ON THE ANALOG CATEGORY OF FINITE GROUPS

BEN KNUDSEN AND SHMUEL WEINBERGER

Abstract. We show that, up to small error, the analog category of a finite group records the size of its largest Sylow subgroup.

1. Introduction

We continue the probabilistic reimagining of the foundations of topological robotics [Far03] begun simultaneously in [KW24] and [DJ24], in which motion planning is conducted according to continuously varying probability measures on the relevant space of paths. The resulting "analog" invariants—which bound their classical counterparts, the Lusternik–Schnirelmann category and topological complexity, from below—display surprisingly subtle behavior.

For example, the analog category of an aspherical space with torsion-free fundamental group is equal to the cohomological dimension of that group [KW24, Thm. 1.1], a direct analogue of the Eilenberg–Ganea theorem. Thus, in this case, analog category equals category. On the other hand, in the case of a finite fundamental group, the classical category is always infinite, while in our setting we have the following [KW24, Thm. 7.2]—see Section 2 for the definition of $\operatorname{acat}(G)$.

Universal upper bound. *If G is finite, then* $\operatorname{acat}(G) < |G|$.

We show here that this bound is arbitrarily far from sharp and almost never holds. Provisionally, let us call G *a-special* if $|G| = ap^s$ with p a prime, $(a, p) = 1$, and $p^s > a$ (thus, a 1-special group is simply a p-group).

Theorem 1.1. *Let G be a finite group.*

1. *If G is not a-special for some $a \in \{1, 2, 3\}$, then the universal upper bound does not hold for G.*
2. *If G is 1- or 2-special, then the universal upper bound holds for G.*[1]
3. *For any $N \geq 0$, there is a group G such that* $\operatorname{acat}(G) < |G| - N$.

In other words, the universal upper bound holds for p-groups and fails for almost all groups not of prime power order. For groups of the latter type, we show that, up to small error, the analog category is recording the size of the largest Sylow subgroup.

Theorem 1.2. *Let G be a finite group not of prime power order. If $P \leq G$ is a Sylow subgroup of maximal order, then*

$$\min\left\{2, \tfrac{|N(P)|}{|P|}\right\} \leq \frac{\operatorname{acat}(G) + 1}{|P|} \leq 3.$$

[1] Unfortunately, we do not know whether the universal upper bound holds for 3-special groups.

In more prosaic terms, the lower bound is 1 when the largest Sylow subgroup of G is self-normalizing[2], and otherwise it is 2; for example, the lower bound of 2 obtains for all nilpotent groups.

Prior to our work here, the quantity $\mathrm{acat}(BG)$ was almost completely unknown apart from the universal upper bound and a calculation for cyclic groups of prime order [Dra]. Strictly speaking, this last calculation was of an a priori different invariant, the distributional category; we show here that the two coincide for finite groups, a special case of [KW24, Conj. 1.2].

Theorem 1.3. *For any finite group G, the analog and distributional category of G coincide.*

In fact, the same argument may be used to show that the analog and distributional versions of the rth sequential topological complexity TC_r coincide for every r.

1.1. **Conventions.** We use the following non-standard notational convention for topological simplices:

$$\Delta^{n-1} := \left\{ (t_1, \ldots, t_n) \in [0,1]^n : \sum_{i=1}^{n} t_i = 1 \right\}.$$

We work in Steenrod's convenient category of topological spaces [Ste67]—see [KW24, Appendix A] for a summary of relevant facts about these spaces. Topological spaces are implicitly convenient, as are limits, including products, and mapping spaces. Convenient colimits, when they exist, are the same as ordinary colimits. The adjective "compact" refers to the definition in terms of open covers. We write BG for the classifying space of the (discrete) group G, i.e., the geometric realization of its nerve. We write EG for the universal cover of BG and $X^{hG} = \mathrm{Map}^G(EG, X)$ for the space of homotopy fixed points of the G-space X.

2. The analog category of a group

The purpose of this section is to establish the following formula, which the reader may take as a definition of the analog category $\mathrm{acat}(G)$.

