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Research paper

Regularization estimates of the Landau–Coulomb diffusion ☆

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ABSTRACT

The Landau–Coulomb equation is an important model in plasma physics featuring both nonlinear diffusion and reaction terms. In this manuscript we focus on the diffusion operator within the equation by dropping the potentially nefarious reaction term altogether. We show that the diffusion operator in the Landau–Coulomb equation provides a much stronger $L^1 \to L^\infty$ rate of regularization than its linear counterpart, the Laplace operator. The result is made possible by a nonlinear functional inequality of Gressman, Krieger, and Strain together with a De Giorgi iteration. This stronger regularization rate illustrates the importance of the nonlinear nature of the diffusion in the analysis of the Landau equation and raises the question of determining whether this rate also happens for the Landau–Coulomb equation itself.

1. Introduction and main result

We consider the nonlinear quadratic equation

$$u_t = \mathrm{div}(A[u]\nabla u), \quad x \in \mathbb{R}^d, \quad t > 0, \tag{1.1}$$

where $A[u]$ is a $d \times d$ matrix defined as

$$A[u](x,t) := c_d \int_{\mathbb{R}^d} \frac{\mathbb{P}(x-y)}{|x-y|^{d-2}} u(y,t)\, dy,$$

and $\mathbb{P}(z)$ is the projection matrix over the space perpendicular to z, defined as

$$\mathbb{P}(z) := \mathbb{I} - \frac{z \otimes z}{|z|^2}, \quad z \in \mathbb{R}^d \setminus \{0\}, \quad \text{and} \quad \mathbb{I} \quad \text{identity matrix}$$

Hereafter the dimension d is greater than or equal to 3. Eq. (1.1) represents the "diffusive" part of the homogeneous Landau–Coulomb equation, in divergence form, which reads as

$$u_t = \mathrm{div}(A[u]\nabla u - u\, \mathrm{div} A[u]), \tag{1.2}$$

or alternatively in the non-divergence form

$$u_t = \mathrm{Tr}(A[u]D^2 u) + u^2. \tag{1.3}$$

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The term in (1.2) that is missing in (1.1), namely $-\mathrm{div}(u \ \mathrm{div}A[u])$, had long been known to be the term that fights against the regularization effect from the diffusive term. Accordingly, a complete analysis of the regularization rate for (1.1) allows for understanding the smoothing effects of this term on its own right, and might provide clues on the regularization rate for the homogeneous Landau–Coulomb equation (1.2).

The homogeneous Landau equation (1.2) is a fundamental equation in kinetic theory, playing the role of the Boltzmann equation when the Coulomb force (as in the case of a plasma) is involved (see for instance the classical reference by Villani [1] for a comprehensive introduction). The mathematical analysis of (1.2) has received considerable attention in the past two decades, and we highlight here (organized by topic) some of the contributions to the study of this equation. The collective investigation for (1.2) of the past years has advanced forward the knowledge in several directions: (i) The existence and uniqueness of smooth solutions for short times have been extensively explored. Notably, Golding and Loher's work [2] has established the most comprehensive result for initial data in $L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$. (ii) Global existence and uniqueness of smooth solutions with initial data close to equilibrium has been addressed across various contexts. For initial data small in Sobolev spaces, contributions can be found in [3–5] and related references. The first work that considers initial data in L^∞ is the one by Kim, Guo and Hwang [6]. Golding, Gualdani and Loher in [7] encompassed the problem in all L^p spaces with $p \geq \frac{3}{2}$. Very recently the case $p \geq 1$ was solved in [8,9]. (iii) Conditional regularity. This line of research concerns the investigations of conditions that guarantee global well-posedness of solutions for arbitrarily large times. Silvestre [10] and Gualdani, Guillen [11] showed that if the function $u(x,t) \in L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$ uniformly in time, then it is smooth. Recently, Alonso, Bagland, Desvillettes, and Lods [12] showed that if $u(x,t) \in L^q(0,T,L^p(\mathbb{R}^3))$ for a certain range of q and p, then it is automatically in $L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$ and therefore smooth. Regarding conditional uniqueness, Fournier in [13] showed that solutions which have L^∞ norm integrable in time are unique. Chern and Gualdani [14] showed that uniqueness holds in the class of high integrable functions. (iv) Partial regularity. This line of research for the Landau equation started with Golse, Gualdani, Imbert and Vasseur [15]. In this work it is shown that, if singularities occur, they are concentrated in a time interval that has Hausdorff measure at most $\frac{1}{2}$. Most recently, Golse, Imbert and Vasseur showed that the spatial and temporal domain for singularities to happen has Hausdorff measure $1+\frac{5}{2}$ [16]. (v) Study of modified models that pertain the same difficulties of the Landau equation but seem analytically more tractable. This line of research started with the work of Gressmann, Krieger and Strain [17,18] and their analysis of an isotropic version of (1.2),

