathcal{R}, \Lambda)$  above.<sup>59</sup>

We now introduce concepts needed for the proof of Claim 6.21. In essence, the proof of Claim 6.21 is a proof that a Boolean comparison between hod pairs  $\{(\mathcal{P}_q, \Lambda_q) : q \in \text{Coll}(\omega, \omega_1^V)\}$ <sup>60</sup> terminates in  $V[G]$  (in less than  $\omega_1$  many steps), where for each  $q$ ,  $\mathcal{P}_q$  is a hod mouse such that  $\lambda^{\mathcal{P}_q}$  is a limit ordinal and  $\Lambda_q$  is an  $(\omega_1, \omega_1)$ -strategy with branch condensation and for each  $(\mathcal{Q}, \Psi) \in B(\mathcal{P}_q, \Lambda_q)$ ,  $\Psi$  is a  $(\omega_1, \omega_1 + 1)$ -strategy and  $\Psi \restriction HC$  belongs to an  $\text{AD}^+$  model. Furthermore, we assume that for  $p \neq q$ ,  $(\mathcal{P}_q, \Lambda_q), (\mathcal{P}_p, \Lambda_p)$  are hod pairs of the “same kind”; this means whenever  $(\mathcal{Q}_1, \Psi_1) \in B(\mathcal{P}_q, \Lambda_q)$  and  $(\mathcal{Q}_2, \Psi_2) \in B(\mathcal{P}_p, \Lambda_p)$  (see Section 3.2), and suppose there is  $\alpha < \min(\lambda^{\mathcal{Q}_1}, \lambda^{\mathcal{Q}_2})$  such that  $(\mathcal{Q}_1(\alpha), (\Psi_1)_{\mathcal{Q}_1(\alpha)}) = (\mathcal{Q}_2(\alpha), (\Psi_2)_{\mathcal{Q}_2(\alpha)})$ , then there are normal trees  $\mathcal{T}_i$  according to  $\Psi_i$  on the window  $(\delta_\alpha^{\mathcal{Q}_i}, \delta_{\alpha+1}^{\mathcal{Q}_i})$  such that letting  $\mathcal{R}_i$  be the final model of  $\mathcal{T}_i$  and  $\Lambda_i = (\Psi_i)_{\mathcal{T}_i, \mathcal{R}_i}$ , then  $(\mathcal{R}_1(\alpha + 1), (\Lambda_1)_{\mathcal{R}_1(\alpha+1)}) = (\mathcal{R}_2(\alpha + 1), (\Lambda_2)_{\mathcal{R}_2(\alpha+1)})$ .

Typically,  $\Psi_1, \Psi_2$  are Suslin co-Suslin in an  $\text{AD}^+$  model  $X$  (e.g.  $X$  is of the form  $L(A, \mathbb{R})$  for  $A \in \Gamma$ ). We let  $(N, \delta, \Sigma)$  be a coarse  $\Omega$ -Woodin mouse for some inductive-like pointclass  $\Omega \in X$  that contains all projective sets in  $(\Psi_1, \Psi_2)$  and  $(N, \delta, \Sigma)$  Suslin captures  $\Psi_1, \Psi_2$ .<sup>61</sup> More precisely,  $(N, \delta, \Sigma)$  has the following properties:

- $N \models \text{ZFC}$ .
- $\delta$  is the unique Woodin cardinal of  $N$ .
- $\Sigma$  is an iteration strategy for  $N$ .

<sup>57</sup>Another context, where the conditions for  $\Lambda_q$ 's below may not be satisfied, occurs in the proof of Claim 5.6. We will show in that case the Boolean comparison still succeeds.

<sup>58</sup>We note that the fatal drop cases can be ruled out in the boolean comparison.

<sup>59</sup>Notice we never referred to the strategies  $\Lambda_q$  in the above process.  $\Lambda_q$  is used to define  $\Lambda_q(\mathcal{T}_q)$  at the end (i.e. picking the last, maximal branch of  $\mathcal{T}_q$ ).  $\Lambda_q$  is not in  $j(\Gamma_g)$  a priori.

<sup>60</sup>More generally, we compare pairs  $(\mathcal{P}_q, \Lambda_q)$  for  $q \leq p$ , for some fixed condition  $p$ .

<sup>61</sup>See [23] for more details on coarse Woodin mice.

- $\mathcal{Q}_1, \mathcal{Q}_2 \in N$ .
- For each  $i \in \{1, 2\}$ , there are trees  $(T_i, U_i) \in N$  that witnesses  $(N, \delta, \Sigma)$  Suslin captures  $\Psi_i$  at  $\delta$ , i.e. for any countable  $\Sigma$ -iterate  $N'$  of  $N$  such that there is an iteration map  $i : N \rightarrow N'$ , for any  $h \subset \text{Coll}(\omega, i(\delta))$  such that  $h \in V$  is  $N'$ -generic,  $p[i(T_i)] \cap N'[h] = \Psi_i \cap N'[h]$  and  $p[i(U_i)] \cap N'[h] = \mathbb{R}^{N'[h]} - \Psi_i$ .<sup>62</sup>

The existence of  $\mathcal{T}_i$  is then easy to see. Let  $\Lambda = (\Psi_1)_{\mathcal{Q}_1(\alpha)} = (\Psi_2)_{\mathcal{Q}_2(\alpha)}$ . In  $N$ , iterate  $(\Psi_1)_{\mathcal{Q}_1(\alpha+1)}$  and  $(\Psi_2)_{\mathcal{Q}_2(\alpha+1)}$  into the  $\Lambda$ -hod mouse construction of  $V_\delta^N$ . Since these two strategies have branch condensation, there are normal trees  $\mathcal{T}_i$  (as specified above) and iteration maps  $k_i : \mathcal{Q}_i \rightarrow \mathcal{R}_i$  according to  $\Psi_i$  such that

- (a)  $(\mathcal{R}_1(\alpha+1), (\Lambda_1)_{\mathcal{R}_1(\alpha+1)}) = (\mathcal{R}_2(\alpha+1), (\Lambda_2)_{\mathcal{R}_2(\alpha+1)})$ .
- (b)  $\mathcal{R}_1(\alpha+1)$  is model in the  $\Lambda$ -hod mouse construction of  $V_\delta^N$  and  $(\Lambda_1)_{\mathcal{R}_1(\alpha+1)}$  is the background induced strategy.

See [7] for more details. The above argument generalizes easily to countably many hod pairs (as in the proof of Claim 6.21).

The comparisons described above are the building blocks of the “diamond comparison” described in Claim 6.21. The “diamond comparison” of all pairs of the form  $(\mathcal{P}_q, \Lambda_q)$  for  $q \in \text{Coll}(\omega, \omega_1)$  must end in  $< \omega_1$  steps in  $V[G]$ ; see the proof of Claim 6.21 for more details.

#### 4. OUTLINE OF THE PROOF OF THEOREM 1.4

We outline the proof of Theorem 1.4. In  $V$ , define the maximal pointclass

$$\Gamma = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \models \text{AD}^+\}.$$

The goal is to show that  $\Gamma$  is sufficiently rich in that there is a  $\Omega \subseteq \Gamma$  such that  $L(\Omega, \mathbb{R}) \models \text{“AD}_{\mathbb{R}} + \Theta$  is regular”}. So suppose not. We assume:

- ( $\dagger$ ) : For any transitive  $N$  such that  $\text{Ord} \cup \mathbb{R} \subset N$  and  $N \models \text{ZF} + \text{AD}^+$ ,  $N$  does not satisfy “ $\text{AD}_{\mathbb{R}} + \Theta$  is regular.”

As part of the induction, we maintain:

- ( $\dagger$ ): For all  $\Sigma \in \Gamma$ , all  $\Sigma$ -cmi operators are  $\omega_1$ -UB.

We will analyze the complexity of  $\Gamma$ , ultimately showing that there is some Wadge initial segment  $\Omega$  of  $\Gamma$  (possibly  $\Omega = \Gamma$ ) such that  $L(\Omega, \mathbb{R}) \models \text{“AD}_{\mathbb{R}} + \Theta$  is regular.” There are two major cases. We summarize the key points of each case below before jumping into the details.

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<sup>62</sup>Here we fix a canonical coding of elements of  $HC$  by reals identify  $\Psi_i$  with its code.

- (i) The successor case (Section 5): we first show that if  $(\mathcal{P}, \Sigma) \in \Gamma$  ( $(\mathcal{P}, \Sigma)$  may be  $\emptyset$ ) is a hod pair such that  $\Sigma$  is  $\Gamma$ -fullness preserving and has branch condensation, then  $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+$ , and therefore  $\wp(\mathbb{R}) \cap \text{Lp}^\Sigma(\mathbb{R}) \subset \Gamma$ . This is via a standard core model induction argument similar to that showing  $\text{AD}$  holds in  $L(\mathbb{R})$  ([14, 34]). One wrinkle that appears in the case that  $\Sigma \neq \emptyset$  is that one needs to show  $\mathcal{M}_1^{\Sigma, \sharp}$  exists before being able to define  $\text{Lp}^\Sigma(\mathbb{R})$  as done in [16]. The argument showing that  $\mathcal{M}_1^{\Sigma, \sharp}$  exists is given in Theorem 5.1.

As part of the induction, we maintain  $(\dagger)$ , the hypothesis that for every  $\Sigma$ -cmi operator  $J$  (including the operator induced by  $\Sigma$ ),  $J$  is  $\omega_1$ -UB. This is what we need to carry out the proof of Theorem 5.1. This then allows us to adapt the standard arguments in [14, 34] to show  $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+$ .

In Section 5 (see in particular Theorem 5.5), we adapt the argument in [34] to show that there is a self-justifying system  $\mathcal{A}$  consisting of sets Wadge cofinal in  $\text{Lp}^\Sigma(\mathbb{R})$ , and a  $\Sigma$ -suitable pair  $(\mathcal{Q}, \Lambda)$  where  $\Lambda$  is the strategy guided by  $\mathcal{A}$ .<sup>63</sup> Therefore,  $\Lambda$  is  $\Gamma$ -fullness preserving and has branch condensation and  $\Lambda \notin \text{Lp}^\Sigma(\mathbb{R})$ .

We can then show  $\text{Lp}^\Lambda(\mathbb{R}) \models \text{AD}^+$  and therefore  $\Lambda \in \Gamma$ . To do this, we first need to show some such  $\Lambda$  can be extended to an  $\omega_2$ -strategy in  $V$  and is  $\omega_1$ -UB (Theorem 5.12). Crucially, we use Lemma 3.36 in this argument.

- (ii) The limit case (Section 6): assuming  $(\ddagger)$  and letting  $\mathcal{H}, \mathcal{H}^+$  and  $\Sigma$  be defined as in Section 6, we use the generic embedding  $j : V \rightarrow M$  induced by a  $V$ -generic  $G \subset \text{Coll}(\omega, \omega_1)$  to derive a nice strategy  $\Lambda$  for  $\mathcal{H}^+$  in  $M$ . The strategy  $\Lambda$  is  $j(\Gamma)$ -fullness preserving, has branch condensation, and most importantly, if  $\Gamma(\mathcal{H}^+, \Lambda) \subsetneq j(\Gamma)$ , then letting  $\mathcal{M}_\infty(\mathcal{H}^+, \Lambda)$  be the direct limit of non-dropping iterates of  $(\mathcal{H}^+, \Lambda)$  in  $j(\Gamma)$ , we have  $\mathcal{M}_\infty(\mathcal{H}^+, \Lambda) = \mathcal{H}(\delta)$  where  $\delta = \delta^{\mathcal{M}_\infty(\mathcal{H}^+, \Lambda)}$ , and there is a factor map  $\sigma : \mathcal{M}_\infty(\mathcal{H}^+, \Lambda) \rightarrow j(\mathcal{H}^+)$  such that  $\text{crt}(\sigma) = \delta$ . This property is a consequence of the *j-condensation lemma*, Theorem 6.17. This result is crucial here and its variations are important in many other arguments (cf. [8, 10, 32]).

Again, Lemma 3.36 will be useful in proving Theorem 6.17 and Lemma 6.11. Part of the proof of Lemma 6.11 is to show that  $j$  is continuous at  $o(\mathcal{H}^+)$ . This continuity property is also important in the proof of Theorem 6.17.

Now there are two cases. Suppose first that  $\Gamma(\mathcal{H}^+, \Lambda) = j(\Gamma)$ . Then by elementarity, in  $V$  there is a hod pair  $(\mathcal{P}, \Sigma)$  such that  $\Gamma(\mathcal{P}, \Sigma) = \Gamma$ ; in particular,  $\Sigma \notin \Gamma$ . By a core model induction as in the successor case,  $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+$ . To show this, we again have to show we can extend  $\Sigma$  to  $H_{\omega_2}^V$  and that  $\Sigma$  is  $\omega_1$ -UB (see Lemma 6.20). This implies  $\Sigma \in \Gamma$ . Contradiction. Otherwise,  $\Gamma(\mathcal{H}^+, \Lambda) \subsetneq j(\Gamma)$ . Therefore  $\sigma$  exists and  $\delta$  is a regular cardinal which is a limit of Woodin cardinals in  $\mathcal{M}_\infty(\mathcal{H}^+, \Lambda)$ . By standard arguments,  $L(j(\Gamma) \upharpoonright \delta, \mathbb{R}^M) \models \text{“AD}_\mathbb{R} + \Theta \text{ is regular.”}$  This is again a contradiction, so  $(\ddagger)$  fails. This completes the outline of the proof.

<sup>63</sup>This argument allows us to construct  $(\mathcal{Q}, \Lambda)$  without the technical hypothesis  $\text{HI}(\text{c})$  in Ketchersid’s thesis. See [14, 5] for an alternative argument constructing  $(\mathcal{Q}, \Lambda)$  that uses a seemingly stronger hypothesis.

## 5. SUCCESSOR STEP

We assume throughout this section CH and there is a dense ideal  $\mathcal{I}$  on  $\omega_1$ . Let  $G \subset \text{Coll}(\omega, \omega_1^V)$  be  $V$ -generic and  $g \subseteq \mathbb{P}_{\mathcal{I}}$  be the corresponding generic induced by  $G, \pi$ , where  $\pi$  is as in Fact 2.4. Let  $j = j_g : V \rightarrow \text{Ult}(V, g) = M$  be the corresponding generic embedding.

Recall the pointclass  $\Gamma$  defined in the previous section. Suppose  $(\mathcal{P}, \Sigma) \in \Gamma$  is a reasonable hod pair such that  $\Sigma$  is  $\Gamma$ -fullness preserving, has branch condensation, and  $\Sigma$  is  $\omega_1$ -UB (i.e. we assume the hypothesis  $(\dagger)$  holds for  $\Sigma$ ). This includes the case  $(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$ . We show that  $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+$ . See Sections 3.1, 3.3 for a summary of mouse operators and  $\Sigma$ -cmi operators and related concepts like the definition of  $\mathcal{F}_{\Sigma, \varphi}, \varphi_{\text{all}}$  etc.

**Theorem 5.1.** *Suppose  $\mathcal{F}$  is a nice mouse operator (or a  $\Sigma$ -cmi operator) on  $H_{\omega_1}^V$  that is  $\omega_1$ -UB, then  $\mathcal{M}_1^{F, \#}$  is a nice operator (or a  $\Sigma$ -cmi operator) and is  $\omega_1$ -UB.*

*Proof.* We assume that  $\mathcal{F}$  is a  $\Sigma$ -cmi operator where  $\Sigma \in \Gamma$  has branch condensation, is  $\Gamma$ -fullness preserving, and is  $\omega_1$ -UB. We consider the case  $\mathcal{F} = \mathcal{F}_{\Sigma, \varphi}$ , where  $\mathcal{F}_{\Sigma, \varphi}$  is the operator induced by  $\Sigma$  and with  $\varphi = \varphi_{\text{all}}$  and  $\mathcal{F}^+$  the canonical extension of  $\mathcal{F}$  in  $V[g]$ . The case  $\mathcal{F} \neq \mathcal{F}_{\Sigma, \varphi}$  is similar. The operator  $\mathcal{F}$  codes up the same information as  $\Sigma$  does; the reader will lose little by pretending  $\mathcal{F} = \Sigma$ .

The proof that  $\mathcal{F}^\#$  exists and is  $\omega_1$ -UB is standard. Details have been given in [34, 14]. We only mention some key points here. The operator  $\mathcal{F}^+$ , the unique extension of  $\mathcal{F}$  in  $V$ , is simply  $j(\mathcal{F}) \restriction V$ . Since  $\mathcal{F}$  satisfies  $(\dagger)$ , the hypothesis DI will imply that  $j(\mathcal{F}) \restriction V$  is in  $V$  and doesn't depend on  $G$ ; this follows from Lemma 2.5 and homogeneity of the forcing  $\text{Coll}(\omega, \omega_1)$ . We will write  $\mathcal{F}$  for  $\mathcal{F}^+$  for brevity.

To see  $\mathcal{F}^\#(x)$  is defined for each  $x \in \text{dom}(\mathcal{F})$ , note that from  $j$ , one can define an ultrafilter  $\mu$  over  $L^{\mathcal{F}}[x]$ <sup>64</sup> as follows: for each  $A \in \wp(\omega_1^V) \cap L^{\mathcal{F}}[x]$ ,

$$A \in \mu \iff \omega_1 \in j(A).$$

By a standard argument,  $\mu$  is a countably complete, normal measure over  $L^{\mathcal{F}}[x]$  that is amenable to  $L^{\mathcal{F}}[x]$  in the sense that for any  $Y$  of size  $\omega_1^V$  in  $L^{\mathcal{F}}[x]$ , we have  $\mu \cap Y \in L^{\mathcal{F}}[x]$ . Furthermore, by condensation properties of  $\mathcal{F}$  we have  $\text{Ult}(L^{\mathcal{F}}[x], \mu) = L^{\mathcal{F}}[x]$  as it embeds into  $j(L^{\mathcal{F}}[x])$ . By standard arguments due to Kunen, the amenable structure  $(L^{\mathcal{F}}[x], \mu)$  is iterable. This implies  $\mathcal{F}^\#(x)$  exists.

To prove  $\mathcal{M}_1^{F, \#}$  exists, we need to build the  $K^{c, \mathcal{F}}$ -construction inside  $N = L^{\mathcal{F}^\#}(\mathbb{R})$  and run the proof of [14, Theorem 2.10.2]. For contradiction, we get for some  $x \in \mathbb{R}$ , the core model (relative to  $\mathcal{F}$ )  $K = K^{\mathcal{F}}(x)$  exists (and iterable) in  $N$ . We need that  $j(K) \in V$ . To show this, we need to show  $j(N)$  is definable in  $V[g]$  from parameters in  $V$ . Here are some details that execute this plan.

We define the following model  $W$  by induction on  $\alpha < \omega_2^V$ :  $W_0 = (HC^V, \in)$ ,

$$W_{\alpha+1} = J_\omega(\text{tr.cl.}(W_\alpha \cup \{(\mathcal{T}, b) : b = \Sigma(\mathcal{T}) \wedge \mathcal{T} \in W_\alpha \wedge \mathcal{T} \text{ is according to } \Sigma\})), \quad ^{65}$$

<sup>64</sup>This is the model  $L_{c+}^{\mathcal{F}}[x]$ .

<sup>65</sup>Equivalently,  $W_{\alpha+1} = J_\omega(\text{tr.cl.}(W_\alpha \cup \{(x, \mathcal{F}(x)) : x \in W_\alpha\}))$ .

and for  $\alpha$  limit,  $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ . Finally, let  $W_{\mathcal{F}} = W = \bigcup_{\alpha < \omega_2^V} W_\alpha$ . Note that  $W \in V$  and  $\Sigma \upharpoonright W_\alpha \in W$  for all  $\alpha < \omega_2^V$ .

By the proof of [7, Lemma 3.35], we have the following.

**Fact 5.2.** *For any poset  $\mathbb{P} \in W$  and any  $W$ -generic  $h \subset \mathbb{P}$  such that  $h \in V$  (or  $h \in M$ ),  $W[h]$  is closed under  $\Sigma$  (respectively  $j(\Sigma)$ ).<sup>66</sup>*

Let  $\Sigma^+$  be the canonical extension of  $\Sigma$  to  $V[g]$ . We fix trees  $T, U \in V$  witnessing  $\Sigma$  is  $\omega_1$ -UB. So in  $V[g]$ ,  $\text{Code}(\Sigma^+) = p[T] = \mathbb{R} \setminus p[U]$ . Note also that  $j(\Sigma) \upharpoonright V = \Sigma^+ \upharpoonright V$ . Suppose  $h \in V$  (or in  $M$ ) is a generic enumeration of  $\mathbb{R}^V$  in order type  $\omega_1^V$ , let  $X_h = \bigcup_{\alpha < \omega_2^V} X_\alpha$ , where  $X_0 = \text{tr.cl.}(h \cup \{h\})$ ,  $X_1 = \mathcal{F}_0^+(X_0)$ , and for  $\alpha \geq 1$ ,  $X_{\alpha+1} = \mathcal{F}_1^+(X_\alpha)$ , and  $X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha$  for  $\lambda$  a limit ordinal; here  $\mathcal{F}^+$  codes  $\Sigma^+$  the same way  $\mathcal{F}$  codes  $\Sigma$ . We note that  $X_h$  contains  $\mathbb{R}^V$  and is closed under  $\Sigma^+$ . Now, if we let  $W^{X_h}$  be the structure  $W_{\mathcal{F}}$  defined as above, but the definition is carried out inside  $X_h$ , then

$$W^{X_h} = W.$$

This means that the model  $W$  is independent of  $h$ .

In a similar manner, letting  $\mathcal{G} = (\mathcal{F}^+)^\sharp$ , we define  $X_h = \bigcup_{\alpha < \omega_2^V} X_\alpha$ , where  $X_0 = \text{tr.cl.}(h \cup \{h\})$ ,  $X_1 = \mathcal{G}_0(X_0)$ , and for  $\alpha \geq 1$ ,  $X_{\alpha+1} = \mathcal{G}_1(X_\alpha)$ , and  $X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha$  for  $\lambda$  a limit ordinal<sup>67</sup> and let  $W$  be the model  $W_{\mathcal{G}}$  defined in  $X_h$  as above, but using  $\mathcal{G}$  instead of  $\mathcal{F}$ . In particular, it is easy to verify that  $W$  now has the following properties:

- $o(W) = \omega_2$  and  $W$  is a transitive model over  $\mathbb{R}^V$ ;
- for any  $a \in H_{\omega_2} \cap W$ , we have  $\mathcal{G}(a) \in W$ ; in particular,  $W$  is closed under  $\Sigma^+$  and if  $h \in V$  (or in  $M$ ) is  $W$ -generic, then  $W[h]$  is closed under  $\Sigma^+$ ;
- $W$  is independent of  $h$ ; in other words, suppose  $h_1, h_2 \in V$  (or in  $M$ ) are two enumerations of  $\mathbb{R}$  in order type  $\omega_1$ , then  $W^{X_{h_1}} = W^{X_{h_2}}$ .
- If  $h \in V$  (or in  $M$ ) is  $\text{Coll}(\omega_1, \mathbb{R}^V)$ -generic over  $W$ , then the universe of  $X_h$  is just the universe of  $W[h]$ .

Suppose that on a cone of  $x \in HC$ ,  $\mathcal{M}_1^{\mathcal{F}, \sharp}(x)$  does not exist. Then in  $W[h]$  where  $h \in V$  is  $\text{Coll}(\omega_1, \mathbb{R}^V)$ -generic over  $W$ , the core model  $K =_{\text{def}} K^{\mathcal{F}^+}(x)$  exists<sup>68</sup>. Here  $K$  is a  $\mathcal{F}$ -mouse and is in  $W$ .

**Claim 5.3.**  $j(K) \in V$ .

*Proof of Claim 5.3.* To see that  $j(K) \in V$ , it suffices to show that  $j(W)$  is definable in  $V[G]$  from parameters in  $V$ . To see this, first note that  $j(j(\Sigma) \upharpoonright V)$  is definable in  $\text{Ult}(V, g) = M$  as the unique

<sup>66</sup>[7, Lemma 3.35] indeed implies that  $\mathcal{F}$  determines itself on generic extensions. It is also easy to see that  $\mathcal{F}$  relativizes well.

<sup>67</sup> $X_h$  is a potential  $\mathcal{G}$ -premouse over  $h$  and it is closed under  $\mathcal{G}$  because  $\mathcal{G}$  relativizes well.

<sup>68</sup>Here the core model relative to  $\Sigma^+$  is defined in the sense of [4] and  $o(K) = \xi < o(W)$  and  $\omega_1^V < \xi$  is a sufficiently large indiscernible relative to  $\mathcal{G}$ .



extension of  $j(\Sigma)$  to  $H_{\omega_2}$  that has branch condensation. Let  $\Lambda = j(j(\Sigma) \upharpoonright V)$ . Note that  $\Lambda$  may not be definable in  $V[G]$ ; the main wrinkle is that  $H_{\omega_2}^{V[G]}$  may differ from  $H_{\omega_2}^{\text{Ult}(V,g)}$ . But in  $V[G]$  we can define  $\Psi$ , the maximal (partial) strategy on  $H_{\omega_2}$  that extends  $j(\Sigma)$  with the property that whenever  $\mathcal{T}$  is according to  $\Psi$ , the branch  $\Psi(\mathcal{T})$  (if defined) is the unique well-founded branch  $b$  such that whenever  $(\mathcal{U}, c)$  is a countable hull of  $(\mathcal{T}, b)$ , then  $\mathcal{U}$  is according to  $j(\Sigma)$  and  $c = j(\Sigma)(\mathcal{U})$ . Note that if  $\Lambda_1$  and  $\Lambda_2$  are partial strategies extending  $j(\Sigma)$  satisfying the above properties, then whenever  $\mathcal{T} \in \text{dom}(\Lambda_1) \cap \text{dom}(\Lambda_2)$ , we have  $\Lambda_1(\mathcal{T}) = \Lambda_2(\mathcal{T})$ . As a result,  $\Psi$  is simply the union of all such partial strategies, and since  $\Lambda$  is one such partial strategy,

$$\Lambda \subseteq \Psi.$$

This easily implies that  $j(W)$  is definable in  $V[G]$  from  $\Psi$  as  $\Psi \upharpoonright W = \Lambda \upharpoonright W$ . Hence  $j(W)$  is definable in  $V[G]$  from  $j(\Sigma)$ , but  $j(\Sigma) = p[j(T)] \cap V^{\text{Ult}(V,g)} = p[T] \cap V[g]$  (by Lemma 2.5); so  $j(W)$  is definable in  $V[G]$  from  $(T, U)$ .<sup>69</sup> By homogeneity,  $j(W) \in V$ .  $\square$

Given this claim, the rest of the proof proceeds as in [14, Theorem 2.10.2] by showing that for the  $(\omega_1^V, \omega_2^V)$ -extender  $E$  derived from  $j$ , we have  $E \upharpoonright \alpha \in j(K)$  for all  $\alpha < \omega_2^V$ . This implies that  $\omega_1^V$  is Shelah in  $j(K)$ , contradiction. Fixing  $\alpha < \omega_2^V$ , we give a sketch of  $E \upharpoonright \alpha \in j(K)$ . We note again that  $W$  is closed under  $\mathcal{G}$ . We need to see that the phalanx  $(j(K), \text{Ult}(j(K), E \upharpoonright \alpha), \alpha)$  is iterable in  $j(W)$ .<sup>70</sup> Otherwise in  $j(W)$  there is a countable  $\mathcal{F}$ -premouse  $\bar{K}$  and a map  $\sigma : \bar{K} \rightarrow \text{Ult}(K, E \upharpoonright \alpha)$  with  $\text{crt}(\sigma) = \alpha$  and

$$j(W) \models (j(K), \bar{K}, \alpha) \text{ is not } \omega_1\text{-iterable}.$$

We have a factor map  $k : \text{Ult}(K, E \upharpoonright \alpha) \rightarrow j(j(K))$  with  $k \upharpoonright \alpha = \text{id}$  and

$$k \circ \sigma : \bar{K} \rightarrow j(j(K))$$

such that  $k \circ \sigma \upharpoonright \alpha = \text{id}$ . Note that  $j(j(K))$  makes sense by the claim above.