Proposition 2.1. *For any group G, we have* $\mathrm{acat}(G) = \min\{n \mid \left(\Delta_n^G\right)^{hG} \neq \varnothing\}$.

We begin with a brief review of the invariants of [KW24], which are defined in terms of the set $\mathcal{P}(X)$ of probability measures with finite support on the topological space X. We view $\mathcal{P}(X)$ as a topological space with the quotient topology inherited from the various maps

$$X^n \times \Delta^{n-1} \longrightarrow \mathcal{P}(X)$$
$$(x, t) \mapsto \sum_{i=1}^{n} t_i \delta_{x_i}.$$

We write $\mathcal{P}_n(X) \subseteq \mathcal{P}(X)$ for the subspace of measures with support of cardinailty at most n.

[2]As shown in [GMN03], the admission of a self-normalizing Sylow subgroup places strong constraints on a group.

Classically, spaces of probability measures are often topologized using the Lévy–Prokhorov metric, which metrizes the topology of weak convergence when the background space X is a separable metric space. We direct the reader to Section 5 below for some comparisons between the two approaches. The twin advantages of ours are its generality, as we do not even require the background space to be metrizable, and its excellent technical features, which are summarized in the following result.

Theorem 2.2 ([KW24, Thm. 2.7])**.** *The functor $\mathcal{P}$ is an endofunctor on the category of convenient spaces, which preserves homotopy, sifted colimits, quotient maps, and closed embeddings.*

Given a map $f : X \to Y$, we may consider the space of probability measures on X with fiberwise support over Y, namely

$$\mathcal{P}(f) = \left\{ \sum_{i=1}^{n} t_i \delta_{x_i} \in \mathcal{P}(X) : f(x_1) = f(x_2) = \cdots = f(x_n) \right\},$$

and we set $\mathcal{P}_n(f) = \mathcal{P}_n(X) \cap \mathcal{P}(f)$. Sending an element of $\mathcal{P}(f)$ to the point in whose fiber it is supported defines a continuous map to Y.

Definition 2.3. The *analog sectional category* of the map $f : X \to Y$ is the least n such that $\mathcal{P}_{n+1}(f) \to Y$ admits a section. The *analog category* of the space X, denoted $\operatorname{acat}(X)$, is the analog sectional category of the evaluation map $(X, x_0)^{([0,1],\{0\})} \to X$, where $x_0 \in X$ is any basepoint.

As our interest here lies solely in the aspherical context, we permit ourselves the abusive abbreviation $\operatorname{acat}(G) = \operatorname{acat}(BG)$.

For the proof of the proposition stated above, we require the following standard fact.

Lemma 2.4. *For any G-space X, the homotopy fixed point space X^{hG} is canonically weakly equivalent to the space of sections of the canonical map $EG \times_G X \to BG$.*

Proof. Consider the following commutative diagram of mapping spaces:

$$\begin{array}{ccccc}
\mathrm{Map}^G(EG, EG \times X) & \longrightarrow & \mathrm{Map}^G(EG, EG \times_G X) & \overset{\sim}{=\!=} & \mathrm{Map}(BG, EG \times_G X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}^G(EG, EG) & \longrightarrow & \mathrm{Map}^G(EG, BG) & \overset{\sim}{=\!=} & \mathrm{Map}(BG, BG)
\end{array}$$

The section space in question is the fiber of the righthand vertical map over id_{BG}, while X^{hG} is the fiber of the lefthand vertical map over id_{EG}. The claim follows after noting that the vertical maps are fibrations and the lefthand square a homotopy pullback. □

