

LOWER SEMI-CONTINUITY OF LAGRANGIAN VOLUME*

BY

ERMAN ÇİNELİ

*Department of Mathematics, ETH Zürich
Rämistrasse 101, 8092 Zürich, Switzerland
e-mail: erman.cineli@math.eth.ch*

AND

VIKTOR L. GINZBURG

*Department of Mathematics, University of California Santa Cruz
Santa Cruz, CA 95064, USA
e-mail: ginzburg@ucsc.edu*

AND

BAŞAK Z. GÜREL

*BG: Department of Mathematics, University of Central Florida
Orlando, FL 32816, USA
e-mail: basak.gurel@ucf.edu*

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ABSTRACT

We study lower semi-continuity properties of the volume, i.e., the surface area, of a closed Lagrangian manifold with respect to the Hofer- and γ -distance on a class of monotone Lagrangian submanifolds Hamiltonian isotopic to each other. We prove that volume is γ -lower semi-continuous in two cases. In the first one the volume form comes from a Kähler metric with a large group of Hamiltonian isometries, but there are no additional constraints on the Lagrangian submanifold. The second one is when the volume is taken with respect to any compatible metric, but the Lagrangian submanifold must be a torus. As a consequence, in both cases, the volume is Hofer lower semi-continuous.

1. Introduction and main results

1.1. INTRODUCTION. In this paper we are concerned with lower semi-continuity properties of the volume, i.e., the surface area, of a closed Lagrangian manifold with respect to the distance of a purely symplectic topological nature, e.g., the Hofer- and γ -distance, on a class of monotone Lagrangian manifolds Hamiltonian isotopic to each other.

We conjecture that volume is γ -lower semi-continuous in general, and we prove this in two situations. The first one is fairly close to the standard setting of integral geometry. This is the case where the volume form comes from a Kähler metric with a very large group of Hamiltonian isometries, but there are no additional constraints on the Lagrangian submanifold. The second one is in some sense much more general: the volume form is taken with respect to any compatible metric, but the Lagrangian submanifold must be a torus. As a consequence, in both cases, the volume is lower semi-continuous with respect to the Hofer metric.

The question is inspired by the key result from [AM] asserting that in dimension two the topological entropy of a Hamiltonian diffeomorphism is Hofer lower semi-continuous. We find results of this type quite interesting because they connect seemingly unrelated entities existing in completely different realms: pure dynamics or metric invariants such as topological entropy or volume on one side and symplectic topological features on the other.

The second motivation for the question comes from [CGG24a] where topological entropy of compactly supported Hamiltonian diffeomorphisms is connected with Hamiltonian or Lagrangian Floer theory via the so-called barcode entropy which is determined by the growth of the number of not-too-short bars in the filtered Floer complex of the iterates. That paper also provides a natural framework to study the question by connecting Crofton's type (in)equalities from integral geometry with Floer theory. Here we use the notion of Lagrangian tomograph introduced in that paper to show that for a large class of n -densities on a $2n$ -dimensional symplectic manifold the integral over a Lagrangian submanifold is γ -lower semi-continuous. Then these densities are used to match or at least approximate from below the metric n -density.

One can pose a similar question about other metric (or dynamics) invariants, but the choice of volume is quite natural for it already enjoys strong C^0 -lower semi-continuity properties; see [Ce, Fe52] and also [BI, Iv]. It is then only reasonable to ask if there is an analogue in the symplectic setting. In some situations the γ -norm is known to be continuous with respect to the C^0 -topology (see [BHS, Ka, KS]), and hence, at least on the conceptual level, γ -lower semi-continuity is a refinement of C^0 -lower semi-continuity in the symplectic framework.

Another interpretation of our results is that the volume function extends to a lower semi-continuous function on the Humilière completion, i.e., the completion with respect to the γ -distance (see [Hu]), of the class of Lagrangians Hamiltonian isotopic to each other, whenever γ -lower semi-continuity is established.

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1.2. MAIN RESULTS. Let (M^{2n}, ω) be a monotone symplectic manifold which is either closed or sufficiently nice at infinity (e.g., convex) to ensure that the relevant filtered Floer homology is defined; see Section 2.1 and Remark 2.1. Furthermore, let \mathcal{L} be a class of closed monotone Lagrangian submanifolds L of M , Hamiltonian isotopic to each other. We require in addition that the minimal Chern number of L is at least 2.

Recall that the γ -norm of a compactly supported Hamiltonian diffeomorphism φ is

$$\gamma(\varphi) := \inf_H (c(H) + c(H^{\text{inv}})),$$

where the infimum is taken over all compactly supported Hamiltonians H generating φ as the time-one map φ_H of the Hamiltonian isotopy φ_H^t , the Hamiltonian H^{inv} generates the isotopy $(\varphi_H^t)^{-1}$ and c is the spectral invariant associated with the fundamental class $[M]$ (relative infinity when M is not compact); see [Oh05, Sc, Vi92].

The **(ambient) γ -distance** on \mathcal{L} is defined as

$$d_\gamma(L, L') := \inf\{\gamma(\varphi) \mid \varphi(L) = L'\}.$$

This is indeed a distance on \mathcal{L} ; see, e.g., [KS] and references therein.

Example 1.1: Two loops L and L' which are d_γ -close need not be close with respect to the Hausdorff distance. This is the case for instance when L' is obtained from L by growing long but narrow tongues (or tentacles) and, if needed, by a small perturbation to keep L and L' Hamiltonian isotopic.

Alternatively, when the Lagrangian submanifolds L from \mathcal{L} are wide in the sense of [BiC], i.e., $\text{HF}(L) = H(L) \otimes \Lambda$, where Λ is the Novikov ring, one has the **(interior) γ -distance**. It is defined in a similar fashion but now by using Lagrangian spectral invariants; see, e.g., [Le, LZ, KS, Vi92] for further details and references. Among wide Lagrangian submanifolds are the zero section of a cotangent bundle and the “equator” $\mathbb{RP}^n \subset \mathbb{CP}^n$. On the other hand, displaceable Lagrangian submanifolds have $\text{HF}(L) = 0$ and hence are not wide.

