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Sharp bounds for multiplicities of Bianchi modular forms

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Abstract

We prove a degree-one saving bound for the dimension of the space of cohomological automorphic forms of fixed level and growing weight on SL_2 over any number field that is not totally real. In particular, we establish a sharp bound on the growth of cuspidal Bianchi modular forms. We transfer our problem into a question over the completed universal enveloping algebras by applying an algebraic microlocalization of Ardakov and Wadsley to the completed homology. We prove finitely generated Iwasawa modules under the microlocalization are generic, solving the representation theoretic question by estimating growth of Poincaré–Birkhoff–Witt filtrations on such modules.

Contents

1. Introduction

Let F be a number field of degree $r = r_1 + 2r_2$, with r_1 real places and r_2 complex places. Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, so that $\mathrm{SL}_2(F_\infty) = \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$. Let Z_∞ be the centre of $\mathrm{SL}_2(F_\infty)$, K_f be a compact open subgroup of $\mathrm{SL}_2(\mathbb{A}_F^\infty)$, and let

$$X(K_f) := \mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)/K_f Z_\infty.$$

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If $\mathbf{k} = (k_1, \dots, k_{r_1+r_2})$ is an $(r_1 + r_2)$-tuple of positive even integers, we define $W_\mathbf{k}$ to be the representation of $\mathrm{SL}_2(F_\infty)$ obtained by taking the tensor product of the representation Sym^{k_i-2} of $\mathrm{SL}_2(F_{v_i})$ when v_i is a real place and the representation $\mathrm{Sym}^{k_i/2-1} \otimes \overline{\mathrm{Sym}}^{k_i/2-1}$ of $\mathrm{SL}_2(F_{v_i})$ when v_i is a complex place. We also use $W_\mathbf{k}$ to denote the local system on $X(K_f)$ coming from the representation $W_\mathbf{k}$. We let $S_\mathbf{k}(K_f)$ be the space of cohomological cusp forms on $X(K_f)$ with weight $\mathbf{k}$. We define $\Delta(\mathbf{k})$ to be

$$\Delta(\mathbf{k}) = \prod_{1\le i\le r_1} k_i \times \prod_{r_1<i\le r_1+r_2} k_i^2.$$

In this paper, we will adapt p-adic algebraic methods to study the growth of dimension of $S_\mathbf{k}(K_f)$ as $\mathbf{k}$ varies and K_f is fixed.

When F is totally real, Shimizu [Shi63] has proven that

$$\dim_\mathbb{C} S_\mathbf{k}(K_f) \sim C \cdot \Delta(\mathbf{k})$$

for some constant C independent of $\mathbf{k}$.

When F is not totally real, the growth rate of $\dim_\mathbb{C} S_\mathbf{k}(K_f)$ is wildly open. The first non-trivial bound is given by a trace formula method:

$$\dim_\mathbb{C} S_\mathbf{k}(K_f) = o(\Delta(\mathbf{k})). \tag{1}$$

Conjecture 1.1. *If F is imaginary quadratic, $\mathbf{k} = (k)$, there exists a constant c depending only on K_f such that for $k \ge 1$,*

$$\dim_\mathbb{C} S_\mathbf{k}(K_f) \le c \cdot k.$$

This conjecture is supported by experimental data of Finis–Grunewald–Tirao [FGT10] and the work of Calegari–Mazur [CM09] (for Hida families). Under mild conditions, such an upper bound of linear growth rate is sharp from the base change of classical elliptic modular forms.

In this paper, we prove this conjecture by giving a polynomial saving improvement of (1).

If F is imaginary quadratic, Finis, Grunewald and Tirao [FGT10] established the bounds

$$k \ll \dim_\mathbb{C} S_\mathbf{k}(K_f) \ll_{K_f} \frac{k^2}{\ln k}, \quad \mathbf{k} = (k)$$

for suitable K_f using base change and the trace formula respectively. In [Mar12], Marshall improved (1) by a power saving bound: supposing $\mathbf{k} = (k, \dots, k)$ is parallel,

$$\dim_\mathbb{C} S_\mathbf{k}(K_f) \ll_{\epsilon,K_f} k^{r-1/3+\epsilon}.$$

Later on in [Hu21], Hu proved a better power saving bound

$$\dim_\mathbb{C} S_\mathbf{k}(K_f) \ll_{\epsilon,K_f} k^{r-1/2+\epsilon}.$$

Both of them use mod p representation theory methods applied to Emerton's completed homology and relate the completed homology back to $S_{\mathbf{k}}(K_f)$ via a spectral sequence and the Eichler–Shimura isomorphism summarized as

$$\dim_{\mathbb{C}} H_c^{r_1+r_2}(Y(K_f), W_{\mathbf{k}}) = 2^{r_1} \dim_{\mathbb{C}} S_{\mathbf{k}}(K_f) \tag{2}$$

in [Mar12]. Here $Y(K_f)$ is defined to be

$$Y(K_f) := X(K_f)/K_\infty,$$

and K_∞ is the maximal compact subgroup of $\mathrm{SL}_2(F_\infty)$.

If F is imaginary quadratic, then $Y(K_f)$ is a hyperbolic 3-manifold. The space $S_{\mathbf{k}}(K_f)$ consists of cuspidal Bianchi modular forms of parallel weight (k,k). This space corresponds to the first compactly supported cohomology with coefficient local system $W_{\mathbf{k}} = \mathrm{Sym}^{k/2-1} \otimes \overline{\mathrm{Sym}}^{k/2-1}$ by the Eichler–Shimura isomorphism (2).

The main result of our paper is to bound $\dim_{\mathbb{C}} S_{\mathbf{k}}(K_f)$:

THEOREM 1.2. *If F is not totally real, then for any fixed K_f, we have*

$$\dim_{\mathbb{C}} S_{\mathbf{k}}(K_f) \leq_{K_f} (\min_{1\leq i\leq r} k_i)^{-1} O(\Delta(\mathbf{k})). \tag{3}$$

If, moreover, $\mathbf{k} = (k,\dots,k)$ is parallel, we have

$$\dim_{\mathbb{C}} S_{\mathbf{k}}(K_f) \leq_{K_f} O(k^{r-1}).$$

COROLLARY 1.3. *Conjecture 1.1 is correct. Suppose K_f is sufficiently small. For the arithmetic hyperbolic 3-manifold $Y(K_f)$ and cohomological degree $n = 1, 2$, we have the sharp bounds*

$$\dim_{\mathbb{C}} H_c^n(Y(K_f), W_{\mathbf{k}}) \sim_{K_f} k.$$

Compared to [Mar12, Th. 1, Cor. 2] and [Hu21, Th. 1.1], we get a degree-one saving bound and we do not need ϵ weakening. Let $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with ring of integers $\mathcal{O}_p$. It is very worth noting that both [Mar12, Th. 1, Cor. 2] and [Hu21, Th. 1.1] crucially use the $\mathrm{SL}_2(F_p)$-action on the completed homology, but we only make use of the group action of the first congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_p)$.

If F only admits one complex place, or equivalently $r_2 = 1$, it seems likely (3) gives a sharp upper bound by heuristics from the Calegari–Emerton conjecture.

Let K be a finite extension of $\mathbb{Q}_p$. Now we fix a compact open level subgroup $G \subset \mathrm{SL}_2(\mathcal{O}_p)$. If K_f further decomposes as

$$K_f = K_p K^p \text{ for } K_p \subset G,\ K^p \subset \mathrm{SL}_2(\mathbb{A}^{p,\infty}),$$

we introduce completed homology of tame level K^p (see [CE09]) as

$$\widetilde{H}_\bullet(K^p) := \varprojlim_s \varprojlim_{K_p\subset G} H_\bullet(Y(K_pK^p), \mathbb{Z}/p^s\mathbb{Z}) \otimes_{\mathbb{Z}_p} K. \tag{4}$$

They are finitely generated modules over the Iwasawa algebra $K[[G]]$ (see [Eme06]).

To prove Theorem 1.2, we establish the following theorem on sub-polynomial growth of algebraic quotients and their higher cohomology of a finitely generated Iwasawa $\mathbb{Q}_p[[G]]$-module for G being a product of uniform (see [DdSMS99, §4]) pro-p compact open subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$. Therefore, the main theorem can be obtained by using a fundamental spectral sequence due to Emerton [Eme06], [CE09] (also see [Mar12] and [Hu21]).

By [Ven02], the Iwasawa algebra $K[[G]]$ is Auslander regular. In particular, it is of finite global dimension. The G-homology of any Iwasawa module should be vanishing above a degree that only depends on G. A multiplicity-free polynomial $p_{\widetilde{M}}$ of $\mathbf{k}$ is a polynomial such that for each monomial term $k_1^{g_1}\cdots k_r^{g_r}$ of $p_{\widetilde{M}}$, each g_i is at most 1 for $1 \le i \le r$.

For each $1 \le i \le r$, let

$$G_i := (I_2 + p\mathrm{M}_2(\mathbb{Z}_p)) \cap \mathrm{SL}_2(\mathbb{Z}_p).$$

Let $G = \prod_{i=1}^r G_i$, $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, and $W_{\mathbf{k}}$ be the algebraic representation $\boxtimes_{i=1}^r \mathrm{Sym}^{k_i}$ of G.

THEOREM 1.4. *Let $\widetilde{M}$ be a finitely generated $\mathbb{Q}_p[[G]]$-module of rank d. There exists a multiplicity-free polynomial $p_{\widetilde{M}}$ of $\mathbf{k}$ of degree at most $r-1$ associated to $\widetilde{M}$ such that for any $\mathbf{k} \in \mathbb{N}^r$ and any $i \ge 1$,*

$$|\dim_{\mathbb{Q}_p} H_0(G, \widetilde{M} \otimes W_{\mathbf{k}}) - d\prod_{i=1}^r (k_i+1)| \le p_{\widetilde{M}}(\mathbf{k}),$$
$$\dim_{\mathbb{Q}_p} H_i(G, \widetilde{M} \otimes W_{\mathbf{k}}) \le p_{\widetilde{M}}(\mathbf{k}).$$

Let us first consider the growth of algebraic representations for some simplest $K[[G]]$-modules.

If $\widetilde{M}$ is of canonical dimension 0, then $\widetilde{M}$ is finite dimensional by [AW13, Lemma 10.13]. The polynomials can be chosen to be constants (of degree 0).

If $\widetilde{M} \simeq K[[G]]$ as the module over itself, in Section 6, we explicitly exhibit

$$\mathrm{Hom}_{K[[G]]}(K[[G]], W_{\mathbf{k}}) \simeq \mathrm{Hom}_{K[[G]]}(\mathrm{End}_K(W_{\mathbf{k}}), W_{\mathbf{k}}).$$

Therefore

$$\dim_K \mathrm{Hom}_{K[[G]]}(K[[G]], W_{\mathbf{k}}) = \dim_K W_{\mathbf{k}} = \prod_{i=1}^r (k_i+1).$$

This is an analogue of the classical algebraic Peter–Weyl theorem for SL_2.

Let $\mathfrak{g}_K$ be the Lie algebra of G with coefficients in K. To deduce Theorem 1.4, we pass our problem to an algebraic microlocalization of the Iwasawa algebra via a completed universal enveloping algebra $K[[G]] \to \widehat{U(\mathfrak{g}_K)}$ introduced in [AW13]. In Section 6, we further apply homological degree-shifting arguments to reduce to only treat the degree-zero case for a cyclic torsion Iwasawa module $\widetilde{M}$.

The completed enveloping algebra $\widehat{U(\mathfrak{g}_K)}$ is the p-adic completion of the usual universal enveloping algebra $U(\mathfrak{g}_K)$. Note that $W_{\mathbf{k}}$ admits actions of $\widehat{U(\mathfrak{g}_K)}$ and $U(\mathfrak{g}_K)$.

Although the Iwasawa algebra has a very small center [Ard04], $\widehat{U(\mathfrak{g}_K)}$ has a larger center containing Casimir operators $\Delta_1, \dots, \Delta_r$. If $K = \mathbb{Q}_p$ and $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}_p^r$, we may specialize Δ_i to be λ_i for $1 \leq i \leq r$. We use $\widehat{U(\mathfrak{g})}_\lambda$ to denote this specialization. The goal of Section 5 is to prove the following theorem.

THEOREM 1.5. *For any $\lambda \in \mathbb{Z}_p^r$, the following map is injective*:

$$\mathbb{Q}_p[[G]] \hookrightarrow \widehat{U(\mathfrak{g})}_\lambda. \tag{5}$$

The right-hand side of (5) as a Noetherian algebra has a smaller dimension compared to the left-hand side, a priori the kernel of it is a two-sided ideal. But it surprisingly turns out to be zero.