Let  $\psi = k \circ \sigma$  and  $\psi = [\beta \mapsto \psi_\beta]_G$ . Let  $\bar{K} = [\beta \mapsto K_\beta]_G$  and  $\alpha = [\beta \mapsto \alpha_\beta]_G$ . We need to see that for  $G$ -almost all  $\beta$ ,

$$W \models (K, K_\beta, \alpha_\beta) \text{ is } \omega_1\text{-iterable}.$$

By absoluteness, in  $j(W)$  there is some  $\psi'_\beta : K_\beta \rightarrow j(K)$  such that  $\psi'_\beta \upharpoonright \alpha_\beta = \text{id}$ . Then in  $W$  there is some  $\bar{\psi} : K_\beta \rightarrow K$  such that  $\bar{\psi} \upharpoonright \alpha_\beta = \text{id}$ . But this means  $(K, K_\beta, \alpha_\beta)$  is iterable in  $W$ . We have reached a contradiction.

Finally, the operator  $H : x \mapsto \mathcal{M}_1^{\mathcal{F}, \sharp}(x)$  is definable from  $\Sigma$ . Since  $j(\Sigma) \upharpoonright V \in V$ , we have  $j(H) \upharpoonright V \in V$  also. It is then standard to show  $H$  is  $\omega_1$ -UB. One shows that for club many countable  $X \prec (H_{\omega_2}, \in, (T, U))$ ,  $X$  is generically correct about  $H$ , namely letting  $\pi_X : M_X \rightarrow X$  be the uncollapse map, for any forcing  $\mathbb{P} \in M_X$  such that

$$M_X \models “|\mathbb{P}| \leq \omega_1”,$$

<sup>69</sup>This is the crucial point and is the reason we maintain that operators we construct in this core model induction are  $\omega_1$ -UB.

<sup>70</sup>Iterability here is with respect to trees of length  $< j(\xi)$  in  $j(W)$ .

for any  $M_X$ -generic  $g \subset \mathbb{P}$  such that  $g \in V$ , then for any  $x \in HC \cap M_X[g]$ ,

$$V \models \varphi[x, (T, U)] \Leftrightarrow M_X[g] \models \varphi[x, \pi_X^{-1}(T, U)],$$

here  $\varphi(x, (T, U))$  is the natural formula that defines  $H(x)$  from  $\mathcal{F}$ . We give an informal definition of  $\varphi(x, y)$  here.  $\varphi(x, y)$  is the statement: there is a unique  $z$  such that

- (a)  $z$  has the first order properties of  $\mathcal{M}_1^{\mathcal{F}, \#}(x)$ , where  $Code(\mathcal{F}) = p[(y)_0] = \mathbb{R} - p[(y)_1]$ .
- (b)  $z$  has a unique  $(\omega_1, \omega_1 + 1)$ -iteration strategy  $\Lambda$  with the property that whenever  $\mathcal{T}$  is according to  $\Lambda$  with limit length (or  $\mathcal{T}$  is a stack with last normal component with limit length),  $\Lambda(\mathcal{T})$  is the unique  $b$  such that  $\mathcal{M}_b^{\mathcal{T}} \trianglelefteq \mathcal{F}^{\#}(\mathcal{M}(\mathcal{T}))$ .

It is easy to verify that  $H$  is a  $\Sigma$ -cmi operator (i.e.  $H$  condenses finely, relativizes well, and determines itself on generic extension); this is because  $\mathcal{F}$  has these properties. We leave the details to our reader. □

The induction through  $Lp^{\Sigma}(\mathbb{R})$  proceeds as usual and is organized by the scales pattern in  $Lp^{\Sigma}(\mathbb{R})$  (see [16]). The above theorem takes care of the successor steps in the induction in  $Lp^{\Sigma}(\mathbb{R})$ . The limit step is non-trivial and requires the use of our hypothesis when we reach an inductive-like  $\Omega$ ; recall here that a pointclass  $\Omega$  is *inductive-like* if it is  $\omega$ -parametrized, closed under  $\forall^{\mathbb{R}}, \exists^{\mathbb{R}}$ , recursive substitution, and has the scale property. We need to construct an operator that is beyond  $\underline{Env}(\Gamma)$  to continue the induction.<sup>71</sup> We start with a useful lemma.

**Lemma 5.4.** *Suppose  $(\mathcal{P}, \Sigma)$  is a reasonable hod pair such that  $\Sigma$  is  $\omega_1$ -UB. Suppose  $(\mathcal{P}', \Sigma')$  is a pair such that  $\mathcal{P}'$  is a countable  $\Sigma$ -premouse that is  $\Omega$ -suitable for some inductive-like pointclass  $\Omega \subset Lp^{\Sigma}(\mathbb{R})$  and  $\Sigma'$  has branch condensation and is a  $\Omega$ -fullness preserving strategy for  $\mathcal{P}'$  (as a  $\Sigma$ -mouse) that can be uniquely extended to an  $(\omega_2, \omega_2)$ -strategy, then  $\Sigma'$  is  $\omega_1$ -UB.*

*Proof.* We identify  $\Sigma'$  with its unique extension to stacks in  $H_{\omega_2}^V$ . Let  $i : \mathcal{P}' \rightarrow \mathcal{M}_{\infty}$  be the direct limit map of all non-dropping  $\Sigma'$ -iterates via stacks in  $H_{\omega_2}^V$ . For a club of countable  $Y \prec (H_{\omega_3}, \in, (\mathcal{P}', \Sigma'), i, \mathcal{M}_{\infty})$ , let  $\pi_Y : M_Y \rightarrow Y$  be the uncollapse, let  $\kappa_Y = crt(\pi_Y)$ , and let  $a^Y = \pi_Y^{-1}(a)$  for any  $a \in Y$ . Let  $h \subseteq Coll(\omega, \kappa_Y)$  be a  $M_Y$ -generic in  $V$ . Let  $\mathcal{T}, b \in M_Y[h]$ , let  $a = (i, \mathcal{M}_{\infty})$  where  $\mathcal{T}$  is a normal tree, and let  $\varphi(\mathcal{T}, b, a^Y)$  say:

- $\mathcal{T}$  is correctly guided i.e. all strict initial segments of  $\mathcal{T}$  are given by the  $\mathcal{Q}$ -structures in  $C_{j(\Omega)}(\mathcal{M}(\mathcal{T}))$ .<sup>72</sup>

<sup>71</sup>In fact, we need the hypothesis in the construction of the “next” operator when  $\Gamma$  is the last scaled pointclass in  $Lp^{\Sigma}(\mathbb{R})$ .

<sup>72</sup>At this point, we know  $C_{j(\Omega)}(\mathcal{M}(\mathcal{T}))$  is independent of generics  $g$ . To see this, suppose  $g_1, g_2$  are such that letting  $j_i : V \rightarrow M_i$  be the corresponding generic embeddings, and  $Lp^{\Sigma, j_1(\Omega)}(\mathcal{M}(\mathcal{T})) \triangleleft Lp^{\Sigma, j_2(\Omega)}(\mathcal{M}(\mathcal{T}))$ . Let  $\mathcal{M}$  be the least in  $Lp^{\Sigma, j_2(\Omega)}(\mathcal{M}(\mathcal{T})) - Lp^{\Sigma, j_1(\Omega)}(\mathcal{M}(\mathcal{T}))$  and  $\Lambda_{\mathcal{M}}$  be its unique strategy. Note that  $\Lambda_{\mathcal{M}} \in j_2(\Omega)$ ; therefore,  $\Lambda_{\mathcal{M}} \upharpoonright V \in \Omega$  as  $j_2(\Lambda_{\mathcal{M}} \upharpoonright V) = \Lambda_{\mathcal{M}}$ . This means  $j_1(\Lambda_{\mathcal{M}}) \in j_1(\Omega)$ . This contradicts the choice of  $\mathcal{M}$ .

- If  $\mathcal{T}$  is short then  $b$  is the unique cofinal branch such that  $\mathcal{Q}(b, \mathcal{T})$  exists and the phalanx  $\Phi(\mathcal{T} \cap \mathcal{Q}(b, \mathcal{T}))$  is iterable with unique strategy in  $C_{j(\Omega)}(\mathcal{M}(\mathcal{T}))$ .
- If  $\mathcal{T}$  is maximal then  $b$  is the unique non-dropping branch such that there is a map  $\sigma : \mathcal{M}_b^{\mathcal{T}} \rightarrow \mathcal{M}_\infty^Y$  such that  $i^Y = \sigma \circ i_b^{\mathcal{T}}$ .

We need to see that  $M_Y[h] \models \varphi(\mathcal{T}, b, a^Y)$  if and only if  $\Sigma'(\mathcal{T}) = b$ . Suppose first  $\mathcal{T}$  is short. Note that  $H_{\omega_2}^{M_Y}$  is closed under  $\Sigma'$ , so we let  $W \in H_{\omega_2}^{M_Y}$  be transitive such that  $\mathcal{T} \in W[h]$ . Let  $\pi : \mathcal{P} \rightarrow \mathcal{Q}$  be the iteration map given by the generic genericity iteration according to  $\Sigma' \upharpoonright H_{\omega_2}^{M_Y}$  that makes  $W$  generically generic. So  $\mathcal{T} \in \mathcal{Q}[W, h]$  and  $C_{j(\Omega)}(\mathcal{M}(\mathcal{T})) \in \mathcal{Q}[W, h]$  by  $j(\Omega)$ -fullness of  $\mathcal{Q}$  and the fact that the operator  $C_{j(\Omega)}$  relativizes well. Therefore,  $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T}) \in M_Y[h]$ . This shows that  $M_Y[h]$  is correct about the shortness of  $\mathcal{T}$  and can compute the correct  $\mathcal{Q}$ -structure and hence the branch  $\Sigma'(\mathcal{T})$ .

Suppose  $\mathcal{T}$  is maximal. The above calculation shows that this is equivalent to  $\mathcal{T}$  being maximal in  $M_Y[h]$ . If  $M_Y[h] \models \varphi(\mathcal{T}, b, a^Y)$  then there is a  $\sigma : \mathcal{M}_b^{\mathcal{T}} \rightarrow \mathcal{M}_\infty^Y$  such that  $i^Y = \sigma \circ i_b^{\mathcal{T}}$ . In  $V$ , let  $\vec{\mathcal{T}}$  be according to  $\Sigma'$  with last model  $\mathcal{M}_\infty^Y$  such that  $i^Y = i^{\vec{\mathcal{T}}}$ . Then by branch condensation of  $\Sigma'$ ,  $b = \Sigma'(\mathcal{T})$ . Conversely, suppose  $\Sigma'(\mathcal{T}) = b$ . Let  $c = j_h(\Sigma' \upharpoonright M_Y)(\mathcal{T})$ . Then by boolean comparisons, it is easy to see there is a  $\sigma : \mathcal{M}_c^{\mathcal{T}} \rightarrow \mathcal{M}_\infty^Y$  such that  $\sigma \circ i_c^{\mathcal{T}} = i^Y$ . But  $i^Y$  is an iteration map according to  $\Sigma'$  (in  $V$ ), by branch condensation of  $\Sigma'$ ,  $c = \Sigma'(\mathcal{T})$ . So  $b = c$ .

The argument for stacks is similar. We leave the details to the reader. This completes the proof of the lemma. □

**Theorem 5.5.** *Suppose  $\Gamma^* \subset Lp^\Sigma(\mathbb{R})$  is an inductive-like pointclass such that:*

$$\Gamma^* \models \text{AD}^+ + \text{MC}(\Sigma).$$

*Then*

- (a) *for any  $A \in \text{Env}(\Gamma^*)$ , there is a scale on  $A$  whose norms are in  $\text{Env}(\Gamma^*)$ ;*
- (b) *there is a self-justifying system (sjs)  $(A_i : i < \omega)$  sealing  $\text{Env}(\Gamma^*)$ .*

*Proof.* We assume for simplicity that  $\Sigma = \emptyset$ , so  $Lp^\Sigma(\mathbb{R}) = Lp(\mathbb{R})$ ; the general case is just more notationally complicated. We assume  $\Gamma^* = \Sigma_1^{Lp(\mathbb{R})}$ , i.e. the largest scaled pointclass in  $Lp(\mathbb{R})$ . The other cases are taken care of by the scales analysis in  $Lp(\mathbb{R})$  (see [14, 19, 25]). Let  $T$  be the tree of a  $\Gamma^*$ -scale on a universal  $\Gamma^*$  set;  $T$  is a tree on  $\omega \times \kappa$ , where  $\kappa$  is the largest Suslin cardinal of  $P = Lp(\mathbb{R})$ .<sup>73</sup> Let  $A = p[T]$  be the universal  $\Gamma^*$ -set induced by  $T$ . We note that at this point, we know that  $P \models \text{AD}^+$ ; this is because by essentially the Kechris-Woodin transfer theorem (see [34] for more discussions in this particular context),  $\text{AD}^+$  holds for sets in  $\text{Env}(\Gamma^*)$  and  $\text{Env}(\Gamma^*) = \wp(\mathbb{R}) \cap P$  by arguments in [34, Lemma 4.5.1]. We assume for contradiction that (a) (and hence (b)) fails.

<sup>73</sup>The following argument works for  $P = Lp^\Sigma(\mathbb{R})$ . One just needs to put the trees  $T, U$  witnessing  $\Sigma$  is  $\omega_1$ -UB into the parameters that define all the relevant objects below.

**Claim 5.6.** *For any  $V$ -generic  $g \in \mathbb{P}_{\mathcal{I}}$ , suppose  $j_g : V \rightarrow \text{Ult}(V, g) = M$  is the associated ultrapower map and  $G \subseteq \text{Coll}(\omega, \omega_1^V)$  is the  $V$ -generic filter associated with  $g$ , then*

(i)  $j_g \upharpoonright \kappa$  is independent of  $g$

(ii)  $j_g(\kappa)$  is the largest Suslin cardinal of  $(\text{Lp}(\mathbb{R}))^{V[G]}$  and hence is independent of  $g$ .

(iii)  $j_g \upharpoonright (\wp(\kappa))^P$  is independent of  $g$ .

*Proof.* To see (i), let  $\gamma < \kappa$  be arbitrary and let  $A \in \text{Lp}(\mathbb{R})$  be of Wadge rank  $\gamma$ . Note that since  $\gamma < \kappa$ , by our induction hypothesis,  $A$  is  $\omega_1$ -UB as witnessed by  $(S, W)$ . Notice then that by Lemma 2.5,

$$j_g(A) = p[S] \cap V[g].$$

If (i) fails at  $\gamma$ , then there exist  $p \Vdash j_g(\gamma) = \gamma_0$  and  $q \Vdash j_g(\gamma) = \gamma_1$  with  $\gamma_0 \neq \gamma_1$ . Let  $g_0, g_1 \subset \mathbb{P}_{\mathcal{I}}$  be  $V$ -generic such that  $p \in g_0, q \in g_1$ , and  $V[g_1] = V[g_0]$ ; such  $g_0, g_1$  can be easily obtained using the homogeneity of the forcing <sup>74</sup>. Let  $M_0 = j_{g_0}(\text{Lp}(\mathbb{R}))$ ,  $M_1 = j_{g_1}(\text{Lp}(\mathbb{R}))$ . Note that  $M_0 \trianglelefteq M_1$  or  $M_1 \trianglelefteq M_0$ . Write  $j_i$  for  $j_{g_i}$  and note that

$$j_0(\gamma) = \gamma_0 \neq \gamma_1 = j_1(\gamma). \quad (5.1)$$

Note also by the fact that  $V[g_0] = V[g_1]$ ,

$$j_0(A) = j_1(A) = p[S] \cap V[g_1] = p[S] \cap V[g_0]. \quad (5.2)$$

The fact that the Wadge hierarchies of  $M_0, M_1$  are compatible gives us

$$j_0(A) = j_1(A) \in M_0 \cap M_1. \quad (5.3)$$

5.2 and 5.3 give us the Wadge rank of  $j_0(A) = j_1(A)$  is  $j_0(\gamma) = j_1(\gamma)$ , which clearly contradicts 5.1. So (i) holds.

We now show (ii). We do not claim here that  $(\text{Lp}(\mathbb{R}))^{V[G]} \models \text{AD}^+$ . Suppose the statement of the claim is false. Fix  $G, g$  as above witnessing the failure of the claim. Then there is  $\mathcal{M} \triangleleft (\text{Lp}(\mathbb{R}))^{V[G]}$  such that  $j(\Gamma^*)$  is Suslin co-Suslin in  $\mathcal{M}$  and  $\mathcal{M} \models \text{AD}^+$ .

By the scales analysis and MC in  $\mathcal{M}$  (MC holds in  $\mathcal{M}$  by our smallness assumption  $(\dagger)$  and results in [7]), there is a sjs  $\vec{A}$  sealing  $\text{Env}(\widehat{j(\Gamma^*)})$  in  $V[G]$ . Let  $(\mathcal{P}, \Sigma) \in \mathcal{M}$  be guided by  $\vec{A}$ . By Boolean-valued comparisons (described in the previous section), there is an iterate  $(\mathcal{R}, \Lambda)$  of  $(\mathcal{P}, \Sigma)$  such that  $\mathcal{R} \in V$  and  $\Lambda \upharpoonright H_{\omega_2}^V \in V$ . Now,  $\Lambda$  has branch condensation and is  $j(\Gamma^*)$ -fullness preserving and hence by Lemma 3.34 has strong hull condensation. By Lemma 3.36,  $\Lambda = j(\Lambda)^j$ . Therefore,  $\Lambda \in M$  and is  $j(\Gamma^*)$ -fullness preserving.

<sup>74</sup>Given  $p \in g_0$ , we can find an automorphism  $\tau : \mathbb{P}_{\mathcal{I}} \rightarrow \mathbb{P}_{\mathcal{I}}$  such that  $\tau(p_0) \leq q$ . Then let  $g_1 = \tau[g_0]$ .  $g_0, g_1$  are as desired.

Now note that  $\Lambda$  is  $\omega_1$ -UB in  $M$  by Lemma 5.4 and hence self-scaled, and so by the core model induction similar to the above,  $\text{Lp}^\Lambda(\mathbb{R}^M) \models \text{AD}^+$  (here by density,  $\mathbb{R}^M = \mathbb{R}^{V[G]}$ ). This implies that  $L(\Lambda, \mathbb{R}^M) \models \Theta > \theta_0$  since  $\Lambda \notin \text{Lp}(\mathbb{R})^M$ . This in particular implies, via standard results (cf. [34]), that conclusions (a) and (b) hold for  $\text{Env}(j(\Gamma^*))$  in  $M$ . By elementarity, (a) and (b) hold for  $\text{Env}(\Gamma^*)$ . This contradicts our assumption that (a), (b) fail.

To see that  $j_g \upharpoonright \wp^P(\kappa)$  is independent of the choice of  $g$  in (iii), fix a  $\Gamma^*$ -prewellorder  $\preceq$  of  $\mathbb{R}$  of length  $\kappa$ ; by choosing a minimal definition, we can assume  $\preceq$  is definable from a real  $y$  and  $\kappa$  in  $P$ . More precisely, we choose the least  $\xi$  such that  $\text{Lp}(\mathbb{R})|\xi$  ordinal defines such a  $\preceq$  from a real  $y$ . By minimizing the ordinal parameters, we can then get that  $\preceq$  is definable over  $\text{Lp}(\mathbb{R})|\xi$  from  $\{y, \kappa\}$ , say by formula  $\varphi$ . Note that any  $X \in \wp^P(\kappa)$  is  $\Sigma_1^1(\preceq, z)$  for some real  $z$  by the Coding Lemma. Suppose  $X$  witnesses the failure of (c) and  $X$  is  $\Sigma_1^1(\preceq, z)$  for some real  $z$ . Let  $g_0, g_1$  be such that  $V[g_0] = V[g_1]$  and  $j_i = j_{g_i}$  be the associated generic embeddings with the property that  $j_1(X) \neq j_0(X)$ . By part (ii),  $j_0(\kappa) = j_1(\kappa)$ , so let  $\kappa^* = j_0(\kappa) = j_1(\kappa)$ . By the choice of  $\preceq$  and part (i),  $j_0(\preceq) = j_1(\preceq)$ ; this is because  $j_0(\preceq), j_1(\preceq)$  are both definable from  $\{\kappa^*, y\}$  via formula  $\varphi$  over the least  $\mathcal{M} \triangleleft (\text{Lp}(\mathbb{R}))^{V[g_0]}$  that defines a prewellorder of  $\mathbb{R}^{V[g_0]}$  of length  $\kappa^*$ . Since  $j_0(X), j_1(X)$  are  $\Sigma_1^1$ -definable from  $j_0(\preceq)$  from  $y$  via the same formula,  $j_0(X) = j_1(X)$ . Contradiction.  $\square$

**Remark 5.7.** In the proof of Claim 5.6(ii), it appears that we need to assume the failure of Theorem 5.5(a). However, one can show

(ii')  $j_g(\kappa)$  is independent of  $g$

without assuming the failure of Theorem 5.5(a). Suppose (ii') fails. We can then find  $g_1, g_2$  such that  $V[g_1] = V[g_2]$  and  $j_{g_1}(\kappa) < j_{g_2}(\kappa)$ . Let  $j_i = j_{g_i}$  for  $i \in 2$  and  $j_i : V \rightarrow M_i$ . We can run the argument in the proof of Claim 5.6(ii) to get  $(\mathcal{R}, \Lambda)$  as there, where  $\Lambda$  is  $\omega_1$ -UB in  $M_1$ . By elementarity, there is such a pair  $(\mathcal{R}, \Lambda) \in V$  such that  $\Lambda$  is  $\omega_1$ -UB,  $\text{Lp}^\Lambda(\mathbb{R}) \models \text{AD}^+$ , and  $\Lambda \notin \text{Lp}(\mathbb{R})$ . Since  $j_0(\kappa) \neq j_1(\kappa)$ , it is easy to see that  $j_0(\Lambda) \neq j_1(\Lambda)$ . But since  $\Lambda$  is  $\omega_1$ -UB as witnessed by trees  $(T, U)$  and  $V[g_0] = V[g_1]$ ,  $j_0(\Lambda) = p[T] \cap V[g_0] = p[j_1(T)] \cap M_1 = j_1(\Lambda)$ . Contradiction.

From the claim above and homogeneity, we easily see that the value of  $j_g(\kappa), j_g(T)$  is independent of  $g$ ; from now on, we will write  $j(\kappa)$  for  $j_g(\kappa)$  etc. We let  $\text{meas}^{\Gamma^*}(\kappa^{<\omega})$  be the set of countably complete measures on  $(\wp(\kappa^{<\omega}))^P$  in  $P$  and

$$\sigma = j'' \text{meas}^{\Gamma^*}(\kappa^{<\omega}).$$

Note also that  $\sigma$  is independent of  $g$ . Let  $\lambda$  be the length of the well-ordering of  $\text{Env}(\Gamma)$ . We have  $\lambda < j(\omega_1^V) = \omega_2^V$ . It follows that  $j''\lambda$  (and hence also  $\sigma$ ) is in  $\text{Ult}(V, g)$  and is countable there. This then implies that  $\sigma \in M$ .

Let  $\mu \in \sigma$ . Suppose  $\mu$  concentrates on  $j(\kappa)^n$  and let  $\langle \mu_i \mid i \leq n \rangle$  be the projections of  $\mu$ , meaning  $A \in \mu_i \iff \{s \in j(\kappa)^n \mid s \upharpoonright i \in A\} \in \mu$ . Note that  $\mu_0$  is the trivial measure.

In  $\text{Ult}(V, g)$ , we define the following putative scale  $\{\varphi_\mu : \mu \in \sigma\}$  on  $\mathbb{R} \setminus p[j(T)]$  as follows. For each  $\mu \in \sigma$ , and for each  $x \in \mathbb{R} \setminus p[j(T)]$  (so  $j(T)_x$  is well-founded),

$$\varphi_\mu(x) = [\text{rank}_{j(T)_x}]_\mu. \text{ }^{75}$$

We now define the following closed game  $G_{j(T)}^{\sigma, \mu}$  in  $\text{Ult}(V, g)$  (equivalently in  $V[G]$ , recalling that  $\mathbb{R}^{V[G]} = \mathbb{R}^{\text{Ult}(V, g)}$  and the pointclass  $j(\Gamma^*)$  is ordinal definable in  $V[G]$ ): player I starts by playing  $m_0, \dots, m_n$  and  $s_n, h_n$ , and player II responds by playing a measure  $\mu_{n+1}$ . In each subsequent move (numbered  $i > n$ ), player I plays  $m_i, s_i, h_i$ , and player II plays a measure  $\mu_{i+1}$ .

**Rules for player I:**

- $m_k < \omega$  for all  $k < \omega$
- $j(T)_{(m_0, \dots, m_{n-1})} \in \mu = \mu_n$
- $s_i \in j_{\mu_i}(j(T)_{(m_0, \dots, m_i)})$ , and in particular  $s_i \in j_{\mu_i}(j(\kappa))^{i+1}$  for all  $i \geq n$
- $s_n \supsetneq [\text{id}]_{\mu_n}$
- $j_{\mu_i, \mu_{i+1}}(s_i) \subsetneq s_{i+1}$  for all  $i \geq n$
- $h_i \in \text{OR}$  for all  $i \geq n$
- $j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1}$  for all  $i \geq n$

**Rules for player II:**

- $\mu_{i+1} \in \sigma$  is a measure on  $j(\kappa)^{i+1}$  projecting to  $\mu_i$
- $\mu_{i+1}$  concentrates on the set  $j(T)_{(m_0, \dots, m_i)} \subset j(\kappa)^{i+1}$ .

The first player that violates one of these rules loses, and if both players follow the rules for all  $\omega$  moves, then player I wins.

The game is closed, hence determined by the Gale–Stewart theorem. Intuitively, player I is building a real  $x = (m_0, m_1, \dots)$ , player II is trying to build a tower  $\vec{\mu}$  of measures in  $\sigma$  concentrating on  $j(T)_x$ , and player I is trying to build a continuous witness  $\vec{h}$  to the illfoundedness of  $\vec{\mu}$  as well as a special kind of branch  $(j_{i, \infty}(s_i) : i \geq n)$  through the direct limit  $j_{0, \infty}(j(T)_x)$  of  $j(T)_x$  along  $\vec{\mu}$ . The following is the main lemma.

**Lemma 5.8.** *Player II has a winning strategy in the game  $G_{j(T)}^{\sigma, \mu}$ .*

*Proof.* First note that  $j(T) \in V$ ; this is because  $T$  is ordinal definable in  $V$ . The parameter defining  $j(T)$  in  $V[G]$  has the form  $j(s)$  for some finite sequence of ordinals  $s \in V$ . Therefore,  $j(s) \in V$  and  $j(T) \in V$  by homogeneity.<sup>76</sup> In fact, by Claim 5.6 and the remark after,  $j(\kappa), j(T), j \restriction \wp^P(\kappa^{<\omega})$  are independent of  $g$ .

