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Relating flat connections and polylogarithms on higher genus Riemann surfaces

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Abstract

In this work, we relate two recent constructions that generalize classical (genus-zero) polylogarithms to higher-genus Riemann surfaces. A flat connection valued in a freely generated Lie algebra on a punctured Riemann surface of arbitrary genus produces an infinite family of homotopy-invariant iterated integrals associated to all possible words in the alphabet of the Lie algebra generators. Each iterated integral associated to a word is a higher-genus polylogarithm. Different flat connections taking values in the same Lie algebra on a given Riemann surface may be related to one another by the composition of a gauge transformation and an automorphism of the Lie algebra, thus producing closely related families of polylogarithms. In this paper we provide two methods to explicitly construct this correspondence between the meromorphic multiple-valued connection introduced by Enriquez in e-Print 1112.0864 and the non-meromorphic single-valued and modular-invariant connection introduced by D'Hoker, Hidding and Schlotterer, in e-Print 2306.08644.

Contents

1 Introduction

Perturbative computations in quantum field theory and string theory involve complicated multidimensional integrals. It has become increasingly clear over the past few decades that these integrals may profitably be organized in terms of *polylogarithms*, and various generalizations thereof. Broadly defined, the term polylogarithm is being used here for a *multiple-valued function* on a manifold M defined as a *homotopy-invariant iterated integral* over an integration path $\gamma : [0,1] \to M$ starting at some fixed integration base-point $\gamma(0) = y$ and depending on the integration end-point $\gamma(1) = x$. By homotopy-invariant iterated integral we mean a suitable linear combination [1] of iterated integrals along the path γ over differential 1-forms $\phi_1, \cdots, \phi_k$,

$$\int_0^1 \phi_1\big(\gamma(t_1)\big) \int_0^{t_1} \phi_2\big(\gamma(t_2)\big) \cdots \int_0^{t_{k-1}} \phi_k\big(\gamma(t_k)\big), \tag{1.1}$$

whose value depends on the base-point y, on the end-point x and on the homotopy class $[\gamma]$ of the integration path γ, but not upon the specific path γ in a given homotopy class $[\gamma]$.

Polylogarithms on the punctured sphere, also known as *hyperlogarithms* [2, 3, 4, 5], complete the space of rational functions to a space of multiple-valued functions that is closed under addition, multiplication, differentiation and the taking of primitives. Polylogarithms were generalized to the elliptic case [6, 7, 8, 9], where they complete the space of elliptic functions to a function space on the punctured torus with the same properties as its genus-zero analogue [10, 11]. When evaluated at special points, genus-zero polylogarithms produce multiple zeta values while their genus-one counterparts produce elliptic multiple zeta values [12, 13, 14, 15] (see also section 1.3 of this introduction). Recently, different further generalizations of polylogarithms to Riemann surfaces of arbitrary genus have been proposed with the goal of providing spaces of functions that are closed under the taking of primitives [16, 17, 18]. There is good evidence that these higher genus polylogarithms may provide a useful organizational set-up for perturbative string theory calculations [19, 20, 21, 22] while also promising relevance to higher loop Feynman integrals in quantum field theory [23, 24, 25, 26, 27, 28].

A general and efficient geometrical construction of polylogarithms is by taking the path-ordered exponential solution of the differential equation induced by a flat connection valued in a (completed) free Lie algebra. Indeed, the coefficient of the path-ordered exponential corresponding to a given word in the Lie algebra generators is an iterated integral which is homotopy invariant, due to the flatness of the connection. At genus zero classical polylogarithms arise from the Knizhnik–Zamolodchikov connection, whose holonomy is closely related to the Drinfeld associator [29, 30]; the significance of the latter to number theory has been discussed in [31, 32, 12], whereas its relevance to string amplitudes was proposed in [33, 34, 35, 36].

The approaches to generalize polylogarithms to arbitrary punctured Riemann surfaces of genus $h \geq 1$ in terms of flat connections can be divided into three different categories. Chronologically, the first is via a holomorphic multiple-valued[1] connection with a regular singularity at the puncture: the connection $d - \mathcal{K}_{\rm E}$ introduced by Enriquez [37]. The second is via a holomorphic single-valued connection with an irregular singularity at the puncture, which is the case of the families of connections introduced by Enriquez and Zerbini in [38, 16]. The third is via a non-holomorphic single-valued and modular-invariant connection with a regular singularity at the puncture: the connection $d - \mathcal{J}_{\rm DHS}$ introduced by D'Hoker, Hidding, and Schlotterer (DHS) in [17]. All these connections take values in the (completed) freely generated Lie algebra $\mathfrak{g}$ on $2h$ generators and give rise to different, but closely related, families of polylogarithms. The relation between the first two approaches was partly discussed in [38] and will be the subject of the forthcoming article [39].

In the present paper, we shall relate the first and the third approaches by showing that $d - \mathcal{K}_{\rm E}$ can be obtained from $d - \mathcal{J}_{\rm DHS}$ by combining a gauge transformation and an automorphism of the Lie algebra $\mathfrak{g}$. As a corollary, we can deduce relations between the function spaces generated by the two corresponding families of polylogarithms. We shall prove this correspondence via two different constructions, provide explicit formulas for the relations between connections and generating functions for polylogarithms, and evaluate the general formulas to low orders.

1.1 Flat connections, iterated integrals, and polylogarithms

An efficient geometric construction of polylogarithms starts from a flat connection $d_x - \mathcal{J}(x;c)$, with $\mathcal{J}(x;c)$ a differential 1-form in x on a Riemann surface Σ, possibly multiple-valued, taking values in the completion[2] $\mathfrak{g}$ of the Lie algebra that is freely generated by a set $c = \{c_1, \cdots, c_n\}$. Flatness of the connection, which is expressed in terms of the Maurer–Cartan equation

$$d_x \mathcal{J}(x;c) - \mathcal{J}(x;c) \wedge \mathcal{J}(x;c) = 0, \tag{1.2}$$

guarantees the integrability of the differential equation

$$d_x \mathbf{\Gamma}(x,y;c) = \mathcal{J}(x;c)\, \mathbf{\Gamma}(x,y;c), \tag{1.3}$$

[1]The expression "multiple-valued connection" (resp. "single-valued connection") is an abuse of terminology, and expresses here the fact that, if a connection ∇ is written as $d - \mathcal{J}$, then $\mathcal{J}$ is a multiple-valued (resp. single-valued) differential 1-form.

[2]If one assigns degree 1 to each variable c_i, the Lie algebra freely generated by c is a graded Lie algebra (of Lie polynomials); its completion with respect to this grading is the Lie algebra $\mathfrak{g}$ of Lie series.

subject to the initial condition $\mathbf{\Gamma}(y,y;c)=1$, for a function $\mathbf{\Gamma}(x,y;c)$ that is a scalar in x and y and which takes values in the Lie group[3] $\exp(\mathfrak{g})$ of $\mathfrak{g}$. The solution of (1.3) can be written in terms of the path-ordered exponential of $\mathcal{J}$,[4]

$$\mathbf{\Gamma}(x,y;c)=\mathrm{P}\exp\int_y^x \mathcal{J}(t;c). \tag{1.4}$$

Taylor expanding the path-ordered exponential in powers of $\mathcal{J}$ gives an explicit expression for $\mathbf{\Gamma}(x,y;c)$ in terms of a series of iterated integrals, each of which takes values in $\mathbb{C}\langle\langle c\rangle\rangle$ (see footnote 3),

$$\mathbf{\Gamma}(x,y;c)=1+\sum_{k=1}^{\infty}\int_y^x \mathcal{J}(t_1;c)\int_y^{t_1}\mathcal{J}(t_2;c)\cdots\int_y^{t_{k-1}}\mathcal{J}(t_k;c). \tag{1.5}$$

Flatness of $\mathcal{J}(x;c)$ further guarantees that $\mathbf{\Gamma}(x,y;c)$ is homotopy invariant, namely, it depends[5] only on the homotopy class of the integration path from y to x and is independent of the specific representative in the class. Also, $\mathbf{\Gamma}$ satisfies the path-concatenation formula

$$\mathbf{\Gamma}(x,z;c)=\mathbf{\Gamma}(x,y;c)\,\mathbf{\Gamma}(y,z;c), \tag{1.6}$$

where the product on the right is taken in the group $\exp(\mathfrak{g})$ and concatenates the words in c from the two factors. This property can be used to deduce useful formulas for the monodromy of the solution, namely for the value of $\mathbf{\Gamma}(\gamma\cdot x,y;c)$ in terms of $\mathbf{\Gamma}(x,y;c)$, with γ an element of the fundamental group[6] $\pi_1(\Sigma,y)$. Indeed, using (1.6) to write $\mathbf{\Gamma}(\gamma\cdot x,y;c)=\mathbf{\Gamma}(\gamma\cdot x,\gamma\cdot y;c)\,\mathbf{\Gamma}(\gamma\cdot y,y;c)$, setting $\mu(\gamma,y;c)=\mathbf{\Gamma}(\gamma\cdot y,y;c)$, and assuming that $\mathcal{J}(x;c)$ is single-valued, we obtain the following formula

$$\mathbf{\Gamma}(\gamma\cdot x,y;c)=\mathbf{\Gamma}(x,y;c)\,\mu(\gamma,y;c). \tag{1.7}$$

[3]This is defined to be the group of group-like elements in the (Hopf) algebra $\mathbb{C}\langle\langle c\rangle\rangle$ of formal series in non-commutative generators $c_1,\cdots,c_n$. The latter is the free associative algebra generated by c equipped with the Hopf algebra structure associated with word concatenation, and should be interpreted here as the degree completion of the universal enveloping algebra of the free Lie algebra generated by c.

[4]It will often be convenient to indicate the variable over which a given integration is being carried out, especially so when several integrations are involved; we shall reserve the letter t for this purpose.

[5]The dependence on the path is suppressed from the notation, because we will rather consider x and y as variables on the universal cover $\tilde{\Sigma}$ of Σ, which is equivalent to specifying the class of the path between points x and y on Σ.

[6] Here and elsewhere we are denoting by y both a chosen integration base-point in the universal cover $\tilde{\Sigma}$ and its image in Σ. The notation $\gamma\cdot y$, inspired by the induced isomorphism $\pi_1(\Sigma,y)\simeq\mathrm{Aut}(\tilde{\Sigma}/\Sigma)$, stands then for the endpoint of the (unique) lift to $\tilde{\Sigma}$ of the path γ which starts at y. There is then a unique element γ of $\mathrm{Aut}(\tilde{\Sigma}/\Sigma)$ which takes y to $\gamma\cdot y$, and $\gamma\cdot x$ denotes the image of x under this automorphism.

For fixed y and c, and assuming that $\mathcal{J}(x;c)$ is single-valued, one verifies using (1.6) that μ is a homomorphism from $\pi_1(\Sigma, y)$ to the group $\exp(\mathfrak{g})$, namely that[7]

$$\mu(\gamma_1, y; c)\mu(\gamma_2, y; c) = \mu(\gamma_1 \star \gamma_2, y; c), \tag{1.8}$$

thus yielding a monodromy representation.

While $\mathbf{\Gamma}$ is homotopy invariant, the contribution to the series in (1.5) from a single value of $k \geq 1$ is not homotopy invariant. Thus, it is understood that the iterated integrals for all values of k are taken along the same path in a given homotopy class. The expansion in powers of $\mathcal{J}$ may be expressed as a series over *words* $\mathfrak{w} \in \mathcal{W}(c)$,

$$\mathbf{\Gamma}(x, y; c) = \sum_{\mathfrak{w} \in \mathcal{W}(c)} \mathfrak{w}\, \Gamma(\mathfrak{w}; x, y), \tag{1.9}$$

where $\mathcal{W}(c)$ is the set of all words in the alphabet of letters $\{c_1, \cdots, c_n\}$, which is the *monoid* freely generated by c, where the sum in (1.9) includes the empty word $\emptyset$ for which $\Gamma(\emptyset; x, y) = 1$. For each non-empty word $\mathfrak{w}$, the coefficient $\Gamma(\mathfrak{w}; x, y)$ is a homotopy-invariant iterated integral, to which we shall generally refer as a *polylogarithm associated with the connection* $\mathcal{J}$. It follows from the fact that $\mathbf{\Gamma}(x, y; c)$ takes values in $\exp(\mathfrak{g})$ that the product of polylogarithms for words $\mathfrak{w}_1$ and $\mathfrak{w}_2$ and identical endpoints x, y may be expressed as a sum of polylogarithms associated with words $\mathfrak{w}$ belonging to the shuffle product[8] $\mathfrak{w}_1 \sqcup\!\sqcup\, \mathfrak{w}_2$ (see [40] and [41] for proofs and further discussions).

$$\Gamma(\mathfrak{w}_1; x, y)\Gamma(\mathfrak{w}_2; x, y) = \sum_{\mathfrak{w} \in \mathfrak{w}_1 \sqcup\!\sqcup\, \mathfrak{w}_2} \Gamma(\mathfrak{w}; x, y). \tag{1.10}$$

1.2 Relating flat connections and polylogarithms

We shall from now on consider connections on a once-punctured surface $\Sigma_p = \Sigma \setminus \{p\}$, where Σ is a compact Riemann surface of arbitrary genus $h \geq 1$ (not necessarily hyperelliptic as those higher-genus surfaces encountered in the current particle-physics literature

[7]Here $\gamma_1 \star \gamma_2$ stands for the composition of two paths $\gamma_1, \gamma_2 \in \pi_1(\Sigma, y)$ with the convention that the composed path traverses first γ_1 followed by γ_2. This implies that $(\gamma_1 \star \gamma_2) \cdot y = \gamma_2 \cdot (\gamma_1 \cdot y)$ and therefore $\mu(\gamma_1 \star \gamma_2, y; c) = \mathbf{\Gamma}(\gamma_2 \cdot (\gamma_1 \cdot y), y; c) = \mathbf{\Gamma}(\gamma_2 \cdot (\gamma_1 \cdot y), \gamma_2 \cdot y; c)\mathbf{\Gamma}(\gamma_2 \cdot y, y; c)$ which then leads to the concatenation order on the left side of (1.8) since $\mathbf{\Gamma}(\gamma_2 \cdot (\gamma_1 \cdot y), \gamma_2 \cdot y; c) = \mathbf{\Gamma}(\gamma_1 \cdot y, y; c)$ and $\mathbf{\Gamma}(\gamma_j \cdot y, y; c) = \mu(\gamma_j, y; c)$.

[8]We recall that the shuffle product $\mathfrak{w}_1 \sqcup\!\sqcup\, \mathfrak{w}_2$ is the subset of $\mathcal{W}(c)$ containing the words $\mathfrak{w}$ obtained from all possible ways of interlacing the letters of $\mathfrak{w}_1$ and $\mathfrak{w}_2$ such that the order of the letters in each word is preserved, see appendix B.2. The shuffle product extends to a commutative and associative operation which turns the $\mathbb{C}$-vector space of linear combination of words $\mathfrak{w} \in \mathcal{W}(c)$ into a ring, with neutral element given by the empty word $\emptyset$. A useful reference on free Lie algebras and the shuffle product is [41].

[23, 24, 25, 26, 27, 28]). The fundamental group $\pi_1(\Sigma_p, y)$, with base-point $y \in \Sigma_p$, is freely generated by closed loops $\mathfrak{A}^I, \mathfrak{B}_I$ for $I = 1, \cdots, h$ which can be chosen such that the intersection pairing $\mathfrak{J}$ of their images (denoted with the same symbol) in the homology group $H_1(\Sigma_p, \mathbb{Z})$ satisfies $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{A}^J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ and $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{B}_J) = \delta^I_J$. The completed free Lie algebra $\mathfrak{g}$ in which the connections take their values will be chosen to be generated by $2h$ independent elements. We shall denote a choice of such generators of $\mathfrak{g}$ by $a \cup b$, where $a = \{a^1, \cdots, a^h\}$ and $b = \{b_1, \cdots, b_h\}$; these are elements of $\mathfrak{g}^h$ and can be interpreted as a basis of the dual of $H^1_{\mathrm{dR}}(\Sigma_p)$, corresponding to the cycles $\mathfrak{A}^1, \cdots, \mathfrak{A}^h, \mathfrak{B}_1, \cdots, \mathfrak{B}_h$ via the isomorphism induced by the period pairing (see [16]).

As shown in appendix A, two flat connections $\mathcal{J}_1, \mathcal{J}_2$ on the trivial $\exp(\mathfrak{g})$-principal bundle over Σ_p are necessarily related by a combination of a *Lie algebra automorphism* of $\mathfrak{g}$ and a *gauge transformation*, as long as the respective monodromy representations μ_1, μ_2 are such that the families $([\log(\mu_i(\mathfrak{A}^1))]_1, \cdots, [\log(\mu_i(\mathfrak{A}^h))]_1, [\log(\mu_i(\mathfrak{B}_1))]_1, \cdots, [\log(\mu_i(\mathfrak{B}_h))]_1)$ (where $[\cdot]_1$ is the projection to the degree-one part of $\mathfrak{g}$) for $i = 1, 2$ are bases of the vector space generated by $a \cup b$.

In the remainder of this paper we shall consider two specific connections on Σ_p, both taking values in $\mathfrak{g}$. The first is a holomorphic multiple-valued connection $d_x - \mathcal{K}_{\mathrm{E}}(x, p; a, b)$ given by specializing a more general construction from [37], while the second is the non-holomorphic, single-valued and modular-invariant connection $d_x - \mathcal{J}_{\mathrm{DHS}}(x, p; a, b)$ introduced in [17]. Their definition and properties will be reviewed in section 2 below. More precisely, we will uniquely characterize the $\mathfrak{g}$-valued differentials $\mathcal{K}_{\mathrm{E}}$ and $\mathcal{J}_{\mathrm{DHS}}$ through their functional properties in Theorems 2.1 and 2.5, respectively. Generalizations of both connections to acquire simple poles in x at an arbitrary number of punctures may be found in [37, 17].

By construction, the connection $d - \mathcal{J}_{\mathrm{DHS}}$ is defined on the trivial $\exp(\mathfrak{g})$-principal bundle over Σ_p, and it follows from Thm. 2.33 in [38] (applied to the case $n = 1$) that this is the case also for $d - \mathcal{K}_{\mathrm{E}}$. Since both connections satisfy the above-mentioned technical condition on their monodromy representations, it follows that $d - \mathcal{J}_{\mathrm{DHS}}$ and $d - \mathcal{K}_{\mathrm{E}}$ must be related by combining a gauge transformation with a Lie algebra automorphism of $\mathfrak{g}$. The key result of this paper is to provide two different methods for the explicit construction of the gauge transformation and of the automorphism which relate the two connections. In both constructions, the gauge transformation will be a smooth multiple-valued function on $\Sigma \times \Sigma$ which takes values in $\exp(\mathfrak{g}_b)$, where $\mathfrak{g}_b$ is the Lie sub-algebra of $\mathfrak{g}$ given by the (completed) free Lie algebra generated by b.

1.2.1 First construction: $\mathcal{K}_{\mathbf{E}}$ from $\mathcal{J}_{\mathbf{DHS}}$

More specifically, section 3 is devoted to the construction (see Theorem 3.9) of a gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$, which will be a smooth function in x on the universal cover of Σ_p, and an automorphism mapping $a \cup b$ to an alternative set of generators $\hat a \cup \hat b$ of $\mathfrak{g}$ such that

$$\begin{aligned}\mathcal{K}_{\rm E}(x,p;a,b) = &\,\mathcal{U}_{\rm DHS}(x,p)^{-1}\mathcal{J}_{\rm DHS}(x,p;\hat a,\hat b)\,\mathcal{U}_{\rm DHS}(x,p)\\ &-\mathcal{U}_{\rm DHS}(x,p)^{-1}d_x\,\mathcal{U}_{\rm DHS}(x,p),\end{aligned} \tag{1.11}$$

which is equivalent to the following relation between the two connections

$$d_x - \mathcal{K}_{\rm E}(x,p;a,b) = \mathcal{U}_{\rm DHS}(x,p)^{-1}\big(d_x - \mathcal{J}_{\rm DHS}(x,p;\hat a,\hat b)\big)\,\mathcal{U}_{\rm DHS}(x,p). \tag{1.12}$$

Here, the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ is obtained from the path-ordered exponential

$$\mathbf{\Gamma}_{\rm DHS}(x,y,p;\xi,\eta) = {\rm P}\exp\int_y^x \mathcal{J}_{\rm DHS}(t,p;\xi,\eta) \tag{1.13}$$

by specializing the free non-commutative variables $\xi = \{\xi^1,\cdots,\xi^h\}$ and $\eta = \{\eta_1,\cdots,\eta_h\}$ to appropriate values $\hat\xi,\hat\eta \in \mathfrak{g}_b^h$ which satisfy[9] $[\hat\eta_I,\hat\xi^I] = 0$. This in turn implies that $\mathcal{J}_{\rm DHS}(t,p;\hat\xi,\hat\eta)$ is regular at $t=p$, and therefore that $\mathbf{\Gamma}_{\rm DHS}(x,y,p;\hat\xi,\hat\eta)$ is regular when y approaches p; we are therefore allowed to set

$$\mathcal{U}_{\rm DHS}(x,p) = \mathbf{\Gamma}_{\rm DHS}(x,p,p;\hat\xi,\hat\eta). \tag{1.14}$$

Throughout we shall suppress the variables $\hat\xi$ and $\hat\eta$ in writing $\mathcal{U}_{\rm DHS}(x,p)$. The notation $\hat a,\hat b$ stands for an alternative set of generators $\hat a = \{\hat a^1,\ldots,\hat a^h\}$ and $\hat b = \{\hat b_1,\ldots,\hat b_h\}$ of $\mathfrak{g}$, so that each hatted element is a Lie series in the original unhatted elements from the set $a\cup b$, and the map $a\cup b \to \hat a\cup\hat b$ from unhatted to hatted elements can be viewed as an automorphism of the Lie algebra $\mathfrak{g}$.

We outline a procedure to determine the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ and the automorphism $a\cup b\to\hat a\cup\hat b$, constructed above, in a series expansions in powers of the generators b. This procedure leads to explicit formulas relating the expansion coefficients $g^{I_1\cdots I_r}{}_J(x,p)$ and $f^{I_1\cdots I_r}{}_J(x,p)$ of $\mathcal{K}_{\rm E}(x,p;a,b)$ and $\mathcal{J}_{\rm DHS}(x,p;a,b)$, respectively, which furnish the integration kernels for the associated polylogarithms [37, 17]. For example, to

[9] Throughout, unless otherwise indicated, we will follow the Einstein convention in which a pair of identical upper and lower indices are summed over $1,2,\cdots,h$, without writing the summation sign.

low degree, the formulas of Proposition 3.14 imply the relations

$$\begin{aligned} g^I{}_J(x,p) &= f^I{}_J(x,p) + \mathcal{T}^I(x,p)\omega_J(x) + \omega_K(x)\mathcal{M}^{KI}{}_J(p), \\ g^{I_1I_2}{}_J(x,p) &= f^{I_1I_2}{}_J(x,p) + \mathcal{T}^{I_1}(x,p) f^{I_2}{}_J(x,p) \\ &\quad + f^{I_1}{}_K(x,p)\mathcal{M}^{KI_2}{}_J(p) - \mathcal{M}^{I_1I_2}{}_K(p) f^K{}_J(x,p) \\ &\quad + \mathcal{T}^{I_1I_2}(x,p)\omega_J(x) + \mathcal{T}^{I_1}(x,p)\omega_K(x)\mathcal{M}^{KI_2}{}_J(p) \\ &\quad - \mathcal{M}^{I_1I_2}{}_K(p)\mathcal{T}^K(x,p)\omega_J(x) + \omega_K(x)\mathcal{M}^{KI_1I_2}{}_J(p). \end{aligned} \tag{1.15}$$

The smooth functions $\mathcal{T}^I(x,p), \mathcal{T}^{I_1I_2}(x,p)$ and $\mathcal{M}^{KI}{}_J(p), \mathcal{M}^{KI_1I_2}{}_J(p)$ arise as expansion coefficients of the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ and the automorphism $a \cup b \to \hat{a} \cup \hat{b}$ in powers of b, respectively, and may be algorithmically computed to arbitrary rank.

1.2.2 Second construction: $\mathcal{J}_{\rm DHS}$ from $\mathcal{K}_{\rm E}$

Section 4 is devoted to the construction (see Theorem 4.8) of another gauge transformation $\mathcal{U}_{\rm E}(x,p)$ and automorphism mapping $a \cup b$ to alternative generators $\check{a} \cup \check{b}$ of $\mathfrak{g}$ such that

$$\mathcal{J}_{\rm DHS}(x,p;a,b) = \mathcal{U}_{\rm E}(x,p)^{-1}\mathcal{K}_{\rm E}(x,p;\check{a},\check{b})\,\mathcal{U}_{\rm E}(x,p) - \mathcal{U}_{\rm E}(x,p)^{-1} d_x\, \mathcal{U}_{\rm E}(x,p), \tag{1.16}$$

which is equivalent to the following relation between the two connections

$$d_x - \mathcal{J}_{\rm DHS}(x,p;a,b) = \mathcal{U}_{\rm E}(x,p)^{-1}\big(d_x - \mathcal{K}_{\rm E}(x,p;\check{a},\check{b})\big)\,\mathcal{U}_{\rm E}(x,p). \tag{1.17}$$

Here, the full gauge transformation

$$\mathcal{U}_{\rm E}(x,p) = \mathbf{\Gamma}_{\rm E}(x,p,p;\check{\xi},\check{\eta})\,\mathbf{\Gamma}_-(x,p;b)^{-1} \tag{1.18}$$

takes the form of a product. The second factor is the inverse of the (anti-holomorphic) path-ordered exponential

$$\mathbf{\Gamma}_-(x,p;b) = \mathrm{P}\exp\int_p^x \big(-\pi\bar{\omega}^I(t) b_I\big), \tag{1.19}$$

with suitably normalized anti-holomorphic Abelian differentials $\bar{\omega}^I$ defined in (2.3) and (2.5) below (also, see footnote 11 for the raising of indices of $\bar{\omega}^I$ in (1.19) through the inverse of the imaginary part of the period matrix). The first factor in (1.18) is obtained by specializing the arguments ξ,η of the path-ordered exponential

$$\mathbf{\Gamma}_{\rm E}(x,y,p;\xi,\eta) = \mathrm{P}\exp\int_y^x \mathcal{K}_{\rm E}(t,p;\xi,\eta) \tag{1.20}$$

to appropriate values $\check{\xi}, \check{\eta} \in \mathfrak{g}_b^h$ which satisfy $[\check{\eta}_I, \check{\xi}^I] = 0$. Since the residue of the pole of $\mathcal{K}_{\rm E}(t,p;\check{\xi},\check{\eta})$ in t at p is proportional to $[\check{\eta}_I, \check{\xi}^I]$, this in turn implies that $\mathcal{K}_{\rm E}$ in the integrand of (1.20) is regular at $t = p$, and therefore that we are allowed to take $y = p$ as integration base-point. As in the first construction from (1.11), the dependence of $\mathcal{U}_{\rm E}(x,p)$ on the Lie algebra elements is omitted, and the notation $\check{a}, \check{b}$ stands for an alternative set of generators $\check{a} = \{\check{a}^1, \ldots, \check{a}^h\}$ and $\check{b} = \{\check{b}_1, \ldots, \check{b}_h\}$ of $\mathfrak{g}$ which are Lie series in the original generators a, b, so that the map $a \cup b \to \check{a} \cup \check{b}$ induces an automorphism of the Lie algebra $\mathfrak{g}$.

The expansion of this second construction in the generators b leads to an equivalent set of explicit formulae between the integration kernels $g^{I_1 \cdots I_r}{}_J(x,p)$ and $f^{I_1 \cdots I_r}{}_J(x,p)$. The resulting analogues of the low-order relations (1.15) feature an alternative formulation of its coefficients $\mathcal{T}$ and $\mathcal{M}$ which can also be algorithmically computed to any desired order.

1.2.3 Relating the polylogarithms obtained from $\mathcal{J}_{\rm DHS}$ and $\mathcal{K}_{\rm E}$

The path-ordered exponentials $\mathbf{\Gamma}_{\rm DHS}$ and $\mathbf{\Gamma}_{\rm E}$ exploited in the construction of the gauge transformations are the generating series of polylogarithms associated to the connections $d - \mathcal{J}_{\rm DHS}$ and $d - \mathcal{K}_{\rm E}$, respectively. These two series are related by

$$\mathbf{\Gamma}_{\rm E}(x,y,p;a,b) = \mathcal{U}_{\rm DHS}(x,p)^{-1}\, \mathbf{\Gamma}_{\rm DHS}(x,y,p;\hat{a},\hat{b})\, \mathcal{U}_{\rm DHS}(y,p) \tag{1.21}$$

as well as by

$$\mathbf{\Gamma}_{\rm DHS}(x,y,p;a,b) = \mathcal{U}_{\rm E}(x,p)^{-1}\, \mathbf{\Gamma}_{\rm E}(x,y,p;\check{a},\check{b})\, \mathcal{U}_{\rm E}(y,p), \tag{1.22}$$

because it follows from (1.11) (resp. (1.16)) that both sides of (1.21) (resp. (1.22)) satisfy the same differential equation with the same initial condition. The expansion of $\mathcal{U}_{\rm DHS}$ and $\mathcal{U}_{\rm E}$ in words, analogous to the expansion of $\mathbf{\Gamma}$ in (1.9), together with the expansion of the Lie series $\hat{\xi}, \hat{\eta}, \hat{a}, \hat{b}$ and $\check{\xi}, \check{\eta}, \check{a}, \check{b}$, leads to two different families of explicit formulas which can be used to relate the polylogarithms for the two connections. By comparing these formulas one can obtain non-trivial identities among functions of y, p and of the moduli of the surface.

Moreover, one can use (1.21) and (1.22) to deduce information about the spaces of functions generated by the respective polylogarithms. Suppose we denote by $\mathcal{H}(\mathcal{J})$ the algebra of polylogarithms associated with a flat connection $d - \mathcal{J}$, namely the ring of functions on the universal cover of Σ_p generated by the coefficients $\Gamma(\mathfrak{w};x,y)$ in (1.9) (for a fixed punctured Riemann surface Σ_p and any[10] fixed $y \in \tilde{\Sigma}_p$) of the path-ordered

[10]Notice that, by the path-concatenation formula (1.6), changing the base point y does not change the space of polylogarithms $\mathcal{H}(\mathcal{J})$.

exponential $\mathbf{\Gamma}(x,y;c) = \mathrm{P}\exp\int_y^x \mathcal{J}(t;c)$. Then, combining the two gauge transformations, we will prove (see Theorem 5.2 below) the relation

$$\mathcal{H}(\mathcal{J}_{\rm DHS}) \;=\; \mathcal{H}(\mathcal{K}_{\rm E}) \cdot \mathcal{H}(\mathcal{J}^{(0,1)}_{\rm DHS}), \tag{1.23}$$

where $\mathcal{J}^{(0,1)}_{\rm DHS}(x;b) = -\pi\bar\omega^I(x)b_I$ is the (purely anti-holomorphic) $(0,1)$-part of the differential form $\mathcal{J}_{\rm DHS}$. In other words, the polylogarithms generated from the single-valued but non-meromorphic connection $d - \mathcal{J}_{\rm DHS}$ in (2.31) are polynomials in the polylogarithms constructed from the meromorphic connection $d - \mathcal{K}_{\rm E}$ in (2.16) and in the iterated integrals of $\bar\omega^I$, whose coefficients will in general depend on y and on the moduli of Σ_p. Moreover, we deduce from (1.23) that $\mathcal{H}(\mathcal{K}_{\rm E})$ is given by the intersection of $\mathcal{H}(\mathcal{J}_{\rm DHS})$ with the algebra of holomorphic multiple-valued functions (see Corollary 5.3).

1.3 Further directions

The results of this work suggest a variety of follow-up questions and future lines of investigation that should be relevant to both mathematicians and physicists.

First, various iterated integrals over $\mathfrak{A}^I$ and $\mathfrak{B}_I$ cycles that arise as expansion coefficients of the gauge transformations and automorphisms constructed in this work may be viewed as higher-genus analogues of elliptic multiple zeta values [12, 13, 14, 15]. Their detailed structure and interrelations remain to be explored and their evaluation remains to be further simplified beyond the genus-one case. More generally, an improved understanding of the relations among higher-genus multiple zeta values may shed light on the generalization of Tsunogai's derivation algebra [42] and analogues beyond genus one of the elliptic associator [43, 44, 45] (see e.g. [46] for associators at higher genus).

Second, our investigations into the monodromies of higher-genus polylogarithms have implications for the construction of their single-valued counterparts, which is relegated to future work. At genus one already, special values of single-valued analogues of elliptic polylogarithms, known as modular graph functions [47, 48] and modular graph forms [49], are playing a key role in organizing the low energy expansion of string theory (recent overviews may be found in [50, 51]; see also [52, 53]). Number theoretic properties of modular graph forms were further studied in [54, 55, 56, 57, 58]. At higher genus, single-valued versions of the polylogarithms encountered in this work generalize the modular tensors introduced in [59, 60, 61, 62] and the higher genus modular graph forms introduced in [19, 20] to depend on various marked points on the surface. They provide powerful tools for string-amplitude computations, offer a novel perspective on the Fay identities for products of Szegö kernels [22, 63, 64] and provide promising new angles on the theory of single-valued periods [65, 66].

Organization

The remainder of this paper is organized as follows. Section 2 is dedicated to a review of flat connections on Riemann surfaces of arbitrary genus. We discuss the meromorphic multiple-valued connection $d - \mathcal{K}_{\mathrm{E}}$ introduced in [37], and the non-meromorphic but single-valued and modular-invariant connection $d - \mathcal{J}_{\mathrm{DHS}}$ of [17], and their respective restriction to genus-one surfaces. In section 3 and section 4 we present the two announced constructions of a relation between $d - \mathcal{K}_{\mathrm{E}}$ and $d - \mathcal{J}_{\mathrm{DHS}}$ given by the composition of a gauge transformation and an automorphism of the Lie algebra $\mathfrak{g}$. In section 5 we relate the two associated spaces of polylogarithms. For some of the results in the main text, the proofs are relegated to appendices A, B and C. Explicit expressions for the relation between the connections are worked out to low orders in appendix D, and the structure of the general construction is provided in appendix E.