Proof of Proposition 2.1. It is well-known that, for any group G, there is a commutative diagram of topological spaces

$$\begin{array}{ccc} G & \longrightarrow & (BG, x_0)^{([0,1],\{0,1\})} \\ \downarrow & & \downarrow \\ EG & \longrightarrow & (BG, x_0)^{([0,1],\{0\})} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ BG & = & BG \end{array}$$

in which the horizontal arrows are homotopy equivalences and the vertical columns are (Hurewicz) fiber sequences. It follows from [KW24, Cor. 5.3] that $\mathrm{acat}(G)$ is the analog sectional category of π, which is a fiber bundle with structure group G; therefore, by [KW24, Cor. 5.8], we have

$$\mathcal{P}_{n+1}(\pi) \cong EG \times_G \mathcal{P}_{n+1}(G)$$

as spaces over BG, and it is easy to see that $\mathcal{P}_{n+1}(G) \cong \Delta_n^G$ as G-spaces, so the claim follows from Lemma 2.4. ☐

Corollary 2.5. *If X is a contractible G-space with an equivariant map $X \to \Delta_n^G$, then* $\mathrm{acat}(G) \leq n$.

Proof. Since X is contractible, an equivariant map $EG \to X$ exists, which is to say that $X^{hG} \neq \varnothing$. It follows that $(\Delta_n^G)^{hG}$ receives a map from a non-empty space, hence is itself non-empty. The claim follows from Proposition 2.1. ☐

Remark 2.6. Essentially the same argument shows that the rth analog topological complexity of BG, as defined in [KW24], is equal to the least n such that the G^r-space $\Delta_n^{G^r/G}$ admits a homotopy fixed point.

3. Designer complexes

This section is concerned with the construction of certain contractible equivariant cell complexes, which, via Corollary 2.5 and obstruction theory, will be the key to proving our main results. The ideas here are mostly taken from the work of Assadi [Ass82], following Oliver, Conner–Floyd, Smith, and others, but the specificity of our situation permits some simplification and hence a relatively self-contained account.

In what follows, the group G is always finite. As a matter of terminology, we say that a space is p-acyclic if its mod p reduced homology vanishes.

Definition 3.1. Let X be a G-complex. Given a prime p, we say that X is *Smith p-acyclic* if X^P is p-acyclic for every nontrivial p-subgroup P. We say that X is *Smith acyclic* if X is Smith p-acyclic for every prime p.

The relevance of this definition lies in its connection to obstruction theory.

Proposition 3.2. *Let X be a G-complex of dimension m.*

1. *If X is Smith (p-)acyclic, then a (p-)acyclic G-complex may be obtained from X by attaching free cells of dimension at most* $m+1$.
2. *If X is acyclic, then a contractible G-complex may be obtained by attaching free cells of dimension at most* 3.

Proof. For the first claim, if X is Smith acyclic, then [Ass82, Prop. I.1.6] guarantees that we may achieve ayclicity below degree m and $\mathbb{Z}[G]$-projectivity in degree m by attaching free cells of dimension at most m. Thus, by the Eilenberg swindle, we may achieve acyclicity by further cell attachments of dimension m and $m+1$. The Smith p-acyclic case is similar, invoking [Ass82, Lem. II.1.5] instead, and obtaining instead a degree m mod p homology group projective over $\mathbb{F}_p[G]$.

For the second claim, note first that X is path connected by acyclicity. We first kill the fundamental group of X by attaching free 2-cells indexed by a set I. Calling the resulting complex Y, the long exact sequence for the pair (Y, X) shows that

$$\widetilde{H}_i(Y) \cong \begin{cases} \bigoplus_I \mathbb{Z}[G] & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a simply connected acyclic G-complex may be obtained by attaching free 3-cells, and any such complex is contractible by Whitehead's theorem. □

Our main construction will proceed inductively and one prime at a time. In order to state the main result, we require the following definition, which will form the basis for our induction.

Definition 3.3. Let G be a finite group and p a prime dividing $|G|$.

(1) A subgroup $H \leq G$ is called a *p-intersection* if it is an intersection of p-Sylow subgroups.
(2) The *(p-Sylow)* depth of the p-intersection H is the largest d for which there is a chain $H = H_d < H_{d-1} < \cdots < H_1 = P$ of proper inclusions with P a p-Sylow subgroup and each H_i a p-intersection.
(3) The *(p-Sylow) depth* of a p-subgroup $H \leq G$ is the maximal depth of a p-intersection containing H.
(4) The *(p-Sylow) depth* of G, denoted $d_p(G)$, is the maximal depth of a p-intersection in G.