After we completed this paper, Konstantin Ardakov pointed out to us that Theorem 1.5 can be deduced from the main results of [AW14, Ths. 4.6 and 5.4]. Since our proof is different and the intermediate results may be of independent interest, we still include Section 5 as part of the paper.

A *generic* element $\delta \in \widehat{U(\mathfrak{g})}$ is an element such that the image of δ under the specialization $\widehat{U(\mathfrak{g})} \twoheadrightarrow \widehat{U(\mathfrak{g})}_\lambda$ is non-zero for any $\lambda \in \mathbb{Z}_p^r$. Theorem 1.5 asserts that the image of the Iwasawa algebra via the microlocalization consists of generic elements (Section 3) of the completed enveloping algebra.

THEOREM 1.6. *Let $\widehat{M}$ be a cyclic torsion module over $\widehat{U(\mathfrak{g}_K)}$ with a generator killed by a generic element. There exists a multiplicity-free polynomial $p_{\widehat{M}}$ in r variables of degree at most $r-1$ such that for any $\mathbf{k} \in \mathbb{N}^r$,*

$$\dim_K \operatorname{Hom}_{\widehat{U(\mathfrak{g}_K)}}(\widehat{M}, W_{\mathbf{k}}) \leq p_{\widehat{M}}(\mathbf{k}) \text{ for } \mathbf{k} \in \mathbb{N}^r.$$

We will prove some comparison results identifying $W_{\mathbf{k}}$-quotients of the original Iwasawa module $\widetilde{M}$ and $W_{\mathbf{k}}$-quotients of its microlocalization $\widehat{M} = \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M}$. Theorem 1.4 will be deduced from combining Theorems 1.6 and 1.5.

Finally, we prove Theorem 1.6 by estimating the growth of dimension of a Poincaré–Birkhoff–Witt filtration on $\widehat{M}$. There is a natural integral model

$\widehat{U(\mathfrak{g}_0)}$ of $\widehat{U(\mathfrak{g}_K)}$. It is important for us to consider the image of $\widehat{U(\mathfrak{g}_0)}$ in $\mathrm{End}(W_{\mathbf{k}})$. The PBW filtration is linked to multiplicities by Proposition 3.4.

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2. **Results on algebra**

The rings of interest will always be Noetherian.

Definition 2.1. Let A be an integral domain (not necessarily commutative), and let M be a finitely generated A-module. The field of fractions $\mathcal{L}$ of A is a division ring that is flat over A on both sides. We use $\dim_{\mathcal{L}} \mathcal{L} \otimes_A M$ to denote the *rank* of M.

Let Λ be a partially ordered abelian group. A Λ-filtration $F_\bullet A$ on a ring A is a set $\{F_\lambda A | \lambda \in \Lambda\}$ of additive subgroups of A such that

- $1 \in F_0 A$;
- $F_\lambda A \subset F_\mu A$ whenever $\lambda \leq \mu$;
- $F_\lambda A \cdot F_\mu A \subset F_{\lambda+\mu} A$ for all $\lambda, \mu \in \Lambda$.

The filtration on A is said to be *separated* if $\bigcap_{\lambda\in\Lambda} F_\lambda A = \{0\}$, and it is said to be *exhaustive* if $\bigcup_{\lambda\in\Lambda} F_\lambda A = A$.

In a similar way, given a Λ-filtered ring $F_\bullet A$ and an A-module M, a filtration of M is a set $\{F_\lambda M | \lambda \in \Lambda\}$ of additive subgroups of M such that

- $F_\lambda M \subset F_\mu M$ whenever $\lambda \leq \mu$;
- $F_\lambda A \cdot F_\mu M \subset F_{\lambda+\mu} M$ for all $\lambda, \mu \in \Lambda$.

Again, the filtration on M is said to be *separated* if $\bigcap_{\lambda\in\Lambda} F_\lambda M = \{0\}$, and it is said to be *exhaustive* if $\bigcup_{\lambda\in\Lambda} F_\lambda M = M$.

If $\Lambda \subset \mathbb{R}$, we can define graded rings and modules for the Λ-filtration. Let A be a Λ-filtered ring. For any $\lambda \in \Lambda$, we put

$$F_{\lambda-} A := \bigcup_{s<\lambda} F_s A.$$

The *associated graded ring* is defined to be

$$\mathrm{gr}(A) := \bigoplus_{\lambda\in\Lambda} F_\lambda A / F_{\lambda-} A.$$

Given a filtered $F_\bullet A$ module $F_\bullet M$, we similarly define *associated graded module* ($F_{\lambda-} M$ is similarly defined):

$$\mathrm{gr}(M) := \bigoplus_{\lambda\in\Lambda} F_\lambda M / F_{\lambda-} M.$$

Here $\mathrm{gr}(M)$ is a natural $\mathrm{gr}(A)$-module. For any $m \in F_\lambda M \backslash F_{\lambda-} M$, we use

$$\mathrm{gr}(m) \in \frac{F_\lambda M}{F_{\lambda-} M} \subset \mathrm{gr}(M)$$

to denote the corresponding principal symbol.

LEMMA 2.2. *Let A be a Λ-filtered ring with $\Lambda \subset \mathbb{R}$. Suppose that for any non-zero $x \in A$, there exists $\lambda_x \in \Lambda$ such that $x \in F_{\lambda_x} A \backslash F_{\lambda_x -} A$. If $\mathfrak{a} \in F_a A \backslash F_{a-} A$ for $a \in \Lambda$, and $\mathrm{gr}(\mathfrak{a})$ is a non-zero divisor of $\mathrm{gr}(A)$, we have*

$$\mathrm{gr}(A/A \cdot \mathfrak{a}) \simeq \mathrm{gr}(A)/\mathrm{gr}(A) \cdot \mathrm{gr}(\mathfrak{a}),$$

where the filtration on $A/A \cdot \mathfrak{a}$ is induced from the filtration on A.

Proof. By the assumptions, $\mathfrak{a} \in A$ is a non-zero divisor. For any $\lambda \in \Lambda$, we have

$$A \cdot \mathfrak{a} \cap F_\lambda A = F_{\lambda - a} A \cdot \mathfrak{a}, \;\; A \cdot \mathfrak{a} \cap F_{\lambda-} A = F_{(\lambda-a)-} A \cdot \mathfrak{a}.$$

By definition of the induced filtration,

$$F_\lambda(A/A \cdot \mathfrak{a}) = F_\lambda A / F_{\lambda-a} A \cdot \mathfrak{a}.$$

For saving notation, let

$$\mathrm{gr}_\lambda A := F_\lambda A / F_{\lambda-} A \text{ and } \mathrm{gr}_\lambda(A/A \cdot \mathfrak{a}) := F_\lambda(A/A \cdot \mathfrak{a}) / F_{\lambda-}(A/A \cdot \mathfrak{a}).$$

We have the following commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_{\lambda-}(A/A \cdot \mathfrak{a}) & \longrightarrow & F_\lambda(A/A \cdot \mathfrak{a}) & \longrightarrow & \mathrm{gr}_\lambda(A/A \cdot \mathfrak{a}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_{\lambda-} A & \longrightarrow & F_\lambda A & \longrightarrow & \mathrm{gr}_\lambda A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_{(\lambda-a)-} A \cdot \mathfrak{a} & \longrightarrow & F_{(\lambda-a)} A \cdot \mathfrak{a} & \longrightarrow & \mathrm{gr}_{\lambda-a} A \cdot \mathrm{gr}(a) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

The claim follows from the last vertical short exact sequence applying the snake lemma. ☐

Lemma 2.3. *Let A be a Noetherian ring with a two-sided ideal I such that A/I is p-torsion free. The p-adic completion $\hat{A}$ of A has a two-sided ideal*

$$\hat{I} := \varprojlim_a \left(\frac{(p^a)+I}{(p^a)} \right) = \hat{A} \cdot I = I \cdot \hat{A} \subset \hat{A},$$

and $\hat{A}/\hat{I}$ is isomorphic to the p-adic completion of A/I.

Proof. We have to prove $\varprojlim_a \left(\frac{(p^a)+I}{(p^a)}\right) \subset \hat{A}\cdot I$; the other inclusion is clear.

Suppose I is generated by $\{m_1, \ldots, m_l\} \subset I$. Choose any compatible system

$$(\overline{m^{(a)}}) \in \varprojlim_a \left(\frac{(p^a)+I}{(p^a)} \right),\ m^{(a)} \in I.$$

We use induction to choose coefficients of m_i in the expression of $\overline{m^{(a)}}$.

For $a = 1$, we pick coefficients $\{x_i^{(1)} \in A\}$ such that $\overline{m^{(1)}} = \sum_{i=1}^{l} \overline{x_i^{(1)}} \bar{m}_i$. For a given $a \in \mathbb{N}$, suppose we have constructed $\{x_i^{(a)} \in A\}$, and express

$$\overline{m^{(a)}} = \sum_{i=1}^{l} \overline{x_i^{(a)}} \bar{m}_i \in \frac{(p^a)+I}{(p^a)}.$$

We want to inductively construct $\{x_i^{(a+1)} \in A\}$. We lift $\overline{m_i^{(a+1)}}$ to $\tilde{m} \in I$. Since A/I is p-torsion free, there exists $\{d_i^{(a+1)} | 1 \le i \le l\}$ such that

$$\tilde{m} - \left(\sum_{i=1}^{l} x_i^{(a)} m_i \right) = p^a \sum_{i=1}^{l} d_i^{(a+1)} m_i.$$

Let $\overline{x_i^{(a+1)}} := \overline{x_i^{(a)}} + p^a \overline{d_i^{(a+1)}} \bmod p^{a+1}$ for $1 \le i \le l$. The lift $\overline{m_i^{(a+1)}}$ can be expressed as

$$\overline{m_i^{(a+1)}} = \sum_{i=1}^{l} \overline{x_i^{(a+1)}} \bar{m}_i \in \frac{(p^{a+1})+I}{(p^{a+1})}.$$

Therefore for each $1 \le i \le l$, $(\overline{x_i^{(a)}})$ defines an element in $\hat{A}$, $\hat{I} \subset \hat{A}\cdot I$. Similarly, $\hat{I} = I \cdot \hat{A}$.

Consider the short exact sequence of inverse systems:

$$0 \to \left(\frac{(p^a)+I}{(p^a)} \right) \to \left(\frac{A}{(p^a)} \right) \to \left(\frac{A}{(p^a)+I} \right) \to 0.$$

The system $\left(\frac{(p^a)+I}{(p^a)}\right)$ satisfies the Mittag-Leffler condition, and the inverse limits give a short exact sequence by [Sta, 02MY Lemma 12.31.3]. Therefore $\hat{A}/\hat{I}$ is isomorphic to the p-adic completion of A/I. □

3. Growth of algebraic quotients for (completed) enveloping algebras

Let K be a field with a subring R such that 2 is invertible in R. Let $\mathfrak{g}_0$ be a direct sum of R-Lie algebras with K-valued extension $\mathfrak{g}$:

$$\mathfrak{g}_0 := \bigoplus_{i=1}^{r} \mathfrak{sl}_{2,R}, \ \mathfrak{g} := \bigoplus_{i=1}^{r} \mathfrak{sl}_{2,K} = K \otimes_R \mathfrak{g}_0.$$

The universal enveloping algebra $U(\mathfrak{g}_0) = \bigotimes_{i=1}^{r} U(\mathfrak{sl}_{2,R})$ (and similarly for $U(\mathfrak{g})_{/K}$) is a multi-filtered R-algebra with index group $\Lambda = \mathbb{Z}^r$. To be precise, let h_i, e_i, f_i be the basis of $\mathfrak{sl}_{2,R}$ with the relations

$$[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad [e_i, f_i] = h_i. \tag{6}$$

Letting

$$\Delta_i := \frac{1}{2}h_i^2 + e_i f_i + f_i e_i = \frac{1}{2}h_i^2 - h_i + 2e_i f_i$$

be the Casimir operator for i-th component, we are interested in the polynomial ring $K[\Delta_1, \ldots, \Delta_r] \subset Z(U(\mathfrak{g}))$.