$$u_t = a[u]\Delta u + \alpha u^2, \quad \alpha > 0. \tag{1.4}$$

In [17,18] they show that (1.4) is globally well-posed if initial data are radially symmetric and monotonically decreasing and $\alpha \in (0, \frac{74}{75})$. Later, Gualdani and Guillen [19] proved global well-posedness for $\alpha = 1$ also in the case when initial data are radially symmetric and monotonically decreasing. These works where the first to establish that, unlike what happens in the semilinear heat or porous media equations, the nonlinear diffusion $a[u]\Delta u$ is strong enough to overcome the reaction u^2. Later, Gualdani and Guillen [20] showed the isotropic Landau equation with less singular potentials ($\gamma \in (-2.5,-2]$) is also globally well-posed.

These findings lead us to the motivation behind the present manuscript. The proofs in groups (i), (iii) and (iv) primarily rely on the ellipticity estimates provided by the lower bound of $A[u]$. Specifically, if the function u has mass, second moment and entropy bounded, the matrix $A[u]$ is uniformly bounded from below by

$$A[u] \geq \frac{c}{1+|x|^d}\mathbb{I}, \quad x \in \mathbb{R}^d,$$

where c only depends on mass, second moment and entropy of u. While a weighted Laplacian operator is analytically more tractable then the full nonlinear nonlocal diffusion $\mathrm{div}(A[u]\nabla u)$, one might argue that by using the lower bound on $A[u]$ we discard an important element that could actually prevent singularities following the intuition that when u is big so is the diffusion coefficient $A[u]$ and this strength could prevent formation of singularities. However, this intuition has been very difficult to apply in practice. Interestingly, however, is the fact that all the global-well-posedness results for general data mentioned above use the full power of the diffusion operator. These include the results in [17,18], which are a consequence of a novel weighted Poincare inequality

$$\int_{\mathbb{R}^d} u^{p+1}\,dx \leq \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^d} A[u]|\nabla u^{\frac{p}{2}}|^2\,dx,$$

the ones in [19], which use a geometric argument in which the coefficient $a[u]$ plays a pivotal role, and lastly, the ones in [20], proven via new weighted Hardy inequalities of the form

$$\int_{\mathbb{R}^d} (u * |x|^\gamma)u^p\,dx \leq c_{d,\gamma,p} \int_{\mathbb{R}^d} (u * |x|^{\gamma+2})|\nabla u^{\frac{p}{2}}|^2\,dx, \quad \gamma > -d.$$

Lastly, it was already noted in [11] that conditional regularity result for (1.2) shows a rate of regularization much stronger than what is usually expected for regular parabolic equations.

In this manuscript we provide a new and precise quantification of the regularization power of the Landau diffusion operator. Notably, this regularization exhibits a significantly faster rate than that achieved by the Laplacian operator.

Theorem 1.1. *Let $u(t,x)$ be a solution to (1.1) with initial data $0 \leq u_{in}$ that belongs to $L^1_m(\mathbb{R}^d)$ for some $m > 3d(d-2)$ and $d \geq 3$. Then the following estimate holds for any small $\varepsilon > 0$ and any times $0 < t < T$,*

$$\|u\|_{L^\infty(t,\,T,\,L^\infty(\mathbb{R}^d))} \leq \frac{c_\varepsilon}{t^{1+\varepsilon}}, \tag{1.5}$$

Here $c_\varepsilon > 0$ is a constant depending only on ε, d, and the L^1_m-norm of the initial data u_{in}.

Our best bound for the constant c_ε in (1.5) is one that goes to infinity as $\varepsilon \to 0+$, so we cannot obtain a rate exactly with $\varepsilon = 0$. Whether an estimate with $\varepsilon = 0$ holds is an interesting question but one that might not be possible with our approach. We remind the reader that in contrast to this $t^{-1-\varepsilon}$ regularization rate, the heat equation in $\mathbb{R}^d$ has a (sharp) regularization rate for the L^∞ norm of $t^{-d/2}$. This bound when $d \geq 3$ is much larger than the $t^{-1-\varepsilon}$ rate (ε small) when t is small, indicating that the L^∞ norm of f regularizes (that is, comes down from $+\infty$) much faster for (1.1).