We adopt the convention that $d_p(G) = 0$ if and only if $(p, |G|) = 1$.

It is easy to see that the depth of H as a p-subgroup coincides with its depth as a p-intersection. It is also easy to see that the $d_p(G)$ is bounded above by the number of distinct p-Sylow subgroups of G, as well as by the exponent of p in $|G|$.

Lemma 3.4. *Let H be a p-intersection of depth d and K any p-subgroup containing H. The depth of K is at most d, with equality if and only if $H = K$.*

Proof. We may assume that $H \neq K$. Supposing that K has depth $s \geq d$, we obtain the chain of inclusions

$$H < K \leq H_s < H_{s-1} < \cdots < H_1 = P,$$

implying that the depth of H is at least $s + 1 > d$, a contradiction. □

Corollary 3.5. *Let H_1 and H_2 be p-intersections. If $H_1 < H_2$, then the depth of H_1 is greater than the depth of H_2.*

Corollary 3.6. *If K is a p-subgroup of depth d, then K is contained in a unique p-intersection of depth d.*

Proof. Let $H_1 \neq H_2$ be p-intersections of depth d containing K. Then $H_1 \cap H_2$ is a p-intersection properly contained in H_1, hence of strictly greater depth by Corollary 3.5. It follows that K has depth greater than d, a contradiction. □

We write $\mathcal{I}_d = \mathcal{I}_d(p)$ for the set of p-intersections (in G, implicitly) of depth d, regarded as a G-set under conjugation.

Convention 3.7. For the remainder of this section, we assume that G is a finite group not of prime power order.

We come now to the main construction (compare [Ass82, Thm. II.1.4]).

Theorem 3.8. *Let p be a prime dividing $|G|$. For $0 \leq d \leq d_p(G)$, there are G-complexes $X_p(G)_d$ with the following properties:*

(1) $X_p(G)_0$ is free of dimension 1;
(2) $X_p(G)_{d+1}$ is obtained from $X_p(G)_d$ by attaching cells of dimension at most $d+2$ with isotropy in $\mathcal{I}_{d+1}$;
(3) $X_p(G)_d^P$ is p-acyclic for every nontrivial p-subgroup P of depth at most d.

Proof. We proceed by simultaneous induction on d and $d_p(G)$. In the case $d = d_p(G) = 0$, we let $X_p(G)_0$ be any connected 1-dimensional free G-complex. Notice that the third condition is vacuous in this case. For $d = 0$ and general G, we choose a p-Sylow subgroup P and set

$$X_p(G)_0 = G \times_{N(P)} X_p(N(P)/P)_0.$$

For $d = 1$, let P be as above and consider $X_p(N(P)/P)_0$. Since P is p-Sylow, the group ring $\mathbb{F}_p[N(P)/P]$ is semisimple by Maschke's theorem. It follows that $\widetilde{H}_1(X_p(N(P)/P)_0; \mathbb{F}_p)$ is projective over $\mathbb{F}_p[N(P)/P]$; therefore, by Proposition 3.2 and the Eilenberg swindle, we may attach free cells of dimension 1 and 2 to obtain a p-acyclic $N(P)/P$-complex $\overline{X}_p(N(P)/P)_0$. Finally, we define

$$X_p(G)_1 = G \times_{N(P)} \overline{X}_p(N(P)/P)_0.$$

The second property holds by construction, and the third follows from the observation that the fixed set of any p-Sylow subgroup is homeomorphic to $\overline{X}_p(N(P)/P)_0$, which is p-acyclic by construction.