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2 Review of the flat connections

Let Σ be a compact Riemann surface of genus h. We denote by $\tilde{\Sigma}$ the universal (simply-connected) cover of Σ, and the associated canonical projection by $\pi : \tilde{\Sigma} \to \Sigma$. For any $p \in \Sigma$, let us consider also the once-punctured surface $\Sigma_p = \Sigma \setminus \{p\}$. Its fundamental group $\pi_1(\Sigma_p, y)$ of Σ_p is freely generated by $\mathfrak{A}^I, \mathfrak{B}_I$, $I = 1, \cdots, h$, which are $2h$ closed loops in Σ based at y that do not contain the point p. We assume that, when viewed as generators of $\pi_1(\Sigma, y)$, they satisfy the relation

$$\prod_{I=1}^{h} \mathfrak{A}^I \star \mathfrak{B}_I \star (\mathfrak{A}^I)^{-1} \star (\mathfrak{B}_I)^{-1} = 1. \tag{2.1}$$

Let $\tilde{\Sigma}_p$ denote the universal cover of Σ_p. Then, a choice of preferred pre-image of $y \in \Sigma_p$ in $\tilde{\Sigma}_p$ induces a canonical identification of the fundamental group $\pi_1(\Sigma_p, y)$ with the automorphism group $\mathrm{Aut}(\tilde{\Sigma}_p/\Sigma_p)$. This canonical identification sets the action of an element $\gamma \in \pi_1(\Sigma_p, y)$ on $\tilde{\Sigma}_p$ (see footnote 6). The image of $x \in \tilde{\Sigma}_p$ under γ will be denoted by $\gamma \cdot x$. The preferred pre-image of y in $\tilde{\Sigma}_p$, as well as its image in $\tilde{\Sigma} \setminus \pi^{-1}(p)$ via the the natural map $\tilde{\Sigma}_p \to \tilde{\Sigma} \setminus \pi^{-1}(p)$, will also be denoted by y, and is part of the topological setup of our construction. Part of this setting is illustrated in figure 1 for a surface of genus two.

The homology groups $H_1(\Sigma, \mathbb{Z})$ and $H_1(\Sigma_p, \mathbb{Z})$ are both isomorphic to $\mathbb{Z}^{2h}$, and one can choose the loops $\mathfrak{A}^I, \mathfrak{B}_I$ in such a way that their image in $H_1(\Sigma, \mathbb{Z}) = \pi_1^{\mathrm{ab}}(\Sigma, y)$ (resp. $H_1(\Sigma_p, \mathbb{Z}) = \pi_1^{\mathrm{ab}}(\Sigma_p, y)$) is a symplectic basis with respect to the canonical intersection pairing $\mathfrak{J}$, i.e. $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{A}^J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ and $\mathfrak{J}(\mathfrak{A}^I, \mathfrak{B}_J) = \delta^I_J$ for $I, J = 1, \cdots, h$. The symplectic group $\mathrm{Sp}(2h, \mathbb{Z})$ takes symplectic bases to symplectic bases as follows,[11]

$$M : \begin{cases} \mathfrak{B}_I \to A_I{}^J \, \mathfrak{B}_J + B_{IJ} \, \mathfrak{A}^J \\ \mathfrak{A}^I \to C^{IJ} \, \mathfrak{B}_J + D^I{}_J \, \mathfrak{A}^J \end{cases} \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2h, \mathbb{Z}). \tag{2.2}$$

The choice of a symplectic basis of $H_1(\Sigma, \mathbb{Z})$ induces a canonical choice of representatives for the h generators of $H^{1,0}(\Sigma)$, namely the holomorphic Abelian differentials ω_I, with $I = 1, \cdots, h$. These differentials are normalized on the $\mathfrak{A}^J$ cycles, and their integrals on the $\mathfrak{B}_J$ cycles give the components of the period matrix Ω:

$$\oint_{\mathfrak{A}^J} \omega_I = \delta^J_I, \qquad \oint_{\mathfrak{B}_J} \omega_I = \Omega_{IJ}, \qquad Y_{IJ} = \mathrm{Im}\,(\Omega_{IJ}). \tag{2.3}$$

[11]We reiterate that, unless otherwise indicated, we will follow the Einstein convention in which a pair of identical upper and lower indices are summed over, without writing the summation sign. Indices may be lowered or raised with the help of the *metric* $Y = \mathrm{Im}\,(\Omega)$, whose components Y_{IJ} are defined in (2.3), or its inverse Y^{-1}, whose components are denoted by Y^{IJ}. Following these conventions we have, for example, $\omega_I = Y_{IJ}\, \omega^J$, $\omega^I = Y^{IJ} \omega_J$, $\overline{\omega}^I = Y^{IJ}\, \overline{\omega}_J$ and $Y_{IJ} Y^{JK} = \delta^K_I$.

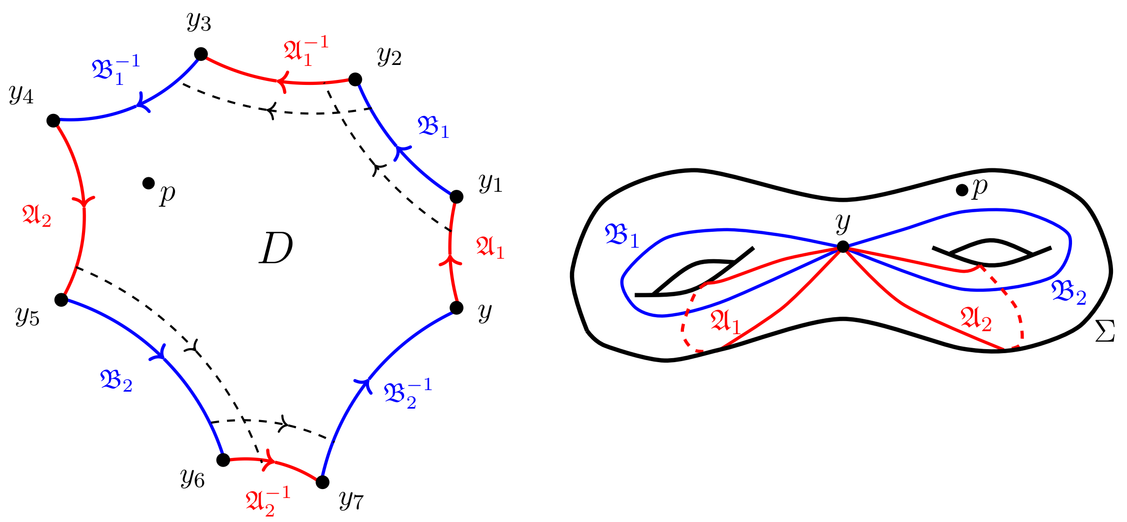

Figure 1: The left panel represents a genus-two Riemann surface Σ in terms of a fundamental domain $D \subset \tilde{\Sigma}$ for the action of $\mathrm{Aut}(\tilde{\Sigma}/\Sigma) \simeq \pi_1(\Sigma, y)$, which can be obtained by cutting Σ along the cycles in the right panel. The surface Σ may be reconstructed from D by pairwise identifying inverse boundary components with one another under the dashed arrows; the projection $\pi : \tilde{\Sigma} \to \Sigma$ maps all the vertices y and y_i for $i = 1, \cdots, 7$ of D to the same point y in Σ. The points $y_i \in \tilde{\Sigma}$ are related to $y \in \tilde{\Sigma}$ by $y_1 = \mathfrak{A}_1 \cdot y$, $y_2 = \mathfrak{B}_1 \cdot y_1$, $y_3 = \mathfrak{A}_1^{-1} \cdot y_2$, $y_4 = \mathfrak{B}_1^{-1} \cdot y_3$, $y_5 = \mathfrak{A}_2 \cdot y_4$, $y_6 = \mathfrak{B}_2 \cdot y_5$ and $y_7 = \mathfrak{A}_2^{-1} \cdot y_6$, the product of loops being understood here as a composition of the corresponding elements in $\mathrm{Aut}(\tilde{\Sigma}/\Sigma)$ with $\mathfrak{B}_2^{-1} \cdot y_7 = y$ in view of (2.1).

By the Riemann relations we have $\Omega^t = \Omega$ and the matrix $Y = \mathrm{Im}\,(\Omega)$ is positive definite. Equivalently, the following pairing holds,

$$\frac{i}{2} \int_\Sigma \omega_J \wedge \overline{\omega}^I = \delta^I_J, \tag{2.4}$$

where, as mentioned in the footnote 11,

$$\overline{\omega}^I = Y^{IJ}\, \overline{\omega}_J. \tag{2.5}$$

Under a modular transformation $M \in \mathrm{Sp}(2h, \mathbb{Z})$ of (2.2), the row matrix ω of holomorphic Abelian differentials and the period matrix Ω transform by $M : \omega \to \omega(C\Omega + D)^{-1}$ and $M : \Omega \to (A\Omega + B)(C\Omega + D)^{-1}$, respectively.

We recall from section 1 that the Lie algebra $\mathfrak{g}$, in which the connections that we will consider take their values, is freely generated by a set of $2h$ elements $a \cup b$ where

$a = \{a^1, \cdots, a^h\}$ and $b = \{b_1, \cdots, b_h\}$ and is completed with respect to the degree, while the (completed) sub-algebra of $\mathfrak{g}$ that is freely generated by the elements of b alone is denoted $\mathfrak{g}_b$. In the remainder of this section, we will review the construction of the two $\mathfrak{g}$-valued connections $d - \mathcal{K}_{\rm E}$ and $d - \mathcal{J}_{\rm DHS}$, and compare them with their more classical genus-one analogues. More precisely, in sections 2.1 and 2.2 we will uniquely characterize $\mathcal{K}_{\rm E}$ and $\mathcal{J}_{\rm DHS}$, respectively, through their functional properties, and review some further features. In section 2.3 we will compare them with their genus-one analogues, introduced by Calaque–Enriquez–Etingof and Brown–Levin, respectively.

2.1 The Enriquez connection $d - \mathcal{K}_{\mathbf{E}}$

In this subsection we shall give the definition and list some basic properties of the Enriquez connection $d - \mathcal{K}_{\rm E}$ needed in this paper. The following result, which defines $\mathcal{K}_{\rm E}$ through a functional characterization, essentially follows from [37] but was never stated in this form. We therefore include also its proof for completeness. The flatness of the corresponding connection trivially follows from the meromorphicity of $\mathcal{K}_{\rm E}$.

Theorem 2.1. *For any fixed $p \in \Sigma$ there exists a unique differential form (in the variable x) $\mathcal{K}_{\rm E}(x, p; a, b)$ which is multiple-valued on Σ, meromorphic on $\tilde{\Sigma}$ with simple poles at all points in $\pi^{-1}(p)$ and holomorphic elsewhere, takes values in $\mathfrak{g}$ and satisfies:*

1. *the monodromy conditions*

$$\begin{aligned} \mathcal{K}_{\rm E}(\mathfrak{A}^K \cdot x, p; a, b) &= \mathcal{K}_{\rm E}(x, p; a, b), \\ \mathcal{K}_{\rm E}(\mathfrak{B}_K \cdot x, p; a, b) &= e^{-2\pi i b_K} \mathcal{K}_{\rm E}(x, p; a, b)\, e^{2\pi i b_K}; \end{aligned} \tag{2.6}$$

2. *the residue condition (where a preferred pre-image of p in $\tilde{\Sigma}$ is also denoted by p)*[12]

$$\operatorname{Res}_{x=p} \mathcal{K}_{\rm E}(x, p; a, b) = [b_I, a^I]; \tag{2.7}$$

3. *it is linear in the generators a^I.*

Proof. It follows from [37], Lemma[13] 6, that for fixed $p \in \Sigma$ there exists a family of multiple-valued differentials (in the variable x) $g^{I_1 \cdots I_r}{}_J(x, p)$, meromorphic on $\tilde{\Sigma}$ with

[12]The residues of the poles at the points in $\pi^{-1}(p)$, other than p itself, may be obtained by combining (2.7) with the monodromy relations of (2.6).

[13]Condition (a) in the original reference (which is (2.11) in this work) is not necessary, as it follows from the other properties (see our proof of the uniqueness of the family).

simple poles at (a subset of) points in $\pi^{-1}(p)$ and holomorphic elsewhere, which satisfy the monodromy conditions[14]

$$\begin{aligned} g^{I_1\cdots I_r}{}_J(\mathfrak{A}^K\cdot x,p) &= g^{I_1\cdots I_r}{}_J(x,p),\\ g^{I_1\cdots I_r}{}_J(\mathfrak{B}_K\cdot x,p) &= g^{I_1\cdots I_r}{}_J(x,p)+\sum_{s=1}^{r}\frac{(-2\pi i)^s}{s!}\,\delta_K^{I_1\cdots I_s}\,g^{I_{s+1}\cdots I_r}{}_J(x,p), \end{aligned} \tag{2.8}$$

and which are holomorphic at (the preferred pre-image of) p for $r\geq 2$ (see footnote 12), whereas for $r=1$ they satisfy the residue condition

$$\mathrm{Res}_{x=p}\, g^I{}_J(x,p)=\delta^I_J. \tag{2.9}$$

It follows that, if we define $\mathcal{K}_{\mathrm{E}}$ as the $\mathfrak{g}$-valued generating series of this family,

$$\mathcal{K}_{\mathrm{E}}(x,p;a,b)=\sum_{r=0}^{\infty} g^{I_1\cdots I_r}{}_J(x,p)B_{I_1}\cdots B_{I_r}a^J, \tag{2.10}$$

where we set $B_I X=[b_I,X]$ for arbitrary $X\in\mathfrak{g}$, then $\mathcal{K}_{\mathrm{E}}$ satisfies all the required properties, which proves the existence part of the statement.

For the uniqueness, notice first that, by properties of Lie brackets, linearity in the generators a^I is equivalent for $\mathcal{K}_{\mathrm{E}}$ to have an expansion like (2.10), hence it is enough to verify the uniqueness of a family of differentials $g^{I_1\cdots I_r}{}_J(x,p)$ as above. By Cauchy's residue theorem, one can prove, using (2.8) and (2.9), that

$$\oint_{\mathfrak{A}^K} g^{I_1\cdots I_r}{}_J(t,p)=(-2\pi i)^r\frac{B_r}{r!}\,\delta_J^{I_1\cdots I_r K}, \tag{2.11}$$

where $\mathfrak{A}^K$ denote the pre-images of the $\mathfrak{A}$-cycles in a fundamental domain $D\subset\tilde{\Sigma}$ containing p (see figure 1) and B_r denote the Bernoulli numbers. For $r=0$ the coefficients

$$g_J(x,p)=\omega_J(x), \tag{2.12}$$

are the holomorphic Abelian differentials on Σ. This implies that this family is uniquely determined by (2.8) and (2.9), which in turn implies the uniqueness of $\mathcal{K}_{\mathrm{E}}(x,p;a,b)$. □

Remark 2.2. *We have departed from the conventions in the original reference [37], as well as [38] and [18], by factors of $-2\pi i$ to align the genus-one instance of the Enriquez*

[14]Here the generalized Kronecker symbol $\delta_K^{I_1\cdots I_s}$ is defined for arbitrary $s\geq 1$ by the product of standard Kronecker symbols: $\delta_K^{I_1\cdots I_s}=\delta_K^{I_1}\cdots\delta_K^{I_s}$.

connections and its expansion coefficients with the common conventions of the particle-physics and string-theory literature reviewed in section 2.3 below. More specifically, the conventions of this work are obtained by rescaling the generators $b_I \to -2\pi i b_I$ in[15] *[38, 18] (while leaving the a^J unchanged) which leads to slightly different conditions for the monodromies* (2.6) *and the residue* (2.7)*, and to the dictionary*

$$g^{I_1\cdots I_r}{}_J(x,p) = (-2\pi i)^r \omega^{I_1\cdots I_r}{}_J(x,p)$$

between the differentials $g^{I_1\cdots I_r}{}_J(x,p)$ in (2.10) and the $\omega^{I_1\cdots I_r}{}_J(x,p)$ in [37].

Remark 2.3. *It will be useful in the sequel to rephrase the residue conditions* (2.7) *and* (2.9) *in terms of distributions. Consider the Dirac δ-function $\delta(x,y)$ which is of type* $(1,1)$ *in x and type* $(0,0)$ *in y, normalized by $\int_\Sigma \delta(x,y)\phi(x) = \phi(y)$ for an arbitrary scalar test function ϕ. It is given in local coordinates by $\delta(x,y) = \frac{i}{2} dx \wedge d\bar{x}\, \delta^{(2)}(x,y)$, where $\delta^{(2)}(x,y)$ is the standard coordinate δ-function. Then* (2.7) *is equivalent to*[16]

$$\bar{\partial}_x \, \mathcal{K}_E(x,p;a,b) = 2\pi i \, [b_I, a^I] \, \delta(x,p), \tag{2.13}$$

and (2.9) *is equivalent to*

$$\bar{\partial}_x \, g^I{}_J(x,p) = 2\pi i \, \delta^I_J \, \delta(x,p). \tag{2.14}$$

Corollary 2.4 (See [37], Lemma 9)**.** *For $r = 0$ or if $I_r \neq J$ the differential form $g^{I_1\cdots I_r}{}_J(x,p)$ is independent of p, otherwise it is a meromorphic multiple-valued function of p, whose monodromies are given by*

$$\begin{aligned} g^{I_1\cdots I_r}{}_J(x, \mathfrak{A}^K \cdot p) &= g^{I_1\cdots I_r}{}_J(x,p), \\ g^{I_1\cdots I_r}{}_J(x, \mathfrak{B}_K \cdot p) &= g^{I_1\cdots I_r}{}_J(x,p) + \delta^{I_r}_J \sum_{s=0}^{r-1} \frac{(2\pi i)^{r-s}}{(r-s)!} \, g^{I_1\cdots I_s}{}_K(x,p) \, \delta_K^{I_{s+1}\cdots I_{r-1}}. \end{aligned} \tag{2.15}$$

[15] The notation used for these generators in the original reference [37] is actually $x_1, \ldots, x_h$, and the translation into our notation is $x_I = -2\pi i b_I$. Also, rather than considering a $\mathfrak{g}$-valued differential with a simple pole at p, one considers in [37] a differential valued in the (completion of) the quotient $\mathrm{Lie}(a,b)/[a^I, b_I]$, which turns out to be independent of p, but whose analytic construction is exactly the same as that presented here. One may also think of the differential $\mathcal{K}_{\mathrm{E}}$ as obtained as the restriction from $\Sigma^2 \setminus \{z_1 = z_2\}$ to Σ_p of the two-variable version of the Enriquez connection of [37], fixing the second component to $z_2 = p$ and restricting the target Lie algebra accordingly.

[16] Throughout, we shall use the notations $\partial_x = dx\, \partial/\partial x$ and $\bar{\partial}_x = d\bar{x}\, \partial/\partial \bar{x}$, so that the total differential is given by $d_x = \partial_x + \bar{\partial}_x$.

Following (1.4) and (1.9), the path-ordered exponential and its expansion in terms of words $\mathfrak{w}$ in non-commutative letters from the set $a \cup b$ and associated polylogarithms $\Gamma_{\rm E}(\mathfrak{w}; x, y, p)$ for the connection $\mathcal{K}_{\rm E}$ are given by

$$\boldsymbol{\Gamma}_{\rm E}(x, y, p; a, b) = {\rm P}\exp \int_y^x \mathcal{K}_{\rm E}(t, p; a, b) = \sum_{\mathfrak{w} \in \mathcal{W}(a \cup b)} \mathfrak{w}\, \Gamma_{\rm E}(\mathfrak{w}; x, y, p). \tag{2.16}$$

The resulting Enriquez polylogarithms $\Gamma_{\rm E}(\mathfrak{w}; x, y, p)$ are multiple-valued functions of x, y, $p \in \Sigma$, which for certain choices of $\mathfrak{w}$ have a logarithmic singularities at $x = p$ and $y = p$. They can be straightforwardly expressed in terms of the iterated integrals introduced in [18, 63]

$$\tilde{\Gamma}\left(\begin{smallmatrix} \vec{I}_1 & \vec{I}_2 & \cdots & \vec{I}_\ell \\ J_1 & J_2 & \cdots & J_\ell \\ p_1 & p_2 & \cdots & p_\ell \end{smallmatrix} ; x, y \right) = \int_y^x dt\, g^{\vec{I}_1}{}_{J_1}(t, p_1)\, \tilde{\Gamma}\left(\begin{smallmatrix} \vec{I}_2 & \cdots & \vec{I}_\ell \\ J_2 & \cdots & J_\ell \\ p_2 & \cdots & p_\ell \end{smallmatrix} ; t, y \right), \qquad \tilde{\Gamma}\left(\begin{smallmatrix} \emptyset \\ \emptyset \\ \emptyset \end{smallmatrix} ; x, y \right) = 1 \tag{2.17}$$

upon specializing $p_1 = p_2 = \cdots = p_\ell = p$, where the multi-indices $\vec{I}_j = I_j^1 I_j^2 \ldots I_j^r$ may be empty to recover the integration kernels $\omega_J(t)$. It should be possible to define by tangential base point regularization [67, 5, 68] also the value at $x = p$ or $y = p$, where the integral is otherwise logarithmically divergent, see [9, 14, 11] for the genus-one case.

Formulas for the integration kernels $g^{I_1 \cdots I_r}{}_J(x, p)$ of the higher-genus polylogarithms $\Gamma_{\rm E}(\mathfrak{w}; x, y, p)$ in terms of the fundamental form of the third kind and of iterated integrals of Abelian differentials, or in terms of averages on the Schottky cover in the case of real hyperelliptic curves, can be deduced from [38] (see section 5) and [18], respectively.

2.2 The DHS connection $d - \mathcal{J}_{\rm DHS}$

The DHS connection $\mathcal{J}_{\rm DHS}(x, p; a, b)$ is a single-valued smooth differential form in $x \in \Sigma_p$, which is the sum of a $(1, 0)$-form $\mathcal{J}^{(1,0)}_{\rm DHS}$ that has a regular singularity at $x = p$, and a $(0, 1)$-form $\mathcal{J}^{(0,1)}_{\rm DHS}$ that is purely anti-holomorphic and single-valued on the whole Σ. The connection $\mathcal{J}_{\rm DHS}(x, p; a, b)$ takes values in the Lie algebra $\mathfrak{g}$. It was defined in [17] via an expansion, similar to the one given for $\mathcal{K}_{\rm E}$ in (2.10), which we shall repeat below, and may also be characterized by its functional properties as follows.

Theorem 2.5. *For any fixed $p \in \Sigma$ there exists a unique differential form (in the variable x) $\mathcal{J}_{\rm DHS}(x, p; a, b)$ which is smooth and single-valued on Σ_p, takes values in $\mathfrak{g}$, and satisfies:*

1. *the Maurer–Cartan equation for $x \neq p$, and it has a simple pole in x at p:*

$$d_x \mathcal{J}_{\rm DHS}(x, p; a, b) - \mathcal{J}_{\rm DHS}(x, p; a, b) \wedge \mathcal{J}_{\rm DHS}(x, p; a, b) = 2\pi i\, \delta(x, p)\, [b_I, a^I]; \tag{2.18}$$

2. *its* $(0,1)$ *component is given by* $\mathcal{J}^{(0,1)}_{\rm DHS}(x,p;a,b) = -\pi b_I \,\overline{\omega}^I(x)$ *with* $\overline{\omega}^I$ *given in (2.4);*
3. *its* $(1,0)$ *component* $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b)$ *is linear in the generators* a^J*.*

Proof. The existence follows from the explicit construction of [17] of a family of differentials $f^{I_1\cdots I_r}{}_J$ whose generating series, similar to the expansion of $\mathcal{K}_{\rm E}$ in (2.10), is a $(1,0)$ differential $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b)$ which satisfies the required properties. Let us show here how the properties in the statement uniquely characterize the construction of [17]. The combination of item 1 and item 2 gives a differential equation for the $(1,0)$ component,

$$\bar\partial_x \,\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b) + \pi\,\overline{\omega}^I(x)\wedge \big[b_I, \mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b)\big] = 2\pi i\,\delta(x,p)\,[b_I,a^I]. \tag{2.19}$$

Linearity of $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b)$ in the generators of a, as prescribed by item 3, combined with the structure of the Maurer–Cartan equation given in (2.19) is equivalent to the condition that $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b)$ admits the following expansion

$$\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b) = \omega_J(x)a^J + \sum_{r=1}^\infty f^{I_1\cdots I_r}{}_J(x,p)B_{I_1}\cdots B_{I_r}a^J, \tag{2.20}$$

where we recall that $B_I X = [b_I, X]$ for $X\in\mathfrak{g}$, and where $f^{I_1\cdots I_r}{}_J(x,p)$ are $(1,0)$ forms in x and scalars in p that are single-valued for $(x,p)\in\Sigma\times\Sigma$, with a simple pole at $x=p$ for $r=1$ and smooth otherwise. The equation (2.19) translates into the following set of differential equations for $f^{I_1\cdots I_r}{}_J(x,p)$, referred to as a Massey system,

$$\begin{aligned} \bar\partial_x f^I{}_J(x,p) &= -\pi\,\overline{\omega}^I(x)\wedge\omega_J(x) + 2\pi i\,\delta^I_J\,\delta(x,p), \\ \bar\partial_x f^{I_1\cdots I_r}{}_J(x,p) &= -\pi\,\overline{\omega}^{I_1}(x)\wedge f^{I_2\cdots I_r}{}_J(x,p), \qquad\qquad r\geq 2. \end{aligned} \tag{2.21}$$

Note that integrability of the Massey system requires that $f^{I_1\cdots I_r}{}_J(x,p)\in \mathrm{Range}(\partial_x)$ for all $r\geq 1$ since the integral of the left side of each equation vanishes and therefore so must the right side. The solution of this system is the content of the following lemma.

Lemma 2.6. *The solution for* $f^I{}_J(x,p)$ *of the first equation in the Massey system of (2.21) is unique and given in terms of the Arakelov Green function* $\mathcal{G}(x,y)$ *as follows,*

$$f^I{}_J(x,p) = \partial_x\int_\Sigma \mathcal{G}(x,t)\Big(-\frac{i}{2}\,\overline{\omega}^I(t)\wedge\omega_J(t) - \delta^I_J\,\delta(t,p)\Big). \tag{2.22}$$

The solution for $f^{I_1\cdots I_r}{}_J(x,p)$ *with* $r\geq 2$ *of the second equation of the Massey system of (2.21) is unique and given recursively in* r *by the absolutely convergent integrals for* $x\neq p$

$$f^{I_1\cdots I_r}{}_J(x,p) = \partial_x\int_\Sigma \mathcal{G}(x,t)\Big(-\frac{i}{2}\,\overline{\omega}^{I_1}(t)\wedge f^{I_2\cdots I_r}{}_J(t,p)\Big). \tag{2.23}$$

Proof of the lemma. To prove the lemma, we use the Arakelov Green function $\mathcal{G}(x,y)$, which was introduced in mathematics in [69] and applied in physics in [70, 19]. It is the unique real-valued symmetric function of $(x,y) \in \Sigma \times \Sigma$ which is smooth for $y \neq x$ and satisfies the following equations,

$$\bar{\partial}_x \partial_x \, \mathcal{G}(x,y) = 2\pi i \Big(\kappa(x) - \delta(x,y) \Big), \qquad \qquad \int_\Sigma \kappa(t) \, \mathcal{G}(t,y) = 0, \tag{2.24}$$

where κ is the canonical volume form on Σ, given by,

$$\kappa = \frac{i}{2h} \, \omega_I \wedge \overline{\omega}^I, \qquad \qquad \int_\Sigma \kappa = 1. \tag{2.25}$$

Thus, $\mathcal{G}(x,y)$ is a smooth function away from the diagonal $x = y$, where it has a logarithmic singularity given by $\mathcal{G}(x,y) = -\ln |x-y|^2 + \text{regular}$, as a result of which its differential has a simple pole,

$$\partial_x \, \mathcal{G}(x,y) = -\frac{dx}{x-y} + \text{regular}, \tag{2.26}$$

and the integrals in (2.23) are absolutely convergent for $x \neq p$. The limit $x \to p$ of (2.23) and its regularization in case of $r = 2$ were discussed in section 8 of [63].

Returning to establishing the solution to the Massey system, we readily obtain the solution (2.22) to the equation for $r = 1$ in terms of $\mathcal{G}(x,y)$ since the right side of the first equation in (2.21) integrates to zero. By construction, the solution for $f^I{}_J(x,p)$ given in (2.22) belongs to the range of the operator ∂_x, as argued already earlier. As a result the integral over x of the right side of the $r = 2$ equation in (2.23) vanishes, so that one may solve the $r = 2$ equation for $f^{I_1 I_2}{}_J(x,p)$ in terms of the Arakelov Green function as well. By induction on r one establishes (2.23) for all values of r. This concludes the proof of Lemma 2.6. □

The family of differentials $f^{I_1 \cdots I_r}{}_J(x,p)$ obtained from the lemma, which coincides with the family originally defined in [17], is therefore uniquely determined by items 1–3, thus concluding the proof of the theorem. □

Remark 2.7. *Note that linearity of $\mathcal{J}^{(1,0)}_{\text{DHS}}(x,p;a,b)$ in a^J which is required in item 3 is consistent with the linearity of the right side of (2.18).*

2.2.1 Modular invariance of the DHS connection

The holomorphic Abelian differentials ω_I, and their conjugates $\overline{\omega}^I$ introduced in (2.4), transform under non-linear realizations of the modular group $P, Q : \text{Sp}(2h, \mathbb{Z}) \times \mathfrak{H}_h \to$

GL$(h,\mathbb{C})$, where $\mathfrak{H}_h$ denotes the Siegel upper half-space, whose action is given as follows,

$$\begin{cases}\omega_I \to P(M,\Omega)_I{}^J\omega_J \\ \overline{\omega}^I \to Q(M,\Omega)^I{}_J\,\overline{\omega}^J\end{cases} \qquad \begin{cases}P(M,\Omega) = \big(Q(M,\Omega)^t\big)^{-1} \\ Q(M,\Omega) = C\Omega + D\end{cases} \qquad M = \begin{pmatrix} A & B \\ C & D\end{pmatrix}, \tag{2.27}$$

for $M \in \mathrm{Sp}(2h,\mathbb{Z})$ and $\Omega \in \mathfrak{H}_h$. The composition law for $M_1, M_2 \in \mathrm{Sp}(2h,\mathbb{Z})$ is given by,

$$\begin{cases}P(M_1M_2,\Omega) = P(M_1,\Omega')P(M_2,\Omega) \\ Q(M_1M_2,\Omega) = Q(M_1,\Omega')Q(M_2,\Omega)\end{cases} \qquad \Omega' = (A_2\Omega + B_2)(C_2\Omega + D_2)^{-1}. \tag{2.28}$$

The equations defining the Arakelov Green function in (2.24) are modular invariant, and so is $\mathcal{G}(x,y)$. Therefore, the functions $f^{I_1\cdots I_r}{}_J(x,p)$ are actually modular tensors in the sense introduced in [62], whose transformation law may be deduced from that of the Abelian differentials in (2.27),

$$M : f^{I_1\cdots I_r}{}_J(x,p) \to Q^{I_1}{}_{K_1}\cdots Q^{I_r}{}_{K_r}P_J{}^L f^{K_1\cdots K_r}{}_L(x,p), \tag{2.29}$$

where we have abbreviated $P = P(M,\Omega)$ and $Q = Q(M,\Omega)$ for $M \in \mathrm{Sp}(2h,\mathbb{Z})$. Note that the dependence of both $P(M,\Omega)$ and $Q(M,\Omega)$ on Ω is holomorphic, as is the entire transformation factor of the modular tensors $f^{I_1\cdots I_r}{}_J(x,p)$. The factors P and Q and their tensor products generalize the automorphy factors $(c\tau+d)^k$ associated with $\mathrm{SL}(2,\mathbb{Z})$ modular transformations of genus one.

The result may be summarized by the following proposition (see Theorem 3.2 of [17]).

Proposition 2.8. *The connection $\mathcal{J}_{\rm DHS}(x,p;a,b)$ is invariant under the action of the modular group Sp$(2h,\mathbb{Z})$ provided the generators of the Lie algebra $\mathfrak{g}$ transform as follows,*

$$M : a^I \to Q(M,\Omega)^I{}_J\, a^J, \qquad\qquad M : b_I \to P(M,\Omega)_I{}^J\, b_J, \tag{2.30}$$

where $P(M,\Omega)$ and $Q(M,\Omega)$ are defined in (2.27) for $M \in$ Sp$(2h,\mathbb{Z})$.

2.2.2 Modular properties of the DHS polylogarithms

Following (1.4) and (1.9), the path-ordered exponential and its expansion in terms of words $\mathfrak{w}$ in non-commutative letters from the set $a \cup b$ and associated polylogarithms $\Gamma_{\rm DHS}(\mathfrak{w};x,y,p)$ for the connection $\mathcal{J}_{\rm DHS}$ are introduced as follows,

$$\mathbf{\Gamma}_{\rm DHS}(x,y,p;a,b) = \mathrm{P}\exp\int_y^x \mathcal{J}_{\rm DHS}(t,p;a,b) = \sum_{\mathfrak{w}\,\in\,\mathcal{W}(a\cup b)} \mathfrak{w}\,\Gamma_{\rm DHS}(\mathfrak{w};x,y,p). \tag{2.31}$$

The implications for the modular transformation law of the polylogarithms $\Gamma_{\rm DHS}(\mathfrak{w};x,y,p)$ are summarized by the following proposition (see section 4.4 of [17]).

Proposition 2.9. *The polylogarithms $\Gamma_{\rm DHS}(\mathfrak{w};x,y,p)$ associated with words composed of the alphabet $a\cup b$ with $a=\{a^1,\cdots,a^h\}$ and $b=\{b_1,\cdots,b_h\}$ map to modular tensors,*

$$\Gamma_{\rm DHS}(a^{I_1}\cdots a^{I_m}b_{J_1}\cdots b_{J_n}\cdots;x,y,p)=\Gamma_{I_1\cdots I_m}{}^{J_1\cdots J_n}{}_{\cdots}{}^{\cdots}(x,y,p), \tag{2.32}$$

whose transformation law under $M\in Sp(2h,\mathbb{Z})$ is given by,

$$M:\ \Gamma_{I_1\cdots I_m}{}^{J_1\cdots J_n}{}_{\cdots}{}^{\cdots}\to P_{I_1}{}^{K_1}\cdots P_{I_m}{}^{K_m}Q^{J_1}{}_{L_1}\cdots Q^{J_n}{}_{L_n}\cdots\Gamma_{K_1\cdots K_m}{}^{L_1\cdots L_n}{}_{\cdots}{}^{\cdots}, \tag{2.33}$$

where again $P=P(M,\Omega)$ and $Q=Q(M,\Omega)$ for brevity.