For $\lambda = (\lambda_1, \ldots, \lambda_r) \in R^r$, we define

$$\mathrm{Ann}_Z(\lambda)^\circ := \sum_{i=1}^{r} Z(U(\mathfrak{g}_0)) \cdot (\Delta_i - \lambda_i), \ \mathrm{Ann}_Z(\lambda) := K \cdot \mathrm{Ann}_Z(\lambda)^\circ$$

as ideals of $Z(U(\mathfrak{g}_0))$ and $Z(U(\mathfrak{g}))$, giving rise to the extension ideal and quotient ring of $U(\mathfrak{g}_0)$ and $U(\mathfrak{g})$:

$$\begin{aligned} U_Z(\lambda)^\circ &:= U(\mathfrak{g}_0) \cdot \mathrm{Ann}_Z(\lambda)^\circ, \ U_Z(\lambda) := U(\mathfrak{g}) \cdot \mathrm{Ann}_Z(\lambda), \\ U_\lambda^\circ &:= U(\mathfrak{g}_0)/U_Z(\lambda)^\circ, \ U_\lambda := U(\mathfrak{g})/U_Z(\lambda). \end{aligned} \tag{7}$$

Let p be an odd prime. If K is a finite extension of $\mathbb{Q}_p$ with ring of integers R, we define

$$\widehat{U(\mathfrak{g}_0)} := \varprojlim_a \left(\frac{U(\mathfrak{g}_0)}{p^a U(\mathfrak{g}_0)} \right), \ \widehat{U(\mathfrak{g})} := K \otimes_R \widehat{U(\mathfrak{g}_0)}.$$

Similarly for $\widehat{U(\mathfrak{g}_0)}$ and $\widehat{U(\mathfrak{g})}$, we define $\widehat{U}_\lambda^\circ$, $\widehat{U}_\lambda$ to be the quotient rings by nullifying the relations $\{(\Delta_i - \lambda_i) | 1 \leq i \leq r\}$.

For $\delta \in \widehat{U(\mathfrak{g})}$, we say that δ is *generic* if the image of δ via $\widehat{U(\mathfrak{g})} \twoheadrightarrow \widehat{U}_\lambda$ is non-zero for all $\lambda \in \mathbb{Z}_p^r$. An equivalent torsion condition is given by compactness of $\mathbb{Z}_p$ via the following lemma.

LEMMA 3.1. *If $\delta \in \widehat{U(\mathfrak{g}_0)}$ is generic, there exists a natural number $n_\delta \geq 1$ such that the image of δ via $\widehat{U(\mathfrak{g}_0)} \twoheadrightarrow \widehat{U}_\lambda^\circ / p^{n_\delta}$ is non-zero for all $\lambda \in \mathbb{Z}_p^r$.*

Proof. We define a function $f_\delta : \mathbb{Z}_p^r \to \mathbb{Z}_{\geq 1}$: for each $\lambda \in \mathbb{Z}_p^r$, $f_\delta(\lambda)$ is defined to be the minimal positive integer such that the image of δ via $\widehat{U(\mathfrak{g}_0)} \twoheadrightarrow \widehat{U}_\lambda^\circ / p^{f_\delta(\lambda)}$ is non-zero. We have

$$\sum_{i=1}^{r} \langle \Delta_i - \lambda_i \rangle + \langle p^{f_\delta(\lambda)} \rangle = \sum_{i=1}^{r} \langle \Delta_i - (\lambda_i + p^{f_\delta(\lambda)} x_i) \rangle + \langle p^{f_\delta(\lambda)} \rangle \; \forall x_i \in \mathbb{Z}_p,$$

therefore

$$f_\delta(\lambda + x) \leq f_\delta(\lambda) \; \forall x \in p^{f_\delta(\lambda)} \mathbb{Z}_p^r.$$

Since $\mathbb{Z}_p^r$ is compact and f_δ is upper-semicontinuous, f_δ is bounded above. □

For $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, we use $W_\mathbf{k}^\circ$ (resp. $W_\mathbf{k}$) to denote the $U(\mathfrak{g}_0)$-module $\boxtimes_{i=1}^r \mathrm{Sym}^{k_i} R^2$ (resp. $U(\mathfrak{g})$-module $\boxtimes_{i=1}^r \mathrm{Sym}^{k_i} K^2$). We use $U_\mathbf{k}$, $\widehat{U}_\mathbf{k}^\circ$, $\widehat{U}_\mathbf{k}$ for U_λ, $\widehat{U}_\lambda^\circ$, $\widehat{U}_\lambda$ when $\lambda = (\frac{1}{2}k_1(k_1+2), \ldots, \frac{1}{2}k_r(k_r+2))$.

The goal of this section is to prove the following statement.

THEOREM 3.2. *If $\widehat{M}$ is cyclic, torsion and a given generator is annihilated by a generic element $\delta \in \widehat{U(\mathfrak{g})}$, there exists a multiplicity-free polynomial $p_{\widehat{M}}$ in r variables $k_1, \ldots, k_r$ of degree at most $r-1$ associated to $\widehat{M}$ such that for all $\mathbf{k} \in \mathbb{N}^r$, we have*

$$H^0_{\widehat{M}}(\mathbf{k}) \leq p_{\widehat{M}}(\mathbf{k}).$$

To prove such a result, we will prove that the image of $\widehat{U(\mathfrak{g}_0)} \cdot \delta$ in $\mathrm{End}_{\mathbb{Q}_p}(W_\mathbf{k})$ modulo p^{n_δ} (here n_δ is a positive integer given by Lemma 3.1) requires suitably many generators over $\mathbb{Z}_p$. This motivates us to estimate the number of generators using the filtration on $U(\mathfrak{g}_0)$, which we illustrate below. Since the natural $\widehat{U(\mathfrak{g}_0)}$ action on $W_\mathbf{k}$ factors through $\widehat{U}_\mathbf{k}^\circ$, our observations that $\widehat{U}_\mathbf{k}^\circ/p$ is an integral domain in Lemma 3.3 and this image is generated by the first $\mathbf{k}$-th filtered piece of $U(\mathfrak{g}_0)$ (Proposition 3.4, Remark 3.5) will achieve the desired estimate.

To prevent any confusion, we emphasize that our argument is mostly integral. But we also include corresponding rational statements for completeness.

Going back to the Lie algebra $\mathfrak{g}_0$, for each $\mathfrak{sl}_{2,R}$-component, by the PBW theorem there is a $\mathbb{Z}$-filtration with Fil_l generated by polynomials of $\{h_i, e_i, f_i\}$ up to degree l. The $\mathbb{Z}^r$ filtration is supported on $\mathbb{N}^r \subset \mathbb{Z}^r$, so sometimes we write $\mathbb{N}^r$ instead of $\mathbb{Z}^r$.

We equip $U(\mathfrak{g}_0)$ with the product filtration indexed by Λ. The abelian group $\Lambda = \mathbb{Z}^r$ comes with the partial order

$$\lambda \leq \mu \text{ if } \lambda_1 \leq \mu_1, \ldots, \lambda_r \leq \mu_r.$$

We write

$$0 \ll \lambda \text{ if } 0 \ll \min_{1 \leq i \leq r} \lambda_i, \text{ and } \lambda \to \infty \text{ if } \min_{1 \leq i \leq r} \lambda_i \to \infty. \tag{8}$$

We have $F_\mu U(\mathfrak{g}_0) = 0$ if $\mu \notin \mathbb{N}^r \subset \mathbb{Z}^r$, and we have $\mathrm{rank}_R\ F_\lambda U(\mathfrak{g}_0) < \infty$ for all $\lambda \in \Lambda$. There exists a polynomial $p_{U(\mathfrak{g}_0)}$ in r variables such that

$$\mathrm{rank}_R\ F_\lambda U(\mathfrak{g}_0) = p_{U(\mathfrak{g}_0)}(\lambda) \text{ for } 0 \ll \lambda \in \mathbb{N}^r.$$

But to form a graded ring, we use the $\mathbb{Z}$-filtration, and its associated graded ring is

$$S_R := \mathrm{gr}(U(\mathfrak{g}_0)) \simeq R[h_1, e_1, f_1, \ldots, h_r, e_r, f_r].$$

Let M be a finitely generated $U(\mathfrak{g})$-module (resp. $U(\mathfrak{g}_0)$-module), with a set of generators $\{m_1, \ldots, m_l\}$. We define two filtrations on M valued in $\Lambda = \mathbb{Z}^r$ or $\Lambda = \mathbb{Z}$, both given by the formula (and similarly for $U(\mathfrak{g}_0)$-modules)

$$F_\lambda M := \sum_{i=1}^{l} F_\lambda U(\mathfrak{g}) \cdot m_i,\ \ \lambda \in \Lambda.$$

If $\Lambda = \mathbb{Z}$, we have the associated graded module $\mathrm{gr}(M)$ over $S = \mathrm{gr}(U(\mathfrak{g}))$. Let I_Z be the ideal of S generated by $\{\frac{1}{2}h_i^2 + 2e_i f_i \mid 1 \le i \le r\}$.

We prove some useful properties of the rings U_λ,$\widehat{U}_\lambda^\circ$ and $\widehat{U}_\lambda$:

LEMMA 3.3. *For $\lambda \in \mathbb{Z}_p^r$, $\mathfrak{r} \in R$, the following hold*:

- *Let h, e, f be the basis of $\mathfrak{sl}_{2,R}$ with $\Delta = \frac{1}{2}h^2 - h + 2ef$ and the same commutation relations as (6). Then $\{e^a f^b h^c | a, b \in \mathbb{N}, c \in \{0, 1\}\}$ is a basis of $U(\mathfrak{sl}_{2,R})/(\Delta - \mathfrak{r})$.*
- *$F_\bullet U(\mathfrak{sl}_{2,R})$ induces a $\mathbb{Z}$-filtration on $U(\mathfrak{sl}_{2,R})/(\Delta - \mathfrak{r})$. Therefore both $\mathbb{Z}^r$ and $\mathbb{Z}$ filtrations induce corresponding filtrations on U_λ° and U_λ, and for the $\mathbb{Z}^r$ filtration,*

$$\mathrm{rank}_R\ F_d U_\lambda^\circ = \dim_K F_d U_\lambda = \prod_{i=1}^{r} (d_i + 1)^2$$

  is a polynomial in $d = (d_1, \ldots, d_r) \in \mathbb{N}^r$.
- *The Noetherian ring U_λ is an integral domain.*

Moreover, if K is a finite extension of $\mathbb{Q}_p$ for $p \ge 3$,

- *the ring $\widehat{U}_\lambda^\circ$ is isomorphic to the p-adic completion of U_λ°, and*

$$K \otimes_R U_\lambda^\circ \simeq U_\lambda,\ \ K \otimes_R \widehat{U}_\lambda^\circ \simeq \widehat{U}_\lambda;$$

- *the rings $\widehat{U}_\lambda^\circ$, $\widehat{U}_\lambda$ are also Noetherian integral domains.*

Proof. The first part follows from the commutation relations and an induction on the total degree.

The second part follows from the first part.

The $\overline{K}$-algebra $\overline{K}[h, e, f]/(\frac{1}{2}h^2 + 2ef)$ is an integral domain over the algebraic closure $\overline{K}$ of K. Furthermore, so is the tensor product of r-copies of $\overline{K}[h, e, f]/(\frac{1}{2}h^2 + 2ef)$. Hence S_K/I_Z is also an integral domain. For the last part, consider the $\mathbb{Z}$-filtration on U_λ. Its graded ring $\mathrm{gr}(U_\lambda)$ is isomorphic to the integral domain S_K/I_Z by Lemma 2.2. For any non-zero $a, b \in U_\lambda$, let i_a, i_b

be the minimal natural numbers such that $a \in F_{i_a}U_\lambda, b \in F_{i_b}U_\lambda$. The image of ab in

$$F_{i_a+i_b}U_\lambda/F_{i_a+i_b-1}U_\lambda$$

is non-zero, and so is $ab \in U_\lambda$.

The fourth part follows from Lemma 2.3 and the flat base change $R \hookrightarrow K$.