In the general case, choose a p-intersection $H \leq G$ of depth $d+1$ and consider the $N(H)/H$-space $X_p(G)_d^H$. We claim that this space is Smith p-acyclic; indeed, given a nontrivial p-subgroup $P \leq N(H)/H$, we have $(X_p(G)_d^H)^P = X_p(G)_d^{\widetilde{P}}$, where $H \leq \widetilde{P}$ is the subgroup of $N(H)$ corresponding to P, and the depth of $\widetilde{P}$ is at most d by Lemma 3.4, since P was assumed nontrivial, so the claim follows by induction. Therefore, by Proposition 3.2, we may achieve p-acyclicity after attaching free $N(H)/H$-cells of dimension at most $d+2$, and, indexing these cells by $i \in I$, we achieve the same result G-equivariantly for all conjugates of H at once via the construction

$$G \times_{N(H)} \left(\coprod_{i \in I} N(H)/H \times D^{n_i} \right) \coprod_{G \times \coprod_{i \in I} N(H)/H \times S^{n_i - 1}} X_p(G)_d.$$

By Corollary 3.5, this construction does not alter the fixed set of any member of $\mathcal{I}_{d+1}$ not conjugate to H; therefore, we may define $X_p(G)_{d+1}$ to be the result of iterating the construction over $\mathcal{I}_{d+1}/G$.

The second condition holds by construction. To check the third, we note that, if P has depth less than d, then $X_p(G)_d^P = X_p(G)_{d-1}^P$ by construction, since P is contained in no member of $\mathcal{I}_d$ by definition, and the latter is p-acyclic by induction. On the other hand, if P has depth d, then P is contained in a *unique* $H \in \mathcal{I}_d$ by Corollary 3.6, so $X_p(G)_d^P = X_p(G)_d^H$, which was constructed to be p-acyclic. $\square$

We write $X_p(G) = X_p(G)_{d_p(G)}$.

Corollary 3.9. *There is a G-complex $X(G)$ with the following properties:*

1. *$X(G)$ is obtained from $\bigsqcup_{p||G|} X_p(G)$ by attaching free cells of dimension at most $\max_p d_p(G) + 2$*
2. *$X(G)$ is contractible.*

Proof. By construction, given $p \neq q$ dividing $|G|$, every p-subgroup of G acts without fixed points on $X_q(G)$, so the third condition of Theorem 3.8 implies that the disjoint union in question is Smith acyclic. Since the dimension of $X_p(G)$ is $d_p(G)+1$, and since $\max_p d_p(G)+2 \geq 3$, Proposition 3.2 shows that we may achieve first acyclicity, then contractibility after the indicated type of cell attachment. $\square$

4. Proofs of the main results

Our strategy will be to exploit Corollary 2.5 by applying obstruction theory to the complex $X(G)$ constructed in the previous section. In order to proceed, we require information on the connectivity of fixed point sets.

Proposition 4.1. *For any $H \leq G$ and any $n \in \mathbb{Z}$, there is a canonical $N(H)/H$-equivariant homeomorphism*

$$\left(\Delta^G_n\right)^H \cong \Delta^{G/H}_{\lfloor \frac{n+1}{|H|} \rfloor - 1}$$

Corollary 4.2. *For any $H \leq G$ and $n < |G|$, the connectivity of $(\Delta^G_n)^H$ is exactly $\lfloor \frac{n+1}{|H|} \rfloor - 2$.*

Proof of Proposition 4.1. Since H is finite, we may define a function $f : \Delta^{G/H} \to \Delta^G$ by the formula

$$f(t)_g = \frac{1}{|H|} t_{gH}.$$

As the restriction of a linear map, this function is continuous, and its image is H-fixed by inspection; thus, we may we view f as a map to $(\Delta^G)^H$. As such, it is $N(H)/H$-equivariant and injective by inspection, and we claim that it is also surjective. To see why, note that a point in Δ^G is fixed by H if and only if the barycentric coordinate of gh is independent of $h \in H$ for every $g \in G$. Thus, given $t \in (\Delta^G)^H$, the assignment $gH \mapsto |H| t_g$ is a well-defined element of $f^{-1}(t)$. The homeomorphism $(\Delta^G)^H \cong \Delta^{G/H}$ follows, since both sides are compact and Hausdorff.