In general, the interior γ -distance is bounded from above by the ambient γ -distance which in turn is bounded from above by the Hofer distance. We are not aware of any example where the two γ -distances are different. Our results hold for both the ambient γ -distance and the interior γ -distance, when the latter is defined. We will not distinguish the two distances and will use the same notation d_γ .

Finally, fix a Riemannian metric compatible with ω . Then we have the volume or, to be more precise, the surface area function:

$$\text{vol}: \mathcal{L} \rightarrow (0, \infty)$$

sending L to its surface area, which we refer to as the Lagrangian volume.

We conjecture that vol is lower semi-continuous on \mathcal{L} with respect to the γ -distance, and here we prove this conjecture in two disparate cases. The first of these is where M is Kähler and has a large symmetry group (e.g., $M = \mathbb{C}^n$ or \mathbb{CP}^n).

THEOREM 1.2: *Let M be Kähler and the Riemannian metric be the real part of the Kähler form. Assume furthermore that the group of Hamiltonian Kähler isometries acts transitively on the Lagrangian Grassmannian bundle over M . Then vol is lower semi-continuous on \mathcal{L} with respect to the γ -distance.*

This theorem is proved in Section 3.

When $L = \mathbb{T}^n$, the restrictive condition that M has a large symmetry group can be dropped. In fact, we have a more precise result asserting roughly speaking that for a fixed Lagrangian submanifold L_0 from \mathcal{L} and another Lagrangian submanifold $L \in \mathcal{L}$, which is d_γ -close to L_0 (depending on L_0), the part of L situated C^0 -close to L_0 is at least almost as large as L_0 .

THEOREM 1.3: *Assume that $L_0 = \mathbb{T}^n \in \mathcal{L}$ and let U be an arbitrary open subset containing L_0 . Then the function*

$$\text{vol}_U: \mathcal{L} \rightarrow [0, \infty)$$

sending L to the surface area of $U \cap L$ is lower semi-continuous on \mathcal{L} at L_0 with respect to the γ -distance.

The proofs of Theorems 1.2 and 1.3 rely on a result of independent interest, Theorem 3.1, asserting d_γ -lower semi-continuity of the integral of certain densities and based on a connection between Floer barcodes and Lagrangian tomographs; cf. [CGG24a, CGG24b].

In Theorem 1.3, U and L_0 are tied up by the requirement that $L_0 \subset U$. Although we do not have a proof of this, we expect this requirement to be unnecessary, i.e., that the function vol_U is lower semi-continuous at every point of \mathcal{L} for any open set $U \subset M$. In any event, as an immediate consequence of Theorem 1.3, we have

COROLLARY 1.4: *Assume that $L = \mathbb{T}^n$. Then vol is lower semi-continuous on \mathcal{L} with respect to the γ -distance.*

Note that since the γ -distance is bounded from above by the Hofer distance, in the setting of this corollary or of Theorem 1.2, the vol function is also lower semi-continuous with respect to the Hofer distance.

Remark 1.5: Another consequence of Theorem 1.3 is that $L \cap U \neq \emptyset$. Here, however, a much more precise and general result is available, which, in particular, does not require L to be a torus. Namely, when L is d_γ -close to L_0 , for every point of L_0 the submanifold L intersects a small ball centered at that point. This is an immediate consequence of, for example, [KS, Thm. F]; see also, e.g., [BaC, BiC, Vi22b] for some relevant results.

Example 1.6: The function vol is automatically lower semi-continuous at L when $L \subset M$ is a global volume minimizer in \mathcal{L} . For instance, as is easy to see, this is the case for the zero section of the cotangent bundle equipped with the Sasaki metric. Likewise, the standard $\mathbb{RP}^n \subset \mathbb{CP}^n$ is a volume minimizer with respect to the Fubini–Studi metric on \mathbb{CP}^n ; [Oh90]. In contrast, Theorem 1.2 and Corollary 1.4 assert lower semi-continuity at every point of \mathcal{L} and in the case of the corollary for a broad class of metrics on M .

Denote by $\hat{\mathcal{L}}$ the Humilière completion of \mathcal{L} , i.e., its completion with respect to the γ -distance; cf., [Hu]. Corollary 1.4 and Theorem 1.2 are equivalent to the following result.

COROLLARY 1.7: *Assume that $L = \mathbb{T}^n$ or that M is as in Theorem 1.2. Then vol extends to a lower semi-continuous function on $\hat{\mathcal{L}}$.*

Remark 1.8: We do not know if in general the function vol is bounded away from zero on \mathcal{L} or equivalently on $\hat{\mathcal{L}}$. This is obviously so when vol has a global minimizer as in the setting of Example 1.6. Furthermore, lower bounds for $\text{vol}(L)$ in terms of the displacement energy of L are obtained in [Vi00] when $M = \mathbb{R}^{2n}$ or \mathbb{CP}^n or a cotangent bundle.

2. Preliminaries

2.1. NOTATION AND CONVENTIONS. Throughout this paper we use conventions and notation from [ÇGG24a] and [ÇGG24b]. Referring the reader to [ÇGG24a, Sect. 3] for a much more detailed discussion, here we only touch upon several key points.

All Lagrangian submanifolds L are assumed to be closed and monotone, and in addition we require that the minimal Chern number of L is at least 2. The ambient symplectic manifold M is also assumed to be monotone but not necessarily compact. In the latter case, we assume that M is sufficiently well-behaved

at infinity (e.g., convex) so that the filtered Floer complex and homology can be defined for the pair (L, L') of Hamiltonian isotopic Lagrangians; see Remark 2.1 below and [CGG24a, Rmk. 2.8] for more details.

For the sake of simplicity Floer complexes and homology and also the ordinary homology are taken over the ground field $\mathbb{F} = \mathbb{F}_2$. When L and L' are Hamiltonian isotopic and intersect transversely, we denote by $\text{CF}(L, L')$ the Floer complex of the pair (L, L') . This complex is generated by the intersections $L \cap L'$ over the **universal Novikov field** Λ . This is the field of formal sums

$$\lambda = \sum_{j \geq 0} f_j T^{a_j},$$

where $f_j \in \mathbb{F}$ and $a_j \in \mathbb{R}$ and the sequence a_j (with $f_j \neq 0$) is either finite or $a_j \rightarrow \infty$.