2.3 Restriction to genus one

We follow the customary notation $\tau=\Omega_{11}$ for the restriction of the period matrix (2.3) to genus $h=1$, and we identify Σ with its Jacobian, the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for $\operatorname{Im}\tau>0$. The restriction of both the Enriquez connection $\mathcal{K}_{\rm E}$ and the DHS connection $\mathcal{J}_{\rm DHS}$ to genus one can be explicitly expressed in terms of the odd Jacobi theta function,

$$\vartheta_1(x)=2q^{1/8}\sin(\pi x)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi ix}q^n)(1-e^{-2\pi ix}q^n)\,,\qquad q=e^{2\pi i\tau}. \tag{2.34}$$

Specifically, the genus-one connections involve the Kronecker function $F(x;\alpha)$ (also known as Kronecker-Eisenstein series), which can be defined as [71, 72]

$$F(x;\alpha)=\frac{\vartheta_1'(0)\vartheta_1(x+\alpha)}{\vartheta_1(x)\vartheta_1(\alpha)}. \tag{2.35}$$

The function $F(x;\alpha)$ is meromorphic on $\mathbb{C}\times\mathbb{C}$ and, viewed as a function on $\Sigma\times\Sigma$, is multiple-valued and has the following monodromies in the variable x,

$$F(x+1;\alpha)=F(x;\alpha),\qquad\qquad F(x+\tau;\alpha)=e^{-2\pi i\alpha}F(x;\alpha). \tag{2.36}$$

These monodromies cancel if one considers the modified version[17]

$$\Omega(x;\alpha)=\exp\left(2\pi i\,\frac{\operatorname{Im}x}{\operatorname{Im}\tau}\,\alpha\right)F(x;\alpha), \tag{2.37}$$

which is doubly periodic but non-meromorphic in x. The Laurent expansions at $\alpha=0$ of (2.35) and its single-valued analogue (2.37) produce the integration kernels $g^{(r)}$ and $f^{(r)}$, respectively:

$$F(x;\alpha)=\sum_{r=0}^{\infty}\alpha^{r-1}g^{(r)}(x),\qquad \Omega(x;\alpha)=\sum_{r=0}^{\infty}\alpha^{r-1}f^{(r)}(x), \tag{2.38}$$

[17]The notation $\Omega(x;\alpha)$ is customary and not to be confused with the period matrix.

with $g^{(0)}(x) = f^{(0)}(x) = 1$, $g^{(1)}(x) = \frac{\partial}{\partial x} \log \vartheta_1(x)$ and $f^{(1)}(x) = \frac{\partial}{\partial x} \log \vartheta_1(x) + 2\pi i \frac{\mathrm{Im}\, x}{\mathrm{Im}\, \tau}$.

At genus one, the Lie algebra $\mathfrak{g}$ in which the Enriquez and DHS connections take values is generated by two elements a and b, with $a = a^1$ and $b = b_1$ in our earlier notation. In terms of the Kronecker function defined in (2.35), the Enriquez and DHS connections in Theorems 2.1 and 2.5 reduce to the following expressions at genus one,

$$\begin{aligned} \mathcal{K}_{\mathrm{E}}(x,p;a,b)\Big|_{h=1} &= dx\, F(x-p;B)Ba, \\ \mathcal{J}_{\mathrm{DHS}}(x,p;a,b)\Big|_{h=1} &= dx\, \Omega(x-p;B)Ba - d\bar{x}\, \frac{\pi\, b}{\mathrm{Im}\, \tau}, \end{aligned} \tag{2.39}$$

where $BX = [b, X]$ for any $X \in \mathfrak{g}$. The right side of the first line is a connection introduced and generalized to multiple variables by[18] Calaque–Enriquez–Etingof [43], whereas the right side of the second line coincides (upon adding $dx\, \pi\, b/\mathrm{Im}\, \tau$) with a connection introduced and generalized to multiple variables by Brown–Levin [8]. The Enriquez kernels $g^{I_1 \cdots I_r}{}_J(x,p)$ introduced in (2.10) and the DHS kernels $f^{I_1 \cdots I_r}{}_J(x,p)$ introduced in (2.20) on a Riemann surface of arbitrary genus, reduce at genus one to the genus-one kernels $g^{(r)}(x-p)$ and $f^{(r)}(x-p)$ introduced in (2.38) as follows,

$$\begin{aligned} g^{I_1 \cdots I_r}{}_J(x,p)\Big|_{h=1} &= g^{(r)}(x-p)\, dx, \\ f^{I_1 \cdots I_r}{}_J(x,p)\Big|_{h=1} &= f^{(r)}(x-p)\, dx. \end{aligned} \tag{2.40}$$

In (2.39) and (2.40) the restrictions to genus one of $\mathcal{K}_{\mathrm{E}}(x,p;a,b)$ and $\mathcal{J}_{\mathrm{DHS}}(x,p;a,b)$ become dependent only on the difference $x - p$ thanks to translation invariance on the torus Σ. Without loss of generality one may fix p to be at the origin of Σ. The monodromy conditions (2.8), (2.15) both specialize to the genus one case as follows,

$$g^{(r)}(x+1) = g^{(r)}(x), \qquad\qquad g^{(r)}(x+\tau) = g^{(r)}(x) + \sum_{s=1}^{r} \frac{(-2\pi i)^s}{s!} g^{(r-s)}(x), \tag{2.41}$$

consistently with the Laurent expansion of (2.36) with respect to α.

The meromorphic multiple-valued connection of Calaque–Enriquez–Etingof and the non-meromorphic single-valued connection of Brown–Levin on the right side of the two equations in (2.39) may be related by a gauge transformation and an automorphism of the Lie algebra $\mathfrak{g}$. This relation provides an explicit realization of the correspondence (1.11) in the introduction. Indeed, one readily verifies that the gauge transformation

$$\mathcal{U}_{\mathrm{BL}}(x) = \exp\left(2\pi i \, \frac{\mathrm{Im}\, x}{\mathrm{Im}\, \tau}\, b \right) \tag{2.42}$$

[18]This connection was independently introduced in the one-variable case by Levin–Racinet [7].

together with the automorphism $\hat{a} = a + \pi b/(\mathrm{Im}\,\tau)$ and $\hat{b} = b$ reproduce the relation

$$\begin{aligned}\mathcal{K}_{\mathrm{E}}(x,p;a,b)\Big|_{h=1} = &\,\mathcal{U}_{\mathrm{BL}}(x{-}p)^{-1}\mathcal{J}_{\mathrm{DHS}}(x,p;\hat{a},\hat{b})\Big|_{h=1}\mathcal{U}_{\mathrm{BL}}(x{-}p) \\ &-\mathcal{U}_{\mathrm{BL}}(x{-}p)^{-1}d_x\,\mathcal{U}_{\mathrm{BL}}(x{-}p),\end{aligned} \tag{2.43}$$

which implies that the two connections are related by

$$d - \mathcal{K}_{\mathrm{E}}(x,p;a,b)\Big|_{h=1} = \mathcal{U}_{\mathrm{BL}}(x{-}p)^{-1}\left(d - \mathcal{J}_{\mathrm{DHS}}(x,p;\hat{a},\hat{b})\Big|_{h=1}\right)\mathcal{U}_{\mathrm{BL}}(x{-}p). \tag{2.44}$$

The anti-holomorphic dependence of (2.42) ensures that the $(0,1)$-form components cancel between the two terms on the right side of (2.43).

3 Gauge transforming $\mathcal{J}_{\rm DHS}$ to $\mathcal{K}_{\rm E}$

In this section, we give the first explicit construction of a gauge transformation and an automorphism of the Lie algebra $\mathfrak{g}$ that relate the connection $d - \mathcal{K}_{\rm E}$ with the connection $d - \mathcal{J}_{\rm DHS}$. More precisely, we will relate the differential forms $\mathcal{K}_{\rm E}(x, p; a, b)$ and $\mathcal{J}_{\rm DHS}(x, p; \hat{a}, \hat{b})$ by a gauge transformation $\mathcal{U}_{\rm DHS}(x, p)$, where a, b and $\hat{a}, \hat{b}$ are two distinct sets of generators of the algebra $\mathfrak{g}$, whose elements are given by

$$
\begin{aligned}
a &= \{a^1, \cdots, a^h\}, & \hat{a} &= \{\hat{a}^1, \cdots, \hat{a}^h\}, \\
b &= \{b_1, \cdots, b_h\}, & \hat{b} &= \{\hat{b}_1, \cdots, \hat{b}_h\}.
\end{aligned}
\tag{3.1}
$$

The replacement $a \cup b \to \hat{a} \cup \hat{b}$ corresponds to an automorphism of $\mathfrak{g}$ whose explicit form we shall construct. The construction of the full relation between $\mathcal{K}_{\rm E}(x, p; a, b)$ and $\mathcal{J}_{\rm DHS}(x, p; \hat{a}, \hat{b})$ may be conveniently decomposed into two parts: first the construction of the gauge transformation $\mathcal{U}_{\rm DHS}(x, p)$ and second the construction of the automorphism. We shall now proceed to each part in turn.

3.1 Construction of the gauge transformation $\mathcal{U}_{\rm DHS}$

To prove the existence of the relation of (1.11), in this section we shall construct a suitable gauge transformation $\mathcal{U}_{\rm DHS}(x, p)$ in terms of the connection $\mathcal{J}_{\rm DHS}$ subject to (1.11), which we repeat here for convenience,

$$
\begin{aligned}
\mathcal{K}_{\rm E}(x, p; a, b) = \,& \mathcal{U}_{\rm DHS}(x, p)^{-1} \mathcal{J}_{\rm DHS}(x, p; \hat{a}, \hat{b})\, \mathcal{U}_{\rm DHS}(x, p) \\
& - \mathcal{U}_{\rm DHS}(x, p)^{-1} d_x\, \mathcal{U}_{\rm DHS}(x, p).
\end{aligned}
\tag{3.2}
$$

The key role of $\mathcal{U}_{\rm DHS}(x, p)$ is to produce the monodromy of $\mathcal{K}_{\rm E}$, given in equation (2.6) of Theorem 2.1, starting from the connection $\mathcal{J}_{\rm DHS}$ which has trivial monodromy. Thus, we seek to construct a gauge transformation with the following monodromy,

$$
\begin{aligned}
\mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot x, p) &= \mathcal{U}_{\rm DHS}(x, p), \\
\mathcal{U}_{\rm DHS}(\mathfrak{B}_K \cdot x, p) &= \mathcal{U}_{\rm DHS}(x, p)\, e^{2\pi i b_K}.
\end{aligned}
\tag{3.3}
$$

To obtain $\mathcal{U}_{\rm DHS}$ in terms of $\mathcal{J}_{\rm DHS}$, we begin by considering the solution $\mathbf{\Gamma}_{\rm DHS}$ to the following differential equation in the variable x,

$$
d_x\, \mathbf{\Gamma}_{\rm DHS}(x, y, p; \xi, \eta) = \mathcal{J}_{\rm DHS}(x, p; \xi, \eta)\, \mathbf{\Gamma}_{\rm DHS}(x, y, p; \xi, \eta),
\tag{3.4}
$$

along with the initial condition $\mathbf{\Gamma}_{\rm DHS}(y, y, p; \xi, \eta) = 1$. Here, ξ and η are taken to be arbitrary elements of $\mathfrak{g}^h$ so that $\mathcal{J}_{\rm DHS}$ and $\mathbf{\Gamma}_{\rm DHS}$ take values in $\mathfrak{g}$ and $\exp(\mathfrak{g})$, respectively. While $\mathcal{J}_{\rm DHS}(x, p; \xi, \eta)$ is single-valued in x on Σ_p, the function $\mathbf{\Gamma}_{\rm DHS}(x, y, p; \xi, \eta)$ is multiple-valued in x, y on Σ_p. Therefore, we shall consider (3.4) for $x, y \in \tilde{\Sigma}_p$, and represent Σ by the fundamental domain D illustrated in figure 1, where the points y in the left and right panels of figure 1 denote the image of y under the natural maps $\tilde{\Sigma}_p \to \tilde{\Sigma} \setminus \pi^{-1}(p)$ and $\tilde{\Sigma}_p \to \Sigma_p$, respectively. The presence of the pole in x at the point p,

$$\mathcal{J}_{\rm DHS}(x,p;\xi,\eta) = \frac{[\eta_J, \xi^J]}{x-p} dx + \text{ regular}, \tag{3.5}$$

implies that, for generic ξ, η, the path-ordered exponential $\mathbf{\Gamma}_{\rm DHS}(x,y,p;\xi,\eta)$ is singular in x at p and has non-trivial monodromy as $x \in D$ circles around p. The solution to (3.4) and the initial condition is given by the path-ordered exponential,

$$\mathbf{\Gamma}_{\rm DHS}(x,y,p;\xi,\eta) = \mathrm{P}\exp \int_y^x \mathcal{J}_{\rm DHS}(t,p;\xi,\eta). \tag{3.6}$$

3.1.1 Implementing the monodromy conditions

In the remainder of this subsection, and in subsection 3.1.2 below, we will develop a systematic method to determine special elements $\hat{\xi}$ and $\hat{\eta}$ of $\mathfrak{g}^h_b$ such that $\mathbf{\Gamma}_{\rm DHS}(x,y,p;\hat{\xi},\hat{\eta})$ specializes for $y = p$ to the desired gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ in (3.2).

To proceed, recall from section 1.1 that the monodromy action of an element $\gamma \in \pi_1(\Sigma_p, y)$ on the end-point $x \in \tilde{\Sigma}_p$ of the path-ordered exponential $\mathbf{\Gamma}_{\rm DHS}$ for arbitrary elements $\xi, \eta \in \mathfrak{g}^h$ is such that

$$\mathbf{\Gamma}_{\rm DHS}(\gamma \cdot x,y,p;\xi,\eta) = \mathbf{\Gamma}_{\rm DHS}(x,y,p;\xi,\eta)\, \mu_{\rm DHS}(\gamma,y,p;\xi,\eta), \tag{3.7}$$

where we set $\mu_{\rm DHS}(\gamma,y,p;\xi,\eta) = \mathbf{\Gamma}_{\rm DHS}(\gamma \cdot y,y,p;\xi,\eta)$, inspired by the notations of (1.7). The monodromy conditions of (3.3) are implemented using the following lemma.

Lemma 3.1. *For any $y \neq p$ there exist unique elements $\hat{\xi}(y,p) = \{\hat{\xi}^1(y,p), \cdots, \hat{\xi}^h(y,p)\}$ and $\hat{\eta}(y,p) = \{\hat{\eta}_1(y,p), \cdots, \hat{\eta}_h(y,p)\}$ of $\mathfrak{g}^h_b$ such that the monodromy of the path-ordered exponential $\mathbf{\Gamma}_{\rm DHS}(x,y,p;\hat{\xi}(y,p),\hat{\eta}(y,p))$ is given by*

$$\begin{aligned}
&\mu_{\rm DHS}\big(\mathfrak{A}^K,y,p;\hat{\xi}(y,p),\hat{\eta}(y,p)\big) = 1, \\
&\mu_{\rm DHS}\big(\mathfrak{B}_K,y,p;\hat{\xi}(y,p),\hat{\eta}(y,p)\big) = e^{2\pi i b_K}.
\end{aligned} \tag{3.8}$$

Before proceeding to the proof of this lemma we notice that, intuitively, its statement is justified by the fact that, given b_K for $K = 1, \cdots, h$, there are $2h$ equations for $2h$ unknowns $\hat{\xi}, \hat{\eta}$.

Proof. By construction, for any γ the element $\mu_{\rm DHS}(\gamma, y, p; \xi, \eta)$ belongs to $\mathbb{C}\langle\langle \xi, \eta \rangle\rangle$, namely, it is a formal series in $2h$ non-commutative variables ξ^I, η_I. The first order in ξ and η is obtained by expanding the path-ordered exponential of (3.7) to first order in $\mathcal{J}_{\rm DHS}$, and then retaining from the latter only its part linear in ξ and η, namely $\mathcal{J}_{\rm DHS}(x, p; \xi, \eta) = \omega_J(x)\xi^J - \pi\,\overline{\omega}^I(x)\eta_I + \mathcal{O}(\xi\eta)$, which gives the following contributions,

$$
\begin{aligned}
\mu_{\rm DHS}(\mathfrak{A}^K, y, p; \xi, \eta) &= 1 - \xi^K + \pi\eta^K + \mathcal{O}(\xi^2, \eta^2, \xi\eta), \\
\mu_{\rm DHS}(\mathfrak{B}_K, y, p; \xi, \eta) &= 1 - \Omega_{KJ}\xi^J + \pi\bar{\Omega}_{KJ}\eta^J + \mathcal{O}(\xi^2, \eta^2, \xi\eta).
\end{aligned} \tag{3.9}
$$

Combining this with the desired monodromy relations of (3.8) leads to simple first order relations between ξ, η and b,

$$
\begin{aligned}
\xi^K - \pi\eta^K + \mathcal{O}(\xi^2, \eta^2, \xi\eta) &= 0, \\
\Omega_{KJ}\xi^J - \pi\bar{\Omega}_{KJ}\eta^J + \mathcal{O}(\xi^2, \eta^2, \xi\eta) &= 2\pi i b_K,
\end{aligned} \tag{3.10}
$$

whose unique solution $\hat{\xi}, \hat{\eta}$ to this order is given by

$$
\hat{\eta}_I(y,p) = b_I + \mathcal{O}(b^2), \qquad \hat{\xi}^I(y,p) = \pi b^I + \mathcal{O}(b^2). \tag{3.11}
$$

This can be used to prove inductively that there is a unique solution $\hat{\xi}(y,p), \hat{\eta}(y,p)$ to all orders whose components belong to $\mathbb{C}\langle\langle b \rangle\rangle$. The fact that these components actually belong to the subset $\mathfrak{g}_b$ of $\mathbb{C}\langle\langle b \rangle\rangle$ follows from the fact that both sides of (3.8) belong to $\exp(\mathfrak{g}_b)$, because one can take the logarithm on both sides and inductively deduce that the components of $\hat{\xi}(y,p)$ and $\hat{\eta}(y,p)$ are Lie series. □

3.1.2 Regularity of the gauge transformation

Although for generic $\xi, \eta \in \mathfrak{g}^h$ the differential $\mathcal{J}_{\rm DHS}(x, p; \xi, \eta)$ has a pole in x at p, which produces a logarithmic singularity in $\mathbf{\Gamma}_{\rm DHS}$, the lemma below guarantees that, for $\hat{\xi} = \hat{\xi}(y,p)$, $\hat{\eta} = \hat{\eta}(y,p)$ as in Lemma 3.1, the coefficient $[\hat{\eta}_J, \hat{\xi}^J]$ of this singularity vanishes, so that in this case $\mathbf{\Gamma}_{\rm DHS}$ is smooth and single-valued on $\tilde{\Sigma}$.

Lemma 3.2. *The quantity $[\hat{\eta}_J, \hat{\xi}^J]$ vanishes when $\hat{\xi} = \hat{\xi}(y,p)$, $\hat{\eta} = \hat{\eta}(y,p) \in \mathfrak{g}_b^h$ are such that $\mathbf{\Gamma}_{\rm DHS}(x, y, p; \hat{\xi}, \hat{\eta})$ satisfies the monodromy conditions (3.8).*

Proof. To prove this lemma, we observe that $[\hat{\eta}_J, \hat{\xi}^J]$ is the residue of $\mathcal{J}_{\rm DHS}(x, p; \hat{\xi}, \hat{\eta})$ at the pole at p and consider the fundamental domain $D_p \subset \tilde{\Sigma}$ for Σ depicted in figure 2, whose boundary curve ∂D_p is the $4h$-gon obtained from the union of the curves $\mathfrak{A}^I$, $\mathfrak{B}_I$ and their inverses, with vertices in $\pi^{-1}(\pi(y))$, as illustrated in figure 2. The curves are chosen such that a preferred preimage $p \in \tilde{\Sigma}$ of the point $p \in \Sigma$ is in the interior of D_p. The closed boundary curve $\partial D_p \subset \tilde{\Sigma} \setminus \pi^{-1}(p)$ is homotopic to a small circle $\mathfrak{C}_p$ around the point p (see figure 2), so that the homotopy invariance of the integral defining $\mathbf{\Gamma}_{\rm DHS}$ implies the relation[19]

$$\mathbf{\Gamma}_{\rm DHS}(\partial D_p \cdot x, y, p; \hat{\xi}, \hat{\eta}) = \mathbf{\Gamma}_{\rm DHS}(\mathfrak{C}_p \cdot x, y, p; \hat{\xi}, \hat{\eta}). \tag{3.12}$$

Using the composition law (1.8) of the map $\mu_{\rm DHS}$, considered here as a map on $\pi_1(\Sigma_p, y)$ with every other dependence omitted, we evaluate $\mathbf{\Gamma}_{\rm DHS}(\partial D_p \cdot x, y, p; \hat{\xi}, \hat{\eta})$ explicitly,

$$\begin{aligned}\mathbf{\Gamma}_{\rm DHS}(\partial D_p \cdot x, y, p; \hat{\xi}, \hat{\eta}) &= \mathbf{\Gamma}_{\rm DHS}(x, y, p; \hat{\xi}, \hat{\eta}) \\ &\quad \times \prod_{K=1}^{h} \mu_{\rm DHS}(\mathfrak{A}^K)\mu_{\rm DHS}(\mathfrak{B}_K)\mu_{\rm DHS}(\mathfrak{A}^K)^{-1}\mu_{\rm DHS}(\mathfrak{B}_K)^{-1}.\end{aligned} \tag{3.13}$$

Since $\hat{\xi}$ and $\hat{\eta}$ are the solutions to the monodromy relations of (3.8), the factors of $\mu_{\rm DHS}(\gamma) = \mu_{\rm DHS}(\gamma, y, p; \hat{\xi}, \hat{\eta})$ take the values $\mu_{\rm DHS}(\mathfrak{A}^K) = 1$ and $\mu_{\rm DHS}(\mathfrak{B}_K) = e^{2\pi i b_K}$, so that their product on the right side of (3.13) cancels. As a result, the monodromy of $\mathbf{\Gamma}_{\rm DHS}(x, y, p; \hat{\xi}, \hat{\eta})$ around the point p is trivial,

$$\mathbf{\Gamma}_{\rm DHS}(\mathfrak{C}_p \cdot x, y, p; \hat{\xi}, \hat{\eta}) = \mathbf{\Gamma}_{\rm DHS}(x, y, p; \hat{\xi}, \hat{\eta}). \tag{3.14}$$

Finally, the monodromy may also be evaluated via explicit calculation by choosing local coordinates x in a neighborhood of p, parametrizing the circle $\mathfrak{C}_p$ by polar coordinates $x_\varepsilon(\theta) - p = \varepsilon\, e^{i\theta}$ for $\varepsilon > 0$, and deriving from (3.4) a differential equation in θ,

$$\frac{\partial}{\partial \theta}\,\mathbf{\Gamma}_{\rm DHS}(x_\varepsilon(\theta), y, p; \hat{\xi}, \hat{\eta}) = \Big(i[\hat{\eta}_J, \hat{\xi}^J] + \mathcal{O}(\varepsilon)\Big)\,\mathbf{\Gamma}_{\rm DHS}(x_\varepsilon(\theta), y, p; \hat{\xi}, \hat{\eta}). \tag{3.15}$$

Integrating this equation in θ from 0 to 2π, in the limit $\varepsilon \to 0$, we obtain,

$$\mathbf{\Gamma}_{\rm DHS}(\mathfrak{C}_p \cdot x, y, p; \hat{\xi}, \hat{\eta}) = e^{2\pi i[\hat{\eta}_J, \hat{\xi}^J]}\,\mathbf{\Gamma}_{\rm DHS}(x, y, p; \hat{\xi}, \hat{\eta}). \tag{3.16}$$

Combining equations (3.14) and (3.16) imposes the condition $e^{2\pi i\,[\hat{\eta}_J, \hat{\xi}^J]} = 1$. Since the Lie algebra $\mathfrak{g}_b$ is freely generated this implies that the residue of $\mathcal{J}_{\rm DHS}(x, p; \hat{\xi}, \hat{\eta})$ vanishes,

$$[\hat{\eta}_J, \hat{\xi}^J] = 0, \tag{3.17}$$

and therefore that $\mathcal{J}_{\rm DHS}(x, p; \hat{\xi}, \hat{\eta})$ is smooth for $x \in \Sigma$. □

It is instructive to verify that the relation $[\hat{\eta}_J, \hat{\xi}^J] = 0$ is obeyed to low orders of $\hat{\xi}$ and $\hat{\eta}$ in b. To first order, the result readily follows from (3.11). The verification of the relation to second order is relegated to appendix D.

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[19]Here and elsewhere, ∂D_p and $\mathfrak{C}_p$ denote also the lifts of ∂D_p and $\mathfrak{C}_p$ from $\tilde{\Sigma} \setminus \pi^{-1}(p)$ to $\tilde{\Sigma}_p$ which are uniquely determined by the choice of a preferred preimage of y in $\tilde{\Sigma}_p$.

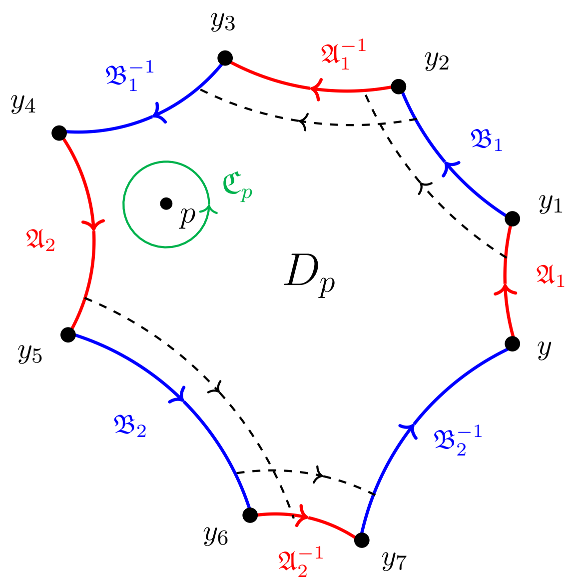

Figure 2: A genus-two Riemann surface Σ_p with puncture p can be represented in terms of a (punctured) fundamental domain $D_p \subset \tilde{\Sigma} \setminus \pi^{-1}(p)$. The surface Σ_p may be reconstructed from D_p by pairwise identifying inverse cycles with one another under the dashed arrows. The points $y_i \in \tilde{\Sigma}$ are related to $y \in \tilde{\Sigma}$ as detailed in the caption of figure 1. The curve $\mathfrak{C}_p$ is homotopic to the boundary curve ∂D_p.

Lemma 3.3. *1. If $\Phi^{I_1 \cdots I_r}{}_J(x)$ is given by equation (3.21) of [17] then one has*

$$\partial_x \Phi^{I_1 \cdots I_r}{}_J(x) = f^{I_1 \cdots I_r}{}_J(x,p) - \frac{1}{h} f^{I_1 \cdots I_{r-1} K}{}_K(x,p)\, \delta^{I_r}_J , \tag{3.18}$$

where the differentials $f^{I_1 \cdots I_r}{}_J(x,p)$ are as in Lemma 2.6. In particular, the right side in this equation is independent of p.

2. If $\lambda = \{\lambda^1, \cdots, \lambda^h\}$ and $\mu = \{\mu_1, \cdots, \mu_h\}$ are in $\mathfrak{g}^h$ such that $[\mu_I, \lambda^I] = 0$, then the differential $\mathcal{J}_{\rm DHS}(x,p;\lambda,\mu)$ is independent of p, and has the expansion

$$\mathcal{J}_{\rm DHS}(x,\cdot\,;\lambda,\mu) = \omega_J(x)\lambda^J - \pi\, \overline{\omega}^I(x)\mu_I + \sum_{r=1}^{\infty} \partial_x \Phi^{I_1 \cdots I_r}{}_J(x) M_{I_1} \cdots M_{I_r} \lambda^J, \tag{3.19}$$

where $M_I X = [\mu_I, X]$ for all $X \in \mathfrak{g}$.

Proof. 1. We first prove the case $r = 1$ and then show how the case for arbitrary $r \geq 2$ can be derived from the case $r = 1$ via a recursion relation. Equations (3.19) and (3.21) in [17] imply the following relations,

$$\begin{aligned}\partial_x \Phi^I{}_J(x) &= -\frac{i}{2}\int_\Sigma \partial_x \mathcal{G}(x,z)\, \overline{\omega}^I(z)\wedge \omega_J(z),\\ \partial_x \Phi^{I_1\cdots I_r}{}_J(x) &= -\frac{i}{2}\int_\Sigma \partial_x \mathcal{G}(x,z)\, \overline{\omega}^{I_1}(z)\wedge \partial_z \Phi^{I_2\cdots I_r}{}_J(z),\end{aligned} \tag{3.20}$$

while the analogous equations for $f^I{}_J(x,p)$ and $f^{I_1\cdots I_r}{}_J(x,p)$ are given in (2.22) and (2.23) of Lemma 2.6, respectively. Expressing the combination on the right side of (3.18) with the help of (2.22), we observe that the term proportional to δ^I_J under the parentheses in the integrand of (2.22) cancels so that we obtain,

$$f^I{}_J(x,p) - \frac{1}{h} f^K{}_K(x,p)\,\delta^I_J = \partial_x \int_\Sigma \mathcal{G}(x,t)\Big(-\frac{i}{2}\,\overline{\omega}^I(t)\wedge\omega_J(t)\Big) \tag{3.21}$$

which is readily seen to coincide with $\partial_x \Phi^I{}_J(x)$, thereby establishing item 1 for the case $r=1$. For the case $r\geq 2$, we express the combination on the right side of (3.18) with the help of (2.23) and obtain the following recursion relation,

$$\begin{aligned}&f^{I_1\cdots I_r}{}_J(x,p) - \frac{1}{h} f^{I_1\cdots I_{r-1}K}{}_K(x,p)\,\delta^{I_r}_J\\ &\quad= -\frac{i}{2}\int_\Sigma \partial_x\mathcal{G}(x,t)\left(\overline{\omega}^{I_1}(t)\wedge\Big\{ f^{I_2\cdots I_r}{}_J(t,p) - \frac{1}{h}\,\delta^{I_r}_J f^{I_2\cdots I_{r-1}K}{}_K(t,p)\Big\}\right).\end{aligned} \tag{3.22}$$

This recursion relation is identical to the second line in (3.20), so that the identity for $r=1$ implies the identity for any r. The result is manifestly independent of p, thereby completing the proof of item 1 of the lemma.[20]

2. Expressing the coefficients $f^{I_1\cdots I_r}{}_J(x,p)$ in (2.20) in terms of $\partial_x\Phi^{I_1\cdots I_r}{}_J(x)$ and $f^{I_1\cdots I_{r-1}K}{}_K(x,p)\,\delta^{I_r}_J$ using (3.18) and setting $a=\lambda$ and $b=\mu$ we obtain,

$$\begin{aligned}\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;\lambda,\mu) = \omega_J(x)\lambda^J + \sum_{r=1}^\infty \Big(&\partial_x\Phi^{I_1\cdots I_r}{}_J(x)\\ &+\frac{1}{h} f^{I_1\ldots I_{r-1}K}{}_K(x,p)\delta^{I_r}_J\Big) M_{I_1}\cdots M_{I_r}\lambda^J.\end{aligned} \tag{3.23}$$

Using the assumption $[\mu_I,\lambda^I]=0$ of the lemma, we see that the effect of the Kronecker $\delta^{I_r}_J$ factor is to produce the combination $\delta^{I_r}_J M_{I_r}\lambda^J = [\mu_J,\lambda^J]=0$, so that the terms in $1/h$ cancel. Combining the result with the $\mathcal{J}^{(0,1)}_{\rm DHS}(x,p;\lambda,\mu) = -\pi\,\overline{\omega}^J(x)\mu_J$ part proves the expression claimed in (3.19). □

[20] An alternative proof may be obtained by combining formula (30) of [22] with its trace over the indices I_r and J and using the tracelessness of $\partial_x\Phi^{I_1\cdots I_r}{}_J(x)$ which follows from the tracelessness of $\partial_x\Phi^I{}_J(x)$ with the recursion relation in the second line of (3.20).

Corollary 3.4. *For $\hat{\xi} = \hat{\xi}(y,p)$ and $\hat{\eta} = \hat{\eta}(y,p)$ obeying the monodromy conditions (3.8) of Lemma 3.1, the differential $\mathcal{J}_{\mathrm{DHS}}(x,p;\hat{\xi},\hat{\eta})$ is given by*

$$\mathcal{J}_{\mathrm{DHS}}(x,\cdot\,;\hat{\xi},\hat{\eta}) = \omega_J(x)\hat{\xi}^J - \pi\,\overline{\omega}^I(x)\hat{\eta}_I + \sum_{r=1}^{\infty} \partial_x \Phi^{I_1\cdots I_r}{}_J(x)\hat{H}_{I_1}\cdots\hat{H}_{I_r}\hat{\xi}^J. \tag{3.24}$$

where $\hat{H}_I X = [\hat{\eta}_I, X]$ for $X \in \mathfrak{g}$.