For the last part, we may assume $K = \mathbb{Q}_p$. We have that $\widehat{U}^\circ_\lambda/p \simeq U^\circ_\lambda/p$ is isomorphic to U_λ over $\mathbb{F}_p$ (defined in (7) for $K = \mathbb{F}_p$). The claim then follows from the third part for $\mathbb{F}_p$. ☐

For the rest of the section, K is a finite extension of $\mathbb{Q}_p$ with the ring of integers R. Let $\mathrm{Ann}(\mathbf{k})$ (resp. $\widehat{\mathrm{Ann}(\mathbf{k})}$) be the annihilator ideal of $U(\mathfrak{g})$ (resp. $\widehat{U(\mathfrak{g})}$) for $W_{\mathbf{k}}$. As will be discussed in Section 4, there is an algebraic isomorphism (see (19))

$$U(\mathfrak{g})/\mathrm{Ann}(\mathbf{k}) \xrightarrow{\sim} \widehat{U(\mathfrak{g})}/\widehat{\mathrm{Ann}(\mathbf{k})} \xrightarrow{\sim} \mathrm{End}_K(W_{\mathbf{k}}).$$

If $\widehat{M}$ is a finitely generated $\widehat{U(\mathfrak{g})}$-module, we define

$$\widehat{M}_{\mathbf{k}} := \widehat{M}/\widehat{\mathrm{Ann}(\mathbf{k})} \cdot \widehat{M}$$

and the $W_{\mathbf{k}}$-*multiplicity* $H^0_{\widehat{M}}(\mathbf{k})$ to be

$$H^0_{\widehat{M}}(\mathbf{k}) := \dim_K \mathrm{Hom}_{\widehat{U(\mathfrak{g})}}(\widehat{M}, W_{\mathbf{k}}) = \frac{\dim_K \widehat{M}_{\mathbf{k}}}{\dim_K W_{\mathbf{k}}}. \tag{9}$$

Note that the natural $U(\mathfrak{g})$, $\widehat{U(\mathfrak{g})}$ actions on $W_{\mathbf{k}}$ factor through $U_{\mathbf{k}}$, $\widehat{U}_{\mathbf{k}}$.

PROPOSITION 3.4. *For any $\mathbf{k} \in \mathbb{N}^r$, as R-modules, the image of $F_{\mathbf{k}}U^\circ_{\mathbf{k}}$ equals to the full image of $U^\circ_{\mathbf{k}}$ in $\mathrm{End}_R(W^\circ_{\mathbf{k}})$. As a corollary, there are isomorphisms of K-vector spaces*:

$$F_{\mathbf{k}}U_{\mathbf{k}} \xrightarrow{\sim} U(\mathfrak{g})/\mathrm{Ann}(\mathbf{k}) \xrightarrow{\sim} \mathrm{End}_K(W_{\mathbf{k}}).$$

Proof. It suffices to prove the case $r = 1$, $\mathbf{k} = (k)$. We first observe that for any $a \geq 0$, $e^a f^a$ can be expressed as an R-linear combination of h^i as an operator in $U^\circ_{\mathbf{k}}$. For example, let $\lambda_k = \frac{1}{4}k(k+2)$. Then we have

$$\begin{aligned} e^2f^2 &= e\left(\lambda_k + \frac{1}{2}h - \frac{1}{4}h^2\right)f \\ &= \lambda_k \cdot ef + \frac{1}{2}(he-2e)f - \frac{1}{4}(he-2e)(fh-2f) \\ &= (\lambda_k + h - 2)ef + \frac{1}{2}efh - \frac{1}{4}hefh \\ &= \left(\lambda_k - 2 + \frac{3}{2}h - \frac{1}{4}h^2\right)\left(\lambda_k + \frac{1}{2}h - \frac{1}{4}h^2\right). \end{aligned}$$

Let $v_0, \ldots, v_k$ be the weight vectors of W°_k of increasing weights, with $W^\circ_{\mathbf{k}} = \oplus_{i=0}^k Rv_i$, f raising a weight and e lowering a weight. Let $W_{\geq i} := \sum_{j\geq i}^k Rv_j$. Note that we want to prove that each monomial $e^af^bh^c$ (c is either zero or one by the first part of Lemma 3.3) with total degree strictly larger than k can be reduced to an R-linear combination of lower degree terms. For explanation, we only show the reduction for the monomial $e^a f^b$, where $a+b>k$, $a \le b$, the monomial $e^a f^b$ is equal to $e^a f^a f^{b-a}$. (Other cases can be similarly obtained.) The operator $e^a f^a$ can be viewed as an endomorphism of $W_{\ge b-a}$ since f^{b-a} maps W to $W_{\ge b-a}$. We know it is an R-linear combination of powers of h. By Cayley-Hamilton,

$$e^a f^a = \sum_{i=0}^{k-b+a} c_i h^i \in \mathrm{End}_R(W_{\ge b-a}),\ c_i \in R.$$

As $k-b+a < 2a$, the degree of $e^a f^b$ is reduced. We can similarly argue the other cases.

The corollary for K-isomorphisms follows from counting the dimensions of both sides by Lemma 3.3. ☐

Remark 3.5. Let $E^\circ_{\mathbf{k}}$ be the image of $U^\circ_{\mathbf{k}}$ in $\mathrm{End}_R(W^\circ_{\mathbf{k}})$. As it is p-adic complete, it coincides with the image of $\widehat{U(\mathfrak{g}_0)}$ as well. Moreover, we have

$$F_{\mathbf{k}}(U^\circ_{\mathbf{k}}/p^m) \simeq E^\circ_{\mathbf{k}}/p^m$$

for all $m \ge 0$.

Proof of Theorem 3.2. Let $\widehat{M_0}$ be a cyclic $\widehat{U(\mathfrak{g}_0)}$-lattice inside $\widehat{M}$ such that $\widehat{M_0} \otimes_R K \xrightarrow{\sim} \widehat{M}$. We may assume $\delta \in \widehat{U(\mathfrak{g}_0)}$. The surjection $\widehat{U(\mathfrak{g}_0)} \twoheadrightarrow \widehat{M_0}$ corresponding to the generator factors through $\widehat{U(\mathfrak{g}_0)}/\widehat{U(\mathfrak{g}_0)}\delta$.

We pick $n_\delta \ge 1$ satisfying Lemma 3.1. Under the natural identification $U(\mathfrak{g}_0)/p^{n_\delta} \simeq \widehat{U(\mathfrak{g}_0)}/p^{n_\delta}$, there exists $\alpha \in \mathbb{N}^r$ such that $\delta \in F_\alpha(\widehat{U(\mathfrak{g}_0)}/p^{n_\delta}) \simeq F_\alpha(U(\mathfrak{g}_0)/p^{n_\delta})$.

If $\mathbf{k} \ge \alpha$, then

$$F_{\mathbf{k}-\alpha}(\widehat{U}^\circ_{\mathbf{k}}/p^{n_\delta}) \cdot \delta \subset F_{\mathbf{k}}(\widehat{U}^\circ_{\mathbf{k}}/p^{n_\delta}).$$

Let ϖ be a uniformizer of R. For any $\mathbf{k} \in \mathbb{N}^r$, and $e \in U^\circ_{\mathbf{k}}$ such that $e \notin \varpi U^\circ_{\mathbf{k}}$, we have $e \cdot \delta \ne 0$ in $U^\circ_{\mathbf{k}}/p^{n_\delta}$ since $U^\circ_{\mathbf{k}}/\varpi$ is an integral domain by Lemma 3.3. The composition of maps of vector spaces

$$F_{\mathbf{k}-\alpha}U_{\mathbf{k}} \to F_{\mathbf{k}-\alpha}U_{\mathbf{k}} \cdot \delta \to \mathrm{End}_K(W_{\mathbf{k}})$$

is injective; otherwise, there exists $e \in F_{\mathbf{k}-\alpha}U^\circ_{\mathbf{k}} \backslash \varpi U^\circ_{\mathbf{k}}$ such that $e \cdot \delta$ maps to

$$0 \in E^\circ_{\mathbf{k}} \hookrightarrow \mathrm{End}_K(W_{\mathbf{k}}),$$

contradicting Remark 3.5 of Proposition 3.4 since $e \cdot \delta \in F_{\mathbf{k}}(U^\circ_{\mathbf{k}}/p^{n_\delta})$ is non-zero modulo p^{n_δ}.

Therefore the image of $\widehat{U(\mathfrak{g})}\delta$ in $\widehat{U(\mathfrak{g})}_{\mathbf{k}} \simeq \mathrm{End}_K(W_{\mathbf{k}})$ has dimension at least $\dim_K F_{\mathbf{k}-\alpha} U_{\mathbf{k}}$, and we have the following bound for $(\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\delta)_{\mathbf{k}}$:

$$H^0_{\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\delta}(\mathbf{k}) = \frac{\dim_K(\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\delta)_{\mathbf{k}}}{\dim_K W_{\mathbf{k}}} \leq \frac{\prod_{i=1}^r (k_i+1)^2 - \prod_{i=1}^r (k_i - \alpha_i + 1)^2}{\prod_{i=1}^r (k_i+1)}.$$

Regarding each (k_i+1) as a variable, each term $c_{S,S'} \cdot \frac{\prod_{i\in S}(k_i+1)}{\prod_{i\in S'}(k_i+1)}$ is bounded by $|c_{S,S'}| \cdot \prod_{i\in S}(k_i+1)$ for $S \sqcup S' \subset \{1,\dots,r\}$. We get the desired bound for $\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\delta$. □

Remark 3.6. If M is a finitely generated module over $U(\mathfrak{g})$, in general, the rank of M does not have to agree with the rank of $U_{\mathbf{k}} \otimes_{U(\mathfrak{g})} M$ over $U_{\mathbf{k}}$ without the genericity condition. If $r=2, \mathfrak{g} = \mathfrak{sl}_{2,K} \oplus \mathfrak{sl}_{2,K}$, and Δ_1, Δ_2 are Casimir operators for the two components, the algebraic representations $W_{k,k}$ of parallel weights grow in quadratic order in the cyclic torsion $U(\mathfrak{g})$-module $U(\mathfrak{g})/(\Delta_1 - \Delta_2)$.

These Casimir operators do not exist in $K[[G]]$, and we will see elements in $\widehat{U(\mathfrak{g})}$ obtained from base change over the microlocalization (18) are generic.

4. Comparison of algebraic quotients

Let K be a finite extension of $\mathbb{Q}_p$ with the ring of integers R, residue field k. Let G be a uniform pro-p group of dimension $d = \dim G$. We define the completed group rings by

$$R[[G]] := \varprojlim R[G/N],\ K[[G]] := K \otimes_R R[[G]],$$

where N runs over all the open normal subgroups N of G.

Lazard [Laz65] defines a $\mathbb{Z}_p$-Lie algebra L_G associated to G (see also [DdSMS99, §4.5]). We briefly recall some basic facts about L_G here. We fix a minimal topological generating set $\{g_1,\dots,g_d\}$ of G. Each element of G can be written uniquely in the form $g_1^{\lambda_1}\cdots g_d^{\lambda_d}$ for some $\lambda_1,\dots,\lambda_d \in \mathbb{Z}_p$. By [DdSMS99, Th. 4.30], the operations

$$\lambda \cdot x = x^{\lambda}, \tag{10}$$

$$x + y = \lim_{i\to\infty}\left(x^{p^i} y^{p^i}\right)^{p^{-i}}, \tag{11}$$

$$[x,y] = \lim_{i\to\infty}\left(x^{-p^i} y^{-p^i} x^{p^i} y^{p^i}\right)^{p^{-2i}} \tag{12}$$

define a Lie algebra structure L_G on G over $\mathbb{Z}_p$. Note that L_G is a *powerful Lie algebra* in the sense that it is free of rank $d = \dim G$ over $\mathbb{Z}_p$ and satisfies $[L_G, L_G] \leq pL_G$. Letting

$$\mathfrak{g}_R = \frac{1}{p} L_G \otimes_{\mathbb{Z}_p} R,\ \mathfrak{g}_K = \mathfrak{g}_R \otimes_R K,$$

the *completed universal enveloping algebras* $\widehat{U(\mathfrak{g}_R)}$, $\widehat{U(\mathfrak{g}_K)}$ are defined to be

$$\widehat{U(\mathfrak{g}_R)} := \varprojlim_a \left(\frac{U(\mathfrak{g}_R)}{p^a U(\mathfrak{g}_R)} \right), \ \widehat{U(\mathfrak{g}_K)} := \widehat{U(\mathfrak{g}_R)} \otimes_R K,$$

following [ST03] (appearing as the "largest" distribution algebra $D_{1/p}(G, K)$) and [AW13].

LEMMA 4.1. *If G is a compact open uniform pro-p subgroup of* $\mathrm{SL}_n(\mathbb{Q}_p)$, *then the associated Lie algebra*

$$\mathfrak{g}_K \simeq \mathfrak{sl}_{n,K}$$

is isomorphic to the Lie algebra of $\mathfrak{sl}_n$ over K.