Due to our choice of the Novikov field, the complex $\text{CF}(L, L')$ is not graded. However, fixing a Hamiltonian isotopy from L to L' and “cappings” of intersections we obtain a filtration on $\text{CF}(L, L')$ by the Hamiltonian action. The differential on the complex is defined in the standard way.

Note that the complex breaks down into a direct sum of subcomplexes over homotopy classes of paths from L to L' . Then to define the action filtration on $\text{CF}(L, L')$ we also need to pick a reference path in every homotopy class.

To an \mathbb{R} -filtered, finite-dimensional complex \mathcal{C} over Λ one then associates its barcode \mathcal{B} . In the most refined form this is a collection of finite or semi-infinite intervals, defined in general up to some shift ambiguity. The number of semi-infinite intervals is equal to $\dim_{\Lambda} \text{H}(\mathcal{C})$. A construction of \mathcal{B} most suitable for our purposes is worked out in detail in [UZ] and also briefly discussed in [CGG24a]. For our goals, it is convenient to forgo the location of the intervals and treat \mathcal{B} as a collection (i.e., a multiset) of positive numbers including ∞ . Setting $\mathcal{C} = \text{CF}(L, L')$ we obtain the barcode $\mathcal{B}(L, L')$. With this convention the barcode $\mathcal{B}(L, L')$ is independent of the choices involved in the definition of the action filtration.

The actual definition of \mathcal{B} is not essential for our purposes and its only feature that matters is that it is continuous in the Hamiltonian or the Lagrangian with respect to the C^∞ -topology and even the Hofer norm or the γ -norm. To be more precise, denote by $b_\epsilon = b_\epsilon(L, L')$ the number of bars in the barcode of length greater than ϵ . This is the main ingredient in the definition of barcode entropy; see [CGG24a]. Assume furthermore that Lagrangian submanifolds L , L' and L''

are Hamiltonian isotopic, and $d_\gamma(L', L'') < \delta/2$, and L' and L'' are transverse to L . Then

$$(2.1) \quad b_\epsilon(L, L'') \geq b_{\epsilon+\delta}(L, L').$$

This is a consequence of [KS, Thm. G]; see also [Vi22a]. This property allows one to extend the definition of the barcode and of b_ϵ “by continuity” to the case where the manifolds are not transverse.

Furthermore, assuming that $L \pitchfork L'$ note that

$$(2.2) \quad \dim_\Lambda \text{CF}(L, L') \geq 2b_\epsilon(L, L') - \dim_\Lambda \text{HF}(L).$$

This inequality turns into an equality when ϵ is smaller than the shortest bar, i.e., b_ϵ is the total number of bars $b(L, L')$:

$$(2.3) \quad \dim_\Lambda \text{CF}(L, L') = 2b(L, L') - \dim_\Lambda \text{HF}(L).$$

Remark 2.1 (Conditions on M at infinity): In this remark we touch upon the conditions, in addition to being monotone, that M must satisfy at infinity when it is not compact. If L is wide in the sense of [BiC], i.e., $\text{HF}(L) = \text{H}(L) \otimes \Lambda$, we can work with the interior γ -norm and it is sufficient to assume that M is geometrically bounded. Otherwise, we use the ambient γ -norm. In this case we need to have the filtered Floer homology and the fundamental class spectral invariant defined for compactly supported Hamiltonians H on M . To this end, we can require M to be geometrically bounded and wide in the sense of [Gü], i.e., admitting a proper function $F: M \rightarrow [0, \infty)$ without non-trivial contractible periodic orbits of period less than or equal to one. Indeed, such a function can then be found vanishing on any compact set and used to perturb H at infinity. The resulting Floer (co)homology is well-defined and isomorphic to $\text{H}_c^*(M) \otimes \Lambda$. Hence the required spectral invariant is also defined. Alternatively, we may require M to be convex at infinity; [FS].

2.2. INPUT FROM INTEGRAL GEOMETRY.

2.2.1. *Densities.* Let P be the Stiefel bundle over a manifold M^m , i.e., P is formed by k -frames $\bar{v} = (v_1, \dots, v_k)$. Recall that a k -density \mathfrak{d} on M is a function $\mathfrak{d}: P \rightarrow \mathbb{R}$ such that

$$(2.4) \quad \mathfrak{d}(\bar{v}') = |\det A| \mathfrak{d}(\bar{v}),$$

where A is the linear transformation of the span of \bar{v} sending \bar{v} to \bar{v}' ; see, e.g., [APF98]. Sometimes it is convenient to drop the condition that the

vectors from $\bar{v} = (v_1, \dots, v_k)$ are linearly independent by setting $\mathfrak{d}(\bar{v}) = 0$ otherwise.

Here are several examples of densities: A Riemannian or Finsler metric on M or, more generally, any homogeneous degree-one function $TM \rightarrow \mathbb{R}$ is a one-density. For instance, in self-explanatory notation, the functions $|dx|$, $|dy|$, $|dx| + |dy|$ and $\sqrt{dx^2 + dy^2}$ are 1-densities on \mathbb{R}^2 . Furthermore, for every $k \leq m$ a Riemannian metric gives rise to a k -density \mathfrak{g}_k defined by the condition that $\mathfrak{g}_k(\bar{v})$ is the volume of the parallelepiped spanned by \bar{v} . Thus \mathfrak{g}_m is the Riemannian volume. For a differential k -form α its absolute value $|\alpha|$ is a k -density. The sum of two k -densities is again a k -density.

A k -density \mathfrak{d} can be integrated over a compact k -dimensional submanifold L without requiring L to be oriented or even orientable. Similarly to differential forms, densities can be pulled back and, under suitable additional conditions, pushed forward. When it is defined, the push-forward $\Psi_* \mathfrak{d}$ of \mathfrak{d} by a map $\Psi: M' \rightarrow M$ is characterized by the condition that

$$\int_L \Psi_* \mathfrak{d} = \int_{\Psi^{-1}(L)} \mathfrak{d}.$$

2.2.2. Lagrangian tomographs and Crofton's formula. Among the key tools entering the proofs of our results are Lagrangian tomographs. In this section we briefly discuss the notion following with minor modifications [CGG24a, CGG24b], which in turn is loosely based on [APF98, APF07, GeSm] and also [GuSt05, GuSt13].