Proof. This follows from combining item 2 of Lemma 3.3 and Lemma 3.2. □

Lemma 3.5. *If $\lambda, \mu \in \mathfrak{g}^h$ are such that $[\mu_I, \lambda^I] = 0$, then for any $y \in \tilde{\Sigma}$ and any $K \in \{1,\cdots,h\}$, the functions $p \mapsto \mu_{\mathrm{DHS}}(\mathfrak{A}^K, y, p; \lambda, \mu)$ and $p \mapsto \mu_{\mathrm{DHS}}(\mathfrak{B}_K, y, p; \lambda, \mu)$ (defined for p in the complement in $\tilde{\Sigma}$ of $\mathrm{Aut}(\tilde{\Sigma}/\Sigma)\cdot y$) are constant. The maps taking (y,λ,μ), where $y \in \tilde{\Sigma}$ and $\lambda,\mu \in \mathfrak{g}^h$ such that $[\mu_I,\lambda^I] = 0$, to these constant values will be denoted $(y,\lambda,\mu) \mapsto \mu_{\mathrm{DHS}}(\mathfrak{A}^K, y, \cdot\,;\lambda,\mu)$ and $(y,\lambda,\mu) \mapsto \mu_{\mathrm{DHS}}(\mathfrak{B}_K, y, \cdot\,;\lambda,\mu)$.*

Proof. It follows from Lemma 3.3 and from the assumption on λ, μ that $\mathcal{J}_{\mathrm{DHS}}(x,p;\lambda,\mu)$ is independent on p. These functions, being defined as the holonomies of the connection $d_x - \mathcal{J}_{\mathrm{DHS}}(x,p;\lambda,\mu)$ based at y, are therefore also independent of p. □

Corollary 3.6. *For any $y \in \tilde{\Sigma}$, the maps $p \mapsto \hat{\xi}(y,p)$, $\hat{\eta}(y,p) \in \mathfrak{g}_b^h$, where $p \in \tilde{\Sigma} \smallsetminus \mathrm{Aut}(\tilde{\Sigma}/\Sigma)\cdot y$ given by Lemma 3.1, are constant; we denote by $\hat{\xi}(y,\cdot)$, $\hat{\eta}(y,\cdot)$ these constant values. The maps $y \mapsto \hat{\xi}(y,\cdot)$, $\hat{\eta}(y,\cdot)$ are smooth functions of $y \in \tilde{\Sigma}$, and therefore well-defined for $y = p$; their values at this point satisfy (and are uniquely determined by) the monodromy conditions*

$$\begin{aligned} &\mu_{\mathrm{DHS}}\big(\mathfrak{A}^K, p, \cdot\,; \hat{\xi}(p,\cdot), \hat{\eta}(p,\cdot)\big) = 1, \\ &\mu_{\mathrm{DHS}}\big(\mathfrak{B}_K, p, \cdot\,; \hat{\xi}(p,\cdot), \hat{\eta}(p,\cdot)\big) = e^{2\pi i b_K}. \end{aligned} \tag{3.25}$$

Proof. The identity $[\hat{\eta}_I(y,p), \hat{\xi}^I(y,p)] = 0$ follows from Lemma 3.2. This fact, combined with Lemma 3.5, enables us to rewrite the system (3.8) as follows

$$\begin{aligned} &\mu_{\mathrm{DHS}}\big(\mathfrak{A}^K, y, \cdot\,; \hat{\xi}(y,p), \hat{\eta}(y,p)\big) = 1, \\ &\mu_{\mathrm{DHS}}\big(\mathfrak{B}_K, y, \cdot\,; \hat{\xi}(y,p), \hat{\eta}(y,p)\big) = e^{2\pi i b_K}. \end{aligned} \tag{3.26}$$

The self-map of $\mathfrak{g}_b^{2h}$ given by

$$(\lambda,\mu) \mapsto \Big(\log \mu_{\mathrm{DHS}}\big(\mathfrak{A}^K, y, \cdot\,;\lambda,\mu\big), \log \mu_{\mathrm{DHS}}\big(\mathfrak{B}_K, y, \cdot\,;\lambda,\mu\big)\Big)_{K=1,\cdots,h} \tag{3.27}$$

is smooth in y, triangular with respect to the degree filtration of $\mathfrak{g}_b^{2h}$, and bijective. The inverse of this self-map is therefore also smooth in y. The above system implies that the pair $(\hat{\xi}(y,p), \hat{\eta}(y,p))$ is the image of $(0, 2\pi i b)$ by this inverse map, which proves at the same time its independence in p and its smoothness in y. □

Definition 3.7. *Henceforth, the notation* $\hat{\xi}, \hat{\eta}$ *will refer to the elements* $\hat{\xi}(p, \cdot), \hat{\eta}(p, \cdot)$ *of* $\mathfrak{g}_b^h$ *which solve the monodromy conditions* (3.25) *from Corollary 3.6 (i.e. abandoning the notation of Lemma 3.2 and Corollary 3.4).*

An explicit evaluation of $\hat{\xi}$ and $\hat{\eta}$ to second order in b is relegated to appendix D. Unlike the contributions linear in b_I spelled out in (3.11) which are independent of the moduli[21] of Σ_p (besides $b^I = Y^{IK} b_K$), all the higher order contributions in b to $\hat{\eta}_I$ and $\hat{\xi}^I$ will be non-trivial functions of the moduli of Σ_p.

3.1.3 Construction of the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$

Finally, we define the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ as follows.

Definition 3.8. *For* $\hat{\xi}$ *and* $\hat{\eta}$ *as in Definition 3.7, the gauge transformation* $\mathcal{U}_{\rm DHS}(x,p) = \mathcal{U}_{\rm DHS}(x,p;\hat{\xi},\hat{\eta})$ *is defined to be the specialization* $\mathbf{\Gamma}_{\rm DHS}(x,p,p;\hat{\xi},\hat{\eta})$ *of* $\mathbf{\Gamma}_{\rm DHS}$*, namely the unique solution of the differential equation*

$$d_x \mathcal{U}_{\rm DHS}(x,p;\hat{\xi},\hat{\eta}) = \mathcal{J}_{\rm DHS}(x,\cdot\,;\hat{\xi},\hat{\eta})\, \mathcal{U}_{\rm DHS}(x,p;\hat{\xi},\hat{\eta}) \tag{3.28}$$

with the boundary condition $\mathcal{U}_{\rm DHS}(p,p;\hat{\xi},\hat{\eta}) = 1$.

It follows from Lemma 3.2 that the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ is a smooth function of $x, p \in \tilde{\Sigma}$. It can be written as a path-ordered exponential,

$$\mathcal{U}_{\rm DHS}(x,p;\hat{\xi},\hat{\eta}) = \mathrm{P}\exp \int_p^x \mathcal{J}_{\rm DHS}(t,\cdot\,;\hat{\xi},\hat{\eta}), \tag{3.29}$$

with $\mathcal{J}_{\rm DHS}(t,\cdot\,;\hat{\xi},\hat{\eta})$ as in (3.24). Notice that $\mathcal{U}_{\rm DHS}(x,p;\hat{\xi},\hat{\eta})$ obeys the following monodromy conditions equivalent to (3.25) which determine both $\hat{\xi}^I$ and $\hat{\eta}_I$ as a Lie series in b_K (see section 3.3 below for details),

$$\begin{aligned} &\mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot p, p;\hat{\xi},\hat{\eta}) = 1, \\ &\mathcal{U}_{\rm DHS}(\mathfrak{B}_K \cdot p, p;\hat{\xi},\hat{\eta}) = e^{2\pi i b_K}. \end{aligned} \tag{3.30}$$

3.2 Relating the connections $d - \mathcal{J}_{\rm DHS}$ and $d - \mathcal{K}_{\rm E}$

In this subsection we shall show that the gauge transformation $\mathcal{U}_{\rm DHS}(x,p)$ introduced in Definition 3.8, combined with a suitable automorphism of the Lie algebra $\mathfrak{g}$, relates $\mathcal{J}_{\rm DHS}$ and $\mathcal{K}_{\rm E}$. The result is summarized in the following theorem.

---

[21] Among the moduli of Σ_p, we include here and elsewhere also the topological datum of a choice of a preferred preimage $y \in \tilde{\Sigma}_p$ which determines the action of the fundamental group $\pi_1(\Sigma_p, y)$ on $\tilde{\Sigma}_p$.

Theorem 3.9. *The flat connections $d_x - \mathcal{K}_{\rm E}(x,p;a,b)$ and $d_x - \mathcal{J}_{\rm DHS}(x,p;\hat a,\hat b)$ are related by the gauge transformation $\mathcal{U}_{\rm DHS}(x,p) = \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)$ defined by (3.29), whose arguments $\hat\xi,\hat\eta \in \mathfrak{g}^h_b$ are the uniquely determined solutions of (3.30), so that*

$$d_x - \mathcal{K}_{\rm E}(x,p;a,b) = \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)^{-1}\Big(d_x - \mathcal{J}_{\rm DHS}(x,p;\hat a,\hat b)\Big)\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta). \quad (3.31)$$

The elements $\hat a,\hat b \in \mathfrak{g}^h$ are uniquely determined in terms of a,b, and the moduli of Σ_p, by the linearity of $\mathcal{K}_{\rm E}(x,p;a,b)$ in a, the linearity of $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;\hat a,\hat b)$ in $\hat a$ and the following residue matching relations,

$$\hat b_I = \hat\eta_I, \qquad\qquad [b_I, a^I] = [\hat\eta_I, \hat a^I - \hat\xi^I]. \quad (3.32)$$

These conditions for $\hat a^I$ and $\hat b_I$ may be recursively solved as formal series in (the non-commutative components of) b, the leading order solution being given by $\hat a^I = a^I + \mathcal{O}(b)$ and $\hat b_I = b_I + \mathcal{O}(b^2)$.

Proof. We begin by noting that the right side of (3.31) has the same monodromy as $\mathcal{K}_{\rm E}$ given in (2.6) for *arbitrary* $\hat a,\hat b$, thanks to the monodromy condition on the gauge transformation $\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)$ of (3.30). Next, requiring the right side of (3.31) to be a form of type $(1,0)$ in x, as indeed $\mathcal{K}_{\rm E}$ should be, is equivalent to requiring the vanishing on its $(0,1)$ component, which is equivalent to the following relation

$$\bar\partial_x \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)\,\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)^{-1} = \mathcal{J}^{(0,1)}_{\rm DHS}(x,p;\hat a,\hat b). \quad (3.33)$$

By construction in (3.28), we have

$$\bar\partial_x \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)\,\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)^{-1} = \mathcal{J}^{(0,1)}_{\rm DHS}(x,p;\hat\xi,\hat\eta), \quad (3.34)$$

while we have $\mathcal{J}^{(0,1)}_{\rm DHS}(x,p;\hat a,\hat b) = -\pi\,\overline{\omega}^I(x)\hat b_I$ in view of item 2 of Theorem 2.5. Combining these results with (3.33) reduces to $\hat b_I = \hat\eta_I$, which gives the first relation in (3.32).

To determine $\hat a$, we substitute the $(1,0)$ component of (3.28) into (3.31) and use the linearity of $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;\hat a,\hat\eta)$ in $\hat a$, to arrive at the following simplified relation,

$$\mathcal{K}_{\rm E}(x,p;a,b) = \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)^{-1}\,\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;\hat a-\hat\xi,\hat\eta)\,\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta). \quad (3.35)$$

By construction, the right side has the monodromies required on $\mathcal{K}_{\rm E}(x,p;a,b)$ and is meromorphic in x, because it is a $(1,0)$ form which satisfies the Maurer–Cartan equation (1.2) (recall that gauge transformations preserve flatness). Moreover, it is clear by construction that its only poles in x are placed at all pre-images $\pi^{-1}(p)$ of p on $\tilde\Sigma$, and are all simple. In view of Theorem 2.1, to identify the right side with $\mathcal{K}_{\rm E}(x,p;a,b)$ it remains to match the residues of their poles in x, which leads to the condition

$$[b_I, a^I] = [\hat\eta_I, \hat a^I - \hat\xi^I] \tag{3.36}$$

and gives the second relation in (3.32). Using the lowest-order solution of the monodromy conditions for $\hat\xi$ and $\hat\eta$, given in (3.11), reduces the conditions of (3.32) to $\hat b_I = b_I + \mathcal{O}(b^2)$ and $[b_I, a^I] = [b_I, \hat a^I] + \mathcal{O}(b)$ whose solution is the one stated in the last line of Theorem 3.9, thereby completing its proof. □

Remark 3.10. *Equation (3.31) of Theorem 3.9 is invariant under right-multiplication of the gauge transformation by an x-independent element $\mathcal{V} \in \exp(\mathfrak{g})$ accompanied by a conjugation of a and b, while leaving $\hat\xi^I, \hat\eta_I, \hat a^I$ and $\hat b_I$ invariant,*

$$\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta) \to \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)\,\mathcal{V}, \qquad \begin{cases} a^J \to \mathcal{V}\,a^J\,\mathcal{V}^{-1}, \\ b_I \to \mathcal{V}\,b_I\,\mathcal{V}^{-1}. \end{cases} \tag{3.37}$$

The residue-matching conditions transform covariantly as follows,

$$\hat b_I = \hat\eta_I, \qquad \mathcal{V}[b_I, a^I]\mathcal{V}^{-1} = [\hat\eta_I, \hat a^I - \hat\xi^I]. \tag{3.38}$$

Linearity of $\mathcal{K}_{\rm E}(x,p;a,b)$ in the generators a^I requires restricting to $\mathcal{V} \in \exp(\mathfrak{g}_b)$. In particular, changing the base point of the path-ordered integral in (3.29) from p to an arbitrary point $p' \in \tilde\Sigma$ is equivalent to multiplying $\mathcal{U}(x,p;\hat\xi,\hat\eta)$ to the right by the x-independent factor $\mathcal{V} = \mathcal{U}(p',p;\hat\xi,\hat\eta)^{-1} \in \exp(\mathfrak{g}_b)$.

Remark 3.11. *Upon restriction to genus $h=1$, the results in Theorem 3.9 and its proof reproduce the automorphism $(\hat a, \hat b) = (a + \pi b/(\mathrm{Im}\,\tau), b)$ and the gauge element (2.42) mapping the Brown–Levin connection $d - \mathcal{J}_{\rm DHS}|_{h=1}$ to that of Calaque–Enriquez–Etingof $d - \mathcal{K}_{\rm E}|_{h=1}$ via (2.43). This can be seen from the fact that any bracket $[b_I, b_J]$ vanishes at $h=1$, which truncates the Lie-series expansion of $\hat\xi$ and $\hat\eta$ to their first order $\pi b/(\mathrm{Im}\,\tau)$ and b, respectively, see (3.11). As a consequence, the gauge transformation (3.29) reduces to the path-ordered exponential of $\pi(dx - d\bar x)b/(\mathrm{Im}\,\tau)$ obtained from (3.24) which matches $\exp\{2\pi i\, b\,(\mathrm{Im}\,x)/(\mathrm{Im}\,\tau)\}$ in (2.42).*

The first part $\hat a = a + \pi b/(\mathrm{Im}\,\tau)$ of the automorphism at $h=1$ follows from comparison of a and $\hat a$ in (3.35) and recalling the restriction $\hat\xi = \pi b/(\mathrm{Im}\,\tau)$ at genus one. Note that the genus-one generators correspond to the placement of uppercase and lowercase indices according to $a = a^1$, $b = b_1$ as well as $\hat\xi = \hat\xi^1$, $\hat\eta = \hat\eta_1$.

3.3 Iterative construction relating $\mathcal{K}_{\mathrm{E}}$ and $\mathcal{J}_{\mathrm{DHS}}$

In this last subsection, we shall outline an iterative procedure for the explicit construction of the gauge transformation $\mathcal{U}_{\mathrm{DHS}}(x,p)$ in (3.35) and the automorphism $a \cup b \to \hat{a} \cup \hat{b}$ to arbitrary order as a Lie series in b. This procedure will express the Enriquez differentials $g^{I_1 \cdots I_r}{}_J(x,p)$ in (2.10) in terms of the DHS differentials $f^{I_1 \cdots I_r}{}_J(x,p)$ in (2.6) and their iterated integrals. Calculations to low orders are relegated to appendices D and E.

3.3.1 Solving for $\hat{\xi}$ in terms of $\hat{\eta}$

The starting point is the path-ordered exponential $\mathcal{U}_{\mathrm{DHS}}(x,p;\hat{\xi},\hat{\eta})$ given in (3.29), where $\hat{\xi},\hat{\eta} \in \mathfrak{g}^h_b$ satisfy the relation $[\hat{\eta}_I,\hat{\xi}^I]=0$. The monodromy conditions of (3.30) may be solved in two step. First, the $\mathfrak{A}$-monodromy conditions are solved for $\hat{\xi}$ in terms of $\hat{\eta}$, as stated in the lemma below. Second, the $\mathfrak{B}$-monodromy conditions may subsequently be solved for both $\hat{\xi}$ and $\hat{\eta}$ as functions of b, which will be done in subsection 3.3.3 below.

Lemma 3.12. *The solution to the monodromy conditions $\mathcal{U}_{\mathrm{DHS}}(\mathfrak{A}^K \cdot p, p; \hat{\xi},\hat{\eta}) = 1$ of (3.30) for $\hat{\xi}$ as a function of $\hat{\eta}$ is unique and given by the following associative series*

$$\hat{\xi}^I = \sum_{r=1}^{\infty} \mathcal{X}^{IJ_1 \cdots J_r} \hat{\eta}_{J_1} \cdots \hat{\eta}_{J_r}. \tag{3.39}$$

1. *The coefficients $\mathcal{X}$ depend on the moduli of Σ but are independent of the point p.*
2. *They obey the following shuffle relations (see (B.8), (B.9) for the shuffle product)*

$$\mathcal{X}^{I(J_1 \cdots J_r \sqcup K_1 \cdots K_s)} = 0, \qquad\qquad r,s \geq 1. \tag{3.40}$$

3. *They are invariant under cyclic permutations of their indices,*

$$\mathcal{X}^{IJ_1J_2 \cdots J_r} = \mathcal{X}^{J_1J_2 \cdots J_r I}. \tag{3.41}$$

The proof of the lemma is presented in appendix B. The leading order solution of (3.11) gives $\mathcal{X}^{IJ} = \pi Y^{IJ}$ while higher orders will be evaluated in appendices D and E.

3.3.2 Applying the gauge transformation $\mathcal{U}_{\mathrm{DHS}}$ to $\mathcal{J}^{(1,0)}_{\mathrm{DHS}}$

The $\hat{\xi}$ dependence of $\mathcal{U}_{\mathrm{DHS}}(x,p;\hat{\xi},\hat{\eta})$ may be eliminated using the expansion (3.39) of Lemma 3.12. The resulting gauge transformation may be expanded in an associative power series in $\hat{\eta}$ as follows,

$$\mathcal{U}_{\mathrm{DHS}}\big(x,p;\hat{\xi},\hat{\eta}\big)^{-1} = 1 + \sum_{r=1}^{\infty} \mathcal{T}^{I_1 \cdots I_r}(x,p)\, \hat{\eta}_{I_1} \cdots \hat{\eta}_{I_r}. \tag{3.42}$$

The path-ordered exponential (3.29) determines the coefficients $\mathcal{T}^{I_1\cdots I_r}(x,p)$ in terms of DHS polylogarithms and the coefficients $\mathcal{X}^{IJ_1\cdots J_r}$ defined in (3.39), see (E.3) for ranks $r=1,2$. The key relation (3.35) is expressed instead in terms of the following Lie series

$$\mathcal{U}_{\rm DHS}\big(x,p;\hat\xi,\hat\eta\big)^{-1} X\, \mathcal{U}_{\rm DHS}\big(x,p;\hat\xi,\hat\eta\big) = X + \sum_{r=1}^\infty \mathcal{T}^{I_1\cdots I_r}(x,p)\, \hat H_{I_1}\cdots \hat H_{I_r} X, \tag{3.43}$$

where again $\hat H_I X = [\hat\eta_I, X]$ for all $X\in\mathfrak{g}$.

Applying the gauge transformation $\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)$ to $\mathcal{J}_{\rm DHS}(x,p;X+\hat\xi,\hat\eta)$ produces a connection, denoted by $\mathcal{K}(x,p;X,\hat\eta)$, for an arbitrary $X\in\mathfrak{g}^h$,

$$\mathcal{K}(x,p;X,\hat\eta) = \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)^{-1} \mathcal{J}^{(1,0)}_{\rm DHS}(x,p;X,\hat\eta)\, \mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta) \tag{3.44}$$

which is meromorphic in x and whose Lie series in powers of $\hat H_I$ may be written as follows,

$$\mathcal{K}(x,p;X,\hat\eta) = \omega_J(x) X^J + \sum_{r=1}^\infty h^{I_1\cdots I_r}{}_J(x,p) \hat H_{I_1}\cdots \hat H_{I_r} X^J. \tag{3.45}$$

The quantities $h^{I_1\cdots I_r}{}_J(x,p)$ are $(1,0)$-forms in x and scalars in p, which may be viewed as intermediate objects between the f-tensors and the g-kernels. They are meromorphic in x but neither in p nor in the moduli of Σ, and are simply related to the f-tensors, as may be seen by combining the expansions for $\mathcal{U}_{\rm DHS}(x,p;\hat\xi,\hat\eta)^{-1}$ and $\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;X,\hat\eta)$,

$$h^{I_1\cdots I_r}{}_J(x,p) = f^{I_1\cdots I_r}{}_J(x,p) + \sum_{j=1}^r \mathcal{T}^{I_1\cdots I_j}(x,p) f^{I_{j+1}\cdots I_r}{}_J(x,p), \tag{3.46}$$

where we set $f^{\emptyset}{}_J(x,p) = \omega_J(x)$, as usual.

3.3.3 Implementing the automorphism $a\cup b\to \hat a\cup \hat b$

To extract the connection $\mathcal{K}_{\rm E}$ from $\mathcal{K}$ in (3.44), it remains to implement four operations

1. solving the $\mathfrak{B}$-monodromy condition $\mathcal{U}_{\rm DHS}(\mathfrak{B}_K\cdot p,p;\hat\xi,\hat\eta) = e^{2\pi i b_K}$ of (3.30), after having eliminated $\hat\xi$ using (3.39), for $\hat\eta$ in terms of Lie series in b as follows

$$\hat\eta_I = b_I + \sum_{r=1}^\infty \mathcal{L}_I{}^{J_1\cdots J_r K} B_{J_1}\cdots B_{J_r} b_K, \tag{3.47}$$

   where $B_I X = [b_I, X]$ for all $X\in\mathfrak{g}$, the coefficients $\mathcal{L}_I{}^{J_1\cdots J_rK}$ are non-holomorphic functions of the moduli of Σ_p, but this dependence is suppressed throughout;

2. setting $\hat{\eta} = \hat{b}$ so that the last argument of $\mathcal{J}^{(1,0)}_{\rm DHS}$ equals $\hat{b}$;
3. setting $X^J = \hat{a}^J - \hat{\xi}^J$ in (3.45) and solving the residue matching equation $[b_I, a^I] = [\hat{b}_I, \hat{a}^I - \hat{\xi}^I]$ of (3.36), with the help of the linearity of $\mathcal{K}_{\rm E}(x,p;a,b)$ in a, the linearity of $\mathcal{K}(x,p;X,\hat{\eta})$ in X, and the fact that $\hat{a}, \hat{\xi}, \hat{\eta} \in \mathfrak{g}^h$, to obtain a Lie series in b

$$\hat{a}^I - \hat{\xi}^I = a^I + \sum_{r=1}^{\infty} \mathcal{M}^{II_1\cdots I_r}{}_J B_{I_1} \cdots B_{I_r} a^J, \tag{3.48}$$

which determines the coefficients $\mathcal{M}^{II_1\cdots I_r}{}_J$ in terms of the $\mathcal{L}_I{}^{J_1\cdots J_r K}$ in (3.47);
4. identifying

$$\mathcal{K}_{\rm E}(x,p;a,b) = \mathcal{K}(x,p;\hat{a}-\hat{\xi},\hat{b}). \tag{3.49}$$

The residue matching condition provides a convenient relation between $\hat{H}$ and B,

$$\hat{H}_J = B_J - \sum_{r=1}^{\infty} \mathcal{M}^{KI_1\cdots I_r}{}_J \hat{H}_K B_{I_1} \cdots B_{I_r}, \tag{3.50}$$

which may be solved iteratively for $\hat{H}$ as a series in B.

Corollary 3.13. *The coefficients $\mathcal{L}$ in the Lie series expansion (3.47) of $\hat{\eta}_I$ may alternatively be expressed in terms of the following associative series expansion*

$$\hat{\eta}_J = b_J - \sum_{r=2}^{\infty} \mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J b_{I_1} \cdots b_{I_r}, \tag{3.51}$$

where the coefficients $\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J$ are given in terms of the coefficients $\mathcal{M}^{I_1\cdots I_r}{}_J$ in the expansion (3.48) of $\hat{a}^I - \hat{\xi}^I$ by

$$\begin{aligned}\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J = \mathcal{M}^{I_1\cdots I_r}{}_J + \sum_{\ell=1}^{r-2} (-1)^\ell \sum_{2\leq j_1<j_2<\cdots<j_\ell}^{r-1} & \mathcal{M}^{I_1 I_2\cdots I_{j_1}}{}_{K_1} \mathcal{M}^{K_1 I_{j_1+1}\cdots I_{j_2}}{}_{K_2} \times \cdots \\ & \times \cdots \mathcal{M}^{K_{\ell-1} I_{j_{\ell-1}+1}\cdots I_{j_\ell}}{}_{K_\ell} \mathcal{M}^{K_\ell I_{j_\ell+1}\cdots I_r}{}_J\end{aligned} \tag{3.52}$$

and obey shuffle relations similar to those in (3.40),

$$\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r \sqcup\!\sqcup K_1\cdots K_s}{}_J = 0, \qquad r,s \geq 1. \tag{3.53}$$

The proof of the corollary is given in appendix C. We reiterate that, even though this is not exposed in our notation, all of $\mathcal{L}_J{}^{I_1\cdots I_r}, \mathcal{M}^{I_1\cdots I_r}{}_J, \mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J$ with $r \geq 2$ depend non-holomorphically on p and the moduli of Σ.

3.3.4 Expressing the g-differentials in terms of the f-differentials

The explicit relations between the g-differentials and the f-differentials may be obtained by combining the relation (3.49) with the expansion (3.45),

$$\mathcal{K}_{\mathrm{E}}(x,p;a,b) = \omega_J(x)(\hat{a}^J - \hat{\xi}^J) + \sum_{r=1}^{\infty} h^{I_1\cdots I_r}{}_J(x,p)\hat{H}_{I_1}\cdots\hat{H}_{I_r}(\hat{a}^J - \xi^J) \tag{3.54}$$

and then expressing $\hat{a} - \hat{\xi}$ in terms of a, b using (3.48), eliminating $\hat{H}$ in favor of B using (3.50), and expressing the functions h in terms of the tensors f using (3.46).

Proposition 3.14. *Up to rank four, the expressions for g in terms of f are determined by combining the relations between g and h given to rank four by*

$$\begin{aligned}
g^{I_1}{}_J(x,p) &= h^{I_1}{}_J(x,p) + \omega_K(x)\mathcal{M}^{KI_1}{}_J(p),\\
g^{I_1I_2}{}_J(x,p) &= h^{I_1I_2}{}_J(x,p) + h^{I_1}{}_K(x,p)\mathcal{M}^{KI_2}{}_J(p)\\
&\quad - h^K{}_J(x,p)\mathcal{M}^{I_1I_2}{}_K(p) + \omega_K(x)\mathcal{M}^{KI_1I_2}{}_J(p),\\
g^{I_1I_2I_3}{}_J(x,p) &= h^{I_1I_2I_3}{}_J(x,p) - \mathcal{M}^{I_1I_2}{}_K(p)h^{KI_3}{}_J(x,p) - \mathcal{M}^{I_2I_3}{}_K(p)h^{I_1K}{}_J(x,p)\\
&\quad + h^{I_1I_2}{}_K(x,p)\mathcal{M}^{KI_3}{}_J(p) - \mathcal{M}^{I_1I_2}{}_K(p)h^K{}_L(x,p)\mathcal{M}^{LI_3}{}_J(p)\\
&\quad + \big\{\mathcal{M}^{I_1I_2}{}_L(p)\mathcal{M}^{LI_3}{}_K(p) - \mathcal{M}^{I_1I_2I_3}{}_K(p)\big\}h^K{}_J(x,p)\\
&\quad + h^{I_1}{}_K(x,p)\mathcal{M}^{KI_2I_3}{}_J(p) + \omega_K(x)\mathcal{M}^{KI_1I_2I_3}{}_J(p),
\end{aligned} \tag{3.55}$$

and the relations (3.46) between h and f expanded to rank four

$$\begin{aligned}
h^{I_1}{}_J(x,p) &= f^{I_1}{}_J(x,p) + \mathcal{T}^{I_1}(x,p)\omega_J(x),\\
h^{I_1I_2}{}_J(x,p) &= f^{I_1I_2}{}_J(x,p) + \mathcal{T}^{I_1}(x,p)f^{I_2}{}_J(x,p) + \mathcal{T}^{I_1I_2}(x,p)\omega_J(x),\\
h^{I_1I_2I_3}{}_J(x,p) &= f^{I_1I_2I_3}{}_J(x,p) + \mathcal{T}^{I_1}(x,p)f^{I_2I_3}{}_J(x,p)\\
&\quad + \mathcal{T}^{I_1I_2}(x,p)f^{I_3}{}_J(x,p) + \mathcal{T}^{I_1I_2I_3}(x,p)\omega_J(x),
\end{aligned} \tag{3.56}$$

resulting, for example, in the lowest two ranks in the formulas of (1.15).

The detailed proof for rank two, including the calculation of the coefficients $\mathcal{X}$, $\mathcal{M}$, $\mathcal{T}$ and $\mathcal{L}$ involved in the derivation, will be given in appendix D. The proof for rank three will be presented in appendix E, while the derivation for rank four is left to the reader.

Further details on the implementation of the procedure outlined in this subsection may be found in appendix E, including

- a more extensive discussion of the symmetry properties of the coefficients $\mathcal{L}_I{}^{J_1\cdots J_rK}$ and $\mathcal{M}^{II_1\cdots I_r}{}_J$ in section E.3 (see (3.40), (3.41) and section B.2 for those of $\mathcal{X}^{IJ_1\cdots J_r}$);
- the explicit form of the gauge transformation (3.42) in sections E.1 and E.2; and
- the detailed construction of the automorphism in sections E.3 and E.4.

4 Gauge transforming $\mathcal{K}_{\mathrm{E}}$ to $\mathcal{J}_{\mathrm{DHS}}$

In this section we give the second explicit construction of a gauge transformation and of an automorphism of the Lie algebra $\mathfrak{g}$ that relate the connection $d - \mathcal{K}_{\mathrm{E}}$ with the connection $d - \mathcal{J}_{\mathrm{DHS}}$. The construction of the gauge transformation in section 4.1 is divided into two steps. In the first step we exploit the $(0,1)$ component of $\mathcal{J}_{\mathrm{DHS}}$, which is purely anti-holomorphic, to reproduce the anti-holomorphic part of $d - \mathcal{J}_{\mathrm{DHS}}$. In the second step, we then exploit the differential $\mathcal{K}_{\mathrm{E}}$ to complete the construction of a gauge transformation $\mathcal{U}_{\mathrm{E}}(x,p)$ which reproduces the desired monodromies. Finally, in section 4.2 we construct a Lie algebra automorphism in the form of an appropriate redefinition $a \cup b \to \check{a} \cup \check{b}$ to match the residue of the gauge transformation of $\mathcal{K}_{\mathrm{E}}(x,p;\check{a},\check{b})$ with that of $\mathcal{J}_{\mathrm{DHS}}(x,p;a,b)$. Explicit formulas at low degree for the gauge transformation and for the automorphism are given in section 4.3.

4.1 Construction of the gauge transformation $\mathcal{U}_{\mathrm{E}}$

We now proceed with presenting the two steps of the construction of the gauge transformation $\mathcal{U}_{\mathrm{E}}(x,p)$, which will be combined to define $\mathcal{U}_{\mathrm{E}}(x,p)$ in subsection 4.1.3.

4.1.1 Step 1: holomorphicity

Recall from Theorem 2.5 that $\mathcal{J}^{(0,1)}_{\mathrm{DHS}}(x,p;a,b) = -\pi b_I \bar{\omega}^I(x)$. It is therefore independent of the point p and the generators a^J of $\mathfrak{g}$, which will be removed from the notation throughout this section. We define $\mathbf{\Gamma}_-(x,y;b)$ as the unique solution of

$$d_x \mathbf{\Gamma}_-(x,y;b) = \mathcal{J}^{(0,1)}_{\mathrm{DHS}}(x;b)\, \mathbf{\Gamma}_-(x,y;b), \tag{4.1}$$

such that $\mathbf{\Gamma}_-(y,y;b) = 1$. It can be explicitly constructed as a path-ordered exponential,

$$\mathbf{\Gamma}_-(x,y;b) = \mathrm{P}\exp \int_y^x \mathcal{J}^{(0,1)}_{\mathrm{DHS}}(t;b). \tag{4.2}$$

The role of $\mathbf{\Gamma}_-(x,y;b)$ for the construction of the gauge transformation $\mathcal{U}_{\mathrm{E}}(x,p)$ will be to recast the anti-holomorphic part of $\mathcal{J}_{\mathrm{DHS}}$ out of a differential which is purely holomorphic in $x \in \tilde{\Sigma}_p$. This can be intuitively understood as follows: consider the differential $\mathcal{J}(x,p;a,b)$, defined by[22]

$$\mathcal{J}(x,p;a,b) = \mathbf{\Gamma}_-(x,p;b)^{-1} \mathcal{J}^{(1,0)}_{\mathrm{DHS}}(x,p;a,b)\, \mathbf{\Gamma}_-(x,p;b). \tag{4.3}$$

[22]Note that $\mathbf{\Gamma}_-(x,y;b)$ is well-defined also when taking the base-point $y = p$ as will be done in Definition 4.6 below.

It follows from its definition that $\mathcal{J}$ is a $(1,0)$-form. Moreover, one has

$$\begin{aligned}\mathbf{\Gamma}_-(x,p;b)^{-1}&(d_x - \mathcal{J}_{\mathrm{DHS}}(x,p;a,b))\,\mathbf{\Gamma}_-(x,p;b)\\ &= d_x - \mathcal{J}(x,p;a,b) + \mathbf{\Gamma}_-(x,p;b)^{-1} d_x \mathbf{\Gamma}_-(x,p;b)\\ &\quad - \mathbf{\Gamma}_-(x,p;b)^{-1}\mathcal{J}^{(0,1)}_{\mathrm{DHS}}(x,p;a,b)\mathbf{\Gamma}_-(x,p;b),\end{aligned} \tag{4.4}$$

and since by the definition (4.1) of $\mathbf{\Gamma}_-$ the last two terms cancel each other, we obtain

$$d_x - \mathcal{J}(x,p;a,b) = \mathbf{\Gamma}_-(x,p;b)^{-1}\left(d_x - \mathcal{J}_{\mathrm{DHS}}(x,p;a,b)\right)\mathbf{\Gamma}_-(x,p;b). \tag{4.5}$$

Since gauge transformations preserve flatness, it follows that $d_x - \mathcal{J}(x,p;a,b)$ is flat which, combined with the fact that $\mathcal{J}$ is a $(1,0)$-form, implies that $\mathcal{J}$ must be purely holomorphic in $x \in \tilde{\Sigma}_p$ (but not in p and the moduli of the surface), namely $\mathbf{\Gamma}_-(x,p;b)$ can be used to gauge transform $\mathcal{J}_{\mathrm{DHS}}$ to a holomorphic $(1,0)$-form in $x \in \tilde{\Sigma}_p$.