Now, a point of Δ^G lies in Δ^G_n if and only if at most $n+1$ of its barycentric coordinates are nonzero. We conclude that f identifies $(\Delta^G_n)^H$ with the subspace of $\Delta^{G/H}$ in which at most $\frac{n+1}{|H|}$ barycentric coordinates are nonzero, as desired. $\square$

In the following, we write $X^\wedge_p$ for the completion of the space X at the prime p—see [MP11], for example. We recall that, according to the Sullivan conjecture, the natural map $(X^P)^\wedge_p \to (X^\wedge_p)^{hP}$ is a weak equivalence for any p-group P and finite P-CW complex X.

Proposition 4.3. *For any p-subgroup $P \leq G$, we have* $\operatorname{acat}(G) \geq |P| - 1$. *If P is not self-normalizing, then* $\operatorname{acat}(G) \geq 2|P| - 1$.

Proof. For the first claim, it suffices to show that $(\Delta_n^G)^{hP} = \varnothing$, since $(\Delta_n^G)^{hG}$ has a canonical map to this space. By the Sullivan conjecture, the second of the following canonical maps is a weak equivalence:

$$(\Delta_n^G)^P \to ((\Delta_n^G)^P)_p^\wedge \to ((\Delta_n^G)_p^\wedge)^{hP} \leftarrow (\Delta_n^G)^{hP}.$$

If $n < |P| - 1$, then $(\Delta_n^G)^P = \varnothing$ by Proposition 4.1. In particular, this space is p-complete, so the first map above is also a weak equivalence. We conclude that the target of the rightmost map is empty, so its source must be so as well.

If $|P| - 1 \leq n < 2|P| - 1$, then Proposition 4.1 instead identifies $(\Delta_n^G)^P$ with the discrete $N(P)/P$-space G/P, which is also p-complete. We consider the result of applying homotopy fixed points for $N(P)/P$ to the above string of arrows. Since P is not self-normalizing, we again obtain the empty space in the leftmost position, implying this time that $(\Delta_n^G)^{hN(P)} = ((\Delta_n^G)^{hP})^{hN(P)/P} = \varnothing$. Since $(\Delta_n^G)^{hG}$ maps to this space, the conclusion follows. $\square$

Proof of Theorem 1.2. The lower bound follows from Proposition 4.3. For the upper bound, it suffices by Corollary 2.5 to construct an equivariant map $X(G) \to \Delta_n^G$ for $n \geq 3q - 1$, where q is the largest prime power dividing $|G|$. To begin, since $X_p(G)_0$ is free of dimension 1 for each prime p, there is no obstruction to constructing a map to Δ_n^G provided $n \geq 1$, which certainly holds in our situation. Proceeding inductively, we may extend an equivariant map from $X_p(G)_d$ to $X_p(G)_{d+1}$ provided $(\Delta_n^G)^H$ is $(d+1)$-connected for every $H \in \mathcal{I}_{d+1}(p)$; we will consider this question presently. Finally, equivariant maps from the various $X_p(G)$ may be extended to $X(G)$ provided $n \geq \max_p d_p(G) + 2$, which certainly holds in our situation, since $d_p(G)$ is bounded above by the exponent of p in $|G|$.

Now, for $H \in \mathcal{I}_{d+1}(p)$, the connectivity of $(\Delta_n^G)^H$ is $\lfloor \frac{n+1}{|H|} \rfloor - 2$ by Corollary 4.2, and $|H| \leq p^{s-d}$, where p^s is the largest power of p dividing $|G|$. Thus, it suffices to establish the inequality

$$1 + \frac{d}{3} \leq \frac{q}{p^{s-d}}$$

for every prime p dividing $|G|$. Since $q \geq p^s$ by definition, the claim follows from the obvious inequality