For our purposes, a **tomograph** \mathcal{T} comprises the following data:

- a fiber bundle $\pi: E \rightarrow B$ with fiber K ;
- a map $\Psi: E \rightarrow M$, which is required to be a submersion onto its image and an embedding of every fiber $\pi^{-1}(s)$, $s \in B$;
- a smooth measure ds on B .

Here the fiber K is required to be a closed manifold; the base B may have boundary and need not be compact, but if it is not, and hence E is not compact, the submersion Ψ must be proper. Finally, the measure ds is required to be supported away from ∂B . It is essential for what follows that ds is a genuine measure, but not a signed measure, i.e., the integral of a non-negative function is non-negative. The key difference of this definition from the references above is that there Ψ is also a fiber bundle and hence a tomograph is a double-fibration. (The term “tomograph” is not used there.)

Set $L_s := \Psi(\pi^{-1}(s))$ and $\Psi_s := \Psi|_{\pi^{-1}(s)}$ for $s \in B$. Then L_s is a smooth closed submanifold of M and $\dim L_s = \dim K$. We call $d = \dim B$ the dimension of the tomograph. The pull-back/push-forward density

$$(2.5) \quad \mathfrak{d}_{\mathcal{T}} := \Psi_* \pi^* ds$$

is a smooth k -density on M with $k = \text{codim } L_s$. We call $\text{supp } \mathfrak{d}_{\mathcal{T}} \subset \Psi(E)$ the support of the tomograph \mathcal{T} .

Next, let L be a closed submanifold of M such that $\text{codim } L = \dim K$. Set

$$N(s) := |L_s \cap L| \in [0, \infty].$$

Since Ψ is a submersion, $\Psi_s \pitchfork L$ for almost all $s \in B$. Hence $N(s) < \infty$ almost everywhere and N is an integrable function on B . We refer the reader to, e.g., [APF98] for the proof of the following simple but important result.

PROPOSITION 2.2 (Crofton's formula; [APF98]): *We have*

$$\int_B N(s) ds = \int_L \mathfrak{d}_{\mathcal{T}}.$$

Remark 2.3: It is useful to keep in mind that the measure ds does not essentially enter in any of the tomograph requirements: it can be any smooth measure on B supported away from ∂B . The latter condition is imposed to ensure that $\mathfrak{d}_{\mathcal{T}}$ is smooth. In particular, one can always localize the support of \mathcal{T} near L_s with s in the interior of B by localizing the support of ds near s .

Remark 2.4: This interpretation of classical Crofton's formula, which utilizes densities and ultimately goes back to [GeSm], is conceptually quite different from the one based on the surface area or, more generally, the Hausdorff measure associated with a metric as in [Fe69, Thm. 3.2.26]. The latter, of course, can be applied to a much bigger class of subsets than submanifolds. We also note that the term tomograph might be misappropriated here because our tomographs have a limited functionality, determining only the volume but not the shape of the subset; cf. [Ga].

Assume now that M is symplectic of dimension $m = 2n$. We call \mathcal{T} a **Lagrangian tomograph** when all submanifolds $L_s = \Psi(\pi^{-1}(s))$ are Lagrangian and Hamiltonian isotopic to each other. Thus a Lagrangian tomograph is a family of Lagrangian submanifolds L_s which are parametrized by B and meet some additional requirements. Note that $\dim L_s = \dim K = n = k$.

Example 2.5 (Classical Lagrangian tomographs): Assume that M is Kähler and that the group G of Hamiltonian Kähler isometries acts transitively on M . Let K be any closed Lagrangian submanifold of M . Set $E = K \times G$ with π being the projection to the second factor $B = G$ and $\Psi(x, s) := s(x)$, where $x \in K$ and $s \in G$. Finally, we let ds be a Haar measure on G . Then, as is easy to see, we obtain a Lagrangian tomograph. This is essentially the classical setting of Crofton's formula—see the reference cited above. (Usually one replaces the base $B = G$ by the space $G/\text{Stab}(K)$ formed by all images of K in M under G . Here, however, we prefer to keep $B = G$.) This construction applies to $M = \mathbb{C}^n$ and \mathbb{CP}^n and, more generally, to any simply connected homogeneous Kähler manifold.

Example 2.5 produces tomographs with support equal to M and requires M to have a large symmetry group. Here, as in [CGG24a, CGG24b], we are also interested in tomographs supported in a small tubular neighborhood U of a Lagrangian submanifold $L \subset M$ and having L as one of the submanifolds L_s . (Hence, $K \cong L$.) In the setting of the example, this can be achieved by localizing ds as in Remark 2.3. However, local tomographs exist in a much more general setting. Indeed, first note that by the Weinstein tubular neighborhood theorem we can set $M = T^*L$. Furthermore, to construct a local tomograph near L we can, essentially without loss of generality by shrinking the support of ds , assume that $E = L \times B$, where B is a d -dimensional ball, and $L = L_0$ is the image of the fiber over center $0 \in B$. It turns out that such localized Lagrangian tomographs always exist.

LEMMA 2.6 (Lemma 5.6; [CGG24a]): *A Lagrangian tomograph of dimension d supported in U exists if and only if L admits an immersion into \mathbb{R}^d .*

Remark 2.7: In fact, the proof of the lemma shows slightly more: on the infinitesimal level such tomographs (without ds fixed) are in one-to-one correspondence with immersions of L into \mathbb{R}^d .

3. Tomographs and barcodes

The proofs of Theorems 1.2 and 1.3 are based on automatic d_γ -lower semi-continuity of the integral of the pull-back/push-forward density associated with a Lagrangian tomograph, which is in turn a consequence of a connection between Floer barcodes and tomographs.