In a preprint following the completion of this manuscript [21], the third author and Luis Silvestre prove that the Fisher information for solutions of the Landau–Coulomb equation (1.1) does not increase, and thus solutions starting from smooth data with fast decay must remain smooth for all times. Also recently [22], Chen proved that blow up occurs for the equation obtained by increasing the coefficient in the reaction term in (1.3) by an arbitrary positive amount. These results together with Theorem 1.1 raise the interesting and worthwhile question of whether the fast regularization rate in Theorem 1.1 continues to hold for solutions of the full Landau–Coulomb Eq. (1.2). Similarly it would be worthwhile to investigate similar regularization rates for the Landau equation for very soft potentials.

Lastly, we clarify some notation. In what follows we will denote by $L^1_m(\mathbb{R}^d)$ the space of all $L^1(\mathbb{R}^d)$ functions such that

$$\int_{\mathbb{R}^d} |f|(1+|x|^2)^{m/2}\,dx < +\infty.$$

The proof of Theorem 1.1 can be divided in two stages. In the first one we show a $L^1 \to L^p$ gain of integrability for u, solution to (1.1). This is possible thanks to the nonlinear Sobolev inequality involving $A[u]$ discovered by Gressmann, Krieger and Strain [18] and which captures the strong regularization effects of the Dirichlet form associated to $A[u]$ – this is explained in Section 3. The second stage is a proof in the style of De Giorgi–Nash–Moser theory, which we use to obtain the $L^p \to L^\infty$ part of the estimate — this is the content of Section 4. The combination of these two steps yields (1.5). Some preparatory lemmas are discussed in the next Section 2.

2. Some technical lemmas

We first recall well-known results on the bounds of the diffusion matrix $A[u]$. For the proof of the following lemma, see for example [23, Lemma 3.1].

Lemma 2.1. *There exist positive constants C_0 and c_0 depending on the dimension $d \geq 3$ such that*

$$\|A[u]\|_{L^\infty(\mathbb{R}^d)} \;\leq\; C_0 \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p(d-2)}{d(p-1)}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{2p-d}{d(p-1)}}, \quad p > \frac{d}{2},$$

and

$$\|divA[u]\|_{L^\infty(\mathbb{R}^d)} \;\leq\; c_0 \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p(d-1)}{d(p-1)}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{p-d}{d(p-1)}}, \quad p > d.$$

We will also use the following weighted Sobolev inequality: for f smooth enough and any $1 \leq s \leq \frac{2d}{d-2}$ we have

$$\left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} \langle x\rangle^{-3d}\,dx\right)^{\frac{d-2}{d}} \leq c_1 \int_{\mathbb{R}^d} |\nabla f|^2 \langle x\rangle^{-d}\,dx + c_2 \left(\int_{\mathbb{R}^d} |f|^s\,dx\right)^{2/s}. \tag{2.1}$$

Here, $\langle x\rangle := (1+|x|^2)^{1/2}$. The derivation of (2.1) follows the steps in [7]; where the authors prove it for the case $d = 3$. Furthermore, the constants c_1 and c_2 depend only on the dimension d. We apply (2.1) in order to prove the following interpolation inequality:

Lemma 2.2. *Let $p > 1$ and q such that $p + \frac{2}{d} < q < p\left(1+\frac{2}{d}\right)$. Let m be defined as*

$$m := \frac{3d(d-2)(p-1)}{(d+2)p - dq}.$$

For any g smooth function the following bound holds:

$$\|g\|^q_{L^q(\mathbb{R}^d)} \leq C\left(\|\langle\cdot\rangle^{-\frac{d}{2}} \nabla g^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^d)} + \|g\|^p_{L^p(\mathbb{R}^d)}\right) \|g\|_{L^p(\mathbb{R}^d)}^{p\left(\frac{q-p-\frac{2}{d}}{p-1}\right)} \|g\langle\cdot\rangle^m\|_{L^1(\mathbb{R}^d)}^{\frac{(d+2)p-dq}{d(p-1)}}. \tag{2.2}$$

Proof. We first establish the following interpolation inequality

$$\|g\|^q_{L^q} \leq \|\langle\cdot\rangle^{-\frac{3(d-2)}{p}} g\|^p_{L^{\frac{dp}{d-2}}} \|g\|_{L^p}^{p\left(\frac{q-p-\frac{2}{d}}{p-1}\right)} \|g\,\langle\cdot\rangle^m\|_{L^1}^{\frac{(d+2)p-dq}{d(p-1)}}, \tag{2.3}$$

that holds for any $p + \frac{2}{d} < q < p\left(1+\frac{2}{d}\right)$ and $m = \frac{3d(d-2)(p-1)}{(d+2)p-dq}$. The lemma follows easily once we prove (2.3): use (2.1) with $f = g^{\frac{p}{2}}$ and $s = 2$ to bound the first term on the right hand side of (2.3) and get

$$\|g\|^q_{L^q} \leq C\left(\|\langle\cdot\rangle^{-\frac{d}{2}} \nabla g^{\frac{p}{2}}\|^2_{L^2} + \|g\|^p_{L^p}\right) \|g\|_{L^p}^{p\left(\frac{q-p-\frac{2}{d}}{p-1}\right)} \|g\langle\cdot\rangle^m\|_{L^1}^{\frac{(d+2)p-dq}{d(p-1)}}.$$