$$1 + \frac{d}{3} \leq p^d.$$

$\square$

Proof of Theorem 1.1. Writing $|G| = qr$ with q as above and appealing to Theorem 1.2, we obtain the inequality

$$\frac{\mathrm{acat}(G) + 1}{|G|} \leq \frac{3}{r}.$$

If $r \notin \{1, 2, 3\}$, then the righthand side of this inequality is strictly less than 1, implying the first claim. The second claim is immediate from Proposition 4.3 and the universal upper bound. For the third claim, fix $N > 0$ and a prime $p > 5$, and take s sufficiently large so that $N \leq 2p^s$. In this case, we have $\mathrm{acat}(C_{5p^s}) < 3p^s = |G| - 2p^s \leq |G| - N$. $\square$

5. Analog vs. distributional

The goal of this section is to prove Theorem 1.3, claiming that the analog category of the finite group G coincides with its distributional category in the sense of [DJ24]. We begin by recalling the relevant definitions.[3]

Throughout, we will use the subscript LP to refer to the Lévy–Prokhorov metric; thus, we have $\mathcal{P}_n(X)_{\mathrm{LP}}$ for X metric, and we have $\mathcal{P}_n(f)_{\mathrm{LP}}$ for continuous f with metric source. The rule for turning a definition of an analog invariant into that of a distributional invariant is to add this subscript; thus, the distributional sectional category of $f : X \to Y$ with metric source is the least n for which $\mathcal{P}_{n+1}(f)_{\mathrm{LP}} \to Y$ admits a section, and this definition specializes to the definition of distributional category as in Definition 2.3.

Remark 5.1. Strictly speaking, the definition of distributional sectional category is not contained in [DJ24], but it should be regarded as implicit. An issue requiring slightly more care is that the distributional category is not defined if X is not metrizable—for example, $X = BG$ with G infinite. One possible workaround is to appeal to the fact that any CW complex is metrizable up to homotopy equivalence [Cau74]. Fortunately, since we confine our discussion here to finite groups, the issue does not arise.

Our main technical result comparing these notions is the following.

Proposition 5.2. *Let $f : X \to Y$ be a map with X metric and Y convenient. If f is proper, then the analog and distributional sectional category of f coincide.*

For the proof, we require the following.

Lemma 5.3. *For a metric space X, the identity function $\mathcal{P}_n(X) \to \mathcal{P}_n(X)_{\mathrm{LP}}$ is continuous for every finite $n \geq 0$. If X is compact, then each of these maps is a homeomorphism.*

Note that the first claim of Lemma 5.3 is simply the claim that the quotient topology on $\mathcal{P}(X)$ is finer than the Lévy–Prokhorov topology when both are defined.

Proof of Lemma 5.3. For the first claim, let X be a metric space, and consider the commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_K \mathcal{P}(K) & \longrightarrow & \operatorname{colim}_K \mathcal{P}(K)_{\mathrm{LP}} \\ \downarrow & & \downarrow \\ \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X)_{\mathrm{LP}}, \end{array}$$

where K ranges over compact subspaces of X. The collection of such is filtered, hence sifted, so the lefthand arrow is a homeomorphism by Theorem 2.2. Thus, in order to establish continuity of the bottom arrow, it suffices to establish continuity of the top arrow; in other words, we may assume that X itself is compact. From the definition of $\mathcal{P}(X)$, continuity is equivalent to continuity of each of the maps

$$X^n \times \Delta^{n-1} \longrightarrow \mathcal{P}(X)_{\mathrm{LP}}$$
$$(x, k) \mapsto \sum_{i=1}^{n} t_i \delta_{x_i}.$$

[3] For the sake of easier reading, we depart from the notation of [DJ24].

which is to say sequential continuity, since the source is metric. Since X is separable, because compact, the topology on the target is the topology of weak convergence of measures, so sequential continuity follows from continuity of the composite

$$X^n \times \Delta^{n-1} \xrightarrow{f^n \times \iota} \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\langle -,- \rangle} \mathbb{R},$$

where ι is the inclusion and $f : X \to \mathbb{R}$ is an arbitrary continuous function.