Thus fix a class \mathcal{L} of Lagrangian submanifolds of M as in Section 1.2 and let \mathcal{T} be a Lagrangian tomograph such that $L_s \in \mathcal{L}$. Denote by $\mathfrak{d}_{\mathcal{T}}$ the pull-back/push-forward density of \mathcal{T} given by (2.5).

THEOREM 3.1: *The function*

$$I_{\mathcal{T}}: L \mapsto \int_L \mathfrak{d}_{\mathcal{T}}$$

is d_{γ} -lower semi-continuous on \mathcal{L} .

Proof. Fix $L \in \mathcal{L}$ and $\eta > 0$. Our goal is to show that

$$I_{\mathcal{T}}(\tilde{L}) \geq I_{\mathcal{T}}(L) - \eta$$

when $d_{\gamma}(L, \tilde{L})$ is small.

Let $\Sigma \subset B$ be the set of all points such that Ψ_s is not transverse to L . Since Ψ is a submersion, this is a closed zero-measure subset of B . Thus there exists a compact subset $B' \subset B \setminus \Sigma$ such that in the notation from Section 2.2.2

$$(3.1) \quad \int_{B'} N(s) ds \geq \int_B N(s) ds - \eta = I_{\mathcal{T}}(L) - \eta,$$

where the equality follows from Crofton's formula (Proposition 2.2). The set B' is obtained by removing from B a sufficiently small neighborhood of Σ .

By construction, $L_s \pitchfork L$ for all $s \in B'$. Since B' is compact, for all $s \in B'$ the shortest bar in the barcode of $\text{CF}(L, L_s)$ is bounded away from zero by some constant $\beta > 0$. Furthermore,

$$N(s) = \dim_{\Lambda} \text{CF}(L, L_s).$$

Take now $\epsilon > 0$ and $\delta > 0$ so small that $\epsilon + \delta < \beta$. In particular,

$$b_{\epsilon+\delta}(L, L_s) = b(L, L_s)$$

and thus, by (2.3),

$$(3.2) \quad N(s) = 2b_{\epsilon+\delta}(L, L_s) - h,$$

where we set $h = \dim_{\Lambda} \text{HF}(L)$ for the sake of brevity.

Assume next that $d_\gamma(L, \tilde{L}) < \delta/2$ and set

$$\tilde{N}(s) := |\tilde{L} \cap L_s|.$$

We have $\tilde{L} \pitchfork L_s$, and hence $\tilde{N}(s) = \dim_\Lambda \text{CF}(\tilde{L}, L_s)$, for almost all $s \in B$. Then

$$\begin{aligned} I_{\mathcal{T}}(\tilde{L}) &= \int_B \tilde{N}(s) ds && \text{by Crofton's formula} \\ &\geq \int_B (2b_\epsilon(\tilde{L}, L_s) - h) ds && \text{by (2.2)} \\ &\geq \int_{B'} (2b_\epsilon(\tilde{L}, L_s) - h) ds && \text{since } B' \subset B \\ &\geq \int_{B'} (2b_{\epsilon+\delta}(L, L_s) - h) ds && \text{by (2.1)} \\ &= \int_{B'} N(s) ds && \text{by (3.2)} \\ &\geq I_{\mathcal{T}}(L) - \eta, && \text{by (3.1)} \end{aligned}$$

which completes the proof of the theorem. \blacksquare

Proof of Theorem 1.2. In the setting of the theorem consider the Lagrangian tomograph \mathcal{T} from Example 2.5. Both the push-forward/pull-back density $\mathfrak{d}_{\mathcal{T}}$ and the metric n -density \mathfrak{g} are invariant under the group G of (Hamiltonian) Kähler isometries. Since G acts transitively on the Lagrangian Grassmannian bundle, these two densities agree up to a factor on the frames spanning Lagrangian subspaces; cf. [APF07, Lem. 5.4]. Hence, the function

$$L \mapsto \int_L \mathfrak{g}$$

is also d_γ -lower semi-continuous on \mathcal{L} by Theorem 3.1. \blacksquare

In general, there is no hope to exactly match a Lagrangian metric density by the push-forward/pull-back density of a localized tomograph as in the proof of Theorem 1.2. However, to prove Theorem 1.3 it is sufficient to loosely bound the density from below and this is done in Theorem 3.2 below.

Let $L = \mathbb{T}^n \subset M^{2n}$ be a Lagrangian torus. Fix a compatible metric on M and denote by \mathfrak{g} the metric density. We consider Lagrangian tomographs \mathcal{T} with fiber K diffeomorphic to L and a ball B^d serving as the base B . Thus $E = L \times B$ and π is the projection to the second factor.

THEOREM 3.2: *For any open set $U \supset L$ and any $\eta > 0$, there exists a Lagrangian tomograph \mathcal{T} as above with $L = L_0 = \Psi(\pi^{-1}(0))$, supported in U and such that*

- (i) $\mathfrak{d}_{\mathcal{T}}|_L = \mathfrak{g}|_L$ pointwise,
- (ii) $\mathfrak{d}_{\mathcal{T}} \leq (1 + \eta)\mathfrak{g}$.

We prove this theorem in Section 4.

Proof of Theorem 1.3. Set $\mathfrak{d} := \mathfrak{d}_{\mathcal{T}}$ for the sake of brevity. By Theorem 3.1 and since \mathcal{T} is supported in U , for any $\delta > 0$,

$$(3.3) \quad \int_{L \cap U} \mathfrak{d} = \int_L \mathfrak{d} \geq \int_{L_0} \mathfrak{d} - \delta$$

when $d_{\gamma}(L_0, L)$ is small. Thus we have

$$\begin{aligned} \int_{L \cap U} \mathfrak{g} &\geq (1 + \eta)^{-1} \int_{L \cap U} \mathfrak{d} && \text{by (ii)} \\ &\geq (1 + \eta)^{-1} \left(\int_{L_0} \mathfrak{d} - \delta \right) && \text{by (3.3)} \\ &= (1 + \eta)^{-1} \left(\int_{L_0} \mathfrak{g} - \delta \right) && \text{by (i).} \end{aligned}$$

Hence, for any $\epsilon > 0$,

$$\int_{L \cap U} \mathfrak{g} \geq \int_{L_0} \mathfrak{g} - \epsilon$$

once $\eta > 0$ and $\delta > 0$ and then $d_{\gamma}(L_0, L)$ are small enough. \blacksquare

4. Proof of Theorem 3.2

We carry out the proof in three steps. In the first step we discuss some preliminaries, then in the second step we introduce the Lagrangian tomographs which are used in the proof, and the last step comprises the actual proof of Theorem 3.2.