Let us also consider the monodromy representation (see section 1.1)

$$\mu_-(\gamma,y;b) = \mathbf{\Gamma}_-(\gamma\cdot y, y; b), \tag{4.6}$$

so that one has

$$\mathbf{\Gamma}_-(\gamma\cdot x, y; b) = \mathbf{\Gamma}_-(x,y;b)\,\mu_-(\gamma,y;b). \tag{4.7}$$

It follows from its definition and from the single-valuedness of $\mathcal{J}_{\mathrm{DHS}}$ that $\mathcal{J}$ is multiple-valued in x, with monodromies given by

$$\begin{aligned}\mathcal{J}(\mathfrak{A}^K\cdot x,p;a,b) &= \mu_-(\mathfrak{A}^K,p;b)^{-1}\mathcal{J}(x,p;a,b)\,\mu_-(\mathfrak{A}^K,p;b),\\ \mathcal{J}(\mathfrak{B}_K\cdot x,p;a,b) &= \mu_-(\mathfrak{B}_K,p;b)^{-1}\mathcal{J}(x,p;a,b)\,\mu_-(\mathfrak{B}_K,p;b).\end{aligned} \tag{4.8}$$

The idea of the second step below will then be to gauge transform $\mathcal{K}_{\mathrm{E}}$ into a holomorphic $(1,0)$-form in $x \in \tilde{\Sigma}_p$ with the same monodromy properties (4.8) as $\mathcal{J}$, because then we know that by further applying the gauge transformation $\mathbf{\Gamma}_-(x,p;b)^{-1}$ we would obtain a single-valued smooth differential with the same $(0,1)$ component as $\mathcal{J}_{\mathrm{DHS}}$, thus matching two fundamental properties which uniquely characterize $\mathcal{J}_{\mathrm{DHS}}$ in Theorem 2.5.

4.1.2 Step 2: monodromies

Let us now consider the solution $\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)$ to the differential equation

$$d_x\,\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta) = \mathcal{K}_{\mathrm{E}}(x,p;\xi,\eta)\,\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta), \tag{4.9}$$

along with the initial condition $\mathbf{\Gamma}_{\mathrm{E}}(y,y,p;\xi,\eta) = 1$. As in section 3, ξ and η are arbitrary elements of $\mathfrak{g}^h$, so that $\mathcal{K}_{\mathrm{E}}$ and $\mathcal{U}_{\mathrm{E}}$ take values in $\mathfrak{g}$ and $\exp(\mathfrak{g})$, respectively, and we will later make a specific choice by requiring $\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)$ to satisfy suitable monodromy properties. The function $\mathbf{\Gamma}_{\mathrm{E}}$ can be explicitly constructed as the path-ordered integral

$$\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)=\mathrm{P}\exp\int_y^x \mathcal{K}_{\mathrm{E}}(t,p;\xi,\eta), \tag{4.10}$$

the integral being taken as usual in the universal cover $\tilde{\Sigma}_p$, and can be seen as a multiple-valued function of $x,y,p\in\Sigma$, with logarithmic singularities at $x=p$ and $y=p$ and holomorphic elsewhere.

Combining the path-concatenation formula (1.6) with the monodromies (2.6) of $\mathcal{K}_{\mathrm{E}}$, one obtains that, for every $K=1,\cdots,h$,

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{E}}(\mathfrak{A}^K\cdot x,y,p;\xi,\eta)&=\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)\,\mu_{\mathrm{E}}(\mathfrak{A}^K,y,p;\xi,\eta),\\ \mathbf{\Gamma}_{\mathrm{E}}(\mathfrak{B}_K\cdot x,y,p;\xi,\eta)&=e^{-2\pi i\eta_K}\,\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)\,e^{2\pi i\eta_K}\,\mu_{\mathrm{E}}(\mathfrak{B}_K,y,p;\xi,\eta),\end{aligned} \tag{4.11}$$

where for $\gamma\in\pi_1(\Sigma_p,y)$ we define $\mu_{\mathrm{E}}(\gamma,y,p;\xi,\eta)$ to be the element[23] of $\exp(\mathfrak{g})$ given by

$$\mu_{\mathrm{E}}(\gamma,y,p;\xi,\eta)=\mathbf{\Gamma}_{\mathrm{E}}(\gamma\cdot y,y,p;\xi,\eta). \tag{4.12}$$

As will become clearer in the sequel, one might be tempted to impose the condition that the monodromies of the differential

$$\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)^{-1}d_x\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)=\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta)^{-1}\mathcal{K}_{\mathrm{E}}(x,p;\xi,\eta)\,\mathbf{\Gamma}_{\mathrm{E}}(x,y,p;\xi,\eta) \tag{4.13}$$

should match the monodromies (4.8) of the differential $\mathcal{J}$ defined in (4.3). If the path-ordered exponentials involved had the same base-point, one would readily obtain the desired condition by combining (2.6) with (4.11). However, in the absence of further constraints on ξ and η, the path-ordered exponential in (4.10) diverges for $y=p$, thereby invalidating this approach. Instead, we shall seek to specialize ξ and η to (p and $y\neq p$ dependent) elements $\check{\xi}(y,p),\check{\eta}(y,p)$ of $\mathfrak{g}^h$ which satisfy the monodromy conditions adapted to $\mathcal{J}$ at integration base-point y, namely that, for every $K=1,\cdots,h$,

$$\begin{aligned}\mu_{\mathrm{E}}\big(\mathfrak{A}^K,y,p;\check{\xi}(y,p),\check{\eta}(y,p)\big)&=\mu_-(\mathfrak{A}^K,y;b),\\ e^{2\pi i\check{\eta}_K}\mu_{\mathrm{E}}\big(\mathfrak{B}_K,y,p;\check{\xi}(y,p),\check{\eta}(y,p)\big)&=\mu_-(\mathfrak{B}_K,y;b),\end{aligned} \tag{4.14}$$

with μ_- given by (4.6). Since the right sides of these equations are independent of a, the elements $\check{\xi}(y,p)$ and $\check{\eta}(y,p)$ naturally belong to the subspace $\mathfrak{g}_b^h$.

---

[23] Notice that, since $\mathcal{K}_{\mathrm{E}}$ is multiple-valued, the map $\mu_{\mathrm{E}}:\pi_1(\Sigma_p,y)\to\exp(\mathfrak{g})$ given by setting $\mu_{\mathrm{E}}(\gamma)=\mu_{\mathrm{E}}(\gamma,y,p;\xi,\eta)$ does not preserves multiplication, and is therefore not a homomorphism.

Lemma 4.1. *For any $y \neq p$ there exists a unique pair of elements $\check{\xi} = \check{\xi}(y,p), \check{\eta} = \check{\eta}(y,p)$ of $\mathfrak{g}_b^h$ which satisfy* (4.14)*. Their components satisfy the identity*

$$[\check{\eta}_I, \check{\xi}^I] = 0. \tag{4.15}$$

Similarly to the case of Lemma 3.1, notice that, intuitively, the first statement of this lemma is justified by the fact that there are $2h$ equations for $2h$ unknowns $\check{\xi}, \check{\eta}$.

Proof. The proof of the existence and uniqueness of $\check{\xi} = \check{\xi}(y,p)$ and $\check{\eta} = \check{\eta}(y,p)$ is similar to the proof of Lemma 3.1, because both sides of the equations (4.14) belong to $\exp(\mathfrak{g})$, and one can recursively (on the degree of the Lie monomials in the components of b) prove that their logarithms, which belong to $\mathfrak{g}$, have a unique solution $\check{\xi}, \check{\eta} \in \mathfrak{g}_b^h$. For the first step of the induction, notice that we have, at order one,

$$\begin{aligned} \log\left(\mu_{\mathrm{E}}(\mathfrak{A}^I, y, p; \check{\xi}, \check{\eta})\right) &= \check{\xi}^I + \mathcal{O}([\check{\xi}, \check{\xi}], [\check{\eta}, \check{\eta}], [\check{\xi}, \check{\eta}]), \\ \log\left(e^{2\pi i \check{\eta}_I} \mu_{\mathrm{E}}(\mathfrak{B}_I, y, p; \check{\xi}, \check{\eta})\right) &= \Omega_{IJ}\check{\xi}^J + 2\pi i \check{\eta}_I + \mathcal{O}([\check{\xi}, \check{\xi}], [\check{\eta}, \check{\eta}], [\check{\xi}, \check{\eta}]), \end{aligned} \tag{4.16}$$

as well as

$$\begin{aligned} \log\left(\mu_-(\mathfrak{A}^I, y; b)\right) &= -\pi Y^{IJ} b_J + \mathcal{O}([b,b]), \\ \log\left(\mu_-(\mathfrak{B}_I, y; b)\right) &= -\pi \bar{\Omega}_{IK} Y^{KJ} b_J + \mathcal{O}([b,b]), \end{aligned} \tag{4.17}$$

which we can equate to find that the components of $\check{\xi}, \check{\eta}$ that satisfy (4.14) are constrained at order one to be

$$\check{\xi}^I = -\pi Y^{IJ} b_J + \mathcal{O}([b,b]), \qquad \check{\eta}_I = b_I + \mathcal{O}([b,b]). \tag{4.18}$$

We are now left with proving that these elements $\check{\xi}, \check{\eta} \in \mathfrak{g}_b^h$ must necessarily satisfy the condition (4.15). First of all, even though the map $\mu_{\mathrm{E}} : \pi_1(\Sigma_p, y) \to \exp(\mathfrak{g})$ is not a homomorphism (see footnote 23), one can verify using the path-concatenation property (1.6) of path-ordered exponentials that, for any $y, p \in \Sigma$ and any $\xi, \eta \in \mathfrak{g}^h$, the map $\tilde{\mu}_{\mathrm{E}}$ defined on the generators of $\pi_1(\Sigma_p, y)$ by setting

$$\tilde{\mu}_{\mathrm{E}}(\mathfrak{A}^I) = \mu_{\mathrm{E}}(\mathfrak{A}^I, y, p; \xi, \eta), \qquad \tilde{\mu}_{\mathrm{E}}(\mathfrak{B}_I) = e^{2\pi i \eta_I} \mu_{\mathrm{E}}(\mathfrak{B}_I, y, p; \xi, \eta) \tag{4.19}$$

preserves multiplication and is therefore a homomorphism from $\pi_1(\Sigma_p, y)$ to $\exp(\mathfrak{g}_b)$. It follows from the first part of the statement that $\check{\xi}$ and $\check{\eta}$ are such that

$$\tilde{\mu}_{\mathrm{E}}(\mathfrak{A}^I) = \mu_-(\mathfrak{A}^I), \qquad \tilde{\mu}_{\mathrm{E}}(\mathfrak{B}_I) = \mu_-(\mathfrak{B}_I). \tag{4.20}$$

Because $\tilde{\mu}_{\rm E}$ is a homomorphism, one can use the same argument inspired by Cauchy's theorem exploited in the proof of Lemma 3.2 to deduce that

$$\prod_{I=1}^{h} \tilde{\mu}_{\rm E}(\mathfrak{A}^I)\tilde{\mu}_{\rm E}(\mathfrak{B}_I)\tilde{\mu}_{\rm E}(\mathfrak{A}^I)^{-1}\tilde{\mu}_{\rm E}(\mathfrak{B}_I)^{-1} = e^{2\pi i[\check{\eta}_I,\check{\xi}^I]} \tag{4.21}$$

Similarly, the fact that $\mu_- : \pi_1(\Sigma, y) \to \exp(\mathfrak{g}_b)$ is a homomorphism implies that

$$\prod_{I=1}^{h} \mu_-(\mathfrak{A}^I)\mu_-(\mathfrak{B}_I)\mu_-(\mathfrak{A}^I)^{-1}\mu_-(\mathfrak{B}_I)^{-1} = 1. \tag{4.22}$$

The identity (4.15) then follows from combining equations (4.20), (4.21) and (4.22). □

Remark 4.2. *Similar to the case of $\hat{\xi}(y,p)$ and $\hat{\eta}(y,p)$ from the first construction in section 3, the defining equations (4.14) for the elements $\check{\xi} = \check{\xi}(y,p)$ and $\check{\eta} = \check{\eta}(y,p)$ may in principle lead to a separate dependence on the base-point y of the $\mathfrak{A}$- and $\mathfrak{B}$-cycles and on p (as well as on the moduli of Σ). However, the result (4.15) of Lemma 4.1 will lead to Corollary 4.4 below showing that $\check{\xi}$ and $\check{\eta}$ do not depend on their second argument p.*

4.1.3 Combining the two steps

The fact that, by Lemma 4.1, the elements $\check{\xi}(y,p), \check{\eta}(y,p)$ of $\mathfrak{g}_b^h$ which satisfy (4.14) are such that $[\check{\eta}_I(y,p), \check{\xi}^I(y,p)] = 0$ implies that the residue of the pole of the differential $\mathcal{K}_{\rm E}(x,p;\check{\xi}(y,p),\check{\eta}(y,p))$ at $x = p$ vanishes, and therefore that this differential is holomorphic on the whole $\tilde{\Sigma}$. This implies the following analogue of Corollary 3.4.

Corollary 4.3. *For $\check{\xi} = \check{\xi}(y,p)$ and $\check{\eta} = \check{\eta}(y,p)$ obeying (4.14), $\mathcal{K}_{\rm E}(x,p;\check{\xi},\check{\eta})$ is independent of its second argument p (and henceforth will be denoted $\mathcal{K}_{\rm E}(x,\cdot\,;\check{\xi},\check{\eta})$), so that it may depend on p only through the corresponding dependence of $\check{\xi}$ and $\check{\eta}$.*

Proof. One could set up a proof based on uniqueness arguments of [37], but we shall instead give a constructive proof here: analogously to Corollary 3.4, one can establish the independence of $\mathcal{K}_{\rm E}(x,p;\check{\xi},\check{\eta})$ of its second argument p through the expansion

$$\mathcal{K}_{\rm E}(x,\cdot\,;\check{\xi},\check{\eta}) = \omega_J(x)\check{\xi}^J + \sum_{r=1}^{\infty} \varpi^{I_1\cdots I_r}{}_J(x)\check{H}_{I_1}\cdots\check{H}_{I_r}\check{\xi}^J, \tag{4.23}$$

where $\check{H}_I = [\check{\eta}_I, X]$ for all $X \in \mathfrak{g}$. Moreover, the differentials $\varpi^{I_1\cdots I_r}{}_J(x)$ are the traceless parts of $g^{I_1\cdots I_r}{}_J(x,p)$ in the following sense

$$\varpi^{I_1\cdots I_r}{}_J(x) = g^{I_1\cdots I_r}{}_J(x,p) - \frac{1}{h}\delta^{I_r}_J g^{I_1\cdots I_{r-1}K}{}_K(x,p) \tag{4.24}$$

and independent on p as the notation suggests (see [37] or section 9.1 of [63]). The manifestly p-independent expansion (4.23) of $\mathcal{K}_{\rm E}$ follows from that in (2.10) since the factors of $\delta^{I_r}_J$ produced by the difference $\varpi^{I_1\cdots I_r}{}_J(x) - g^{I_1\cdots I_r}{}_J(x,p)$ give rise to the vanishing commutator $\delta^{I_r}_J \check{H}_{I_r}\check{\xi}^J = [\check{\eta}_I, \check{\xi}^I] = 0$. □

Thanks to the result of Corollary 4.3, we can deduce also the following corollary, whose proof is analogous to that of Corollary 3.6, and will therefore be omitted.

Corollary 4.4. *The elements $\check{\xi}(y,p)$, $\check{\eta}(y,p) \in \mathfrak{g}^h_b$ determined by Lemma 4.1 are independent of p. They extend to smooth functions of $y \in \tilde{\Sigma}$, so that $\check{\xi}(y,\cdot), \check{\eta}(y,\cdot)$ are well-defined elements of $\mathfrak{g}^h_b$ also for $y = p$ which satisfy (and are uniquely determined by) the monodromy conditions*

$$\begin{aligned} \mu_{\rm E}\big(\mathfrak{A}^K, p, \cdot\,; \check{\xi}(p,\cdot\,), \check{\eta}(p,\cdot\,)\big) &= \mu_-(\mathfrak{A}^K, p; b), \\ e^{2\pi i \check{\eta}_K}\mu_{\rm E}\big(\mathfrak{B}_K, p, \cdot\,; \check{\xi}(p,\cdot\,), \check{\eta}(p,\cdot\,)\big) &= \mu_-(\mathfrak{B}_K, p; b). \end{aligned} \tag{4.25}$$

Here again the dot in the argument of a function is meant to stress the independence of the function on that argument.

Definition 4.5. *Henceforth, the notation $\check{\xi}, \check{\eta}$ will be used for the elements $\check{\xi}(p,\cdot\,), \check{\eta}(p,\cdot\,)$ of $\mathfrak{g}^h_b$ which solve the monodromy conditions* (4.25) *from Corollary 4.4 (i.e. abandoning the notation of Lemma 4.1 and Corollary 4.3).*

Definition 4.6. *For $\check{\xi}$ and $\check{\eta}$ as in Definition 4.5, we define the gauge transformation $\mathcal{U}_{\rm E}(x,p) = \mathcal{U}_{\rm E}(x,p;\check{\xi},\check{\eta})$ as the product of the specialization $\mathbf{\Gamma}_{\rm E}(x,p,\cdot\,;\check{\xi},\check{\eta})$ of $\mathbf{\Gamma}_{\rm E}(x,y,p;\xi,\eta)$ from* (4.9) *with the inverse of $\mathbf{\Gamma}_-(x,p;b)$, obtained by specializing* (4.1) *to $y=p$,*

$$\mathcal{U}_{\rm E}(x,p;\check{\xi},\check{\eta}) = \mathbf{\Gamma}_{\rm E}(x,p,\cdot\,;\check{\xi},\check{\eta})\,\mathbf{\Gamma}_-(x,p;b)^{-1} \tag{4.26}$$

with $\check{\xi}^I, \check{\eta}_I$ determined as a Lie series in b_K by the following equivalent of (4.25),

$$\begin{aligned} \mathcal{U}_{\rm E}(\mathfrak{A}^K\cdot p, p; \check{\xi},\check{\eta}) &= 1, \\ \mathcal{U}_{\rm E}(\mathfrak{B}_K\cdot p, p; \check{\xi},\check{\eta}) &= e^{-2\pi i \check{\eta}_K}. \end{aligned} \tag{4.27}$$

Note that (4.27) is the direct analogue of the monodromy conditions (3.30) that determine the generators $\hat{\xi}^I, \hat{\eta}_I$ of $\mathcal{U}_{\rm DHS}$ in terms of b_K.

4.2 Relating the connections $d - \mathcal{K}_{\rm E}$ and $d - \mathcal{J}_{\rm DHS}$

We are left with constructing suitable elements $\check{a}, \check{b} \in \mathfrak{g}^h$ that, combined with the previously constructed gauge transformations, will enable to obtain $\mathcal{J}_{\rm DHS}(x,p;a,b)$ out of $\mathcal{K}_{\rm E}(x,p;\check{a},\check{b})$. As we will see below, a key ingredient is the following statement.

Lemma 4.7. *There is a unique element $\tilde{a}$ of $\mathfrak{g}^h$, linear in a, which satisfies*

$$[b_J, a^J] = [\check{\eta}_J, \tilde{a}^J], \tag{4.28}$$

with $\check{\eta}$ as in Lemma 4.1, and $a \cup b \to \tilde{a} \cup \check{\eta}$ is a Lie automorphism of $\mathfrak{g}$.

Proof. This result can be proven along similar lines to the proof of Theorem 3.9 (see the discussion on the algorithmic determination of $\tilde{a}$ in section 4.3). However, we will take a different route: to prove existence, we will construct an automorphism θ of $\mathfrak{g}$ and show that its inverse ψ satisfies $\psi(b_I) = \check{\eta}_I$ and is such that the element $\tilde{a} = \psi(a)$ satisfies the conditions of the statement. We will conclude the proof by showing the uniqueness of $\tilde{a}$.

Recall from the proof of Lemma 4.1, equation (4.18), that $\check{\eta}_J = b_J + \mathcal{O}([b, b])$ for $J = 1, \cdots, h$, with $\check{\eta} \in \mathfrak{g}_b^h$. This implies that the map $b \to \check{\eta}$ can be inverted and the elements $\check{\eta}_J$ generate $\mathfrak{g}_b$, so it can be viewed as an automorphism of $\mathfrak{g}_b$. One can therefore view each b_J as an element of the completed free Lie algebra $\mathfrak{g}_{\check{\eta}} \simeq \mathfrak{g}_b$ generated by $\check{\eta}$. Since elements of $\mathfrak{g}_{\check{\eta}}$ can also be viewed as elements of the (associative) algebra $\mathbb{C}\langle\langle\check{\eta}\rangle\rangle$ of power series in the (non-commutative) variables $\check{\eta}_1, \cdots, \check{\eta}_h$, we can write

$$b_J = \check{\eta}_J + \sum_{r=2}^{\infty} \sigma^{I_1 \cdots I_r}{}_J \, \check{\eta}_{I_1} \cdots \check{\eta}_{I_r}, \tag{4.29}$$

with $\sigma^{I_1 \cdots I_r}{}_J \in \mathbb{C}$ depending on the moduli of Σ_p. If we set

$$s^I{}_J = \sum_{r=2}^{\infty} \sigma^{I I_2 \cdots I_r}{}_J \, b_{I_2} \cdots b_{I_r} \in \mathbb{C}\langle\langle b \rangle\rangle, \tag{4.30}$$

then since $s^I{}_J = \mathcal{O}(b)$ one can recursively (on the degree in b) construct elements $t^I{}_J \in \mathbb{C}\langle\langle b \rangle\rangle$ that satisfy the relation

$$t^I{}_J + s^I{}_K \, t^K{}_J = \delta^I_J, \tag{4.31}$$

so that the $h \times h$ matrix t with entries $t^I{}_J$ is the inverse of the matrix $I + s$ with entries $\delta^I_J + s^I{}_J$. Up to and including the second order in b_K, we find

$$t^I{}_J = \delta^I_J - \sigma^{IK}{}_J b_K + (\sigma^{IK_1}{}_L \sigma^{LK_2}{}_J - \sigma^{IK_1K_2}{}_J) b_{K_1} b_{K_2} + \mathcal{O}(b^3). \tag{4.32}$$

One can extend the usual adjoint action on $\mathfrak{g}$ given by $\mathrm{ad}(X)(Y) = [X, Y]$, with $X, Y \in \mathfrak{g}$, to elements $X = \sum_{r=0}^{\infty} X^{I_1 \cdots I_r} b_{I_1} \cdots b_{I_r}$ of $\mathbb{C}\langle\langle b \rangle\rangle$ by setting

$$\mathrm{ad}\bigg(\sum_{r=0}^{\infty} X^{I_1 I_2 \cdots I_r} b_{I_1} b_{I_2} \cdots b_{I_r} \bigg)(Y) = \sum_{r=0}^{\infty} X^{I_1 I_2 \cdots I_r} \, [b_{I_1}, [b_{I_2}, \cdots, [b_{I_r}, Y] \cdots]]. \tag{4.33}$$

Notice that, if X is in the first place an element of $\mathfrak{g}$, then one has the non-trivial identity, also known as Dynkin's lemma,

$$\Big[\sum_{r=0}^{\infty} X^{I_1\cdots I_r} b_{I_1}\cdots b_{I_r},\, Y\Big] = \mathrm{ad}\Big(\sum_{r=0}^{\infty} X^{I_1\cdots I_r} b_{I_1}\cdots b_{I_r}\Big)(Y), \tag{4.34}$$

where the right side of the equation is defined by (4.33).

Let us now define a Lie algebra endomorphism θ of $\mathfrak{g}$ by setting $\theta(a^I) = \mathrm{ad}(t^I{}_J)(a^J)$ and $\theta(\check{\eta}_I) = b_I$, with $\mathrm{ad}(t^I{}_J)(a^J)$ as in (4.33). As remarked above, the assignment $\check{\eta} \to b$ induces an automorphism of $\mathfrak{g}_b$, which combined with the invertibility of the matrix $(t^I{}_J)$ implies that θ is invertible, and is therefore an automorphism of $\mathfrak{g}$. Notice that θ preserves the degree in the variables a^I of the generators ($\theta(a^I)$ has degree one, $\theta(b_I)$ has degree zero), hence it preserves the degree of every element. In the remainder we will show that, if we define ψ to be the automorphism given by the inverse of θ, then ψ satisfies all the desired properties.

By construction we immediately obtain $\psi(b_I) = \check{\eta}_I$. Moreover, notice that, since θ preserves the a-degree, then so does ψ, and therefore $\tilde{a} = \psi(a)$ is linear in a, as requested. We are left to show that eq. (4.28) holds for this $\tilde{a}$, namely that

$$\psi([b_J, a^J]) = [b_J, a^J]. \tag{4.35}$$

As a first step, notice that we can combine the formula (4.29) with Dynkin's lemma (4.34) to obtain the identity

$$[b_J, a^J] = [\check{\eta}_J, a^J] + \sum_{r=2}^{\infty} \sigma^{I_1\cdots I_r}{}_J [\check{\eta}_{I_1}\cdots\check{\eta}_{I_r}, a^J] = \mathrm{ad}\Big(\check{\eta}_J + \sum_{r=2}^{\infty} \sigma^{I_1\cdots I_r}{}_J\, \check{\eta}_{I_1}\ldots\check{\eta}_{I_r}\Big)(a^J). \tag{4.36}$$

Applying $\theta(\check{\eta}_I) = b_I$ on these identities we find

$$\begin{aligned}
\theta([b_J, a^J]) &= \mathrm{ad}\Big(b_J + \sum_{r=2}^{\infty} \sigma^{I_1\cdots I_r}{}_J\, b_{I_1}\cdots b_{I_r}\Big)\big(\theta(a^J)\big) \\
&= \mathrm{ad}(b_J)\big(\theta(a^J)\big) + \mathrm{ad}\Big(\sum_{r=2}^{\infty} \sigma^{II_2\cdots I_r}{}_J\, b_I\, b_{I_2}\cdots b_{I_r}\Big)\big(\theta(a^J)\big) \\
&= \mathrm{ad}(b_I)\Big(\theta(a^I) + \mathrm{ad}\Big(\sum_{r=2}^{\infty} \sigma^{II_2\cdots I_r}{}_J\, b_{I_2}\cdots b_{I_r}\Big)\big(\theta(a^J)\big)\Big) \\
&= \mathrm{ad}(b_I)\Big(\theta(a^I) + \mathrm{ad}(s^I{}_J)\big(\theta(a^J)\big)\Big) = \mathrm{ad}(b_I)\Big(\mathrm{ad}(\delta^I_J + s^I{}_J)\big(\theta(a^J)\big)\Big) \\
&= \mathrm{ad}(b_I)\Big(\mathrm{ad}(\delta^I_J + s^I{}_J)\big(\mathrm{ad}(t^J{}_K)(a^K)\big)\Big) = \mathrm{ad}(b_I)\Big(\mathrm{ad}\big((\delta^I_J + s^I{}_J)t^J{}_K\big)(a^K)\Big) \\
&= \mathrm{ad}(b_I)\big(\mathrm{ad}(\delta^I_K)(a^K)\big) = [b_I, a^I].
\end{aligned} \tag{4.37}$$

Since $\psi \circ \theta = \mathrm{Id}$, applying ψ to the first and the last term of this chain of identities we obtain $[b_J, a^J] = \psi([b_J, a^J])$.

Finally, let us prove uniqueness of $\tilde{a} \in \mathfrak{g}^h$ as in the statement. Suppose that $\tilde{a}' \in \mathfrak{g}^h$ satisfies the same properties, then we would get an element $c = \tilde{a}' - \tilde{a} \in \mathfrak{g}^h$ which is linear in a and such that $[\check{\eta}_J, c^J] = 0$, but this is impossible, because the left side would have degree 1 in a, whereas the right side has degree 0. □

We shall now prove that $\mathcal{U}_{\mathrm{E}}(x, p; \check{\xi}, \check{\eta})$ indeed provides the gauge transformation required in the relation (1.16) between $\mathcal{K}_{\mathrm{E}}(x, p; \check{a}, \check{b})$ and $\mathcal{J}_{\mathrm{DHS}}(x, p; a, b)$, for suitable $\check{a}, \check{b} \in \mathfrak{g}^h$ constructed out of the elements $\check{\xi}, \check{\eta} \in \mathfrak{g}^h_b$ from Definition 4.5 and of the element $\tilde{a} \in \mathfrak{g}^h$ from Lemma 4.7. The result is summarized in the following theorem.

Theorem 4.8. *The flat connections $d_x - \mathcal{J}_{\mathrm{DHS}}(x, p; a, b)$ and $d_x - \mathcal{K}_{\mathrm{E}}(x, p; \check{a}, \check{b})$ are related by the gauge transformation $\mathcal{U}_{\mathrm{E}}(x, p) = \mathcal{U}_{\mathrm{E}}(x, p; \check{\xi}, \check{\eta})$ defined by (4.26), whose arguments $\check{\xi}, \check{\eta} \in \mathfrak{g}^h_b$ are the uniquely determined solutions of (4.27), namely we have*

$$d_x - \mathcal{J}_{\mathrm{DHS}}(x, p; a, b) = \mathcal{U}_{\mathrm{E}}(x, p; \check{\xi}, \check{\eta})^{-1}\Big(d_x - \mathcal{K}_{\mathrm{E}}(x, p; \check{a}, \check{b})\Big)\mathcal{U}_{\mathrm{E}}(x, p; \check{\xi}, \check{\eta}). \tag{4.38}$$

The elements $\check{a}$ and $\check{b}$ are defined as $\check{a} = \tilde{a} + \check{\xi}$ and $\check{b} = \check{\eta}$, with $\tilde{a} \in \mathfrak{g}^h$ as in Lemma 4.7 and $\check{\xi}, \check{\eta} \in \mathfrak{g}^h_b$ as in Definition 4.5.

Proof. We want to prove that the connection on the right side of (4.38) satisfies the properties of Theorem 2.5 which uniquely characterize the DHS connection. Recall from its defining equation (4.26) that $\mathcal{U}_{\mathrm{E}}(x, p; \check{\xi}, \check{\eta}) = \mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})\,\mathbf{\Gamma}_{-}(x, p; b)^{-1}$.

First of all, by definition of $\mathbf{\Gamma}_{\mathrm{E}}$ and by the linearity of $\mathcal{K}_{\mathrm{E}}(x, p; \xi, \eta)$ in its argument ξ, one has

$$\begin{aligned}
\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})^{-1}&\mathcal{K}_{\mathrm{E}}(x, p; \tilde{a} + \check{\xi}, \check{\eta})\,\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta}) - \mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})^{-1}\,d_x\,\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta}) \\
&= \mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})^{-1}\Big(\mathcal{K}_{\mathrm{E}}(x, p; \tilde{a} + \check{\xi}, \check{\eta}) - \mathcal{K}_{\mathrm{E}}(x, p; \check{\xi}, \check{\eta})\Big)\,\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta}) \\
&= \mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})^{-1}\mathcal{K}_{\mathrm{E}}(x, p; \tilde{a}, \check{\eta})\,\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta}).
\end{aligned} \tag{4.39}$$

Equation (4.38) is therefore equivalent to proving that

$$\begin{aligned}
\mathcal{J}_{\mathrm{DHS}}(x, p; a, b) = {}& \mathbf{\Gamma}_{-}(x, p; b)\,\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})^{-1}\mathcal{K}_{\mathrm{E}}(x, p; \tilde{a}, \check{\eta})\,\mathbf{\Gamma}_{\mathrm{E}}(x, p, \cdot\,; \check{\xi}, \check{\eta})\,\mathbf{\Gamma}_{-}(x, p; b)^{-1} \\
& - \mathbf{\Gamma}_{-}(x, p; b)\,d_x\,\mathbf{\Gamma}_{-}(x, p; b)^{-1}.
\end{aligned} \tag{4.40}$$

Notice that the first term on the right side of (4.40) is a $(1, 0)$-form, whereas the second term can be rewritten as $(d_x\,\mathbf{\Gamma}_{-}(x, p; b))\,\mathbf{\Gamma}_{-}(x, p; b)^{-1}$, which by definition of $\mathbf{\Gamma}_{-}$ is equal to $\mathcal{J}^{(0,1)}_{\mathrm{DHS}}(x, p; a, b)$. This implies that item 2 of Theorem 2.5 is satisfied. Moreover, putting together eqs. (2.6), (4.7), (4.11) and (4.25), it follows that the first term on the right side of (4.40) is single-valued as a differential in x, and by construction of $\tilde{a}$, together with the fact that $\check{\xi}, \check{\eta} \in \mathfrak{g}^h_b$, it is linear in a, thus verifying item 3. Finally, item 1 follows by combining (4.28) with the flatness of the right side of (4.38), which in turn follows from the flatness of $\mathcal{K}_{\rm E}$ and the fact that gauge transformations preserve flatness. □

As an intermediate step of the proof, we have verified the validity of (4.40), which can be combined with the identity $(d_x \boldsymbol{\Gamma}_-(x,p;b))\boldsymbol{\Gamma}_-(x,p;b)^{-1} = \mathcal{J}^{(0,1)}_{\rm DHS}(x,p;a,b)$ to obtain the following immediate consequence.

Corollary 4.9. *For $\check{\xi}, \check{\eta} \in \mathfrak{g}^h_b$ as in Definition 4.5 and $\tilde{a} \in \mathfrak{g}^h$ as in Lemma 4.7, one has*

$$\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b) = \mathcal{U}_{\rm E}(x,p;\check{\xi},\check{\eta})^{-1}\mathcal{K}_{\rm E}(x,p;\tilde{a},\check{\eta})\,\mathcal{U}_{\rm E}(x,p;\check{\xi},\check{\eta}). \tag{4.41}$$

This identity can be exploited to compare the expansion coefficients $f^{I_1\cdots I_r}{}_J(x,p)$ of $\mathcal{J}^{(1,0)}_{\rm DHS}$ with the expansion coefficients $g^{I_1\cdots I_r}{}_J(x,p)$ of $\mathcal{K}_{\rm E}$ (see section 4.3).

Remark 4.10. *Similar to Remark 3.11, restricting the conjugation (4.41) to genus $h=1$ reproduces the gauge transformation (2.42) relating the Brown–Levin and Calaque–Enriquez–Etingof connections. More specifically, it is convenient to rearrange (2.43) into*

$$\mathcal{J}^{(1,0)}_{\rm DHS}(x,p;a,b)\Big|_{h=1} = \mathcal{U}_{\rm BL}(x-p)\,\mathcal{K}_{\rm E}(x,p;a,b)\Big|_{h=1}\mathcal{U}_{\rm BL}(x-p)^{-1}, \tag{4.42}$$

where $\mathcal{U}_{\rm BL}(x-p) = \exp(\frac{2\pi i b}{\operatorname{Im}\tau}\operatorname{Im}(x-p))$, and the $(1,0)$-form part of $\mathcal{U}_{\rm BL}(x-p)^{-1}d_x\,\mathcal{U}_{\rm BL}(x-p)$ cancels the admixture of b in the genus-one generator $\hat{a} = a + \pi b/(\operatorname{Im}\tau)$. In order to recover (4.42) from the genus-one instance of (4.41), we specialize the factorized form $\mathcal{U}_{\rm E}(x,p;\check{\xi},\check{\eta})^{-1} = \boldsymbol{\Gamma}_-(x,p;b)\,\boldsymbol{\Gamma}_{\rm E}(x,p,\cdot\,;\check{\xi},\check{\eta})^{-1}$ of (4.26) to $h=1$. Since $\check{\xi} = -\pi b/(\operatorname{Im}\tau)$ at genus one, the two factors reduce to $\boldsymbol{\Gamma}_{\rm E}(x,p,\cdot\,;\check{\xi},\check{\eta})^{-1}|_{h=1} = \exp(\frac{\pi b}{\operatorname{Im}\tau}(x-p))$ as well as $\boldsymbol{\Gamma}_-(x,p;b)|_{h=1} = \exp(-\frac{\pi b}{\operatorname{Im}\tau}(\bar{x}-\bar{p}))$, respectively. As a result, we have $\mathcal{U}_{\rm E}(x,p;\check{\xi},\check{\eta})^{-1}|_{h=1} = \mathcal{U}_{\rm BL}(x-p)$. Since the element $\tilde{a} = \psi(a)$ of Lemma 4.7 reduces to a at genus one[24] this completes our derivation of (4.42) from the specialization of (4.41) to $h=1$.