Next, to show (2.3) we start with a weighted interpolation

$$\|g\|^q_{L^q(\mathbb{R}^d)} \le \left(\int_{\mathbb{R}^d} g^{\frac{dp}{d-2}} \langle x\rangle^{-\frac{dp\alpha}{\theta q(d-2)}}\, dx\right)^{\frac{\theta q(d-2)}{dp}} \left(\int_{\mathbb{R}^d} g^r \langle x\rangle^{\frac{r\alpha}{(1-\theta)q}}\, dx\right)^{\frac{(1-\theta)q}{r}},$$

with α, θ and r satisfying

$$\begin{cases} \frac{\theta q(d-2)}{dp} + \frac{(1-\theta)q}{r} = 1, \\ \frac{\theta q(d-2)}{dp} = \frac{d-2}{d}, \\ \frac{dp\alpha}{\theta q(d-2)} = 3d. \end{cases}$$

The above system has solutions $\alpha = 3(d-2)$, $\frac{(1-\theta)q}{r} = \frac{2}{d}$, $r = \frac{d}{2}(q-p)$, and $\theta = \frac{p}{q}$, which yield

$$\int g^q\, dx \le \left(\int g^{\frac{dp}{d-2}} \langle\cdot\rangle^{-3d}\, dx\right)^{\frac{d-2}{d}} \left(\int g^{\frac{d}{2}(q-p)} \langle\cdot\rangle^{\frac{3d(d-2)}{2}}\, dx\right)^{\frac{2}{d}}.$$

Let us focus on the last term: once more, use Hölder's inequality and get

$$\begin{aligned} \left(\int g^{\frac{d}{2}(q-p)} \langle\cdot\rangle^{\frac{3d(d-2)}{2}}\, dx\right)^{\frac{2}{d}} &= \left(\int g^{\alpha} g^{\frac{d}{2}(q-p)-\alpha} \langle\cdot\rangle^{\frac{3d(d-2)}{2}}\, dx\right)^{\frac{2}{d}} \\ &\le \left[\left(\int g^p dx\right)^{\frac{\alpha}{p}} \left(\int g^{(\frac{d}{2}(q-p)-\alpha)\beta} \langle\cdot\rangle^{\frac{3d(d-2)}{2}\beta} dx\right)^{\frac{1}{\beta}}\right]^{\frac{2}{d}}, \end{aligned} \tag{2.4}$$

where $\beta := \frac{p}{p-\alpha}$. We choose α such that $(\frac{d}{2}(q-p)-\alpha)\beta = 1$, which implies

$$\alpha = \left(\frac{d}{2}(q-p) - 1\right)\frac{p}{p-1}.$$

Note that $\alpha > 0$; hence $\beta > 1$, requires $q > p + \frac{2}{d}$. Since

$$\beta = \frac{2(p-1)}{(d+2)p - dq},$$

and as p/α has to be strictly greater than 1, we require $q < \left(\frac{d+2}{d}\right)p$. Substitution of α and β in (2.4) yields

$$\left(\int g^{\frac{d}{2}(q-p)} \langle\cdot\rangle^{\frac{3d(d-2)}{2}}\, dx\right)^{\frac{2}{d}} \le \left(\int g^p\, dx\right)^{\frac{q-p-\frac{2}{d}}{p-1}} \left(\int g\langle x\rangle^m\right)^{\frac{(d+2)p-dq}{d(p-1)}}, \tag{2.5}$$

with $m = \frac{3d(d-2)(p-1)}{(d+2)p-dq}$. This proves (2.3) and finishes the proof. □

Remark 2.3. The condition $p + \frac{2}{d} < q < \frac{d+2}{d}p$ indicates that $m := \frac{3d(d-2)(p-1)}{(d+2)p-dq}$ is such that $m > \frac{3d(d-2)}{2}$.