STEP 1. Let $\mathbb{T}^n = S^1 \times \cdots \times S^1$ (n times), where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, with angular coordinates $x = (x_1, \dots, x_n)$ and let $M = T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ with coordinates (x, y) . We get a similar decomposition of $T_{(x,y)}M = \mathbb{R}^n \times \mathbb{R}^n$ and we denote the resulting coordinates on this space by (w, u) .

For $k \in \mathbb{Z}^+$, set

$$\Pi_k: \mathbb{T}^n \rightarrow \mathbb{T}^n, \quad \Pi_k(x) = kx = (kx_1, \dots, kx_n)$$

and

$$F_k = k(\Pi_k^*)^{-1} : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{T}^n \times \mathbb{R}^n, \quad F_k(x, y) = (kx, y).$$

We note here that Π_k^* as a map from $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ to itself is not defined, but its inverse $(\Pi_k^*)^{-1}$ is.

Let \mathfrak{d} be an n -density on M . Consider the density

$$\frac{1}{k^n} F_k^* \mathfrak{d} =: \mathfrak{d}_k.$$

Explicitly, for an n -frame $\bar{v} = (v_1, \dots, v_n)$ at (x, y) , write

$$v_i = (w_i, u_i) \in \mathbb{R}^n \times \mathbb{R}^n = T_{(x,y)} M,$$

and

$$\mathfrak{d}(\bar{v}) =: h_{(x,y)}(w_1, u_1, \dots, w_n, u_n).$$

Then

$$\mathfrak{d}_k(\bar{v}) = \frac{1}{k^n} h_{(kx,y)}(kw_1, u_1, \dots, kw_n, u_n) = h_{(kx,y)}(w_1, u_1/k, \dots, w_n, u_n/k).$$

Hence, when \mathfrak{d} is \mathbb{T}^n -invariant, i.e., h is independent of x , we have

$$\mathfrak{d}_k(\bar{v}) = h_{(x,y)}(w_1, u_1/k, \dots, w_n, u_n/k).$$

Setting $P_k(v) := (w, u/k)$ for $v = (w, u)$ and $P_k(\bar{v}) := (P_k v_1, \dots, P_k v_n)$, we can rewrite the above expression as

$$\mathfrak{d}_k(\bar{v}) = \mathfrak{d}(P_k \bar{v}) =: (P_k^* \mathfrak{d})(\bar{v}).$$

In other words, for any invariant density \mathfrak{d} ,

$$(4.1) \quad \frac{1}{k^n} F_k^* \mathfrak{d} = P_k^* \mathfrak{d}.$$

In a similar vein, let P_∞ be the projection to the horizontal direction:

$$P_\infty(v) = (w, 0)$$

for $v = (w, u)$. We conclude that

$$(4.2) \quad \mathfrak{d}_k = P_k^* \mathfrak{d} \rightarrow \mathfrak{d}_\infty := P_\infty^* \mathfrak{d}$$

uniformly on compact sets. In the next step, we explain how to obtain such sequences \mathfrak{d}_k from tomographs.

STEP 2. Let \mathcal{T} be the Lagrangian tomograph

$$(B, ds) \xleftarrow{\pi} B \times \mathbb{T}^n \xrightarrow{\Psi} \mathbb{T}^n \times \mathbb{R}^n,$$

given by

$$(4.3) \quad \Psi(s, x) = (x, \rho_1 \sin(x_1 + \phi_1), \dots, \rho_n \sin(x_n + \phi_n)),$$

where

$$s = ((\rho_1, \phi_1), \dots, (\rho_n, \phi_n))$$

is a point in the polydisk $B = (B^2)^n$ with $\rho_i \leq R$, for all i and some $R > 0$. (Here we think of (ρ_i, ϕ_i) as polar coordinates in the i -th copy of the disk B^2 .) In other words, $L_s = \Psi(s, \mathbb{T}^n)$ is the graph of the one-form

$$(4.4) \quad \alpha_s = \sum_{i=1}^n \rho_i \sin(x_i + \phi_i) dx_i.$$

As the measure ds , we can take any smooth rotationally symmetric (i.e., independent of ϕ_i) measure, which is not identically zero and supported away from the boundary of B . Then, in particular, $\int_B ds > 0$ since ds is by definition non-negative. This data gives an "equivariant" tomograph, and, as a result, the pull-back/push-forward density $\mathfrak{d}_{\mathcal{T}} = \Psi_* \pi^* ds$ is \mathbb{T}^n -invariant.

More precisely, consider the \mathbb{T}^n -action on $B \times \mathbb{T}^n$ defined by

$$\theta(s, x) = ((\rho, \phi - \theta), x + \theta)$$

for $\theta \in \mathbb{T}^n$. Since $\Psi\theta = \theta\Psi$ and $\theta_* \pi^* ds = \pi^* ds$, we also have $\theta_* \Psi_* \pi^* ds = \Psi_* \pi^* ds$. Here the identity $\theta_* \pi^* ds = \pi^* ds$ is a consequence of the assumption that ds is rotationally symmetric. Namely, denote by $\bar{\theta}$ the restriction of θ to B . Then,

$$\pi^* \bar{\theta}_* ds = \pi^* ds$$

by the symmetry and $\theta_* \pi^* ds = \pi^* \bar{\theta}_* ds$ is easy to see since θ and $\bar{\theta}$ are diffeomorphisms (see [APF07, Thm. 6.2] for a more general statement).