We define a winning strategy for player II in  $G_{j(T)}^{\sigma, \mu}$  in  $\text{Ult}(V, g)$ . Let  $\mu_0, \dots, \mu_n$  be the projections of  $\mu$  in order (here  $\mu_n = \mu$ ). Let  $j(\vec{\mu}_i) = \mu_i$  for  $i = 0, \dots, n$ . Note that for all  $i$ ,

<sup>75</sup> $\text{rank}_{j(T)_x}(t)$  denotes the rank of the node  $t$  in the tree  $j(T)_x$ , and is considered to be zero if  $t \notin j(T)_x$  and undefined if  $j(T)_x$  is illfounded below  $t$ .

<sup>76</sup>In the case  $P = \text{Lp}^\Sigma(\mathbb{R})$ ,  $T$  is ordinal definable from  $\Sigma$  and there are trees  $(W, S)$  witnessing  $\Sigma$  is  $\omega_1$ -UB. Then  $j(T)$  is ordinal definable in  $V[G]$  from  $(W, S)$  by the fact that  $p[W] = p[j(W)]$  and  $p[S] = p[j(S)]$  (see a similar calculation in the proof of Claim 5.3). Therefore,  $j(T) \in V$  by homogeneity.

$$j_{\mu_i} \circ j = j \circ j_{\bar{\mu}_i}.$$

Suppose player I starts the game by playing integers  $m_0, \dots, m_n$ , a finite sequence of ordinals  $s_n \in j_{\mu_n}(j(T_{m_0, \dots, m_n})) \cap j_{\mu_n}(j(\kappa)^{n+1})$ , and an ordinal  $h_n$ . Define the measure  $\bar{\mu}_{n+1} \in \text{meas}^{\Gamma^*}(\kappa^{<\omega})$  as follows.

$$X \in \bar{\mu}_{n+1} \iff s_n \in j_{\mu_n}(j(X)).$$

$\bar{\mu}_{n+1}$  is  $OD^{V[g]}$  from a finite sequence of ordinals, some real  $x \in \mathbb{R}^{V77}$  and  $j_g \restriction \wp^P(\kappa^{<\omega})$ . Since  $j_g \restriction \wp^P(\kappa^{<\omega})$  is independent of  $g$ ,  $\bar{\mu}_{n+1} \in V$ .<sup>78</sup>

For  $i > n$ , suppose player I has played an integer  $m_i$ , a finite sequence of ordinals  $s_i \in j_{\mu_i}(j(T_{m_0, \dots, m_i})) \cap j_{\mu_i}(j(\kappa)^{i+1})$ , and an ordinal  $h_i$ . Define the measure  $\bar{\mu}_{i+1} \in \text{meas}^{\Gamma^*}(\kappa^{<\omega})$  as follows.

$$X \in \bar{\mu}_{i+1} \iff s_i \in j_{\mu_i}(j(X)).$$

As before, the measure  $\bar{\mu}_{i+1}$  is in  $V$ , concentrates on  $T_{m_0, \dots, m_i}$ , and projects to  $\bar{\mu}_i$ . Let player II play the measure  $\mu_{i+1} = j(\bar{\mu}_{i+1})$ .

Assume for contradiction that player I is able to play  $\omega$  many moves, following all the rules of the game. We get a real  $x = (m_0, m_1, \dots)$ , a tower of measures  $(\mu_i : i < \omega)$  in  $\sigma$ , and a countable sequence of ordinals  $(h_i : i < \omega)$  witnessing the illfoundedness of this tower. By elementarity, the tower  $(\bar{\mu}_i : i < \omega)$  is also illfounded.

Take a wellfounded tree  $W \in \bigcup_{x \in \mathbb{R}} L[T, x]$  on  $\kappa$  on which each measure  $\bar{\mu}_i$  in this tower concentrates, and such that the function  $\bar{h} : \omega \rightarrow \text{Ord}$  defined by  $\bar{h}(i) = [\text{rank}_W]_{\bar{\mu}_i}$  is a pointwise minimal witness to the illfoundedness of the tower  $(\bar{\mu}_i : i < \omega)$  (see [34, Lemma 3.5.9]). Then by the elementarity of  $j$ , the function  $h = j(\bar{h})$  is a pointwise minimal witness to the illfoundedness of the tower  $(\mu_i : i < \omega)$ .<sup>79</sup> Because  $\bar{\mu}_i$  concentrates on  $W$  we have  $s_i \in j_{\mu_i}(j(W))$  for all  $i < \omega$ . Define a function  $h' : \omega \rightarrow \text{Ord}$  by  $h'(i) = \text{rank}_{j_{\mu_i}(j(W))}(s_i)$ . Then from the rules for player I concerning the finite sequences  $s_i$  we have  $j_{\mu_i, \mu_{i+1}}(h'(i)) > h'(i+1)$  and also  $h'(n) < \text{rank}_{j_{\mu_n}(j(W))}([\text{id}]_{\mu_n}) = h(n)$ , contradicting the minimality of  $h(n)$ .  $\square$

**Remark 5.9.** In the above proof, we use CH in a crucial way. CH implies that  $\lambda < \omega_2^V$  and we in turns get that  $\sigma \in M$  and is countable there. These two facts are key for the proof. As mentioned in the introduction, without CH the existence of an  $\omega_1$ -dense ideal on  $\omega_1$  is equiconsistent with AD.

The proof of Claim 5.6 and the argument in the following remark give us the following useful corollary.

**Corollary 5.10.** *Suppose  $A \in \Gamma^*$  is  $\omega_1$ -UB and let  $\gamma = w(A)$  in  $\Gamma^*$ . Then  $j_g(\gamma)$  is independent of  $g$ .*

<sup>77</sup>The real  $x$  can be taken to be the real that appears in the definition of  $j^{-1}(\mu)$ .

<sup>78</sup>In the general case  $P = \text{Lp}^\Sigma(\mathbb{R})$ , we reach the same conclusion because  $\bar{\mu}_{n+1}$  is  $OD^{V[g]}$  from a real, a finite sequence of ordinals,  $j_g \restriction \wp^P(\kappa^{<\omega})$ , and  $(W, S)$ , where  $(W, S)$  witnesses  $\Sigma$  is  $\omega_1$ -UB.

<sup>79</sup>Actually we only need the minimality of  $h(n)$ .



**Lemma 5.11.** *In  $\text{Ult}(V, g)$ , the set of norms  $\{\varphi_\mu : \mu \in \sigma\}$  defined by  $\varphi_\mu(x) = [\text{rank}_{j(T)_x}]_\mu$  (or more precisely, any enumeration of this countable set of norms in order type  $\omega$ ) is a scale on the complement of  $p[j(T)]$ .*

*Proof.* Work in  $\text{Ult}(V, g)$ . Let  $\mu \in \sigma$ . We say that  $\sigma$  *stabilizes*<sup>80</sup>  $\mu$  if, whenever  $(x_k : k < \omega)$  is a sequence of reals in  $\mathbb{R}[p[j(T)]]$  converging to a limit  $x$  and such that for each  $\mu' \in \sigma$ , the ordinals  $\varphi_{\mu'}(x_k)$  are eventually constant, we have  $\varphi_\mu(x) \leq \lim_{k \rightarrow \omega} \varphi_\mu(x_k)$ . (In particular,  $\varphi_\mu(x) < \infty$ .)

It is clear from the definition that if  $\sigma$  stabilizes every  $\mu \in \sigma$ , then  $\{\varphi_\mu : \mu \in \sigma\}$  is a scale. So fix a measure  $\mu \in \sigma$ . We want to show  $\sigma$  stabilizes  $\mu$ . Suppose not. We describe a winning strategy for player I in  $G_{j(T)}^{\sigma, \mu}$ . Let  $(x_k : k < \omega)$  witness that  $\sigma$  does not stabilize  $\mu$ . That is,  $x_k \in \mathbb{R}[p[j(T)]]$  for each  $k < \omega$ , and the sequence of ordinals  $(\varphi_\nu(x_k) : k < \omega)$  has an eventually constant value  $h(\nu)$  for each measure  $\nu \in \sigma$  but the limit  $x$  of the sequence  $(x_k : k < \omega)$  satisfies  $\varphi_\mu(x) > \lim_{k \rightarrow \omega} \varphi_\mu(x_k)$ . (This includes the possibility that  $\varphi_\mu(x) = \infty$ .)

Define  $m_i = x(i)$  and  $h(\nu) = \lim_{k \rightarrow \omega} \varphi_\nu(x_k)$ . Let  $n$  be the unique integer such that  $\mu$  concentrates on  $j(\kappa)^n$  and let  $\mu_i$  be the projection of  $\mu$  onto  $j(\kappa)^i$  for all  $i \leq n$ . In particular,  $\mu_n = \mu$ . By definition,

$$\varphi_{\mu_n}(x) = [s \mapsto \text{rank}_{j(T)_x}(s)]_{\mu_n} = \text{rank}_{j_{\mu_n}(j(T)_x)}([\text{id}]_{\mu_n}) > h(\mu_n).$$

So there is a finite sequence  $s_n \supsetneq [\text{id}]_{\mu_n}$  with  $\text{rank} \geq h(\mu_n)$  in the tree  $j_{\mu_n}(j(T)_x)$ . Let player I play as his first move the integers  $m_0, \dots, m_n$ , the ordinal  $h_n = h(\mu_n)$ , and  $s_n$ , where  $s_n$  is the least such sequence. For  $i \geq n$ , we will show inductively that player I can maintain the inequality

$$\text{rank}_{j_{\mu_i}(j(T)_x)}(s_i) \geq h(\mu_i). \quad (5.4)$$

Whenever player II plays a measure  $\mu_{i+1}$  according to the rules of the game, we have

$$\text{rank}_{j_{\mu_{i+1}}(j(T)_x)}(j_{\mu_i, \mu_{i+1}}(s_i)) = j_{\mu_i, \mu_{i+1}}(\text{rank}_{j_{\mu_i}(j(T)_x)}(s_i)) \geq j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1}.$$

To show the last step  $j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1}$ , we argue as follows. Recall that for each  $l$  we have  $h_l = h(\mu_l) = \lim_{k \rightarrow \omega} \varphi_{\mu_l}(x_k)$ . Since the measure  $\mu_{i+1}$  concentrates on  $j(T)_{x \upharpoonright (i+1)}$  and projects to  $\mu_i$ , for each  $k$  we have

$$j_{\mu_i, \mu_{i+1}}(\varphi_{\mu_i}(x_k)) = j_{\mu_i, \mu_{i+1}}([\text{rank}_{j(T)_{x_k}}]_{\mu_i}) = [\text{ext}_{i, i+1} \text{rank}_{j(T)_{x_k}}]_{\mu_{i+1}},$$

where the “extension” of a function  $F : j(\kappa)^i \rightarrow \text{Ord}$  to  $j(\kappa)^{i+1}$  is defined by  $\text{ext}_{i, i+1} F(s) = F(s \upharpoonright i)$  for all  $s \in j(\kappa)^{i+1}$ . Note that

$$[\text{ext}_{i, i+1} \text{rank}_{j(T)_{x_k}}]_{\mu_{i+1}} > [\text{rank}_{j(T)_{x_k}}]_{\mu_{i+1}} = \varphi_{\mu_{i+1}}(x_k).$$

Finally, since for each  $l$  the ordinal  $h_l$  is the eventual value of  $\varphi_{\mu_l}(x_k)$  as  $k \rightarrow \omega$ , consideration of sufficiently large  $k$  gives  $j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1}$ .

This shows that player I can choose a successor  $s_{i+1} \supsetneq j_{\mu_i, \mu_{i+1}}(s_i)$  of rank at least  $h(\mu_{i+1})$  in the tree  $j_{\mu_{i+1}}(j(T)_x)$ , thereby maintaining the desired inequality (5.4) for one more step. Then

<sup>80</sup>The idea of this definition comes from a similar notion of stability used in unpublished work of S. Jackson.

player I can play the integer  $m_{i+1} = x(i+1)$ , the least such finite sequence  $s_{i+1}$ , and the ordinal  $h_{i+1} = h(\mu_{i+1})$ . By playing in this way, player I can follow the rules forever. This contradicts the previous lemma, which showed that player II has a winning strategy.  $\square$

The previous claims and elementarity establishes (a) for  $A$  being the universal  $\check{\Gamma}^*$ -set, where recall  $\check{\Gamma}^*$  is the dual pointclass of  $\Gamma^*$ . By standard arguments, see [34, Section 4.3], the rest of (a) and (b) follow. This contradicts our assumption. Therefore, (a) and (b) hold after all.  $\square$

**Theorem 5.12.** *There is a hod pair  $(\mathcal{P}', \Sigma')$  in  $V$  such that*

- (1)  $\Sigma'$  is  $Lp^\Sigma(\mathbb{R})$ -fullness preserving and  $\Sigma' \notin Lp^\Sigma(\mathbb{R})$ .
- (2)  $\Sigma'$  has branch condensation.
- (3)  $\Sigma'$  is  $\omega_1$ -UB.

*Proof.* Let  $\Gamma^*$  be the largest Suslin pointclass of  $Lp^\Sigma(\mathbb{R})$ . Let  $\vec{A} = (A_i : i < \omega)$  be the sjs sealing  $\text{Env}(\Gamma^*)$  as in the previous theorem. Let  $(\mathcal{P}'', \Sigma'')$  be a pair such that  $\Sigma''$  is guided by  $\vec{A}$ .  $\Sigma''$  has properties (1) and (2), but (3) may fail for  $\Sigma''$ . Here one can regard  $\mathcal{P}''$  as a  $\Sigma$ -suitable mouse with one Woodin cardinal or a hod mouse. We take the first viewpoint and hence we regard  $\Sigma''$  as an iteration strategy for  $\mathcal{P}''$  as a  $\Sigma$ -mouse (so all  $\vec{T}$  according to  $\Sigma''$  are above  $o(\mathcal{P})$  and iterates of  $\mathcal{P}''$  according to  $\Sigma''$  are  $\Sigma$ -premise).

For each  $p \in \text{Coll}(\omega, \omega_1)$ , let  $G_p$  be the “finite variation” of  $G$  induced by  $p$  and let  $g_p$  be the corresponding  $\mathbb{P}_T$ -generic induced by  $\pi$  and  $G_p$ . We let  $\vec{A}^p = (A_i^p : i < \omega)$  be  $j_{g_p}(\vec{A})$ . Let  $(\mathcal{Q}_p, \Sigma_p)$  be a hod pair in  $V[g] = V[g_p]$  guided by  $\vec{A}^p$ <sup>81</sup> and  $(N, \Lambda)$  be obtained by Boolean comparing all  $(\mathcal{Q}_p, \Sigma_p)$ . So  $N \in V$  and  $\Lambda \restriction V \in V$  is a strategy acting on stacks in  $H_{\omega_2}^V$  such that  $\Lambda$  has branch condensation (and is guided by  $\mathcal{B} = \bigcup_p \text{rng}(\vec{A}^p)$ ), and is  $j(\Gamma^*)$ -fullness preserving. Therefore,  $\Lambda$  also has strong hull condensation by Lemma 3.34. Note that  $\mathcal{R}$  is countable in  $M$  and  $\Lambda \notin j(Lp^\Sigma(\mathbb{R}))$ .

Applying Lemma 3.36, we get that  $\Lambda = j(\Lambda)^j$ . By elementarity, in  $V$ , there is a pair  $(\mathcal{P}', \Sigma')$  and an elementary embedding  $\pi : \mathcal{P}' \rightarrow \mathcal{R}$  such that

- (a)  $\Sigma' = \Lambda^\pi$ .
- (b)  $\Sigma' \restriction HC$  is  $\Gamma^*$ -fullness preserving and has branch condensation.
- (c)  $\mathcal{P}'$  is a countable  $\Sigma$ -mouse (i.e.  $\pi \restriction \mathcal{P} = id$ ) that is  $\Gamma^*$ -suitable.

$(\mathcal{P}', \Sigma')$  satisfies (1) and (2). We note that property (a) above gives that  $\Sigma'$  is an  $(\omega_2, \omega_2)$ -strategy. Now Lemma 5.4 implies that  $\Sigma'$  is  $\omega_1$ -UB. This completes the proof of the theorem.  $\square$

<sup>81</sup>We can take  $(\mathcal{Q}_p, \Sigma_p)$  to be  $(\mathcal{P}'', j_{g_p}(\Sigma''))$ .

## 6. THE LIMIT CASE

Recall we let  $g \subseteq \mathbb{P}_{\mathcal{I}}$  be  $V$ -generic and  $j = j_g : V \rightarrow M = \text{Ult}(V, g)$  be the corresponding ultrapower map; by our hypothesis,  $g$  corresponds to a  $V$ -generic  $G \subset \text{Coll}(\omega, \omega_1)$ . We also let  $k : M \rightarrow N$  be the generic ultrapower map induced by a generic  $h \subset j(\mathbb{P}_{\mathcal{I}})$ . We remind the reader that CH holds, so the continuum  $\mathfrak{c}$  is  $\omega_1$ .

Let  $\langle \theta_\alpha : \alpha < \gamma \rangle$  be the Solovay sequence computed in  $\Gamma$  (our maximal model) and  $\Theta = \sup_\gamma \theta_\gamma$ . By the previous section,  $\gamma$  is a limit ordinal and  $\Theta$  is the Wadge ordinal of  $\Gamma$ . Recall for  $\alpha \leq \Theta$ , by  $\Gamma \restriction \alpha$ , we mean the set of  $B \in \Gamma$  such that the Wadge rank of  $B$  is less than  $\alpha$ . We also remind the reader that our inductive hypothesis implies that every  $B \in \Gamma$  is  $\omega_1$ -UB; in particular, because  $\Theta$  is a limit of Suslin cardinals in  $\Gamma$ , by Corollary 5.10,  $j \restriction \Theta$  is independent of  $g$ . First we claim

$$|\Gamma| \leq \mathfrak{c}.$$

**Lemma 6.1.** *Suppose  $|\Gamma| = \mathfrak{c}^+$ . Then  $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ .*

*Proof.* Suppose not. Let  $\alpha$  be the least such that  $\rho_\omega(J_\alpha(\Gamma, \mathbb{R})) = \mathbb{R}$ , i.e.  $J_\alpha(\Gamma, \mathbb{R})$  defines a set of reals  $A$  such that  $A \notin \Gamma$ . Hence  $\alpha \geq \mathfrak{c}^+$  by our assumption. Let  $f : \alpha \times \Gamma \rightarrow J_\alpha(\Gamma, \mathbb{R})$  be a surjection that is definable over  $J_\alpha(\Gamma, \mathbb{R})$  (from parameters).

We first define a sequence  $\langle H_i \mid i < \omega \rangle$  as follows. Let  $H_0 = \mathbb{R}$ . By induction, suppose  $H_n$  is defined and there is a surjection from  $\mathbb{R}$  onto  $H_n$ . Suppose  $(\psi, a)$  is such that  $a \in H_n$  and  $J_\alpha(\Gamma, \mathbb{R}) \models \exists x \psi[x, a]$ . Let  $(\gamma_{a,\psi}, \beta_{a,\psi})$  be the  $<_{lex}$ -least pair such that there is a  $B \in \Gamma$  with Wadge rank  $\beta_{a,\psi}$  such that

$$J_\alpha(\Gamma, \mathbb{R}) \models \psi[f(\gamma_{a,\psi}, B), a].$$

Let then  $H_{n+1} = H_n \cup \{f(\gamma_{a,\psi}, B) \mid J_\alpha(\Gamma, \mathbb{R}) \models \exists x \psi[x, a] \wedge w(B) = \beta_{a,\psi} \wedge a \in H_n\}$ . It's easy to see that there is a surjection from  $\mathbb{R}$  onto  $H_{n+1}$ . This uses the fact that  $\Theta^\Gamma = \mathfrak{c}^+$  is regular, which implies  $\sup\{\beta_{a,\psi} \mid a \in H_n \wedge L_\alpha(\Gamma, \mathbb{R}) \models \exists x \psi[x, a]\} < \Theta = \mathfrak{c}^+$ . Let  $H = \bigcup_n H_n$ . By construction,  $H \prec J_\alpha(\Gamma, \mathbb{R})$ . Finally, let  $M$  be the transitive collapse of  $H$ .

Say  $M = J_\beta(\Gamma^*, \mathbb{R})$ . By construction, it is easy to see that  $\Gamma^* = \Gamma \restriction \theta_\gamma$  for some  $\gamma$  such that  $\theta_\gamma < \Theta$ . But then  $\rho_\omega(J_\beta(\Gamma^*, \mathbb{R})) = \mathbb{R}$ .<sup>82</sup> This contradicts that  $\Gamma^*$  is constructibly closed.  $\square$

The lemma gives  $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$  and in fact,  $L(\Gamma, \mathbb{R}) \models \text{“AD}_{\mathbb{R}} + \Theta \text{ is regular”}$ . This is because  $\Theta = \mathfrak{c}^+$  in this case. This contradicts  $(\ddagger)$ . Therefore,  $|\Gamma| \leq \mathfrak{c}$  as desired.

Let  $\mathcal{H}$  be the direct limit of hod pairs  $(\mathcal{P}, \Sigma) \in \Gamma$  such that  $\Sigma$  has branch condensation and is fullness preserving under iteration embeddings by  $\Sigma$ . So  $\lambda^{\mathcal{H}}$  is a limit ordinal. For each  $\alpha < \lambda^{\mathcal{H}}$ , let  $\Sigma_\alpha$  be the strategy of  $\mathcal{H}(\alpha)$  in  $j(\Gamma)$  obtained as a tail of some (any)  $j(\Sigma)$ , where  $(\mathcal{P}, \Sigma)$  is a hod pair in  $\Gamma$  with branch condensation and is fullness preserving such that  $\mathcal{M}(\mathcal{P}, \Sigma) = \mathcal{H}(\alpha)$ . Let

$$\Sigma = \oplus_{\alpha < \lambda^{\mathcal{H}}} \Sigma_\alpha.$$

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<sup>82</sup>For instance, to see that  $\Gamma \restriction \theta_0 \subset \Gamma^*$ , let  $A \in \Gamma$  be  $OD$  in  $J_\alpha(\Gamma, \mathbb{R})$  from a real  $x$ . Suppose  $A \notin M$ . By minimizing the Wadge rank of  $A$  and minimizing the ordinal parameters defining  $A$ , we may assume  $A$  is definable in  $J_\alpha(\Gamma, \mathbb{R})$  from  $x$ . By elementarity,  $A$  is definable in  $M$  from  $x$ , so  $A \in \Gamma^*$ . Contradiction.

Now note that

$$j \text{ is continuous at } \lambda^{\mathcal{H}} \text{ if and only if } \text{cof}^V(\lambda^{\mathcal{H}}) = \omega.$$

First note that  $j \upharpoonright \omega_1^V \in M$ . If  $j$  is continuous at  $\lambda^{\mathcal{H}}$  and  $\text{cof}^V(\lambda^{\mathcal{H}}) = \omega_1$ , then  $j(\omega_1^V)$  is singular in  $M$ . This contradicts the fact that  $j(\omega_1^V)$  is a successor cardinal, hence regular, in  $M$ . This implies  $\text{cof}^V(\lambda^{\mathcal{H}}) \neq \omega_1^V$  and hence  $\text{cof}^V(\lambda^{\mathcal{H}}) = \omega$ .

At this point, we can show that  $\text{cof}^V(\lambda^{\mathcal{H}}) > \omega$ . But the following analysis does not assume this. See Remark 6.9.

**Lemma 6.2.**     •  $\Sigma \upharpoonright V \in V$  and  $\Sigma$  does not depend on  $g$ .

•  $j \upharpoonright \Theta^\Gamma$  is independent of  $g$ .

*Proof.* This follows from our induction hypothesis, i.e. for each  $\alpha$ ,  $\Sigma_\alpha$  is  $\omega_1$ -UB by the inductive hypothesis, and hence  $\Sigma_\alpha \upharpoonright V \in V$  and does not depend on  $g$ . This gives the first item. The argument for the second item is given at the beginning of the section.  $\square$

Let

$$\mathcal{H}^+ = \begin{cases} \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H}) & \text{if } \forall \mathcal{M} \triangleleft \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H}) \ \rho_\omega(\mathcal{M}) \geq \Theta \\ \mathcal{P} & \text{where } \mathcal{P} \triangleleft \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H}) \text{ is the least } \mathcal{N} \text{ such that } \rho_\omega(\mathcal{N}) < \Theta. \end{cases} \quad (6.1)$$

To be technically correct, by  $\text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H})$  we mean  $\text{Lp}^\Sigma(\mathcal{H})$  defined inside  $L(j(\mathbb{R}), C)$  for some  $C \in j(\Gamma)$ . This makes sense as  $\Sigma \in j(\Gamma)$  and the Solovay sequence of  $j(\Gamma)$  has limit length. By Lemma 6.2, we get that

$$\mathcal{H}^+ \in V.$$

This is because  $\mathcal{H}^+$  is definable in  $V[G]$  from  $\mathcal{H}, \Sigma \upharpoonright V$  and by Lemma 6.2,  $\Sigma \upharpoonright V \in V$  and does not depend on  $G$ .

**Proposition 6.3.**  $|\mathcal{H}^+| \leq \mathfrak{c}$ . Therefore,  $j \upharpoonright \mathcal{H}^+ \in M$

*Proof.* Suppose we have  $\mathcal{H}^+ = \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H})$ . If  $|\mathcal{H}^+| = \mathfrak{c}^+$ , we would get an  $\omega_1$ -sequence of distinct reals in  $j(\Gamma)$ , noting that  $(\mathfrak{c}^+)^V = \omega_2^V$  is  $\omega_1$  in  $M$  by the density of  $\mathcal{I}$ . Contradiction. Therefore,  $|\mathcal{H}^+| = \mathfrak{c}$ , and hence  $\mathcal{H}^+$  is countable in  $M$ . Again, by density of  $\mathcal{I}$ ,  $j \upharpoonright \mathcal{H}^+ \in M$ . A similar argument also works for the second case of (6.1).  $\square$

Using the embedding  $j$ , the fact that  $j \upharpoonright \mathcal{H}^+ \in M$ , and the construction in [8, Section 11], we obtain a strategy  $\Lambda$  for  $\mathcal{H}^+$  such that

- (i)  $\Lambda$  extends  $\Sigma$ ;
- (ii) for any  $\Lambda$ -iterate  $\mathcal{Q}$  of  $\mathcal{H}^+$  via a stack  $\vec{\mathcal{T}}$  such that  $i^{\vec{\mathcal{T}}}$  exists, there is an embedding  $\sigma : \mathcal{Q} \rightarrow j(\mathcal{H}^+)$  such that  $j \upharpoonright \mathcal{H}^+ = \sigma \circ i^{\vec{\mathcal{T}}}$ . Furthermore, letting  $\Lambda_{\mathcal{Q}}$  be the  $\vec{\mathcal{T}}$ -tail of  $\Lambda$ , for all  $\alpha < \lambda^{\mathcal{Q}}$ ,  $\Lambda_{\mathcal{Q}(\alpha)} \in j(\Gamma)$  has branch condensation.

(iii)  $\Lambda$  is  $\Gamma(\mathcal{H}^+, \Lambda)$ -fullness preserving.

We outline the construction here. First recall definitions and notations related to the analysis of stacks in [7, Section 6.2] summarized in Section 3.2; see [7, Section 6.2] for a more detailed discussion.