Proof. This is an exercise [DdSMS99, Part II, Ch. 9, Ex. 9] following from Lazard's paper [Laz65]. ☐

As G is a uniform pro-p-group, by [DdSMS99, Th. 8.18] it is compact locally $\mathbb{Q}_p$-analytic. Moreover, G satisfies the assumption (HYP) of [ST03, §4] by the remark before [ST03, Lemma 4.4]. Schneider and Teitelbaum have introduced the K-Fréchet–Stein algebra $D(G, K)$ of K-valued locally analytic distributions on G ([ST02b], [ST03]). We briefly recall some basic properties of $D(G, K)$ from [ST03] here.

Let $b_i := g_i - 1 \in R[G]$, and write

$$\mathbf{b}^\alpha = b_1^{\alpha_1} \cdots b_d^{\alpha_d} \in R[G] \tag{13}$$

for any d-tuple $\alpha \in \mathbb{N}^d$. We write $|\alpha| := \sum_{i=1}^d \alpha_i$. It follows from the proof of [DdSMS99, Th. 7.20] that $R[[G]]$ can be naturally identified with the set of non-commutative formal power series in $b_1, \ldots, b_d$ with coefficients in R:

$$R[[G]] = \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{b}^\alpha \mid \lambda_\alpha \in R \right\}.$$

There is a faithfully flat natural map from the Iwasawa algebra to the distribution algebra

$$K[[G]] \to D(G, K) \tag{14}$$

by [ST03, Th. 4.11], such that $D(G, K)$ can be identified with power series in $b_1, \ldots, b_d$ with convergence conditions

$$D(G, K) = \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{b}^\alpha \mid \lambda_\alpha \in K, \text{ and for } \forall\ 0 < r < 1, \ \sup_{\alpha \in \mathbb{N}^d} |\lambda_\alpha| r^{|\alpha|} < \infty \right\}.$$

For G, there is an integrally valued *p-valuation* $\omega : G\backslash\{1\} \to \mathbb{Z}_{\geq 1}$ such that

$$\omega(gh^{-1}) \geq \min(\omega(g), \omega(h)),$$
$$\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h), \text{ and}$$
$$\omega(g^p) = \omega(g) + 1$$

for any $g, h \in G$, with $\omega(1) := \infty$ [Laz65, III 2.1.2].

For G being uniform, we may define $\omega(g)$ to be $n \geq 1$ such that $g \in G^{p^{n-1}} \backslash G^{p^n}$. It is indeed an integrally valued p-valuation on G by, for example, [AW13, Lemma 10.2]. By the discussion in [DdSMS99, §4.2], ω is characterized by

$$\omega(g_i) = 1 \text{ for } 1 \leq i \leq d, \text{ and} \tag{15}$$
$$\omega(g) = 1 + \min_{1 \leq i \leq d} \omega_p(x_i) \; \forall g = g_1^{x_1} \cdots g_d^{x_d} \in G,$$

where ω_p denotes the p-adic valuation on $\mathbb{Z}_p$.

The Fréchet topology of $D(G, K)$ is defined by the family of norms

$$\|\lambda\|_r := \sup_{\alpha \in \mathbb{N}^d} |\lambda_\alpha| \, r^{|\alpha|}$$

for $0 < r < 1$, where the absolute value $|\cdot|$ is normalized as usual by $|p| = p^{-1}$. We let

$$D_r(G, K) := \text{completion of } D(G, K) \text{ with respect to the norm } \| \;\|_r.$$

As a K-Banach space,

$$D_r(G, K) = \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{b}^\alpha \mid \lambda_\alpha \in K, \; \sup_{\alpha \in \mathbb{N}^d} |\lambda_\alpha| r^{|\alpha|} < \infty \right\}. \tag{16}$$

THEOREM 4.2. *If $1/p \leq r < 1$, then $D_r(G, K)$ is a Banach noetherian integral domain with multiplicative norm $\| \;\|_r$. The distribution algebra*

$$D(G, K) = \varprojlim_r D_r(G, K)$$

is a K-Fréchet–Stein algebra.

Proof. This is the main result of [ST03, §4]. □

We remark that Schneider–Teitelbaum's definition of $\| \;\|_r$ is slightly more complicated in general, but it agrees with our $\| \;\|_r$ because of (15) due to the uniform assumption of G.

Let $\mathfrak{m} := \ker(R[[G]] \to k)$ be the unique maximal ideal of $R[[G]]$. Following [AW13, §10], we consider a microlocal Ore set S_0:

$$S_0 := \bigcup_{a \geqslant 0} \left(p^a + \mathfrak{m}^{a+1}\right) \subseteq R[[G]]. \tag{17}$$

Associated to S_0, there is a flat extension (see remarks in [AW13, §1.4])

$$K[[G]] \to \widehat{U(\mathfrak{g}_K)} \tag{18}$$

by construction of [AW13, §10], called the *microlocalization* of Iwasawa algebra. For a finitely generated $K[[G]]$-module $\widetilde{M}$, we use $\widehat{M} := \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M}$ to denote the *microlocalization* of $\widetilde{M}$.

From now on we assume $\mathfrak{g}_K$ is a split, semisimple Lie algebra over K. We refer to [Bou05] for a treatment of such a Lie theory. Let W be an irreducible finite dimensional representation of $U(\mathfrak{g}_K)$ with a $U(\mathfrak{g}_R)$-lattice $W_0 \subset W$.

As W_0 is of finite rank, it is automatically p-adic complete. The $U(\mathfrak{g}_R)$ action on W_0 extends to a $\widehat{U(\mathfrak{g}_R)}$ action, and the $U(\mathfrak{g}_K)$ action on W extends to a $\widehat{U(\mathfrak{g}_K)}$ action (also see [AW13, §9.2]).

We pull back W as a $K[[G]]$-module via the microlocalization (18). Iwasawa modules arising from this way are called *Lie modules* in [AW13, §11.1]. By [AW13, Th. 11.1, Cor. 11.1], W remains irreducible as a $K[[G]]$-module.

Let

$$\mathrm{Ann}(W) \subset U(\mathfrak{g}_K),\ \widehat{\mathrm{Ann}(W)} \subset \widehat{U(\mathfrak{g}_K)},\ \widetilde{\mathrm{Ann}(W)} \subset K[[G]]$$

respectively be the annihilator ideals of $U(\mathfrak{g}_K), \widehat{U(\mathfrak{g}_K)}, K[[G]]$ for W. Similarly, we have

$$U(\mathfrak{g}_K)/\mathrm{Ann}(W) \xrightarrow{\sim} \widehat{U(\mathfrak{g}_K)}/\widehat{\mathrm{Ann}(W)} \xrightarrow{\sim} \mathrm{End}_K(W), \tag{19}$$

$$K[[G]]/\widetilde{\mathrm{Ann}(W)} \hookrightarrow \widehat{U(\mathfrak{g}_K)}/\widehat{\mathrm{Ann}(W)} \xrightarrow{\sim} \mathrm{End}_K(W). \tag{20}$$

The second map of (19) is surjective by [Bou05, Ch. VIII, §6.2, Cor. of Prop. 3].

By [AW13, Th. 11.1], every finite dimensional $K[[G]]$-module is semisimple. For a finitely generated $K[[G]]$-module $\widetilde{M}$, we use

$$\widetilde{M}_W := \widetilde{M}/\widetilde{\mathrm{Ann}(W)} \cdot \widetilde{M} \ (\text{resp. } \widehat{M}_W := \widehat{M}/\widehat{\mathrm{Ann}(W)} \cdot \widehat{M})$$

to denote the maximal quotient of $\widetilde{M}$ (resp. $\widehat{M}$) that is isomorphic to a finite sum of W as a $K[[G]]$-module (resp. $\widehat{U(\mathfrak{g}_K)}$-module).

THEOREM 4.3. *Let $\widetilde{M}$ be a finitely generated $K[[G]]$-module with microlocalization $\widehat{M} = \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M}$. Then the natural map*

$$\widetilde{M}_W \xrightarrow{\sim} \widehat{M}_W$$

is an isomorphism. In particular, $K[[G]]/\widetilde{\mathrm{Ann}(W)} \xrightarrow{\sim} \widehat{U(\mathfrak{g}_K)}/\widehat{\mathrm{Ann}(W)}$ in (20) is an isomorphism.

Proof. We apply the flat base change microlocalization $K[[G]] \to \widehat{U(\mathfrak{g}_K)}$ (18) to the short exact sequence

$$0 \to \widetilde{\mathrm{Ann}(W)} \cdot \widetilde{M} \to \widetilde{M} \to \widetilde{M}_W \to 0$$

to get

$$0 \to \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{\mathrm{Ann}(W)} \cdot \widetilde{M} \to \widehat{M} \to \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M}_W \to 0. \tag{21}$$

The microlocal Ore set S_0 (17) acts invertibly on W as S_0 consists of units in $\widehat{U(\mathfrak{g}_K)}$. (Note that S_0 is inverted to form the microlocalization in Ardakov-Wadsley's construction; also see the proof of part (b) of [AW13, Th. 11.1].) We can apply [AW13, Prop. 11.1] to W taking $n = 0$, which asserts that the natural map $\widetilde{M}_W \xrightarrow{\sim} \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M}_W$ is an isomorphism as $K[[G]]$-modules because $\widetilde{M}_W$ is isomorphic to a direct sum of W. By our assumption on W, the $K[[G]]$ action on $\widetilde{M}_W$ (uniquely) extends to $\widehat{U(\mathfrak{g}_K)}$. Moreover, it is an isomorphism over $\widehat{U(\mathfrak{g}_K)}$ due to the natural $\widehat{U(\mathfrak{g}_K)}$-equivariant reverse map $\widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M}_W \to \widetilde{M}_W$.

By maximality of $\widehat{M}_W$, the exact sequence (21) gives

$$\widehat{\mathrm{Ann}(W)} \cdot \widehat{M} \subset \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{\mathrm{Ann}(W)} \cdot \widetilde{M},$$

both as submodules of $\widehat{M}$. From (20) we get $\widehat{U(\mathfrak{g}_K)} \cdot \widetilde{\mathrm{Ann}(W)} \subset \widehat{\mathrm{Ann}(W)}$, therefore

$$\widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{\mathrm{Ann}(W)} \cdot \widetilde{M} \subset \widehat{\mathrm{Ann}(W)} \cdot \widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{M} = \widehat{\mathrm{Ann}(W)} \cdot \widehat{M}.$$

This forces $\widehat{U(\mathfrak{g}_K)} \otimes_{K[[G]]} \widetilde{\mathrm{Ann}(W)} \cdot \widetilde{M} = \widehat{\mathrm{Ann}(W)} \cdot \widehat{M}$ and thus $\widehat{M}_W = \widetilde{M}_W$. $\square$

5. Infinitesimal specialization

We continue to use notation from Section 4. For simplicity, we take K to be $\mathbb{Q}_p$. The norm $|\ |$ on $\mathbb{Q}_p$ is normalized as usual by $|p| = p^{-1}$. Let $C(G, \mathbb{Q}_p)$ and $C^{\mathrm{la}}(G, \mathbb{Q}_p)$, $C^{\mathrm{sm}}(G, \mathbb{Q}_p)$ be respectively the space of continuous functions on G, the space of locally analytic functions on G, and the space of smooth functions on G, all valued in $\mathbb{Q}_p$. The Lie algebra $\mathfrak{g}$ acts on $C^{\mathrm{la}}(G, \mathbb{Q}_p)$ by continuous endomorphisms defined by

$$\mathfrak{x} f := \lim_{t \to 0} \frac{(t \cdot \mathfrak{x}) f - f}{t} \tag{22}$$

for $\mathfrak{x} \in \mathfrak{g}_0$, $f \in C^{\mathrm{la}}(G, \mathbb{Q}_p)$, where the dot action $t \cdot \mathfrak{x}$ is given by (10). We have a natural inclusion

$$U(\mathfrak{g}) \to D(G, \mathbb{Q}_p). \tag{23}$$

We further assume that G is an open subgroup of the group of $\mathbb{Q}_p$-rational points of a connected split reductive $\mathbb{Q}_p$-group $\mathbb{G}$ with Borel pair $(\mathbb{B}, \mathbb{T})$. The

$\mathbb{Q}_p$-split Lie algebra of $\mathbb{G}$ should be identified with the Lie algebra $\mathfrak{g}$ associated to G in Section 4. We use $\mathfrak{g}_0$ to denote $\mathfrak{g}_{\mathbb{Z}_p}$.