3. $L^1 \to L^p$ gain of integrability

The next theorem shows a $L^1 \to L^p$ gain of integrability for solutions to (1.1). The proof follows almost directly from the nonlinear Poincare's inequality

$$\int_{\mathbb{R}^d} u^{p+1}\, dx \le \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^d} A[u]|\nabla u^{p/2}|^2\, dx, \tag{3.1}$$

first proved in [18]. The gain of integrability we obtain is much faster than the one of the solution to the heat equation, which is of the order of $\frac{1}{t^{\frac{d}{2}\left(1-\frac{1}{p}\right)}}$.

Theorem 3.1. *Let $u(t,x)$ be a solution to (1.1). For any $p > 1$ and for all $T > 0$, we have*

$$\|u\|_{L^\infty(t,T,L^p(\mathbb{R}^d))} \le \frac{c}{t^{1-\frac{1}{p}}},$$

with c a constant depending only on p and $\|u_{in}\|_{L^1(\mathbb{R}^d)}$.

Proof. Multiply (1.1) by $\varphi := u^{p-1}$ and integrate the resulting equation in $\mathbb{R}^d$. Integration by parts yields

$$\partial_t \int u^p\, dx = -\frac{4(p-1)}{p} \int \left\langle A[u]\nabla u^{\frac{p}{2}}, \nabla u^{\frac{p}{2}}\right\rangle\, dx.$$

Inequality (3.1) implies

$$\partial_t \int u^p(x)\,dx \le -\frac{4p(p-1)}{(p+1)^2} \int u^{p+1}(x)\,dx. \tag{3.2}$$

Combining the interpolation inequality

$$\|u\|_{L^p} \le \|u\|_{L^1}^{\theta} \|u\|_{L^{p+1}}^{1-\theta}, \quad \theta = \frac{1}{p^2},$$

with (3.2) yields

$$\partial_t \|u\|_{L^p}^p \le -C\|u\|_{L^p}^{\frac{p^2}{(p-1)}},$$

with $C = \frac{4p(p-1)}{(p+1)^2}\|u_{\text{in}}\|_{L^1}^{-\frac{1}{p-1}}$. Note that the L^1-norm is conserved. Define $y := \|u\|_{L^p}^p$; the solution to the differential inequality

$$y' \le -Cy^{\frac{p}{p-1}},$$

has the bound

$$y \le \frac{1}{\left(y_0^{-\frac{1}{p-1}} + \frac{C}{p-1}t\right)^{p-1}}.$$

This implies that

$$\|u\|_{L^\infty(t,\,T,\,L^p(\mathbb{R}^d))} \le \left(\frac{(p-1)}{C}\right)^{1-\frac{1}{p}} \frac{1}{t^{1-\frac{1}{p}}},$$

and this finishes the proof. □

We also have the following moment estimate:

Lemma 3.2. *Let $u(t,x)$ be a smooth solution to (1.1) in the time interval $[0,T]$ with initial data $u_{in} \in L^1_m(\mathbb{R}^d)$ for some $m \ge 2$. Then there exists a constant c that only depends on T and the L^1_m-norm of u_{in} such that*

$$\sup_{t\in[0,T]} \|u(t,x)\|_{L^1_m(\mathbb{R}^d)} \le c.$$

Proof. We start with $m = 2$. Testing with $\phi = (1+|x|^2)$ and integrating by parts yield

$$\begin{aligned}\partial_t \int u(1+|x|^2)dx &\le 4\int u\,|x||\nabla A[u]|\,dx + 4d\int u\,\mathrm{Tr}(A[u])\,dx\\ &=: \mathcal{J}_1 + \mathcal{J}_2.\end{aligned}$$

Let us first estimate $\mathcal{J}_2$. Applying the first estimate of Lemma 2.1 to $\mathcal{J}_2$, we get

$$\mathcal{J}_2 \le C_0\|u\|_{L^p(\mathbb{R}^d)}^{\frac{p(d-2)}{(p-1)d}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{2p-d}{d(p-1)}+1}. \tag{3.3}$$

Then an application of Theorem 3.1 to the L^p-norm of (3.3), gives

$$\mathcal{J}_2 \lesssim \frac{1}{t^{1-\frac{2}{d}}}\|u\|_{L^1(\mathbb{R}^d)}^{\frac{2p-d}{d(p-1)}+1}. \tag{3.4}$$