**Definition 6.4** (*j-realizable iterations*). Let  $\vec{\mathcal{T}} \in HC^M$  be a stack on  $\mathcal{H}^+$ . We say  $\vec{\mathcal{T}}$  is *j-realizable* if there is a sequence  $\langle \sigma_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}) \rangle$  such that

- (1)  $\sigma_{\mathcal{H}^+} = j \restriction \mathcal{H}^+$ ; for all  $\mathcal{R} \in tn(\vec{\mathcal{T}})$ ,  $\sigma_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{H}^+)$ .
- (2) For  $\mathcal{R}, \mathcal{Q} \in tn(\vec{\mathcal{T}})$  such that  $\mathcal{R} \prec^{\vec{\mathcal{T}}, s} \mathcal{Q}$ ,  $\sigma_{\mathcal{R}} = \sigma_{\mathcal{Q}} \circ \pi_{\mathcal{R}, \mathcal{Q}}^{\vec{\mathcal{T}}}$ .
- (3) For every  $\mathcal{R} \in ntn(\vec{\mathcal{T}})$ , there is a reasonable hod pair  $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}}) \in j(\Gamma)$  that is  $j(\Gamma)$ -fullness preserving and has branch condensation such that  $\sigma_{\mathcal{R}}[\mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1)] \subset \text{rng}(\pi_{\mathcal{S}_{\mathcal{R}}, \infty}^{\Lambda_{\mathcal{R}}})$ .
- (4) For every  $\mathcal{R} \in ntn(\vec{\mathcal{T}})$ , letting  $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$  be as above, and letting  $k_{\mathcal{R}} : \mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1) \rightarrow \mathcal{S}_{\mathcal{R}}$  be given by:  $k_{\mathcal{R}}(x) = y$  if and only if  $\sigma_{\mathcal{R}}(x) = \pi_{\mathcal{S}_{\mathcal{R}}, \infty}^{\Lambda_{\mathcal{R}}}(y)$  and  $k_{\mathcal{R}} \vec{\mathcal{T}}_{\mathcal{R}}$  is according to  $\Lambda_{\mathcal{R}}$ .
- (5) For every  $\mathcal{R} \in ntn(\vec{\mathcal{T}})$ , let  $\mathcal{S}_{\mathcal{R}}^*$  be the last model of  $k_{\mathcal{R}} \vec{\mathcal{T}}_{\mathcal{R}}$  and let  $\mathcal{Q}_{\mathcal{R}}$  be the last model of  $\vec{\mathcal{T}}_{\mathcal{R}}$  (considered as a stack on all of  $\mathcal{R}$ ). Suppose  $\pi^{\vec{\mathcal{T}}_{\mathcal{R}}}$  is defined (hence,  $\mathcal{Q}_{\mathcal{R}} \in tn(\vec{\mathcal{T}})$  and  $\mathcal{R} \prec^{\vec{\mathcal{T}}, s} \mathcal{Q}_{\mathcal{R}}$ ). Let  $k_{\mathcal{R}}^* : \mathcal{Q}_{\mathcal{R}}(\zeta) \rightarrow \mathcal{S}_{\mathcal{R}}^*$  be the natural map that comes from the copying construction, where  $\mathcal{Q}_{\mathcal{R}}(\zeta)$  is the image of  $\mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1)$  under the iteration embedding of  $\vec{\mathcal{T}}_{\mathcal{R}}$ . Then we define  $\sigma_{\mathcal{Q}_{\mathcal{R}}} : \mathcal{Q}_{\mathcal{R}} \rightarrow j(\mathcal{H}^+)$  as follows: for all  $x \in \mathcal{Q}_{\mathcal{R}}$ ,

$$\sigma_{\mathcal{Q}_{\mathcal{R}}}(x) = \sigma_{\mathcal{R}}(f)(\pi_{\mathcal{S}_{\mathcal{R}}^*, \infty}^{\Lambda_{\mathcal{R}}} (k_{\mathcal{R}}^*(a))),$$

where  $f \in \mathcal{R}$ , and  $a \in [\mathcal{Q}(\pi_{\mathcal{R}, \mathcal{Q}_{\mathcal{R}}}^{\vec{\mathcal{T}}_{\mathcal{R}}}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1))]^{<\omega}$  are such that  $x = \pi_{\mathcal{R}, \mathcal{Q}_{\mathcal{R}}}^{\vec{\mathcal{T}}_{\mathcal{R}}}(f)(a)$ ; here  $\Lambda = (\Lambda_{\mathcal{R}})_{k_{\mathcal{R}} \vec{\mathcal{T}}_{\mathcal{R}}, \mathcal{S}_{\mathcal{R}}^*}$ .

- (6) For every trivial terminal node  $\mathcal{R}$ , for every  $\xi < \lambda^{\mathcal{R}}$ , there is a reasonable hod pair  $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}}) \in j(\Gamma)$  where  $\Lambda$  is  $j(\Gamma)$ -fullness preserving, and has branch condensation and  $\sigma_{\mathcal{R}}(\xi + 1) \subset \text{rng}(\pi_{\mathcal{S}_{\mathcal{R}}, \infty}^{\Lambda_{\mathcal{R}}})$ .

The maps  $(\sigma_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}))$  are the *j-realizable embeddings* of  $\vec{\mathcal{T}}$ . In the above, we may also choose  $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$  such that letting  $j(\mathcal{H})(\alpha) = \mathcal{M}_{\infty}(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$ , then  $\alpha$  is minimal.  $\dashv$

Now we define the domain of the strategy  $\Lambda$ . Basically, it consists of *j-realizable stacks*. See [8, Definition 11.5].

**Definition 6.5.** Let  $\vec{\mathcal{T}} \in HC^M$  be a stack of on  $\mathcal{H}^+$ .<sup>83</sup> We let  $\vec{\mathcal{T}} \in \text{dom}(\Lambda)$  iff  $\vec{\mathcal{T}}$  is *j-realizable*. Define  $\Lambda(\vec{\mathcal{T}}) = b$  iff  $\vec{\mathcal{T}} \frown b$  is *j-realizable*.  $\dashv$

**Lemma 6.6.** *Whenever  $\vec{\mathcal{T}} \in \text{dom}(\vec{\mathcal{T}})$ , then  $\Lambda(\vec{\mathcal{T}})$  is defined.*

<sup>83</sup> $\vec{\mathcal{T}}$  either has a strongly linear, closed and cofinal set  $C \subseteq tn(\vec{\mathcal{T}})$  or  $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$  is of limit length.

See [8, Lemma 11.6] for a similar argument. In other words, the lemma states that if  $\vec{\mathcal{T}}$  is  $j$ -realizable and has no last model, then we can find a cofinal branch  $b$  of  $\vec{\mathcal{T}}$  so that  $\vec{\mathcal{T}} \hat{\smallfrown} b$  is  $j$ -realizable. We sketch the argument here.

*Proof.* Suppose  $\vec{\mathcal{T}}$  has a last model. Then it is easy to verify that  $\Lambda(\vec{\mathcal{T}})$  is defined. So we now assume  $\vec{\mathcal{T}}$  does not have a last model.

Suppose there is a strongly closed, cofinal  $C \subset \text{tn}(\vec{\mathcal{T}})$ . In this case  $\vec{\mathcal{T}}$  has a unique, cofinal, non-dropping branch  $b$  determined by  $C$ . Let  $\mathcal{Q} = \mathcal{M}_b^{\vec{\mathcal{T}}}$  and  $\sigma_{\mathcal{Q}} : \mathcal{Q} \rightarrow j(\mathcal{H}^+)$  be the direct limit of the maps  $\{\sigma_{\mathcal{R}} : \mathcal{R} \in C\}$ ; more precisely, let  $\sigma_{\mathcal{Q}}(x) = y$  if and only if there is some  $x^* \in \mathcal{R}$  for some  $\mathcal{R} \in C$  such that  $\pi_{\mathcal{R}, \mathcal{Q}}^{\vec{\mathcal{T}}}(x^*) = x$  and  $\sigma_{\mathcal{R}}(x^*) = y$ . It is easy to see that  $\sigma_{\mathcal{Q}}$  is well-defined and satisfies the clauses of Definition 6.4.

Otherwise, we are looking for a branch of  $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ . Let  $\mathcal{R} = \mathcal{S}_{\vec{\mathcal{T}}}$  and  $\mathcal{U} = \vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ . By our hypothesis, objects like  $\sigma_{\mathcal{R}}, k_{\mathcal{R}}, (\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$  as in (3) and (4) can be defined. Let then  $b = \Lambda_{\mathcal{R}}(k_{\mathcal{R}}\mathcal{U})$ ,  $\mathcal{Q} = \mathcal{M}_b^{\mathcal{U}}$ ,  $\mathcal{S}^* = \mathcal{M}_b^{k_{\mathcal{R}}\mathcal{U}}$ ,  $k : \mathcal{Q}(\zeta) \rightarrow \mathcal{S}^*$ ,  $\sigma_{\mathcal{Q}} : \mathcal{Q} \rightarrow j(\mathcal{H}^+)$  be the objects as described in (5) above. So  $b$  is the branch of  $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$  we are looking for.

In the following, we assume  $\mathcal{Q}$  is a terminal node; otherwise, we're done. We need to verify clause (6) in the case  $\mathcal{Q}$  is a trivial terminal node. The case for non-trivial terminal nodes has been dealt with as above. Without loss of generality, we assume  $\mathcal{Q} \neq \mathcal{R}$  and there is a  $\mathcal{U}$  on  $\mathcal{R}$  with last model  $\mathcal{Q}$  such that  $\pi_{\mathcal{R}, \mathcal{Q}}^{\mathcal{U}}$  exists. We let  $\sigma_{\mathcal{R}}, k_{\mathcal{R}}, (\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}}), \mathcal{S}^*, k$  be the objects associated with  $\mathcal{R}, \mathcal{U}, \mathcal{Q}$  as before. We let  $\Lambda = (\Lambda_{\mathcal{R}})_{k_{\mathcal{R}}\mathcal{U}, \mathcal{S}^*}$  and  $\sigma_{\mathcal{Q}} = \pi_{\mathcal{S}^*, \infty}^{\Lambda} \circ k$ . Fix  $\xi < \lambda^{\mathcal{Q}}$ . Let  $(\mathcal{W}, \Psi) \in j(\Gamma)$  be a reasonable hod pair such that  $\Psi$  is  $j(\Gamma)$ -fullness preserving, and such that  $\mathcal{M}_{\infty}(\mathcal{W}, \Psi) = j(\mathcal{H}^+)(\sigma_{\mathcal{Q}}(\xi + 1))$ . We can then find  $(\mathcal{S}, \Psi_{\mathcal{S}}) \in I(\mathcal{W}, \Psi)$  such that  $\sigma_{\mathcal{Q}}[\mathcal{Q}(\xi + 1)] \subset \text{rng}(\pi_{\mathcal{S}, \infty}^{\Psi_{\mathcal{S}}})$ . We are done.  $\square$

**Remark 6.7.** Suppose  $\vec{\mathcal{T}} \in \text{dom}(\Lambda)$ , then there is at most one  $b$  such that  $\vec{\mathcal{T}} \hat{\smallfrown} b$  is  $j$ -realizable. In the proof of Lemma 6.6, the only case to verify is when  $\mathcal{S}_{\vec{\mathcal{T}}}$  exists. Let  $\mathcal{R}, \mathcal{U}, \sigma_{\mathcal{R}}, k_{\mathcal{R}}, (\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$  be as there. Suppose  $(\mathcal{S}_{\mathcal{R}}^*, \Lambda_{\mathcal{R}}^*)$  and  $l_{\mathcal{R}} : \mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1) \rightarrow \mathcal{S}_{\mathcal{R}}^*$  are such that  $l_{\mathcal{R}}(x) = y$  if and only if  $\sigma_{\mathcal{R}}(x) = \pi_{\mathcal{S}_{\mathcal{R}}^*, \infty}^{\Lambda_{\mathcal{R}}^*}(y)$ ,  $l_{\mathcal{R}}\vec{\mathcal{T}}_{\mathcal{R}}$  is according to  $\Lambda_{\mathcal{R}}^*$  and  $c = \Lambda_{\mathcal{R}}^*(l_{\mathcal{R}}\mathcal{U})$ . To see  $b = c$ , we let  $(\mathcal{S}, \Psi)$  be the common iterate of  $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$  and  $(\mathcal{S}_{\mathcal{R}}^*, \Lambda_{\mathcal{R}}^*)$ . Let  $\sigma_0 : \mathcal{S}_{\mathcal{R}} \rightarrow \mathcal{S}$  and  $\sigma_1 : \mathcal{S}_{\mathcal{R}}^* \rightarrow \mathcal{S}$  be the iteration maps. So  $\Lambda_{\mathcal{R}} = (\Psi)^{\sigma_0}$  and  $\Lambda_{\mathcal{R}}^* = (\Psi)^{\sigma_1}$  because these strategies are pullback consistent. It is also easy to verify that

$$\sigma_0 \circ k_{\mathcal{R}} = \sigma_1 \circ l_{\mathcal{R}};$$

this is because letting  $\tau : \mathcal{S} \rightarrow \sigma_{\mathcal{R}}(\mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1))$  be the direct limit embedding according to  $\Psi$ , then

$$\sigma_{\mathcal{R}} = \tau \circ \sigma_0 \circ k_{\mathcal{R}} = \tau \circ \sigma_1 \circ l_{\mathcal{R}}.$$

So  $\sigma_0 \circ k_{\mathcal{R}} = \sigma_1 \circ l_{\mathcal{R}}$  as desired. Therefore,

$$b = \Psi^{\sigma_0 \circ k_{\mathcal{R}}}(\mathcal{U}) = \Psi^{\sigma_1 \circ l_{\mathcal{R}}}(\mathcal{U}) = c.$$

Clearly, if  $\Lambda$  is a  $j$ -realizable strategy, then  $\Lambda$  satisfies (i) and the first clause of (ii); by basic hod mice theory (cf. [7]),  $\Lambda$  also satisfies the “Furthermore” clause. By the proof of [8, Lemma

11.8], we can choose  $\Lambda$  so that  $\Gamma(\mathcal{H}^+, \Lambda)$  is Wadge minimal (amongst all strategies  $\Lambda$  constructed this way) and this particular choice of  $\Lambda$  satisfies (iii) as well.

**Lemma 6.8.**  $\mathcal{H}^+ = Lp^{\Sigma, j(\Gamma)}(\mathcal{H})$  and if  $j$  is discontinuous at  $\lambda^{\mathcal{H}}$ , then  $\mathcal{H}^+ \models \text{cof}(\lambda^{\mathcal{H}})$  is measurable.

*Proof.* The second clause follows from the first clause and the case assumption that  $j$  is discontinuous at  $\lambda^{\mathcal{H}}$ . To see this, assume the first clause. If  $\mathcal{H}^+ \models \text{“}\lambda^{\mathcal{H}} \text{ is regular”}$ , then by standard results on Vopenka forcing (cf. [31])  $L[\mathcal{H}^+](\Gamma) \cap \wp(\mathbb{R}) = \Gamma$  and therefore,  $L(\Gamma, \mathbb{R}) \models \text{“}\text{AD}_{\mathbb{R}} + \Theta \text{ is regular”}$ , contradicting our smallness assumption ( $\ddagger$ ). If  $\mathcal{H}^+ \models \text{“}\lambda^{\mathcal{H}} \text{ is singular”}$ , then letting  $\kappa = \text{cof}^{\mathcal{H}^+}(\lambda^{\mathcal{H}})$ , then  $\kappa$  must be measurable in  $\mathcal{H}^+$ . This is because  $j \upharpoonright (\kappa + 1)$  is the iteration embedding of  $\mathcal{H}(\alpha)$  according to  $\Psi =_{\text{def}} \Sigma_{\mathcal{H}(\alpha)}$  in  $M$  for some (equivalently any)  $\alpha$  such that  $\kappa \in \mathcal{H}(\alpha)$ ; therefore,  $i_{\mathcal{H}(\alpha), \infty}^{\Psi}$  is discontinuous at  $\kappa$ ,<sup>84</sup> implying  $\kappa$  is measurable in  $\mathcal{H}(\alpha)$ , hence in  $\mathcal{H}^+$ .

Now, suppose for contradiction that there is a  $\mathcal{P} \triangleleft \mathcal{H}^+$  such that  $\rho_{\omega}(\mathcal{P}) < \Theta$ . Let  $\mathcal{P}$  be the least such. Let  $\beta < \lambda^{\mathcal{H}}$  be least such that  $\rho_{\omega}(\mathcal{P}) \leq \delta_{\beta}^{\mathcal{P}}$  and  $\delta_{\beta}^{\mathcal{P}} > \text{cof}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ , here  $\lambda^{\mathcal{P}} = \lambda^{\mathcal{H}}$  and  $\delta_{\alpha}^{\mathcal{P}} = \delta_{\alpha}^{\mathcal{H}}$  for all  $\alpha < \lambda^{\mathcal{P}}$ .  $\mathcal{P}$  can be considered a hod premouse over  $(\mathcal{H}(\beta), \Sigma_{\beta})$ . Using  $j$  and the construction in [8, Section 11] discussed above, we can define a strategy  $\Lambda$  for  $\mathcal{P}$  such that  $\Lambda$  acts on stacks above  $\delta_{\beta}^{\mathcal{P}}$  and extends  $\oplus_{\alpha < \lambda^{\mathcal{P}}} \Sigma_{\alpha}$  (the strategy is simply  $\oplus_{\alpha < \lambda^{\mathcal{P}}} \Sigma_{\alpha}$  for stacks based on  $\mathcal{H}$  (above  $\delta_{\beta}^{\mathcal{P}}$ ), but the point is that it also acts on all of  $\mathcal{P}$  because of  $j$ ). This is because given a stack  $\vec{T}$  according to  $\Lambda$ , there is a map  $\sigma : \mathcal{M}^{\vec{T}} \rightarrow j(\mathcal{P})$  such that  $\sigma \circ i^{\vec{T}} = j \upharpoonright \mathcal{P}$ , where for any  $f \in \mathcal{P}$ , any generator  $a$  used along the main branch of  $\vec{T}$ , say  $a \in \mathcal{M}^{\vec{T}}(\gamma)$  and  $\mathcal{M}^{\vec{T}}(\gamma)$  is the image of  $\mathcal{P}(\gamma^*)$ , then letting  $\Psi = \Sigma_{\gamma^*}$ ,

$$\sigma(i^{\vec{T}}(f)(a)) = j(f)(i^{\Psi_{\vec{T}, \mathcal{M}^{\vec{T}}(\gamma)}}(a)).$$

In the above, we note that  $i^{\vec{T}}$  is continuous at  $\lambda^{\mathcal{P}}$ , so we can find  $\gamma, \gamma^*$ .

Note that  $\Lambda$  has branch condensation. By a core model induction as in the successor case, we get that  $\Lambda \in j(\Gamma)$ .<sup>85</sup> In  $j(\Gamma)$ , let  $\mathcal{F}$  be the direct limit system of  $\Sigma_{\beta}$ -hod pairs  $(\mathcal{Q}, \Psi)$  Dodd-Jensen equivalent to  $(\mathcal{P}, \Lambda)$ .<sup>86</sup>  $\mathcal{F}$  can be characterized as the direct limit system of  $\Sigma_{\beta}$ -hod pairs  $(\mathcal{Q}, \Psi)$  in  $j(\Gamma)$  such that  $\Psi$  is  $\Gamma(\mathcal{P}, \Lambda)$ -fullness preserving and has branch condensation and  $\Gamma(\mathcal{Q}, \Psi) = \Gamma(\mathcal{P}, \Lambda)$ .  $\mathcal{F}$  only depends on  $\Sigma_{\beta}$  and the Wadge rank of  $\Gamma(\mathcal{P}, \Lambda)$  and hence is  $OD_{\Sigma_{\beta}}^{L(j(\mathbb{R}), C)}$  for some  $C \in j(\Gamma)$ .

Fix such a  $C$  and note that  $L(j(\mathbb{R}), C) \models \text{AD}^+ + \text{SMC}$ . See Section 3.4 for a definition of SMC. Let  $A \subseteq \delta_{\beta}^{\mathcal{P}}$  witness  $\rho_{\omega}(\mathcal{P}) \leq \delta_{\beta}^{\mathcal{P}}$ , that is,  $A \notin \mathcal{P}$  and there is a formula  $\phi$  such that for all  $\alpha \in \delta_{\beta}^{\mathcal{P}}$ ,

$$\alpha \in A \Leftrightarrow \mathcal{P} \models \phi[\alpha, p],$$

where  $p$  is the standard parameter of  $\mathcal{P}$ . Now  $A$  is  $OD_{\Sigma_{\beta}}$  in  $L(j(\mathbb{R}), C)$ ; this is because letting  $\mathcal{M}_{\infty}$  be the direct limit of  $\mathcal{F}$  under iteration maps, then in  $L(j(\mathbb{R}), C)$ ,  $\mathcal{M}_{\infty} \in \text{HOD}_{\Sigma_{\beta}}$  and  $A$  witnesses

<sup>84</sup>If  $j$  is continuous at  $\kappa$ , we show that  $j$  is continuous at  $\lambda^{\mathcal{H}}$ . Suppose  $f : \kappa \rightarrow \lambda^{\mathcal{H}}$  is cofinal and increasing and  $f \in \mathcal{H}^+$ . Then  $j(f) \in j(\mathcal{H}^+)$ , and  $j(f) : j(\kappa) \rightarrow j(\lambda^{\mathcal{H}})$  is cofinal and increasing. But  $j(\kappa) = \sup j''\kappa$ , therefore,  $j(\lambda^{\mathcal{H}}) = \sup j''\lambda^{\mathcal{H}}$ .

<sup>85</sup> $\Lambda$  is essentially  $\Sigma$ , acting on stacks above  $\delta_{\beta}^{\mathcal{P}}$ , so it has branch condensation. The core model induction (in  $M$ ) as done so far works for  $\Lambda$ , showing that in  $M$  we can uniquely extend  $\Lambda$  to an  $(\omega_2, \omega_2)$ -strategy and  $\Lambda$  is  $\omega_1$ -UB. We then proceed to show  $\mathcal{M}_1^{\Lambda, \#}$  exists, and  $Lp^{\Lambda}(\mathbb{R}) \models \text{AD}^+$  just like before.

<sup>86</sup> $(\mathcal{P}, \Lambda)$  is an anomalous hod pair in the terminology of [7].  $(\mathcal{Q}, \Psi)$  is Dodd-Jensen equivalent to  $(\mathcal{P}, \Lambda)$  means that there are non-dropping iterates  $(\mathcal{Q}^*, \Psi^*)$  of  $(\mathcal{Q}, \Psi)$  and  $(\mathcal{P}^*, \Lambda^*)$  of  $(\mathcal{P}, \Lambda)$  such that  $(\mathcal{Q}^*, \Psi^*) = (\mathcal{P}^*, \Lambda^*)$ .



that  $\rho_\omega(\mathcal{M}_\infty) \leq \delta_\beta^{\mathcal{P}}$ . By SMC in  $L(j(\mathbb{R}), C)$  and the fact that  $\mathcal{H}(\beta + 1)$  is  $j(\Gamma)$ -full, we get that  $A \in \mathcal{P}$ . This is a contradiction.  $\square$

**Remark 6.9.** The construction of  $\Lambda$  is nontrivial in the case that  $\mathcal{H}^+ \models \text{cof}(\Theta)$  is measurable; otherwise,  $\Lambda$  is simply  $\Sigma$  but because of  $j$ , it acts on all of  $\mathcal{H}^+$  by an argument as in the proof of Lemma 6.8. So from this point on, we assume  $j$  is not continuous at  $\lambda^{\mathcal{H}^+}$ .

**Definition 6.10** (Nice strategies). Suppose  $\pi_{\mathcal{H}^+, \mathcal{R}} : \mathcal{H}^+ \rightarrow \mathcal{R}$ ,  $\sigma : \mathcal{R} \rightarrow j(\mathcal{H}^+)$  are elementary and  $\mathcal{R}$  is countable in  $M$ . Suppose  $j \restriction \mathcal{H}^+ = \sigma \circ \pi_{\mathcal{H}^+, \mathcal{R}}$ . Let  $\alpha < \lambda^{\mathcal{R}}$ . We say that an iteration strategy  $\Lambda_{\mathcal{R}(\alpha)}$  for  $\mathcal{R}(\alpha)$  is **nice** if and only if

- (i)  $\Lambda_{\mathcal{R}(\alpha)}$  is a  $j(\Gamma)$ -fullness preserving strategy for  $\mathcal{R}(\alpha)$  with branch condensation.  $\Lambda_{\mathcal{R}(\alpha)}$  is also positional and commuting.
- (ii)  $\pi_{\mathcal{R}(\alpha), \infty}^{\Lambda_{\mathcal{R}(\alpha)}} = \sigma' \restriction \mathcal{R}(\alpha)$  for some elementary map  $\sigma' : \mathcal{R} \rightarrow j(\mathcal{H}^+)$  such that  $j \restriction \mathcal{H}^+ = \sigma' \circ \pi_{\mathcal{H}^+, \mathcal{R}}$  (so  $\Lambda_{\mathcal{R}(\alpha)}$  acts on all of  $\mathcal{R}$ ).
- (iii) If  $\pi_{\mathcal{H}^+, \mathcal{R}} \in M$ , then  $\Sigma_\alpha \restriction M \in M$ .

We say that a  $j$ -realizable strategy  $\Lambda$  is nice if given  $\pi_{\mathcal{H}^+, \mathcal{R}}, \sigma$  as above, where  $\pi_{\mathcal{H}^+, \mathcal{R}}$  is an iteration map according to  $\Lambda$ , and  $\sigma$  is the realizing map given in the construction of  $\Lambda$ , then for each  $\alpha < \lambda^{\mathcal{R}}$ ,  $\Lambda_{\mathcal{R}(\alpha)}$  is nice.  $\dashv$

We want to show some  $j$ -realizable strategies are nice. This will be accomplished through the next several lemmas. Recall the notion of fullness with respect to a pointclass in Remark 3.12.

**Lemma 6.11.** Let  $\vec{\mathcal{T}}, \mathcal{R}, \sigma_{\mathcal{R}}$  be as above. Then  $\mathcal{R}$  is full in  $j(\Gamma)$ . In fact, letting  $\pi : \mathcal{H}^+ \rightarrow \mathcal{R}$  and  $\sigma : \mathcal{R} \rightarrow j(\mathcal{H}^+)$  be arbitrary elementary embeddings such that  $j \restriction \mathcal{H}^+ = \sigma \circ \pi$ , then  $\mathcal{R}$  is full in  $j(\Gamma)$ . Furthermore,  $j \restriction \mathcal{H}^+$ , and hence  $\pi$ , must be continuous at  $o(\mathcal{H}^+)$ .