Let $T := \mathbb{T}(\mathbb{Q}_p) \cap G$ be a torus of G, and let $B := \mathbb{B}(\mathbb{Q}_p) \cap G$ be a Borel subgroup with unipotent radical N such that $B = TN$. Let $\mathfrak{t}_0$, $\mathfrak{b}_0$, $\mathfrak{n}_0$ respectively be their associated Lie algebras over $\mathbb{Z}_p$, with generic fibres $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{n}$. Suppose $\overline{\mathfrak{n}}_0$ is an opposite nilpotent of $\mathfrak{n}$ such that $\mathfrak{g}_0$ admits a triangular decomposition

$$\mathfrak{g}_0 = \overline{\mathfrak{n}}_0 \oplus \mathfrak{t}_0 \oplus \mathfrak{n}_0.$$

Let $Z(\mathfrak{g}_0)$, $Z(\mathfrak{g})$ respectively be the centers of $U(\mathfrak{g}_0)$, $U(\mathfrak{g})$. Any character of $Z(\mathfrak{g}_0)$ is naturally extended to a character of $Z(\mathfrak{g})$. There is the Harish-Chandra homomorphism [Bou05, Ch. VIII, §6.4] associated to the triangular decomposition

$$\mathrm{HC} : Z(\mathfrak{g}) \to U(\mathfrak{t}). \tag{24}$$

If $\chi : T \to 1+p\mathbb{Z}_p$ is a continuous/locally analytic character of the torus T, it induces a character $d\chi : U(\mathfrak{t}_0) \to \mathbb{Z}_p$ by formula (22), extending to $d\chi : U(\mathfrak{t}) \to \mathbb{Q}_p$. We call an infinitesimal character $\lambda : Z(\mathfrak{g}) \to \mathbb{Q}_p$ *induced* if $\lambda = d\chi \circ \mathrm{HC}$ for a character χ of T.

Our main theorem in this section is the following.

THEOREM 5.1. *Let λ be an induced infinitesimal character. If G is a finite product of first congruence subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$, the composition of microlocalization* (18) *with infinitesimal specialization is injective*:

$$\mathbb{Q}_p[[G]] \hookrightarrow \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p.$$

The corresponding statement for distribution algebra turns out to be much easier, and it serves as a first step to prove Theorem 5.1.

For a proof, we make use of the locally analytic principal series of $\chi : T \to 1 + p\mathbb{Z}_p$:

$$\mathrm{Ind}(\chi) := \{f : G \to \mathbb{Q}_p | f \text{ locally analytic}, \ f(gtn) = \chi(t) f(g) \ \forall t \in T, n \in N, g \in G\}.$$

The locally analytic principal series has an induced infinitesimal character determined by $d\chi$ via the Harish-Chandra homomorphism (24).

THEOREM 5.2. *As $\mathbb{Q}_p$-Fréchet spaces, $D(G, \mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p$ is the strong dual of the locally convex vector space of compact type $C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda]$, where $C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda]$ is the λ-isotypic part of the space of locally analytic functions on G.*

If the infinitesimal character λ is induced, the composition of (14) *with infinitesimal specialization is injective*:

$$\mathbb{Q}_p[[G]] \hookrightarrow D(G, \mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p.$$

Proof. Let $z_1, \dots, z_n$ be a set of generators of the kernel of λ. By [ST02b, Prop. 3.7], these differential operators define G-equivariant endomorphisms of $C^{\mathrm{la}}(G, \mathbb{Q}_p)$. Therefore we have a short left exact sequence of admissible locally analytic representations of G:

$$0 \to C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda] \to C^{\mathrm{la}}(G, \mathbb{Q}_p) \to C^{\mathrm{la}}(G, \mathbb{Q}_p)^n, \tag{25}$$

$$f \mapsto (z_i \cdot f)_i. \tag{26}$$

By taking the strong dual of (25) and the anti-equivalence of categories between admissible locally analytic representations and coadmissible $D(G, \mathbb{Q}_p)$-modules [ST03, Th. 6.3], we identify $D(G, \mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p$ with $C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda]'_b$.

For the second part, as $\mathbb{Q}_p[[G]]$ is dual to $C(G, \mathbb{Q}_p)$, it suffices to prove that $C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda]$ is dense in $C(G, \mathbb{Q}_p)$.

We choose any locally analytic character $\chi : T \to K^\times$ such that $\mathrm{Ind}(\chi)$ has the infinitesimal character λ. Let X be the quotient space $X := G/B$, then there is a splitting $X \hookrightarrow G \twoheadrightarrow X$ as p-adic manifolds such that $\mathrm{Ind}(\chi) \simeq C^{\mathrm{la}}(X, \mathbb{Q}_p)$ as topological K-vector spaces. We choose any nowhere vanishing function $f_0 \in C^{\mathrm{la}}(X, \mathbb{Q}_p)$ so that $f_0 \in C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda]$. The pointwise product of f_0 with any smooth function still has the infinitesimal character λ. We see that $f_0 \cdot C^{\mathrm{sm}}(G, \mathbb{Q}_p) \subset C^{\mathrm{la}}(G, \mathbb{Q}_p)[\lambda] \subset C(G, \mathbb{Q}_p)$ is clearly dense in $C(G, \mathbb{Q}_p)$. ☐

Following [Fro03], [Koh07], we define $U_r(\mathfrak{g})$ to be the closure of $U(\mathfrak{g})$ in $D_r(G, \mathbb{Q}_p)$ with respect to the norm $\| \ \|_r$ for $0 < r < 1$. If $\lambda : Z(\mathfrak{g}) \to \mathbb{Q}_p$ is an infinitesimal character, we define

$$U_r^\lambda(\mathfrak{g}) := U_r(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p, \ D_r^\lambda(G, \mathbb{Q}_p) := D_r(G, \mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p.$$

PROPOSITION 5.3. *If $r = \sqrt[p^n]{1/p}$ for $n \in \mathbb{Z}_{\geq 1}$, then $D_r(G, \mathbb{Q}_p)$ is a crossed product* ([MR01, §1.5.8], [AW13, Proof of Prop. 10.6]) *of $U_r(\mathfrak{g})$ by G/G^{p^n}. Consequently,*

$$D_r^\lambda(G, \mathbb{Q}_p) \simeq U_r^\lambda(\mathfrak{g}) * (G/G^{p^n})$$

is a crossed product of $U_r^\lambda(\mathfrak{g})$ by G/G^{p^n}.

Proof. See [Sch13, (6.8), Cor. 5.13] for a similar statement of the first claim. We give a proof as follows. By (22), for any $g \in G, \mathfrak{x} \in \mathfrak{g}_0, f \in C^{\mathrm{la}}(G, \mathbb{Q}_p)$,

$$\begin{aligned} g(\mathfrak{x} \cdot (g^{-1} f)) &= g\left(\lim_{t \to 0} \frac{(t \cdot \mathfrak{x})(g^{-1} f) - (g^{-1} f)}{t}\right) \\ &= \lim_{t \to 0} \frac{g(t \cdot \mathfrak{x}) g^{-1} f - f}{t} \\ &= \lim_{t \to 0} \frac{(t \cdot g\mathfrak{x}g^{-1}) f - f}{t} \\ &= (g\mathfrak{x}g^{-1}) \cdot f. \end{aligned}$$

We see that $U(\mathfrak{g})$ is stable under the conjugation action of G. By Frommer's theorem [Fro03], [Koh07, Th. (Frommer), Proof of Cor. 1.4.1],

$$U_r(\mathfrak{g}) = \left\{ \sum_{\alpha} d_\alpha \mathfrak{X}^\alpha | d_\alpha \in \mathbb{Q}_p, \lim_{|\alpha|\to\infty} |d_\alpha| \, \|\mathfrak{X}^\alpha\|_r = 0 \right\} \tag{27}$$

for $\mathfrak{X}^\alpha = \log(1+b_1)^{\alpha_1} \cdots \log(1+b_d)^{\alpha_d}$ compared to (13), and $D_r(G, \mathbb{Q}_p)$ is a free (left or right) $U_r(\mathfrak{g})$-module of basis given by representatives of G/G^{p^n}. It suffices to prove that for any $g \in G$, its p^n-th power belongs to $U_r(\mathfrak{g})$, and we may further assume $g = b_i$ for certain $1 \le i \le d$. As a formal power series, we have

$$g^{p^n} = \exp(p^n \mathfrak{X}_i) = \sum_{k=0}^{\infty} \frac{p^{nk}\mathfrak{X}_i^k}{k!}$$

for $\mathfrak{X}_i := \log(1+b_i)$. Here, by Taylor series of $\log(1+x)$, we have

$$\|\mathfrak{X}_i\|_r = \max_{i\ge 1} \frac{r^i}{|i|} = \max_{k\ge 0} \frac{r^{p^k}}{|p^k|} = \max_{k\ge 0} p^{-\frac{p^k}{p^n}} \cdot p^k = \max_{k\ge 0} p^{(k-\frac{p^k}{p^n})} = p^{n-1},$$

and since $\| \ \|_r$ is multiplicative by [ST03, Th. 4.5], $\|\mathfrak{X}_i^k\|_r = p^{(n-1)k}$, we have

$$\lim_{k\to\infty} |\frac{p^{nk}}{k!}| \cdot p^{(n-1)k} \le p^{\frac{k}{p-1}} \cdot p^{-k} = 0.$$

As a consequence, $g^{p^n} \in U_r(\mathfrak{g})$ by characterization (27).

The second claim follows from the first claim since $Z(\mathfrak{g})$ is in the center of $D_r(G, \mathbb{Q}_p)$ by [ST02b, Prop. 3.7] and $Z(\mathfrak{g}) \subset U(\mathfrak{g}) \subset U_r(\mathfrak{g})$. ☐

Remark 5.4. It is pointed out in [AW13, Rem. 10.6] that $D_r(G, \mathbb{Q}_p)$ should be a crossed product of the microlocalization of $\mathbb{Q}_p[[G^{p^n}]]$ by G/G^{p^n} for $r = \sqrt[p^n]{1/p}$. It is quite likely that $U_r(\mathfrak{g})$ coincides with such a microlocalization .

For each $n \ge 0$, G^{p^n} is isomorphic to its (unnormalized) Lie algebra

$$L_{G^{p^n}} \simeq p^{n+1}\mathbb{Z}_p^d \simeq \mathbb{Z}_p^d$$

as p-adic manifolds. We define

$$\begin{aligned} C^{n,\mathrm{an}}(G, \mathbb{Q}_p) &:= \left\{ f \in C(G, \mathbb{Q}_p) \mid f \text{ is analytic on each } G^{p^n} \text{ coset} \right\}, \\ &\text{with } C^{\mathrm{la}}(G, \mathbb{Q}_p) = \varinjlim_{n\ge 0} C^{n,\mathrm{an}}(G, \mathbb{Q}_p). \end{aligned} \tag{28}$$

In particular, $C^{\mathrm{an}}(G, \mathbb{Q}_p) := C^{0,\mathrm{an}}(G, \mathbb{Q}_p)$ is the space of analytic functions on G.