Next, we estimate $\mathcal{J}_1$. We have

$$\mathcal{J}_1 \le \|\nabla A[u]\|_{L^\infty} \int u\,(1+|x|^2)\,dx.$$

Apply once more Lemma 2.1 and Theorem 3.1 to get

$$\mathcal{J}_1 \lesssim \frac{1}{t^{1-\frac{1}{d}}} \int u(1+|x|^2)\,dx.$$

Gathering the estimates of $\mathcal{J}_1$ and $\mathcal{J}_2$ together, we acquire the bound

$$\begin{aligned}\partial_t \int u(1+|x|^2)dx &= \mathcal{J}_1 + \mathcal{J}_2\\ &\le \frac{c}{t^{\frac{d-1}{d}}} \int u(1+|x|^2)\,dx + \frac{C}{t^{\frac{d-2}{d}}},\end{aligned}$$

where $c := \|u\|_{L^1(\mathbb{R}^d)}^{\frac{p-d}{d(p-1)}}$ and $C := \|u\|_{L^1}^{\frac{2p-d}{d(p-1)}+1}$. The last inequality is equivalent to the differential inequality

$$y' \le \frac{c}{t^{\frac{d-1}{d}}}y + \frac{C}{t^{\frac{d-2}{d}}}, \tag{3.5}$$

which, after multiplying by $\mu(s) = e^{-ds^{\frac{1}{d}}}$, reduces to

$$(y\mu(s))' \leq \mu(s)s^{-\frac{d-2}{d}},$$

and has solution:

$$y(t) = e^{dt^{1/d}} \left\{ \int_0^t e^{-ds^{1/d}} s^{\frac{2-d}{d}} ds + y_0 e^{-dt^{\frac{1}{d}}} \right\}.$$

Applying the same argument iteratively, we can get the estimate for any $m > 2$. □

Remark 3.3. Thanks to the bound on the second moments from Lemma 3.2, the conservation of mass and the decay of entropy, the matrix $A[u]$ satisfies the following ellipticity condition:

$$A[u](x,t) \geq \frac{c(T)}{\langle x\rangle^d} \quad \text{for any} \quad x \in \mathbb{R}^d,\ t \in [0,T]. \tag{3.6}$$

4. $L^1 \to L^\infty$ gain of integrability

In this section we first show the $L^p \to L^\infty$ gain of integrability for solutions to (1.1). This, combined with the estimate of Theorem 3.1 will conclude the proof of Theorem 1.1. We follow a modification of the De Giorgi iteration previously used in [2,7]. We start with a technical lemma. Let $M > 0$ and $t > 0$; for each $k \in \mathbb{N}$, define

$$C_k := M(1-2^{-k}), \quad T_k := \frac{t}{2}\left(1 - \frac{1}{2^k}\right).$$

Note that M is considered constant with respect to t once t has been fixed.

We denote with $(u-c)_+$ the maximum between 0 and $(u-c)$.

Lemma 4.1. *Let $p > d/2$, $\gamma > 0$ defined as*

$$\gamma = -1 + \frac{2}{d}p - \frac{3}{m}(d-2)(p-1),$$

and $m \geq 2$ such that

$$m > \frac{3d(d-2)}{2} \max\left\{1, \frac{p-1}{p-\frac{d}{2}}\right\}.$$

For each $k \geq 1$ we have the bound

$$\int_{\mathbb{R}^d} (u-C_k)_+^p dx \leq \left(\frac{c_0 2^k}{M}\right)^{1+\gamma} \left(\|\langle\cdot\rangle^{-d/2}\nabla(u-C_{k-1})_+^{\frac{p}{2}}\|_{L^2}^2 + \|(u-C_{k-1})_+\|_{L^p}^p \right) \cdot \|(u-C_{k-1})_+\|_{L^p}^{p\left(\frac{2}{d}-\frac{3}{m}(d-2)\right)} \|(u-C_{k-1})_+\|_{L^1_m}^{\frac{3}{m}(d-2)},$$

with c_0 dimensionless constant.

Proof. Observe that $0 \leq C_{k-1} < C_k$. From this we have

$$0 \leq (u-C_k)_+ \leq (u-C_{k-1})_+. \tag{4.1}$$

Moreover $u - C_{k-1} = u - C_k + C_k - C_{k-1}$. Dividing by $C_k - C_{k-1}$ we acquire on the set $\{u \geq C_k\}$,

$$\frac{u-C_{k-1}}{C_k-C_{k-1}} = \frac{u-C_k}{C_k-C_{k-1}} + 1 \geq 1.$$

This tells us that

$$\mathbb{1}_{\{u-C_k\geq 0\}} \leq \frac{(u-C_{k-1})_+}{C_k-C_{k-1}}.$$

Hence, for any $a > 0$ we have

$$\mathbb{1}_{\{u-C_k\geq 0\}} \leq \left(\frac{(u-C_{k-1})_+}{C_k-C_{k-1}}\right)^a.$$