*Proof.* We show the last statement of the lemma. The argument is very similar for all the other statements; we briefly indicate the changes at the end of the proof. Suppose  $j$  is not continuous at  $o(\mathcal{H}^+)$ . Suppose without loss of generality that  $\pi$  is not continuous at  $o(\mathcal{H}^+)$ . Indeed the general case can be reduced to this case. Suppose  $j \restriction \mathcal{H}^+ = \sigma' \circ \pi'$ , where  $\sigma' : \mathcal{R}' \rightarrow j(\mathcal{H}^+)$  is discontinuous at  $o(\mathcal{R}')$  and  $\pi' : \mathcal{H}^+ \rightarrow \mathcal{R}'$  is continuous at  $o(\mathcal{H}^+)$ . In  $M$ , let  $\sigma : \mathcal{R} \rightarrow j(\mathcal{H}^+)$  be elementary such that  $\mathcal{R}$  is countable, transitive and  $\text{rng}(\sigma') \subseteq \text{rng}(\sigma)$ ; such a  $\sigma$  can easily be found in  $M$  by considering a countable hull  $X \prec H_{\omega_2}^M$  that contains all relevant objects, then  $\sigma$  can be taken to be the restriction of the uncollapse map associated with  $X$ . Let  $\pi = \sigma^{-1} \circ \sigma'$ . It is easy to see then that  $\pi$  is not continuous at  $o(\mathcal{H}^+)$ .

This means there is a mouse  $\mathcal{M} \triangleleft \text{Lp}^{\oplus_{\beta < \lambda^{\mathcal{R}}} \Lambda_{\mathcal{R}(\beta)}, j(\Gamma)}(\mathcal{R} \restriction \delta^{\mathcal{R}}) = \mathcal{R}$  such that  $\mathcal{M} \notin \mathcal{R} \restriction \gamma$  where  $\gamma = \sup \pi[o(\mathcal{H}^+)]$ . We take  $\mathcal{M}$  to be the least such and let  $\Sigma_{\mathcal{M}}$  be the unique strategy for  $\mathcal{M}$  in  $j(\Gamma)$  (acting on trees on  $\mathcal{M}$  above  $\delta^{\mathcal{R}}$ ).

**Claim 6.12.** *There is a  $\Sigma$ -hod pair  $(\mathcal{P}, \Phi)$  such that*

- (a)  $\mathcal{P} \in V$ ,  $\Phi \restriction V \in V$ ,<sup>87</sup> and  $\Phi \in j(\Gamma)$  is fullness preserving and has branch condensation.
- (b)  $\mathcal{P}$  is countable in  $M$ ,  $\lambda^{\mathcal{P}}$  is limit and  $\text{cof}^{\mathcal{P}}(\lambda^{\mathcal{P}})$  is not measurable in  $\mathcal{P}$ .
- (c)  $\Phi = j(\Phi)^j$ .
- (d) in  $j(\Gamma)$ ,  $\exists \beta$  such that  $\Gamma(\mathcal{P}, \Phi) = j(\Gamma)|_{\theta_{\beta+\omega}}$  and  $\Gamma(\mathcal{P}, \Phi)|_{\theta_{\beta}} \models “(\mathcal{M}, \Sigma_{\mathcal{M}})$  witnesses  $\pi$  is not continuous at  $o(\mathcal{H}^+)$ ”.
- (e)  $o(\mathcal{H}^+)$  is a cardinal of  $\mathcal{P}$ , i.e.  $\mathcal{P} \models “\mathcal{H}^+$  is full.”

*Proof.* First note that in  $M$ , there is some  $\alpha$  such that  $\Sigma_{\mathcal{M}}$ , the canonical strategy of  $\mathcal{M}$ , is in  $j(\Gamma) \restriction \delta_{\beta}^{\mathcal{P}^*}$ , where  $\mathcal{P}^* = \text{HOD}_{\Sigma}^{j(\Gamma)}(\alpha)$ <sup>88</sup> and  $\mathcal{P}^* \models \alpha = \beta + \omega$ . Such  $\mathcal{P}^*$  and  $\alpha$  exist by our assumptions on  $\Gamma$ .  $\mathcal{P}^* \in V$  follows from homogeneity. Let  $\Psi^*$  be the strategy of  $\mathcal{P}^*$  which is the tail of some (equivalently, all)  $\Sigma$ -hod pair  $(\mathcal{R}^*, \Psi) \in j(\Gamma)$  where  $\Psi$  is fullness preserving and has branch condensation in  $j(\Gamma)$  and  $\mathcal{M}_{\infty}(\mathcal{R}, \Psi) = \mathcal{P}^*$ .  $\Psi^*$  is fullness preserving and has branch condensation in  $k(j(\Gamma))$ . It follows that  $\Psi^* \restriction V \in V$ : we can ordinal define  $\Psi^* \restriction V$  in  $V[G]$  from  $\Sigma$  and  $\mathcal{P}$  with the prescription above, using the fact that  $j(\Gamma)$  is *OD* in  $V[G]$ ; so by homogeneity,  $\Psi^* \restriction V \in V$ .

We want to find a countable-in- $M$  version of  $\mathcal{P}^*$  in  $V$ . Let  $(\mathcal{R}, \Psi)$  be a  $\Sigma$ -hod pair in  $j(\Gamma)$  such that  $\mathcal{M}_{\infty}(\mathcal{R}, \Psi) = \mathcal{P}^*$  and  $\Psi$  has strong hull condensation, branch condensation, and is  $j(\Gamma)$ -fullness preserving. By boolean comparisons, there is a  $\Psi$ -iterate  $(\mathcal{P}, \Phi)$  such that  $(\mathcal{P}, \Phi)$  satisfies (a). (b) is clear from the choice of  $\mathcal{P}^*$ . (c) follows from Lemma 3.36. (d) follows from the choice of  $\mathcal{P}^*$  and the fact that  $\Gamma(\mathcal{P}, \Phi) = j(\Gamma) \restriction \delta_{\alpha}^{\mathcal{P}^*}$ .

To see (e), suppose not and for simplicity, let  $\mathcal{H}^+ \trianglelefteq \mathcal{N} \triangleleft \mathcal{P}$  be least such that  $\rho_1(\mathcal{N}) = \Theta$ . Let  $f : \kappa^* \rightarrow \Theta$  be an increasing and cofinal map in  $\mathcal{H}^+$ , where  $\kappa^* = \text{cof}^{\mathcal{H}^+}(\Theta)$ .  $\mathcal{N}$  is intercomputable with the sequence  $g = \langle \mathcal{N}_{\alpha} \mid \alpha < \kappa^* \rangle$ , where  $\mathcal{N}_{\alpha} = \text{Th}_{\Sigma_1}^{\mathcal{N}}(\delta_{f(\alpha)}^{\mathcal{H}^+} \cup \{p_{\mathcal{N}}\})$ . Note that  $\mathcal{N}_{\alpha} \in \mathcal{H}^+$  for each  $\alpha < \kappa^*$ . Now let  $\mathcal{R}_0 = \text{Ult}_0(\mathcal{H}^+, \mu)$ ,  $\mathcal{R}_1 = \text{Ult}_1(\mathcal{N}, \mu)$ , where  $\mu \in \mathcal{H}^+$  is the (extender on the sequence of  $\mathcal{H}^+$  coding a) measure on  $\kappa^*$  with Mitchell order 0. Let  $i_0 : \mathcal{H}^+ \rightarrow \mathcal{R}_0$ ,  $i_1 : \mathcal{N} \rightarrow \mathcal{R}_1$  be the ultrapower maps. Letting  $\delta = \delta_{\lambda^{\mathcal{H}^+}} = \Theta$ , it's easy to see that  $i_0 \restriction (\kappa^* + 1) = i_1 \restriction (\kappa^* + 1)$  and  $\wp(\delta)^{\mathcal{R}_0} = \wp(\delta)^{\mathcal{R}_1}$ . The second equality follows from the fact that  $\mathcal{R}_0$  is full in  $j(\Gamma)$  (and hence in  $k(j(\Gamma))$ ). This means  $\langle i_1(\mathcal{N}_{\alpha}) \mid \alpha < \kappa^* \rangle \in \wp(\delta)^{\mathcal{R}_0}$ . By fullness of  $\mathcal{H}^+$  in  $j(\Gamma)$ ,<sup>89</sup>  $\langle i_1(\mathcal{N}_{\alpha}) \mid \alpha < \kappa^* \rangle \in \mathcal{H}^+$ . Using  $i_0$ ,  $\langle i_1(\mathcal{N}_{\alpha}) \mid \alpha < \kappa^* \rangle \in \mathcal{H}^+$ , and the fact that  $i_0 \restriction \mathcal{H}^+ \restriction \Theta = i_1 \restriction \mathcal{N} \restriction \Theta \in \mathcal{H}^+$ , we can get  $\mathcal{N} \in \mathcal{H}^+$  as follows. For any  $\alpha, \beta < \Theta$ ,  $\alpha \in \mathcal{N}_{\beta}$  if and only if  $i_0(\alpha) \in i_1(\mathcal{N}_{\beta}) = i_0(\mathcal{N}_{\beta})$ . Since  $\mathcal{H}^+$  can compute the right hand side of the equivalence, it can compute the sequence  $\langle \mathcal{N}_{\alpha} \mid \alpha < \kappa^* \rangle$ . Contradiction. □

<sup>87</sup>By  $\Phi \restriction V$ , we mean  $\Phi \restriction H_{c^+}^V$ .

<sup>88</sup>We identify  $\text{HOD}_{\Sigma}^{j(\Gamma)}$  with the direct limit of  $\Sigma$ -hod pairs  $(\mathcal{R}, \Psi)$  and  $\Psi$  is fullness preserving and has branch condensation in  $j(\Gamma)$ .

<sup>89</sup>Any  $A \subset \delta$  in  $\mathcal{R}_0$  is  $OD_{\Sigma}^{j(\Gamma)}$ , and so by Strong Mouse Capturing (SMC, see [7]),  $A \in \mathcal{H}^+$ .

Let  $(\mathcal{P}, \Phi)$  be as in the claim. Let  $\pi^+ : \mathcal{P} \rightarrow \mathcal{S}$  be the ultrapower map derived from the  $\pi$ -extender of length  $\delta^\mathcal{R}$ . We note that  $\pi^+$  is continuous at  $o(\mathcal{H}^+)$  and by elementarity,  $\mathcal{S} \models \text{“}\mathcal{R}|\gamma \text{ is full”}$ . Therefore,  $\mathcal{M} \notin \mathcal{S}$ .

Let  $\tau^+ : \mathcal{S} \rightarrow j(\mathcal{P})$  be the factor map, so  $j \restriction \mathcal{P} = \tau^+ \circ \sigma^+$ . Let  $\Psi = j(\Phi)^{\tau^+}$ . By (c) of the claim,  $\Phi = \Psi^{\sigma^+}$ . Therefore,

$$\Gamma(\mathcal{P}, \Phi) \subset \Gamma(\mathcal{R}, \Psi).$$

Hence,  $\Gamma(\mathcal{R}, \Psi)|\theta_\beta \models \text{“}(\mathcal{M}, \Sigma_\mathcal{M}) \text{ witnesses } \pi \text{ is not continuous at } o(\mathcal{H}^+) \text{”}$ . Now, we iterate  $\mathcal{S}$  using  $\Psi$  at the top  $\omega$  Woodin cardinals of  $\mathcal{S}$  to make  $\mathbb{R}^M$  generic.<sup>90</sup> Let  $\mathcal{S}^*$  be the resulting model. The derived model of  $\mathcal{S}^*$  at  $\delta^{\mathcal{S}^*}$  satisfies:

$$L(\Gamma(\mathcal{S}^*, \Psi_{\mathcal{S}^*}) \restriction \theta_\beta) \models \text{“}\mathcal{M} \text{ is a sound } \oplus_{\beta < \lambda^\mathcal{R}} \Lambda_{\mathcal{R}(\beta)}\text{-mouse such that } \rho_\omega(\mathcal{M}) = \delta^\mathcal{R} \text{ but } \mathcal{M} \text{ is not in } \mathcal{R}|\gamma \text{”}.$$

On the other hand,  $\mathcal{S}^* \models \text{“}\mathcal{R}|\gamma \text{ is full with respect to sound } \oplus_{\beta < \lambda^\mathcal{R}} \Lambda_{\mathcal{R}(\beta)}\text{-mice projecting to } \delta^\mathcal{R} \text{”}$ . This contradicts the displayed line above.

We have shown that  $\pi$  is continuous and that  $\mathcal{R}$  is full “at the top”, i.e. for every  $\mathcal{M} \triangleleft \text{Lp}^{\oplus_{\beta < \lambda^\mathcal{R}} \Lambda_{\mathcal{R}(\beta)}, j(\Gamma)}(\mathcal{R}|\delta^\mathcal{R})$ , there is  $\alpha < \gamma$  such that  $\mathcal{M} \triangleleft \mathcal{R}|\alpha$ . The remaining clause of fullness is proved in an almost identical manner. Suppose there is a strong cut point  $\xi$  such that letting  $\alpha < \lambda^\mathcal{R}$  be the largest such that  $\delta_\alpha^\mathcal{R} \leq \gamma$ , then in  $j(\Gamma)$ , there is a mouse  $\mathcal{M} \triangleleft \text{Lp}^{\Sigma_{\mathcal{Q}(\alpha)}(\mathcal{Q}|\gamma)}(\mathcal{Q}|\gamma)$  such that  $\mathcal{M} \notin \mathcal{R}$ . The argument given above can be carried out verbatim to obtain a contradiction.  $\square$

**Definition 6.13.** In  $M$ , suppose  $X \prec (H_{\mathfrak{c}^+}, \in)$  is countable.<sup>91</sup>  $X$  is *good* if letting  $\pi_X : M_X \rightarrow X$  be the uncollapse map,

- (a)  $j[\mathcal{H}^+] \cup \{j(\mathcal{H}^+)\} \subset \text{rng}(\pi_X)$ ;
- (b)  $\mathcal{H}^+ \cup \{\mathcal{H}^+\} \subset M_X$ ;
- (c) letting  $\mathcal{P}_X = \pi_X^{-1}(j(\mathcal{H}^+))$ , then  $\mathcal{P}_X$  is  $j(\Gamma)$ -full (see Remark 3.12) and for any  $\alpha < \lambda^{\mathcal{P}_X}$ ,  $\pi_X \restriction \mathcal{P}_X(\alpha) = i_{\mathcal{P}_X(\alpha), \infty}^{\Lambda_\alpha^X}$ , where  $\Lambda_\alpha^X$  is a tail of  $\Lambda$  for some (equivalently any) hod pair  $(\mathcal{Q}, \Lambda) \in j(\mathcal{F}) \cap X$  such that  $\Lambda$  is  $j(\Gamma)$ -fullness preserving and has branch condensation and  $(\mathcal{M}_\infty(\mathcal{Q}, \Lambda))^{M_X} = \mathcal{P}_X(\alpha)$ .

⊣

**Remark 6.14.** (a) Note that if  $X$  is good, then  $\mathcal{P}_X$  is the transitive collapse of  $\text{Hull}^{j(\mathcal{H}^+)}(j[\mathcal{H}^+] \cup$

$$\oplus_{\alpha < \lambda^{\mathcal{P}_X}} i_{\mathcal{P}_X(\alpha), \infty}^{\Lambda_\alpha^X}).$$

<sup>90</sup>More precisely, we write  $(\delta_i^\mathcal{S} : i < \omega)$  for the top  $\omega$  Woodin cardinals of  $\mathcal{S}$  and a similar notation applies to iterates of  $\mathcal{S}$ . We work in  $M[L]$  where  $L \subseteq \text{Coll}(\omega, \mathbb{R}^M)$ . We have a generic enumeration  $(x_n : n < \omega)$  of  $\mathbb{R}^M$  and we have a sequence of normal trees and models  $(\mathcal{T}_n, \mathcal{S}_n : n < \omega)$  according to  $\Psi$ , where  $\mathcal{T}_0$  is on  $\mathcal{S} = \mathcal{S}_0$ ,  $\mathcal{T}_n$  is a  $x_n$ -genericity iteration tree on  $\mathcal{S}_n$  on the window  $(\delta_{n-1}^{\mathcal{S}_n}, \delta_n^{\mathcal{S}_n})$  according to the  $\mathcal{T}_{n-1}$ -tail of  $\Psi$ , here  $\delta_{-1}^{\mathcal{S}} = 0$ . Letting  $\mathcal{S}_\infty$  be the direct limit, then  $\mathbb{R}^M$  is the symmetric reals of  $\mathcal{S}_\infty$  for some  $g \subseteq \text{Coll}(\omega, < \lambda)$ , where  $\lambda$  is the supremum of the Woodin cardinals of  $\mathcal{S}_\infty$ .

<sup>91</sup>Sometimes, we just write  $H_{\mathfrak{c}^+}$  for  $(H_{\mathfrak{c}^+}, \in)$  for brevity. Also, note that  $\mathfrak{c}^+ = \omega_2$  in  $M$  by elementarity.

(b) Letting  $X^* = \text{Hull}^{H^V}_{c^+}(\mathcal{H}^+)$  and  $X = j[X^*]$ , then  $X$  is good.

(c) Any good  $X$  is cofinal in  $o(j(\mathcal{H}^+))$  by Lemma 6.11.

**Lemma 6.15.** *In  $M$ , the set of good  $X$  is closed and unbounded; therefore, the set  $\{X \cap \mathbb{R} : X \text{ is good}\}$  is in the dual filter of  $j(\mathcal{I})$ .*

*Proof.* Let  $X$  be as in Remark 6.14(b) and let  $Y \prec (H_{c^+}, \in)$  be countable in  $M$ ,  $X \prec Y$ , and  $\mathcal{H}^+ \cup \{\mathcal{H}^+\} \subset Y$ . Since  $\mathcal{H}^+$  is countable in  $M$ , there is a club of such  $Y$ . Clearly, (a) and (b) in Definition 6.13 hold for  $Y$ . For (c), using the notation above and Lemma 6.11, we have that  $\mathcal{P}_Y$  is  $j(\Gamma)$ -full. Furthermore, for all  $\alpha < \lambda^{\mathcal{P}_Y}$ ,  $\pi_Y \upharpoonright \mathcal{P}_Y(\alpha) = i^{\Lambda_Y}_{\mathcal{P}_Y(\alpha), \infty}$  by elementarity of  $\pi_Y$ .  $\square$

Suppose  $X$  is a good hull, we let  $j_X : \mathcal{H}^+ \rightarrow \mathcal{P}_X$  be  $j_X = \pi_X^{-1} \circ j$ . We let  $\Lambda_X$  be the strategy for  $\mathcal{P}_X$  defined from  $\pi_X$  the same way  $\Lambda$  is defined from  $j$  for  $\mathcal{H}^+$  (again, we take  $\Lambda_X$  with  $\Gamma(\mathcal{P}_X, \Lambda_X)$  minimal). By Lemma 6.11 and the fact that  $X$  is good,  $\Lambda_X$  is  $j(\Gamma)$ -fullness preserving. By [7], there is an iterate  $(\mathcal{T}_X, \mathcal{Q}_X)$  of  $(\mathcal{P}_X, \Lambda_X)$  such that letting  $\Psi_X = (\Lambda_X)_{\mathcal{T}_X, \mathcal{Q}_X}$ ,  $\Psi_X$  has branch condensation, and is commuting (see [7]). Let now  $\mathcal{M}_\infty^X = \mathcal{M}_\infty(\mathcal{Q}_X, \Psi_X)$ . Note that  $\mathcal{M}_\infty^X = j(\mathcal{H}^+)(\gamma)$  for some  $\gamma < j(\lambda^{\mathcal{H}})$  and  $\mathcal{M}_\infty^X$  does not depend on the choice of  $(\mathcal{Q}_X, \Psi_X)$ .

By construction of  $\Lambda_X$ , there is a map  $m_X : \mathcal{M}_\infty^X \rightarrow j(\mathcal{H}^+)$  such that

$$\pi_X \upharpoonright \mathcal{P}_X = m_X \circ i^{\Psi_X}_{\mathcal{Q}_X, \infty} \circ i^{\mathcal{T}_X}.^{92}$$

We need a strong form of condensation to show  $\mathcal{H}^+ \models \text{“}\Theta \text{ is regular”}$ ; basically, this form of condensation will imply that if  $m_X$  is nontrivial, then

$$\text{crt}(m_X) = \delta^{\mathcal{M}_\infty^X}.^{93}$$

Therefore,  $\mathcal{M}_\infty^X \models \text{“}\delta^{\mathcal{M}_\infty^X} \text{ is a regular cardinal which is a limit of Woodin cardinal.”}$  Since  $m_X(\delta^{\mathcal{M}_\infty^X}) = j(\Theta)$ , this gives  $j(\mathcal{H}^+) \models \text{“}j(\Theta) \text{ is regular”}$ . By elementarity,  $\Theta$  is regular in  $\mathcal{H}^+$ .

The following definition originates from [8, Definition 11.14]. Let  $\mathfrak{S}$  be the set of good hulls. For each  $X \in \mathfrak{S}$ , let  $\Theta_X = j_X(\Theta)$ .

**Definition 6.16.** Suppose  $X \in \mathfrak{S}$  and  $A \in \mathcal{P}_X \cap \wp(\Theta_X)$ . We say that  $\pi_X$  has  $A$ -condensation if whenever there are elementary embeddings  $v : \mathcal{P}_X \rightarrow \mathcal{Q}$ ,  $\tau : \mathcal{Q} \rightarrow j(\mathcal{H}^+)$  such that  $\mathcal{Q}$  is countable in  $M$  and  $\pi_X = \tau \circ v$ , then

$$v(T_{\mathcal{P}_X, A}) = T_{\mathcal{Q}, \tau, A},$$

where

$$T_{\mathcal{P}_X, A} = \{(\phi, s) \mid s \in [\Theta_X]^{<\omega} \wedge \mathcal{P}_X \models \phi[s, A]\},$$

and

<sup>92</sup>Recall we assume  $j$  is discontinuous at  $\lambda^{\mathcal{H}^+}$ . Otherwise,  $\mathcal{M}_\infty^X = j(\mathcal{H}^+)$  and  $m_X$  is the identity.

<sup>93</sup>It could be that  $\mathcal{M}_\infty^X = j(\mathcal{H}^+)$  and  $m_X$  is the identity map. In which case, we cannot conclude  $\Theta$  is regular in  $\mathcal{H}^+$ . In this case,  $\Gamma(\mathcal{H}^+, \Lambda) = j(\Gamma)$ . We then simply continue the core model induction. See Section 4.

$$T_{\mathcal{Q}, \tau, A} = \{(\phi, s) \mid s \in [\delta_\alpha^{\mathcal{Q}}]^{<\omega} \text{ for some } \alpha < \lambda_{\mathcal{Q}} \wedge j(\mathcal{H}^+) \models \phi[i_{\mathcal{Q}(\alpha), \infty}^{\Sigma_{\mathcal{Q}}^\tau}(s), \pi_X(A)]\},$$

where  $\Sigma_{\mathcal{Q}}^\tau$  is the  $\tau$ -pullback strategy of  $j(\Sigma)$ .<sup>94</sup>

We say  $\pi_X$  has *condensation* if it has  $A$ -condensation for every  $A \in \mathcal{P}_X \cap \wp(\Theta_X)$ .  $\dashv$

**Theorem 6.17** (*j*-condensation lemma). *Let  $X^* = \text{Hull}^{H^+}(\mathcal{H}^+)$  and  $X = j[X^*]$ ; so  $\mathcal{P}_X = \mathcal{H}^+$ ,  $\Theta_X = \Theta$ , and  $\pi_X \upharpoonright \mathcal{P}_X = j \upharpoonright \mathcal{P}_X$ . Then  $\pi_X$  has condensation.*

*Proof.* Fix  $A \in \mathcal{P}_X \cap \wp(\Theta_X)$ . We show that  $\pi_X$  has  $A$ -condensation. Suppose not.

We first claim that if  $Y \in \mathfrak{S}$  is such that  $X \prec Y$  and  $\pi_Y$  has  $\pi_{X,Y}(A)$ -condensation, then  $\pi_X$  has  $A$ -condensation. Fix such a  $Y$ . Recall  $k : M \rightarrow N$  is the generic ultrapower induced by a generic  $h \subset j(\mathbb{P}_{\mathcal{I}})$  that we fix at the beginning of the section. Note that  $k(\pi_X) = k(\pi_Y) \circ \pi_{X,Y}$  and  $k(\pi_Y) = k \upharpoonright j(\mathcal{P}_X) \circ \pi_Y$ . By elementarity,  $k(\pi_Y)$  has  $\pi_{X,Y}(A)$ -condensation in  $N$  and hence  $k \upharpoonright j(\mathcal{P}_X)$  has  $j(A)$ -condensation in  $N$ , by the following calculations: for any countable  $\mathcal{R}$  in  $N$ , suppose there are embeddings  $i : j(\mathcal{P}_X) \rightarrow \mathcal{R}$  and  $\tau : \mathcal{R} \rightarrow k(j(\mathcal{P}_X))$  such that  $k \upharpoonright j(\mathcal{P}_X) = \tau \circ i$ , then

$$\begin{aligned} i(T_{j(\mathcal{P}_X), j(A)}) &= i(\pi_Y(T_{\mathcal{P}_Y, \pi_{X,Y}(A)})) \\ &= T_{\mathcal{R}, \tau, \pi_{X,Y}(A)} \\ &= T_{\mathcal{R}, \tau, j(A)}; \end{aligned}$$

the second equality uses the fact that  $k(\pi_Y)$  has  $\pi_{X,Y}(A)$ -condensation in  $N$  and  $k(\pi_Y) = \tau \circ i \circ \pi_Y$ . Therefore,  $\pi_X$  has  $A$ -condensation (in  $M$ ) by the elementarity of  $j$ .

Suppose now for every  $Y \in \mathfrak{S}$  such that  $X \prec Y$ ,  $\pi_Y$  does not have  $\pi_{X,Y}(A)$ -condensation. We say that a tuple  $(\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega \rangle, \mathcal{M}_\infty^Y)$  is a **bad tuple** (see Figure 4) if

- (1)  $Y \in \mathfrak{S}$ ;
- (2)  $\mathcal{P}_i = \mathcal{P}_{X_i}$  for all  $i$ , where  $X_i \in \mathfrak{S}$ ;
- (3)  $X_0 = X$  and for all  $i < j$ ,  $X_i \prec X_j \prec Y$ ;
- (4) for all  $i$ ,  $\xi_i : \mathcal{P}_i \rightarrow \mathcal{Q}_i$ ,  $\sigma_i : \mathcal{Q}_i \rightarrow \mathcal{M}_\infty^Y$ ,  $\tau_i : \mathcal{P}_{i+1} \rightarrow \mathcal{M}_\infty^Y$ , and  $\pi_i : \mathcal{Q}_i \rightarrow \mathcal{P}_{i+1}$ ;
- (5) for all  $i$ ,  $\tau_i = \sigma_i \circ \xi_i$ ,  $\sigma_i = \tau_{i+1} \circ \pi_i$ , and  $\pi_{X_i, X_{i+1}} \upharpoonright \mathcal{P}_i =_{\text{def}} \phi_{i, i+1} = \pi_i \circ \xi_i$ ;
- (6)  $\phi_{i, i+1}(A_i) = A_{i+1}$ , where  $A_i = \pi_{X, X_i}(A)$ ;
- (7) for all  $i$ ,  $\xi_i(T_{\mathcal{P}_i, A_i}) \neq T_{\mathcal{Q}_i, \sigma_i, A_i}$ .

In (7),  $T_{\mathcal{Q}_i, \sigma_i, A_{X_i}}$  is computed relative to  $\mathcal{M}_\infty^Y$ , that is

$$T_{\mathcal{Q}_i, \sigma_i, A_i} = \{(\phi, s) \mid s \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega} \text{ for some } \alpha < \lambda^{\mathcal{Q}_i} \wedge \mathcal{M}_\infty^Y \models \phi[i_{\mathcal{Q}_i(\alpha), \infty}^{\Sigma_{\mathcal{Q}_i}^{\sigma_i}}(s), \tau_i(A_i)]\}$$

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<sup>94</sup>  $\Sigma_{\mathcal{Q}}^\tau = \oplus_{\alpha < \lambda_{\mathcal{Q}}} j(\Sigma)_{\mathcal{Q}(\alpha)}^\tau$ .