Next, define \mathcal{T}_k to be the tomograph given by the data

$$(B, 1/k^n ds) \xleftarrow{\pi} B \times \mathbb{T}^n \xrightarrow{\Psi_k} \mathbb{T}^n \times \mathbb{R}^n,$$

where

$$\Psi_k(s, x) = (x, \rho_1 \sin(kx_1 + \phi_1), \dots, \rho_n \sin(kx_n + \phi_n)).$$

In other words, the tomograph \mathcal{T}_k comprises the graphs of the differential forms $\Pi_k^* \alpha_s / k$. Note that we scaled the measure on B by $1/k^n$.

Remark 4.1: In the framework of Lemma 2.6, \mathcal{T} corresponds to the standard embedding $j: \mathbb{T}^n \hookrightarrow \mathbb{R}^{2n}$ and \mathcal{T}_k corresponds to the immersion $(j \circ \Pi_k)/k$.

LEMMA 4.2: We have

$$\mathfrak{d}_{\mathcal{T}_k} = P_k^* \mathfrak{d}_{\mathcal{T}}.$$

Combining this lemma with (4.2), we observe that

$$\mathfrak{d}_{\mathcal{T}_k} \rightarrow P_\infty^* \mathfrak{d}_{\mathcal{T}}$$

uniformly on compact sets. Therefore, for any $\eta > 0$,

$$(4.5) \quad \mathfrak{d}_{\mathcal{T}_k} \leq (1 + \eta) P_\infty^* \mathfrak{d}_{\mathcal{T}} + \eta \mathfrak{g}$$

when k is large enough, by (2.4).

Before turning to the proof of Lemma 4.2, we note another feature of the density $\mathfrak{d}_{\mathcal{T}}$. Namely, by \mathbb{T}^n -invariance,

$$P_\infty^* \mathfrak{d}_{\mathcal{T}} = \sigma(y) |dx_1 \wedge \cdots \wedge dx_n|$$

for some smooth, compactly supported, non-negative function σ on \mathbb{R}^n . The next lemma asserts that this function attains its maximum at the origin.

LEMMA 4.3: *There exists a constant $c > 0$ such that*

$$P_\infty^* \mathfrak{d}_{\mathcal{T}} \leq c |dx_1 \wedge \cdots \wedge dx_n|$$

with equality along the zero section. In other words, $c := \sigma(0) = \max \sigma$.

Proof. Set $\mathbb{T}_y^n := \mathbb{T}^n \times \{y\} \subset \mathbb{T}^n \times \mathbb{R}^n$. Then

$$(4.6) \quad |L_s \cap \mathbb{T}_y^n| \leq |L_s \cap \mathbb{T}_0^n|$$

for all $y \in \mathbb{R}^n$ and $s \in B$. It follows that

$$(4.7) \quad \int_{\mathbb{T}_y^n} P_\infty^* \mathfrak{d}_{\mathcal{T}} = \int_B |L_s \cap \mathbb{T}_y^n| ds \leq \int_B |L_s \cap \mathbb{T}_0^n| ds = \int_{\mathbb{T}_0^n} P_\infty^* \mathfrak{d}_{\mathcal{T}}$$

for all $y \in \mathbb{R}^n$. We have $c := \sigma(0) = \max_{y \in \mathbb{R}^n} \sigma(y)$. \blacksquare

The following observation we will be useful in the proof of Lemma 4.2.

LEMMA 4.4: *For any density \mathfrak{d} on $\mathbb{T}^n \times \mathbb{R}^n$, we have $F_k^* F_{k*} \mathfrak{d} = k \mathfrak{d}$. Moreover, if \mathfrak{d} is \mathbb{T}^n -invariant, then $F_{k*} F_k^* \mathfrak{d} = k \mathfrak{d}$.*

Proof. Let \mathfrak{d} be an m -density on $M = \mathbb{T}^n \times \mathbb{R}^n$. As above, for an m -frame $\bar{v} = (v_1, \dots, v_m)$ at $(x, y) \in M$, we write $v_i = (w_i, u_i) \in \mathbb{R}^n \times \mathbb{R}^n = T_{(x,y)}M$, and

$$\mathfrak{d}(\bar{v}) =: h_{(x,y)}(w_1, u_1, \dots, w_m, u_m).$$

Then

$$F_{k*}\mathfrak{d}(\bar{v}) = \sum_{i=0}^{k-1} h_{((x+i)/k, y)}(w_1/k, u_1, \dots, w_m/k, u_m)$$

and

$$F_k^*F_{k*}\mathfrak{d}(\bar{v}) = \sum_{i=0}^{k-1} h_{(x,y)}(w_1, u_1, \dots, w_m, u_m) = k\mathfrak{d}(\bar{v}).$$

Now suppose that \mathfrak{d} is \mathbb{T}^n -invariant. Then

$$F_{k*}F_k^*\mathfrak{d}(\bar{v}) = \sum_{i=0}^{k-1} h_{(x+i/k, y)}(w_1, u_1, \dots, w_m, u_m),$$

which is equal to $k\mathfrak{d}(\bar{v})$ since h is independent of x . \blacksquare

Next, we prove Lemma 4.2. We will partially follow the argument given in [APF07, Thm. 6.2].

Proof of Lemma 4.2. Consider the commutative diagram

$$\begin{array}{ccccc} B & \xleftarrow{\pi} & B \times \mathbb{T}^n & \xrightarrow{\Psi_k} & \mathbb{T}^n \times \mathbb{R}^n \\ \text{id} \downarrow & & \downarrow \text{id} \times \Pi_k & & \downarrow F_k \\ B & \xleftarrow{\pi} & B \times \mathbb{T}^n & \xrightarrow{\Psi} & \mathbb{T}^n \times \mathbb{R}^n. \end{array}$$