Multiplying the above inequality by $(u-C_k)_+$ and using (4.1), we deduce

$$(u-C_k)_+ \leq \frac{(u-C_{k-1})_+^{1+a}}{(C_k-C_{k-1})^a} \quad \text{for any} \quad a > 0. \tag{4.2}$$

Chose $a = \frac{1+\gamma}{p}$ for some $\gamma > 0$ to be defined later. Inequality (4.2) implies

$$\int_{\mathbb{R}^d} (u-C_k)_+^p\, dx \leq \left(\frac{2^k}{M}\right)^{1+\gamma} \int_{\mathbb{R}^d} (u-C_{k-1})_+^{p+1+\gamma}\, dx.$$

Lemma 2.2 with $q = 1 + \gamma + p$ yields

$$\int_{\mathbb{R}^d} (u - C_k)_+^p \, dx \le c_0 \left(\frac{2^k}{M}\right)^{1+\gamma} \left(\left\| \nabla (u - C_{k-1})_+^{\frac{p}{2}} \langle \cdot \rangle^{-\frac{d}{2}} \right\|_{L^2}^2 + \|(u - C_{k-1})_+\|_{L^p}^p \right) \cdot \|(u - C_{k-1})_+\|_{L^2}^{p\left(\frac{\frac{d-2}{d}+\gamma}{p-1}\right)} \|(u - C_{k-1})_+\|_{L^1_m}^{\frac{2p-d-d\gamma}{d(p-1)}},$$

with c_0 dimensionless constant and $m = \frac{3d(d-2)(p-1)}{(d+2)p-d(1+\gamma+p)}$. Next, we express γ in terms of m, and get $\gamma = \frac{2}{d}p - 1 - \frac{3(d-2)(p-1)}{m}$, which implies, after substitution in the norms,

$$\int_{\mathbb{R}^d} (u - C_k)_+^p \, dx \le c_0 \left(\frac{2^k}{M}\right)^{1+\gamma} \left(\left\| \langle \cdot \rangle^{-d/2} \nabla (u - C_{k-1})_+^{\frac{p}{2}} \right\|_{L^2}^2 + \|(u - C_{k-1})_+\|_{L^p}^p \right) \cdot \|(u - C_{k-1})_+\|_{L^p}^{p\left(\frac{2}{d} - \frac{3}{m}(d-2)\right)} \|(u - C_{k-1})_+\|_{L^1_m}^{\frac{3}{m}(d-2)}.$$

The constraint $\gamma > 0$ implies $m > \frac{3d(d-2)}{2} \frac{(p-1)}{p-\frac{d}{2}}$. The proof of the lemma is complete after recalling Remark 2.3. □

We are now ready to start the De Giorgi iteration. For any $k \ge 1$ let us define the energy $\mathcal{E}_k$ as

$$\mathcal{E}_k(T_{k+1}, t) := \sup_{\tau \in (T_{k+1}, t)} \int (u - C_k)_+^p(\tau, x) dx + C(p) \int_{T_{k+1}}^t \int \langle x \rangle^{-d} \left| \nabla (u - C_k)_+^{\frac{p}{2}} \right|^2 dx d\tau,$$

and $\mathcal{E}_0$ as

$$\mathcal{E}_0 := \sup_{(t/4, t)} \int_{\mathbb{R}^d} u^p \, dx + C(p) \int_{t/4}^t \int_{\mathbb{R}^d} \langle x \rangle^{-d} \left| \nabla u^{\frac{p}{2}} \right|^2 \, dx \, d\tau, \quad C(p) := \frac{4(p-1)}{p}. \tag{4.3}$$

Lemma 4.2. *Given $p > \frac{d}{2}$, $\gamma = -1 + \frac{2}{d}p - \frac{3(d-2)(p-1)}{m}$ and $m > \frac{3d(d-2)}{2} \max\left\{1, \frac{p-1}{p-\frac{d}{2}}\right\}$. For all $k \ge 1$ we have*
$$\mathcal{E}_k(T_{k+1}, t) \lesssim \frac{1}{t M^{1+\gamma}} \, \mathcal{E}_{k-1}(T_k, t)^{\left(1 + \frac{2}{d} - \frac{3}{m}(d-2)\right)}.$$

Proof. We test (1.1) with $(u - C_k)_+^{p-1}$, integrate in $\mathbb{R}^d \times (s, \tau)$ with $0 \le T_k \le s \le T_{k+1} \le \tau$. After averaging on s between T_k and T_{k+1}, and taking the supremum of τ in (T_{k+1}, t) we get

$$\sup_{\tau \in (T_{k+1}, t)} \int (u - C_k)_+^p(\tau, x) dx + C(p) \int_{T_{k+1}}^t \int A[u] \left| \nabla (u - C_k)_+^{\frac{p}{2}} \right|^2 dx ds \le \frac{1}{T_{k+1} - T_k} \int_{T_k}^t \int (u - C_k)_+^p dx \, ds,$$

which can be also written as

$$\mathcal{E}_k(T_{k+1}, T) \le \frac{1}{T_{k+1} - T_k} \int_{T_k}^t \int (u - C_k)_+^p dx \, ds. \tag{4.4}$$