For any $n \ge 0$, let $r_n := \sqrt[p^n]{1/p}$. The transition of spaces of analytic functions of decreasing radius is compact

$$C^{n,\mathrm{an}}(G, \mathbb{Q}_p) \hookrightarrow C^{n+1,\mathrm{an}}(G, \mathbb{Q}_p).$$

For any $n \ge 1$, if $z_1^{n-1}, \dots, z_d^{n-1}$ are coordinates of $G^{p^{n-1}}$, then it follows that $z_1^n = pz_1^{n-1}, \dots, z_d^n = pz_d^{n-1}$ are coordinates of G^{p^n}. For any $g \in G^{p^{n+1}}$, we may pull back

$$g^*(z_i^{n-1}) = \sum_{\alpha\in\mathbb{Z}_{\geq 0}^d} c_\alpha^i (z^{n-1})^\alpha \in C^{\mathrm{an}}(G^{p^{n-1}}, \mathbb{Q}_p),\ c_\alpha^i \in \mathbb{Z}_p,$$

and the constant term $c_{\underline{0}}^i$ is divided by p^2. Consider the commutative diagram

$$\begin{array}{ccc} C^{n-1,\mathrm{an}}(G,\mathbb{Q}_p) & \longrightarrow & C^{n,\mathrm{an}}(G,\mathbb{Q}_p) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ C^{n-1,\mathrm{an}}(G,\mathbb{Q}_p) & \longrightarrow & C^{n,\mathrm{an}}(G,\mathbb{Q}_p). \end{array}$$

We have

$$g^*(z_i^n) = \sum_{\alpha\in\mathbb{Z}_{\geq 0}^d} p^{|\alpha|-1} c_\alpha^i (z^n)^\alpha \in C^{\mathrm{an}}(G^{p^n}, \mathbb{Q}_p),\ c_\alpha^i \in \mathbb{Z}_p.$$

Since $g \in G^{p^{n+1}}$ induces the identical map on $G^{p^n}/G^{p^{n+1}}$, we have $c_j^i \equiv \delta_{ij} \bmod p$ for z_j^n's coefficient c_j^i in $g^*(z_i^n)$. We see that the operator $g-1$ has norm at most $1/p$ on the Banach space $C^{n,\mathrm{an}}(G,\mathbb{Q}_p)$ for any $n \geq 1$, $g \in G^{p^{n+1}}$, and so is the operator $(g-1)^{p^{n+1}}$ for any $g \in G$. Any power series of $D_r(G,\mathbb{Q}_p)$ converges as an endomorphism of $C^{n,\mathrm{an}}(G,\mathbb{Q}_p)$,

$$D_{r_{n+1}}(G,\mathbb{Q}_p) \to \mathrm{End}_{\mathbb{Q}_p}(C^{n,\mathrm{an}}(G,\mathbb{Q}_p)),$$

by the description (16). Composed with the evaluation map at identity

$$\begin{aligned} C^{n,\mathrm{an}}(G,\mathbb{Q}_p) &\xrightarrow{\mathrm{ev}} \mathbb{Q}_p \\ f &\mapsto f(\mathrm{id}), \end{aligned}$$

we have a natural map

$$D_{r_{n+1}}(G,\mathbb{Q}_p) \to (C^{n,\mathrm{an}}(G,\mathbb{Q}_p))'_b. \tag{29}$$

PROPOSITION 5.5. *There is an algebraic isomorphism*

$$D(G,\mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p \xrightarrow{\sim} \varprojlim_{n\geq 1} D_{r_n}^\lambda(G,\mathbb{Q}_p).$$

Proof. It suffices to construct the inverse of the natural map

$$D(G,\mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p \to \varprojlim_{n\geq 1} (D_{r_n}(G,\mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p).$$

By Theorem 5.2,

$$D(G,\mathbb{Q}_p) \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p \to (C^{\mathrm{la}}(G,\mathbb{Q}_p)[\lambda])'_b.$$

From (28),

$$(C^{\mathrm{la}}(G,\mathbb{Q}_p)[\lambda])'_b \simeq \varprojlim_{n\geq 0} (C^{n,\mathrm{an}}(G,\mathbb{Q}_p)[\lambda])'_b$$

as $\mathbb{Q}_p$-Fréchet spaces. The inverse

$$\varprojlim_{n\geq 0} D_{r_{n+1}}(G,\mathbb{Q}_p)\otimes_{Z(\mathfrak{g}),\lambda}\mathbb{Q}_p \to \varprojlim_{n\geq 0}(C^{n,\mathrm{an}}(G,\mathbb{Q}_p)[\lambda])'_b$$

is given by (29). □

Let $m \geq 1$ be a positive integer. For each $1 \leq i \leq m$, let

$$G_i := (I_2 + p\mathrm{M}_2(\mathbb{Z}_p)) \cap \mathrm{SL}_2(\mathbb{Z}_p).$$

For the rest of this section, let $G = \prod_{i=1}^{m} G_i \subset \widetilde{G} := \prod_{i=1}^{m} \widetilde{G}_i$ be r copies of the first congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$.

LEMMA 5.6. *The associated Lie algebra for* $G_1 = \ker(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{F}_p))$

$$\mathfrak{g}_0 \simeq \mathfrak{sl}_{2,\mathbb{Z}_p} = \mathbb{Z}_p \cdot h \oplus \mathbb{Z}_p \cdot e \oplus \mathbb{Z}_p \cdot f,$$

has the standard commutator brackets

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Proof. We set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p).$$

We let $\exp(ph), \exp(pe), \exp(pf)$ be a set of minimal generators of G_1. The p-adic manifolds G_1 and $p\mathfrak{sl}_2(\mathbb{Z}_p)$ are identified via the exponential and logarithm maps. The commutator brackets on $p\mathfrak{sl}_2(\mathbb{Z}_p)$ transfer to G_1, for example, by the computations in [DdSMS99, Lemma 7.12]. □

For example, if $m = 1$, the Iwasawa algebra $\mathbb{Z}_p[[G]]$ is identified with a non-commutative formal power series ring $\mathbb{Z}_p[[F,H,E]]$ in three variables for

$$F := \exp(pf) - 1,\ H := \exp(ph) - 1,\ E := \exp(pe) - 1$$

as in the lemma. Actually we may explicitly describe the microlocalization map (18) for our case when we identify $p\mathfrak{sl}_2(\mathbb{Z}_p)$ with $L_G = p\mathfrak{g}_0$ (proof of [AW13, Th. 10.4]):

$$F \mapsto \exp(pf) - 1,\ H \mapsto \exp(ph) - 1,\ E \mapsto \exp(pe) - 1.$$

Under such an identification, the Lie algebra action (22) is equivalent to

$$\mathfrak{x}f := \log(1 + (\mathfrak{x} - 1)) \cdot f \tag{30}$$

for $\mathfrak{x} \in \mathfrak{g}_0 \simeq G$ and $f \in C^{\mathrm{la}}(G,\mathbb{Q}_p)$.

THEOREM 5.7. *Let p be an odd prime. If $1/p < r < 1$, then $D_r^\lambda(G,\mathbb{Q}_p)$ is an integral domain.*

Proof. For saving notation, we prove the case $m = 1$. Let F, H, E be the formal variables with

$$pf = \log(1+F),\ ph = \log(1+H),\ pe = \log(1+E)$$

defined after Lemma 5.6.

For any $1/p < r < 1$, we can find $r' \leq r$ such that $1/p < r' < \sqrt{1/p}$. The sequence $\{\frac{(r')^k}{|k|} | k \in \mathbb{Z}_{\geq 1}\}$ has the property that

$$\max_{k \geq 1} \frac{(r')^k}{|k|} = \max_{i \geq 0} \frac{(r')^{p^i}}{|p^i|} = r'.$$

Note that we can endow the topology of $D_{r'}(G, \mathbb{Q}_p)$ on $D_r(G, \mathbb{Q}_p)$ since

$$D_r(G, \mathbb{Q}_p) \hookrightarrow D_{r'}(G, \mathbb{Q}_p) \tag{31}$$

is naturally a dense subalgebra by characterization (16).

On $R = D_r(G, \mathbb{Q}_p)$, or $D_{r'}(G, \mathbb{Q}_p)$, we have the filtration

$$F^s_{r'} R := \left\{ a \in R : \|a\|_{r'} \leq p^{-s} \right\}.$$

The associated graded ring is denoted by $\mathrm{gr}_{r'}(R)$. By the density of (31), we have the isomorphism

$$\mathrm{gr}_{r'} D_r(G, \mathbb{Q}_p) \simeq \mathrm{gr}_{r'} D_{r'}(G, \mathbb{Q}_p) \simeq \mathbb{F}_p[\epsilon, \epsilon^{-1}][F, H, E]$$

by [ST03, Th. 4.5].

Let $\Delta := \frac{1}{2}h^2 - h + 2ef$ be the Casimir operator. The kernel of λ is generated by $p^2\Delta + \lambda_0$ for $\lambda_0 \in \mathbb{Q}_p$. The associated graded ring for $D^\lambda_r(G, \mathbb{Q}_p)$ is

$$\mathrm{gr}_{r'} D^\lambda_r(G, \mathbb{Q}_p) \simeq \mathrm{gr}_{r'} D_r(G, \mathbb{Q}_p) / \mathrm{gr}_{r'}(p^2\Delta + \lambda_0)$$

by Lemma 2.2, where the principal symbol of generator $\mathrm{gr}_{r'}(p^2\Delta + \lambda_0)$ (Section 2.1) in $\mathrm{gr}_{r'}(D_r(G, \mathbb{Q}_p))$ equals

$$\begin{aligned} &\mathrm{gr}_{r'} \left(\frac{1}{2} \log(1+H)^2 - p\log(1+H) + 2\log(1+E)\log(1+F) + \lambda_0 \right) \\ &\quad = \mathrm{gr}_{r'} \left(\frac{1}{2} H^2 + 2EF + \lambda_0 \right) \end{aligned}$$

by the assumption of r'. If the valuation of λ_0 is at most 1, then

$$\mathrm{gr}_{r'}(p^2\Delta + \lambda_0) = \mathrm{gr}_{r'}(\lambda_0)$$

is a unit in $\mathrm{gr}_{r'} D_r(G, \mathbb{Q}_p)$, making the quotient equal to zero. Otherwise,

$$\begin{aligned} \mathrm{gr}_{r'}(p^2\Delta + \lambda_0) &= \frac{1}{2}H^2 + 2EF, \\ \mathrm{gr}_{r'} D^\lambda_r(G, \mathbb{Q}_p) &\simeq \mathbb{F}_p[\epsilon, \epsilon^{-1}][F, H, E] / \left(\frac{1}{2}H^2 + 2EF \right) \end{aligned}$$

is an integral domain by the proof of the third part of Lemma 3.3. ∎

Remark 5.8.

- Although $\mathrm{gr}_{1/p}D_r(G,\mathbb{Q}_p)$ is non-commutative, it can probably be shown that $\mathrm{gr}_{1/p}D_r^\lambda(G,\mathbb{Q}_p)$ is an integral domain as well.
- The infinitesimal character λ is induced if and only if the valuation λ_0 is at least 2 since for any continuous character $\chi : T \simeq 1+p\mathbb{Z}_p \to 1+p\mathbb{Z}_p$, $d\chi(p^2h^2) \in p^2\mathbb{Z}_p$.

Proof of Theorem 5.1. For any non-zero $\delta \in \mathbb{Q}_p[[G]]$, there exists $n_\delta \geq 1$ such that the image of δ in $D_r^\lambda(G,\mathbb{Q}_p)$ is non-zero for $r = \sqrt[p^{n_\delta}]{1/p}$ by Theorem 5.2 and Proposition 5.5. The right $D_r^\lambda(G,\mathbb{Q}_p)$-module $D_r^\lambda(G,\mathbb{Q}_p)/\delta\cdot D_r^\lambda(G,\mathbb{Q}_p)$ is torsion. Since $D_r^\lambda(G,\mathbb{Q}_p)$ is an integral domain by Theorem 5.7, $D_r^\lambda(G,\mathbb{Q}_p)/\delta\cdot D_r^\lambda(G,\mathbb{Q}_p)$ has positive codimension by [AW13, Prop. 2.5]. By applying Proposition 5.3 and [AB07, Cor. 5.4], $D_r^\lambda(G,\mathbb{Q}_p)/\delta\cdot D_r^\lambda(G,\mathbb{Q}_p)$ also has positive codimension over $U_r^\lambda(\mathfrak{g})$. By Lemma 3.3, $U_r^\lambda(\mathfrak{g})$ is an integral domain, and there exists an element $\delta' \in D_r^\lambda(G,\mathbb{Q}_p)$ such that $\delta\delta' \in U_r^\lambda(\mathfrak{g})$ is non-zero. By the description (27) of $U_r(\mathfrak{g})$, we leave it the reader to prove

$$U_r^\lambda(\mathfrak{g}) = \left\{ \sum_\alpha d_\alpha \prod_{i=1}^m e_i^{\alpha_a^i} f_i^{\alpha_b^i} h_i^{\alpha_c^i} \mid \alpha_c^i \in \{0,1\},\ d_\alpha \in K,\ \lim_{|\alpha|\to\infty} |d_\alpha| \left\| \prod_{i=1}^m e_i^{\alpha_a^i} f_i^{\alpha_b^i} h_i^{\alpha_c^i} \right\|_r = 0 \right\},$$

similar to the first part of Lemma 3.3 giving topological basis of $U_r^\lambda(\mathfrak{g})$, and similarly for $\widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p$. Given the basis for both $U_r^\lambda(\mathfrak{g})$ and $\widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p$, it is direct to see that the natural inclusion

$$U_r^\lambda(\mathfrak{g}) \hookrightarrow \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p$$

is injective, hence $\delta\delta'$ is non-zero in $\widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g}),\lambda} \mathbb{Q}_p$. The completed enveloping algebra is identified with $D_{1/p}(G,\mathbb{Q}_p)$ ([AW13, Rem. 10.5, (c)], [AW14, Lemma 5.2]). The image of δ via the microlocalization (18) is non-zero as well. □

6. Local applications to finitely generated Iwasawa modules

We continue to use notation from the previous sections. Let p be an odd prime, and let $r \geq 1$ be a positive integer. For each $1 \leq i \leq r$, let

$$G_i := (I_2 + p\mathrm{M}_2(\mathbb{Z}_p)) \cap \mathrm{SL}_2(\mathbb{Z}_p).$$

Let $G = \prod_{i=1}^r G_i$, $\mathbf{k} = (k_1,\dots,k_r) \in \mathbb{N}^r$, and let $W_\mathbf{k}$ be the algebraic representation $\boxtimes_{i=1}^r \mathrm{Sym}^{k_i}$ of G. As G is compact, by choosing an integral lattice, $W_\mathbf{k}$ is equipped with a structure of finite dimensional Banach representation of G. We leave it as an exercise for the reader to show that $W_\mathbf{k}$ is irreducible self-dual. By the main result of [ST02a], $W_\mathbf{k}^* \simeq W_\mathbf{k}$ admits an action of $K[[G]]$, and we do not distinguish $W_\mathbf{k}^*$ and $W_\mathbf{k}$.