Observe that π^*ds is \mathbb{T}^n -invariant and $\pi^*ds = (\text{id} \times \Pi_k)^*\pi^*ds$. For the latter we used the commutativity of the first block. Now one can deduce from the proof of Lemma 4.4 that

$$(\text{id} \times \Pi_k)_*\pi^*ds = k\pi^*ds.$$

Also, the commutativity of the second block yields

$$\Psi_*(\text{id} \times \Pi_k)_*\pi^*ds = F_{k*}\Psi_{k*}\pi^*ds.$$

By the above equalities and the definition of pull-back/push-forward density and (2.5),

$$k\mathfrak{d}_{\mathcal{T}} = F_{k*}k^n\mathfrak{d}_{\mathcal{T}_k}.$$

(Recall that in the tomograph \mathcal{T}_k the measure on B is scaled by $1/k^n$.) Applying F_k^* to both sides and using Lemma 4.4, we obtain

$$F_k^* \mathfrak{d}_{\mathcal{T}} = k^n \mathfrak{d}_{\mathcal{T}_k}.$$

On the other hand, since $\mathfrak{d}_{\mathcal{T}}$ is \mathbb{T}^n -invariant,

$$F_k^* \mathfrak{d}_{\mathcal{T}} / k^n = P_k^* \mathfrak{d}_{\mathcal{T}}$$

by (4.1). Thus

$$\mathfrak{d}_{\mathcal{T}_k} = P_k^* \mathfrak{d}_{\mathcal{T}}. \quad \blacksquare$$

Remark 4.5: The homogenization procedure described here is somewhat similar to the one from [Vi23], although we apply it to tomographs and densities rather than Hamiltonians, and the \mathbb{T}^n -invariance condition considerably simplifies the situation. (In the setting of that paper homogenization is trivial for invariant Hamiltonians.) We also note that Lemma 4.2 holds for any equivariant tomograph \mathcal{T} , when \mathcal{T}_k is defined as a tomograph satisfying the condition $F_k \circ \Psi_k = \Psi \circ (\text{id} \times \Pi_k)$ with renormalized measure ds/k^n . It is easy to see that \mathcal{T}_k exists, but it is not unique unless we require that $\Psi_k|_{\pi^{-1}(0)} = \text{id}$. In any case, the density $\mathfrak{d}_{\mathcal{T}_k}$ is independent of the choice of \mathcal{T}_k . On the other hand, Lemma 4.3 relies on (4.6) which is satisfied for the tomograph \mathcal{T} given by (4.3), but not for an arbitrary equivariant tomograph comprising the graphs of exact forms. Finally, the reader has certainly noticed that while \mathcal{T} and \mathcal{T}_k are given by simple and explicit formulas, the proof is quite indirect. The reason is that we do not have an explicit and easy to work with expression for the densities $\mathfrak{d}_{\mathcal{T}}$ and $\mathfrak{d}_{\mathcal{T}_k}$, even for such simple tomographs. (Such an expression would depend on ds .)

Finally, we are in a position to prove Theorem 3.2.

STEP 3. Let now, as in the statement of the theorem, M be the ambient symplectic manifold equipped with a compatible metric and let \mathfrak{g} be the metric n -density. It is clear that in the proof of the theorem we may replace U by any open subset containing $L_0 = \mathbb{T}^n$. Thus, without loss of generality, we can identify U with a neighborhood of the zero section in $T^*\mathbb{T}^n$ so that the fibers are orthogonal to the zero section; this Weinstein tubular neighborhood structure will be used in what follows. Fix some angular coordinates $x = (x_1, \dots, x_n)$ on \mathbb{T}^n and further identify $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ by using these coordinates as above.

Without loss of generality, by Moser's theorem, we can assume that

$$\mathfrak{g}|_{\mathbb{T}^n} = |dx_1 \wedge \cdots \wedge dx_n|,$$

i.e., these two densities agree pointwise on the frames tangent to the zero section. Next, note that, along the zero section (but not necessarily only on the frames tangent to the zero section),

$$|dx_1 \wedge \cdots \wedge dx_n| \leq \mathfrak{g}.$$

To see this, consider the standard Euclidean metric n -density on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Let \bar{v} be the image of the coordinate frame in \mathbb{R}^n under a map of the form

$$(A, B): \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n.$$

Then

$$\mathfrak{g}(\bar{v}) = \sqrt{\det(AA^* + BB^*)} \quad (\text{Pythagorean theorem}).$$

Hence,

$$\mathfrak{g}(\bar{v}) \geq \sqrt{\det(AA^*)} = |\det A| = |dx_1 \wedge \cdots \wedge dx_n|(\bar{v}).$$

We conclude by continuity of the metric density \mathfrak{g} that for all $\eta > 0$ there exists a neighborhood $V \subset U$ of the zero section such that on V we have

$$|dx_1 \wedge \cdots \wedge dx_n| \leq (1 + \eta)\mathfrak{g}.$$

Strictly speaking, continuity only implies that the inequality holds on a compact subset of the Stiefel bundle of TV . However, then it holds for every n -frame tangent to V since densities are homogeneous or, to be more precise, by (2.4).

Let \mathcal{T} be the tomograph discussed in Step 2. Shrink the support of ds so that the density $\mathfrak{d}_{\mathcal{T}}$ is supported in V . (Abusing notation, here and below, as we modify ds and hence \mathcal{T} , we keep the same notation.) Next, divide the measure ds by the constant c provided by Lemma 4.3 so that we have

$$P_{\infty}^* \mathfrak{d}_{\mathcal{T}} \leq |dx_1 \wedge \cdots \wedge dx_n|$$

with equality along the zero section. Combining (4.5) and the previous two inequalities, we see that when k is large

$$\mathfrak{d}_{\mathcal{T}_k} \leq (1 + \eta)P_{\infty}^* \mathfrak{d}_{\mathcal{T}} + \eta \mathfrak{g} \leq (1 + \eta)|dx_1 \wedge \cdots \wedge dx_n| + \eta \mathfrak{g} \leq (1 + 3\eta + \eta^2)\mathfrak{g}$$

on V . Since $\mathfrak{d}_{\mathcal{T}_k}$ vanishes outside V , we have $\mathfrak{d}_{\mathcal{T}_k} \leq (1 + 3\eta + \eta^2)\mathfrak{g}$ on U . Note that $\mathfrak{d}_{\mathcal{T}_k} = \mathfrak{g}$ on n -frames tangent to the zero section as well. To complete the proof, it remains to replace \mathcal{T} by \mathcal{T}_k and change η so that $1 + 3\eta + \eta^2$ becomes $1 + \eta$. ■

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