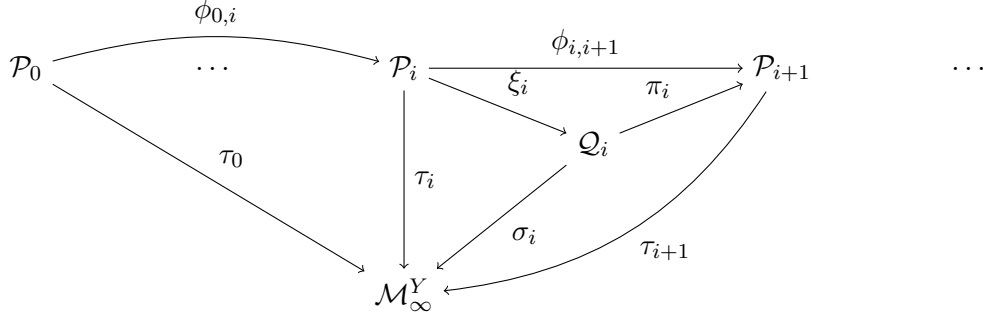


Figure 2: A bad tuple

**Claim 6.18.** *There is a bad tuple.*

*Proof.* For brevity, we first construct a bad tuple  $(\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega \rangle, j(\mathcal{H}^+))$  with  $j(\mathcal{H}^+)$  playing the role of  $\mathcal{M}_\infty^Y$ . We then simply choose a sufficiently large  $Y \in \mathfrak{S}$  and let  $i_Y : \mathcal{P}_Y \rightarrow \mathcal{M}_\infty^Y$  be the direct limit map,  $m_Y : \mathcal{M}_\infty^Y \rightarrow \mathcal{H}^+$  be the natural factor map, i.e.  $m_Y \circ i_Y = \pi_Y$ . It's easy to see that for all sufficiently large  $Y$ , the tuple  $(\langle \mathcal{P}_i, \mathcal{Q}_i, m_Y^{-1} \circ \tau_i, m_Y^{-1} \circ \xi_i, m_Y^{-1} \circ \pi_i, m_Y^{-1} \circ \sigma_i \mid i < \omega \rangle, \mathcal{M}_\infty^Y)$  is a bad tuple. But the existence of such a tuple  $(\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega \rangle, j(\mathcal{H}^+))$  follows from our assumption.  $\square$

Fix a bad tuple  $\mathcal{A} = (\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega \rangle, \mathcal{M}_\infty^Y)$  given by the previous claim.

**Claim 6.19.** *There is a  $\Sigma$ -hod pair  $(\mathcal{P}_0^+, \Pi)$  such that*

- (a)  $\lambda^{\mathcal{P}_0^+}$  is limit ordinal of the form  $\alpha' + \omega$ , and such that  $\Lambda_Y \leq_w \Pi_{\mathcal{P}_0^+(\alpha')}$  (so  $\Lambda_{X_i} \leq_w \Pi_{\mathcal{P}_0^+(\alpha')}$  for all  $i$ ).
- (b)  $(\mathcal{P}_0^+, \Pi \upharpoonright V) \in V$ .
- (c) In  $M$ ,  $\mathcal{P}_0^+$  is countable and  $\Gamma(\mathcal{P}_0^+(\alpha'), \Pi_{\mathcal{P}_0^+(\alpha')}) \models \mathcal{A}$  is a bad tuple.
- (d)  $\Pi$  has branch condensation, strong hull condensation, is  $j(\Gamma)$ -fullness preserving.
- (e)  $\Pi = j(\Pi \upharpoonright V)^j$ .

*Proof.* The properties above for  $(\mathcal{P}_0^+, \Pi)$  can be obtained by a proof similar to that of Claim 6.12, with the last clause coming from Lemma 3.36.  $\square$

This type of reflection is possible because we replace  $j(\mathcal{H}^+)$  by  $\mathcal{M}_\infty^Y$ . If  $\mathcal{Z}$  is the result of iterating  $\mathcal{P}_0^+$  via  $\Pi$  above  $\delta_{\alpha'}^{\mathcal{P}_0^+}$  to make  $\mathbb{R}^M$  generic (see Footnote 90), then letting  $h$  be  $\mathcal{Z}$ -generic for the Levy collapse of the supremum of  $\mathcal{Z}$ 's Woodin cardinals such that  $\mathbb{R}^M$  is the symmetric reals of  $\mathcal{Z}[h]$ , then in  $\mathcal{Z}(\mathbb{R}^M)$ ,

$$\Gamma(\mathcal{P}_0^+(\alpha'), \Pi_{\mathcal{P}_0^+(\alpha')}) \models \mathcal{A} \text{ is a bad tuple.}$$

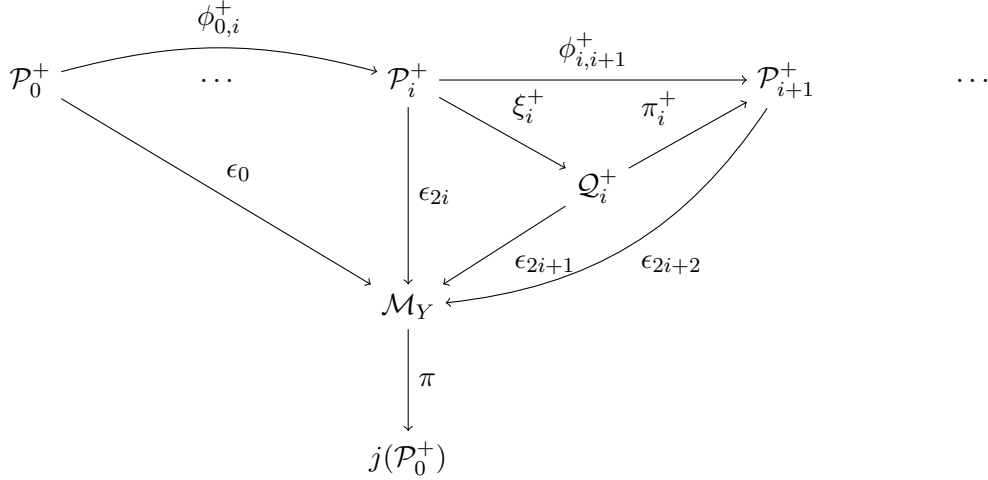


Figure 3: Lift-up maps of a bad tuple

Now we define by induction  $\xi_i^+ : \mathcal{P}_i^+ \rightarrow \mathcal{Q}_i^+$ ,  $\pi_i^+ : \mathcal{Q}_i^+ \rightarrow \mathcal{P}_{i+1}^+$ ,  $\phi_{i,i+1}^+ : \mathcal{P}_i^+ \rightarrow \mathcal{P}_{i+1}^+$  as follows.  $\phi_{0,1}^+ : \mathcal{P}_0^+ \rightarrow \mathcal{P}_1^+$  is the ultrapower map by the extender of length  $\Theta_{X_1}$  derived from  $\pi_{X_0, X_1}$ . Note that  $\phi_{0,1}^+$  extends  $\phi_{0,1}$ . Let  $\xi_0^+ : \mathcal{P}_0^+ \rightarrow \mathcal{Q}_0^+$  extend  $\xi_0$  be the ultrapower map by the  $(\text{crt}(\xi_0), \delta^{\mathcal{Q}_0})$ -extender derived from  $\xi_0$ . Finally let  $\pi_0^+ = (\phi_{0,1}^+)^{-1} \circ \xi_0^+$ . The maps  $\xi_i^+, \pi_i^+, \phi_{i,i+1}^+$  are defined similarly. Let also  $\mathcal{M}_Y = \text{Ult}(\mathcal{P}_0^+, F)$ , where  $F$  is the extender of length  $\Theta_Y$  derived from  $\pi_{X,Y}$ . There are maps  $\epsilon_{2i} : \mathcal{P}_i^+ \rightarrow \mathcal{M}_Y$ ,  $\epsilon_{2i+1} : \mathcal{Q}_i^+ \rightarrow \mathcal{M}_Y$  for all  $i$  such that  $\epsilon_{2i} = \epsilon_{2i+1} \circ \xi_i^+$ ,  $\epsilon_{2i} = \epsilon_{2i+2} \circ \phi_{i,i+1}^+$ , and  $\epsilon_{2i+1} = \epsilon_{2i+2} \circ \pi_i^+$ . Let  $\pi : \mathcal{M}_Y \rightarrow j(\mathcal{P}_0^+)^{95}$  be the factor map. When  $i = 0$ ,  $\epsilon_0$  is simply  $\pi_F$ , the ultrapower map by  $F$ . That these maps are well-defined and the objects  $\mathcal{P}_i^+, \mathcal{Q}_i^+$  end-extend  $\mathcal{P}_i, \mathcal{Q}_i$  respectively come from the fact that  $j \restriction \mathcal{H}^+$  is continuous (see Lemma 6.11). See Figure 3.

Letting  $\Sigma_i = \Sigma_{\mathcal{P}_i}$  and  $\Psi_i = \Sigma_{\mathcal{Q}_i}$ , there is a finite sequence of ordinals  $t$  and a formula  $\theta(u, v)$  such that in  $\Gamma(\mathcal{P}_0^+, \Pi)$

- (8) for every  $i < \omega$ ,  $(\phi, s) \in T_{\mathcal{P}_i, A_i} \Leftrightarrow \theta[i_{\mathcal{P}_i(\alpha), \infty}^{\Sigma_i}(s), t]$ , where  $\alpha$  is least such that  $s \in [\delta_\alpha^{\mathcal{P}_i}]^{<\omega}$ ;
- (9) for every  $i$ , there is  $(\phi_i, s_i) \in T_{\mathcal{Q}_i, \xi_i(A_i)}$  such that  $\neg \theta[i_{\mathcal{Q}_i(\alpha), \infty}^{\Psi_i}(s_i), t]$  where  $\alpha$  is least such that  $s_i \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega}$ .

The pair  $(\theta, t)$  essentially defines a Wadge-initial segment of  $\Gamma(\mathcal{P}_0^+, \Pi)$  that can define the pair  $(\mathcal{M}_\infty^Y, A^*)$ , where  $\tau_i(A_i) = A^*$  for some (any)  $i$ . In fact, these parameters are inside  $\Gamma(\mathcal{P}_0^+(\alpha), \Pi)$ .

Let  $\Pi_i$  be the  $\pi \circ \epsilon_i$ -pullback of  $j(\Pi)$ . Hence,

$$\Sigma_Y \leq_w \Pi_0 = \Pi = j(\Pi \restriction V)^j \leq_w \Pi_1 \cdots \leq_w j(\Pi \restriction V)^\pi.$$

We can use the strategies  $\Pi_i$ 's to simultaneously execute a  $\mathbb{R}^M$ -genericity iterations. We outline the process here. First we rename  $\langle \mathcal{P}_i^+, \mathcal{Q}_i^+, \xi_i^+, \phi_{i,i+1}^+, \pi_i^+ \mid i < \omega \rangle$  to  $\langle \mathcal{P}_i^0, \mathcal{Q}_i^0, \xi_i^0, \phi_i^0, \pi_i^0 \mid i < \omega \rangle$ . We

<sup>95</sup>  $\pi = \sigma_1 \circ \sigma_0$ , where  $\sigma_0 : \mathcal{M}_Y \rightarrow \pi_E(\mathcal{P}_0^+)$  is given by  $\sigma_0(\pi_{X,Y}(f)(a)) = \pi_E(f)(\pi_Y(a))$  for  $f \in \mathcal{P}_0^+$  and  $a \in [\Theta_Y]^{<\omega}$  and  $\sigma_1 : \pi_E(\mathcal{P}_0^+) \rightarrow j(\mathcal{P}_0^+)$  is defined as:  $\sigma_1(\pi_E(f)(a)) = j(f)(a)$  for  $f \in \mathcal{P}_0^+$  and  $a \in [\pi_E(\Theta)]^{<\omega}$ . Here  $E$  is the extender of length  $j(\Theta)$  derived from  $j$ .



fix in  $M^{Col(\omega, \mathbb{R})}$ ,  $\langle x_i \mid i < \omega \rangle$ , a generic enumeration of  $\mathbb{R}^M$ . We get  $\langle \mathcal{P}_i^n, \mathcal{Q}_i^n, \xi_i^n, \phi_i^n, \pi_i^n, \tau_i^n, k_i^n \mid n \leq \omega \wedge i < \omega \rangle$  such that

- (i)  $\mathcal{P}_i^\omega$  is the direct limit of the  $\mathcal{P}_i^n$ 's under maps  $\tau_i^n$ 's for all  $i < \omega$ .
- (ii)  $\mathcal{Q}_i^\omega$  is the direct limit of the  $\mathcal{Q}_i^n$ 's under maps  $k_i^n$ 's for all  $i < \omega$ .
- (iii)  $\mathcal{P}_\omega^n$  is the direct limit of the  $\mathcal{P}_i^n$ 's under maps  $\pi_i^n$ 's.
- (iv) for all  $n \leq \omega$ ,  $i < \omega$ ,  $\phi_i^n : \mathcal{P}_i^n \rightarrow \mathcal{P}_{i+1}^n$ ;  $\xi_i^n : \mathcal{P}_i^n \rightarrow \mathcal{Q}_i^n$ ;  $\pi_i^n : \mathcal{Q}_i^n \rightarrow \mathcal{P}_{i+1}^n$  and  $\phi_i^n = \pi_i^n \circ \xi_i^n$ .

Then we start by iterating  $\mathcal{P}_0^0$  above  $\delta_{\alpha+1}^{\mathcal{P}_0^0}$  to  $\mathcal{P}_0^1$  to make  $x_0$ -generic at  $\delta_{\alpha+1}^{\mathcal{P}_0^1}$ ; say the tree is  $\mathcal{T}_0$ . We let  $\tau_0^0 : \mathcal{P}_0^0 \rightarrow \mathcal{P}_0^1$  be the iteration map. During this process, we lift  $\mathcal{T}_0$  to all  $\mathcal{P}_n^0, \mathcal{Q}_n^0$  for  $n < \omega$  using the maps  $\xi_i^0, \phi_i^0$ . We pick branches for the trees on  $\mathcal{P}_i^0, \mathcal{Q}_i^0$  according to the strategies  $\Pi_i$ . We describe this process for the models  $\mathcal{Q}_0^0, \mathcal{P}_1^0$ . Let  $\mathcal{W}$  be the end model of the lift-up tree  $\xi_0^0 \mathcal{T}$  on  $\mathcal{Q}_0^0$ . Note that the tree  $\xi_0^0 \mathcal{T}_0$  is according to  $\Pi_1$ . We then iterate  $\mathcal{W}$  to  $\mathcal{Q}_0^1$  (using  $(\Pi_1)_{\mathcal{W}}$ ) to make  $x_0$  generic at  $\delta_{\alpha+1}^{\mathcal{Q}_0^1}$ . Let  $\xi_0^1 : \mathcal{P}_0^1 \rightarrow \mathcal{Q}_0^1$  be the natural embedding. Let  $\mathcal{T}_1$  be the  $x_0$ -genericity iteration tree on  $\mathcal{W}$  just described and  $\mathcal{W}^*$  be the last model of  $\phi_0^0 \mathcal{T}_0 \hat{\xi} \mathcal{T}_1$ , where  $\xi$  is the natural map from  $\mathcal{W}$  to the last model of  $\phi_0^0 \mathcal{T}_0$ . We then iterate the end model of the lifted stack  $\phi_0^0 \mathcal{T}_0 \hat{\xi} \mathcal{T}_1$  on  $\mathcal{Q}_0^1$ , noting that this stack is according to  $\Pi_2$ , to  $\mathcal{Q}_1^1$  to make  $x_0$  generic at  $\delta_{\alpha+1}^{\mathcal{Q}_1^1}$ . Let  $k_0^1 : \mathcal{Q}_0^1 \rightarrow \mathcal{Q}_1^1$ ,  $\tau_1^0 : \mathcal{P}_1^0 \rightarrow \mathcal{P}_1^1$  be the iteration embeddings,  $\pi_0^1 : \mathcal{Q}_0^1 \rightarrow \mathcal{P}_1^1$  be the natural map, and  $\phi_0^1 = \pi_0^1 \circ \xi_0^1$ . Continue this process of making  $x_0$  generic for the later models  $\mathcal{Q}_n^0$ 's and  $\mathcal{P}_n^0$ 's for  $n < \omega$ . We then start at  $\mathcal{P}_0^1$  and repeat the above process, iterating above  $\delta_{\alpha+1}^{\mathcal{P}_0^1}$  to make  $x_1$  generic at images of  $\delta_{\alpha+2}^{\mathcal{P}_0^1}$  etc. This whole process defines models and maps  $\langle \mathcal{P}_i^n, \mathcal{Q}_i^n, \xi_i^n, \phi_i^n, \pi_i^n, \tau_i^n, k_i^n \mid n \leq \omega \wedge i < \omega \rangle$  as described above.

The process yields a sequence of models  $\langle \mathcal{P}_{i,\omega}^+ = \mathcal{P}_i^\omega, \mathcal{Q}_{i,\omega}^+ = \mathcal{Q}_i^\omega \mid i < \omega \rangle$  and maps  $\xi_{i,\omega}^+ = \xi_i^\omega : \mathcal{P}_{i,\omega}^+ \rightarrow \mathcal{Q}_{i,\omega}^+, \pi_{i,\omega}^+ = \pi_i^\omega : \mathcal{Q}_{i,\omega}^+ \rightarrow \mathcal{P}_{i+1,\omega}^+$ , and  $\phi_{i,i+1,\omega}^+ = \phi_i^\omega = \pi_{i,\omega}^+ \circ \pi_{i+1,\omega}^+$ . Furthermore, each  $\mathcal{P}_{i,\omega}^+, \mathcal{Q}_{i,\omega}^+$  embeds into a  $j(\Pi \restriction V)^\pi$ -iterate of  $\mathcal{M}_Y$  and hence the direct limit  $\mathcal{P}_\infty$  of  $(\mathcal{P}_{i,\omega}^+, \mathcal{Q}_{j,\omega}^+ \mid i, j < \omega)$  under maps  $\pi_{i,\omega}^+$ 's and  $\xi_{i,\omega}^+$ 's is wellfounded. See the following figure.

[illegible]

Let  $C_i$  be the derived model of  $\mathcal{P}_{i,\omega}^+$ ,  $D_i$  be the derived model of  $\mathcal{Q}_{i,\omega}^+$  (at the sup of the Woodin cardinals of each model), then  $\mathbb{R}^N = \mathbb{R}^{C_i} = \mathbb{R}^{D_i}$ . Furthermore,  $C_i \cap \wp(\mathbb{R}) \subseteq D_i \cap \wp(\mathbb{R}) \subseteq C_{i+1} \cap \wp(\mathbb{R})$  for all  $i$ .

(8), (9) and the construction above give us that there is a  $t \in [\text{OR}]^{<\omega}$ , a formula  $\theta(u, v)$  such that

(10) for each  $i$ , in  $C_i$ , for every  $(\phi, s)$  such that  $s \in \delta^{\mathcal{P}_i}$ ,  $(\phi, s) \in T_{\mathcal{P}_i, A_i} \Leftrightarrow \theta[i_{\mathcal{P}_i(\alpha), \infty}^{\Sigma_i}(s), t]$  where  $\alpha$  is least such that  $s \in [\delta_{\alpha}^{\mathcal{P}_i}]^{<\omega}$ .

Let  $n$  be such that for all  $i \geq n$ ,  $\xi_{i,\omega}^+(t) = t$ . Such an  $n$  exists because the direct limit  $\mathcal{P}_\infty$  is wellfounded.<sup>96</sup> By elementarity of  $\xi_{i,\omega}^+$  and the fact that  $\xi_{i,\omega}^+ \upharpoonright \mathcal{P}_i = \xi_i$ ,

(11) for all  $i \geq n$ , in  $D_i$ , for every  $(\phi, s)$  such that  $s \in \delta^{\mathcal{Q}_i}$ ,  $(\phi, s) \in T_{\mathcal{Q}_i, \xi_i(A_i)} \Leftrightarrow \theta[i^{\Psi_i}_{\mathcal{Q}_i(\alpha), \infty}(s), t]$  where  $\alpha$  is least such that  $s \in [\delta^{\mathcal{Q}_i}_{\alpha}]^{<\omega}$ .

However, using (9), we get

(12) for every  $i$ , in  $D_i$ , there is a formula  $\phi_i$  and some  $s_i \in [\delta^{\mathcal{Q}_i}]^{<\omega}$  such that  $(\phi_i, s_i) \in T_{\mathcal{Q}_i, \xi_i(A_i)}$  but  $\neg \phi[i_{\mathcal{Q}_i(\alpha), \infty}^{\Psi_i}(s_i), t]$  where  $\alpha$  is least such that  $s \in [\delta_{\alpha}^{\mathcal{Q}_i}]^{<\omega}$ .

<sup>96</sup>We can arrange that  $\mathcal{P}_\infty$  embeds into a  $j(\Pi)^+$ -iterate of  $j(\mathcal{P}_0^+)$ , where  $j(\Pi)^+$  is the canonical extension of  $j(\Pi)$  in  $N$ .

Clearly (11) and (12) give us a contradiction. This shows that  $\pi_X$  has  $A$ -condensation. Since  $A$  is arbitrary,  $\pi_X$  has condensation. This completes the proof of the theorem.  $\square$

From the above theorems, we obtain a nice,  $j$ -realizable iteration strategy  $\Lambda$  with the following property:

letting  $\mathcal{M}_\infty(\mathcal{H}^+, \Lambda)$  be the direct limit of (all countable)  $\Lambda$ -iterates of  $\mathcal{H}^+$  in  $M$ , then  
there is an elementary map  $\tau : \mathcal{M}_\infty(\mathcal{H}^+, \Lambda) \rightarrow j(\mathcal{H}^+)$  such that  $\tau \circ \pi_{\mathcal{H}^+, \infty}^\Lambda = j \upharpoonright \mathcal{H}^+$   
and if  $\tau$  is nontrivial, then  $\text{crt}(\tau) = \delta^{\mathcal{M}_\infty(\mathcal{P}^+, \Lambda)}$ .

The map  $\tau$  is defined as follows: for any  $x \in \mathcal{M}_\infty(\mathcal{H}^+, \Lambda)$ , let  $\mathcal{R} \in M$  be a  $\Sigma$ -iterate of  $\mathcal{H}^+$  such that there is some  $y \in \mathcal{R}$  such that  $\pi_{\mathcal{R}, \infty}^{\Lambda_{\mathcal{R}}}(y) = x$ . Now by construction of  $\Lambda$ , there is a map  $\tau_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{H}^+)$  such that  $j \upharpoonright \mathcal{P}^+ = \tau_{\mathcal{R}} \circ \pi_{\mathcal{H}^+, \mathcal{R}}^\Lambda$  and  $\tau_{\mathcal{R}} \upharpoonright \delta^{\mathcal{R}}$  agrees with the iteration map by  $\Lambda$ . We then let  $\tau(x) = \tau_{\mathcal{R}}(y)$ .  $\tau$  is well-defined by the fact that some iterate of  $\Lambda$  has branch condensation and is commuting.

The reason  $\Lambda$  is nice is because by construction and Theorem 6.17, whenever  $i : \mathcal{H}^+ \rightarrow \mathcal{R}$  is according to  $\Lambda$ , letting  $\tau_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{H}^+)$  be given by the construction of  $\Lambda$ , then  $\tau_{\mathcal{R}} \upharpoonright \delta^{\mathcal{R}} = \pi_{\mathcal{R}, \infty}^{\Lambda_{\mathcal{R}}} \upharpoonright \delta^{\mathcal{R}}$  and  $\Lambda_{\mathcal{R}|\delta^{\mathcal{R}}} = j(\Sigma)^\tau$ . So (ii) of Definition 6.10 holds for  $\Lambda_{\mathcal{R}(\alpha)}$  for all  $\alpha < \lambda^{\mathcal{R}}$ . From this and standard theorems in the theory of hod mice, see [7, Theorem 3.26], we get that for all  $\alpha < \delta^{\mathcal{R}}$ ,  $\Lambda_{\mathcal{R}(\alpha)}$  satisfies (i) Definition 6.10. (iii) is also clear. Furthermore, if  $\tau : \mathcal{M}_\infty(\mathcal{H}^+, \Lambda) \rightarrow j(\mathcal{H}^+)$  is as above and is nontrivial, then since  $\tau$  is the “direct limit” of the  $\tau_{\mathcal{R}}$ ’s for non-dropping  $\Lambda$ -iterates  $\mathcal{R}$  of  $\mathcal{H}^+$ , Theorem 6.17 implies that  $\text{crt}(\tau) = \delta^{\mathcal{M}_\infty(\mathcal{P}^+, \Lambda)}$ .

There are two cases. The first case is when  $\tau$  is non-trivial, we then have that  $\delta^{\mathcal{M}_\infty(\mathcal{P}^+, \Lambda)}$  is a regular cardinal which is a limit of Woodin cardinals of  $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda)$ . Furthermore, by fullness preservation of  $\Lambda$ ,  $\delta^{\mathcal{M}_\infty(\mathcal{P}^+, \Lambda)} = \theta_\alpha^{j(\Gamma)}$  for some  $\alpha$  and hence  $L(\mathcal{M}_\infty(\mathcal{P}^+, \Lambda), j(\Gamma)|\theta_\alpha) \models \text{“AD}^+ + \Theta \text{ is regular.”}$  Contradiction to our smallness assumption.

The remaining case is when  $\tau$  is trivial. In other words,  $\Gamma(\mathcal{H}^+, \Lambda) = j(\Gamma)$ . By elementarity, there is a reasonable pair  $(\mathcal{P}, \Lambda)$  in  $V$  such that  $\Lambda$  is fullness preserving, has hull and branch condensation, is pullback consistent, commuting, and  $\Gamma(\mathcal{P}, \Lambda) = \Gamma$ . At this point, we remind the reader that our assumption is  $\lambda^{\mathcal{P}}$  has measurable cofinality in  $\mathcal{P}$  (cf. Remark 6.9). We need to show.

**Lemma 6.20.** *There is a reasonable hod pair  $(\mathcal{Q}, \Psi)$  such that  $\mathcal{Q} \in V$  is countable,  $\Gamma = \Gamma(\mathcal{Q}, \Psi)$ ,  $\Psi$  has a unique extension  $\Psi^+$  that acts on stacks in  $H_{\omega_2}^V$  and  $\Psi$  is  $\omega_1$ -UB.*

*Proof.* Let  $(\mathcal{P}, \Lambda)$  be a reasonable pair in  $V$  such that  $\Lambda$  is fullness preserving, has hull and branch condensation, is pullback consistent, commuting, and  $\Gamma(\mathcal{P}, \Lambda) = \Gamma$ . Let  $(\mathcal{R}, \Lambda')$  be the result of boolean comparing all “finite variations” of  $\Lambda$  i.e. for a  $V$ -generic  $G \subset \text{Coll}(\omega, \omega_1)$ , for a condition  $q$ , let  $G_q = G - G \upharpoonright \text{dom}(q) \cup q$ , let  $\Lambda_q = j_{G_q}(\Lambda)$  and compare in  $V[G]$  all pairs  $(\mathcal{P}, \Lambda_q)$  (see Section 3.6).<sup>97</sup>

**Claim 6.21.** *The boolean comparisons outlined above succeeds and hence  $(\mathcal{R}, \Lambda')$  above exists.*

<sup>97</sup>To be completely precise, we compare all pairs  $(\mathcal{P}, \Lambda_q)$  for  $q \leq p$ , where  $p$  forces all relevant facts in  $V[G]$  mentioned above. But we suppress the condition  $p$  for brevity.