Note that the genus-one generators correspond to the placement of uppercase and lowercase indices according to $a = a^1$, $b = b_1$ and $\check{\xi} = \check{\xi}^1$, $\check{\eta} = \check{\eta}_1$.

[24] This follows from the fact that the expansion coefficients $\mathcal{R}^{II_1\cdots I_r}{}_J$ of $\tilde{a}^I$ in (4.48) below with $r \geq 1$ are polynomials in coefficients $\mathcal{Q}_I{}^{J_1\cdots J_r}$ of a Lie series (4.44) which obey shuffle relations in their upper indices $J_1, \cdots, J_r$ and thereby vanish at genus one.

4.3 Iterative construction relating $\mathcal{J}_{\mathrm{DHS}}$ and $\mathcal{K}_{\mathrm{E}}$

As in the case of the first construction of a gauge transformation, we conclude this section by outlining an algorithmic procedure to iteratively construct the gauge transformation $\mathcal{U}_{\mathrm{E}}$ and the Lie algebra automorphism $a \cup b \to \check{a} \cup \check{b}$. The steps of the procedure are similar to those spelled out in section 3.3 (and associated appendices), therefore we will give fewer details and present just a summary, together with the resulting low-degree formulas.

The main challenge is to obtain expressions for the first orders of the series expansion in the set of generators b of the elements $\check{\xi}, \check{\eta} \in \mathfrak{g}^h_b$ from Lemma 4.1, as well as the series expansion of $\tilde{a} \in \mathfrak{g}^h$ from Lemma 4.7 (we recall that $\check{a} = \tilde{a} - \check{\xi}$, and $\check{b} = \check{\eta}$). Notice that, if we simply want to relate the differentials $f^{I_1 \cdots I_r}{}_J(x,p)$ and $g^{I_1 \cdots I_r}{}_J(x,p)$, we can bypass Theorem 4.8 and use instead Corollary 4.9.

4.3.1 Second order of $\check{\xi}$ and $\check{\eta}$ in b

We have already spelled out in the proof of Lemma 4.1 how to obtain the first order of the solutions $\check{\xi}, \check{\eta}$ to the system of equations (4.25), which we repeat here for convenience,

$$\check{\xi}^I = -\pi Y^{IJ} b_J + \mathcal{O}(b^2), \qquad \check{\eta}_I = b_I + \mathcal{O}(b^2) \tag{4.43}$$

and which takes the same form with the choice of basepoint $y = p$ fixed in Definition 4.5. More generally, we need to determine two families of coefficients $\mathcal{P}^{IJ_1 \cdots J_r}$ and $\mathcal{Q}_I{}^{J_1 \cdots J_r}$, dependent on the moduli of Σ_p, such that

$$\begin{aligned} \check{\xi}^I &= \sum_{r=1}^{\infty} \mathcal{P}^{IJ_1 \cdots J_r} b_{J_1} \cdots b_{J_r}, \\ \check{\eta}_I &= \sum_{r=1}^{\infty} \mathcal{Q}_I{}^{J_1 \cdots J_r} b_{J_1} \cdots b_{J_r} \end{aligned} \tag{4.44}$$

and such that the monodromy conditions (4.25) hold, and we already know by (4.43) that $\mathcal{P}^{IJ} = -\pi Y^{IJ}$ and $\mathcal{Q}_I{}^J = \delta^J_I$. Combining the degree-one terms with the first equation in (4.25), one obtains

$$\begin{aligned} \mathcal{P}^{IJ_1J_2} &= -2\pi^2 i \,\mathrm{Im}\bigg(\int_p^{\mathfrak{A}^I \cdot p} \omega^{J_1}(t_1) \int_p^{t_1} \omega^{J_2}(t_2) \bigg) \\ &\quad + \pi \int_p^{\mathfrak{A}^I \cdot p} \big(\varpi^{J_1}{}_K(t) Y^{KJ_2} - \varpi^{J_2}{}_K(t) Y^{KJ_1} \big) \\ &= i\pi^2 \bigg\{ \delta^{IJ_1}_K Y^{KJ_2} - \delta^{IJ_2}_K Y^{KJ_1} - 2\,\mathrm{Im}\bigg(\int_p^{\mathfrak{A}^I \cdot p} \omega^{J_1}(t_1) \int_p^{t_1} \omega^{J_2}(t_2) \bigg) \bigg\}, \end{aligned} \tag{4.45}$$

where the traceless part $\varpi^{J_1}{}_K(t)$ of the differential $g^{J_1}{}_K(t,p)$ is defined in (4.24). The second equality in (4.45) follows from the formula (2.11) for the $\mathfrak{A}$-cycle integrals of the g-differentials.

Combining the degree-one terms with the second equation in (4.25) one obtains

$$\mathcal{Q}_I{}^{J_1J_2} = -\pi\,\mathrm{Im}\bigg(\int_p^{\mathfrak{B}_I\cdot p}\omega^{J_1}(t_1)\int_p^{t_1}\omega^{J_2}(t_2)\bigg) - \pi i\delta_I^{J_1}\delta_I^{J_2} - \frac{1}{2\pi i}\Omega_{IK}\mathcal{P}^{KJ_1J_2} \\ + \pi\delta_I^{J_1}\Omega_{IK}Y^{KJ_2} - \frac{i}{2}\int_p^{\mathfrak{B}_I\cdot p}\big(\varpi^{J_1}{}_K(t)Y^{KJ_2} - \varpi^{J_2}{}_K(t)Y^{KJ_1}\big), \tag{4.46}$$

which in turn determines $\mathcal{Q}_I{}^{J_1J_2}$ upon substituting (4.45) into (4.46). Similarly, one may recursively obtain the higher-order terms.

4.3.2 Implementing the automorphism $a \cup b \to \check{a} \cup \check{b}$

As for the computation of the element $\tilde{a} \in \mathfrak{g}^h$ from Lemma 4.7, one can either follow the steps of the proof of Lemma 4.7, or directly use the condition

$$[b_I, a^I] = [\check{\eta}_I, \tilde{a}^I], \tag{4.47}$$

together with the Ansatz[25] (implied by imposing linearity of $\tilde{a}$ in a)

$$\tilde{a}^I = \sum_{r=0}^{\infty}\mathcal{R}^{II_1\cdots I_r}{}_J B_{I_1}\cdots B_{I_r}\, a^J, \tag{4.48}$$

where as usual $B_I X = [b_I, X]$ for all $X \in \mathfrak{g}$, and $\mathcal{R}^{II_1\cdots I_r}{}_J$ depend on the moduli of Σ_p. We will take here the second route. It follows immediately from (4.43) and (4.48) that

$$[\check{\eta}_I, \tilde{a}^I] = [b_I, \mathcal{R}^I{}_J\, a^J] + \mathcal{O}(b^2), \tag{4.49}$$

from which we deduce that (4.47) requires $\mathcal{R}^I{}_J = \delta^I_J$ and therefore, at first order,

$$\tilde{a}^I = a^I + \mathcal{O}(b). \tag{4.50}$$

This can in turn be used to compute the second order of the right side of (4.47),

$$[\check{\eta}_I, \tilde{a}^I] = [b_I, a^I] + [\mathcal{Q}_I{}^{J_1J_2}b_{J_1}b_{J_2}, a^I] + [b_I, \mathcal{R}^{II_1}{}_J[b_{I_1}, a^J]] + \mathcal{O}(b^3) \\ = [b_I, a^I] + \mathcal{Q}_I{}^{J_1J_2}[b_{J_1},[b_{J_2}, a^I]] + \mathcal{R}^{II_1}{}_J[b_I,[b_{I_1}, a^J]] + \mathcal{O}(b^3), \tag{4.51}$$

[25]Notice that this is the analogue in this setting of (3.48).

where in the second equality we made use of Dynkin's lemma (4.34). Comparing this with the left side of (4.47) immediately yields

$$\mathcal{R}^{II_1}{}_J = -\mathcal{Q}_J{}^{II_1}, \tag{4.52}$$

with $\mathcal{Q}_J{}^{II_1}$ as in (4.46), and one can recursively iterate this procedure to obtain the coefficient $\mathcal{R}^{II_1\cdots I_r}{}_J$ in terms of $\mathcal{Q}_J{}^{I_1\cdots I_p}$ and of $\mathcal{R}^{II_1\cdots I_q}{}_J$ with $p \leq r$ and $q < r$. In fact, the computations of section 3.3.3 can be straightforwardly adapted from the case of $[\hat{\eta}_I, \hat{a}^I - \hat{\xi}^I] = [b_I, a^I]$ to the present case of $[\check{\eta}_I, \tilde{a}^I] = [b_I, a^I]$. More specifically, the results of Corollary 3.13 translate into the following all-order relation between the expansion coefficients $\mathcal{Q}_J{}^{I_1\cdots I_r}$ of $\check{\eta}_I$ and $\mathcal{R}^{II_1\cdots I_q}{}_J$ of $\tilde{a}^I$:

$$\begin{aligned}\mathcal{Q}_J{}^{I_1\cdots I_r} = -\mathcal{R}^{I_1\cdots I_r}{}_J - \sum_{\ell=1}^{r-2}(-1)^\ell \sum_{2\leq j_1<j_2<\cdots<j_\ell}^{r-1} &\mathcal{R}^{I_1I_2\cdots I_{j_1}}{}_{K_1}\mathcal{R}^{K_1I_{j_1+1}\cdots I_{j_2}}{}_{K_2} \times \cdots \\ &\times \cdots \mathcal{R}^{K_{\ell-1}I_{j_{\ell-1}+1}\cdots I_{j_\ell}}{}_{K_\ell}\mathcal{R}^{K_\ell I_{j_\ell+1}\cdots I_r}{}_J.\end{aligned} \tag{4.53}$$

Note that, as a consequence of the expansion (4.44) and $\hat{\eta}_I \in \mathfrak{g}_b$, the coefficients $\mathcal{Q}_J{}^{I_1\cdots I_r}$ obey shuffle relations (see (B.8), (B.9) for the shuffle product)

$$\mathcal{Q}_J{}^{I_1\cdots I_r ⧢ K_1\cdots K_s} = 0, \qquad r,s \geq 1, \tag{4.54}$$

which via (4.53) imply similar relations with admixtures of lower-rank terms for the coefficients $\mathcal{R}^{II_1\cdots I_q}{}_J$ in (4.48).

4.3.3 Expressing the f-differentials in terms of the g-differentials

We are now in the position to apply Corollary 4.9 to relate the f-differentials and the g-differentials at degree two[26]. To do so, first notice that the degree-one computation of $\check{\xi}, \check{\eta}$ is sufficient to obtain the expression

$$\mathcal{U}_{\mathrm{E}}(x,p) = 1 - 2\pi i\, \mathrm{Im}\bigg(\int_p^x \omega^I(t)\bigg) b_I + \mathcal{O}(b^2). \tag{4.55}$$

[26]At degree one, it can easily be checked that Corollary 4.9 yields the identity $f_J(x,p) = g_J(x,p)$, which is correct because we have seen that both sides are equal to the normalized holomorphic Abelian differential $\omega_J(x)$.

Recall that, if $X, Y \in \mathfrak{g}$, then $\exp(X)Y\exp(X)^{-1} = \exp(\mathrm{ad}(X))(Y)$. The expression $\mathcal{U}_{\mathrm{E}}(x,p;\check{\xi},\check{\eta})^{-1}\mathcal{K}_{\mathrm{E}}(x,p;\tilde{a},\check{\eta})\,\mathcal{U}_{\mathrm{E}}(x,p;\check{\xi},\check{\eta})$ on the right side of (4.41) is therefore equal to

$$\begin{aligned} &g_J(x,p)\tilde{a}^J + \left(g^I{}_J(x,p) + 2\pi i\,\mathrm{Im}\left(\int_p^x \omega^I(t)\right)g_J(x,p)\right)[\check{\eta}_I,\tilde{a}^J] + \mathcal{O}(b^2) \\ &= \omega_J(x)a^J + \left(g^I{}_J(x,p) + 2\pi i\,\mathrm{Im}\left(\int_p^x \omega^I(t)\right)\omega_J(x) - \mathcal{Q}_J{}^{KI}\omega_K(x)\right)[b_I,a^J] + \mathcal{O}(b^2), \end{aligned} \tag{4.56}$$

where to pass from the first to the second line we used the known low-degree expressions for $\check{\eta}_I$ and $\tilde{a}^J$, as well as the fact that $g_J(x,p) = \omega_J(x)$. Corollary 4.9 then implies that

$$f^I{}_J(x,p) = g^I{}_J(x,p) + 2\pi i\,\mathrm{Im}\left(\int_p^x \omega^I(t)\right)\omega_J(x) + \omega_K(x)\mathcal{R}^{KI}{}_J(p), \tag{4.57}$$

where we have exposed the p-dependence of the coefficients $\mathcal{R}^{KI}{}_J(p)$ to highlight the analogy with the relation between $f^I{}_J(x,p)$ and $g^I{}_J(x,p)$ in the first line of (1.15): using the expression $\mathcal{T}^I(x,p) = -2\pi i\mathrm{Im}\int_p^x \omega^I$ by (E.3), we find the relation

$$\mathcal{R}^{KI}{}_J(p) = -\mathcal{M}^{KI}{}_J(p) \tag{4.58}$$

by matching (4.57) with the first line of (1.15), where the explicit form of the coefficient $\mathcal{M}^{KI}{}_J(p)$ in the expansion (3.48) can be found in (E.23) and (E.27). Note that both sides of (4.58) additionally depend on the moduli of Σ.

5 Comparing spaces of higher-genus polylogarithms

To compare the higher-genus polylogarithms that arise from the connections $\mathcal{J}_{\rm DHS}$ and $\mathcal{K}_{\rm E}$, we begin by comparing their generating series $\mathbf{\Gamma}_{\rm E}$ and $\mathbf{\Gamma}_{\rm DHS}$ whose expansions are displayed in (2.16) and (2.31) and given in terms of the path-ordered exponentials of $\mathcal{J}_{\rm DHS}$ and $\mathcal{K}_{\rm E}$, respectively.

Lemma 5.1. *The generating series of higher-genus polylogarithms $\mathbf{\Gamma}_{\rm E}$ and $\mathbf{\Gamma}_{\rm DHS}$ are related by*

$$\mathbf{\Gamma}_{\rm E}(x,y,p;a,b) = \mathcal{U}_{\rm DHS}(x,p)^{-1}\,\mathbf{\Gamma}_{\rm DHS}(x,y,p;\hat{a},\hat{b})\,\mathcal{U}_{\rm DHS}(y,p) \tag{5.1}$$

as well as by

$$\mathbf{\Gamma}_{\rm DHS}(x,y,p;a,b) = \mathcal{U}_{\rm E}(x,p)^{-1}\,\mathbf{\Gamma}_{\rm E}(x,y,p;\check{a},\check{b})\;\mathcal{U}_{\rm E}(y,p), \tag{5.2}$$

with $\mathcal{U}_{\rm DHS}(x,p), \hat{a}, \hat{b}$ as in section 3 and $\mathcal{U}_{\rm E}(x,p), \check{a}, \check{b}$ as in section 4.

Proof. It follows from Theorem 3.9 (resp. Theorem 4.8) that both sides of (5.1) (resp. (5.2)) satisfy the same differential equation with the same initial conditions. □

Recall from section 1.2 that, given a flat connection $d - \mathcal{J}$, one can define a space $\mathcal{H}(\mathcal{J})$ of polylogarithms associated with $\mathcal{J}$, which is the ring of multiple-valued functions generated over $\mathbb{C}$ by all the polylogarithms $\Gamma(\mathfrak{w};x,y)$, namely the coefficients of the path-ordered exponential $\mathbf{\Gamma}(x,y;c) = \mathrm{P}\exp\int_y^x \mathcal{J}(t;c)$, as specified by (1.9). The polylogarithms $\Gamma(\mathfrak{w};x,y)$ are considered here only as functions of x, namely we fix the Riemann surface and we fix a choice of integration base-point $y \in \tilde{\Sigma}_p$, hence the coefficients of the ring $\mathcal{H}(\mathcal{J})$ can be functions of these parameters. As a consequence of the two constructions presented in sections 3 and 4, respectively, we are able to deduce the following precise relation between the spaces of polylogarithms associated with $\mathcal{J}_{\rm DHS}$ and $\mathcal{K}_{\rm E}$.

Theorem 5.2. *One has*

$$\mathcal{H}(\mathcal{J}_{\rm DHS}) = \mathcal{H}(\mathcal{K}_{\rm E}) \cdot \mathcal{H}(\mathcal{J}_{\rm DHS}^{(0,1)}), \tag{5.3}$$

where $\mathcal{H}(\mathcal{J}_{\rm DHS}^{(0,1)})$ denotes the ring of polynomials in the anti-holomorphic iterated integrals $\int_p^x \bar{\omega}_{I_1}(t_1)\int_p^{t_1}\bar{\omega}_{I_2}(t_2)\cdots\int_p^{t_{r-1}}\bar{\omega}_{I_r}(t_r)$, $r \geq 1$, with complex coefficients which may depend on the other fixed parameters, such as y or the moduli of Σ_p.

Proof. The generators of $\mathcal{H}(\mathcal{J}_{\rm DHS}^{(0,1)})$ are immediately deduced from the expansion of the path-ordered exponential of $\mathcal{J}_{\rm DHS}^{(0,1)}(x;b) = -\pi b_I \bar{\omega}^I(x)$, see section 4.1.1.

The inclusion $\mathcal{H}(\mathcal{J}_{\rm DHS}) \subset \mathcal{H}(\mathcal{K}_{\rm E}) \cdot \mathcal{H}(\mathcal{J}^{(0,1)}_{\rm DHS})$ follows by combining the identity (5.2) with the expansions displayed in (2.16) and (2.31), because we recall that $\mathcal{U}_{\rm E}(x,p) = \mathbf{\Gamma}_{\rm E}(x,p,p;\check{\xi},\check{\eta})\,\mathbf{\Gamma}_{-}(x,p;b)^{-1}$, with $\mathbf{\Gamma}_{-}(x,p;b) = \mathrm{P}\exp\int_p^x \mathcal{J}^{(0,1)}_{\rm DHS}(t;b)$ and $\check{\xi},\check{\eta}$ as in Definition 4.5.

To conclude the proof we need to show the opposite inclusion $\mathcal{H}(\mathcal{K}_{\rm E}) \cdot \mathcal{H}(\mathcal{J}^{(0,1)}_{\rm DHS}) \subset \mathcal{H}(\mathcal{J}_{\rm DHS})$. This follows by combining the inclusion $\mathcal{H}(\mathcal{K}_{\rm E}) \subset \mathcal{H}(\mathcal{J}_{\rm DHS})$, which is a consequence of (5.1) and of the fact that $\mathcal{U}_{\rm DHS}(x,p) = \mathbf{\Gamma}_{\rm DHS}(x,p,p;\hat{\xi},\hat{\eta})$ with $\hat{\xi},\hat{\eta}$ as in Definition 3.7, with the obvious inclusion $\mathcal{H}(\mathcal{J}^{(0,1)}_{\rm DHS}) \subset \mathcal{H}(\mathcal{J}_{\rm DHS})$. □

If we denote by $\mathrm{Hol}(\tilde{\Sigma}_p)$ the space of all holomorphic multiple-valued functions on Σ_p, then we have the following consequence of Theorem 5.2.

Corollary 5.3. *One has*

$$\mathcal{H}(\mathcal{K}_{\rm E}) = \mathcal{H}(\mathcal{J}_{\rm DHS}) \cap \mathrm{Hol}(\tilde{\Sigma}_p). \tag{5.4}$$

Proof. The statement follows by combining (5.3) with the fact that $\mathcal{J}^{(0,1)}_{\rm DHS}(x;b)$ is purely anti-holomorphic in x. □

A Relating flat connections on trivial bundles

In this appendix we want to show the claim from the introduction that two flat connections on the trivial principal $\mathfrak{g}$-bundle over Σ_p are necessarily related by a combination of an automorphism of $\mathfrak{g}$ and a gauge transformation, as long as their monodromy representations satisfy a technical condition.

Let us first fix the notation: for M a smooth real manifold, we denote by $E^k(M)$ the space of smooth $\mathbb{C}$-valued differential forms of degree k. We then set $E^k(\Sigma_p, \mathfrak{g}) = E^k(\Sigma_p)\hat{\otimes}_{\mathbb{C}}\,\mathfrak{g}$.

Proposition A.1. *Suppose that $\mathcal{J}_1, \mathcal{J}_2 \in E^1(\Sigma_p, \mathfrak{g})$ give rise to two flat connections $d - \mathcal{J}_i$ on the trivial principal $\mathfrak{g}$-bundle over Σ_p, $i = 1, 2$, and suppose that the associated monodromy representations $\mu_1, \mu_2 : \pi_1(\Sigma_p, y) \to \exp(\mathfrak{g})$ are such that the families $([\log(\mu_i(\mathfrak{A}^1))]_1, \cdots, [\log(\mu_i(\mathfrak{A}^h))]_1,\ [\log(\mu_i(\mathfrak{B}_1))]_1, \cdots, [\log(\mu_i(\mathfrak{B}_h))]_1)$ (where $[\cdot]_1$ is the projection to the degree-one part of $\mathfrak{g}$) for $i = 1, 2$ are bases of the vector space generated by $a \cup b$. Then there exists $\theta \in \mathrm{Aut}(\mathfrak{g})$ and $u \in E^0(\Sigma_p, \mathfrak{g})$ such that*

$$\mathcal{J}_2 \,=\, e^u\,\theta(\mathcal{J}_1)\,e^{-u} - e^u\,d(e^{-u}), \tag{A.1}$$

or equivalently such that

$$d - \mathcal{J}_2 \,=\, e^u \circ \big(d - \theta(\mathcal{J}_1)\big) \circ e^{-u} \tag{A.2}$$

(equality of linear maps $E^0(\Sigma_p, \mathfrak{g}) \to E^1(\Sigma_p, \mathfrak{g})$), so that the two connections are related by a combination of the Lie algebra automorphism θ and the gauge transformation e^u.

Proof. By the properties of μ_1 and μ_2, and by the freeness of $\mathfrak{g}$, there exists an automorphism θ of $\mathfrak{g}$ such that $\mu_2 = \theta \circ \mu_1$ (equality of group morphisms $\pi_1(\Sigma_p, y) \to \exp(\mathfrak{g})$, θ being identified with the induced automorphism of $\exp(\mathfrak{g})$). We want now to construct the function $u \in E^0(\Sigma_p, \mathfrak{g})$ of the statement. To do so, we decompose $u = u_1 + u_2 + \cdots$ according to the degree in $\mathfrak{g}$, and construct u_k ($k \geq 1$) inductively. Assume that $k \geq 1$ and that $(u_1, \ldots, u_{k-1})$ have been constructed such that, if we set $u_{<k} = u_1 + \cdots + u_{k-1}$, one has $u_{<k}(y) = 0$ and $d - \mathcal{J}_2 = e^{u_{<k}} \circ (d - \theta(\mathcal{J}_1)) \circ e^{-u_{<k}} + \alpha_{\geq k}$, where $\alpha_{\geq k}$ is an element of $E^1(\Sigma_p, \mathfrak{g}_{\geq k})$, with $\mathfrak{g}_{\geq k} = \prod_{l \geq k} \mathfrak{g}_l$. Let us denote by $\alpha_k \in E^1(\Sigma_p, \mathfrak{g}_k)$ the class modulo $\mathfrak{g}_{>k}$ of this element.

On the one hand, the monodromy representation of $d - \mathcal{J}_2$ is μ_2. On the other hand, the monodromy representation of $e^{u_{<k}} \circ (d - \theta(\mathcal{J}_1)) \circ e^{-u_{<k}}$ is $\theta \circ \mu_1$ (since $e^{u_{<k}}(y) = 1$), therefore the monodromy representation of $e^{u}_{<k} \circ (d - \theta(\mathcal{J}_1)) \circ e^{-u_{<k}} + \alpha_{\geq k}$ is a morphism $\pi_1(\Sigma_p, y) \to \exp(\mathfrak{g})$, $\gamma \mapsto \theta \circ \mu_1(\gamma)\,(1 - \int_\gamma \alpha_k) \cdot \exp(\mathfrak{g}_{>k})$. Since $\mu_2 = \theta \circ \mu_1$, it follows that for any $\gamma \in \pi_1(\Sigma_p, y)$, one has $\int_\gamma \alpha_k = 0$. But one knows that the complex

$$0 \to \mathbb{C} \to E^0(\Sigma_p) \to E^1(\Sigma_p) \to \mathbb{C}^{2h} \to 0 \tag{A.3}$$

is exact, where the second map is the canonical inclusion, the third map is the differential, and the fourth map is given by $\alpha \mapsto (\int_{\mathfrak{A}^1} \alpha, \ldots, \int_{\mathfrak{B}_h} \alpha)$. Therefore there exists u_k such that $d(u_k) = -\alpha_k$ and $u_k(y) = 0$.

Then $(u_1, \ldots, u_k)$ is such that $u_{<k+1}(y) = 0$ and

$$\begin{aligned} & e^{u_{<k+1}} \circ \big(d - \theta(\mathcal{J}_1)\big) \circ e^{-u_{<k+1}} \\ & = e^{u_k} \circ e^{u_{<k}} \circ \big(d - \theta(\mathcal{J}_1)\big) \circ e^{-u_{<k}} \circ e^{-u_k} \text{ modulo } E^1(\Sigma_p, \mathfrak{g}_{\geq k+1}) \\ & = e^{u_k} \circ (d - \mathcal{J}_2 - \alpha_{\geq k}) \circ e^{-u_k} \text{ modulo } E^1(\Sigma_p, \mathfrak{g}_{\geq k+1}) \\ & = (d - \mathcal{J}_2 - d(u_k) - \alpha_{\geq k}) \text{ modulo } E^1(\Sigma_p, \mathfrak{g}_{\geq k+1}) \\ & = d - \mathcal{J}_2 \text{ modulo } E^1(\Sigma_p, \mathfrak{g}_{\geq k+1}). \end{aligned} \tag{A.4}$$

We conclude that we can define the function u as the infinite sum $\sum_{k\geq 1} u_k$, where $(u_k)_{k\geq 1}$ is the inductively defined family. □

B Proof of Lemma 3.12

This appendix is dedicated to establishing properties of the coefficients $\mathcal{X}^{IJ_1\cdots J_r}$ in the expansion (3.39) of the solution $\hat{\xi}$ to the monodromy conditions $\mathcal{U}_{\rm DHS}(p+\mathfrak{A}^K, p; \hat{\xi}, \hat{\eta}) = 1$ of (3.30). Their independence on p and symmetry properties (3.40), (3.41) under permutations of the indices $I, J_1, \cdots, J_r$ will be proven in sections B.1 and B.2, respectively.

B.1 Proving independence of $\mathcal{X}$ on p

Proof. The proof of item 1. of Lemma 3.12, namely the independence of the coefficients $\mathcal{X}$ on the point p, proceeds via a recursive formula for the derivatives $\partial_p \hat{\xi}^J$ and $\bar{\partial}_p \hat{\xi}^J$. These derivatives are obtained by differentiating the $\mathfrak{A}$-monodromy conditions with respect to p, while keeping $\hat{\eta}$ fixed (the derivative with respect to $\bar{p}$ is obtained analogously),

$$
\begin{aligned}
\partial_p \mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot p, p; \hat{\xi}, \hat{\eta}) = & \int_p^{\mathfrak{A}^K \cdot p} \mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot p, t; \hat{\xi}, \hat{\eta}) \mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \partial_p \hat{\xi}, \hat{\eta}) \mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta}) \\
& + \Big[\mathcal{J}^{(1,0)}_{\rm DHS}(p, \cdot\,; \hat{\xi}, \hat{\eta}), \mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot p, p; \hat{\xi}, \hat{\eta}) \Big].
\end{aligned}
\tag{B.1}
$$

Here we have used the linearity of $\mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \hat{\xi}, \hat{\eta})$ in $\hat{\xi}$ to carry out the derivative of $\mathcal{J}_{\rm DHS}(t, \cdot\,; \hat{\xi}, \hat{\eta})$ and to regroup the result in the form $\mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \partial_p \hat{\xi}, \hat{\eta})$. In view of the monodromy conditions $\mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot p, p; \hat{\xi}, \hat{\eta}) = 1$, the left side of (B.1) vanishes; the commutator on the second line of the right side vanishes for the same reason, and the remaining equation may be simplified as follows,

$$
\int_p^{\mathfrak{A}^K \cdot p} \mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta})^{-1} \mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \partial_p \hat{\xi}, \hat{\eta}) \mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta}) = 0. \tag{B.2}
$$

Both $\mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta})$ and $\mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \partial_p \hat{\xi}, \hat{\eta})$ admit expansions in powers of $\hat{\eta}$,

$$
\begin{aligned}
\mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta})^{-1} X \mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta}) &= X + \sum_{r=1}^{\infty} \mathcal{T}^{I_1 \cdots I_r}(t, p) \hat{H}_{I_1} \cdots \hat{H}_{I_r} X, \\
\mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \partial_p \hat{\xi}, \hat{\eta}) &= \omega_J(t) \partial_p \hat{\xi}^J + \sum_{r=1}^{\infty} \partial_t \Phi^{I_1 \cdots I_r}{}_J(t) \hat{H}_{I_1} \cdots \hat{H}_{I_r} \partial_p \hat{\xi}^J.
\end{aligned}
\tag{B.3}
$$

Combining the two expansions, we may rearrange the result in terms of a single expansion,

$$
\begin{aligned}
& \mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta})^{-1} \mathcal{J}^{(1,0)}_{\rm DHS}(t, \cdot\,; \partial_p \hat{\xi}, \hat{\eta}) \mathcal{U}_{\rm DHS}(t, p; \hat{\xi}, \hat{\eta}) \\
& \quad = \omega_J(t) \partial_p \hat{\xi}^J + \sum_{r=1}^{\infty} \mathcal{S}^{I_1 \cdots I_r}{}_J(t, p) \hat{H}_{I_1} \cdots \hat{H}_{I_r} \partial_p \hat{\xi}^J,
\end{aligned}
\tag{B.4}
$$

where the coefficients $\mathcal{S}$ are functions of $\mathcal{T}$, $\partial_t \Phi$ and ω. Substituting the expansion (B.4) into (B.2) and carrying out the $\mathfrak{A}^K$ integral of the first term in the expansion gives

$$\partial_p \hat{\xi}^K + \sum_{r=1}^{\infty} \mathcal{Z}^{KI_1 \cdots I_r}{}_J(p) \hat{H}_{I_1} \cdots \hat{H}_{I_r} \partial_p \hat{\xi}^J = 0, \tag{B.5}$$

where $\mathcal{Z}$ has been defined by,

$$\mathcal{Z}^{KI_1 \cdots I_r}{}_J(p) = \int_p^{\mathfrak{A}^K \cdot p} \mathcal{S}^{I_1 \cdots I_r}{}_J(t,p). \tag{B.6}$$

Finally, we substitute the derivative $\partial_p \hat{\xi}^J$ obtained from differentiating the Lie series (3.39) into (B.5), and identify terms of degree $r \geq 1$ in $\hat{\eta}$,

$$\partial_p \mathcal{X}^{KI_1 \cdots I_r}(p) \hat{\eta}_{I_1} \cdots \hat{\eta}_{I_r} = -\sum_{\substack{s,t \geq 1 \\ s+t=r}}^{\infty} \mathcal{Z}^{KI_1 \cdots I_s}{}_M(p) \partial_p \mathcal{X}^{MJ_1 \cdots J_t}(p) \hat{H}_{I_1} \cdots \hat{H}_{I_s} \hat{\eta}_{J_1} \cdots \hat{\eta}_{J_t}. \tag{B.7}$$

The key observation is that the rank of the tensor $\partial_p \mathcal{X}^{MJ_1 \cdots J_t}(p)$ under the sum on the right side is strictly smaller than the rank of the tensor $\partial_p \mathcal{X}^{KI_1 \cdots I_r}(p)$ on the left side since $t < r$. Since we have already shown that $\mathcal{X}^{MJ} = \pi Y^{MJ}$ is independent of p, it follows that $\partial_p \mathcal{X}^{KI_1 \cdots I_r}(p) = 0$ for all $r \geq 1$ by induction on r. From implementing the corresponding argument on the derivative $\partial_{\bar{p}} \hat{\xi}$ we conclude analogously that $\partial_{\bar{p}} \mathcal{X}^{KI_1 \cdots I_r}(p) = 0$ for all $r \geq 1$, so that $\mathcal{X}^{KI_1 \cdots I_r}(p)$ is independent of p for all $r \geq 1$. This result can be readily confirmed at rank three by differentiating the expressions (D.18) or (E.11) for $\mathcal{X}^{LIJ}$ below with respect to p and $\bar{p}$. □

B.2 Proving the shuffle property and cyclic invariance

The shuffle product is an associative and commutative binary operation on words for which the empty set $\emptyset$ is the neutral element (see footnote 8). For multi-index words $J_1 \cdots J_r$ and $K_1 \cdots K_s$ of length $r, s \geq 1$, the shuffle product is defined recursively by

$$J_1 \cdots J_r \sqcup\!\sqcup K_1 \cdots K_s = J_1(J_2 \cdots J_r \sqcup\!\sqcup K_1 \cdots K_s) + K_1(J_1 \cdots J_r \sqcup\!\sqcup K_2 \cdots K_s), \tag{B.8}$$

along with the neutrality of the empty set

$$J_1 \cdots J_r \sqcup\!\sqcup \emptyset = J_1 \cdots J_r, \qquad r \geq 0. \tag{B.9}$$

On functions of multi-index words, such as $\mathcal{X}^{J_1 \cdots J_r}$, the shuffle product acts linearly

$$\begin{aligned} \mathcal{X}^{J_1 \cdots J_r \sqcup\!\sqcup K_1 \cdots K_s} &= \mathcal{X}^{J_1(J_2 \cdots J_r \sqcup\!\sqcup K_1 \cdots K_s)} + \mathcal{X}^{K_1(J_1 \cdots J_r \sqcup\!\sqcup K_2 \cdots K_s)}, \\ \mathcal{X}^{J_1 \cdots J_r \sqcup\!\sqcup \emptyset} &= \mathcal{X}^{J_1 \cdots J_r}. \end{aligned} \tag{B.10}$$

Proof. The proof of item 2. of Lemma 3.12, namely the relation (3.40) claiming the vanishing of all shuffles $\mathcal{X}^{I(J_1\cdots J_r \sqcup K_1\cdots K_s)}$ with $r, s \geq 1$, proceeds as follows. The monodromy condition $\mathcal{U}_{\rm DHS}(\mathfrak{A}^K \cdot p, p; \hat{\hat{\xi}}, \hat\eta)= 1$ is a group-like element for $\hat{\hat{\xi}}, \hat\eta \in \mathfrak{g}^h_b$ so that each order $\mathcal{X}^{IJ_1\cdots J_r}\hat\eta_{J_1}\cdots\hat\eta_{J_r}$ in the expansion of $\hat{\hat{\xi}}$ in (3.39) must be a Lie polynomial in $\hat\eta_J$. By Ree's theorem [73], this implies the shuffle properties in (3.40).