Since $(T_{k+1} - T_k) = \frac{t}{2^{k+2}}$, we apply the integral bound of Lemma 4.1 to get

$$\begin{aligned}
\mathcal{E}_k &\lesssim \frac{2^{k+1}}{t} \left(\frac{2^k}{M}\right)^{1+\gamma} \sup_{(T_k, t)} \|(u - C_{k-1})_+\|_{L^1_m}^{\frac{3(d-2)}{m}} \sup_{(T_k, t)} \|(u - C_{k-1})_+\|_{L^p}^{p\left(\frac{2}{d} - \frac{3}{m}(d-2)\right)} \\
&\quad \cdot \left[\sup_{(T_k, t)} \|(u - C_{k-1})_+\|_{L^p}^p + \int_{T_k}^t \|\langle \cdot \rangle^{-d/2} \nabla (u - C_{k-1})_+^{\frac{p}{2}}\|_{L^2}^2 \, ds \right] \\
&\le \frac{2^{k+1}}{t} \left(\frac{2^k}{M}\right)^{1+\gamma} \sup_{(T_k, t)} \|(u - C_{k-1})_+\|_{L^1_m}^{\frac{3(d-2)}{m}} \\
&\quad \cdot \left[\sup_{(T_k, t)} \|(u - C_{k-1})_+\|_{L^p}^p + \int_{T_k}^t \|\langle \cdot \rangle^{-d/2} \nabla (u - C_{k-1})_+^{\frac{p}{2}}\|_{L^2}^2 \, ds \right]^{1 + \frac{2}{d} - \frac{3}{m}(d-2)} \\
&= \frac{C^k C_0}{t M^{1+\gamma}} \mathcal{E}_{k-1}^{\left(1 + \frac{2}{d} - \frac{3}{m}(d-2)\right)},
\end{aligned}$$

with $C_0 := \sup_{(0,T)} \|u\|_{L^1_m}^{\frac{3}{m}(d-2)}$ □

For simplicity in the notation, we define $\beta_1 := \frac{2}{d} - \frac{3}{m}(d-2)$. The inequality of the previous lemma shows that, iteratively,

$$\mathcal{E}_k \lesssim \left(\frac{c_0}{t^{\frac{1}{\beta_1}} M^{\frac{(1+\gamma)}{\beta_1}}} \mathcal{E}_0 \right)^{(1+\beta_1)^k}. \tag{4.5}$$

Recall the definition of $\mathcal{E}_0$:

$$\mathcal{E}_0 = \sup_{(t/4,t)} \int u^p(s,x)\,dx + C(p)\int_{t/4}^{t}\int A[u]\left|\nabla u^{\frac{p}{2}}\right|^2\,dx\,ds, \quad C(p) := \frac{4(p-1)}{p}.$$

Since

$$\mathcal{E}_0 \le c \sup_{(t/4,\infty)} \int u^p(s,x)\,dx,$$

where c is a dimensionless constant greater than 2. Theorem 3.1 implies

$$\mathcal{E}_0 \le \frac{c_p}{t^{p-1}},$$

where $c_p \approx (p-1)^{p-1}$ only depends on p and on the L^1-norm of the initial data. Passing to the limit $k \to +\infty$ in (4.5) we obtain

$$u \le M,$$

provided

$$M \gtrsim \frac{c_p}{t^{1+\varepsilon}},$$

with $\varepsilon = \frac{1-\frac{2}{d}}{1+\gamma}$ and c_p only dependent on the L^1_m-norm of the initial data and $p > 1$. Note that $\varepsilon > 0$ can be as small as one wishes by choosing p arbitrarily large. To see this, first note that if p is greater than $d-1$ then $1 < \max\left\{1, \frac{p-1}{p-\frac{d}{2}}\right\} \le 2$. Then, taking $m > 3d(d-2)$, we get

$$\varepsilon = \frac{1-\frac{2}{d}}{1+\gamma} = \frac{1-\frac{2}{d}}{\frac{2p}{d} - \frac{3(d-2)(p-1)}{m}} \le \frac{d-2}{p+1}.$$

This finishes the proof of Theorem 1.1.

Data availability

No data was used for the research described in the article.

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