*Proof.* The argument is basically from [7, Theorem 2.47]. Suppose the comparison doesn't succeed. We can then build a “diamond sequence” of length  $\omega_1$ . More precisely, we have a sequence  $\mathcal{B} = (\mathcal{R}_\alpha, \mathcal{S}_\alpha, \mathcal{P}_\alpha^q, \vec{\mathcal{T}}_\alpha, \vec{\mathcal{U}}_\alpha, \vec{\mathcal{W}}_\alpha^q, b_\alpha^q, i_\alpha^q, j_\alpha^q, \xi_\alpha : \alpha < \beta < \omega_1 \wedge q \in \text{Coll}(\omega, \omega_1^V))$ , where

- (1)  $\mathcal{R}_0 = \mathcal{P}$ .
- (2)  $\vec{\mathcal{T}}_0 \vec{\mathcal{U}}_0$  is a minimal disagreement between  $\{(\mathcal{R}_0, \Lambda_q) : q \in \text{Coll}(\omega, \omega_1^V)\}$ .  $\vec{\mathcal{T}}_0$  is according to all  $\Lambda_q$  with last model  $\mathcal{S}_0$ . For each  $q$ ,  $b_0^q = (\Lambda_q)_{\vec{\mathcal{T}}_0}(\vec{\mathcal{U}}_0)$  and  $i_0^q : \mathcal{S}_0 \rightarrow \mathcal{P}_\alpha^q$  is the iteration embedding according to  $(\Lambda_q)_{\vec{\mathcal{T}}_0, \mathcal{S}_0}$ , i.e.  $i_0^q = i_{b_0^q}^{\vec{\mathcal{U}}_0}$ . We write  $\Psi_{0,q}$  for  $\Lambda_q$ .
- (3) For  $\alpha > 0$ ,  $\vec{\mathcal{T}}_\alpha \vec{\mathcal{U}}_\alpha$  is a minimal disagreement between  $\{(\mathcal{R}_\alpha, \Psi_{\alpha,q}) : q \in \text{Coll}(\omega, \omega_1^V)\}$ .  $\vec{\mathcal{T}}_\alpha$  is according to all  $\Psi_{\alpha,q}$  with last model  $\mathcal{S}_\alpha$ , where  $\Psi_{\alpha,q}$  is the appropriate tail of  $\Lambda_q$  on  $\mathcal{R}_\alpha$  via the stack  $\oplus_{\beta < \alpha} \vec{\mathcal{T}}_\beta \vec{\mathcal{U}}_\beta \vec{\mathcal{W}}_\beta^q$ . For each  $q$ ,  $b_\alpha^q = (\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\alpha, \mathcal{S}_\alpha}(\vec{\mathcal{U}}_\alpha)$  and  $i_\alpha^q : \mathcal{S}_\alpha \rightarrow \mathcal{P}_\alpha^q$  is the corresponding iteration embedding according to  $(\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\alpha}$ , i.e.  $i_\alpha^q = i_{b_\alpha^q}^{\vec{\mathcal{U}}_\alpha}$ .
- (4)  $j_\alpha^q : \mathcal{P}_\alpha^q \rightarrow \mathcal{R}_{\alpha+1}$  are iteration maps via stack  $\vec{\mathcal{W}}_\alpha^q$  according to  $(\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\alpha \vec{\mathcal{U}}_\alpha b_\alpha^q}$ .
- (5) For any  $\alpha$ , for any  $\beta < \lambda^{\mathcal{R}_{\alpha+1}}$ , for any  $p \neq q$ ,

$$(\Psi_{\alpha,p})_{\vec{\mathcal{T}}_\alpha \vec{\mathcal{U}}_\alpha b_\alpha^p \vec{\mathcal{W}}_\alpha^p, \mathcal{R}_{\alpha+1}(\beta)} = (\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\alpha \vec{\mathcal{U}}_\alpha b_\alpha^q \vec{\mathcal{W}}_\alpha^q, \mathcal{R}_{\alpha+1}(\beta)}.$$

but for some  $p \neq q$ ,

$$(\Psi_{\alpha,p})_{\vec{\mathcal{T}}_\alpha \vec{\mathcal{U}}_\alpha b_\alpha^p \vec{\mathcal{W}}_\alpha^p, \mathcal{R}_{\alpha+1}} \neq (\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\alpha \vec{\mathcal{U}}_\alpha b_\alpha^q \vec{\mathcal{W}}_\alpha^q, \mathcal{R}_{\alpha+1}}.$$

- (6) For each  $\beta < \omega_1$ ,  $\xi_\beta$  is the least  $\xi \in (\delta(\vec{\mathcal{T}}_\beta), \lambda^{\mathcal{S}_\beta})$ <sup>98</sup> such that  $\vec{\mathcal{U}}_\beta$  is a stack on  $\mathcal{S}_\beta(\xi + 1)$  and there are  $p \neq q$  such that  $\vec{\mathcal{U}}_\beta$  witnesses  $(\Psi_{\alpha,p})_{\vec{\mathcal{T}}_\beta, \mathcal{S}_\beta(\xi+1)} \neq (\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\beta, \mathcal{S}_\beta(\xi+1)}$ , so in particular  $b_\alpha^p \neq b_\alpha^q$ , but for all  $p, q$   $(\Psi_{\alpha,p})_{\vec{\mathcal{T}}_\beta, \mathcal{S}_\beta(\xi)} = (\Psi_{\alpha,q})_{\vec{\mathcal{T}}_\beta, \mathcal{S}_\beta(\xi)}$ .

Clause (6) explains the term “minimal disagreement” used in (2) and (3). By our assumption, for each  $\alpha$ , there are  $p \neq q$  such that  $b_\alpha^p \neq b_\alpha^q$ , equivalently  $\vec{\mathcal{U}}_\alpha$  witnesses  $(\Psi_q)_{\vec{\mathcal{T}}_\alpha, \mathcal{S}_\alpha} \neq (\Psi_p)_{\vec{\mathcal{T}}_\alpha, \mathcal{S}_\alpha}$ . For each  $\alpha, q$ , let  $\Sigma_q$  be the appropriate tail of  $\Lambda_q$  on  $\mathcal{P}_\alpha^q$  and  $\lambda^{\alpha,q}$  be the order type of the Woodin cardinals of  $\mathcal{P}_\alpha^q$ . The maps  $j_\alpha^q$  (in (4)) exist by the process of simultaneously comparing all  $(\mathcal{P}_\alpha^q, (\oplus_{\alpha < \lambda^{\alpha,q}} \Sigma_q(\alpha)))$  into a common hod pair construction inside  $j(\Gamma)$  (as described in Section 3.6). Furthermore, the common model of the comparison exists and is called  $\mathcal{R}_{\alpha+1}$ , see [7, Theorem 2.47]. The main point is  $\{q : q \in \text{Coll}(\omega, \omega_1^V)\}$  is countable in  $M$  and the supremum of the Wadge ranks of  $\{\oplus_{\alpha < \lambda^{\alpha,q}} \Sigma_q(\alpha) : q \in \text{Coll}(\omega, \omega_1^V)\}$  is bounded in  $j(\Gamma)$ . That is why we can find a coarse  $\Omega$ -Woodin mouse  $(N, \Psi, \delta^N)$  that Suslin captures  $\{\oplus_{\alpha < \lambda^{\alpha,q}} \Sigma_q(\alpha) : q \in \text{Coll}(\omega, \omega_1^V)\}$  (and a universal  $\Omega$ -set, for  $\Omega \subsetneq j(\Gamma)$ , a Suslin co-Suslin pointclass containing all  $\{\oplus_{\alpha < \lambda^{\alpha,q}} \Sigma_q(\alpha) : q \in \text{Coll}(\omega, \omega_1^V)\}$ ) and performs the above comparison with the hod pair construction done inside  $N$  to guarantee (5); this process is further explained in Section 3.6. The comparison succeeds for each  $\alpha < \omega_1$ . So the sequence is of length  $\omega_1$ .

<sup>98</sup> $\delta(\vec{\mathcal{T}}_\beta)$  is the supremum of generators used along  $\vec{\mathcal{T}}_\beta$ .

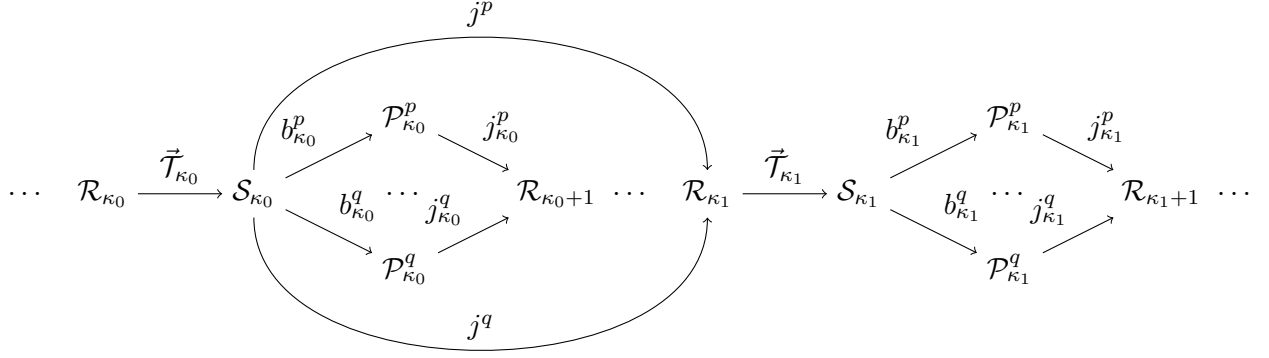


Figure 4: A Diamond sequence

Now, the proof of [7, Theorem 2.49] gives us a contradiction. We sketch the proof here for the reader's convenience. Let  $\mathcal{B}$  be the sequence above and let  $X_0 \prec X_1 \prec H_{\omega_2}$  be countable and contain all relevant objects (recall we work in  $V[g]$ ). Let  $\pi_i : H_i \rightarrow X_i$  be the uncollapse map,  $\kappa_i = crt(\pi_i)$  for  $i \in \{0, 1\}$  and let  $\pi : H_0 \rightarrow H_1$  be the map  $\pi_1^{-1} \circ \pi_0$ . For each  $p \in Coll(\omega, \omega_1^V)$ , let  $j_{\kappa_0, \kappa_1}^p$  be the iteration embedding from  $\mathcal{R}_{\kappa_0}$  to  $\mathcal{R}_{\kappa_1}$  by  $\Psi_{\kappa_0, p}$ . It is easy to see that (see [7, Theorem 2.49] for the simple calculations) for each such  $p$ :

$$j_{\kappa_0, \kappa_1}^p = \pi \upharpoonright \mathcal{R}_{\kappa_0}.$$

Let then  $j^p : \mathcal{S}_{\kappa_0} \rightarrow \mathcal{R}_{\kappa_1}$  be the embeddings according to  $(\Psi_{\kappa_0, p})_{\vec{T}_{\kappa_0}, \mathcal{S}_{\kappa_0}}$ . For each  $x \in \mathcal{S}_{\kappa_0}$ , let  $f \in \mathcal{R}_{\kappa_0}$  and  $a \in \delta(\vec{T}_{\kappa_0})^{<\omega}$  such that  $x = \pi_{\vec{T}_{\kappa_0}}(f)(a)$ , it is easy to see that

$$j^p(x) = \pi(f)(j^p(a)).$$

But note that the maps  $j^p \upharpoonright \delta(\vec{T}_{\kappa_0})$  agree (by property (6)), so indeed, the maps  $j^p$  agree on  $\mathcal{S}_{\kappa_0}$ . Using this and pullback consistency, an argument just as in [7, Theorem 2.48] shows that for all  $p, q$ ,

$$(\Psi_{\kappa_0, p})_{\vec{T}_{\kappa_0}, \mathcal{S}_{\kappa_0}(\xi_{\kappa_0+1})}(\vec{\mathcal{U}}_{\kappa_0}) = (\Psi_{\kappa_0, q})_{\vec{T}_{\kappa_0}, \mathcal{S}_{\kappa_0}(\xi_{\kappa_0+1})}(\vec{\mathcal{U}}_{\kappa_0}). \quad (6.2)$$

This clearly contradicts (6).

The equality (6.2) above holds because, by pullback consistency, for each  $p$

$$(\Psi_{\kappa_0, p})_{\vec{T}_{\kappa_0}, \mathcal{S}_{\kappa_0}(\xi_{\kappa_0+1})}(\vec{\mathcal{U}}_{\kappa_0}) = (\Psi_{\kappa_1, p})_{\mathcal{R}_{\kappa_1}(j^p(\xi_{\kappa_0+1}))}(j^p \vec{\mathcal{U}}_{\kappa_0}),$$

and by (5) and the fact that the maps  $j^p$ 's agree on  $\mathcal{S}_{\kappa_0}$ , for any  $p, q$ ,

$$(\Psi_{\kappa_1, p})_{\mathcal{R}_{\kappa_1}(j^p(\xi_{\kappa_0+1}))}(j^p \vec{\mathcal{U}}_{\kappa_0}) = (\Psi_{\kappa_1, p})_{\mathcal{R}_{\kappa_1}(j^q(\xi_{\kappa_0+1}))}(j^q \vec{\mathcal{U}}_{\kappa_0}).$$

This easily implies (6.2).  $\square$

By the claim,  $\mathcal{R} \in V$  and  $\Lambda' \upharpoonright V \in V$ . By Lemma 3.36,  $\Lambda' = j(\Lambda')^j$ . By elementarity, there is a hod pair  $(\mathcal{Q}, \Psi)$  such that  $\mathcal{Q} \in V$  is countable, an elementary embedding  $\pi : \mathcal{Q} \rightarrow \mathcal{R}$  such

that  $\Psi = (\Lambda')^\pi$  and  $\Gamma(\mathcal{Q}, \Psi) = \Gamma$ .  $\Psi$  is an  $(\omega_2, \omega_2)$ -strategy with branch condensation is  $\Gamma$ -fullness preserving.

**Claim 6.22.**  $\Psi$  is  $\omega_1$ -UB.

*Proof.* Let  $\mathcal{M}_\infty$  be the direct limit of all non-dropping iterates of  $\Psi$  and  $i : \mathcal{Q} \rightarrow \mathcal{M}_\infty$  be the direct limit map. Let  $a = (\mathcal{Q}, i, \mathcal{M}_\infty, \Psi)$ . We want to find a formula  $\varphi[x, a]$  such that for a club of countable  $X \prec H_{\omega_3}$  containing all relevant objects, letting  $\pi_X : M_X \rightarrow X$  be the uncollapse and  $(\omega_1^X, a^X) = \pi_X^{-1}(\omega_1, a)$ , for any  $M_X$ -generic  $h \subset \text{Coll}(\omega, \omega_1^X)$  in  $V$ , for any  $\vec{\mathcal{T}} \in M_X[h]$  according to  $\Psi$  and  $b \in M_X[h]$  a cofinal branch of  $\vec{\mathcal{T}}$ ,

$$M_X[h] \models \varphi[(\vec{\mathcal{T}}, b), a^X] \Leftrightarrow V \models \varphi[(\vec{\mathcal{T}}, b), a]. \quad (6.3)$$

We describe the formula  $\varphi$ .  $\varphi[x, a]$  says:

- If  $x_0 \hat{\ } x_1$  is non-dropping, letting  $\vec{\mathcal{T}} = x_0$  and  $b = x_1$ , then there is a map  $\sigma : \mathcal{M}_b^{\vec{\mathcal{T}}} \rightarrow a_2$  such that  $\sigma \circ i_b^{\vec{\mathcal{T}}} = a_1$ . Here we think of  $x$  as an ordered pair  $(x_0, x_1)$  and  $a$  as an ordered tuple  $(a_0, a_1, a_2, a_3)$ .
- If  $x_0 \hat{\ } x_1$  drops, letting  $\vec{\mathcal{T}} = x_0$ ,  $b = x_1$ , and  $(\vec{\mathcal{T}}_\alpha, \mathcal{P}_\alpha, \xi_\alpha : \alpha \leq \nu \wedge \xi_\alpha < \lambda^{\mathcal{P}_\alpha})$  be the essential components of  $\vec{\mathcal{T}}$ ,<sup>99</sup> then for  $\alpha < \nu$ ,  $\vec{\mathcal{T}}_\alpha$  does not drop and is based on  $\mathcal{P}_\alpha(\xi_\alpha)$ ,  $\vec{\mathcal{T}}_\nu$  is a stack based on  $\mathcal{P}_\nu(\xi_\nu)$ ,  $\xi_\nu$  is a successor ordinal, with cofinal branch  $b$ . There is a  $\mathcal{Q}'$ , a nondropping iterate of  $a_0$  according to  $a_3$ , with iteration map  $\tau : a_0 \rightarrow \mathcal{Q}'$ , and a  $\sigma_1 : \mathcal{P}_\nu \rightarrow \mathcal{Q}'$  such that  $\sigma_1 \circ i^{\vec{\mathcal{T}}_{<\nu}} = \tau$ , where  $\vec{\mathcal{T}}_{<\nu} = \oplus_{\alpha < \nu} \vec{\mathcal{T}}_\alpha$ , and letting  $\Psi' = (a_3)^{\sigma_1}_{\mathcal{Q}}$ ,<sup>100</sup> then  $b = \Psi'(\vec{\mathcal{T}}_\nu)$ .

Now we show 6.3, suppose  $\vec{\mathcal{T}} \hat{\ } b$  does not drop, then the equivalence follows easily. This is because if  $M_X[h] \models \varphi[(\vec{\mathcal{T}}, b), a^X]$ , then  $\pi_X \circ \sigma : \mathcal{M}_b^{\vec{\mathcal{T}}} \rightarrow \mathcal{M}_\infty$  is such that  $\pi_X \circ \sigma \circ i_b^{\vec{\mathcal{T}}} = \pi_X \circ a_1$ .  $\pi_X \circ a_1 = \pi_X(a_1) : \mathcal{Q} \rightarrow \mathcal{M}_\infty$  is the direct limit map. By branch condensation,  $b = \Psi(\vec{\mathcal{T}})$ . If  $V \models \varphi[(\vec{\mathcal{T}}, b), a]$ , then again by branch condensation,  $b = \Psi(\vec{\mathcal{T}})$ .  $M_X[h] \models \varphi[(\vec{\mathcal{T}}, b), a^X]$  by boolean comparisons done inside  $M_X[h]$ .

Suppose  $b$  drops. Then clearly,  $\mathcal{Q}(b, \vec{\mathcal{T}}_\nu)$  exists. If  $M_X[h] \models \varphi[(\vec{\mathcal{T}}, b), a^X]$  then clearly  $V \models \varphi[(\vec{\mathcal{T}}, b), a]$ . Conversely, by boolean comparison, we can find a  $\tau, \mathcal{Q}' \in M_X$  and  $\sigma_1 \in M_X[h]$  that satisfy the second clause above. Letting  $\Psi' = (a_3)^{\sigma_1}_{\mathcal{Q}}$  and  $c = \Psi'(\vec{\mathcal{T}}_\nu)$ , then since  $\Psi'$  is fullness preserving (see [7, Theorem 3.26]),  $\mathcal{Q}(\vec{\mathcal{T}}, c)$  must exist, and therefore  $c = b$  as shown in the previous section. In both cases,  $b = \Psi(\vec{\mathcal{T}})$ . □

<sup>99</sup>See [7, Definition 2.37].

<sup>100</sup>We suppress from the notation the stack  $\vec{\mathcal{U}}$  with iteration map  $\tau$  and last model  $\mathcal{Q}$ , technically we should write  $(a_3)^{\sigma}_{\vec{\mathcal{U}}, \mathcal{Q}}$ . This case includes the case  $\nu = 0$  and  $\tau, \sigma_1$  are the identity maps. □

Let  $(\mathcal{Q}, \Psi)$  be given by the above Lemma. We can then proceed with the CMI and show  $\text{Lp}^{\Psi^+}(\mathbb{R}) \models \text{AD}^+$ . This implies then that  $\Psi \in \Gamma$ , contradicting the fact that  $\Gamma(\mathcal{Q}, \Psi) = \Gamma$ . This completes the proof of Theorem 1.4.

## 7. OUTLINE OF THE PROOF OF THEOREM 1.10

We outline the argument constructing models of “ $\text{AD}_{\mathbb{R}} + \Theta$  is regular” from the assumption that the non-stationary ideal on  $\wp_{\omega_1}(\mathbb{R})$  is strong and pseudo-homogeneous. We let  $\mathcal{I}$  be the non-stationary ideal on  $\wp_{\omega_1}(\mathbb{R})$ . Let  $G \subseteq \mathbb{P}_{\mathcal{I}}$  be  $V$ -generic and  $j = j_G : V \rightarrow M = \text{Ult}(V, G) \subseteq V[G]$  be the generic embedding. Let  $k : M \rightarrow N$  be the generic embedding given by an  $M$ -generic  $H \subset j(\mathbb{P}_{\mathcal{I}})$ . We note that

- $j(\omega_1) = \mathfrak{c}^+$  (by the strength of the ideal).
- The properties in Lemma 2.7 hold for  $j$ .
- Letting  $M = \text{Ult}(V, G)$ .  $M$  need not be closed under  $\omega$ -sequences in  $V[G]$ . In particular,  $\mathbb{R}^M$  may differ from  $\mathbb{R}^{V[G]}$ . Also,  $\mathfrak{c}^+$  may be  $> \omega_2^V$ .

We let  $\Gamma$  be defined as in Section 4 and operate under the smallness assumption  $(\ddagger)$  as before. Our inductive hypothesis in this case is:

(\*\*): if  $J$  is a  $\Sigma$ -cmi operator for some reasonable hod pair  $(\mathcal{P}, \Sigma)$  such that  $\Sigma$  is definable in  $V$  from a countable sequence of ordinals, then  $J$  is definable in  $V$  from a countable sequence of ordinals.

The core model induction is very similar to the one given in the previous section; however, instead of maintaining the inductive hypothesis  $(\dagger)$ , we maintain (\*\*). We mention some key points below. The details are left to the reader. We fix the pair  $(\mathcal{P}, \Sigma)$  as in (\*\*).  $(\mathcal{P}, \Sigma)$  is allowed to be  $(\emptyset, \emptyset)$ .

- If  $J$  is a  $\Sigma$ -cmi operator on (a cone above some  $a$  in)  $H_{\omega_1}^V$  that satisfies (\*\*), then by pseudo-homogeneity, we can show  $j(J) \restriction V \in V$  and by strongness,  $j(J) \restriction V \in V$  has domain the cone above  $a$  over  $H_{\mathfrak{c}^+}^V$ . The definability calculations are done in  $M$  and  $V[G]$  plays no role in the argument. For instance, one can show using pseudo-homogeneity (as the base case) that  $j(\Sigma) \restriction V \in V$ .
- One can then show the existence of  $\mathcal{M}_1^{\mathcal{F}, \sharp}$  whenever  $\mathcal{F}$  is a  $\Sigma$ -cmi operator that satisfies (\*\*). Using pseudo-homogeneity again, one shows the operator  $H : x \mapsto \mathcal{M}_1^{\mathcal{F}, \sharp}(x)$  has the property that  $j(H) \restriction V \in V$  and (\*\*) holds for  $H$ . This is the analog of Theorem 5.1.
- Theorem 5.5 can be proved by a similar argument, though much simpler as Claim 5.6 follows easily from pseudo-homogeneity. The proof of Lemmata 5.8 and 5.11 is also given in [34]. This gives also that  $o(\text{Lp}^{\Sigma}(\mathbb{R})) < j(\omega_1) = \mathfrak{c}^+$ .



- The above gives an analog of Theorem 5.12, namely the existence of a hod pair  $(\mathcal{P}', \Sigma')$  such that  $\Sigma'$  is  $\text{Lp}^\Sigma(\mathbb{R})$ -fullness preserving,  $\Sigma' \notin \text{Lp}^\Sigma(\mathbb{R})$ , and  $\Sigma'$  is definable in  $V$  from a countable sequence of ordinals.
- In the limit case, we can define in  $M$  the model  $\mathcal{H}^+$  (see 6.1) from  $j \restriction \mathcal{H}$ . Since  $j \restriction \mathcal{H}$  is independent of  $G$  and hence  $j \restriction \mathcal{H} \in V$ ,  $\mathcal{H}^+ \in V$  by pseudo-homogeneity.
- By an argument similar to that of Proposition 6.3,  $\mathcal{H}^+$  is countable in  $M$ . We can argue  $j$  is continuous at  $o(\mathcal{H}^+)$  as follows.

**Claim 7.1.** *Let  $\gamma = o(\mathcal{H}^+)$ . Then  $j(\gamma) = \sup_{\alpha < \gamma} j(\alpha)$ .*

*Proof.* We first claim  $j \restriction \mathcal{H}^+ \in V$ . Let  $\prec$  be the canonical well-order of  $\mathcal{H}^+$ ;  $\prec$  is definable over  $\mathcal{H}^+$ . We think of  $\prec$  as a bijection from  $o(\mathcal{H}^+)$  onto  $\mathcal{H}^+$ . Note that  $j(\mathcal{H}^+) \in V$  (equivalently  $j(\prec) \in V$ ) and  $j \restriction o(\mathcal{H}^+) \in V$  (this follows from the above discussion).  $j \restriction \mathcal{H}^+$  can be easily computed from  $j \restriction o(\mathcal{H}^+), j(\mathcal{H}^+), j(\prec)$ . Therefore,  $j \restriction \mathcal{H}^+ \in V$ .

Suppose for contradiction that  $j(\gamma) > \sup_{\alpha < \gamma} j(\alpha)$ . Let  $\nu = \sup j[\gamma]$ . Let  $\vec{C} = (C_\alpha : \alpha < \gamma)$  be the canonical  $\square_\Theta$ -sequence defined over  $\mathcal{H}^+$  (see [13] for a construction of such a sequence). Let  $D = j(\vec{C})_\nu$ . Since  $\nu < j(\gamma)$ ,  $D$  is defined and is club in  $\nu$ . Furthermore, since  $j \restriction \mathcal{H}^+ \in V$ ,

$$\text{cof}^V(\nu) = \text{cof}^V(\gamma) > \omega.$$

Since  $j(\mathcal{H}^+) \in V$ ,  $\text{cof}^{j(\mathcal{H}^+)}(\nu) > \omega$ . This, in particular, implies that the set of limit points of  $D$  is non-empty and in fact a club in  $\nu$ . By the property of  $\square$ -sequences, for each limit point  $\alpha \in D$ ,

$$D \cap \alpha = j(\vec{C})_\alpha.$$

Since  $j \restriction \mathcal{H}^+ \in V$ ,  $E =_{\text{def}} j^{-1}[D] \in V$  is an  $\omega$ -club in  $\nu$  with the property: for all limit point  $\alpha$  of  $E$  with  $\text{cof}^V(\alpha) = \omega$ ,

$$E \cap \alpha = C_\alpha.$$

By the construction of  $\vec{C}$ ,  $E$  induces a  $\mathcal{P} \triangleleft \mathcal{H}^+ = \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H})$ , but also that every  $\mathcal{M} \triangleleft \mathcal{H}^+$  is an initial segment of  $\mathcal{P}$ . So  $\mathcal{P} \notin \mathcal{H}^+$ . Contradiction. □

- We can show the corresponding claim in Section 6 that continuity of  $j$  at  $\lambda^{\mathcal{H}}$  implies  $\text{cof}^V(\lambda^{\mathcal{H}}) = \omega$  as follows. If  $\kappa \in [\omega_1, \mathfrak{c}]$  is a successor cardinal or a weakly inaccessible cardinal, then  $j$  is discontinuous at  $\kappa$ . This is because  $j \restriction \kappa \in M$  and if  $j$  is continuous at  $\kappa$ , then  $j(\kappa)$  is singular in  $M$ . This contradicts the fact that  $j(\kappa)$  is successor or weakly inaccessible, hence regular, in  $M$ . This implies  $\text{cof}^V(\lambda^{\mathcal{H}}) = \omega$ . The proof that  $|\mathcal{H}^+| \leq \mathfrak{c}$ ,  $\Sigma \restriction V \in V$  and does not depend on  $G$ ,  $\mathcal{H}^+ = \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H}) \models \text{“cof}^V(\lambda^{\mathcal{H}}) \text{ is measurable”}$  (if  $j$  is discontinuous at  $\lambda^{\mathcal{H}}$ ) is similar, using pseudo-homogeneity.