We exhibit the $W_{\mathbf{k}}$-quotient of $K[[G]]$ explicitly using the theory of Schneider–Teitelbaum as follows. To ease notation, we do this for $r = 1$, $\mathbf{k} = k \in \mathbb{N}$. As a module over itself, the dual (constructed in [ST02a]) of $K[[G]]$ is the Banach representation of the continuous function $C(G, K)$ on G. Our choice of G can be viewed as an open subgroup of $\mathbb{Z}_p$-points of the group scheme SL_2 over $\mathbb{Z}_p$. We define the space of algebraic vectors $C^{\mathrm{alg}}(G, K)$ of $C(G, K)$ to be the following K-linear vector space of polynomial functions on G,

$$\{K[x, y, z, w]/(xw - yz - 1)\},$$

as $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in G \subset \mathrm{SL}_2(\mathbb{Z}_p)$. There is a natural two sided action of G on $C^{\mathrm{alg}}(G, K)$ with a G-stable filtration $C^{\mathrm{alg}}_{\le n}(G, K)$ by degrees of polynomials in variables x, y, z, w such that

$$\dim_K(C^{\mathrm{alg}}_{\le n}(G, K)/C^{\mathrm{alg}}_{\le n-1}(G, K)) = \binom{n+3}{3} - \binom{n+1}{3} = (n+1)^2.$$

Here the dimension of the space of homogeneous polynomials of degree n (resp. $n-2$) in 4-variables is $\binom{n+3}{3}$ (resp. $\binom{n+1}{3}$).

The space of degree k homogeneous polynomials in variables x, z is isomorphic to W_k as a left representation of G. The vector x^k is left invariant under the lower triangular nilpotent subgroup and right invariant under the lower upper triangular nilpotent subgroup with the highest weight in W_k. It generates an irreducible $G \times G$ subrepresentation $W_k \otimes W_k^*$ inside $C^{\mathrm{alg}}_{\le k}(G, K)/C^{\mathrm{alg}}_{\le k-1}(G, K)$. By counting the dimensions, we have

$$C^{\mathrm{alg}}_{\le k}(G, K)/C^{\mathrm{alg}}_{\le k-1}(G, K) \simeq W_k \otimes W_k^*.$$

By complete reducibility of finite dimensional representations, $W_k \otimes W_k^*$ is identified with W_k-algebraic vectors in $C(G, K)$ as a left or right representation of G. It is automatically closed in $C(G, K)$ by finite dimensionality. Again by the main result of [ST02a], $W_k \otimes W_k^*$ is also identified with the maximal W_k-quotient of $K[[G]]$.

Note that as an algebraic representation of G, $W_{\mathbf{k}}$ receives a Lie algebra action (with respect to the explicit algebraic structure of G above). When restricted to arbitrarily small open subgroups of G, $W_{\mathbf{k}}$ remains irreducible. According to [AW13, Th. 11.3], any finite dimensional simple Iwasawa module is a tensor product of a smooth G-representation and a Lie module (Section 4). Therefore our $W_{\mathbf{k}}^* \simeq W_{\mathbf{k}}$ is a Lie module, and we are entitled to apply results in Section 4.

Let $\widetilde{\mathrm{Ann}}(\mathbf{k}) := \mathrm{Ann}_{K[[G]]}(W_{\mathbf{k}}) = \widetilde{\mathrm{Ann}(W_{\mathbf{k}})}$ be the annihilator ideal of $K[[G]]$ for $W_{\mathbf{k}}$. We have an isomorphism of algebras

$$K[[G]]/\widetilde{\mathrm{Ann}}(\mathbf{k}) \xrightarrow{\sim} \mathrm{End}_K(W_{\mathbf{k}}). \tag{32}$$

Here the surjectivity follows from either directly counting dimensions of both sides given our explicit computation for SL_2, or combining (20) and Theorem 4.3.

Proof of Theorem 1.4. Any $\mathbb{Q}_p[[G]]$-equivariant map from $\widetilde{M}$ to $W_{\mathbf{k}}$ factors through $\widetilde{M}_{W_{\mathbf{k}}}$. We have $\mathrm{Hom}_{\mathbb{Q}_p[[G]]}(\widetilde{M}, W_{\mathbf{k}}) = \mathrm{Hom}_{\mathbb{Q}_p[[G]]}(\widetilde{M}_{W_{\mathbf{k}}}, W_{\mathbf{k}})$. By Theorem 4.3,

$$\mathrm{Hom}_{\mathbb{Q}_p[[G]]}(\widetilde{M}, W_{\mathbf{k}}) \simeq \mathrm{Hom}_{\widehat{U(\mathfrak{g}_K)}}(\widehat{M}, W_{\mathbf{k}}).$$

There are multiple ways to express the multiplicity of $W_{\mathbf{k}}$ in $\widetilde{M}$. We have

$$\mathrm{Hom}_{\mathbb{Q}_p[[G]]}(\widetilde{M}, W_{\mathbf{k}}) \simeq \mathrm{Hom}_{\mathbb{Q}_p[[G]]}(\widetilde{M} \otimes W_{\mathbf{k}}^*, \mathbb{1}) \simeq H_0(G, \widetilde{M} \otimes W_{\mathbf{k}}^*)^*.$$

The dimensions of these K-vector spaces all agree with $H^0_{\widehat{M}}(\mathbf{k})$ in (9).

We choose short exact sequences

$$0 \to (\mathbb{Q}_p[[G]])^d \to \widetilde{M} \to \widetilde{Q} \to 0, \tag{33}$$

$$0 \to \widetilde{N} \to (\mathbb{Q}_p[[G]])^l \to \widetilde{M} \to 0, \tag{34}$$

where $\widetilde{Q}$ is torsion. Let $\widehat{M} = \widehat{U(\mathfrak{g})} \otimes_{K[[G]]} \widetilde{M}$ be the microlocalization of $\widetilde{M}$, and similarly for $\widetilde{Q}$, $\widetilde{N}$. The microlocalization s preserve short exact sequences like (33) and (34) as (18) is flat. We may filter $\widetilde{Q}$ by cyclic subquotients $\widetilde{Q}_1, \ldots, \widetilde{Q}_q$ with $\mathbb{Q}_p[[G]]/\mathbb{Q}_p[[G]] \cdot \delta_i \twoheadrightarrow \widetilde{Q}_i$ with non-zero $\delta_i \in \mathbb{Q}_p[[G]]$. By Theorem 5.1, δ_i are generic (Section 3). Theorem 3.2 produces $p_{\widehat{Q}_i}$ such that $H^0_{\widehat{Q}_i}(\mathbf{k}) \le p_{\widehat{Q}_i}(\mathbf{k})$. And we may choose $p_{\widehat{Q}} := \sum_{i=1}^q p_{\widehat{Q}_i}$ so $H^0_{\widehat{Q}}(\mathbf{k}) \le p_{\widehat{Q}}(\mathbf{k})$. By applying the microlocalization s to the sequence (33) followed by taking $W_{\mathbf{k}}$ quotient

$$(\widehat{U(\mathfrak{g})}/\widehat{\mathrm{Ann}(\mathbf{k})})^d \to \widehat{M}_{\mathbf{k}} \to \widehat{Q}_{\mathbf{k}} \to 0,$$

we have

$$H^0_{\widehat{M}}(\mathbf{k}) \le d \prod_{i=1}^r (k_i + 1) + p_{\widehat{Q}}(\mathbf{k}),$$

thus proving one side of the $i = 0$ case for any finitely generated $\mathbb{Q}_p[[G]]$-module $\widetilde{M}$. Similarly, by applying the microlocalization s to the sequence (34) followed by taking the $W_{\mathbf{k}}$ quotient, we get the other side of inequality for the case $i = 0$ using

$$\mathrm{rank}_{\widehat{U(\mathfrak{g})}} \widehat{N} + \mathrm{rank}_{\widehat{U(\mathfrak{g})}} \widehat{M} = l.$$

Because $(\mathbb{Q}_p[[G]])^l$ is acyclic, the long exact sequence associated to (34) in homology gives

$$\begin{aligned} 0 &\to H_1(G, \widetilde{M} \otimes W_{\mathbf{k}}) \to H_0(G, \widetilde{N} \otimes W_{\mathbf{k}}) \\ &\to H_0(G, (\mathbb{Q}_p[[G]]/\widetilde{\mathrm{Ann}(\mathbf{k})}) \otimes W_{\mathbf{k}})^l \to H_0(G, \widetilde{M} \otimes W_{\mathbf{k}}) \to 0, \end{aligned} \tag{35}$$

$$H_i(G, \widetilde{M} \otimes W_{\mathbf{k}}) \simeq H_{i-1}(G, \widetilde{N} \otimes W_{\mathbf{k}}), \ i \ge 2. \tag{36}$$

The case $i = 1$ is obtained by applying case $i = 0$ to both $\widetilde{M}$ and $\widetilde{N}$ using (35). Furthermore, the $i \geq 2$ cases follow from an induction using (36). □

7. Global automorphic applications

In this final section, we refer to the notation in Section 1. We choose an odd prime p that splits completely in F. Recall that $r := [F : \mathbb{Q}]$. Let

$$G := \prod_{v|p} G_v, \;\; G_v := (I_2 + p\mathrm{M}_2(\mathbb{Z}_p)) \cap \mathrm{SL}_2(\mathbb{Z}_p),$$

as the ring of integers in F_v is identified with $\mathbb{Z}_p$ for all $v|p$.

For $\mathbf{k} \in \mathbb{N}^r$, it defines a $\mathbb{C}$-representation $V_{\mathbf{k}}$ of $\mathrm{SL}_2(F_\infty)$, as well as a $\mathbb{Q}_p$-representation $W_{\mathbf{k}}$ of $\mathrm{SL}_2(F_p)$. These representations give rise to a $\mathbb{C}$-local system $V_{\mathbf{k}}$ and its corresponding $\mathbb{Q}_p$-local system $W_{\mathbf{k}}$. It is explained in [Mar12, §5] that

$$\dim_{\mathbb{C}} H_i(Y(K_f), V_{\mathbf{k}}) = \dim_{\mathbb{Q}_p} H_i(Y(K_f), W_{\mathbf{k}}). \tag{37}$$

We cite some properties of the completed homology (4),

$$\widetilde{H}_\bullet(K^p) := \varprojlim_s \varprojlim_{K_p \subset G} H_\bullet(Y(K_p K^p), \mathbb{Z}/p^s\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

from [Eme06], [CE09]:

- If F is not totally real, $\widetilde{H}_i(K^p)$ is a finitely generated torsion $\mathbb{Q}_p[[G]]$-module.
- There is a spectral sequence

$$E_2^{i,j} = H_i(G, \widetilde{H}_j(K^p) \otimes W_{\mathbf{k}}) \Longrightarrow H_{i+j}(Y(K_f), W_{\mathbf{k}}), \tag{38}$$

  where $K_f = GK^p$.

By (37), the spectral sequence (38) implies an upper bound

$$\dim_{\mathbb{C}} H_q(Y(K_f), V_{\mathbf{k}}) \leq \sum_{i+j=q} \dim_{\mathbb{Q}_p} H_i(G, \widetilde{H}_j(K^p) \otimes W_{\mathbf{k}}).$$

Theorem 1.4 produces a multiplicity-free polynomial p_{K_f} of degree at most $r - 1$ for the right-hand side, giving

$$\dim_{\mathbb{C}} H_q(Y(K_f), V_{\mathbf{k}}) \leq p_{K_f}(\mathbf{k}), \;\; \mathbf{k} \in \mathbb{N}^r.$$

Finally, we apply the Poincaré duality and Eichler–Shimura isomorphism (2) to finish our proof of Theorem 1.2.

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