The proof of item 3. of Lemma 3.12, namely cyclic permutation relation of (3.41), proceeds by exploiting the vanishing of the commutator $[\hat\eta_I, \hat{\hat{\xi}}^I]$ established in Lemma 3.2. Given that all $r^{\rm th}$-order contributions to $\hat{\hat{\xi}}^I$ in $\hat\eta_J$ are bound to separately result in vanishing commutators, we have

$$0 = \mathcal{X}^{IJ_1J_2\cdots J_r}[\hat\eta_I, \hat\eta_{J_1}\hat\eta_{J_2}\cdots\hat\eta_{J_r}] = \hat\eta_I\hat\eta_{J_1}\hat\eta_{J_2}\cdots\hat\eta_{J_r}(\mathcal{X}^{IJ_1J_2\cdots J_r} - \mathcal{X}^{J_1J_2\cdots J_rI}). \quad \text{(B.11)}$$

Linear independence of different words in $\hat\eta_K$ then completes the proof. ☐

B.3 Several remarks

We conclude this appendix with several remarks upon Lemma 3.12.

- By combining the symmetry properties (3.40) and (3.41), the number of independent permutations of $\mathcal{X}^{I_1I_2\cdots I_s}$ in the s indices is $(s-2)!$: the cyclic symmetry (3.41) can be used to move any given index I_k (with fixed $k = 1, 2, \cdots, s$) to the first entry, and the vanishing shuffles (3.40) imply that only $(r-1)!$ out of the $r!$ permutations of $\mathcal{X}^{IJ_1J_2\cdots J_r}$ in $J_1, \cdots, J_r$ are independent.

- As a result of item 2. of Lemma 3.12, we obtain an explicit formula for the Lie series associated with the coefficients $\mathcal{X}$. Upon inserting the $r^{\rm th}$-order contribution to $\hat{\hat{\xi}}^I$ in $\hat\eta_J$ into (3.39), the series may be recast in a Lie series form
$$\mathcal{X}^{IJ_1J_2\cdots J_r}\hat\eta_{J_1}\hat\eta_{J_2}\cdots\hat\eta_{J_r} = \frac{1}{r}\mathcal{X}^{IJ_1J_2\cdots J_r}[\hat\eta_{J_1}, [\hat\eta_{J_2}, \cdots [\hat\eta_{J_{r-1}}, \hat\eta_{J_r}]\cdots]]. \quad \text{(B.12)}$$

- Finally, at low ranks, the shuffle and cyclicity properties, established in Lemma 3.12, imply the following relations. For rank three, the shuffle relation $\mathcal{X}^{I(J\sqcup K)} = 0$ implies anti-symmetry in the last two indices $\mathcal{X}^{IJK} = -\mathcal{X}^{IKJ}$ which, combined with the cyclic property, makes $\mathcal{X}^{IJK}$ totally anti-symmetric,
$$\mathcal{X}^{IJK} = \mathcal{X}^{[IJK]}. \quad \text{(B.13)}$$
For rank four, $\mathcal{X}^{I_iI_jI_kI_\ell}$ with any permutation i, j, k, l of $1, 2, 3, 4$ can be expressed in a basis generated by the functions $\mathcal{X}^{I_1I_2I_3I_4}$ and $\mathcal{X}^{I_1I_2I_4I_3}$ with coefficients 0 or ±1. For example, the shuffle properties imply
$$\begin{aligned} \mathcal{X}^{IJ_1J_2J_3} - \mathcal{X}^{IJ_3J_2J_1} &= 0, \\ \mathcal{X}^{IJ_1J_2J_3} + \text{cycl}(J_1, J_2, J_3) &= 0. \end{aligned} \quad \text{(B.14)}$$

C Proof of Corollary 3.13

In this appendix, we shall prove that the coefficients $\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J$ in the expansion (3.51) of $\hat{\eta}_J$ in b_I are explicitly given in terms of the coefficients $\mathcal{M}^{I_1\cdots I_r}{}_J$ of $\hat{a}^I - \hat{\xi}^I$ by (3.52) and obey shuffle relations (3.53).

Proof. Starting with an Ansatz

$$\hat{H}_J = B_J - \sum_{r=2}^{\infty} \mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J(p) B_{I_1}\cdots B_{I_r} \tag{C.1}$$

for the derivation that implements the adjoint action of $\hat{\eta}_J$, the r^{th} order in the expansion of (3.50) in B_K implies the recursion relation

$$\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_r}{}_J = \mathcal{M}^{I_1\cdots I_r}{}_J - \sum_{\ell=2}^{r-1} \mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_\ell}{}_K \mathcal{M}^{K I_{\ell+1}\cdots I_r}{}_J \tag{C.2}$$

among the coefficients. We shall now prove by induction that this recursion is solved by (3.52). For $r=2$, one readily shows that $\mathcal{M}_{\sqcup\!\sqcup}^{I_1 I_2}{}_J = \mathcal{M}^{I_1 I_2}{}_J$ obeys both (C.2) and (3.52). Assuming that the claim (3.52) holds for $r = 2, 3, \cdots, n$, and substituting these relations into the case of (C.2) at $r = n{+}1$ yields

$$\begin{aligned}\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_{n+1}}{}_J = \mathcal{M}^{I_1\cdots I_{n+1}}{}_J - \sum_{\ell=2}^{n}\sum_{s=0}^{\ell-2}(-1)^s \sum_{2\leq j_1<\cdots<j_s}^{\ell-1} \mathcal{M}^{I_1 I_2\cdots I_{j_1}}{}_{K_1}\mathcal{M}^{K_1 I_{j_1+1}\cdots I_{j_2}}{}_{K_2}\times\cdots\\ \times\cdots\mathcal{M}^{K_{s-1}I_{j_{s-1}+1}\cdots I_{j_s}}{}_{K_s}\mathcal{M}^{K_s I_{j_s+1}\cdots I_\ell}{}_K \mathcal{M}^{K I_{\ell+1}\cdots I_{n+1}}{}_J.\end{aligned} \tag{C.3}$$

This needs to be lined up with the $r = n{+}1$ instance of the claim (3.52),

$$\begin{aligned}\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_{n+1}}{}_J = \mathcal{M}^{I_1\cdots I_{n+1}}{}_J + \sum_{u=1}^{n-1}(-1)^u \sum_{2\leq j_1<j_2<\cdots<j_u}^{n} \mathcal{M}^{I_1 I_2\cdots I_{j_1}}{}_{K_1}\mathcal{M}^{K_1 I_{j_1+1}\cdots I_{j_2}}{}_{K_2}\times\cdots\\ \times\cdots\mathcal{M}^{K_{u-1}I_{j_{u-1}+1}\cdots I_{j_u}}{}_{K_u}\mathcal{M}^{K_u I_{j_u+1}\cdots I_{n+1}}{}_J.\end{aligned} \tag{C.4}$$

We have renamed the summation variable of (3.52) to $\ell \to u$ and note that the first term $\mathcal{M}^{I_1\cdots I_{n+1}}{}_J$ on the right side readily matches that of (C.3). The remaining contributions in the nested sums of (C.3) can be shown to match those of (C.4) by interchanging the sums over ℓ and s and then renaming $s = u{-}1$,

$$\begin{aligned}\mathcal{M}_{\sqcup\!\sqcup}^{I_1\cdots I_{n+1}}{}_J = \mathcal{M}^{I_1\cdots I_{n+1}}{}_J + \sum_{u=1}^{n-1}(-1)^u \sum_{\ell=u+1}^{n}\sum_{2\leq j_1<\cdots<j_{u-1}}^{\ell-1} \mathcal{M}^{I_1 I_2\cdots I_{j_1}}{}_{K_1}\mathcal{M}^{K_1 I_{j_1+1}\cdots I_{j_2}}{}_{K_2}\times\cdots\\ \times\cdots\mathcal{M}^{K_{u-2}I_{j_{u-2}+1}\cdots I_{j_{u-1}}}{}_{K_{u-1}}\mathcal{M}^{K_{u-1}I_{j_{u-1}+1}\cdots I_\ell}{}_{K_u}\mathcal{M}^{K_u I_{\ell+1}\cdots I_{n+1}}{}_J,\end{aligned} \tag{C.5}$$

where the last summation index has been renamed from K to K_u. As a last step, we rename ℓ in (C.5) to j_u and rearrange the nested sum $\sum_{\ell=u+1}^{n}\sum_{2\leq j_1<\cdots<j_{u-1}}^{\ell-1}$ into $\sum_{2\leq j_1<\cdots<j_u}^{n}$ (the lower bound $\ell \geq u+1$ is consistent with $j_i \geq i+1$). This recovers the expression (C.4) for the claim (3.52) at $r = n+1$ and completes both the inductive step and the inductive proof of (3.52).

Finally, the shuffle relations (3.53) are necessary by Ree's theorem to ensure that the expression (3.51) for $\hat{\eta}_J$ is in $\mathfrak{g}$ as exposed by (3.47). □

D Explicit evaluation to low orders

The purpose of this appendix is to illustrate the relation between the connections $d-\mathcal{K}_{\rm E}$ and $d-\mathcal{J}_{\rm DHS}$, stated in (1.12) of the introduction and proven in Theorem 3.9 of section 3, by providing a detailed derivation of explicit formulas for the gauge transformation $\mathcal{U}_{\rm DHS}$ and the automorphism $a\cup b \to \hat{a}\cup\hat{b}$ to lowest and next-to-lowest orders in the generators a and b. Along the way, we shall produce a number of useful properties of the connections and their interrelation. We begin by summarizing the approach suitably readied for practical calculations.

Throughout this appendix, we shall denote the solutions $\hat{\xi}$ and $\hat{\eta}$ of the monodromy equations (3.30) simply by ξ and η in order to avoid cluttering.

D.1 Practical summary of the approach

A convenient starting point for the construction of the gauge transformation is obtained by combining the results of Lemmas 3.1, 3.2 and Corollaries 3.4, 3.6 which guarantee that $\mathcal{U}_{\rm DHS}(x,p)$ is given by the path-ordered exponential,

$$\mathcal{U}_{\rm DHS}(x,p;\xi,\eta) = {\rm P}\exp\int_p^x \mathcal{J}_{\rm DHS}(t,\cdot\,;\xi,\eta). \tag{D.1}$$

Here ξ and η obey $[\eta_I,\xi^I]=0$ so that the connection $\mathcal{J}_{\rm DHS}(t,p;\xi,\eta)$ is independent of p, smooth in x, and given by the simplified formula,

$$\mathcal{J}_{\rm DHS}(t,\cdot\,;\xi,\eta) = \omega_J(t)\xi^J - \pi\bar{\omega}^I(t)\eta_I + \sum_{r=1}^{\infty}\partial_t\Phi^{I_1\cdots I_r}{}_J(t)H_{I_1}\cdots H_{I_r}\xi^J, \tag{D.2}$$

where $H_I X = [\eta_I, X]$ for any $X\in\mathfrak{g}$, the smooth functions $\Phi^{I_1\cdots I_r}{}_J(t)$ were defined in equation (3.21) of [17], and ξ and η are the unique solutions to the system of monodromy conditions,

$$\begin{aligned}\mathcal{U}_{\rm DHS}(\mathfrak{A}^K\cdot p,p;\xi,\eta) &= 1,\\ \mathcal{U}_{\rm DHS}(\mathfrak{B}_K\cdot p,p;\xi,\eta) &= e^{2\pi i b_K}.\end{aligned} \tag{D.3}$$

Solving the above system of equations gives ξ,η and $\mathcal{U}_{\rm DHS}$ in terms of b.

D.2 Expansions in words of the alphabet $\xi\cup\eta$

To solve the monodromy relations (D.3) for ξ and η as a function of b (and the moduli of the surface Σ_p), it will be convenient to obtain first the expansion of $\mathcal{U}_{\rm DHS}(x,p;\xi,\eta)$ in a power series in ξ and η, subject only to the constraint $[\eta_I, \xi^I] = 0$. This expansion may be organized in terms of the integer r which is the sum of the number of letters ξ and the number of letters η (or equivalently the total word length),

$$\mathcal{U}_{\rm DHS}(x,p;\xi,\eta) = 1 + \sum_{r=1}^{\infty} \mathcal{U}_r(x,p;\xi,\eta). \tag{D.4}$$

We refrain from imposing any relation between ξ and η due to the monodromy relations at this stage which in general do not preserve the total word length. While $\mathcal{U}_{\rm DHS}(x,p;\xi,\eta)$ takes values in $\exp(\mathfrak{g}_b)$, each term $\mathcal{U}_r(x,p;\xi,\eta)$ takes values in $\mathbb{C}\langle\langle b\rangle\rangle$. The expansion of (D.4) has the advantage of leading to simple shuffle relations obeyed by the polylogarithms in their coefficients. It will be convenient to also consider the complementary expansion of the Lie series $\mathfrak{u}_{\rm DHS}(x,p;\xi,\eta) \in \mathfrak{g}_b$ defined by

$$\mathcal{U}_{\rm DHS}(x,p;\xi,\eta) = \exp\left\{\mathfrak{u}_{\rm DHS}(x,p;\xi,\eta)\right\} \tag{D.5}$$

in a power series in ξ and η, whose terms $\mathfrak{u}_r(x,p;\xi,\eta)$ have word length r,

$$\mathfrak{u}_{\rm DHS}(x,p;\xi,\eta) = \sum_{r=1}^{\infty} \mathfrak{u}_r(x,p;\xi,\eta). \tag{D.6}$$

Being directly in terms of Lie algebra elements, this expansion[27] provides a convenient way to relate the connections of (3.31) which are valued in $\mathfrak{g}$, but it has the disadvantage of obscuring the role of the shuffle relations and lacking a canonical presentation due to the Jacobi identity. The terms in the two expansions may, of course, be simply related to one another using (D.5), and the lowest three orders are given by

$$\begin{aligned}
\mathfrak{u}_1(x,p;\xi,\eta) &= \mathcal{U}_1(x,p;\xi,\eta),\\
\mathfrak{u}_2(x,p;\xi,\eta) &= \mathcal{U}_2(x,p;\xi,\eta) - \tfrac{1}{2}\mathcal{U}_1(x,p;\xi,\eta)^2,\\
\mathfrak{u}_3(x,p;\xi,\eta) &= \mathcal{U}_3(x,p;\xi,\eta) - \tfrac{1}{2}\mathcal{U}_1(x,p;\xi,\eta)\mathcal{U}_2(x,p;\xi,\eta)\\
&\quad -\tfrac{1}{2}\mathcal{U}_2(x,p;\xi,\eta)\mathcal{U}_1(x,p;\xi,\eta) + \tfrac{1}{3}\mathcal{U}_1(x,p;\xi,\eta)^3.
\end{aligned} \tag{D.7}$$

These relations allow us to pursue the two expansions in parallel. Since $\mathcal{J}_{\rm DHS}(x,\cdot\,;\xi,\eta)$ is a flat connection, the gauge transformation $\mathcal{U}_{\rm DHS}(x,p;\xi,\eta)$, and its expansion components $\mathcal{U}_r(x,p;\xi,\eta)$ and $\mathfrak{u}_r(x,p;\xi,\eta)$ are all homotopy invariant.

In the remainder of this appendix, we shall evaluate $\mathcal{U}_1$ and $\mathcal{U}_2$, or equivalently $\mathfrak{u}_1$ and $\mathfrak{u}_2$, enforce the monodromy relations (D.3) to second order in b, and then use the result to obtain ξ, η and $\mathcal{U}_{\rm DHS}$ to second order in b.

[27]The expansion is often referred to as the *Magnus expansion* of the path-ordered exponential, see for example [74] and references therein.

D.3 Calculating $\mathcal{U}_1, \mathcal{U}_2$ and $\mathfrak{u}_1, \mathfrak{u}_2$

The contributions $\mathcal{U}_1, \mathcal{U}_2$ are obtained by substituting the connection (D.2) into the path-ordered exponential of (D.1) and using the expansion (D.4),

$$\begin{aligned}\mathcal{U}_1(x,p;\xi,\eta) &= \int_p^x \big(\omega_J \xi^J - \pi \bar\omega^I \eta_I\big), \\ \mathcal{U}_2(x,p;\xi,\eta) &= \int_p^x \big(\omega_I(t)\xi^I - \pi\bar\omega_I(t)\eta^I\big) \int_p^t \big(\omega_J\xi^J - \pi\bar\omega_J\eta^J\big) + [\eta_I,\xi^J]\int_p^x \partial_t \Phi^I{}_J(t).\end{aligned} \tag{D.8}$$

Using the first two lines of (D.7), we obtain the second order contributions to $\mathfrak{u}_1, \mathfrak{u}_2$,

$$\begin{aligned}\mathfrak{u}_1(x,p;\xi,\eta) &= \int_p^x \big(\omega_I\xi^I - \pi\bar\omega^J\eta_J\big), \\ \mathfrak{u}_2(x,p;\xi,\eta) &= [\eta_I,\xi^J]\int_p^x \left(\partial_t\Phi^I{}_J(t) - \frac{\pi}{2}\bar\omega^I(t)\int_p^t \omega_J + \frac{\pi}{2}\omega_J(t)\int_p^t \bar\omega^I\right) \\ &\quad + \frac{1}{2}[\xi^I,\xi^J]\int_p^x \omega_I(t)\int_p^t \omega_J + \frac{\pi^2}{2}[\eta_I,\eta_J]\int_p^x \bar\omega^I(t)\int_p^t \bar\omega^J,\end{aligned} \tag{D.9}$$

where we have used the familiar rearrangement formula underlying shuffle relations

$$\int_p^x dt_1 \int_p^x dt_2 = \int_p^x dt_1 \int_p^{t_1} dt_2 + \int_p^x dt_2 \int_p^{t_2} dt_1 \tag{D.10}$$

to present $\mathfrak{u}_2$ in a form that manifestly belongs to $\mathfrak{g}$. The contributions $\mathfrak{u}_1$ and the second line of $\mathfrak{u}_2$ are manifestly homotopy invariant for arbitrary $\xi, \eta \in \mathfrak{g}$. Using the relation,

$$\bar\partial_t \partial_t \Phi^I{}_J(t) = \pi\kappa(t)\,\delta^I_J - \pi\bar\omega^I(t) \wedge \omega_J(t), \tag{D.11}$$

where $\kappa(x)$ was defined in (2.25), we see that the integrand of the first line of $\mathfrak{u}_2$ is a closed 1-form in view of the fact that the term in κ cancels thanks to the relation $[\eta_I, \xi^I] = 0$. Thus, $\mathfrak{u}_2$ is homotopy invariant to this order in ξ, η, as expected on general grounds.

D.4 Solving the monodromy relations to order b^2

The monodromies of $\mathfrak{u}_1$ are p-independent, and we recover the result of (3.10),

$$\begin{aligned}\mathfrak{u}_1(\mathfrak{A}^K \cdot p, p; \xi, \eta) &= \xi^K - \pi\eta^K, \\ \mathfrak{u}_1(\mathfrak{B}_K \cdot p, p; \xi, \eta) &= \Omega_{KI}\xi^I - \pi\bar\Omega_{KI}\eta^I.\end{aligned} \tag{D.12}$$

Setting these monodromies equal to 0 and $2\pi i b_K$, respectively, we recover the result of (3.11) to first order in b, namely $\eta_I = b_I + \mathcal{O}(b^2)$ and $\xi_I = \pi b_I + \mathcal{O}(b^2)$.

The monodromies of $\mathfrak{u}_2$ may be simplified by using the fact that we need their evaluation only to second order in ξ and η so that we are allowed to substitute the first order relation $\xi_I = \pi\eta_I$ into $\mathfrak{u}_2$ in (D.9). The result is as follows,

$$\mathfrak{u}_2(x,p;\pi\eta,\eta) = \pi\, \eta_I \eta_J \left(\int_p^x 2\,\partial_t \Phi^{[IJ]}(t) - 4\pi \,\mathrm{Im} \int_p^x \omega^{[I}(t) \,\mathrm{Im} \int_p^t \omega^{J]} \right), \qquad \text{(D.13)}$$

where the square brackets stand for anti-symmetrization of the indices I, J, such as for example $\eta_{[I}\eta_{J]} = \frac{1}{2}(\eta_I\eta_J - \eta_J\eta_I)$. We have used this property to recast the commutators in (D.9) in terms of a product $\eta_I\eta_J$ in (D.13). The integral of $\partial_t\Phi^{[IJ]}$ may be simplified by using the following rearrangement,

$$2\,\partial_t \Phi^{[IJ]} = d_t \Phi^{[IJ]} + \partial_t \Phi^{[IJ]} + \overline{\partial_t \Phi^{[IJ]}}, \qquad \text{(D.14)}$$

which is obtained with the help of the complex-conjugation property $\overline{\Phi^{[IJ]}(t)} = -\Phi^{[IJ]}(t)$. Integrating the total differential $d_t\Phi^{[IJ]}$ and recasting the remainder as a sum of an integral and its complex conjugate, we obtain the following formula for $\mathfrak{u}_2$ to this order,

$$\mathfrak{u}_2(x,p;\pi\eta,\eta) = \pi\, \eta_I \eta_J \left(\Phi^{[IJ]}(x) - \Phi^{[IJ]}(p) + \int_p^x \lambda^{IJ}(t,p) + \int_p^x \overline{\lambda^{IJ}(t,p)} \right). \qquad \text{(D.15)}$$

The $(1,0)$-form $\lambda^{IJ}(t,p)$ is given by,

$$\lambda^{IJ}(t,p) = \partial_t \Phi^{[IJ]}(t) + 2\pi i\, \omega^{[I}(t) \,\mathrm{Im} \int_p^t \omega^{J]}. \qquad \text{(D.16)}$$

One verifies that $\lambda^{IJ}(t,p)$ is anti-symmetric in its indices I, J as well as holomorphic in t in view of the relation (D.11). As a result, the function $\mathfrak{u}_2(x,p;\pi\eta,\eta)$ is homotopy invariant, as expected. Its monodromies are given by,

$$\begin{aligned} \mathfrak{u}_2(\mathfrak{A}^K \cdot p, p; \pi\eta, \eta) &= -\mathcal{X}^{KIJ}(p)\, \eta_I \eta_J, \\ \mathfrak{u}_2(\mathfrak{B}_K \cdot p, p; \pi\eta, \eta) &= -\mathcal{Y}_K{}^{IJ}(p)\, \eta_I \eta_J, \end{aligned} \qquad \text{(D.17)}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are given by

$$\begin{aligned} \mathcal{X}^{KIJ}(p) &= -\pi \int_p^{\mathfrak{A}^K \cdot p} \lambda^{IJ}(t,p) + \text{ c.c.} \\ \mathcal{Y}_K{}^{IJ}(p) &= -\pi \int_p^{\mathfrak{B}_K \cdot p} \lambda^{IJ}(t,p) + \text{ c.c.} \end{aligned} \qquad \text{(D.18)}$$

By construction, $\mathcal{X}^K{}_{IJ}$ and $\mathcal{Y}_{KIJ}$ are anti-symmetric in I, J and real multiple-valued harmonic functions of p. Using the properties of λ^{IJ} one readily shows that

$$\partial_p \mathcal{X}^{KIJ}(p) = \partial_{\bar{p}} \mathcal{X}^{KIJ}(p) = 0, \tag{D.19}$$

i.e. $\mathcal{X}^{KIJ}$ is independent of p as expected on general grounds by the proof in appendix B.1.

Combining the conditions of (D.3) with (D.5) and the monodromies of $\mathfrak{u}_1$ in (D.12) and $\mathfrak{u}_2$ in (D.17), we obtain the full monodromy relations to second order,

$$\begin{aligned} \xi^K - \pi\eta^K - \mathcal{X}^{KIJ}\,\eta_I\eta_J + \mathcal{O}(\eta^3) &= 0, \\ \Omega_{KJ}\xi^J - \pi\bar{\Omega}_{KJ}\eta^J - \mathcal{Y}_K{}^{IJ}(p)\,\eta_I\eta_J + \mathcal{O}(\eta^3) &= 2\pi i b_K. \end{aligned} \tag{D.20}$$

Solving the first equation of (D.20) for ξ in terms of η, substituting the result into the second equation of (D.20), and then solving for η and ξ up to second order in b gives the final result which may be summarized in terms of the following lemma.

Lemma D.1. *The solution to the monodromy equations (D.3) is given as follows,*

$$\begin{aligned} \xi^K &= \pi b^K + \Big(\mathcal{X}^{KIJ} - \pi\mathcal{M}^{IJK}(p)\Big)\, b_I b_J + \mathcal{O}(b^3), \\ \eta_K &= b_K - \mathcal{M}^{IJ}{}_K(p)\, b_I b_J + \mathcal{O}(b^3), \end{aligned} \tag{D.21}$$

where $\mathcal{X}^{KIJ}$ is independent of p and $\mathcal{M}^{IJ}{}_K(p)$ is given by

$$\mathcal{M}^{IJ}{}_K(p) = \frac{i}{2\pi}\Big(\mathcal{Y}_K{}^{IJ}(p) - \Omega_{KL}\mathcal{X}^{LIJ}\Big) \tag{D.22}$$

in terms of $\mathcal{X}$ and $\mathcal{Y}$ given in (D.18).

Remark D.2. *An alternative way to express the first line in (D.21) is as follows,*

$$\xi_K = \pi b_K - \frac{i}{2}\Big(\mathcal{Y}_K{}^{IJ}(p) - \bar{\Omega}_{KL}\mathcal{X}^{LIJ}\Big) b_I b_J + \mathcal{O}(b^3). \tag{D.23}$$

D.5 Matching residues and relating the connections

We are now ready to state and prove the main proposition of this appendix.

Proposition D.3. *The relation between the connections $d - \mathcal{K}_E$ and $d - \mathcal{J}_{DHS}$ which, to the lowest and next-to-lowest orders in b are given by*

$$\begin{aligned} \mathcal{K}_E(x,p;a,b) &= \omega_J(x)a^J + g^I{}_J(x,p)[b_I, a^J] + \mathcal{O}(ab^2), \\ \mathcal{J}^{(1,0)}_{DHS}(x,p;\hat{a},\hat{b}) &= \omega_J(x)\hat{a}^J + f^I{}_J(x,p)[\hat{b}_I, \hat{a}^J] + \mathcal{O}(\hat{a}\hat{b}^2), \end{aligned} \tag{D.24}$$

is determined by the automorphism $a \cup b \to \hat{a} \cup \hat{b}$ of $\mathfrak{g}$,

$$\begin{aligned}\hat{a}^K &= a^K + \xi^K + \mathcal{M}^{KI}{}_J(p)[b_I, a^J] + \mathcal{O}(ab^2, b^3), \\ \hat{b}_K &= b_K - \mathcal{M}^{IJ}{}_K(p) b_I b_J + \mathcal{O}(b^3),\end{aligned} \tag{D.25}$$

and the following relation between the forms $f^I{}_J$ and $g^I{}_J$,

$$g^I{}_J(x,p) = f^I{}_J(x,p) - 2\pi i\, \omega_J(x)\, \mathrm{Im} \int_p^x \omega^I + \omega_K(x)\mathcal{M}^{KI}{}_J(p). \tag{D.26}$$

The coefficients $\mathcal{X}$ and $\mathcal{M}$ are given by the first equation in (D.18) as well as (D.22) and ξ is given in terms of b to this order in (D.21).

Proof. To prove the proposition, we begin by using the first equation in (3.32) to establish $\hat{b}_I = \eta_I$, which by the second line in (D.21) readily proves the second equation in (D.25). To prove the first equation in (D.25), we substitute the expression for η, obtained in the second line of (D.21) to second order in b, into the residue matching condition (3.36),

$$[b_K, a^K] = \big[b_K - \mathcal{M}^{IJ}{}_K(p)\, b_I b_J, \hat{a}^K - \xi^K\big] + \mathcal{O}(b^3). \tag{D.27}$$

Since the Lie algebra $\mathfrak{g}$ is freely generated, the unique solution to first order in a and zeroth order in b is given by $\hat{a}^K = a^K + \mathcal{O}(b)$. To first order in b we obtain uniquely,

$$\hat{a}^K - \xi^K = a^K + \mathcal{M}^{KI}{}_J(p)[b_I, a^J] + \mathcal{O}(b^2), \tag{D.28}$$

which establishes the first line in (D.25). Finally, using the relation (3.35) to first order in b by expanding $\mathcal{U}_{\mathrm{DHS}}$ accordingly, we obtain the condition

$$\mathcal{K}_{\mathrm{E}}(x,p;a,b) = \mathcal{J}^{(1,0)}_{\mathrm{DHS}}(x,p;\hat{a}-\xi,\eta) - \Big[\mathfrak{u}_1(x,p;\xi,\eta), \mathcal{J}^{(1,0)}_{\mathrm{DHS}}(x,p;\hat{a}-\xi,\eta)\Big] + \mathcal{O}(b^2), \tag{D.29}$$

where $\mathcal{K}_{\mathrm{E}}(x,p;a,b)$ is given in the first line of (D.24), $\mathfrak{u}_1(x,p;\xi,\eta)$ can be found in the first line of (D.9), and $\mathcal{J}^{(1,0)}_{\mathrm{DHS}}(x,p;\hat{a}-\xi,\eta)$ is given by,

$$\mathcal{J}^{(1,0)}_{\mathrm{DHS}}(x,p;\hat{a}-\xi,\eta) = \omega_K(x)(\hat{a}^K - \xi^K) + f^I{}_J(x,p)[\eta_I, \hat{a}^J - \xi^J] + \mathcal{O}(b^2). \tag{D.30}$$

Using the relation $\hat{b}_I = \eta_I$ and the solution of (D.28) for $\hat{a}^I$, we readily obtain the relation (D.26), thereby completing the proof of Proposition D.3.

D.6 Evaluation of $\mathcal{X}^{KIJ}$ and $\mathcal{M}^{IJ}{}_K$

To evaluate the coefficients $\mathcal{X}^{KIJ}$ and $\mathcal{Y}_K{}^{IJ}$ and thus $\mathcal{M}^{IJ}{}_K$ in Lemma D.1 we begin by simplifying the integral of the differential form λ_{IJ} defined in (D.16). To do so, we recall the expression for $\partial_t \Phi^I{}_J(t)$, raise the index J and anti-symmetrize in the indices I, J,

$$\partial_t \Phi^{[IJ]}(t) = -\frac{i}{4} \int_\Sigma \partial_t \mathcal{G}(t,t') \Big(\bar\omega^I(t') \wedge \omega^J(t') - \bar\omega^J(t') \wedge \omega^I(t') \Big). \tag{D.31}$$

The derivative of $\mathcal{G}(t,t')$ may be expressed in terms of the prime form $E(x,y)$,

$$\partial_t \mathcal{G}(t,t') = -\partial_t \ln E(t,t') - \partial_t \gamma(t) - 2\pi i \, \omega_K(t) \, \mathrm{Im} \int_{t'}^t \omega^K, \tag{D.32}$$

where the expression for $\gamma(t)$ may be found in [19], but will not be needed here as its contribution is y-independent and integrates to zero against $\bar\omega^I(t') \wedge \omega^J(t') - \bar\omega^J(t') \wedge \omega^I(t')$. To proceed, we denote the imaginary part of the Abelian integral on Σ_p by

$$\phi^I(x) = \mathrm{Im} \int_p^x \omega^I. \tag{D.33}$$

One verifies that ϕ^I has vanishing $\mathfrak{A}$-monodromies while its $\mathfrak{B}$-monodromies are given by $\phi^I(\mathfrak{B}_K \cdot x) = \phi^I(x) + \delta^I_K$. We also have the following relation,

$$\bar\omega^I(t') \wedge \omega^J(t') - \bar\omega^J(t') \wedge \omega^I(t') = 4 \, d\phi^I(t') \wedge d\phi^J(t'). \tag{D.34}$$

Expressing (D.31) in terms of ϕ_I we have

$$\partial_t \Phi^{[IJ]}(t) = \int_\Sigma \Big(i \partial_t \ln \frac{E(t,t')}{E(t,p)} + 2\pi \, \omega_K(t) \phi^K(t') \Big) d\phi^I(t') \wedge d\phi^J(t'), \tag{D.35}$$

where we have used the fact that the contributions inside the parentheses proportional to $\phi^K(t)$ and $\partial_t \ln E(t,p)$ integrate to zero against $d\phi^I \wedge d\phi^J$. The term in $\partial_t \ln E(t,p)$ has been included to render the integrand monodromy free in t, while its monodromy in t' integrates to zero against $d\phi^I \wedge d\phi^J$. Using the fact that the first term inside the parentheses has vanishing $\mathfrak{A}$-periods in t and that the $\mathfrak{B}$-periods are given by

$$\oint_{\mathfrak{B}_K} \partial_t \ln \frac{E(t,t')}{E(t,p)} = 2\pi i \int_p^{t'} \omega_K(t), \tag{D.36}$$

we evaluate the periods of $\partial_t \Phi_{[IJ]}$ as follows,

$$\begin{aligned} \oint_{\mathfrak{A}^K} \partial_t \Phi^{[IJ]} &= 2\pi \int_\Sigma \phi^K \, d\phi^I \wedge d\phi^J, \\ \oint_{\mathfrak{B}_K} \partial_t \Phi^{[IJ]} &= 2\pi \int_\Sigma \left(- \int_p^{t'} \omega_K + \Omega_{KL} \phi^L(t') \right) d\phi^I(t') \wedge d\phi^J(t'). \end{aligned} \tag{D.37}$$

As a result, we find

$$\mathcal{X}^{KIJ} = -4\pi^2 \int_\Sigma \phi^K \, d\phi^I \wedge d\phi^J + 2\pi^2 \oint_{\mathfrak{A}^K} \Big(\phi^J \, d\phi^I - \phi^I \, d\phi^J\Big),$$
$$\mathcal{Y}_K{}^{IJ}(p) = -4\pi^2 \int_\Sigma \left(-\mathrm{Re} \int_p^{t'} \omega_K + \mathrm{Re}\,(\Omega)_{KL} \phi^L(t')\right) d\phi^I(t') \wedge d\phi^J(t'),$$
$$+2\pi^2 \int_p^{\mathfrak{B}_K \cdot p} \Big(\phi^J \, d\phi^I - \phi^I \, d\phi^J\Big). \tag{D.38}$$

Both expressions are manifestly real-valued and antisymmetric in I and J as expected on general grounds. The combination $\mathcal{M}^{IJ}{}_K(p)$ then follows from (D.22) and is given by

$$\mathcal{M}^{IJ}{}_K(p) = -2\pi i \int_\Sigma d\phi^I(t') \wedge d\phi^J(t') \int_p^{t'} \omega_K \tag{D.39}$$
$$+\pi i \int_p^{\mathfrak{B}_K \cdot p} \Big(\phi^J \, d\phi^I - \phi^I \, d\phi^J\Big) - \pi i \, \Omega_{KL} \oint_{\mathfrak{A}^L} \Big(\phi^J \, d\phi^I - \phi^I \, d\phi^J\Big).$$

For later use, we record the following simplification

$$\oint_{\mathfrak{A}^K} \Big(\phi^J \, d\phi^I - \phi^I \, d\phi^J\Big) = 2 \oint_{\mathfrak{A}^K} \phi^J \, d\phi^I \tag{D.40}$$

in view of the fact that the integrand has vanishing $\mathfrak{A}$-monodromies, so that $d(\phi^I \phi^J)$ has vanishing $\mathfrak{A}$-period. Note, however, that no such simplification applies to the corresponding integral over the $\mathfrak{B}_K$ cycles since ϕ^I does have non-vanishing $\mathfrak{B}$-monodromy. For the same reason the integrals over the $\mathfrak{B}$ cycles depend on the base-point of the cycle, which is why we have left the base point p exposed in the above formulas for $\mathfrak{B}$ periods.