- From this point on, we assume  $j$  is discontinuous at  $\lambda^{\mathcal{H}}$  and hence  $\mathcal{H}^+ = \text{Lp}^{\Sigma, j(\Gamma)}(\mathcal{H}) \models \text{“cof}^V(\lambda^{\mathcal{H}}) \text{ is measurable”}$ . Otherwise, the argument is much easier.
- Claim 7.1 and the above argument show  $\text{cof}(o(\mathcal{H}^+)) = \omega$ .
- By Lemma 2.7, arguments in Proposition 6.3 and the fact that  $j$  is continuous at  $o(\mathcal{H}^+)$ , we get that  $j \restriction \mathcal{H}^+ \in V \cap M$ .
- The analog of Lemma 6.11 is the following.

**Lemma 7.2.**  $\Lambda$  is  $j(\Gamma)$ -fullness preserving .

*Proof.* Suppose not. Let  $\vec{T}$  be according to  $\Lambda$  with end model  $\mathcal{Q}$  such that  $\mathcal{Q}$  is not  $j(\Gamma)$ -full. This means there is a strong cut point  $\gamma$  such that letting  $\alpha \leq \lambda^{\mathcal{Q}}$  be the largest such that  $\delta_{\alpha}^{\mathcal{Q}} \leq \gamma$ , then without loss of generality, in  $j(\Gamma)$ , there is a mouse  $\mathcal{M} \triangleleft \text{Lp}^{\Sigma_{\mathcal{Q}(\alpha)}}(\mathcal{Q}|\gamma)$ <sup>101</sup> such that  $\mathcal{M} \notin \mathcal{Q}$ . Let  $l : \mathcal{Q} \rightarrow j(\mathcal{H}^+)$  be such that  $j \restriction \mathcal{H}^+ = l \circ i^{\vec{T}}$ ; here by the above discussions,  $j(\mathcal{H}^+) = \pi_E(\mathcal{H}^+) = \{j(f)(a) : a \in [j(\Theta)]^{<\omega} \wedge f \in \mathcal{H}^+\}$  and  $l$  is defined as:

$$l(i^{\vec{T}}(f)(a)) = j(f)(i_{\mathcal{Q}, \infty}^{\Sigma_{\mathcal{Q}}}(a)),$$

where  $f \in \mathcal{H}^+$ ,  $a \in [\delta^{\mathcal{Q}}]^{<\omega}$ . Here  $E$  is the (long) extender of length  $o(j(\mathcal{H}))$  derived from  $j$ . We use  $i$  to denote  $i^{\vec{T}}$  from now on.

**Claim 7.3.** *There is a  $\Sigma$ -hod pair  $(\mathcal{P}, \Phi)$  such that*

- (a)  $\mathcal{P} \in V$ ,  $\Phi \restriction V \in V$ ,<sup>102</sup> and  $\Phi \in j(\Gamma)$  is fullness preserving and has branch condensation.
- (b)  $\mathcal{P}$  is countable in  $M$ ,  $\lambda^{\mathcal{P}}$  is limit and  $\text{cof}^{\mathcal{P}}(\lambda^{\mathcal{P}})$  is not measurable in  $\mathcal{P}$ .
- (c) in  $j(\Gamma)$ ,  $\Gamma(\mathcal{P}, \Phi)$  witnesses  $\Lambda$  is not fullness preserving.

*Proof.* First note that in  $M$ , there is some  $\alpha$  such that  $\Sigma_{\mathcal{M}}$ , the canonical strategy of  $\mathcal{M}$ , is in  $j(\Gamma)|\delta_{\alpha}^{\mathcal{P}^*}$ , where  $\mathcal{P}^* = \text{HOD}_{\Sigma}^{j(\Gamma)}(\alpha)$ <sup>103</sup> and  $\mathcal{P}^* \models \exists \beta \alpha = \beta + \omega$ . Such  $\mathcal{P}^*$  and  $\alpha$  exists by our assumptions on  $\Gamma$ .  $\mathcal{P}^* \in V$  follows from pseudo-homogeneity. Let  $\Psi$  be the strategy of  $\mathcal{P}^*$  which is the tail of some (equivalently, all)  $\Sigma$ -hod pair  $(\mathcal{R}, \Psi^*) \in j(\Gamma)$   $\Psi$  is fullness preserving and has branch condensation in  $j(\Gamma)$  and  $\mathcal{M}_{\infty}(\mathcal{R}, \Psi^*) = \mathcal{P}^*$ . Note that  $\Psi$  is fullness preserving and has branch condensation in  $k(j(\Gamma))$ . It follows that  $\Psi \restriction V \in V$ . From pseudo-homogeneity, we can ordinal define  $\Psi \restriction V$  in  $M$  from  $\Sigma$  and  $\mathcal{P}$  with the prescription above, using the fact that  $j(\Gamma)$  is  $OD$  in  $M$  and  $j(\Theta)$ , the Wadge rank of  $j(\Gamma)$ , doesn't depend on the choice of  $G$ .

<sup>101</sup>The case where  $\gamma = \delta_{\alpha}$  and  $\mathcal{M} \triangleleft \text{Lp}^{\oplus_{\beta < \alpha} \Sigma_{\mathcal{Q}(\beta)}}(\mathcal{Q}|\gamma)$  is similar.

<sup>102</sup>By  $\Phi \restriction V$ , we mean  $\Phi \restriction H_{c^+}^V$ .

<sup>103</sup>We identify  $\text{HOD}_{\Sigma}^{j(\Gamma)}$  with the direct limit of  $\Sigma$ -hod pairs  $(\mathcal{R}, \Psi)$  and  $\Psi$  is fullness preserving and has branch condensation in  $j(\Gamma)$ .

We also have that  $j(\mathcal{P}^*) \in V$ . This is because  $j(\mathcal{P}^*)$  is definable in  $M$  from  $\{j(\alpha), j(\mathcal{I}), j(\mathcal{H}^+)\}$ , but  $j(\mathcal{I})$  and  $j(\mathcal{H}^+)$  are both definable in  $M$ .<sup>104</sup> By an argument similar to that of Claim 7.1,  $j \restriction \mathcal{P}^* \in V$ . We want to find a countable-in- $M$  version of  $\mathcal{P}^*$  in  $V$ .

Let  $(\dot{T}, \dot{\mathcal{Q}}, \dot{\mathcal{M}}, \dot{\Lambda})$  be  $\mathbb{P}_{\mathcal{I}}$ -names for  $(\vec{T}, \mathcal{Q}, \mathcal{M}, \Lambda)$  and let  $p \in \mathbb{P}_{\mathcal{I}}$  force all relevant facts about these objects. Let  $X \prec (H_\lambda, \in)$  where

- $\lambda > \mathfrak{c}^+$  is regular,
- $X^\omega \subset X$ ,
- $\mathfrak{c} \cup \Gamma \cup \mathcal{H}^+ \cup \{\dot{T}, \dot{\mathcal{Q}}, \dot{\mathcal{M}}, \Gamma, (\mathcal{P}^*, \Psi \restriction V), (j(\mathcal{P}^*), j \restriction \mathcal{P}^*)\} \subset X$ , and
- $|X| \leq \mathfrak{c}$ .

Let  $\pi : M_X \rightarrow X$  be the transitive uncollapse map and for any  $x \in X$ , let  $\bar{x} = \pi^{-1}(x)$ . Note that

$$\overline{\mathcal{H}^+} = \mathcal{H}^+.$$

Let  $\mathbb{P} = \mathbb{P}_{\mathcal{I}}$  and  $h \subset \bar{\mathbb{P}}$  be  $M_X$ -generic such that  $h \in M$ . Such an  $h$  exists by the properties of  $X$ .<sup>105</sup>

Work in  $M_X[h]$ , let  $(\bar{\mathcal{T}}, \bar{\mathcal{Q}}, \bar{\mathcal{M}}, \bar{\Lambda})$  be the interpretation of  $(\dot{\mathcal{T}}, \dot{\mathcal{Q}}, \dot{\mathcal{M}}, \dot{\Lambda})$ . Let  $\sigma = j \restriction \mathcal{P}^*$ ; so  $\bar{\sigma} : \bar{\mathcal{P}}^* \rightarrow \bar{j}(\mathcal{P}^*)$ . Let  $\bar{R}$  be the image of  $\bar{\mathcal{P}}^*$  under the extender  $F$  derived from  $i^{\bar{\mathcal{T}}}$ , i.e.

$$\bar{R} = \{i^{\bar{\mathcal{T}}}(f)(a) : f \in \bar{\mathcal{P}}^* \wedge a \in [\delta \bar{\mathcal{Q}}]^{<\omega}\}.$$

Let  $i_F : \bar{\mathcal{P}}^* \rightarrow \bar{R}$  be the associated ultrapower map, and let  $\bar{l} : \bar{R} \rightarrow \bar{j}(\mathcal{P}^*)$ . Let  $\tau : \bar{R} \rightarrow j(\mathcal{P}^*)$  be  $\tau = \pi \circ \bar{l}$ . Note that  $\sigma \circ \pi = \tau \circ i_F$ .

Let  $\Upsilon = j(\Psi \restriction V)$  and  $\Psi^* = \pi^{-1}(\Psi \restriction V)$ . In  $M_X[h]$ ,  $\bar{\Lambda}$  is not full as witnessed by  $\bar{T}, \bar{\mathcal{Q}}, \bar{\mathcal{M}}$  inside  $\bar{j}(\bar{\Gamma})|_{\bar{\alpha}}$ , where  $\bar{j}$  is the generic ultrapower induced by  $h$ . Therefore, letting  $j(\Psi \restriction V)^{\tau \circ i_F} = \Sigma_1$  and  $j(\Psi \restriction V)^\tau = \Sigma_2$ , we note that

$$\Sigma_1 \leq_w \Sigma_2.$$

In  $M$ ,

$$\Gamma(\bar{\mathcal{P}}^*, \Sigma_1) \subset \Gamma(\bar{\mathcal{R}}, \Sigma_2),$$

and letting  $\Sigma_3 = j(\Sigma)^\tau$ ,

$$L(\Gamma(\bar{\mathcal{P}}^*, \Sigma_1)) \models \text{“}\bar{\mathcal{M}} \text{ is a } \Sigma_3\text{-mouse and } \neg(\bar{\mathcal{M}} \triangleleft \bar{\mathcal{Q}})\text{.”}$$

<sup>104</sup>This is one place where we use the ideal  $\mathcal{I}$  is the non-stationary ideal, or just that it is definable in  $V$ . Technically,  $j(\mathcal{H}^+)$  is definable in  $M$  from  $j(\mathcal{H})$  and a countable sequence of ordinals, namely any sequence  $(j(\gamma_n) : n < \omega)$ , where  $(\gamma_n : n < \omega)$  is cofinal in  $o(\mathcal{H}^+)$  and there is  $j(\mathcal{H}) \triangleleft \mathcal{M}_n \triangleleft j(\mathcal{H}^+)$  such that  $o(\mathcal{M}_n) = j(\gamma_n)$ .

<sup>105</sup>We do not have a way of lifting  $\pi$  to all of  $M_X[h]$ . This creates complications and forces us to argue as below.

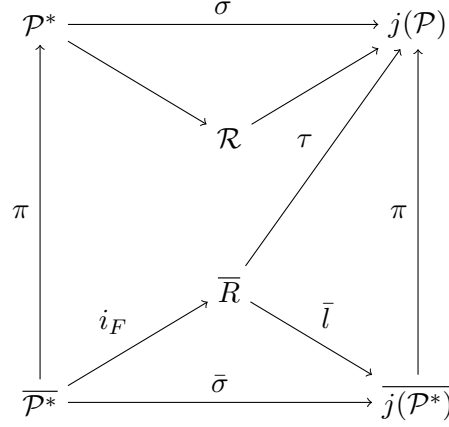


Figure 5: Diagram for the proof of Claim 6.12. Here  $\tau = \pi \circ \bar{l}$ .

Finally, note that  $\bar{\mathcal{T}}$  is according to  $\Lambda$  as  $\bar{\mathcal{T}}$  is  $j$ -realizable. It is easy then to see that (a),(b), (c) hold for  $(\bar{\mathcal{P}}^*, \Sigma_1)$ . Therefore, the pair  $(\bar{\mathcal{P}}^*, \Sigma_1)$  is the desired  $(\mathcal{P}, \Phi)$ . See Figure 5 for an illustration of the argument above.

□

Now we proceed to finish the proof of Lemma 7.2. Let  $(\mathcal{P}, \Phi)$  be as in the claim. We assume that  $L(\Gamma(\mathcal{P}, \Phi))$  satisfies the statement: “ $\mathcal{Q}$  is not full as witnessed by  $\mathcal{M}$ ”, i.e. we reuse the notation for  $\vec{\mathcal{T}}, \mathcal{Q}, \mathcal{M}, l$ . By arguments similar to that used in Lemma 6.8, we see that no levels of  $\mathcal{P}$  projects across  $\Theta$  and in fact, by the proof of Claim 6.12(e),  $o(\mathcal{H}^+)$  is a cardinal of  $\mathcal{P}$ .

In other words,  $\mathcal{P}$  thinks  $\mathcal{H}^+$  is full. Let  $\Psi = \Phi \upharpoonright V$  and let

$$i^* : \mathcal{P} \rightarrow \mathcal{R}$$

be the ultrapower map by the extender induced by  $i$  of length  $\delta^{\mathcal{Q}}$ . Note that  $\mathcal{Q} \triangleleft \mathcal{R}$  and  $\mathcal{R}$  is wellfounded since there is a natural map

$$l^* : \mathcal{R} \rightarrow \mathcal{P}_E$$

extending  $l$  and  $\pi_E \upharpoonright \mathcal{P} = l^* \circ i^*$ ; here  $l^*(i^*(f)(a)) = \pi_E(f)(i_{\mathcal{Q}, \infty}^{\Sigma_{\mathcal{Q}}}(a))$  for  $f \in \mathcal{P}$  and  $a \in [\delta^{\mathcal{Q}}]^{<\omega}$  and  $\mathcal{P}_E = \{\pi_E(f)(a) : f \in \mathcal{P} \wedge a \in [j(\Theta)]^{<\omega}\}$ . We note here that since  $\pi_E$  is continuous at  $o(\mathcal{H}^+)$ ,  $j(\mathcal{H}^+)$  is a cardinal initial segment of  $\mathcal{P}_E$ . Furthermore, there is a natural embedding  $\sigma^- : \mathcal{P}_E \rightarrow j(\mathcal{P})$  such that

$$j \upharpoonright \mathcal{P} = \sigma^- \circ l^* \circ i^*.$$

Here  $\sigma^-(\pi_E(f)(a)) = j(f)(a)$  for all  $f \in \mathcal{P}$  and  $a \in [j(\Theta)]^{<\omega}$ . The equality above just comes from the fact that  $E$  is an extender derived from  $j$ .

By the choice of  $(\mathcal{P}, \Phi)$ ,  $\mathcal{M}$ 's unique strategy  $\Sigma_{\mathcal{M}} \leq_w \Phi$  and  $\Sigma_{\mathcal{M}} \in L(\Gamma(\mathcal{P}, \Phi))$ ; so in particular,  $L(\Gamma(\mathcal{P}, \Phi))$  knows  $\mathcal{Q}$  is not full as witnessed by  $(\mathcal{M}, \Sigma_{\mathcal{M}})$ .

Let  $\mathcal{W} = \mathcal{M}_{\omega}^{\Phi, \sharp}$  and  $\Lambda^*$  be the unique strategy of  $\mathcal{W}$ ; again  $\mathcal{W} \in V$ ,  $\mathcal{W}$  is countable in  $M$ , and  $\Lambda^* \restriction V \in V$ . Furthermore, by fullness of  $\mathcal{P}$ ,  $o(\mathcal{P})$  is a cardinal of  $\mathcal{W}$ . Let  $\mathcal{W}^*$  be a  $\Lambda^*$ -iterate of  $\mathcal{W}$  below its first Woodin cardinal that makes  $(\mathcal{Q}, \vec{\mathcal{T}})$  generic via the  $(\mathcal{Q}, \vec{\mathcal{T}})$ -genericity iteration. Letting  $K$  be the generic for the extender algebra of  $\mathcal{W}^*$  at its first Woodin cardinal such that  $(\mathcal{Q}, \vec{\mathcal{T}}) \in \mathcal{W}^*[K]$ , then the derived model  $D(\mathcal{W}^*[K])$  (at the supremum of the Woodin cardinals of  $\mathcal{W}^*[K]$ ) satisfies

$$L(\Gamma(\mathcal{P}, \Phi), \mathbb{R}) \models \mathcal{Q} \text{ is not full.}^{106107}$$

So the above fact is forced over  $\mathcal{W}^*[K]$ .

Now further extend  $i^*$  to  $i^+ : \mathcal{W} \rightarrow \mathcal{Y}$  and extend  $l^*$  to  $l^+ : \mathcal{Y} \rightarrow \mathcal{W}_E$  so that  $\pi_E \restriction \mathcal{W} = l^+ \circ i^+$ ;  $i^+, l^+, \mathcal{W}_E$  are defined in a similar manner as above. Again, there is a natural map  $\sigma : \mathcal{W}_E \rightarrow j(\mathcal{W})$  such that  $\sigma \circ l^+ \circ i^+ = j \restriction \mathcal{W}$ . Note that  $(\mathcal{Y}, \sigma \circ l^+)$  are countable in  $M$ ; this is the key reason we need  $\mathcal{P}$  is countable in  $M$ . Therefore, it makes sense to pullback in  $M$  via  $\sigma \circ l^+$ . Let

$$\Psi^* = j(\Lambda^*)^{\sigma \circ l^+}.$$

Now note that  $\Phi = (\pi_E(\Psi)^{l^*})^{i^*}$  and  $\Lambda^* = (\Psi^*)^{i^+}$ , so

$$\Gamma(\mathcal{P}, \Phi) \subseteq \Gamma(\mathcal{R}, \pi_E(\Psi)^{l^*}) \quad (7.1)$$

and

$$\Lambda \leq_w \Psi^*. \quad (7.2)$$

Now iterate  $\mathcal{Y}$  using  $\Psi^*$  to  $\mathcal{Y}^*$  above  $\mathcal{Q}$  to make  $\mathbb{R}^M$  generic (see Footnote 90). From 7.1 and 7.2, we get that in  $D(\mathcal{Y}^*)$ ,

$$L(\Gamma(\mathcal{R}, \pi_E(\Psi)^{l^*})) \models \mathcal{Q} \text{ is not full as witnessed by } \mathcal{M}.$$

This gives  $\mathcal{M}$  is  $OD_{\Sigma_{\mathcal{Q}}}^{D(\mathcal{Y}^*)}$ , so  $\mathcal{M} \in \mathcal{Y}^*$  and so  $\mathcal{M} \in \mathcal{R}$  since  $\mathcal{R}$  is a cardinal initial segment of  $\mathcal{Y}^*$ . This contradicts the internal fullness of  $\mathcal{Q}$  inside  $\mathcal{R}$  ( $\mathcal{P}$  thinks  $\mathcal{H}^+$  is full, so by elementarity,  $\mathcal{R}$  thinks  $\mathcal{Q}$  is full). See Figure 6 for an illustration of the argument above.

□

<sup>106</sup>Here we abuse notations a bit, by using the same notation for  $\Phi$  and its various restrictions.

<sup>107</sup>This is because we can continue iterating  $\mathcal{W}^*$  above the first Woodin cardinal to  $\mathcal{W}^{**}$  such that letting  $\lambda$  be the sup of the Woodin cardinals of  $\mathcal{W}^{**}$ , then there is a  $Col(\omega, < \lambda)$ -generic  $h$  such that  $\mathbb{R}^{V[G]}$  is the symmetric reals for  $\mathcal{W}^{**}[h]$ . And in  $\mathcal{W}^{**}(\mathbb{R}^{V[G]})$ , the derived model satisfies that  $L(\Gamma(\mathcal{P}, \Phi)) \models \mathcal{Q}$  is not full. In the above, we have used the fact that the interpretation of the UB-code of the strategy for  $\mathcal{P}$  in  $\mathcal{W}^{**}$  to its derived model is  $\Phi \restriction \mathbb{R}^{V[G]}$ ; this key fact is proved in [7, Theorem 3.26].

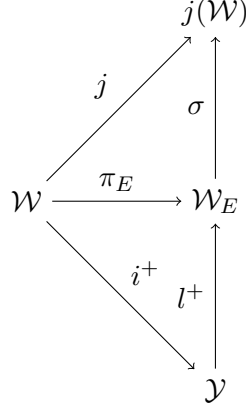


Figure 6: Diagram for the proof of Lemma 7.2.

- Regarding the proof of the  $j$ -condensation lemma (Theorem 6.17), the following are the main changes we need. Fix a bad tuple  $\mathcal{A} = (\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega \rangle, \mathcal{M}_\infty^Y)$  in  $M$  as in the proof of Theorem 6.17; note that  $k(\mathcal{A}) = (\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega \rangle, k(\mathcal{M}_\infty^Y))$  is also a bad tuple in  $N$ .

We let  $(\mathcal{P}_0^+, \Pi)$  be such that

- (a)  $\mathcal{P}_0^+ = \text{HOD}_\Sigma^{j(\Gamma)}(\alpha' + \omega)$  for some limit ordinal  $\alpha'$  such that  $\mathcal{A} \in j(\Gamma)|\theta_{\alpha'}$ . Note that  $\mathcal{P}_0^+$  is countable in  $N$  and  $\{k(\mathcal{P}_0^+), k \restriction \mathcal{P}_0^+\} \in M$ .
- (b)  $\Pi$  is the natural strategy of  $\mathcal{P}_0^+$  and is the tail of any  $\Sigma$ -hod pair  $(\mathcal{R}, \Psi)$  such that  $\mathcal{M}_\infty(\mathcal{R}, \Psi) = \mathcal{P}_0^+$ .
- (c)  $\Pi \restriction M \in M$  and  $\Pi \restriction M \subseteq k(\Pi \restriction M)^k$ . The latter property follows from the fact that  $(\mathcal{P}_0^+, \Pi)$  is a hod pair of limit type,  $\Pi$  has branch condensation and is  $k(j(\Gamma))$  fullness preserving; therefore, basic theory of hod mice, e.g. the proof of [7, Theorem 3.26], implies  $\Pi = k(\Pi)^k$ . We do not know if the conclusion of Lemma 3.36 holds for all hod pairs constructed in the core model induction here, but fortunately, we do not need it.<sup>108</sup> We will also write  $\Pi$  for  $k(\Pi \restriction M)^k$  when interpreted in  $N$ .
- (d)  $\Lambda_Y \leq_w \Pi_{\mathcal{P}_0^+(\alpha')}$  (so  $\Lambda_{X_i} \leq_w \Pi_{\mathcal{P}_0^+(\alpha')}$  for all  $i$ ) in  $N$ . Note that we can extend  $\Lambda_Y$  (similarly  $\Lambda_{X_i}$  for all  $i$ ) in  $N$  as the realizable strategy (which we also call  $\Lambda_Y$ ) of  $\mathcal{P}_Y$  into  $k(j(\mathcal{H}^+))$  using the map  $k \circ \pi_Y$ .
- (e) In  $N$ ,  $\mathcal{P}_0^+$  is countable and  $\Gamma(\mathcal{P}_0^+(\alpha'), \Pi_{\mathcal{P}_0^+(\alpha')}) \models \mathcal{A}$  is a bad tuple.

The rest of the proof is essentially the same as before, but now we run the “three dimensional argument” using  $k$  (instead of  $j$ ) and the argument takes place in  $N$  (instead of in  $M$ ). We leave the details to the reader.

This completes our outline.

<sup>108</sup>In the context of DI, we need Lemma 3.36 in situations where the hod pair has successor type. In the case where hod pairs are of limit type, we can argue as above.

**Remark 7.4.** If one strengthens DI to the hypothesis in Footnote 8, then an easy modification of the outline given above will also give the consistency of “ $\text{AD}_{\mathbb{R}} + \Theta$  is regular”. In particular, one maintains the inductive hypothesis (\*\*), instead of (†), and one does not need Lemma 3.36. We leave the details to our reader.

## 8. OPEN PROBLEMS AND QUESTIONS

As mentioned above, there are various important and intriguing questions concerning ideals on  $\omega_2$ . Woodin has conjectured that (see Theories (a) and (c) in [35, Question 12])

**Conjecture 8.1.** *The following theories are equiconsistent.*

1.  $\text{ZFC} + \text{MM}(\mathfrak{c}) + J_{NS}$  is weakly presaturated.
2.  $\text{ZF} + \text{AD}_{\mathbb{R}} + “\Theta$  is regular”.

In the above  $\text{MM}(\mathfrak{c})$  is Martin’s Maximum for posets of size at most  $\mathfrak{c}$ , and  $J_{NS}$  is the non-stationary ideal on  $\omega_2$  concentrating on ordinals of cofinality  $\omega$ .  $J_{NS}$  is weakly presaturated if for every function  $f : \omega_2 \rightarrow \omega_2$ , for every  $S \in \wp(\omega_2)/J_{NS}$ , there exists a canonical function  $h : \omega_2 \rightarrow \omega_2$  such that

$$\{\alpha \in S : f(\alpha) \leq h(\alpha)\} \notin J_{NS}.$$

See [35, Section 9.7] for a more detailed discussions on ideals on  $\omega_2$ . [35, Theorem 9.137] has established one direction of the conjecture. The converse most likely requires new techniques in the core model induction.

**Question 8.2.** *What is the consistency strength of the theory “ $\text{ZFC} +$  there is a dense ideal on  $\omega_2$ ”?*

Finally, as mentioned in the previous section, we do not know the exact consistency strength of the theory “ $\text{ZFC} +$  there is a strong, pseudo-homogeneous ideal on  $\wp_{\omega_1}(\mathbb{R})$ ”, but we conjecture

**Conjecture 8.3.** *The following theories are equiconsistent.*

1. “ $\text{ZFC} +$  there is a strong, pseudo-homogeneous ideal on  $\wp_{\omega_1}(\mathbb{R})$ ”.
2.  $\text{ZF} + \text{AD}_{\mathbb{R}} + “\Theta$  is regular”.

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