For the second claim, if X is compact, then so is $\mathcal{P}_n(X)$. Since $\mathcal{P}_n(X)_{\mathrm{LP}}$, as a metric space, is Hausdorff, and since the map in question is a continuous bijection, the claim follows. $\square$

Proof of Proposition 5.2. It follows from Lemma 5.3 that the top arrow in the commutative diagram

$$\begin{array}{ccc} \mathcal{P}_n(f) & \xrightarrow{\mathrm{id}} & \mathcal{P}_n(f)_{\mathrm{LP}} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

is continuous, so a section of the lefthand map determines a section of the right. Thus, the distributional sectional category bounds the analog from below. For the reverse inequality, we will show that any section σ of the righthand map is continuous when considered as a map to $\mathcal{P}_n(f)$. By assumption, the space Y is the colimit of its compact subsets; therefore, since $\sigma|_K$ factors through $\mathcal{P}_n(f|_{f^{-1}(K)})_{\mathrm{LP}}$, we may assume without loss of generality that Y itself is compact. In this case, since f is proper, it follows that X is also compact, and Lemma 5.3 implies the claim. $\square$

Lemma 5.4. *For any $n \geq 0$, the functor $\mathcal{P}_n(-)_{\mathrm{LP}}$ preserves homotopy, hence homotopy equivalence.*

Proof. Using the topological basis given in [DJ24, §3.1], it is easy to check that, for any metric space Y, the assignment

$$\left(\sum_{i=1}^{m} t_i \delta_{x_i}, y \right) \mapsto \sum_{i=1}^{m} t_i \delta_{(x_i, y)}$$

defines a continuous map $\mathcal{P}_n(X)_{\mathrm{LP}} \times Y \to \mathcal{P}_n(X \times Y)_{\mathrm{LP}}$. The claim follows easily after taking $Y = [0, 1]$. $\square$

Proof of Theorem 1.3. It suffices to show that the distributional category of BG is the distributional sectional category of the map $\pi : EG \to BG$; indeed, the corresponding analog statement is true, as shown in the course of proving Proposition 2.1, and π is proper, so Proposition 5.2 applies. Considering the diagram

$$\begin{array}{ccc} G & \longrightarrow & (BG, x_0)^{([0,1],\{0,1\})} \\ \downarrow & & \downarrow \\ EG & \longrightarrow & (BG, x_0)^{([0,1],\{0\})} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ BG & = & BG \end{array}$$

from the proof of Proposition 2.1, the claim follows by noting that the construction $f \mapsto \mathcal{P}_n(f)_{\mathrm{LP}}$ preserves Hurewicz fibrations by [DJ24, Prop. 5.1] and that the construction $X \mapsto \mathcal{P}(X)_{\mathrm{LP}}$ preserves homotopy equivalence by Lemma 5.4. □

References

[Ass82] A. Assadi, *Finite Group Actions on Simply-Connected Manifolds and CW Complexes*, American Mathematical Society, 1982.

[Cau74] R. Cauty, *Rétractions dans les espaces stratifiables*, Bull. Soc. Math. Fr. **102** (1974).

[DJ24] A. Dranishnikov and E. Jauhari, *Distributional topological complexity and LS-category*, Topology and AI, European Mathematical Society, 2024.

[Dra] A. Dranishnikov, *Distributional topological complexity of groups*, arXiv:2404.03041.

[Far03] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003).

[GMN03] R. Guralnick, G. Malle, and G. Navarro, *Self-normalizing Sylow subgroups*, Proc. Amer. Math. Soc (2003).

[KW24] B. Knudsen and S. Weinberger, *Analog category and complexity*, SIAM J. Appl. Algebra Geom. (2024).

[MP11] J. P. May and K. Ponto, *More Concise Algebraic Topology*, University of Chicago Press, 2011.

[Ste67] N. Steenrod, *A convenient category of topological spaces*, Mich. Math. J. **14** (1967), no. 2, 133–152.