D.7 Verifying the cyclic property of $\mathcal{X}_{IJK}$

The relation $[\eta_K, \xi^K] = 0$ requires the following identity on $\mathcal{X}$,

$$\mathcal{X}^{KIJ} \big[b_K, [b_I, b_J]\big] = 0. \tag{D.41}$$

Since the algebra $\mathfrak{g}_b$ is freely generated, the only relation the commutators can satisfy is the Jacobi identity. As a result, the above relation implies the following condition on $\mathcal{X}$,

$$\mathcal{X}^{KIJ} = \mathcal{X}^{JKI} = \mathcal{X}^{IJK} \tag{D.42}$$

which is proven on general grounds in appendix B.2. Here, we shall verify that this condition holds by explicit calculation. We regroup the difference as follows,

$$\mathcal{X}^{JKI} - \mathcal{X}^{KIJ} = -4\pi^2 \bigg\{ \int_\Sigma d\Big(\phi^J \phi^K d\phi^I\Big) + \oint_{\mathfrak{A}^J} \phi^K \, d\phi^I + \oint_{\mathfrak{A}^K} \phi^J \, d\phi^I \bigg\} \tag{D.43}$$

by using the simplifications of (D.40) to convert $\phi^I \, d\phi^K \to -\phi^K \, d\phi^I$ and $\phi^I \, d\phi^J \to -\phi^J \, d\phi^I$ in the integrand of $\mathfrak{A}^J$ and $\mathfrak{A}^K$, respectively. The first term may be recast as a line integral using Stokes's theorem,

$$\int_D d\Big(\phi^J \phi^K d\phi^I\Big) = \oint_{\partial D} \phi^J \phi^K d\phi^I, \tag{D.44}$$

which in turn may be evaluated using the canonical decomposition of the boundary ∂D of the fundamental domain depicted in figure 2,

$$\begin{aligned} \oint_{\partial D} \phi^J \phi^K d\phi^I &= \sum_{L=1}^{h} \int_y^{\mathfrak{A}^L \cdot y} \Big(\phi^J(t)\phi^K(t) - \phi^J(\mathfrak{B}_L \cdot t)\phi^K(\mathfrak{B}_L \cdot t)\Big) d\phi^I(t) \\ &= -\sum_{L=1}^{h} \int_y^{\mathfrak{A}^L \cdot y} \Big(\delta^J_L \phi^K + \delta^K_L \phi^J + \delta^J_L \delta^K_L\Big) d\phi^I \\ &= -\oint_{\mathfrak{A}^J} \phi^K d\phi^I - \oint_{\mathfrak{A}^K} \phi^J d\phi^I. \end{aligned} \tag{D.45}$$

The last term in the integrand on the second line above cancels by itself and the remaining integrals cancel the second and third integrals on the right side of (D.43), thus confirming the identity (D.42).

E Explicit construction of Enriquez kernels

This appendix describes the explicit construction of the Enriquez kernels $g^{I_1\cdots I_r}{}_J(x,p)$ in (2.10) in terms of the f-tensors in Lemma 2.6, Abelian differentials as well as their iterated integrals. In particular, we spell out detailed examples and intermediate results in the procedure of section 3.3 including the coefficients of the contributing series expansions.

Throughout this appendix, we shall denote the solutions $\hat{\xi}$ and $\hat{\eta}$ of the monodromy equations (3.30) simply by ξ and η in order to avoid cluttering.

E.1 Expanding the gauge transformation

Before determining the explicit form of the $\mathcal{X}^{IJ_1\cdots J_r}$ coefficients in the expansion (3.39) of ξ^I, it is convenient to first compute the expansion of the path-ordered exponential $\mathcal{U}_{\rm DHS}(x,p;\xi,\eta)^{-1}$ in (3.42) for generic x. By the vanishing of $[\eta_I,\xi^I]$ and Corollary 3.4, the DHS polylogarithms in this expansion boil down to iterated integrals involving the $\delta^J_{I_r}$-traceless and p-independent parts $\partial_x \Phi^{I_1\cdots I_r}{}_J(x)$ of the $f^{I_1\cdots I_r}{}_J(x,p)$-tensors in (3.18). Homotopy invariance at the subleading order in η_J relies on the combination of terms in

$$\Gamma^{\langle IJ\rangle}(x,p) = \int_p^x \left(\partial_t \Phi^{IJ}(t) - \partial_t \Phi^{JI}(t) + \pi\omega^J(t_1)\int_p^{t_1}\bar{\omega}^I(t_2) - \pi\omega^I(t_1)\int_p^{t_1}\bar{\omega}^J(t_2)\right) \tag{E.1}$$

with $\Phi^{IJ}(t) = \Phi^I{}_K(t)Y^{KJ}$ and manifest antisymmetry $\Gamma^{\langle IJ\rangle}(x,p) = -\Gamma^{\langle JI\rangle}(x,p)$. The expression (E.1) enters $\mathcal{U}_{\rm DHS}(x,p;\xi,\eta)^{-1}$ in combination with the following iterated Abelian integrals realized as DHS polylogarithms associated with $\mathfrak{w} = a^{I_1}\cdots a^{I_r}$ in (2.31)

$$\Gamma_{I_1I_2\cdots I_r}(x,p) = \int_p^x \omega_{I_1}(t_1)\int_p^{t_1}\omega_{I_2}(t_2)\cdots\int_p^{t_{r-1}}\omega_{I_r}(t_r). \tag{E.2}$$

In slight abuse of notation, we shall write $\Gamma^{J_1J_2\cdots J_r}(x,p)$ for their contraction with $Y^{I_1J_1}\cdots Y^{I_rJ_r}$ (which does not refer to the DHS polylogarithms associated with $\mathfrak{w} = b_{J_1}\cdots b_{J_r}$) and $\overline{\Gamma^{J_1J_2\cdots J_r}(x,p)}$ for the complex conjugates of these contractions.

With these prerequisites in place, we can write the coefficients $\mathcal{T}^{I_1\cdots I_r}(x,p)$ at the order $r\leq 2$ of the expansion (3.42) of $\mathcal{U}_{\rm DHS}(x,p;\xi,\eta)^{-1}$ in the following form:

$$\begin{aligned}
\mathcal{T}^I(x,p) &= \pi\big(\overline{\Gamma^I(x,p)} - \Gamma^I(x,p)\big), \\
\mathcal{T}^{IJ}(x,p) &= -\mathcal{X}_M{}^{IJ}\Gamma^M(x,p) - \pi\Gamma^{\langle IJ\rangle}(x,p) \\
&\quad + \pi^2\big(\Gamma^{JI}(x,p) - \overline{\Gamma^J(x,p)}\Gamma^I(x,p) + \overline{\Gamma^{JI}(x,p)}\big).
\end{aligned} \tag{E.3}$$

Together with the results for the coefficients $\mathcal{X}_M{}^{IJ}$ in section E.2 below, the expressions (E.3) determine all instances of the key ingredients (3.46) at $r \leq 2$, e.g.

$$\begin{aligned} h^I{}_J(x,p) &= f^I{}_J(x,p) + \mathcal{T}^I(x,p)\omega_J(x), \\ h^{I_1I_2}{}_J(x,p) &= f^{I_1I_2}{}_J(x,p) + \mathcal{T}^{I_1}(x,p)f^{I_2}{}_J(x,p) + \mathcal{T}^{I_1I_2}(x,p)\omega_J(x). \end{aligned} \tag{E.4}$$

This is an important step towards expressing the Enriquez kernels $g^{I_1\cdots I_r}{}_J(x,p)$ at $r \leq 2$ in terms of f-tensors, Abelian differentials and DHS polylogarithms.

E.2 Determining the explicit form of the $\mathcal{X}$-coefficients

We now proceed to determining the $\mathcal{X}$-coefficients from the $\mathfrak{A}$-monodromy condition (3.30) which translates into the vanishing

$$\mathcal{T}^{I_1\cdots I_r}(\mathfrak{A}^L \cdot p, p) = 0\,, \qquad r \geq 1 \tag{E.5}$$

of the coefficients in the expansion (3.43) at the special value $x = p + \mathfrak{A}^L$. By their explicit form (E.3) at $r \leq 2$ and for generic endpoints, the solution to (E.5) will express $\mathcal{X}_M{}^{IJ}$ in terms of $\mathfrak{A}$-periods of iterated integrals of Abelian differentials (E.2) and antisymmetric kernels in (E.1). In view of the similar $\mathfrak{B}$ periods to be encountered in section E.4 below, we introduce the shorthand notation

$$\begin{aligned} \alpha^{L\langle IJ\rangle}(p) &= \Gamma^{\langle IJ\rangle}(\mathfrak{A}^L \cdot p, p) \\ \beta_L{}^{\langle IJ\rangle}(p) &= \Gamma^{\langle IJ\rangle}(\mathfrak{B}_L \cdot p, p) \end{aligned} \tag{E.6}$$

and more generally write the special values $x = \mathfrak{A}^L \cdot p$ or $x = \mathfrak{B}_L \cdot p$ of the iterated Abelian integrals in (E.2) as follows

$$\begin{aligned} \alpha^L{}_{I_1\cdots I_r}(p) &= \Gamma_{I_1\cdots I_r}(\mathfrak{A}^L \cdot p, p) \\ \beta_{L|I_1\cdots I_r}(p) &= \Gamma_{I_1\cdots I_r}(\mathfrak{B}_L \cdot p, p) \end{aligned} \tag{E.7}$$

with the usual raising of indices via Y^{JK} to $\alpha^{L|I_1\cdots I_r}(p) = \Gamma^{I_1\cdots I_r}(\mathfrak{A}^L \cdot p, p)$. While the periods (2.3) lead to simple, p-independent expressions at $r = 1$,

$$\alpha^L{}_I = \delta^L_I\,, \qquad \beta_{L|I} = \Omega_{LI} \tag{E.8}$$

equivalent to $\alpha^{L|I} = Y^{LI}$, $\beta_L{}^I = Y^{IK}\Omega_{KL}$, generic instances of (E.7) depend non-trivially on p as exemplified by $\partial_p \alpha^L{}_{IJ}(p) = \omega_I(p)\delta^L_J - \omega_J(p)\delta^L_I$. The shuffle relations (1.10) straightforwardly propagate to (see (B.8) and (B.9) for $\sqcup\!\sqcup$)

$$\begin{aligned} \alpha^L{}_{I_1\cdots I_r \sqcup\!\sqcup J_1\cdots J_s}(p) &= \alpha^L{}_{I_1\cdots I_r}(p)\,\alpha^L{}_{J_1\cdots J_s}(p) \\ \beta_{L|(I_1\cdots I_r \sqcup\!\sqcup J_1\cdots J_s)}(p) &= \beta_{L|I_1\cdots I_r}(p)\,\beta_{L|J_1\cdots J_s}(p) \end{aligned} \tag{E.9}$$

and reduce certain permutation sums of (E.7) in the I_j to combinations of (E.8). For instance, the fact that both of $\alpha^L{}_{IJ}(p) + \alpha^L{}_{JI}(p)$ and its complex conjugate simplify to $\delta_I^L \delta_J^L$ implies the antisymmetry of $\mathrm{Im}\, \alpha^L{}_{IJ}(p) = -\mathrm{Im}\, \alpha^L{}_{JI}(p)$.

In this setting, we can express the contribution of $\mathcal{T}^{IJ}(x,p)$ in (E.3) to the $\mathfrak{A}$-monodromy of $\mathcal{U}_{\mathrm{DHS}}(x,p;\xi,\eta)^{-1}$ as

$$\mathcal{T}^{IJ}(\mathfrak{A}^L \cdot p, p) = -\mathcal{X}^{LIJ} - \pi \alpha^{L\langle IJ\rangle}(p) + \pi^2\big(\alpha^{L|JI}(p) - Y^{LJ}Y^{LI} + \overline{\alpha^{L|JI}(p)}\big). \tag{E.10}$$

Hence, the monodromy condition $\mathcal{T}^{IJ}(\mathfrak{A}^L \cdot p, p) = 0$ determines

$$\mathcal{X}^{LIJ} = -\pi \alpha^{L\langle IJ\rangle}(p) + \frac{\pi^2}{2}\big(\alpha^{L|JI}(p) + \overline{\alpha^{L|JI}}(p) - (I \leftrightarrow J)\big), \tag{E.11}$$

where we have rewritten $\alpha^{L|JI}(p) = \frac{1}{2}(\alpha^{L|JI}(p) - \alpha^{L|IJ}(p) + Y^{LJ}Y^{LI})$ and similarly for its complex conjugate to expose the antisymmetry $\mathcal{X}^{LIJ} = -\mathcal{X}^{LJI}$. Even though individual terms on the right side of (E.11) depend on p, one can verify that $\partial_p \mathcal{X}^{LIJ} = \partial_{\bar{p}} \mathcal{X}^{LIJ} = 0$ by combining the derivatives in (E.25) below. Note that both of $\alpha^{L\langle IJ\rangle}(p)$ and $\beta_L{}^{\langle IJ\rangle}(p)$ in (E.6) are antisymmetric in I, J since $\Gamma^{\langle IJ\rangle}(x,p)$ is.

Inserting the result (E.11) for $\mathcal{X}^{LIJ}$ into the expressions (E.3) makes the intermediate objects $h^{I_1\cdots I_r}{}_J(x,p)$ in (3.46) for the relations between Enriquez kernels $g^{I_1\cdots I_r}{}_J(x,p)$ and f-tensors fully explicit for $r \leq 2$.

E.3 Implementing the automorphism

The next step is to express the Enriquez kernels $g^{I_1\cdots I_r}{}_J(x,p)$ in terms of the above $h^{I_1\cdots I_r}{}_J(x,p)$ and the expansion coefficients $\mathcal{L}_I{}^{J_1\cdots J_r K}(p)$ and $\mathcal{M}^{II_1\cdots I_r}{}_J(p)$ in (3.47) and (3.48) that implement the automorphism $a \cup b \to \hat{a} \cup \hat{b}$. This is most conveniently done after eliminating one of the infinite families of coefficients $\mathcal{M}$ or $\mathcal{L}$ in terms of the other by exploiting the matching (3.36) of residues in the connections (3.35).

The order-by-order computations amount to inserting the expansions (3.47) and (3.48) into $[b_I, a^I] = [\eta_I, \hat{a}^I - \xi^I]$ and bringing all the nested brackets into the standard form $B_{I_1} \cdots B_{I_r} a^J$ by exhaustive use of Jacobi identities, for instance $[[b_I, b_J], a^K] = [b_I, [b_J, a^K]] - [b_J, [b_I, a^K]]$. Matching the coefficients of $B_{I_1} \cdots B_{I_r} a^J$ in (3.36) then implies identities such as

$$\begin{gathered}\mathcal{L}_I{}^{JK} - \mathcal{L}_I{}^{KJ} = -\mathcal{M}^{JK}{}_I, \\ \mathcal{L}_I{}^{JKP} - \mathcal{L}_I{}^{JPK} - \mathcal{L}_I{}^{PJK} + \mathcal{L}_I{}^{PKJ} = \mathcal{M}^{JK}{}_Q \mathcal{M}^{QP}{}_I - \mathcal{M}^{JKP}{}_I.\end{gathered} \tag{E.12}$$

Their generalization to higher order is most conveniently obtained by iterative use of (3.50) and leads to the closed formula (3.52) for the coefficients $\mathcal{M}_{\sqcup\!\sqcup}^{II_1\cdots I_r}{}_J(p)$ of the reorganized expansion (3.51) of η_I. By the shuffle symmetries

$$\mathcal{L}_I{}^{J_1\cdots J_r \sqcup\!\sqcup K_1\cdots K_s} = 0\,, \qquad r,s \geq 1 \tag{E.13}$$

following from Ree's theorem and $\eta_I \in \mathfrak{g}$, we have

$$r\mathcal{L}_I{}^{J_1\cdots J_r} = -\mathcal{M}_{\sqcup\!\sqcup}^{J_1\cdots J_r}{}_I\,, \qquad r \geq 2, \tag{E.14}$$

where (E.12) for instance identifies $\mathcal{M}_{\sqcup\!\sqcup}^{I_1 I_2}{}_K = \mathcal{M}^{I_1 I_2}{}_K$ and

$$\mathcal{M}_{\sqcup\!\sqcup}^{I_1 I_2 I_3}{}_K = \mathcal{M}^{I_1 I_2 I_3}{}_K - \mathcal{M}^{I_1 I_2}{}_L \mathcal{M}^{L I_3}{}_K. \tag{E.15}$$

With the relations in (E.12), one can now express the Enriquez kernels $g^{I_1\cdots I_r}{}_J(x,p)$ solely in terms of the expansion coefficients $h^{I_1\cdots I_r}{}_J(x,p)$ in (3.46) and the quantities $\mathcal{M}^{I I_1\cdots I_r}{}_J(p)$: Inserting all of (3.47), (3.48) and (E.12) into (3.35) and comparing the coefficients of $B_{I_1}\cdots B_{I_r} a^J$ determines

$$\begin{aligned} g^I{}_J(x,p) &= h^I{}_J(x,p) + \omega_K(x)\mathcal{M}^{KI}{}_J(p), \\ g^{I_1 I_2}{}_J(x,p) &= h^{I_1 I_2}{}_J(x,p) + h^{I_1}{}_K(x,p)\mathcal{M}^{K I_2}{}_J(p) \\ &\quad - h^K{}_J(x,p)\mathcal{M}^{I_1 I_2}{}_K(p) + \omega_K(x)\mathcal{M}^{K I_1 I_2}{}_J(p). \end{aligned} \tag{E.16}$$

Now the only missing piece of information in the above relations between $g^{I_1\cdots I_r}{}_J$ and $h^{I_1\cdots I_r}{}_J$ is the explicit form of the $\mathcal{M}^{J K_1\cdots K_r}{}_I$ which will be determined next.

E.4 Explicit form of the automorphism and Enriquez kernels

The last step in our explicit construction of the Enriquez kernels is the extraction of the coefficients $\mathcal{M}^{J K_1\cdots K_r}{}_I$ in the automorphism (3.48) from the $\mathfrak{B}$-monodromy condition in (3.30). The latter is particularly suitable for order-by-order computations when rewritten in the form

$$e^{-2\pi i b_K} = \mathcal{U}_{\rm DHS}\big(\mathfrak{B}_K\cdot p, p; \xi, \eta\big)^{-1} = 1 + \sum_{r=1}^{\infty} \eta_{I_1}\cdots\eta_{I_r}\mathcal{T}^{I_1\cdots I_r}(\mathfrak{B}_K\cdot p, p) \tag{E.17}$$

that incorporates the expansion in (3.43) with explicit results for its coefficients $\mathcal{T}^{I_1\cdots I_r}$ in (E.3). The desired $\mathcal{M}^{J K_1\cdots K_r}{}_I$ are determined by lining up the Lie-algebra valued expansion variables η_I and b_K on the two sides of (E.17) which can be done in a variety of ways. We found it convenient to invert the expansion of η_I in (3.51) while eliminating the coefficients $\mathcal{M}_{⧢}$ in favor of $\mathcal{M}$ using (3.52), e.g.[28]

$$\begin{aligned} b_J = \eta_J &+ \eta_{I_1}\eta_{I_2}\mathcal{M}^{I_1I_2}{}_J(p) \\ &+\eta_{I_1}\eta_{I_2}\eta_{I_3}\big(\mathcal{M}^{I_1I_2I_3}{}_J(p) + \mathcal{M}^{I_3I_2}{}_K(p)\mathcal{M}^{KI_1}{}_J(p)\big) + \mathcal{O}(\eta^4), \end{aligned} \tag{E.18}$$

where the coefficient of $\eta_{I_1}\eta_{I_2}\eta_{I_3}$ is another shuffle-symmetric combination different from $\mathcal{M}_{⧢}^{I_1I_2I_3}{}_I(p)$ in (E.15). After inserting the expansion (E.18) into the exponentials of (E.17),

$$\begin{aligned} e^{-2\pi i b_K} = 1 &- 2\pi i\eta_I\delta^I_K + \eta_{I_1}\eta_{I_2}\big(\tfrac{1}{2}(2\pi i)^2\delta^{I_1I_2}_K - 2\pi i\mathcal{M}^{I_1I_2}{}_K(p)\big) \\ &+ \eta_{I_1}\eta_{I_2}\eta_{I_3}\big(-\tfrac{1}{6}(2\pi i)^3\delta^{I_1I_2I_3}_K + \tfrac{1}{2}(2\pi i)^2\mathcal{M}^{I_1I_2}{}_K(p)\delta^{I_3}_K + \tfrac{1}{2}(2\pi i)^2\delta^{I_1}_K\mathcal{M}^{I_2I_3}{}_K(p) \\ &\qquad - 2\pi i\big[\mathcal{M}^{I_1I_2I_3}{}_K(p) - \mathcal{M}^{I_2I_3}{}_L(p)\mathcal{M}^{LI_1}{}_K(p)\big]\big) + \mathcal{O}(\eta^4), \end{aligned} \tag{E.19}$$

we can equate the unknown $\mathcal{M}$-dependent coefficients of $\eta_{I_1}\cdots\eta_{I_r}$ with the computable quantities $\mathcal{T}^{I_1\cdots I_r}(p+\mathfrak{B}_K,p)$. Among the resulting conditions at rank $r\leq 3$,

$$\begin{aligned} \mathcal{T}^I(\mathfrak{B}_K\cdot p,p) &= -2\pi i\delta^I_K, \\ \mathcal{T}^{I_1I_2}(\mathfrak{B}_K\cdot p,p) &= \tfrac{1}{2}(2\pi i)^2\delta^{I_1I_2}_K - 2\pi i\mathcal{M}^{I_1I_2}{}_K(p), \\ \mathcal{T}^{I_1I_2I_3}(\mathfrak{B}_K\cdot p,p) &= -\tfrac{1}{6}(2\pi i)^3\delta^{I_1I_2I_3}_K + \tfrac{1}{2}(2\pi i)^2\mathcal{M}^{I_1I_2}{}_K(p)\delta^{I_3}_K + \tfrac{1}{2}(2\pi i)^2\delta^{I_1}_K\mathcal{M}^{I_2I_3}{}_K(p) \\ &\quad + 2\pi i\mathcal{M}^{I_2I_3}{}_N(p)\mathcal{M}^{NI_1}{}_K(p) - 2\pi i\mathcal{M}^{I_1I_2I_3}{}_K(p), \end{aligned} \tag{E.20}$$

the first line is trivially satisfied since the leading order $b_I = \eta_I + \mathcal{O}(\eta^2)$ in the expansions (3.47) and (E.18) already takes our computations in (3.10) into account. The second line of (E.20) in turn can be solved for $\mathcal{M}^{IJ}{}_K(p)$ in terms of the following $\mathfrak{B}$-periods

$$\begin{aligned} \mathcal{T}^{IJ}(\mathfrak{B}_K\cdot p,p) = &-\Omega_{KR}\mathcal{X}^{RIJ} - \pi\beta_K{}^{\langle IJ\rangle}(p) \\ &+ \pi^2\big[\beta_K{}^{JI}(p) - Y^{JR}\bar{\Omega}_{RK}Y^{IS}\Omega_{SK} + \overline{\beta_K{}^{JI}(p)}\big] \end{aligned} \tag{E.21}$$

obtained from (E.3) at $x=\mathfrak{B}_K\cdot p$, with $\mathcal{X}$ given by the $\mathfrak{A}$-periods (E.11), and the α,β notation introduced in (E.6) as well as (E.7). One can view the second line of (E.20) as providing both an expression for the unknown coefficient $\mathcal{M}^{IJ}{}_K(p)$ and as a crosscheck of the $\mathfrak{B}$-periods (E.21): the antisymmetry of the solution

$$\mathcal{M}^{IJ}{}_K(p) = i\pi\delta^{IJ}_K + \frac{i}{2\pi}\mathcal{T}^{IJ}(\mathfrak{B}_K\cdot p,p) \tag{E.22}$$

[28]The coefficient of $\eta_{I_1}\eta_{I_2}\eta_{I_3}\eta_{I_4}$ at the next order of the expansion (E.18) of b_J is given by $\mathcal{M}^{I_1I_2I_3I_4}{}_J + \mathcal{M}^{I_1I_2K}{}_J\mathcal{M}^{I_3I_4}{}_K + \mathcal{M}^{I_1K}{}_J\mathcal{M}^{I_2I_3I_4}{}_K + \mathcal{M}^{I_1KI_4}{}_J\mathcal{M}^{I_2I_3}{}_K + \mathcal{M}^{I_1K}{}_J\mathcal{M}^{I_2L}{}_K\mathcal{M}^{I_3I_4}{}_L$ and does not line up with a closed formula where the number of terms doubles at each order as in the coefficients (3.52) of the inverse expansion of η_J in terms of $b_{I_1}\cdots b_{I_r}$.

in $I \leftrightarrow J$ derived from (E.12) is not manifest term by term and relies on finding the symmetric part $\mathcal{T}^{IJ}(\mathfrak{B}_K\cdot p,p)+\mathcal{T}^{JI}(\mathfrak{B}_K\cdot p,p)=(2\pi i)^2\delta^{IJ}_K$ in (E.21). This can be verified using the consequence $\beta_{K|IJ}(p)+\beta_{K|JI}(p)=\Omega_{KI}\Omega_{KJ}$ of the shuffle relations (E.9), so the desired coefficient can be alternatively expressed as the antisymmetric part

$$
\begin{aligned}
\mathcal{M}^{IJ}{}_K(p) &= \frac{i}{4\pi}\big[\mathcal{T}^{IJ}(\mathfrak{B}_K\cdot p,p)-\mathcal{T}^{JI}(\mathfrak{B}_K\cdot p,p)\big] \\
&= \frac{i}{2}\Big(\Omega_{KR}\alpha^{R\langle IJ\rangle}(p)-\beta_K{}^{\langle IJ\rangle}(p)\Big) \\
&\quad+\frac{i\pi}{4}\Big(\beta_K{}^{JI}(p)-\Omega_{KR}\alpha^{R|JI}(p)+Y^{IR}\bar\Omega_{RK}Y^{JS}\Omega_{SK} \\
&\qquad+\overline{\beta_K{}^{JI}(p)}-\Omega_{KR}\overline{\alpha^{R|JI}(p)}-(I\leftrightarrow J)\Big),
\end{aligned}
\tag{E.23}
$$

where the antisymmetrization prescription $-(I\leftrightarrow J)$ applies to the last two lines.

One can similarly solve the third equation of (E.20) for $\mathcal{M}^{I_1I_2I_3}{}_K(p)$ in terms of $\mathfrak{B}$-monodromies $\mathcal{T}^{I_1I_2I_3}(\mathfrak{B}_K\cdot p,p)$ and lower-order expressions. Again, the shuffle symmetry of $\mathcal{M}^{I_1I_2I_3}{}_K(p)-\mathcal{M}^{I_2I_3}{}_L(p)\mathcal{M}^{LI_1}{}_K(p)$ will not be obvious from the resulting expression but can be verified using a sequence of shuffle relations (E.9) of the $\mathfrak{A}$- and $\mathfrak{B}$ periods and their special cases (E.8). An alternative approach towards $\mathcal{M}^{I_1I_2I_3}{}_K(p)$ is to project the third equation of (E.20) to those symmetry components with respect to the permutation group of I_j that eliminate all the admixtures of Kronecker deltas as done in (E.23).

E.5 Summary and further simplifications

The explicit form of low rank Enriquez kernels may be obtained by combining (E.4) with (E.16), resulting in equation (1.15) of the Introduction. The simplest components $\mathcal{T}^I(x,p)$ and $\mathcal{T}^{I_1I_2}(x,p)$ of the gauge transformation are explicitly available by combining (E.3) with the expression (E.11) for $\mathcal{X}^{KIJ}$. The coefficients $\mathcal{M}^{KI_1\ldots I_r}{}_J(p)$ of the automorphism are explicitly given in (E.23) for $r=1$ and implicitly given in (E.20) for $r=2$, resulting from the second- and third-order expansion of $\mathcal{U}_{\rm DHS}(\mathfrak{B}_K\cdot p,p;\xi,\eta)^{-1}$, respectively.

The analogous expressions for $g^{I_1\cdots I_r}{}_J(x,p)$ at higher order $r\geq 3$ require the contributions to the automorphism up to and including the rank of $\mathcal{M}^{KI_1\cdots I_r}{}_J(p)$ which is computed from expansions of $\mathcal{U}_{\rm DHS}\big(\mathfrak{B}_K\cdot p,p;\xi,\eta\big)^{-1}$ to the $(r+1)^{\rm th}$ order in η_I or b_I. With the expression for $\mathcal{M}^{KI}{}_J(p)$ in (E.23), we have the fully explicit form of the Enriquez kernel $g^I{}_J(x,p)$, and the solution $\mathcal{M}^{KI_1I_2}{}_J(p)$ of the last equation in (E.20) completely fixes $g^{I_1I_2}{}_J(x,p)$ in (1.15).

However, we have not yet attempted an exhaustive simplification of the $\mathfrak{A}$- and $\mathfrak{B}$-periods appearing in the expressions for $\mathcal{M}^{KI_1\cdots I_r}{}_J(p)$ due to the procedure in section 3.3.

The expression (E.23) for $\mathcal{M}^{KI}{}_J(p)$ admits a more minimal form that can be anticipated from the differential equations

$$\begin{aligned}\partial_p \mathcal{M}^{IJ}{}_K(p) &= \pi\big(\delta^J_K \omega^I(p) - \delta^I_K \omega^J(p)\big),\\ \partial_{\bar p} \mathcal{M}^{IJ}{}_K(p) &= \pi\big(\delta^I_K \bar\omega^J(p) - \delta^J_K \bar\omega^I(p)\big),\end{aligned} \tag{E.24}$$

which follow from

$$\begin{aligned}\partial_p \alpha^{K\langle IJ\rangle}(p) &= \pi\big(\omega^J(p) Y^{KI} - \omega^I(p) Y^{KJ}\big),\\ \partial_{\bar p} \alpha^{K\langle IJ\rangle}(p) &= \pi\big(\bar\omega^J(p) Y^{KI} - \bar\omega^I(p) Y^{KJ}\big),\\ \partial_p \alpha^{K|IJ}(p) &= \omega^I(p) Y^{JK} - \omega^J(p) Y^{IK},\end{aligned} \tag{E.25}$$

as well as

$$\begin{aligned}\partial_p \beta_K{}^{\langle IJ\rangle}(p) &= \pi\big(\omega^J(p) Y^{IR} - \omega^I(p) Y^{JR}\big)\bar\Omega_{RK},\\ \partial_{\bar p} \beta_K{}^{\langle IJ\rangle}(p) &= \pi\big(\bar\omega^J(p) Y^{IR} - \bar\omega^I(p) Y^{JR}\big)\Omega_{RK},\\ \partial_p \beta_K{}^{IJ}(p) &= \big(\omega^I(p) Y^{JR} - \omega^J(p) Y^{IR}\big)\Omega_{RK}.\end{aligned} \tag{E.26}$$

Matching the p- and $\bar p$-derivatives of (E.24) with those of $2\pi i\,\mathrm{Im}\,\alpha_K{}^{IJ}(p)$, we establish

$$\mathcal{M}^{IJ}{}_K(p) = 2\pi i\,\mathrm{Im}\,\alpha_K{}^{IJ}(p) + \mathcal{Z}^{IJ}{}_K, \tag{E.27}$$

where $\mathcal{Z}^{IJ}{}_K = -\mathcal{Z}^{JI}{}_K$ is independent on p but may depend non-meromorphically on the moduli of Σ. Hence, the final form of the first non-trivial Enriquez kernel to be provided in this work is given by

$$g^I{}_J(x,p) = f^I{}_J(x,p) - 2\pi i\,\omega_J(x)\mathrm{Im}\int_p^x \omega^I + \omega_K(x)\Big(2\pi i\,\mathrm{Im}\,\alpha_J{}^{KI}(p) + \mathcal{Z}^{KI}{}_J\Big), \tag{E.28}$$

which follows from (1.15) and (E.27). While meromorphicity in x is a consequence of the absence of $(0,1)$-form components in (3.31), meromorphicity of $g^I{}_J(x,p)$ in p relies on the interplay of the first three terms in (E.28) and is guaranteed by the uniqueness of the Enriquez kernels based on the defining properties of $\mathcal{K}_{\mathrm{E}}$. We leave obtaining a direct derivation of (E.27), an explicit formula for $\mathcal{Z}^{IJ}{}_K$, and the generalizations to simplify higher-rank contributions $\mathcal{M}^{KI_1\ldots I_r}{}_J(p)$ to the automorphism to future work.

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